Exact results for one-dimensional many-particle systems

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Some exact results for one-dimensional many-body systems, both classical and quantum, are reviewed.

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INTRODUCTION

It is well known that neither the classical nor the quantum problem of three or more particles with a realistic interaction between them (Coulomb or nuclear) admits an explicit solution. This explains the interest in simpler but in consequence exactly solvable models of n interacting particles. There are reasons to believe that some qualitative features of such models are also preserved in the real case. In addition, these models can be helpful for estimating the accuracy of the different approximate methods widely used in nuclear physics.

In the three-dimensional case, an exact solution is known only for a system of n particles coupled by oscillator forces:

$$U(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n) = \sum_{j<k} V(\mathbf{r}_j - \mathbf{r}_k); \quad V(\mathbf{r}) = \omega^2 r^2/2.$$

However, after the introduction of Jacobi coordinates this model goes over into a model of $n-1$ particles, each of them moving independently in the field of the common oscillator well. It therefore hardly differs from the model of a single particle.

In the one-dimensional case (classical and quantum) one can obtain exact results for a larger class of potentials. As the first example of an exactly solvable one-dimensional quantum model of n interacting particles one can take the model with a δ-function interaction between the particles: $V(q) = g\delta(q)$.[1-5] We also mention the model with an interaction of the form $V(q) = g|q|$ (Ref. 6). In the present paper, we shall consider neither of these models, nor the classical model with an interaction between only nearest neighbors of the form $V(q) = g\exp(-q)$, the so-called Toda chain.[7-11]

In the present paper, we study in detail the one-dimensional classical and quantum models of n interacting particles with potentials of the following five forms[1)]:

I. $V(q) = g^2 q^{-2}$;
II. $V(q) = g^2 a^2 \operatorname{sh}^{-2} aq$;
III. $V(q) = g^2 a^2 \sin^{-2} aq$;
IV. $V(q) = g^2 a^2 \wp(aq)$;
V. $V(q) = g^2 q^{-2} + \omega^2 q^2$.

Here, $\wp(q)$ is the Weierstrass function.

A number of exact results on the form of the wave functions, the spectrum and the nature of the scattering for the quantum one-dimensional many-particle problem with potentials of the form (I) and (V) (Calogero model)[12-17] and potential of the form (III) (Sutherland model)[18] were obtained in 1969–1975. Not far removed from this cycle are the papers of Refs. 19 and 20, in which a three-particle system with a nonbinary interaction of the type

$$U(q_1, q_2, q_3) = g_1^2 \sum_{i<j} V(q_i - q_j) + g_2^2 \sum_{\substack{i\neq j,\, j\neq k \\ i\neq k}} V(2q_i - q_j - q_k) \quad (1)$$

with a function $V(q)$ of the form (I) or (V) was studied.

Further progress in this field is associated with the use (in the classical case) of a new method-the method of isospectral deformation, which was first applied to the Korteweg—de Vries equation, and then to the nonlinear Schrödinger equation and the sine–Gordon equation. Application of this technique to many-particle problems[9,10,21] made it possible to establish the complete integrability of the classical many-particle problems with potentials of the form (I)–(V),[21-27] Toda chains,[9-11] and one further system of the type (II).[28]

It was shown in Ref. 24 that these systems have a high hidden symmetry and are a special case of a larger class of systems associated with semisimple Lie algebras. Subsequently, in Refs. 29 and 30, the principles of complete integrability of classical systems of such type were established. In particular, it was shown that to systems with potentials of the form (I)–(III) there corresponds free motion (motion along a geodesic) in definite spaces (symmetric spaces) of more than n dimensions. This connection made it possible to find a natural generalization of this class of systems and completely integrate the equations of motion by means of the method of projection of free motion.[29,30]

The idea of the method is to consider the free motion along a geodesic in a definite (symmetric) space of zero, negative, or positive curvature, the space having more than n dimensions. After special projection onto a space of fewer dimensions (n-dimensional space) one no longer has a free motion, but motion in the potential field

$$U(q_1, \ldots, q_n) = \sum_{j<k} V(q_j - q_k),$$

where $V(q)$ has the form (I)–(III).

