# Path integrals in collective fields and applications to nuclear and hadron physics

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**Path integrals in collective fields are used to reformulate various models of field theory (nonrelativistic nuclear many-particle model, Abelian gauge theory of massless quarks and gluons, two-dimensional quantum chromodynamics). Observable bound states are taken as the collective fields. The first and second variations of the collective actions give equations for the fermion and boson spectrum of the theories. New diagram rules of perturbation theory are obtained.**

PACS numbers: 11.10.St

## INTRODUCTION

Path integrals[1-3] are widely used in nonrelativistic many-body physics and in relativistic quantum field theory, including its use for quantizing non-Abelian gauge theories.[4] In particular, these integrals give a natural generalization to field theory of the quasi-classical methods of quantum mechanics.[5]

The functional (path integral) technique becomes especially effective in the investigation of collective phenomena for which the ordinary methods of perturbation theory are inapplicable and must be modified. This is the case, for example, in descriptions of collective oscillations in a superconductor,[6] plasma oscillations of electrons interacting through Coulomb forces,[7] or collective motions in a nuclear many-nucleon system.[8]

In relativistic field theory, path integrals can be used for the transition from the local quark-gluon theory to a completely equivalent field theory formulated in terms of observable hadrons,[9,10] which are interpreted as collective degrees of freedom. An important feature of the method considered here of using path integrals in collective fields is the transition to new dynamical degrees of freedom, which, as a rule, are characterized by different quantum numbers. In the majority of cases, the introduction of the collective degrees of freedom considerably simplifies the treatment of many-particle systems. This comes about because the major part of the interaction between the fundamental fields of the theory can be included in the collective modes, so that the residual interaction can be treated perturbatively.

The functional technique is particularly convenient for deriving the Feynman rules of the modified perturbation theory which includes the collective excitations of the system as basic fields of the theory (for example, plasmons in an electron gas, surface and pairing phonons in a nucleus, mesons and diquarks in quark-gluon theory).

The aim of the present review is to demonstrate the effectiveness of the use of functional methods for describing collective degrees of freedom in different branches of modern physics. For clarity, the methods are demonstrated in simplified models (see Secs. 1 and 3). For better understanding, we briefly list the main aspects of the method of functional integration in collective variables. Our point of departure is a Lagrangian of fermions with a two-particle interaction. Such an interaction can arise, for example, through exchange of fundamental particles (gluons). This two-particle interaction (of fourth order in the fermion fields) is then linearized by the introduction of collective (boson) variables. For particle-hole (quark-antiquark) excitations, real fields, which are bilocal in relativistic theory, are used. Particle-particle excitations (diquarks) are described by complex fields. Integration over the fermion variables gives finally a new "effective" action in terms of collective fields, which (to a high degree) is nonlinear and nonlocal. From the priniciple of least action, we then find "classical" equations of motion of the collective fields.

These equations coincide with the Schwinger-Dyson equations[11,12] for the fermion Green's functions in the lowest nontrivial approximation of perturbation theory. The collective action is then expanded around the classical solutions. The second-order term gives the Bethe-Salpeter equation in the ladder approximation, which determines the free collective (boson) excitations. Finally, a modified perturbation theory is derived in terms of the collective fields (loop expansion).

In Sec. 1, this method is applied to a schematic model of a nuclear many-particle system with pairing and particle-hole forces. As will be shown below, the functional method gives a rigorous justification (without the use of perturbation theory) of the so-called nuclear field theory,[13-16] which has hitherto been derived heuristically. Naturally, the corresponding graphical rules of nuclear field theory are obtained. In the framework of this model, one can readily study various many-particle effects, for example, phase transitions and concomitant phenomena. In addition, this method reveals some interesting connections between collective excitation modes in phase transitions.

In Sec. 2, we describe the Abelian gauge theory of interacting massless quarks and vector gluons. In this model, the radiative corrections result in dynamical spontaneous breaking of the chiral $\gamma_5$ invariance.

Finally, in Sec. 3, we investigate the two-dimensional non-Abelian gauge theory of colored quarks and gluons,

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which is known as quantum chromodynamics. It is demonstrated in this model that local quark-gluon theory can be transformed into an infinite-component nonpolynomial field theory in terms of colorless bound states-mesons. The modified perturbation theory arises in the form of a $1/N_c$ expansion ($N_c$ is the number of colors), which is formally very close to the $1/\Omega$ expansion ($\Omega$ is the degeneracy of the single-particle levels) in nuclear field theory.

## 1. SCHEMATIC MODEL FOR NUCLEAR MANY-PARTICLE SYSTEM

We begin by applying the functional technique to the nonrelativistic many-body problem. The effectiveness of the functional method can be demonstrated fairly clearly in a simple schematic model of the nucleus.[8] This comparatively simple model reflects the characteristic features of the nuclear many-particle system.[17] In addition, it reveals interesting relationships between different phase transitions in nuclei (superfluid phases, deformation of the nucleus).

This model makes it possible to explain in a simple manner the basic features of a new type of field theory; this is a field theory for composite fields and is known as *nuclear field theory* (Refs. 13–16, 19, and 21). This theory correctly describes the interaction between elementary and composite (collective) fields. It also overcomes all the problems associated with the identity of the fermions occurring simultaneously in single-particle and collective excitations (Pauli principle). In recent years, nuclear field theory has been successfully used to study the physics of nuclear structure.[19,22,23]

*Model and Method.* We consider a model consisting of $N$ fermions distributed over two single-particle levels, each of which has degeneracy $2\Omega=N$. The single-particle levels are separated by the energy $\bar{\varepsilon}$. The state of the particles is characterized by the quantum numbers $(\sigma, m)$, where the index $\sigma$ takes the two values $\sigma=-1$ and 1 for the lower and upper level, respectively, and $m$ denotes the degenerate states in each level. The fermions interact pairwise through monopole forces [particle-hole and particle-particle]:

$$\left.\begin{array}{l} H=H_{\rm sp}+H_{\rm tb};\quad H_{\rm sp}=\bar{\varepsilon}\sum\limits_{m,\,\sigma}\sigma a^+_{m\sigma}a_{m\sigma}/2;\\ H_{\rm tb}=-k(P^+P+PP^+)/2-\rho\,(A+A^+)^2, \end{array}\right\} \quad (1)$$

where

$$P^+=\sum_{m\sigma}a^+_{m\sigma}a^+_{\overline{m}\sigma};\quad A^+=\sum_{m}a^+_{m,\,1}a_{m,\,-1}. \quad (2)$$

The operators $a^+_{m\sigma}$ and $a_{m\sigma}$ create and annihilate a particle in the state $(\sigma, m)$, respectively. By definition, $|(\sigma,\overline{m})\rangle=T|(\sigma,m)\rangle$, where $T$ is the operation of time reversal. In the unperturbed ground state $|0\rangle$, the lower level is filled (the $N$ fermions occupy the $2\Omega$ substates of the lower level), the upper level is empty, and

$$a_{m1}|0\rangle=a^+_{m,\,-1}|0\rangle=0. \quad (3)$$

Since our system can undergo a phase transition to a superfluid ground state, one could include in the Hamiltonian (1) the term $-\lambda\hat{N}$ in order to guarantee conservation of the particle number $N$ on the average. However, in the considered model the chemical potential $\lambda$ can be set equal to zero.[8] If we include Hartree-Fock self-energy contributions arising from $H_{\rm tb}$ in the renormalization of the single-particle energy $\varepsilon=(\bar{\varepsilon}+2\rho+k)/2$, we can rewrite the Hamiltonian in the form

$$H=\varepsilon\sum_{m\sigma}\sigma a^+_{m\sigma}a_{m\sigma}-k:P^+P:-\rho:(A+A^+)^2:, \quad (4)$$

where the normal product (denoted : :) is defined relative to the Hartree-Fock ground state (3).

The generating functional for the fermion Green's functions of the system is given by the path integral

$$Z[\eta,\eta^+]=N\int DaDa^+\exp\left[{\rm i}\int dt\,\{\mathscr{L}_f(t)+\eta^+a+a^+\eta\}\right], \quad (5)$$

where

$$\mathscr{L}_f(t)=\sum_{m\sigma}a^+_{m\sigma}(t)\,({\rm i}\partial_t-\sigma\varepsilon)\,a_{m\sigma}(t)+k:P^+P:+\rho:(A+A^+)^2: \quad (6)$$

is the Lagrangian corresponding to the Hamiltonian (4); $N$ is an unimportant normalization factor, which is fixed by the requirement $Z[0,0]=1$. The fermion operators $a_{m\sigma}(t)$ and $a^+_{m\sigma}(t)$ and the external sources $\eta_{m\sigma}(t)$ are regarded as anticommuting variables (Grassman variables), which are introduced in the Appendix. The integration over the fermion variables in (5) can be readily performed by linearizing the interaction by means of new dynamical variables (collective fields).

Using the functional identities (see Appendix 3):

$$\exp\left[{\rm i}\int dt\rho\,(A+A^+)^2\right]$$
$$=c_1\int D\Phi\exp\left[{\rm i}\int dt\left\{-\frac{1}{4\rho}\Phi^2(t)+\Phi(t)\,(A+A^+)\right\}\right]; \quad (7)$$

$$\exp\left[{\rm i}\int dtkP^+P\right]$$
$$=c_2\int D\Psi D\Psi^+\exp\left[{\rm i}\int dt\,\{-(1/k)\,\Psi^+(t)\,\Psi(t)+P^+\Psi(t)+\Psi^+(t)\,P\}\right], \quad (8)$$

where $\Phi(t)$ and $\Psi(t)$ are commuting (Bose) variables, we represent the generating functional $Z[\eta,\eta^+]$ in the form

$$Z[\eta,\eta^+]=N_1\int DhDh^+\int D\Phi\int D\Psi D\Psi^+$$
$$\times\exp\left[{\rm i}\int dt\,\{h^+(t)\,G^{-1}h(t)+Q^+h(t)\right.$$
$$\left.+h^+(t)\,Q-(1/k)\,\Psi^+(t)\,\Psi(t)-(1/4\rho)\,\Phi^2(t)\}\right]. \quad (9)$$

Here, for convenience, we have introduced the matrix notation

$$h^+_{m\sigma}=(a^+_{m\sigma};\ a^-_{\overline{m}\sigma});\quad Q^+_{m\sigma}=(\eta^+_{m\sigma};\ -\eta_{\overline{m}\sigma});$$
$$G^{-1}_{\sigma\sigma'}(m;\,t,\,t')=G^{-1}_{\sigma\sigma'}(m;\,t)\,\delta(t-t');$$
$$G^{-1}_{\sigma\sigma'}(m;\,t)=\begin{pmatrix}({\rm i}\partial_t-\sigma\varepsilon)\,\delta_{\sigma\sigma'}+(1-\delta_{\sigma\sigma'})\,\Phi(t);\ \Psi(t)\,\delta_{\sigma\sigma'}\\ \Psi^+(t)\,\delta_{\sigma\sigma'};\ ({\rm i}\partial_t+\sigma\varepsilon)\,\delta_{\sigma\sigma'}-(1-\delta_{\sigma\sigma'})\,\Phi(t)\end{pmatrix}. \quad (10)$$

Integrating over the fermion fields in (9), we obtain

$$Z[Q,Q^+]=N_2\int D\Phi\int D\Psi D\psi^+\exp\,[{\rm i}\,\{S[\Phi,\Psi,\Psi^+]-Q^+GQ\}], \quad (11)$$

where the new effective action $S$ depends only on the collective variables $\Phi$ and $\Psi$:

$$S[\Phi,\Psi,\Psi^+]=\int dt\left\{-\frac{1}{k}\Psi^+(t)\,\Psi(t)\right.$$
$$\left.-\frac{1}{4\rho}\Phi^2(t)-{\rm i}\Omega{\rm tr}\,({\rm lg}\,G^{-1})\,(t,t)\right\}. \quad (12)$$

Here, $G$ is the Green's function of a fermion moving in the collective fields $\Phi(t)$ and $\Psi(t)$. The explicit expression for $G$ can be found in Appendix 1. The equations of motion of the collective fields follow from variation of the action:

$$\Phi_0(t) = -i2\Omega\rho tr\left\{\begin{pmatrix}\hat{I} & 0\\ 0 & -\hat{I}\end{pmatrix} G(t, t')\right\}_{t'=t+0}; \tag{13}$$

$$\Psi_0(t) = -i\Omega k tr\left\{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} G(t, t')\right\}_{t'=t+0}, \tag{14}$$

where

$$\hat{I} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

*Single-Particle Excitations*. We solve Eqs. (13) and (14). For simplicity, we restrict ourselves to the static solution $\Phi_0, \Psi_0$. In this case, Eqs. (13) and (14) reduce to the so-called *Hartree-Fock-Bogolyubov equations*.[25] Using the explicit expression for the Green's function $G(t, t')$ given in Appendix 1, we represent Eqs. (13) and (14) in the form

$$\Phi_0 = \Phi_0\tilde{\rho}/E;\quad \Psi_0 = \Psi_0\tilde{k}/E;\quad (\tilde{\rho} = 4\Omega\rho,\ \tilde{k} = \Omega k), \tag{15}$$

where the quasiparticle energy is

$$E = \sqrt{\varepsilon^2 + \Phi_0^2 + |\Psi_0|^2}. \tag{16}$$

It follows from (16) that $\Phi_0$ and $\Psi_0$ are the energy gaps arising due to the phase transitions in the particle-hole and (or) particle-particle channels. Depending on the values of the coupling constants $\tilde{\rho}$ and $\tilde{k}$, we can distinguish the following cases:

(i) $\tilde{\rho} < \varepsilon, \tilde{k} < \varepsilon, \Phi_0 = \Psi_0 = 0$; the static field configuration $\Phi_0 = \Psi_0 = 0$ is stable;

(ii) $\tilde{\rho} = E > \varepsilon, \tilde{k} < E, \Phi_0 \neq 0, \Psi_0 = 0$; there exists a phase transition in the particle-hole channel, and the field configuration $\Phi_0 = \Psi_0 = 0$ becomes unstable;

(iii) $\tilde{k} = E > \varepsilon, \tilde{\rho} < E, \Phi_0 = 0, \Psi_0 \neq 0$; there exists a phase transition of superconducting type in the particle-particle channel;

(iv) $\tilde{k} = \tilde{\rho} = E > \varepsilon, \Phi_0^2 + |\Psi_0|^2 > 0$; the system undergoes a simultaneous phase transition in both channels.