Let us illustrate this by very simple examples.

1. Suppose a particle of unit mass moves freely with momentum $\mathbf{p}$ in the two-dimensional plane $\mathbf{x} = (x_1, x_2)$. Then the radial motion is described by the Hamiltonian

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[1)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

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$$H = p^2/2 + g^2 q^{-2}, \tag{2}$$

where $p = |\mathbf{p}|, q = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2}, E = \mathbf{p}^2/2$.

2. If we consider a two-dimensional isotropic oscillator with frequency ω, then similarly we arrive at the Hamiltonian

$$H = p^2/2 + g^2 q^{-2} + \omega^2 q^2/2. \tag{3}$$

3. For free motion of a particle on the upper sheet of the two-sheeted hyperboloid $H^2 = \{\mathbf{x} | \mathbf{x}^2 = x_0^2 - x_1^2 - x_2^2 = 1, x_0 > 1\}$ we obtain

$$H = p^2/2 + g^2 \operatorname{sh}^{-2} q \quad (x_0 = \operatorname{ch} q). \tag{4}$$

4. For free motion of a particle on the single-sheeted hyperboloid $H^2 = \{\mathbf{x} | \mathbf{x}^2 = x_0^2 - x_1^2 - x_2^2 = -1\}$ we have

$$H = p^2/2 - g^2 \operatorname{ch}^{-2} q \quad (x_0 = \operatorname{sh} q). \tag{5}$$

5. For free motion of a particle on the two-dimensional sphere

$$S^2 = \{\mathbf{x} | \mathbf{x}^2 = x_0^2 + x_1^2 + x_2^2 = 1\}$$

we obtain the Hamiltonian

$$H = p^2/2 + g^2 \sin^{-2} q \quad (x_0 = \cos q). \tag{6}$$

It was found that a similar treatment for more complicated symmetric spaces makes it possible to integrate effectively the equations of motion in the many-particle case as well.

This approach was subsequently extended to the quantum case,[31-33] which was made possible by, first, the use of the results obtained for the classical case in Ref. 24, and, second, by the establishment of a connection between the many-particle Hamiltonian and the so-called *Laplace-Beltrami operator* on the corresponding symmetric space.[32] Because of this connection, many mathematical results can be translated into the language of the quantum systems considered here. On the other hand, a number of the formulas obtained in the physical investigations mentioned above are new mathematical results.

1. THE CLASSICAL CASE

Completely Integrable Hamiltonian Systems. We consider a dynamical system with n degrees of freedom and Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + U(\mathbf{q}), \quad \mathbf{q} = (q_1, \ldots, q_n). \tag{7}$$

Here and in what follows, p_j and q_k are the momenta and coordinates, and the dot denotes the derivative with respect to the time. Such a system is described by the Hamilton equations

$$\dot{p}_j = -\partial U / \partial q_j, \quad \dot{q}_j = p_j \tag{8}$$

and is said to be completely integrable if there exist variables $I_j(\mathbf{p}, \mathbf{q})$ and $\varphi_k(\mathbf{p}, \mathbf{q})$ *of the action–angle type defined globally on the complete phase space*[2)] (see, for example, Ref. 34).

[2)] It should not be thought that such variables always exist. Usually, variables of the action–angle type have a local nature and cannot be defined on the complete phase space.