One can show exactly that these solutions realize a minimum of the collective action $S[\Phi, \Psi, \Psi^+]$ (see, for example, Ref. 8).¹⁾ Solving (13) and (14), we can formulate a modified perturbation theory, which uses the Green's function $G_0 = G(\Phi_0, \Psi_0, \Psi_0^+)$ as an "unperturbed" propagator. For this, we expand the integrand of the functional (11) around the static solution $\Phi_0$:

$$\Phi(t) = \Phi_0 + \Phi'(t);\quad \Phi(t) = \begin{pmatrix}\Phi(t)\\ \Psi(t)\\ \Psi^+(t)\end{pmatrix}. \tag{17}$$

Then the third term in (12) takes the form (in what follows, $\Phi' \equiv \Phi$)²⁾

$$-i\Omega\int dt\,\mathrm{tr}\,(\lg G^{-1}(t, t)) = -i\Omega\int dt\,\mathrm{tr}\,(\lg G_0^{-1})(t, t) + \sum_n L_n[\Phi], \tag{18}$$

where the term

$$L_n[\varphi] = -i\Omega\,\mathrm{tr}\int dt\left\{\frac{(-)^{n+1}}{n}\left[G_0\begin{pmatrix}\Phi\hat{I} & \Psi\\ \Psi^+ & -\Phi\hat{I}\end{pmatrix}\right]^n\right\}(t, t) \tag{19}$$

represents closed fermion loops with the emission or absorption of the collective lines $\Phi$ and $\Psi$. The loop processes described by the term $L_2[\Phi]$ are usually included in the free action:

$$S_{\text{free}}[\Phi] = \int dt\left\{-\frac{1}{k}\Psi^+(t)\Psi(t) - \frac{1}{4\rho}\Phi^2(t)\right\} + L_2[\Phi]. \tag{20}$$

Thus,

$$S[\Phi] = S_{\text{free}}[\Phi] + S_{\text{int}}[\Phi];\quad S_{\text{int}}[\Phi] = \sum_{n=3}^{\infty} L_n[\Phi]. \tag{21}$$

The free action can be rewritten in the form

$$S_{\text{free}}[\Phi] = \frac{1}{2}\int dt\,dt'\Phi(t)\,\hat{T}^{-1}(tt')\,\Phi(t'), \tag{22}$$

where the propagator of the collective fields

$$\hat{T}(t, t') = \langle\Phi(t)\,\Phi(t')\rangle_{S_{\text{free}}}$$

[$\langle\ldots\rangle_{S_{\text{free}}}$ denotes functional averaging with weight factor $\exp(iS_{\text{free}})$] is determined by the matrix

$$\hat{T} = -[1 + \hat{V}\hat{B}]^{-1}\hat{V}/2, \tag{23}$$

where

$$\hat{V} = \begin{pmatrix}4\rho & 0 & 0\\ 0 & 0 & 2k\\ 0 & 2k & 0\end{pmatrix} \tag{24}$$

is the matrix of coupling constants, and $\hat{B}$ is the matrix whose elements are given by loop graphs composed of normal or anomalous Green's functions (see Appendix 2). The collective propagator $\hat{T}$ is identical with the two-channel amplitude for particle-particle and particle-hole scattering in the ladder approximation. This is readily seen by rewriting (23) as an inhomogeneous Bethe-Salpeter equation:

$$\hat{T} = -\hat{V}/2 - \hat{V}\hat{B}\hat{T}. \tag{25}$$

The Euler-Lagrange equation of motion for the free collective fields $\Phi(t)$ follows from variation of the free action

$$\Phi(t) = -\hat{V}\int dt'\,\hat{B}(t, t')\,\Phi(t') \tag{26}$$

and is identical with the homogeneous Bethe-Salpeter equation.

*Modes of Collective Excitations*. We discuss the spectrum of collective (boson) excitations. The eigenfrequencies $\omega_n$ of the collective modes are obtained by solving the homogeneous Bethe-Salpeter equation (26). Fourier transformation of (26) gives the eigenvalue equation

$$\det[1 + \hat{V}\hat{B}(\omega)] = 0, \tag{27}$$

where

$$\hat{B}(\omega) = \frac{1}{\omega^2 - 4E^2}\hat{b} \tag{28}$$

and the matrix $\hat{b}$ is defined as (see Appendix 2)

$$\hat{b} = \frac{\Omega}{2E}\begin{pmatrix}8(\varepsilon^2 + |\Psi_0|^2) & -4\Psi_0^+\Phi_0 & -4\Psi_0\Phi_0\\ -4\Psi_0^+\Phi_0 & -2\Psi_0^{+2} & 2(E^2 + \varepsilon^2 + \Phi_0^2)\\ -4\Psi_0\Phi_0 & 2(E^2 + \varepsilon^2 + \Phi_0^2) & -2\Psi_0^2\end{pmatrix}. \tag{29}$$

Equation (27) has the roots

$$\omega_1 = 2\sqrt{E^2 + x_1/4E};\quad \omega_2 = 2\sqrt{E(E - \tilde{k})};\quad \omega_3 = 2\sqrt{E^2 + x_3/4E},$$

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¹⁾In the functional treatment of superconductivity,[6] variation of the corresponding collective action gives an equation for the gap analogous to (14). In this case, the Bogolyubov–BCS theory[26,27] is simply the classical limit of quantum theory in collective variables.

²⁾Matrix multiplication here includes integration over intermediate times.

where

$$x_{3,1} = -2[\tilde{\rho}(\varepsilon^2 + |\Psi_0|^2) + \tilde{k}(\varepsilon^2 + \Phi_0^2)] \pm 2\sqrt{[\tilde{\rho}(\varepsilon^2 + |\Psi_0|^2) + \tilde{k}(\varepsilon^2 + \Phi_0^2)]^2 - 4\tilde{\rho}\tilde{k}\varepsilon^2 E^2} \quad (x_{3,1} < 0).$$

Depending on the various possibilities of phase transitions discussed above, we obtain the eigenfrequencies

$$(i)\ \omega_1 = 2\sqrt{\varepsilon(\varepsilon - \tilde{\rho})} \text{ — is a surface vibration;}$$

$$\omega_2 = \omega_3 = 2\sqrt{\varepsilon(\varepsilon - \tilde{k})} \text{ — is a pairing vibration;}$$

$$(ii)\ \omega_1 = 2|\Phi_0|,\ \omega_2 = \omega_3 = 2\sqrt{E(E - \tilde{k})};$$

$$(iii)\ \omega_1 = 2\sqrt{E(E - \tilde{\rho})},\ \omega_2 = 0,\ \omega_3 = 2|\Psi_0|;$$

$$(iv)\ \omega_1 = \omega_2 = 0,\quad \omega_3 = 2\sqrt{\Phi_0^2 + |\Psi_0|^2}.$$

Case (i) gives the modes (phonons) well known in the random phase approximation that describe surface and pairing vibrations of the system with the normal ground state. If $\tilde{\rho} > \varepsilon$ and (or) $\tilde{k} > \varepsilon$, the frequencies $\omega_n$ become imaginary, which leads to exponential growth of the wave function ($\exp(-i\omega_n t) \to \infty$). The probability for a phase transition of the system to a new ground state containing Cooper pairs and (or) particle-hole pairs increases infinitely, and the normal ground state becomes unstable. This picture agrees with the simultaneous appearance of a gap in the single-particle spectrum if $\tilde{\rho} > \varepsilon$ and (or) $\tilde{k} > \varepsilon$. Case (ii) corresponds to a phase transition in the particle-hole channel. The frequency of the surface phonons is twice the gap $|\Phi_0|$. Note that a phase transition in the particle-hole channel decreases the ratio $\tilde{k}/E(\tilde{k}/E < \tilde{k}/\varepsilon)$, which characterizes the extent to which the pairing phonons are collective. Thus, the phase transition in the particle-hole channel removes the collectivization from the particle-particle channel. Similar results are also valid for phase transitions in the particle-particle channel [case (iii)]. However, in this case there is a solution with zero frequency $\omega = 0$ (Goldstone boson), which corresponds to azimuthal motion with zero energy in the complex plane of $\Psi_0: |\Psi_0|^2 = R^2$. As is well known, such a solution indicates spontaneous symmetry breaking in the new ground state (in the present case, breaking of the conservation of the particle number). In case (iv), we observe a simultaneous phase transition in both channels, which leads to azimuthal motion with zero energy on the sphere $\Phi_0^2 + |\Psi_0|^2 = R^2$.

*Loop Expansion.* We formulate the modified perturbation theory, which uses the collective propagator $\hat{T}$ (23) and the quasiparticle propagator $G_0$ as "free" propagators of the theory. For this purpose, it is convenient to introduce the term $ij\Phi$ ($j$ is the source of the field $\Phi$) in the exponential of Eq. (11) and write the generating functional in the form

$$Z[Q, Q^+] = N_3 \exp\left\{i\left[S_{\text{int}}\left[\frac{\delta}{i\delta j^t}\right] - Q^+ G\left(\Phi_0 + \frac{\delta}{i\delta j^t}\right) Q\right]\right\} \times \int D\Phi \exp\{i[S_{\text{free}}[\varphi] + j^t\Phi]\}_{j=0}.$$

Then the integral over $\Phi$ is of Gaussian type and can be calculated, which gives

$$Z[Q, Q^+] = N_4 \exp\left\{i\left[S_{\text{int}}\left[\frac{\delta}{i\delta j^t}\right] - Q^+ G\left(\Phi_0 + \frac{\delta}{i\delta j^t}\right) Q\right]\right\} \exp[-ij^t \hat{T} j/2]_{j=0}. \quad (30)$$

Using the identity

$$F\left(-i\frac{\partial}{\partial x}\right) G(x) = \left[G\left(-i\frac{\partial}{\partial y}\right) F(y) \exp(ixy)\right]_{y=0},$$

where $F$ and $G$ are two arbitrary functions, we write the generating functional in the form

$$Z[Q, Q^+] = N_4 \exp\left[\frac{1}{2} i \frac{\delta}{\delta M^t} \hat{T} \frac{\delta}{\delta M}\right] \times \exp\left[-i\frac{\delta}{\delta N} G(\Phi_0 + M)\frac{\delta}{\delta N^+}\right] \exp[i\{S_{\text{int}}(M) + Q^+ N + N^+ Q\}_{\substack{M=0 \\ N=N^+=0}}]. \quad (31)$$

Equation (31) expresses in compact form the Feynman rules of the loop expansion discussed above. In this expansion, $G(\Phi_0 + M)$ and $S_{\text{int}}$ (M) are given by open and closed fermion lines, which emit and absorb collective lines described by the propagator $\hat{T}$ (Fig. 1). Note that in such a picture all the fermion loops of the type of self-energy corrections are absent. There remain only fermion loops from $S_{\text{int}}$, which represent the effective interaction $\Phi^n$ of the collective fields.

Such interaction terms lead to anharmonic effects in the spectrum of collective excitations. It should be noted that the frequencies of the phonons and the magnitude of the gap contain only the scale-invariant coupling constants $\tilde{k} = \Omega k$ and $\tilde{\rho} = 4\Omega\rho$, which are assumed to be fixed. If we bear in mind the fact that the residue of the collective propagator at the pole $\omega = \omega_n$ contains the factor $\Omega$ [see (29)], we find an effective coupling constant at the particle-phonon vertex of order $\Omega^{-1/2}$. Further, for each closed fermion loop there is an additional factor $\Omega$, which arises from the summation over the index $m$. Thus, we obtain an expansion in a perturbation series in $1/\Omega$, which for sufficiently large values of the degeneracy $\Omega$ (or, equivalently, the fermion number $N = 2\Omega$) converges much better than ordinary perturbation theory in $k$ and $\rho$.

*Nuclear Field Theory.* In the modified perturbation theory based on the generating functional (31) ("loop expansion"), the fermion variables are completely

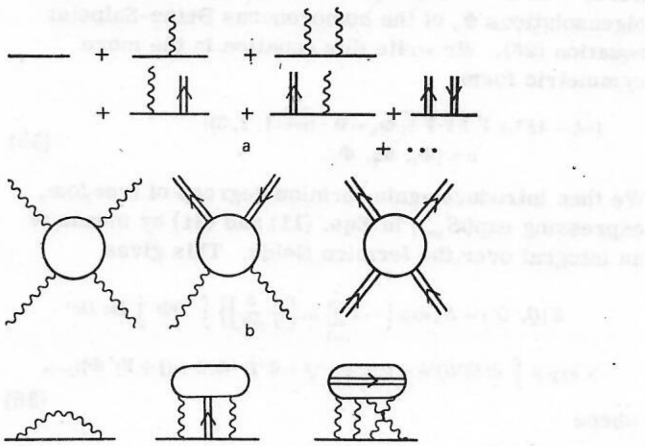


FIG. 1. Diagrammatic representation of the single-particle Green's function in terms of the collective fields $\Phi(t)$ and $\Psi(t)$ (a); some typical loop graphs of order $1/\Omega$ resulting from the expansion of $\exp(iS_{\text{int}})$ (b); and diagrams that contribute to the total single-particle Green's function (c). The continuous line denotes the "free" Green's function $G_0 = G(\Phi_0, \Psi_0, \Psi_0^+)$; $\Phi(t)$ is denoted by the wavy line, and $\Psi(t)$ by the continuous line.

eliminated. A memory of them is contained only in the external sources. On the other hand, it is well known that single-particle and collective excitations (phonons) are observed in the spectrum of nuclear excitations. The interconnection between the single-particle and collective degrees of freedom is important for the low-energy behavior of the nucleus. It is therefore desirable to have a theory that includes explicitly the elementary fermion field and the composite phonon field together with their coupling. The phonon-particle coupling constant can be determined directly from experiment (see Refs. 19 and 23). Therefore, in what follows we shall reformulate the theory in such a way that the unperturbed basis is given by the product of the spaces of the quasiparticles (Fock space) and the collective bound states (space of phonons). This basis is undoubtedly overcomplete, and the Pauli principle is also violated. Both these defects are eliminated in the so-called *nuclear field theory* (see Refs. 13, 14, 19, and 21). In what follows, we obtain for this model the Lagrangian of nuclear field theory together with the corresponding Feynman rules. (The derivation of nuclear field theory for the general case is given, for example, in Refs. 21 and 28.)