Such variables have a simple time dependence:

$$I_j(t) = \text{const}, \quad \varphi_k(t) = \varphi_k^0 + \omega_k t.$$

If $I_j(\mathbf{p}, \mathbf{q})$ and $\varphi_k(\mathbf{p}, \mathbf{q})$ are known and the coordinates and momenta can be expressed in terms of them in the form

$$q_j = q_j(\mathbf{I}, \boldsymbol{\varphi}), \quad p_k = p_k(\mathbf{I}, \boldsymbol{\varphi}),$$

it is then possible to integrate the equations of motion of the system. Thus, for completely integrable systems there is the basic possibility of integrating the equations of motion, but this possibility can be realized in practice only in rare cases. A criterion for complete integrability of a system is given by Liouville's theorem (see, for example, Ref. 34):

If there exist n functionally independent global integrals of the motion $I_l(\mathbf{p}, \mathbf{q})$ that are in involution, i.e., such that the Poisson brackets of any two integrals vanish,

$$\{I_j, I_k\} = \sum_l \left(\frac{\partial I_j}{\partial p_l} \frac{\partial I_k}{\partial q_l} - \frac{\partial I_j}{\partial q_l} \frac{\partial I_k}{\partial p_l} \right) = 0, \tag{9}$$

then the system is completely integrable.

Thus, completely integrable systems have *n global* integrals of the motion and, therefore, a definite hidden symmetry.

Until recently, only a few such systems were known. We mention some of them:

1) motion in a central (Newtonian) potential $U(\mathbf{q}) = U(|\mathbf{q}|)$;

2) motion in the field of two fixed centers (Euler), with

$$U(\mathbf{q}) = \alpha_1 |\mathbf{q} - \mathbf{a}_1|^{-1} + \alpha_2 |\mathbf{q} - \mathbf{a}_2|^{-1};$$

3) free motion of a point on the surface of a triaxial ellipsoid (Jacobi);

4) one-dimensional motion of three particles with a two-body interaction of the form

$$U(\mathbf{q}) = \sum_{j<k}^{3} g_{jk}^2 (q_j - q_k)^{-2}$$

(Jacobi[35]).

For four and five particles, see Refs. 14 and 15;

5) motion of a rigid body with a fixed point in a number of special cases (Euler, Lagrange, Kovalevskaya).

Recently, considerable progress has been made in this field following the discovery of a new method of integrating nonlinear equations.[36] This method was applied for the first time to dynamical systems of mechanics in Refs. 9, 10, and 21. There are now known a large number of completely integrable mechanical systems, some of which have been completely integrated. We shall consider here only completely integrable many-particle systems. We now find the possible form of the potential energy for such systems.

Many-Particle Systems Having Additional Integrals of the Motion. To find the form of the two-particle potential for which the systems (7) have additional integrals of the motion, we use, following Refs. 21–23, the method of isospectral deformation, which is frequently called Lax's device.[37] We attempt to find a pair of Hermitian $n\times n$ matrices L and M, a so-called L-M pair, whose elements depend on the coordinates q_j and momenta p_k and are such that the Hamilton equations

$$\dot{q}_j = p_j, \quad \dot{p}_j = -\partial U/\partial q_j \tag{10}$$

are equivalent to the single matrix equation

$$i\dot{L} = [M, L] \tag{11}$$

It follows from (11) that the matrix $L(t)$ undergoes a unitary transformation:

$$L(t) = U(t)\, L(0)\, U^{-1}(t); \quad U^{-1} = U^{+}; \quad M = i\dot{U}U^{-1}. \tag{12}$$

Thus, the eigenvalues of the matrix $L(t)$ do not depend on the time or, which is the same thing, the matrix $L(t)$ undergoes an isospectral deformation with the course of time. Therefore, the eigenvalues of the matrix L or, which is more convenient, symmetric functions of the eigenvalues, for example,

$$I_k = \mathrm{Sp}\,(L^k)/k, \tag{13}$$

are integrals of the motion.

If one can find n independent integrals of the motion and show that they are in involution, i.e., that the Poisson brackets vanish, then the system under consideration is completely integrable.

We now implement this program for a system characterized by the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j<k}^{n} V(q_j - q_k); \quad V(-q) = V(q). \tag{14}$$

For the matrices L and M we use the ansatz

$$L_{jk} = p_j\delta_{jk} + i\,(1-\delta_{jk})\, x\,(q_j - q_k); \tag{15}$$

$$M_{jk} = \delta_{jk}\Big(\sum_{l\neq j} z\,(q_j - q_l)\Big) - (1-\delta_{jk})\, y\,(q_j - q_k), \tag{16}$$

where $x(-q) = -x(q), y(-q) = y(q), z(-q) = z(q)$.

Substituting L and M in Lax's equation (11) and requiring that this equation be equivalent to the Jacobi equations (10), we obtain an expression for the function $y(\xi)$:

$$y(\xi) = -x'(\xi) \tag{17}$$

and a functional equation for the functions $x(\xi)$ and $z(\xi)$:

$$x(\xi)\, x'(\eta) - x(\eta)\, x'(\xi) = x(\xi+\eta)\,[z(\xi) - z(\eta)]. \tag{18}$$

The potential energy $V(\xi)$ has the form

$$V(\xi) = x^2(\xi) + \mathrm{const}. \tag{19}$$

The solution of the functional equation (18) is given in Refs. 24, 42, and 43 (see the Appendix).

It is found that

$$z(\xi) = g x''(\xi)/2x(\xi), \tag{20}$$

and the function $x(\xi)$ has the form

$$x(\xi) = \begin{cases} g\xi^{-1}; & (21)\\ ga\,\mathrm{cth}\, a\xi, \quad ga\,\mathrm{sh}^{-1} a\xi; & (22)\\ ga\,\mathrm{ctg}\, a\xi, \quad ga\,\sin^{-1} a\xi; & (23)\\ ga\dfrac{\mathrm{cn}\,(a\xi)}{\mathrm{sn}\,(a\xi)}, \quad ga\dfrac{\mathrm{dn}\,(a\xi)}{\mathrm{sn}\,(a\xi)}. & (24)\end{cases}$$

It follows from (21)–(24) and (19) that we obtain systems with potential energy (I)–(IV).

Knowledge of the functions $x(\xi)$, $y(\xi)$, and $z(\xi)$ enables us to construct an L–M pair for the considered many-particle system and, thus, to find the integrals of the motion in accordance with (13). A small modification of the method of Ref. 25 enables us to include a potential of the form (V) in the treatment. For this, instead of Lax's equation (11), we consider the equations

$$i\dot{L}^{\pm} = [M, L^{\pm}] \pm \omega L^{\pm}. \tag{25}$$

It follows from these equations that although

$$B_k^{\pm} = \mathrm{Sp}\,(L^{\pm})^k/k \tag{26}$$

are not integrals of the motion, they do depend very simply on the time. Namely,

$$B_k^{\pm}(t) = B_k^{\pm}(0)\exp(\mp ik\omega t). \tag{27}$$

From (25) we can also readily obtain integrals of the motion. For example, the matrices

$$N_1 = L^{+}L^{-}; \quad N_2 = L^{-}L^{+}, \tag{28}$$

as is readily seen, satisfy the ordinary Lax equation

$$i\dot{N}_j = [M, N_j], \quad j = 1, 2. \tag{29}$$

The eigenvalues of either of these matrices or the traces of their powers are integrals of the motion.

As is shown in Ref. 25, the matrices $L^{\pm}$ in (25) have the form

$$L^{\pm} = L \pm i\omega Q, \tag{30}$$

where

$$Q = \mathrm{diag}\,(q_1, \ldots, q_n), \tag{31}$$

and the matrix L is defined by (15); here $x(q) = gq^{-1}$. This follows from the simple identity

$$[Q, M] = X; \quad X_{jk} = g\,(1-\delta_{jk})\,(q_j - q_k)^{-1}. \tag{32}$$

Thus, for systems of all five types we have found a series of integrals of the motion.

Proof of Complete Integrability of Many-Particle Systems Having Additional Integrals of the Motion. We show that the considered systems are completely integrable. For this, it is sufficient to show that the integrals of the motion I_k (13) ($k = 2, \ldots, n$) are functionally independent and are in involution: $\{I_k, I_l\} = 0$.