We begin by remarking that the collective field $\Phi$ used in the loop approximation is not actually the phonon field itself, since the propagator $\hat{T}$ contains the contact term $\hat{V}$. We therefore subtract this contact term from the $\hat{T}$ matrix. This gives

$$\hat{T} = -\hat{V}/2 + \hat{T}_c, \tag{32}$$

where[3)]

$$\hat{T}_c = K^t \hat{D} K; \quad K = \sqrt{1/2}\, u^t \sqrt{\hat{b}}\, \hat{V}; \tag{33}$$

$$\hat{D} = \begin{bmatrix} \frac{1}{\omega^2-\omega_1^2} & 0 & 0 \\ 0 & \frac{1}{\omega^2-\omega_2^2} & 0 \\ 0 & 0 & \frac{1}{\omega^2-\omega_3^2} \end{bmatrix}. \tag{34}$$

Here, $u$ is an orthogonal matrix constructed from the eigensolutions $\Phi_n$ of the homogeneous Bethe-Salpeter equation (26). We write this equation in the more symmetric form

$$[\omega_n^2 - 4E^2 + \sqrt{\hat{b}}\,\hat{V}\sqrt{\hat{b}}]\,\Phi_n = 0 \quad (n=1,\,2,\,3); \qquad u = (\Phi_1,\ \Phi_2,\ \Phi_3). \tag{35}$$

We then introduce again fermion degrees of freedom, expressing $\exp(iS_{\text{int}})$ in Eqs. (11) and (21) by means of an integral over the fermion fields. This gives

$$Z[Q,\ Q^+] = N_5 \exp\Big\{-\mathrm{i}\sum_{i=1}^{2} L_i\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta \mathbf{j}^t}\Big]\Big\} \int D\Phi \int Da\, Da^+ \times \exp\Big[\mathrm{i}\int dt\, \{h^+G_0^{-1}h + Q^+h + h^+Q + \Phi^t \hat{T}^{-1}\Phi/2 + (\mathbf{j}+\mathbf{P})^t\,\Phi\}\Big]_{\mathbf{j}=0}, \tag{36}$$

where

$$\mathbf{P} = \begin{pmatrix} A+A^+ \\ P^+ \\ P \end{pmatrix}.$$

We use here the original fermion variables corresponding to the free single-particle states [see Eq. (1)]. However, for practical application, it is more convenient to make a unitary transformation with respect to the fermion variables (similar to the Bogolyubov transformation to quasiparticles[27,25]) to make the Green's function of the quasiparticles (A.5), (A.9), and (A.11) a diagonal matrix.

After integration over $\Phi$, we obtain

$$Z[Q,\ Q^+] = N_6 \exp\Big\{-\mathrm{i}\sum_{i=1}^{2} L_i\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta \mathbf{j}^t}\Big]\Big\}\int Da\, Da^+ \times \exp\Big[\mathrm{i}\int dt\,\Big\{\mathscr{L}_f + \frac{1}{4}\mathbf{j}^t\hat{V}\mathbf{j} + \frac{1}{2}\mathbf{j}^t\hat{V}\mathbf{P} + Q^+h + h^+Q - \frac{1}{2}[(\mathbf{j}+\mathbf{P})^t K^t]\,\hat{D}\,[K(\mathbf{j}+\mathbf{P})]\Big\}\Big]_{\mathbf{j}=0}, \tag{37}$$

where

$$\mathscr{L}_f = h^+G_0^{-1}h + P^t\hat{V}P/4 \tag{38}$$

is the Lagrangian for the fermions. To linearize the last term in the exponential (37), we can by analogy with (7) introduce new collective fields $\varphi(t)$, whose propagator is the Green's function $\hat{D}$ [see (34)]. This gives

$$Z[Q,Q^+] = N_7 \exp\Big\{-\mathrm{i}L_2\Big[\frac{1}{\mathrm{i}}\frac{\delta}{\delta\mathbf{j}^t}\Big]\Big\}\int D\varphi\int Da Da^+ \times \exp\Big[\mathrm{i}\int dt\,\Big\{\mathscr{L}_{NFT} + \frac{1}{4}\mathbf{j}^t\hat{V}\mathbf{j} + j^t\Big(\frac{1}{2}\hat{V}\mathbf{P} + K^t\varphi\Big) + Q^+h + h^+Q\Big\}\Big]_{\mathbf{j}=0}$$

or, equivalently,

$$Z[Q,\ Q^+] = \overline{N}\exp\Big\{-\mathrm{i}\sum_{i=1}^{2} L_i\Big[\frac{1}{2}\hat{V}\overline{P} + \frac{1}{\mathrm{i}}\frac{\delta}{\delta\mathbf{j}^t}\Big]\Big\}\int D\varphi\int Da Da^+ \times \exp\Big[\mathrm{i}\int dt\,\{\mathscr{L}_{NFT} + Q^+h + h^+Q + \mathbf{j}^tK^t\varphi\}\Big]_{\mathbf{j}=0}, \tag{39}$$

where the Lagrangian

$$\mathscr{L}_{NFT} = \mathscr{L}_f + \mathscr{L}_b + \mathscr{L}_{pv} \tag{40}$$

is the effective Lagrangian of nuclear field theory. It includes the total fermion Lagrangian $\mathscr{L}_f$, which is defined by (38) or, in the absence of phase transitions, by (6), the free boson Lagrangian $\mathscr{L}_b$, and the interaction part $\mathscr{L}_{pv}$:

$$\mathscr{L}_b = \frac{1}{2}\varphi^t\hat{D}^{-1}\varphi; \tag{41}$$

$$\mathscr{L}_{pv} = \mathbf{P}^tK^t\varphi. \tag{42}$$

The expressions $\overline{P}$ appearing in the argument of $L_2[\dots]$ are obtained from P by replacing the variables $a$ and $a^+$ by the functional derivatives $\delta/i\delta\eta^+$ and $\delta/i\delta\eta$; $K$ is the vertex describing the interaction of the vibrations and the particles. The collective fields are quantized as in Ref. 30. The functional derivation of the Lagrangian of nuclear field theory given above gives simultaneously the graphical rules for the diagrammatic perturbation theory based on this Lagrangian. The factor in front of the integral in (39) eliminates the Hartree-Fock-Bogolyubov self-energy insertions ($i=1$) and the loop diagrams ($i=2$) (Figs. 2a and 2b). These graphs are already included in the

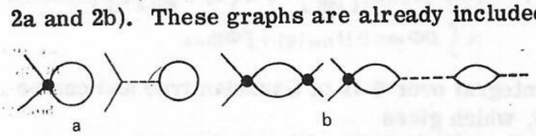


FIG. 2. Diagrams that must be eliminated from the diagrams of the nuclear loop approach: a) Hartree-Fock-Bogolyubov insertion; b) loop diagrams. The black dot denotes the two-particle interaction, and the dashed line denotes the phonon.

3)The square root of the matrix $\hat{b}$ is defined as usual by the equation $\sqrt{\hat{b}} = W^t\sqrt{\hat{b}_{\text{diag}}}\,W$, where $W$ is the matrix that diagonalizes $\hat{b}$. The index $t$ denotes transposition.

definition of a particle (quasiparticle) and the collective (phonon) excitations.

Thus, we have eliminated from the theory the double allowance (overallowance) for the corrections that are already included in the effective Lagrangian (40). Because of the condition $j=0$, the generating functional (40) describes bosons that appear only in the intermediate state. The natural generalization of (40) that also takes into account external boson states is obtained by giving up the condition $\mathbf{j}=0$. It is then necessary to restrict oneself to the use of external fermion sources in such a manner that fermion configurations do not arise in the external states that could be replaced by phonon states. This restriction is necessary to eliminate the overcompleteness of the basis, which includes fermion and phonon fields (for a more detailed discussion of the problem associated with external states in nuclear field theory, see Refs. 21 and 29).

## 2. DYNAMICAL BREAKING OF CHIRAL SYMMETRY IN THE ABELIAN GAUGE THEORY OF MASSLESS QUARKS AND GLUONS

According to current ideas, the observed hadrons, the particles which participate in strong interactions, are bound states of unobservable quarks and gluons. By analogy with the nonrelativistic case, one can construct quark field models in which the hadrons play the part of collective fields. In the present section, we consider one such model—*the Abelian gauge theory of massless quarks and gluons*. As was shown above, the introduction of collective fields may be accompanied by spontaneous breaking of the symmetry of the theory introduced at the level of the elementary fields. This symmetry breaking occurs dynamically without the introduction of elementary Higgs fields,[31] whose part is played by the collective excitations.

Such a point of view was expressed many years ago by Nambu and Jona-Lasinio[32] (for the model with a four-fermion interaction) in the description of spontaneous breaking of chiral symmetry dynamically by analogy with the theory of superconductivity. The papers of Ref. 32 stimulated a large number of investigations in this direction.[33]

In this section, we also study the problem of dynamical breaking of chiral $\gamma_5$ symmetry in the Abelian gauge model of interacting quarks and vector gluons. The mass-like quark term that can arise through radiative corrections and break the original chiral symmetry is a scalar quantity, and the main idea is to go over to new bilocal dynamical variables containing scalar components. For this, path integrals are again used.

Calculation of the path integrals with respect to the bilocal variables by the method of stationary phase leads to "classical" equations, which are identical with the Schwinger-Dyson equation in differential form. We discuss the solution of this equation that was proposed some years ago as the point of departure for constructing "finite" quantum electrodynamics.[34] We then formulate a modified perturbation theory with respect to the bilocal variables, this having a nontrivial classical solution as the zeroth approximation.

*New Dynamical Variables.* We consider massless fermions ("quarks") $q$ interacting with a vector field $A_\mu$. The original expression is the generating functional for the Green's function:[4)]

$$Z[j,\bar{\eta},\eta]=N\int DA_\mu Dq D\bar{q}\,\delta[\partial_\mu A_\mu]$$
$$\times\exp\left\{i\int d^4x\,[\bar{q}i\partial\!\!\!/ q-F^2_{\mu\nu}/4+g\bar{q}A\!\!\!/ q+j_\mu A_\mu+\bar{q}\eta+\bar{\eta}q]\right\};$$
$$F_{\mu\nu}=\partial_\mu A_\mu-\partial_\nu A_\mu;\quad A\!\!\!/=\gamma_\mu A_\mu,\ \text{etc.},\tag{43}$$

where $j_\mu$, $\eta$, and $\bar{\eta}$ are external sources.

The normalization $N$ is chosen such that $Z[0,0,0]=1$. The Lagrangian in (43) is invariant under 1) chiral and 2) scale transformations: 1) $q\to\exp(i\gamma_5\lambda)q$; 2) $q(x)\to\lambda^{3/2}q(\lambda x)$; $A_\mu(x)\to\lambda A_\mu(\lambda x)$. The dynamical variables are here a spinor and a vector field, whereas the mass of the spinor field is a scalar quantity. Therefore, to investigate the problem of dynamical occurrence of fermion mass, we intend to go over to new dynamical variables containing scalar components.

After integration over the vector field $A_\mu$, we obtain

$$Z[j,\bar{\eta},\eta]=N'\int DqD\bar{q}\exp\Big\{i\int d^4x\Big[\bar{q}(x)\Big(i\partial\!\!\!/$$
$$-g\gamma_\mu\int d^4y D_{\mu\nu}(x-y)j_\nu(y)\Big)q(x)+\bar{\eta}q+\bar{q}\eta\Big]$$
$$-\frac{i}{2}\int d^4x\,d^4y\,[j_\mu(x)D_{\mu\nu}(x-y)j_\nu(y)$$
$$+g^2\bar{q}(x)\gamma_\mu q(x)D_{\mu\nu}(x-y)\bar{q}(y)\gamma_\nu q(y)],\tag{44}$$

where $D_{\mu\nu}(x)$ is the gluon propagator in the Landau gauge:

$$D_{\mu\nu}(x)=(g_{\mu\nu}-\partial_\mu\partial_\nu/\partial^2)D^c(x);$$
$$D^c(x)=-\int\frac{d^4q}{(2\pi)^4}\frac{\exp(-iqx)}{q^2+i\varepsilon}.\tag{45}$$

(We shall use the notation of the book of Ref. 11.) We introduce the abbreviation

$$K_{(\alpha_1\beta_1;\,\alpha_2\beta_2)}(x_1y_1;\ x_2y_2)$$
$$=(\gamma_\mu)_{\alpha_1\beta_2}D_{\mu\nu}(x_1-x_2)(\gamma_\nu)_{\alpha_2\beta_1}\delta^{(4)}(x_1-y_2)\delta^{(4)}(y_1-x_2)\tag{46}$$

and rewrite the four-fermion term in the argument of the exponential of (44) in the form

$$\int d^4x\,d^4y\,\bar{q}(x)\gamma_\mu q(x)D_{\mu\nu}(x-y)\bar{q}(y)\gamma_\nu q(y)$$
$$=-\int d^4x_1\,d^4y_1\,d^4x_2\,d^4y_2\,[q_{\beta_1}(y_1)\bar{q}_{\alpha_1}(x_1)]$$
$$\times K_{(\alpha_1\beta_1;\,\alpha_2\beta_2)}(x_1y_1;\ x_2y_2)[q_{\beta_2}(y_2)\bar{q}_{\alpha_2}(x_2)]$$

or

$$-q_{b_1}\bar{q}_{a_1}K_{(a_1b_1;\,a_2b_2)}q_{b_2}\bar{q}_{a_2}=-(q\bar{q},\ Kq\bar{q}).\tag{47}$$

The minus sign in (47) arises from the permutation of the fields $q$ and $\bar{q}$. Further, we denote the pair $(\alpha_i, x_i)$ of a discrete and a continuous variable by the Latin index $a_i$. Summation over Latin indices denotes simultaneously integration over the continuous variables.

We can linearize (47), using an integral over the bilocal variables (see Appendix 3):

$$\exp\frac{i}{2}g^2(q\bar{q},\ Kq\bar{q})=C\int D\chi\exp\left\{-\frac{i}{2g^2}(\chi,\ K^{-1}\chi)+iq_b\bar{q}_a\chi_{ab}\right\};$$
$$K^{-1}_{(ab;\,cd)}K_{dc,\,ef}=\delta_{(ab,\,ef)}\equiv\delta_{af}\delta_{be},\tag{48}$$

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[4)]We shall not here consider renormalization questions. For simplicity, we do not take into account the internal degrees of freedom of the quarks (see Sec. 3, where the internal degrees of freedom are included).

where the bilocal field $\chi_{ab}=\chi_{\alpha\beta}(x, y)$ is treated as a boson field. It can be decomposed into scalar, pseudoscalar, vector, axial, and tensor components:

$$\chi(x, y)=S(x, y)+\gamma_5 P(x, y)+\gamma_\mu V_\mu(x, y) + i\gamma_\mu\gamma_5 A_\mu(x, y)+\frac{i}{2}[\gamma_\mu, \gamma_\nu]T_{\mu\nu}(x, y). \quad (49)$$

In accordance with (49), $D\chi = DSDPDV_\mu DA_\mu DT_{\mu\nu}$. Substituting (48) in (44) and integrating over the fermion fields [see Eq. (A.14)], we obtain an expression for the generating functional (43):

$$Z[j, \bar{\eta}, \eta]=\bar{N}\int D\chi \exp(iS[\chi])Z[j, \bar{\eta}, \eta|\chi]. \quad (50)$$

Here

$$S[\chi]=-\frac{1}{2g^2}(\chi, K^{-1}\chi)-i\,\mathrm{tr}\ln(1+G_0\chi); \quad (51)$$

$$Z[j, \bar{\eta}, \eta|\chi]=\exp\left[-\frac{i}{2}j_\mu D_{\mu\nu}j_\nu+i\bar{\eta}G(\chi-g\hat{A}_{\mathrm{ext}})\eta + \mathrm{tr}\ln(1-gG(\chi)\hat{A}_{\mathrm{ext}})\right]. \quad (52)$$

The Green's functions $G_0$ and $G(\chi-g\hat{A}_{\mathrm{ext}})$ are determined by the equations

$$\left.\begin{array}{l} i\hat{\partial}_x G_0(x, y)=-\delta^{(4)}(x-y); \\ (i\hat{\partial}_x+g\hat{A}_{\mathrm{ext}})G(x, y|\chi-g\hat{A}_{\mathrm{ext}}) \\ -\int d^4z\chi(x, z)G(z, y|\chi-g\hat{A}_{\mathrm{ext}})=-\delta^{(4)}(x-y), \end{array}\right\} \quad (53)$$

where

$$\hat{A}_{\mathrm{ext}}(x)=-\gamma_\mu\int d^4y D_{\mu\nu}(x-y)j_\nu(y).$$

We use the abbreviation $j_\mu D_{\mu\nu}j_\nu = \int d^4x d^4y j_\mu(x)D_{\mu\nu}(x-y)j_\nu(y)$, etc. In deriving (52), we have used the identity[5)] $\mathrm{Det}A=\exp\mathrm{tr}\ln A$ and $\mathrm{tr}\ln[G_0^{-1}+\chi-g\hat{A}_{\mathrm{ext}}]=\mathrm{tr}\ln G_0^{-1}[1+G_0\chi] +\mathrm{tr}\ln[1-gG(\chi)\hat{A}_{\mathrm{ext}}]$. The expression $Z[0,\bar{\eta},\eta|\chi]$ can be interpreted as the generating functional of the fermion fields in the external bilocal field $\chi(x, y)$ with probability distribution $\exp iS[\chi]$.