To prove the functional independence, we note that the integrals I_k have the form

$$I_k = \frac{1}{k}\sum_{j=1}^{n} p_j^k + \text{terms of lower degree in the momenta} \tag{33}$$

Therefore, the functional independence of the I_k follows from the functional independence of the $S_k = \sum_{i=1}^{n} p_j^k$, which is readily proved. The proof that the integrals I_k are in involution is a more difficult problem.

As was shown by Moser,[21] for the systems (I) this follows immediately from the circumstance that as $t \to \pm\infty$ the distance between any two particles tends to infinity, $|q_j(t) - q_k(t)| \to \infty$. Indeed, we then have $I_k(t) \to \sum_{j=1}^{n} p_j^k(t)$ and, using the fact that the quantities $\{I_k, I_l\}$ are integrals of the motion, they do not depend on the time. For the systems (II), this proof remains valid.[22]

For the systems (V), it follows directly that $B_k(\mathbf{p}, \mathbf{q})$ (26) are in involution, and this is sufficient for the complete integrability of the systems.

With regard to the systems (III), they, as is noted in Ref. 22, are obtained from the systems (II) by the substitution $a \to ia$, and they are therefore also completely integrable.

However, the proof that the integrals of the motion for the systems (IV) are in involution is a much more difficult problem. Two different proofs were given in Refs. 26 and 27. We give here the proof from Ref. 26. Note that it is valid for the potentials (I)–(IV), and also for some of the systems with nonbinary interaction considered in Ref. 24.

We consider the systems given above with a function $V(q)$ of the form (I)–(IV). Let the $n \times n$ Hermitian matrix $L = P + iX$ be constructed in accordance with (15), where the function $x(\xi)$ satisfies the functional equation (18). Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\psi = (\psi_1, \ldots, \psi_n)$ be the eigenvectors of L corresponding to the eigenvalues λ and $\mu (\lambda \neq \mu)$:

$$L\varphi = (P + iX)\varphi = \lambda\varphi; \quad L\psi = (P + iX)\psi = \mu\psi. \tag{34}$$

We show that if the function $x(\xi)$ satisfies the functional equation (18), then λ and μ are in involution, i.e.,

$$\{\lambda, \mu\} = \sum_{j=1}^{n} \left(\frac{\partial \lambda}{\partial p_j} \frac{\partial \mu}{\partial q_j} - \frac{\partial \lambda}{\partial q_j} \frac{\partial \mu}{\partial p_j} \right) = 0. \tag{35}$$

Conceptually, the method of proof of (35) is close to that of Refs. 44 and 10, in which it was shown that the integrals of the motion are in involution for the nonlinear Schrödinger equation and the Toda chain. Note first that

$$\frac{\partial \lambda}{\partial p_k} = \left(\varphi, \frac{\partial L}{\partial p_k} \varphi \right) = \bar{\varphi}_k \varphi_k; \tag{36}$$

$$\frac{\partial \lambda}{\partial q_k} = \left(\varphi, \frac{\partial L}{\partial q_k} \varphi \right) = i \sum_l x'(q_k - q_l)(\bar{\varphi}_k \varphi_l - \bar{\varphi}_l \varphi_k). \tag{37}$$

Using these relations, we can cast the expression for the Poisson brackets in the form

$$\{\lambda, \mu\} = i \sum_{k, l} (\bar{\varphi}_k \bar{\psi}_k R_{kl} - \varphi_k \psi_k \bar{R}_{kl}) x'(q_k - q_l), \tag{38}$$

where

$$R_{kl} = \varphi_k \psi_l - \varphi_l \psi_k; \quad R_{lk} = -R_{kl}. \tag{39}$$

On the other hand, from Eqs. (34),

$$\varphi_k \psi_k = i(\lambda - \mu)^{-1} \sum_l x(q_k - q_l) R_{lk}. \tag{40}$$

Substituting the expressions for $\varphi_k \psi_k$ and $\bar{\varphi}_k \bar{\psi}_k$ in Eq. (38), we obtain

$$\{\lambda, \mu\} = (\lambda - \mu)^{-1} \sum_{k, l \neq j} \bar{R}_{lk} R_{kj} [x'(q_j - q_k) x(q_k - q_l) - x(q_j - q_k) x'(q_k - q_l)]. \tag{41}$$

Using the functional equation (18), we transform this relation to

$$\{\lambda, \mu\} = (\mu - \lambda)^{-1} \sum_{k, l \neq j} \bar{R}_{lk} R_{kj} [z(q_j - q_k) - z(q_k - q_l)] x(q_j - q_l). \tag{42}$$

Equation (42) contains two sums. In the first of them, we sum over l, and in the second over j. We then use the relation

$$\sum_l x(q_j - q_l) R_{lk} = -i\lambda \psi_k \varphi_j + i\mu \varphi_k \psi_j + i p_j R_{jk} \tag{43}$$

and its complex conjugate. Noting that the function $z(q)$ is even, we obtain

$$\{\lambda, \mu\} = i(\mu - \lambda)^{-1} \lambda \sum_{j \neq k} (\bar{\psi}_k \bar{\varphi}_j R_{kj} + \psi_k \varphi_j \bar{R}_{jk}) z(q_j - q_k) - i(\mu - \lambda)^{-1} \mu \sum_{j \neq k} (\bar{\varphi}_k \bar{\psi}_j R_{kj} + \varphi_k \psi_j \bar{R}_{jk}) z(q_j - q_k). \tag{44}$$

It is easy to see that the expressions for the first and the second sum are antisymmetric. Therefore, $\{\lambda, \mu\} = 0$ and these systems are completely integrable.

Explicit Integration of the Equations of Motion for Potentials V(q) of the Form (I) *and* (V). It was shown above that these systems for potentials of all five types are completely integrable. However, Liouville's theorem, from which this assertion follows, does not give a constructive method for integrating the equations of motion; that is a difficult problem.

Following Ref. 28, we show here how, using the new method, the so-called *projection method*, we can integrate the equations of motion for systems with the potentials (I) and (V).

The idea is to go over from the n-dimensional space to a space of more dimensions, an $N = (n^2 - 1)$-dimensional space, in which the equations of motion take a simpler form and can be readily integrated. Projecting the obtained solution onto the subspace in which we are interested so as to obtain the necessary system, we obtain an explicit solution of the equations of motion:

$$\dot{q}_j = p_j; \quad \dot{p}_j = -\partial U / \partial q_j. \tag{45}$$

1. We consider first the potential (*I*) ($V(q) = g^2 q^{-2}$). As the extended space, we take the space $X^0 = \{x\}$ of Hermitian $n \times n$ matrices with vanishing trace, and in this space we consider free motion. The equations of motion are then

$$\ddot{x} = 0, \tag{46}$$

and the general solution is

$$x(t) = at + b, \tag{47}$$

where $a, b \in X^0$.

Using a unitary transformation U, we reduce the Hermitian matrix x to diagonal form:

$$x(t) = U(t) Q(t) U^{-1}(t). \tag{48}$$

Here

$$Q(t) = \operatorname{diag}(q_1(t), \ldots, q_n(t)) \tag{49}$$

and without loss of generality we can assume that the quantities q_j are ordered $q_1 \leqslant q_2 \leqslant \ldots \leqslant q_n$. Note that in the simplest case $n = 2$, $x = \sum_{j=1}^{3} x_j \sigma_j$ (σ_j are the Pauli matrices), $Q = \operatorname{diag}(-q, q)$ and $q = |\mathbf{x}|$, i.e., the transition from x to Q can be called *spherical projection*.