*"Classical" Equation of Motion of the Bilocal Field and Discussion of the Solutions.* The functional $S[\chi]$ in the integrand of the path integral (50) can be naturally interpreted as the action function for the bilocal fields. Then the classical equation of motion that follows from variation of the action has the form

$$\delta S[\chi]/\delta\chi=-(g^2K)^{-1}\chi-i\;\;G(\chi)=0. \quad (54)$$

Let $M=\chi$ denote the solution of (54). Then, multiplying (54) by the operator $g^2K$, we obtain[6)]

$$M_{\alpha\beta}(x, y)=-ig^2D_{\mu\nu}(x-y)[\gamma_\mu G(x, y|M)\gamma_\nu]_{\alpha\beta}, \quad (55)$$

where the Green's function $G(x, y|M)$ satisfies the equation

$$i\hat{\partial}_x G(x, y|M)-\int d^4z M(x, z)G(z, y|M)=-\delta^{(4)}(x-y). \quad (56)$$

Since Eqs. (54) and (55) are obtained without allowance for the external currents, it is natural to assume translational invariance:

$$M(x, y)=M(x-y), \quad G(x, y|M)=G(x-y|M).$$

Then in the momentum representation, we have instead of (55)

$$\left.\begin{array}{l} M(p)=-ig^2\int\frac{d^4q}{(2\pi)^4}D_{\mu\nu}(p-q)\gamma_\mu\frac{1}{M(q)-\hat{q}}\gamma_\nu; \\ G(p|M)=1/[M(p)-\hat{p}]. \end{array}\right\} \quad (57)$$

It follows from $PT$ and $T$ invariance that $M(p)$ has only a scalar and a vector part, and it is easy to show that in the transverse Landau gauge the vector part of $M$ is zero. Thus, we finally obtain

$$M(p)=-i\frac{\lambda}{\pi^2}\int\frac{d^4q}{(p-q)^2}\frac{M(q)}{q^2-M^2(q)}; \quad \lambda=\frac{3g^2}{(4\pi)^2}. \quad (58)$$

Equations (57) and (58) are identical with the Schwinger-Dyson equation for the mass operator in the lowest nontrivial order of perturbation theory.[11] The original equation (54) is a differential form of the Schwinger-Dyson equation:

$$(\partial/\partial p_\mu)^2 M(p)=-4\lambda M(p)/[p^2-M^2(p)]. \quad (59)$$

This equation is equivalent to (58) under the boundary conditions $(M(p)=M(p^2))$

$$\left.\begin{array}{l} \left[p^2\frac{dM(p)}{dp^2}+M(p)\right]_{p^2\to-\infty}=0; \\ \left[p^4\frac{dM(p)}{dp^2}\right]_{p^2\to 0}=0, \end{array}\right\} \quad (60)$$

which follow from (58) after integration over the angular variables.

Equation (58) is satisfied, in particular, by the trivial symmetric solution $M(p)\equiv 0$. The nontrivial solution of the linearized equation (58) was studied in "finite" quantum electrodynamics[34,35] for large values of the momentum in a space-like region. Recently, a solution of the nonlinear equation (59) was also studied in space- and time-like regions.[36] Substituting the asymptotic expression $M(p)\sim(-p^2)^c$ (or $\sqrt{p^2}$) for $p^2\to-\infty$ $(+\infty)$ in (59) and (60), we readily obtain nontrivial asymptotic solutions:

$$M(p)\sim\left\{\begin{array}{ll} M_0(-p^2)^{-(1-\sqrt{1-4\lambda})/2}, & -p^2\to\infty; \\ \sqrt{p^2(1+(4/3)\lambda)}, & p^2\to\infty. \end{array}\right. \quad (61)$$

As was pointed out in Ref. 32, an equation of the type (58) is the relativistic analog of the equation for the gap in superconductivity theory.[26,27] And just as a quasiparticle in a superconductor is a mixture of bare electrons with opposite signs of the charge, the massive Dirac particle must be a mixture of bare fermions with opposite chirality. [Cf. also (58) with the "gap" equation (15) in the nonrelativistic case.]

Decomposing the Green's function $G(M)$ into normal and anomalous parts:

$$G(M)=[G_{RR}(M)+G_{LL}(M)]+[G_{RL}(M)+G_{LR}(M)], \quad (62)$$

where

$$\left.\begin{array}{l} G_{\substack{RR\\(LL)}}(M)=\frac{1\overset{+}{(-)}\gamma_5}{2}G(M)\frac{1\overset{(+)}{-}\gamma_5}{2}; \\ G_{\substack{RL\\(LR)}}(M)=\frac{1\overset{+}{(-)}\gamma_5}{2}G(M)\frac{1\overset{+}{(-)}\gamma_5}{2}, \end{array}\right\} \quad (63)$$

we obtain

$$\left.\begin{array}{l} G_{\substack{RR\\(LL)}}(M)=\frac{1\overset{+}{(-)}\gamma_5}{2}\hat{p}\frac{1}{p^2-M^2(p)}; \\ G_{\substack{RL\\(LR)}}(M)=\frac{1\overset{+}{(-)}\gamma_5}{2}M(p). \end{array}\right\} \quad (64)$$

Thus, the anomalous Green's function vanishes iden-

---

[5)]The operation tr denotes the taking of the trace of the operator in the functional sense and the ordinary trace with respect to the matrix indices; $G(\chi)\equiv G(\chi-g\hat{A}_{\mathrm{ext}})_{A_{\mathrm{ext}}=0}$.

[6)]Note that the class of solutions may be changed by such multiplication.

tically in the symmetric case $M(p)\equiv 0$, just as the anomalous Green's function of the nonrelativistic model vanishes in the absence of superfluidity [see Eq. (A.11)]. It follows from the invariance of the vacuum under chiral transformations that

$$\exp(i\gamma_5\lambda)\,G\exp(i\gamma_5\lambda)=G, \tag{65}$$

as can be readily seen by expressing $G$ in the form of the vacuum expectation value of the $T$ product of the field operators. The nontrivial solution $M(p)\neq 0$ violates (65) and is associated with a noninvariant vacuum.

Finally, we note that (58), rewritten in terms of the functions $M(p)/[p^2-M^2(p)]$, is equivalent to the pseudoscalar sector of the homogeneous fermion-antifermion Bethe-Salpeter equation for $P^2=0$. Thus, for $M(p)\neq 0$ there also exists a nontrivial solution of the homogeneous Bethe-Salpeter equation $\Gamma(p, P=0)\sim\gamma_5 M(p)/[p^2-M^2(p)]$. This solution (see below), which is usually identified with a Goldstone boson, nevertheless leads to a continuous spectrum and cannot be normalized.[38]

*Loop Expansion.* The classical solution of Eq. (54) corresponds to summation of a definite class of diagrams of ordinary perturbation theory. To calculate the corrections to the Green's functions, it is convenient to consider a perturbation theory in the bilocal dynamical variables that uses a nontrivial classical solution as the zeroth approximation. By analogy with the nonrelativistic case (see Sec. 1), we make for this purpose in the integral (50) the change of variables

$$\chi(x, y)=M(x-y)+\Phi(x, y), \tag{66}$$

where $M$ is the solution of Eq. (54), and we expand the integrand of (50) with respect to the bilocal field $\Phi(x,y)$. Then the generating functional (50) can be represented in the form

$$Z[j, \bar\eta, \eta]=\overline{N}'\int D\Phi\exp[iS_{\rm free}[\Phi])\times(\exp(iS_{\rm int}[\Phi])\,Z[j, \bar\eta, \eta\,|\,M+\Phi]), \tag{67}$$

where $S_{\rm free}[\Phi]=-(\Phi, S^{(2)}(M)\Phi)/2$;

$$S_{\rm free}[\Phi]=-(\Phi,\ S^{(2)}(M)\,\Phi)/2;$$
$$S^{(2)}_{(a_1b_1;\,a_2b_2)}(M)=\frac{1}{g^2}K^{-1}_{(a_1b_1;\,a_3b_3)}\times[\delta_{(b_3a_3;\,a_2b_2)}-ig^2G_{a_2k}(M)\,K_{(b_3a_3;\,kl)}G_{lb_2}(M)] \tag{68}$$

and

$$S_{\rm int}[\Phi]=i\sum_{n=3}^{\infty}\frac{(-)^n}{n}\,{\rm tr}\,[G(M)\,\Phi]^n. \tag{69}$$

The term linear in $\Phi$ in the expansion of $S[M+\Phi]$, which can lead to spontaneous rearrangement of the vacuum, is not present in (67) by virtue of Eq. (54). The bilocal path integral (67) again gives a modified perturbation theory with free action $S_{\rm free}[\phi]$ for the bilocal fields. The higher powers of the expansion in $\Phi$ of the action $S[M+\Phi]$ determine the interactions $S_{\rm int}[\Phi]$ and the vertices. In what follows, the functional averaging with weight factor $\exp[-i/2(\Phi, S^{(2)}(M)\Phi]$ will be denoted by $\langle\ldots\rangle_{S_{\rm free}}$. (The definition of the averaging $\langle\ldots\rangle_{S_{\rm free}}$ already includes the normalization factor $\langle 1\rangle^{-1}_{S_{\rm free}}$, which we omit for brevity.) We define the propagator $T_{\alpha\beta;\gamma\delta}(xy;x'y')$ for the bilocal field $\Phi_{\alpha\beta}(x,y)$:[7)]

$$T_{(\alpha\beta;\,\gamma\delta)}(xy;\ x'y')=i\,\langle\Phi_{\alpha\beta}(x,\ y)\,\Phi_{\gamma\delta}(x',\ y')\rangle_{S_{\rm free}}=[S^{(2)}(M)^{-1}]_{\alpha\beta;\,\gamma\delta}(xy;\ x'y'). \tag{70}$$

Note that $T$ is the relativistic analog of (23) and satisfies the following inhomogeneous integral equation of Bethe-Salpeter type:

$$T_{(ab;\,gd)}=g^2K_{(ab;\,gd)}+ig^2G_{jh}(M)\,K_{(ab;\,kl)}G_{li}(M)\,T_{(ij;\,gd)}. \tag{71}$$

The graphical representation of (70) and the expansion in the perturbation series of the propagator $T$ is given in Fig. 3, in which the dashed line denotes the propagator of the bilocal field $\Phi$, for which we can introduce the notation $\overline{\Phi\Phi}$. In the second row of Fig. 3, we give the expression for the propagator of the bilocal field in terms of ordinary Feynman graphs, the continuous line denoting the quark Green's function $G(x,y|\,M)$. This class of graphs will be denoted by a broken line, symbolizing the exchange of a large number of gluons.

For completeness, we also give the field equation of motion that arises from variation of the free (quadratic) part of the bilocal action $S_{\rm free}[\Phi]$:

$$\frac{\delta S_{\rm free}[\Phi]}{\delta\Phi}=0\rightarrow S^{(2)}(M)\,\Phi=0 \tag{72}$$

or, using Eqs. (68),

$$\Phi(x,\ y)=ig^2D_{\mu\nu}(x-y)\int d^4x'd^4y'\gamma_\mu G(x,\ x'|M)\times\Phi(x',\ y')\,G(y',\ y|M)\,\gamma_\nu. \tag{73}$$

We go over to the momentum space:

$$\Phi(p_2,\ p_1)=\Phi(q\,|\,P)=\int dx_2\,dx_1\exp[i(x_2p_2-x_1p_1)]\,\Phi(x_2,\ x_1),$$

where $q$ and $P$ are the relative and the total momentum of the quark-antiquark pair: $q=(p_2+p_1)/2$ and $P=p_2-p_1$.

Equation (73) takes the form

$$\Phi(q\,|\,P)=ig^2\int\frac{d^4q'}{(2\pi)^4}D_{\mu\nu}(q-q')\,\gamma_\mu G\left(q'+\frac{P}{2}(M)\,\Phi(q'\,|\,P)\times G\left(q'-\frac{P}{2}\right|M\right)\gamma_\nu. \tag{74}$$

In this equation, we readily recognize the homogeneous Bethe-Salpeter equation in the ladder approximation for the vertex function $\Gamma_m(q|\,P)$ of the bound state of the quark and antiquark,

$$\Gamma_m(q\,|\,P)=N_mG\left(q+\frac{P}{2}\right|M\Big)\int d^4z\exp(iqz)\times\langle 0\,|\,Tq(z/2)\,\bar q(-z/2)\,|\,B_m\rangle\,G\left(q-\frac{P}{2}\right|M\Big), \tag{75}$$

where $N_m$ is a normalization factor. The solution of (74) can be normalized in accordance with the condition

$$-i\int\frac{d^4q}{(2\pi)^4}{\rm tr}\left[G\left(q+\frac{P}{2}\right|M\Big)\,\Gamma_m(q\,|\,P)\times G\left(q-\frac{P}{2}\right|M\Big)\,\overline{\Gamma}_{m'}(q\,|-P)\right]=\varepsilon_m\delta_{mm'}, \tag{76}$$

where $\varepsilon_m$ is a factor that takes the values +1, −1, 0;

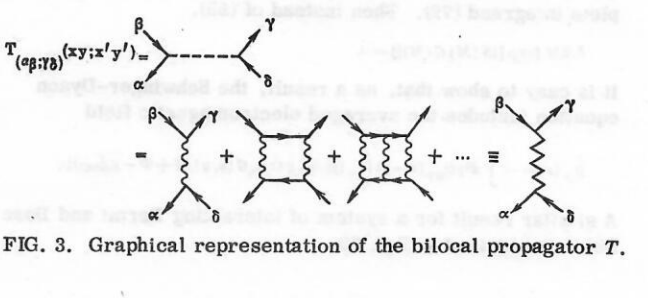


FIG. 3. Graphical representation of the bilocal propagator $T$.