We now attempt to derive equations for $q_j(t)$ and $p_j(t) = \dot{q}_j(t)$. Differentiating Eq. (48) with respect to the time, we obtain

$$U(t) L(t) U^{-1}(t) = a, \tag{50}$$

where

$$L = P + i[M, Q]. \quad P = \dot{Q}; \tag{51}$$

$$M = -iU^{-1}\dot{U}; \tag{52}$$

L and M are Hermitian $n \times n$ matrices.

Differentiating Eq. (50) with respect to t, we obtain

$$\dot{L}+i[M,L]=0, \tag{53}$$

i.e., Lax's equation (11).

Thus, the pair of matrices L and M must satisfy Eqs. (51) and (53). For matrices L and M having the form (15) and (16), Eq. (53) is satisfied. It is easy to show, by direct substitution of (15) and (16) in (51), that Eq. (51) is also satisfied for the systems (I), i.e., for $V(q) = q^{-2}$.

It should, however, be borne in mind that the matrices a and b in (47) cannot be arbitrary. Indeed, the matrix of the angular momentum

$$N=i[x,\dot{x}]=i[b,a]=U[Q,L]U^{-1} \tag{54}$$

cannot be arbitrary, since $n-1$ eigenvalues of this matrix are equal. Without loss of generality, we can assume that $U(0)=1$; at the same time, the matrices a and b in (47) can be expressed in terms of the initial conditions in accordance with the formulas

$$a=L(0);\quad b=Q(0), \tag{55}$$

where the matrix L is given by (15).

Thus, we have obtained the final result: *the coordinates $q_j(t)$-the solutions of the equations of motion for the system* (I)*-are eigenvalues of the matrix*

$$Q(0)+L(0)t. \tag{56}$$

We now discuss the scattering process. The potential $U(\mathbf{q})$ in (14) vanishes as $q_j-q_k\to\infty$, so that

$$q_j(t)\sim p_j^{\pm}t+q_j^{\pm},\quad \text{as}\quad t\to\pm\infty. \tag{57}$$

Thus, the scattering process is determined by a canonical transformation from the variables (p_i^-, q_k^-) to the variables (p_i^+, q_k^+).

Further, it is easy to see that

$$I_k=\frac{1}{k}\operatorname{Sp}(L^k)=\frac{1}{k}\sum_j(p_j^-)^k=\frac{1}{k}\sum_j(p_j^+)^k. \tag{58}$$

It follows from this that p_j^+ differ from p_k^- only be a permutation:

$$\mathbf{p}^+=S\mathbf{p}^-. \tag{59}$$

Using the condition $q_1<q_2<\ldots<q_n$, we obtain

$$p_1^+<p_2^+<\ldots<p_n^+;\quad p_1^->p_2^->\ldots>p_n^-$$

and, therefore,

$$p_1^-=p_n^+;\quad p_2^-=p_{n-1}^+,\ \ldots,\ p_n^-=p_1^+. \tag{60}$$

We show that q_j^+ and q_k^- also satisfy the analogous condition

$$q_1^-=q_n^+;\quad q_2^-=q_{n-1}^+,\ \ldots;\quad q_n^-=q_1^+. \tag{61}$$

Indeed, it follows from (50) that

$$a=U(\infty)L(\infty)U^{-1}(\infty)=U(-\infty)L(-\infty)U^{-1}(-\infty). \tag{62}$$

In addition,

$$L(\infty)=P^+=\operatorname{diag}(p_1^+,\ldots,p_n^+);$$
$$L(-\infty)=P^-=\operatorname{diag}(p_1^-,\ldots,p_n^-). \tag{63}$$

Hence,

$$P^+=SP^-S^{-1},$$

where

$$S=U^{-1}(\infty)U(-\infty). \tag{64}$$

We now use the equation

$$Q(t)=U^{-1}(t)x(t)U(t)=Pt+i[M,Q]t+U^{-1}(t)bU(t). \tag{65}$$

It follows from this that

$$Q^{\pm}=U^{-1}(\pm\infty)bU(\pm\infty),\quad Q^+=SQ^-S^{-1},$$

and thus the relation (61) is proved.