[7)] Here, we include the factor $i$ in the definition of the propagaor.

$\overline{\Gamma}_m$ is the conjugate vertex function. Note that the complete set of states (75) can in principle be used to expand the collective variable $\Phi(x, y)$ with respect to this basis. These states represent bare quanta of the Hilbert space in the interaction representation.

The expectation value of the product of bilocal fields can be readily calculated in accordance with Wick's theorem; for example,

$$i^2 \langle \Phi_1\Phi_2\Phi_3\Phi_4\rangle_{S_{\rm free}} = \overline{\Phi_1\Phi_2}\,\overline{\Phi_3\Phi_4} + \Phi_1\Phi_2\Phi_3\Phi_4 + \Phi_1\Phi_2\Phi_3\Phi_4;$$
$$\langle \Phi_1\Phi_2, \ldots, \Phi_{2n+1}\rangle_{S_{\rm free}} = 0\text{, etc.} \quad (77)$$

For Wick's theorem, we can use the compact form of expression

$$\langle f(\Phi)\rangle_{S_{\rm free}} = \exp\left[-\frac{i}{2}\left(\frac{\delta}{\delta B}, T\frac{\delta}{\delta B}\right)\right] f(B)\Big|_{B=0}. \quad (78)$$

*Corrections to the Green's Functions*. As an illustration, we consider some examples of the corrections to the quark propagator:[8)]

$$G_{\alpha\beta}(x, y) = \frac{1}{iZ}\frac{\delta}{\delta\eta_\beta(y)}\frac{\delta Z}{\delta\bar{\eta}_\alpha(x)}\Big|_{\eta=\bar{\eta}=0}$$
$$= R^{-1}\langle R[\Phi]\, G_{\alpha\beta}(x, y) \mid (M+\Phi-gA_{\rm ext})\rangle_{S_{\rm free}};$$
$$R[\Phi] = \exp(iS_{\rm int}[\Phi])\, Z[J, 0, 0 \mid M+\Phi]; \quad R = \langle R[\Phi]\rangle_{S_{\rm free}} \quad (79)$$

or, using (78), we obtain

$$G_{\alpha\beta}(x, y) = R^{-1}\exp\left[-\frac{i}{2}\left(\frac{\delta}{\delta B}, T\frac{\delta}{\delta B}\right)\right]$$
$$\times\{R[B]\, G_{\alpha\beta}(x, y) \mid M+B-gA_{\rm ext})\}_{B=0}. \quad (80)$$

For simplicity, we shall consider in what follows only the case $J_\mu = 0$.

The corresponding terms of the expansion in the modified perturbation theory for the Green's function can be obtained by expanding $\exp(iS_{\rm int}[\Phi])$, $G(M+\Phi)$ in a Taylor series with respect to the bilocal field $\Phi(x, y)$ and performing functional averaging.

Substituting

$$\exp(iS_{\rm int}[\Phi]) = \left\{1 + \frac{1}{3}\,{\rm tr}\,[G(M)\Phi]^3 + 0(\Phi^4)\right\}$$

in (79) and using (70), (77), and (78), we obtain

$$G = G(M) + R^{-1}\langle \exp(iS_{\rm int}[\Phi])\,[-G(M)\Phi G(M)$$
$$+ G(M)\Phi G(M)\Phi G(M) - G(M)\Phi G(M)\Phi G(M)\Phi G(M) + \ldots]\rangle_{S_{\rm free}}$$

and

$$G = G(M) - iG_3(M)\Phi G_2(M)\Phi G_1(M)$$
$$+ \frac{1}{3}\{{\rm tr}\,\Phi G_5(M)\Phi G_4(M)\Phi G_3(M)\}\, G_2(M)\Phi G_1(M)$$
$$-\frac{i}{3}\{{\rm tr}\, G_7(M)\Phi G_6(M)\Phi G_5(M)\Phi\} G_4(M)\Phi G_3(M)\Phi G_2(M)\Phi G_1(M) + \ldots. \quad (81)$$

Here and in what follows, we omit the vacuum contribution, which arises from the normalization factor

$$R^{-1} = \langle \exp(iS_{\rm int}[\Phi])\rangle^{-1}_{S_{\rm free}}.$$

The second, third, and fourth terms in (81) are shown graphically in Fig. 4. The graphs on the left-hand side of Fig. 4 are the expressions obtained by expansion in the integrand of the functional integral with respect to the bilocal field $\Phi(x, y)$ (dashed line). (The closed quark loops with external dashed lines arise from $S_{\rm int}[\Phi]$.) If we substitute for the dashed lines of the bilocal propagator $\overline{\Phi\Phi}$ their corresponding zigzag lines, symbolizing many-particle exchange (see Fig. 3), we obtain the class of Feynman diagrams on the right-hand side of Fig. 4. In Eq. (81), we have numbered the Green's functions, to indicate their position in the diagrams. Thus, the loop expansion gives in principle a new perturbation theory, in which the fundamental particles (the dressed quarks) of the original theory interact with their bound states.[9)] Dynamical spontaneous symmetry breaking was also investigated by functional methods in Ref. 40.

## 3. TWO-DIMENSIONAL QUANTUM CHROMODYNAMICS

We describe here the application of a path integral with respect to collective fields to a model with internal degrees of freedom (such as color and flavor of quarks), namely, to a two-dimensional model of quantum chromodynamics.

It is currently hoped that quantum chromodynamics, *the gauge theory of colored quarks and gluons*, is a good candidate for a realistic theory of hadrons.[41] It is expected that infrared singularities will confine the quarks within the observed hadrons, which are themselves colorless. An important step toward the understanding of these ideas was the investigation of two-dimensional quantum chromodynamics.[42] This model

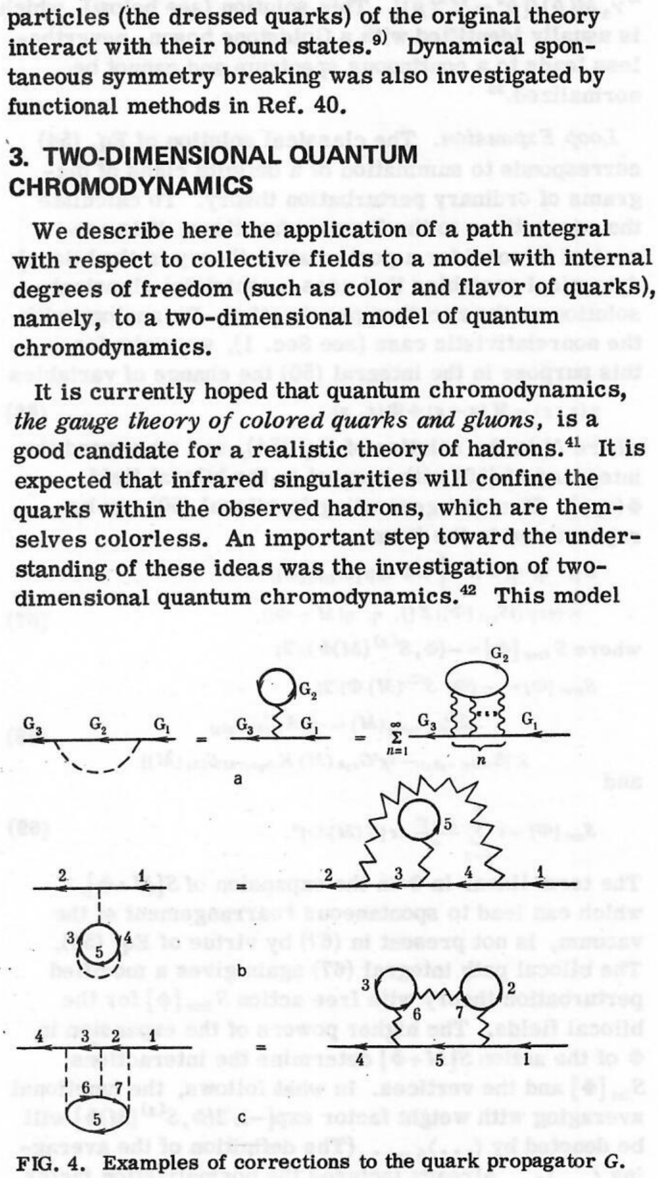


FIG. 4. Examples of corrections to the quark propagator $G$.

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[8)] One can also obtain a different modified expansion of the quark Green's function around the stationary point of the complete integrand (79). Then instead of (55),

$$\delta/\delta M\,\{\exp[iS(M)\, G(M)]\} = 0.$$

It is easy to show that, as a result, the Schwinger–Dyson equation includes the averaged electromagnetic field

$$\mathfrak{A}_\mu(x) = -\int d^4y D_{\mu\nu}(x-y)\,[J_\nu(y) + i\,g\,{\rm tr}\,\gamma_\nu G(y, y \mid M+\Phi-g\mathfrak{A}_{\rm ext})].$$

A similar result for a system of interacting Fermi and Bose fields is obtained in Ref. 39.

[9)] The convergence of such a modified perturbation theory can be significantly improved by including the internal degrees of freedom of the quarks and gluons. For the group $SU(N)$, an effective expansion in $1/N$ is obtained (see Sec. 3).

has been used to study many questions such as the Regge behavior of the amplitudes of bound states,[43] the asymptotic behavior of the form factors, deep inelastic scattering, and $e^+e^-$ annihilation.[44,45] Our aim is to formulate two-dimensional quantum chromodynamics directly in terms of the fields of bound states. As is shown above, this can be achieved by the introduction of bilocal fields as new dynamical variables. In terms of these variables, the quasiclassical approximation is identical with the approximation of planar diagrams. As a result, we obtain an infinite-component nonpolynomial field theory for bound states.[46] The loop expansion in the theory considered here can be rewritten in a form completely analogous to nuclear field theory (see Sec. 1). The effective expansion parameter is $1/N_c$ ($N_c$ is the number of colors), instead of $1/\Omega$ in nuclear field theory.

*Model and Method.* The Lagrangian for two-dimensional chromodynamics has the form

$$L = G_{\mu\nu,\,\alpha\beta}G^{\mu\nu}_{\beta\alpha}/4 + \bar{q}_{a\alpha}(\mathrm{i}\gamma^{\mu}D_{\mu} - m_a)\,q_{a\alpha}, \tag{82}$$

where the intensity tensor $G_{\mu\nu,\alpha\beta}$ and the covariant derivative are defined by

$$\begin{aligned} G_{\mu\nu,\,\alpha\beta} &= \partial_{\mu}A_{\nu,\,\alpha\beta} - \partial_{\nu}A_{\mu,\,\alpha\beta} + g\,[A_{\mu}, A_{\nu}]_{\alpha\beta};\\ D_{\mu}q_{a\alpha} &= [\partial_{\mu}\delta_{\alpha\beta} + gA_{\mu,\,\alpha\beta}]\,q_{a\beta};\\ A_{\mu,\,\alpha\beta} &= \sum_{n=1}^{N_c^2-1} A_{\mu}^{(n)}\lambda_{\alpha\beta}^{(n)}. \end{aligned} \tag{83}$$

Here, $q$ and $A_{\mu}$ are the quark and gluon fields. The indices $\alpha, \beta = 1, 2, \ldots, N_c$ denote the color, and the indices $a = 1, 2, \ldots, N_f$ the flavor. The local gauge group $SU(N_c)$ is conserved exactly, whereas the global $SU(N_f)$ symmetry is broken by the quark masses $\bar{q}_{a\alpha}m_a q_{a\alpha}$, if there are no $m_a = m$; $\lambda^{(n)}$ is an $SU(N_c)$ representation of the generators with normalization condition $\mathrm{tr}\lambda^{(n)}\lambda^{m} = T\delta^{nm}$; $q_{a\alpha} = (q_1/q_2)_{a\alpha}$ is a two-component Dirac spinor. Two-dimensional chromodynamics takes its simplest form in the light-cone gauge:

$$A_{-} = \frac{1}{\sqrt{2}}(A_0 - A_1) = A^{+} = 0$$

$$(x^{\pm} = x_{\mp} = (x^0 \pm x^1)/\sqrt{2};\ ab = a^+b^+ + a_-b^- = a^+b_- + a_-b_+). \tag{84}$$

In such a gauge, only $\hat{q}_2 = \binom{0}{q_2}$ is an independent dynamical variable,[42,47,48] the self-interaction of the gluon fields disappears, and Faddeev-Popov ghosts[4] are absent.

After elimination of the dependent dynamical variables $A_+$, $\binom{q_1}{0}$, the generating functional for the Green's functions takes a form analogous to (44) ($\hat{q}_2 \to q$):

$$\begin{aligned} &Z[J, \eta^+, \eta]\\ &= N'\int DqDq^+\exp\Big\{\mathrm{i}\int d^2x d^2y\Big[q^+(x)\,\mathrm{i}G^{-1}(x, y)\mid A_{\mathrm{ext}})\,q(y)\\ &\quad + (\eta^+(x)\,q(y) + q^+(x)\,\eta(y))\,\delta^2(x-y)\\ &\quad + \frac{\mathrm{i}}{2}\sum_{n=1}^{N_c^2-1} TJ^{(n)}(x)\,D(x-y)\,J^{(n)}(y)\\ &\quad + \mathrm{i}\frac{(2gi)^2}{2}\sum_{n=1}^{N_c^2-1} q^+(x)\,\lambda^{(n)}q(x)\,T^{-1}D(x-y)\,q^+(y)\,\lambda^{(n)}q(y)\Big]\Big\}. \end{aligned} \tag{85}$$

The quark and gluon Green's functions are determined by the equation

$$\left.\begin{aligned} &\int d^2y\,(G_0^{-1}(x, y) + 2\mathrm{i}gA_{\mathrm{ext}}(x)\,\delta^2(x-y))\,G(y, z\mid A_{\mathrm{ext}})\\ &\qquad = \delta^2(x-z);\\ &A_{\mathrm{ext}}(x) = \int d^2yD(x-y)\,J(y); \end{aligned}\right\} \tag{86}$$

$$\begin{aligned} G_0(x, y) &= \frac{-\partial_-}{-2\partial_+\partial_- - m_a^2 + \mathrm{i}\varepsilon}\,\delta^2(x-y)\\ &= \int\frac{d^2k}{(2\pi)^2}\left[\frac{\mathrm{i}k_-}{2k_+k_- - m_a^2 + \mathrm{i}\varepsilon}\right]\exp[-\mathrm{i}k(x-y)]; \end{aligned} \tag{87}$$

$$D(x) = \frac{\mathrm{i}}{\partial_-^2}\,\delta^2(x) = \int\frac{d^2k}{(2\pi)^2}\left[-\frac{\mathrm{i}}{k_-^2}\right]\exp(-\mathrm{i}kx). \tag{88}$$

The generating functional (85) together with the propagators (86)–(88) reproduces the Feynman rules of two-dimensional quantum chromodynamics formulated in Ref. 42 (we choose the normalization $Z[0, 0, 0] = 1$).

*Introduction of Bilocal Variables.* Our task is to integrate over the quark fields in the expression (85). For this purpose, it is convenient to rewrite the four-quark term in (85) (the factor $i(2gi)^2/2$ is omitted) in the abbreviated notation [cf. (47)]

$$F = -q_B q_A^+ K_{AB;\,GD}\,q_D q_G^+, \tag{89}$$

where

$$\underset{(a\alpha b\beta;\ g\gamma d\delta)}{K(xy; y'x')} = \sum_{n=1}^{N_c^2-1}\lambda_{\alpha\delta}^{(n)}\lambda_{\gamma\beta}^{(n)}\delta_{ad}\delta_{gb}T^{-1}D(x-y)\,\delta(x-x')\,\delta(y'-y). \tag{90}$$

Here, the index $A$ denotes the triplet of two discrete and one continuous index $(a, \alpha, x)$ (summation over $A$ also means integration over the continuous index).

Using the decomposition $\{N_c\}\times\{N_c^*\} = \{1\} + \{N_c^2 - 1\}$ and denoting the projection operators onto the singlet and $(N_c^2 - 1)$-plet quark-antiquark channels by $P_1$ and $P_{N_c^2-1}$, we can write

$$k_{\alpha\beta,\,\gamma\delta} \equiv \sum_{n=1}^{N_c^2-1}\lambda_{\alpha\delta}^{(n)}\lambda_{\gamma\beta}^{(n)} = T\left\{\frac{N_c^2-1}{N_c}P_1 - \frac{1}{N_c}P_{N_c^2-1}\right\}_{\alpha\beta,\gamma\delta}. \tag{91}$$

It can be seen from (91) that in the singlet channel attractive forces exist. We shall consider only this channel, since we are interested in only bound states. We rewrite (89) in the form

$$F = -q_B q_A^+(P_1KP_1)_{AB,\,GD}\,q_D q_G^+ + W(q, q^+). \tag{92}$$

Here, $W$ contains the projection of the interaction onto the nonresonance channel $(N_c^2 - 1)$. We write $W(q, q^+)$ in the form

$$W\left(\frac{1}{i}\frac{\delta}{\delta\eta^+}, \frac{1}{i}\frac{\delta}{\delta\eta}\right)$$

and take this part of the interaction outside the path integral.

To integrate over the quark fields, we again linearize the four-quark interaction by introducing an additional Gaussian functional integration over the bilocal fields[46] as in the case of Eq. (48). Integrating over $q$ and $q^+$, we obtain the following expression[46] for $Z(J=0)$:

$$Z[\eta, \eta^+, R] = N''\int D\chi\exp(\mathrm{i}S[\chi])\,Z[\eta, \eta^+, R\mid\chi], \tag{93}$$

where

$$S[\chi] = -\frac{\mathrm{i}}{2(2gi)^2}(\chi, K^{-1}\chi) - \mathrm{i}\,\mathrm{tr}\ln[iG_0^{-1} - \chi_1]; \tag{94}$$

$$\begin{aligned} &Z[\eta, \eta^+, R\mid\chi]\\ &= \exp\Big\{-\eta^+G(\chi_1)\,\eta + \frac{1}{2(2gi)^2}(P_1R, K^{-1}P_1R) - \frac{1}{(2gi)^2}(P_1R, K^{-1}\chi)\Big\};\\ &\mathrm{i}G^{-1}(\chi_1) = \mathrm{i}G_0^{-1} - \chi_1;\quad \chi_1 = P_1\chi; \end{aligned}$$

$G(\chi)$ is the quark Green's function in the external bilocal field $\chi(x, y)$.[10)] For convenience, we have also included the bilocal source $(qq^{+}P_1)_{A\,B}R_{BA}$ in the integral (85) in order to have the possibility of varying with respect to $R$.

*Quark Spectrum.* To obtain the quark spectrum, we consider the stationary point of the phase of the integrand in (93) in the absence of external sources. The equation for the stationary point of the action $S[\chi]$ has the form

$$M = (2gi)^2 KP_1 \frac{G(M)}{i} \qquad (95)$$

and is identical with the Schwinger-Dyson equation in the first nontrivial order of perturbation theory. In the momentum representation,

$$M(p_-) = (2g)^2 \frac{N_c^2-1}{N_c} \int \frac{d^2k}{(2\pi)^2} \Theta(|k_-|-\lambda) \frac{1}{k_-^2} G(p+k\,|\,M), \qquad (96)$$

where $\Theta(|k_-|-\lambda)(\lambda\to 0)$ is an infrared cutoff.

This equation corresponds to allowance for only planar diagrams and was studied by 't Hooft,[42] who showed that the approximation of planar diagrams is valid in the limit $N_c \to \infty$ and $g^2N_c$ fixed.

Equation (96) has the solution[11)]

$$M(p_-) = \frac{g^2}{\pi} \frac{N_c^2-1}{N_c} \left( \frac{\operatorname{sgn} p_-}{\lambda} - \frac{1}{p_-} \right); \qquad (97)$$

$$G(p\,|\,M) = \frac{ip_-}{2p_+p_- - m^2 + i\varepsilon - p_- M(p_-)}. \qquad (98)$$

As can be seen from (97) and (98), the poles of the quark propagator become infinite if $\lambda \to 0(M(p_-)\to\infty)$:

$$G(p\,|\,M) \sim -\lambda i \left[ \frac{g^2}{\pi} \frac{N_c^2-1}{N_c} \operatorname{sgn} p_- \right]^{-1}. \qquad (99)$$

*Meson Spectrum.* We give here a derivation of the equation that describes the meson spectrum of the theory. Since this derivation is completely analogous to that of the equation for the boson spectrum in Sec. 2, we give only the main results.

The shift $\chi = M + \Phi$ of the bilocal field and expansion of the path integral around the stationary point $M(x-y)$ of the action leads to the propagator of the bilocal field $\Phi(x, y)$:

$$T = \left\{ \frac{\delta^2 S[\chi]}{\delta\chi^2} \Big|_{\chi=M} \right\}^{-1},$$

which satisfies the inhomogeneous Bethe-Salpeter equation for $q\bar{q}$ scattering.

The free-field equations

$$\frac{\delta S_{\rm free}}{\delta\Phi}\Big|_{\Phi=\Gamma} = 0$$

correspond to the homogeneous Bethe-Salpeter equation for the bound states of a quark and an antiquark. Using the momentum representation, we obtain

$$\Gamma(p, r) = -i(2g)^2 \frac{N_c^2-1}{N_c}$$
$$\times \int \frac{d^2k}{(2\pi)^2} \frac{1}{k_-^2} G(p+k\,|\,M)\,\Gamma(p+k, r)\,G(p+k-r\,|\,M). \qquad (100)$$

The arguments $p$ and $r$ in the vertex function denote the momentum of the ingoing quark and the total momentum of the quark-antiquark pair (Fig. 5). Equation (100) can be solved by introducing the wave function

$$h(p_-, r) = \int dp_+ G(p\,|\,M)\,\Gamma(p, r)\,G(p-r\,|\,M). \qquad (101)$$

The equation for $h$ takes the form

$$\mu_k^2 h_k(x) = Hh_k(x) \equiv \left( \frac{\alpha_1}{x} + \frac{\alpha_2}{1-x} \right) h_k(x) - P \int_0^1 dy \frac{h_k(y)}{(y-x)^2}, \qquad (102)$$

where

$$\left. \begin{aligned} & x = p_-/r_-; \quad \alpha_{1,2} = m_{1,2}^2 \Big/ \left[ \frac{g^2}{\pi} \frac{N_c^2-1}{N_c} - 1 \right]; \\ & M_k^2 = (2r_+r_-)_k = \frac{g^2}{\pi} \frac{N_c^2-1}{N_c} \mu_k^2, \end{aligned} \right\} \qquad (103)$$

and $P$ is the principal-value symbol.

The following results are obtained[42,43]:

i) $H$ is a positive definite and self-adjoint operator on the space of functions that vanish for $x=0$ $[x=1]$ like $x^{\beta\alpha}[(1-x)^{\beta b}]$, where $\pi\beta_\alpha \cot\pi\beta_\alpha = -\alpha_a$. The operator $H$ has only a discrete spectrum. The eigenfunctions are complete and orthogonal:

$$\sum_k h_k(x)\,h_k(x') = \delta(x-x'); \qquad \int_0^1 dx h_m(x)\,h_n(x) = \delta_{mn}. \qquad (104)$$

ii) For large $k$, the eigenfunctions and eigenvalues can be written approximately in the form

$$h_k(x) \approx \sqrt{2} \sin \pi kx \ (k \gg 1); \quad \mu_k^2 \approx \pi^2 k. \qquad (105)$$

From the inhomogeneous and the homogeneous Bethe-Salpeter equations for $T$ and $\Gamma$ we obtain in the standard manner orthogonality and normalization relations (Ref. 46):[12)]

$$i\,{\rm tr} \int \frac{d^2q}{(2\pi)^2} \bar{\Gamma}_l(q, -r) \left[ \frac{1}{2r_-} \frac{\partial}{\partial r_+} \frac{G(q+r/2\,|\,M)}{i} P_1 \frac{1}{i} \right.$$
$$\left. \times G\left(q - \frac{r}{2} \Big| M\right) P_1 \right]_{r^2=M_l^2} \Gamma_l(q, r) = 1; \qquad (106)$$

$$i\,{\rm tr} \int \frac{d^2q}{(2\pi)^2} \bar{\Gamma}_l(q, -r) \left[ \frac{1}{i} G\left(q + \frac{r_l}{2} \Big| M\right) P_1 \frac{1}{i} G\left(q - \frac{r_l}{2} \Big| M\right) P_1 \right.$$
$$\left. - \frac{1}{i} G\left(q + \frac{r_k}{2} \Big| M\right) P_1 \frac{1}{i} G\left(q - \frac{r_k}{2} \Big| M\right) P_1 \right] \Gamma_k(q, r) = 0, \quad l \neq k. \qquad (107)$$

Using the explicit expression for the normalized vertex

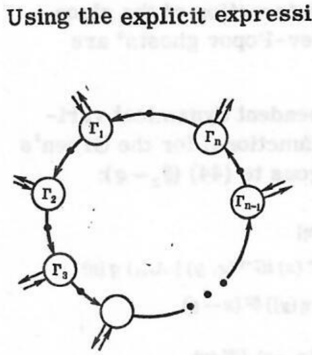


FIG. 5. Graphical representation of the vertex part $V_{k_n\cdots k_1}$.

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[10)] The definition of the propagator $G$ here differs by the factor $-i$ from the definition in Sec. 2.

[11)] This vanishing of the pole of the quark propagator can be interpreted as infrared confinement of quarks. The infrared regularization of the integral (96) by means of the step $\Theta$ function or the principal value[43] was questioned in Ref. 49. For the quark propagator one can then obtain a different result by making a Wick rotation and using a symmetric cutoff.[49] The obtained propagator contains a cut in the complex plane. However, the part played by this cut for the Wick rotation is not clear. We use here 't Hooft's solution.

[12)] In deriving (106) and (107), we have used the fact that here the argument $q$ of the function $\Gamma$ is the relative momentum of the quark–antiquark pair. For real $h_k(x)$, we also have $\overline{\Gamma_k(q, -r)/i} = \Gamma_k(q, -r)/i$.

functions, we can rewrite the bilocal propagator $T$ in the final form

$$T(x, x'; r) = (2g)^2 k_{\alpha\beta,\gamma\delta} \frac{\mathrm{i}}{r_-^2 (x'-x)^2} - \sum_k \frac{\mathrm{i}}{r^2 - M_k^2} \left[ \frac{1}{r_-} \Gamma_k(x) \right] \left[ \frac{1}{r_-} \overline{\Gamma}_k(x') \right]. \tag{108}$$

The bilocal propagator describes the propagation of bound states. In the calculation of the scattering amplitude of the bound states, the factor $\Theta(x(1-x))/\lambda$ in the vertex compensates the factor $\lambda$ in the quark propagator, which gives a finite nontrivial result in the limit $\lambda \to 0$.

*Infinite-Component Field Theory for Bound States.* We show that quantum field theory in bilocal fields can be interpreted as an infinite-component field theory for bound states.

For this we expand the bilocal fields with respect to a complete set of solutions of the homogeneous Bethe-Salpeter equation. These last can be interpreted like the plane-wave solutions of the free field equations $(\Phi_1 P_1 \Phi)$

$$\Phi_1(x, y) = \sum_k \int \frac{d^2 r}{(2\pi)^2} \int \frac{d^2 q}{(2\pi)^2} \exp\left\{ -\mathrm{i} \left[ r \left( \frac{x+y}{2} \right) + q(x-y) \right] \right\} \Gamma_k(q, r) \varphi_k(r), \tag{109}$$

where the mesons are described by the Hermitian fields $\varphi_k(X)$ $(X = (x+y)/2)$. Substituting the expansion (109) in the free action function and using (106) and (107), we obtain

$$S_{\text{free}} \sim \frac{1}{2} \sum_k \int \frac{d^2 r}{(2\pi)^2} \varphi_k(-r) [r^2 - M_k^2] \varphi_k(r). \tag{110}$$

Equation (110) can be interpreted as the free effective action of an infinite-component field theory. The mass spectrum is determined by the Klein-Gordon operator $(-\Box_X - M_k^2)$ [see (103)].

The Lagrangian of the interaction of the mesons is essentially nonpolynomial and nonlocal:

$$S_{\text{int}}[\Phi] = \mathrm{i} \sum_{n=3}^{\infty} \frac{1}{n} \operatorname{tr} \left[ \frac{G(M)}{i} \Phi_1 \right]^n$$
$$= \sum_{n=3}^{\infty} \sum_{k_1, \ldots, k_n} \frac{(i)^{1-n}}{n} \int \prod_{i=1}^{n} \frac{d^2 r_i}{(2\pi)^2} (2\pi)^2 \delta^{(2)}\left( \sum_1^n r_i \right) \times V_{k_n, \ldots, k_1}(r_n \ldots r_1) \prod_{i=1}^{n} \varphi_{k_i}(r_i), \tag{111}$$

where

$$V_{k_n, \ldots, k_1}(r_n, \ldots, r_1) = \int \frac{d^2 q}{(2\pi)^2} \operatorname{tr} \Gamma_{k_n}(\xi_n, r_n) G\left( \xi_{n-1} + \frac{r_{n-1}}{2} \middle| M \right) \times \Gamma_{k_{n-1}}(\xi_{n-1}, r_{n-1}), \ldots, G\left( \xi_1 + \frac{r_1}{2} \middle| M \right) \Gamma_{k_1}(\xi_1, r_1) G(q \mid M);$$

$$\xi_k = q + \sum_{i=1}^{k-1} r_i + \frac{r_k}{2}.$$

Thus, starting from the local theory of quarks and gluons, we have finally obtained a nonpolynomial, nonlocal, infinite-component field theory for bound states. It is easy to show that this loop expansion can be rewritten in the equivalent form of a theory describing the direct interaction of dressed quarks with their bound states, i.e., mesons. Using the same technique as for the nonrelativistic nuclear field theory, we obtain the equivalent representation[13)]

$$Z[\eta^+, \eta] = \overline{N} \exp\left[ 2g^2 W\left( \frac{1}{\mathrm{i}} \frac{\delta}{\delta\eta^+}, \frac{1}{\mathrm{i}} \frac{\delta}{\delta\eta} \right) \right]$$
$$\times \exp\left\{ -\mathrm{i} \sum_{i=1}^{2} L_i \left[ (2g)^2 k_{\alpha\beta,\gamma\delta} \frac{1}{r_-^2 (x'-x)^2} \frac{\delta}{\mathrm{i}\delta\eta} P_1 \frac{\delta}{\mathrm{i}\delta\eta^+} + \frac{1}{r_-} \Gamma_k \frac{\delta}{\delta jk} \right] \right\}$$
$$\times \int \prod_k D\varphi_k \int Dq Dq^+ \exp \int \left\{ \frac{\mathrm{i}}{2} \sum_k \varphi_k (r^2 - M_k^2) \varphi_k + \mathrm{i} \sum_k j_k \varphi_k \right.$$
$$+ \mathrm{i} q^+ \mathrm{i} G^{-1}(M) q + \mathrm{i}(\eta^+ q + q^+ \eta) - \mathrm{i} \sum_k (q^+ P_1 q) \left( \frac{1}{r_-} \Gamma_k \right) \varphi_k$$
$$\left. + \frac{\mathrm{i}}{2} (q^+ P_1 q)(2g)^2 k_{\alpha\beta,\gamma\delta} \frac{1}{r_-^2 (x'-x)^2} (q^+ P_1 q) \right\}_{j_k = 0}, \tag{112}$$

where the momentum integration is written out only symbolically. Here, $\exp\{-i\sum_{r=1}^{2} L_i\}$ is the projection operator that subtracts the ladder diagrams already taken into account in the formation of the meson spectrum. Thus, double allowance is eliminated. It can be seen from (112) that the quarks interact with the mesons nonlocally in the form of the Yukawa interaction $(q^+ P_1 q)(1/r_- \Gamma_k)\varphi_k$. With allowance for the explicit expression for the normalized vertex function,

$$\frac{1}{r_-} \Gamma_k(x) = -2\mathrm{i} \left( \frac{N_c^2 - 1}{N_c^2} \right)^{1/2} \left( g^2 \frac{N_c^2 - 1}{N_c \pi} \right)^{1/2} \frac{2g}{\lambda}$$
$$\times \left[ \Theta(x(1-x)) + \frac{\lambda}{2|r_-|} \left( \frac{\alpha_1}{x} + \frac{\alpha_2}{1-x} - \mu_k^2 \right) h_k(x) \sim O\left( \frac{1}{\sqrt{N_c}} \right), \right.$$

we can see that the representation (112) ensures a modified perturbation theory with effective expansion parameter $1/N_c$. It is also easy to include the diquark sector[46] in our treatment. In this case, the stationary point of the collective action determines the equations for the normal and anomalous Green's functions in complete analogy with the equations of superconductivity.[12,26,27] Moreover, as was shown in Ref. 46, there exists a bound state in the $qq$ (diquark) channel. The mass of this state tends to infinity when the infrared cutoff is lifted.

## CONCLUSIONS

We have presented the use of path integrals for describing collective phenomena in nonrelativistic many-body physics and in relativistic quantum field theory. The principal feature of this functional technique is the transition to dynamical variables with different quantum numbers not present in the original formulation of the theory in terms of elementary fields.

The main result of the path-integral method in collective fields is the derivation of the Feynman rules for the quantum field theory that describes the interaction of the elementary and the composite fields. In particular, in this way one can rigorously justify *nuclear field theory* (Refs. 13–16 and 19–21).

Functional methods could also adequately describe the low-energy limit of quantum chromodynamics, non-Abelian gauge theory, by means of which it is hoped to construct a theory of the strong interactions. One of the main results of quantum chromodynamics is

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13) The representation (112) for the infrared cutoff with the $\Theta$ function demonstrates clearly once more quark confinement. Indeed, introducing the new variables $q = \lambda q'$ for $\lambda \to 0$, we find that the terms with external sources vanish and the quarks contribute only to the loop diagrams.

asymptotic freedom, i.e., a small value of the coupling constant at short distances, which agrees with the idea that at short distances hadrons consist of noninteracting point quark-partons. Feynman's quark-parton model, which is based on this idea, successfully describes the data on deep inelastic processes,[50] and thus indicates that hadrons have the structure of composite particles. Thus, the hadrons could be regarded as collective excitations of elementary fields of quarks and gluons. The unobservability of the quarks and gluons (quark confinement) means that they are not present in the complete spectrum of elementary excitations.

An important argument for the correctness of this interpretation of hadrons is the chiral symmetry of the strong interactions, which is exact at high energies (at short distances for the quarks and gluons) and is spontaneously broken at low energies. In this connection, it should be noted that the low-energy physics of hadrons can be satisfactorily described by the theorems of current algebra and PCAC.[51] The results of these theorems are reproduced by chiral phenomenological Lagrangians[52] with spontaneous breaking of chiral symmetry and Goldstone pions. From the point of view of "collective fields," quantum chromodynamics must appear as a "microscopic theory," explaining dynamically (without the Higgs effect) the spontaneous breaking of chiral symmetry, just as the microscopic Bogolyubov-BCS theory of superfluidity explains the phenomenology of superconductivity and the spontaneous breaking of gauge symmetry (another similar example has been considered above). From this point of view, it is to be expected that chiral phenomenological Lagrangians, written down directly in terms of *collective excitations (hadrons)* represent the low-energy limit of quantum chromodynamics. The description of hadrons as collective fields in the framework of quantum chromodynamics for $D=4$ space-time dimensions encounters serious difficulties. In contrast to Abelian theory or $QCD_2$ we cannot, because of the existence of gluon self-interaction terms $\sim A_\mu^3$ and $\sim A_\mu^4$, exactly calculate the path integrals over the Yang-Mills fields $A_\mu$.

During the last few years, there have been several new discoveries in the quantum theory of non-Abelian gauge fields. The most important of them is the discovery of a periodic structure of the classical vacuum in Yang-Mills theory.[53] It is the periodic vacuum that provides the solution of the problem of the number of Goldstone mesons.[54] We discuss in conclusion this question.

Historically, the periodic structure of the vacuum in Yang-Mills theory was discovered through the requirement of finiteness of the action[53] [we here consider the group $SU(2)$]:

$$S=\frac{1}{4}\int d^4x G^a_{\mu\nu}G^{\mu\nu,a};\quad G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-g\varepsilon^{abc}A^b_\mu A^c_\nu.$$

This requirement follows from the quasiclassical treatment of the Euclidean path integral $\int DA\exp(-S)$. All fields with finite action must be purely longitudinal at infinity:

$$A_\mu=v^{-1}\partial_\mu v/g;\quad A_\mu=\mathrm{i}\frac{\tau^a}{2}A^a_\mu.$$

With each matrix $v(x)$ there is associated a mapping of $S^3$ (of the boundary of four-space) into the group $SU(2)$. This mapping is characterized by an integer (the Pontryagin index), which indicates how many times the boundary of $S^3$ is wrapped around the $SU(2)$ sphere by the mapping. Therefore, all fields with finite action are divided into different topologically inequivalent classes with index $\nu$ calculated in accordance with the formula

$$\nu=-\frac{g^2}{16\pi^2}\int d^4x\,\mathrm{tr}\,\widetilde{G}_{\mu\nu}G^{\mu\nu}=-\frac{g^2}{32\pi^2}\int d^4x\widetilde{G}^a_{\mu\nu}G^{\mu\nu,a}=-\frac{1}{16\pi^2}\int ds_\mu X^\mu;$$
$$X^\mu=2\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{tr}\left(A^\nu\partial^\alpha A^\beta+\frac{2}{3}A^\nu A^\alpha A^\beta\right);\quad \widetilde{G}_{\alpha\beta}=\varepsilon_{\alpha\beta\mu\nu}G^{\mu\nu}/2.$$

The minimum of the action in each class is attained for fields that are called pseudoparticles or instantons, these satisfying the (Euclidean) self-duality equation $G_{\mu\nu}=\pm\widetilde{G}_{\mu\nu}$. The instantons have zero energy and are interpreted as classical solutions that interpolate from one classical vacuum to another. In the field theory of interacting massless quarks with Yang-Mills fields, the divergence of the axial current, which corresponds to $U(1)$ chiral symmetry, is not zero. This is a consequence of the Adler anomalies[55] and the existence of topologically nontrivial configurations of the Yang-Mills fields with nonzero Pontryagin index $\nu\neq 0$:

$$\partial_\mu J^A_\mu\sim G^a_{\mu\nu}\widetilde{G}^{\mu\nu,a}\sim\nu\neq 0.$$

From the nonconservation of the chiral $U(1)$ current one can deduce absence of the ninth pseudoscalar meson.[54] A quasiclassical expansion[56] around the instantons has led to interesting results. However, a discussion of these results would go far beyond the ambit of the present review.

## APPENDIX 1

### Single-particle Green's function

In this Appendix, we give some formulas for the Green's functions used in the review.

We introduce the "free" fermion Green's function

$$\left.\begin{aligned}&G^{-1}_{\pm}(t,t')=(\mathrm{i}\partial_t\mp\varepsilon)\,\delta(t-t');\\&\mathrm{i}G_{\pm}(t,t')=\pm\Theta(\pm(t-t'))\exp[\mp\mathrm{i}\varepsilon(t-t')];\\&\overline{G}^{-1}_{\pm}(t,t')=(\mathrm{i}\partial_t\pm\varepsilon)\,\delta(t-t');\\&\mathrm{i}\overline{G}_{\pm}(t,t')=-\mathrm{i}G_{\pm}(t',t),\end{aligned}\right\}\tag{A.1}$$

where $\Theta(t)$ is the step function. By definition,

$$\int dxG^{-1}_{(\dots)}(t,x)G_{(\dots)}(x,t')=\delta(t-t').$$

It is convenient to rewrite the inverse Green's function defined by Eq. (10) in the form

$$G^{-1}(m;t,t')=\begin{pmatrix}G^{-1}_a(t,t') & \Psi(t)\,\delta(t-t')\\ \Psi^+(t)\,\delta(t-t') & G^{-1}_b(t,t')\end{pmatrix},\tag{A.2}$$

where

$$G^{-1}_a(t,t')=\begin{pmatrix}G_+(t,t') & \Phi(t)\,\delta(t-t')\\ \Phi(t)\,\delta(t-t') & G^{-1}_-(t,t')\end{pmatrix};\tag{A.3}$$

$$G^{-1}_b(t,t')=\begin{pmatrix}\overline{G}^{-1}_+(t,t') & -\Phi(t)\,\delta(t-t')\\ -\Phi(t)\,\delta(t-t') & \overline{G}^{-1}_-(t,t')\end{pmatrix}.\tag{A.4}$$

The matrix operator that is the inverse of (A.2) has the form

$$G(m;\,t,\,t')=\begin{pmatrix} G_N(t,\,t') & \overline{G}_A(t,\,t') \\ G_A(t,\,t') & \overline{G}_N(t,\,t') \end{pmatrix}, \tag{A.5}$$

where the normal and anomalous Green's functions are defined by the equations[14)]

$$\left.\begin{aligned} G_N &= G_a(1-\Psi G_b\Psi^+ G_a)^{-1};\\ \overline{G}_N &= G_b(1-\Psi^+ G_a\Psi G_b)^{-1};\\ G_A &= -G_b\Psi^+ G_a(1-\Psi G_b\Psi^+ G_a)^{-1};\\ \overline{G}_A &= -G_a\Psi G_b(1-\Psi^+ G_a\Psi G_b)^{-1}, \end{aligned}\right\} \tag{A.6}$$

where

$$G_a=\begin{pmatrix} G_+(1-\Phi G_-\Phi G_+)^{-1} & -G_+\Phi G_-(1-\Phi G_+\Phi G_-)^{-1} \\ -G_-\Phi G_+(1-\Phi G_-\Phi G_+)^{-1} & G_-(1-\Phi G_+\Phi G_-)^{-1} \end{pmatrix}; \tag{A.7}$$

$$G_b=\begin{pmatrix} \overline{G}_+(1-\Phi\overline{G}_-\Phi\overline{G}_+)^{-1} & \overline{G}_+\Phi\overline{G}_-(1-\Phi\overline{G}_+\Phi\overline{G}_-)^{-1} \\ \overline{G}_-\Phi\overline{G}_+(1-\Phi\overline{G}_-\Phi\overline{G}_+)^{-1} & \overline{G}_-(1-\Phi\overline{G}_+\Phi\overline{G}_-)^{-1} \end{pmatrix}. \tag{A.8}$$

The Green's functions for configurations of the static fields $\Phi_0$ and $\Psi_0$ can be readily calculated by means of (A.6)–(A.8). Setting $\tau=t-t'$, for the normal Green's function we obtain the expression

$$G_N(\tau)=-\mathrm{i}\,\{\Theta(\tau)\,\hat{a}(E)\exp(-\mathrm{i}E\tau)-\Theta(-\tau)\,\hat{a}(-E)\exp(\mathrm{i}E\tau)\}, \tag{A.9}$$

where

$$\hat{a}(\pm E)=\pm\frac{1}{2E}\begin{pmatrix} \pm E+\varepsilon & -\Phi_0 \\ -\Phi_0 & \pm E-\varepsilon \end{pmatrix} \tag{A.10}$$

and

$$E=\sqrt{\varepsilon^2+\Phi_0^2+|\Psi_0|^2}$$

are the quasiparticle energies determined in (16). Similarly, for the anomalous Green's functions we find the expression

$$\overline{G}_A(\tau)=\mathrm{i}\Psi_0\,\{\Theta(\tau)\,\hat{b}(E)\exp(-\mathrm{i}E\tau)-\Theta(-\tau)\,\hat{b}(-E)\exp(\mathrm{i}E\tau)\}, \tag{A.11}$$

where

$$\hat{b}(\pm E)=\pm\frac{1}{2E}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Green's functions satisfy the symmetry relations

$$\overline{G}_N(\tau)=-G_N(-\tau);\quad \overline{G}_A(\tau)/\Psi_0=G_A(\tau)/\Psi_0^{\dagger}. \tag{A.12}$$

## APPENDIX 2

### Definitions of the Matrix B

The elements of the "loop" matrix $\hat{B}$ are defined as $\hat{B}=-\mathrm{i}\Omega\hat{A}/2$, where

$$\begin{aligned}
A_{11}(t,\,t') &= \mathrm{tr}\,\{\hat{G}_N(t,\,t')\,\hat{G}_N(t',\,t)+\hat{\overline{G}}_N(t,\,t')\,\hat{\overline{G}}_N(t',\,t)\\
&\quad -[\hat{\overline{G}}_A(t,\,t')\,\hat{G}_A(t',\,t)+\hat{G}_A(t,\,t')\,\hat{\overline{G}}_A(t',\,t)]\};\\
A_{12}(t,\,t') &= \mathrm{tr}\,\{G_N(t,\,t')\,\hat{G}_A(t',\,t)-G_A(t,\,t')\,\hat{\overline{G}}_N(t',\,t)\};\\
A_{13}(t,\,t') &= \mathrm{tr}\,\{\overline{G}_A(t,\,t')\,\hat{G}_N(t',\,t)-\overline{G}_N(t,\,t')\,\hat{\overline{G}}_A(t',\,t)\};\\
A_{21}(t,\,t') &= \mathrm{tr}\,\{\hat{G}_A(t,\,t')\,G_N(t',\,t)-\hat{\overline{G}}(t,\,t')\,G_A(t',\,t)\};\\
A_{22}(t,\,t') &= \mathrm{tr}\,\{G_A(t,\,t')\,G_A(t',\,t)\};\\
A_{23}(t,\,t') &= \mathrm{tr}\,\{\overline{G}_N(t,\,t')\,G_N(t',\,t)\};\\
A_{31}(t,\,t') &= \mathrm{tr}\,\{\hat{G}_N(t,\,t')\,\overline{G}_A(t',\,t)-\hat{\overline{G}}_A(t,\,t')\,\overline{G}_N(t',\,t)\};\\
A_{32}(t,\,t') &= \mathrm{tr}\,\{G_N(t,\,t')\,\overline{G}_N(t',\,t)\};\\
A_{33}(t,\,t') &= \mathrm{tr}\,\{\overline{G}_A(t,\,t')\,\overline{G}_A(t',\,t)\}.
\end{aligned} \tag{A.13}$$

We have used here the abbreviated notation

$$\hat{G}_A(t,\,t')=G_A(t,\,t')\,\hat{I},\quad \hat{I}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \text{etc.}$$

## APPENDIX 3

In this Appendix, we give some useful equations that are frequently used.[1-4]

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[14)]The matrix multiplication here includes integration over the intermediate times.

We consider the real and complex Gaussian integrals

$$\int_{-\infty}^{\infty}\frac{d\varphi}{\sqrt{2\pi\mathrm{i}}}\exp\left[\mathrm{i}\left(\frac{1}{2}\varphi A\varphi+j\varphi\right)\right]=A^{-1/2}\exp\left(-\frac{\mathrm{i}}{2}jA^{-1}j\right); \tag{A.14}$$

$$\int\frac{d\varphi}{\sqrt{2\pi\mathrm{i}}}\frac{d\varphi^*}{\sqrt{2\pi\mathrm{i}}}\exp\left[\mathrm{i}\left(\varphi^*A\varphi+j^*\varphi+\varphi^*j\right)\right]=A^{-1}\exp\left(-\mathrm{i}j^*A^{-1}j\right), \tag{A.15}$$

where $\int d\varphi\, d\varphi^*$ is a symbolic representation of $2\int d(\mathrm{Re}\varphi)d(\mathrm{Im}\varphi)$; $A$ has a positive imaginary part that does not vanish. (In the case of real $A$, we define the integral by taking $A\to A+i\varepsilon$ $(\varepsilon>0)$, $\varepsilon\to 0$.) Suppose $\varphi$ is the $n$-component vector $\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_n)$, and $A$ is a nonsingular symmetric or Hermitian matrix. Then (A.14) and (A.15) are generalized by

$$\int D\varphi\exp\left\{\mathrm{i}\left[\frac{1}{2}(\varphi,\,A\varphi)+(j,\,\varphi)\right]\right\}=(\det A)^{-1/2}\exp\left[-\frac{\mathrm{i}}{2}(j,\,A^{-1}j)\right]; \tag{A.16}$$

$$\int D\varphi D\varphi^+\exp\left\{\mathrm{i}\left[(\varphi^+,\,A\varphi)+(j^+,\varphi)+(\varphi^+,j)\right]\right\}=(\det A)^{-1}\exp\left[-\mathrm{i}\,(j^+,A^{-1}j)\right], \tag{A.17}$$

where $(j,\varphi)$ denotes the scalar product $\sum_{i=1}^{n} j_n\varphi_n$, etc., and we have used the abbreviation

$$D\varphi=\prod_{k=1}^{n} d\varphi_k/\sqrt{2\pi i}.$$

When $A$ is diagonal, $\det A=A_1A_2\ldots A_n$, and Eqs. (A.16) and (A.17) follow trivially from (A.14) and (A.15). For a nondiagonal matrix, (A.16) and (A.17) can be proved by diagonalization by means of an orthogonal or unitary matrix using the invariance of the scalar product and the measure of integration. Note that (A.17) is a special case of the more general integral

$$\int D\varphi D\varphi^+\exp\left\{\mathrm{i}\left[(\varphi^+,\,A\varphi)+(b^+,\varphi)+(\varphi^+,a)\right]\right\}=(\det A)^{-1}\exp\left[-\mathrm{i}(b^+A^{-1}a)\right]. \tag{A.18}$$

We also give a helpful matrix generalization of Eqs. (A.16) and (A.17) (Ref. 3):

$$\begin{aligned}
&\int D\Phi\exp\left\{\mathrm{i}\left[\frac{1}{2}(\varphi\varphi^+)\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \varphi \\ \varphi^+ \end{pmatrix}+(j_1j_2)\begin{pmatrix} \varphi \\ \varphi^+ \end{pmatrix}\right]\right\}\\
&=\left[\det\begin{pmatrix} A_{21} & A_{22} \\ A_{11} & A_{12} \end{pmatrix}\right]^{-1/2}\exp\left\{-\frac{\mathrm{i}}{2}(j_1j_2)\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1}\begin{pmatrix} j_1 \\ j_2 \end{pmatrix}\right\},
\end{aligned} \tag{A.19}$$

where

$$D\Phi=D\varphi D\varphi^+.$$

Using the ordinary Gaussian integrals given above, we define the path integrals used in the review.

i) *The case of Bose statistics:*

$$\begin{aligned}
&\int D\varphi\exp\left\{\mathrm{i}\int d^4x\,d^4y\,[\varphi(x)A(x,\,y)\varphi(y)+j(x)\varphi(y)\delta^4(x-y)]\right\}\\
&[\det A(xy)]^{-1/2}\exp\left\{-\frac{\mathrm{i}}{2}\int d^4x\,d^4y\,j(x)A^{-1}(x,\,y)\,j(y)\right\};
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
&\int D\varphi D\varphi^+\exp\left\{\mathrm{i}\int d^4x\,d^4y\,[\varphi^+(x)A(x,\,y)\varphi(y)\right.\\
&\left.\qquad+(j^+(x)\varphi(y)+\varphi^+(x)j(y))\,\delta^4(x-y)]\right\}\\
&=[\det A(x,\,y)]^{-1}\exp\left\{-\mathrm{i}\int d^4x\,d^4y\,j^+(x)A^{-1}(x,y)\,j(y)\right\},
\end{aligned} \tag{A.21}$$

where $\det A(x,y)$ is the functional determinant of the operator $A(x,y)$; the inverse operator satisfies the equation

$$\int d^4y A(x,\,y)\,A^{-1}(y,\,z)=\delta^{(4)}(x-z). \tag{A.22}$$

Equations (A.20) and (A.21) are understood as the $n\to\infty$, $\varepsilon\to 0$ limit of the ordinary integrals. Here, $\varepsilon$ is the volume of the cells of the lattice space with points $x_j=j\varepsilon$ $(j=0,\pm 1,\pm 2,\ldots)$ and one makes the substitution $\varphi(x)\to(\varphi_1,\varphi_2,\ldots,\varphi_n)$ with field values $\varphi_j=\sqrt{\varepsilon}\varphi(x_j)$ ($n$ is the number of cells).

ii) *Case of Fermi statistics.* Classical Fermi fields are regarded as anticommuting elements of a Grassmann algebra with involution, i.e.,

$$\{\psi(x),\ \psi(y)\}_+ = \{\psi(x),\ \psi^+(y)\}_+ = \{\psi^+(x),\ \psi^+(y)\}_+ = 0. \tag{A.23}$$

The path integral can be defined as the $n\to\infty$ limit for an integral with respect to $n$-dimensional vectors in the space of Grassmann variables: $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$, $\psi^+ = (\psi_1^+, \psi_2^+, \ldots, \psi_n^+)$. Using the following definitions for integrals over Grassmann variables,

$$\int d\psi_i = \int d\psi_i^+ = 0;\quad \int \psi_i\, d\psi_i = \int \psi_i^+\, d\psi_i^+ = 1, \tag{A.24}$$

we can obtain a formula analogous to (A.17):

$$\int D\psi D\psi^+ \exp\{i\,[(\psi^+,\ A\psi) + (\eta^+,\ \psi) + (\psi^+,\ \eta)]\} = [\det A]\exp[i\,(\eta^+,\ A^{-1}\eta)]. \tag{A.25}$$

Note that $[\det A]$ here occurs in a positive power, in contrast to Bose statistics. It is also easy to obtain the generalization

$$\int D\psi D\psi^+ \exp\left\{i\left[(\psi_1^+\psi_2)\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2^+\end{pmatrix} + (\eta_1^+\eta_2)\begin{pmatrix}\psi_1\\ \psi_2^+\end{pmatrix} + (\psi_1^+\psi_2)\begin{pmatrix}\eta_1\\ \eta_2^+\end{pmatrix}\right]\right\}$$
$$= \left[\det\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\right]\exp\left\{-i\left[(\eta_1^+\eta_2)\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}^{-1}\begin{pmatrix}\eta_1\\ \eta_2^+\end{pmatrix}\right]\right\}, \tag{A.26}$$

where $\psi^+ = (\psi_1^+, \psi_2)$.

Finally, we write down the functional integral obtained by going to the limit in (A.25):

$$\int D\psi D\psi^+ \exp\left\{i\int d^4x\, d^4y\, [\psi^+(x)\, A(x,\ y)\, \psi(y) + (\eta^+(x)\,\psi(y) + \psi^+(x)\,\eta(y))\,\delta^4(x-y)]\right\}$$
$$= [\det A(x,\ y)]\exp\left\{-i\int d^4x\, d^4y\, \eta^+(x)\, A^{-1}(x,\ y)\,\eta(y)\right\}. \tag{A.27}$$

iii) *Bilocal path integral.* We consider the path integral over the bilocal matrix functions $\Phi_{\alpha\beta}(x, y)$, where the pair $(\alpha, \beta)$ ranges over the Dirac indices and (or) internal symmetry indices; $\Phi$ is a Bose variable. The corresponding bilocal Gaussian integral has the form $(d^4x d^4y \equiv d^4(x, y))$:

$$\int D\Phi(x,\ y)\exp\left\{i\left[\frac{1}{2}\int d^4(x,\ y)\, d^4(x',\ y')\,\Phi_{\alpha\beta}(x,\ y)\, A_{(\beta\alpha;\,\gamma\delta)}(yx;\ x'y') \times \Phi_{\delta\gamma}(y',x') + \int d^4(x,\ y)\,\Phi_{\alpha\beta}(x,\ y)\, J_{\beta\alpha}(y,\ x)\right]\right\}$$
$$= [\mathrm{Det}\, A_{(mn,\ m'n')}]^{-1/2}\exp\left\{-\frac{i}{2}\int d^4(x,\ y)\, d^4(x',\ y')\, J_{\alpha\beta}(x,\ y) \times A^{-1}_{(\beta\alpha;\,\gamma\delta)}(yx;\ x'y')\, J_{\delta\gamma}(y',\ x')\right\}. \tag{A.28}$$

To see this, we introduce the orthogonal system of matrix functions $[H_{mn}(x, y)]_{\alpha\beta} \equiv (\Gamma_m)_{\alpha\beta} h_n(x, y)$; here, $h_n(x, y)$ is Hermitian: $h_n(x, y) = h_n^*(y, x)$; $\Gamma_m$ is a complete set of Dirac and internal-symmetry matrices. The functions $H$ satisfy the completeness and orthogonality relations

$$\mathrm{tr}\int d^4x\, d^4y\, H_{mn}(x,\ y)\, H_{kl}(y,\ x) = \delta_{mk}\delta_{nl}; \tag{A.29}$$

$$\sum_{m,n}[H_{mn}(x,\ y)]_{\alpha\beta}\,[H_{mn}(x',\ y')]_{\gamma\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma}\delta^4(x-y')\,\delta^4(y-x'). \tag{A.30}$$

Using the Fourier expansion

$$\left.\begin{aligned}
\Phi(x,\ y) &= \sum_{m,n}\Phi_{mn}H_{mn}(x,\ y);\\
A(yx;\ x'y') &= \sum_{(m,n),\,(m'n')} H_{mn}(y,\ x)\, A_{mn,\ m'n'}H_{m'n'}(x',\ y');\\
J(x,\ y) &= \sum_{m,n} J_{mn}H_{mn}(x,\ y)
\end{aligned}\right\} \tag{A.31}$$

and substituting Eqs. (A.31) in the left-hand side of (A.28), we obtain $(D\Phi(x, y) \equiv \Pi_{(m,n)} D\Phi_{mn})$:

$$\int\prod_{(m,\ n)} D\Phi_{mn}\exp\left\{i\left[\frac{1}{2}\sum_{(mn),\,(m'n')}\Phi_{mn}A_{mn,\ m'n'}\Phi_{m'n'} + \Phi_{mn}J_{mn}\right]\right\}$$
$$= [\mathrm{Det}\, A_{mn,\ m'n'}]^{-1/2}\exp\left\{-\frac{i}{2}\sum_{(mn),\,(m'n')} J_{mn}A^{-1}_{mn,\ m'n'}J_{m'n'}\right\}. \tag{A.32}$$

The right-hand side of Eq. (A.28) follows from (A.32) by means of the substitution

$$A^{-1}_{mn,\ m'n'} = \int d^4(xy)\, d^4(x'y')\, H_{mn}(x,\ y)\, A^{-1}(yx;\ x'y')\, H_{m'n'}(y'x').$$


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Translated by Julian B. Barbour