A different proof of (61) is given in Ref. 38. The relations (60) and (61) mean that in this problem scattering reduces to successive scatterings of individual pairs of particles. As will be shown below, the situation is analogous in the quantum case.

2. We now consider the potential (V):

$$V(q)=g^2q^{-2}+\omega^2q^2.$$

In this case, we consider, not free motion, but harmonic motion in the space X^0:

$$\ddot{x}+\omega^2x=0,\quad x\in X^0. \tag{66}$$

The solution of this equation is

$$x(t)=\frac{a}{\omega}\sin\omega t+b\cos\omega t,\quad a,b\in X^0. \tag{67}$$

Representing this expression in the form

$$x(t)=U(t)Q(t)U^{-1}(t), \tag{68}$$

where $Q(t)=\operatorname{diag}[q_1(t),\ldots,q_n(t)]$ and U is a unitary matrix and differentiating (68) twice with respect to the time, we arrive as in the previous case at the following assertion: *the coordinates $q_j(t)$ of the considered system are eigenvalues of the matrix*

$$Q(0)\cos\omega t+\omega^{-1}L(0)\sin\omega t. \tag{69}$$

Further, it follows from (48) and (68) that

$$\operatorname{Sp}[Q(t)]^k=\operatorname{Sp}[x(t)]^k, \tag{70}$$

but $\operatorname{Sp}[Q(t)]$ is a polynomial in q_j of degree k that is invariant under permutations. Hence, we obtain:

COROLLARY 1. The polynomial of degree k in q invariant under permutations is a polynomial of degree k in $t(\omega=0)$ or $\sin\omega t$ and $\cos\omega t(\omega\neq 0)$.

Note that the explicit solutions of the euqations of motion for the systems (I) and (V) [see (56) and (69)] make it possible to establish a simple relationship between these solutions.

Let $q_j(t)$ be the solution of the equations of motion for the system (I) $(\omega=0)$. Then it follows from (56) and (69) that

$$\tilde{q}_j(t)=q_j\left(\frac{1}{\omega}\operatorname{tg}\omega t\right)\cos\omega t \tag{71}$$

are a solution of the corresponding system of type (V) $(\omega\neq 0)$. Of course, the converse is also true. A similar connection for systems of more general form is given in Ref. 39.

Note also that $\operatorname{Tr}(Q^{k_1}L^{l_1}Q^{k_2}L^{l_2}\ldots)$ have a simple time dependence. Namely, quantities of this kind are polynomials of degree $k=\sum k_j$ in t for $\omega=0$ or in $\cos\omega t$ and $\sin\omega t$ for $\omega\neq 0$. The algebra of the Poisson brackets for such quantities is studied in Ref. 40.

Explicit Integration of the Equations of Motion for the Potentials (II) *and* (III). To integrate similarly the equations of motion for the systems (II) ($V(q)=a^2\sinh^{-2}aq$) and (III) ($V(q)=a^2\sin^{-2}aq$), we note first that the considered set X^0—the set of Hermitian $n\times n$ matrices with vanishing trace—is a symmetric Riemannian space of zero curvature (see Ref. 41) if the natural metric $ds^2=\mathrm{Tr}(dxdx)$ is chosen. We now consider its Cartan-associated spaces of negative curvature X^- and positive curvature X^+ and motion along geodesics in thesespaces. As above, we obtain a solution of the equations of motion for the systems (II) and (III), respectively.

We consider first the system (II). Let X^- be the space of negative curvature corresponding to the space X^0, which is the space of Hermitian, positive-definite $n\times n$ matrices with determinant equal to unity.

Let $x(t)$ be a curve in X^-. Then the matrices $x^{-1}(t)\dot{x}(t)$ and $\dot{x}(t)x^{-1}(t)$ can be regarded as two vector fields on the group $G=SL(n,C)$, which is the group of complex matrices with unit determinant. These fields are not vector fields on the space X^-; their half-sum, as is readily seen, is a vector field on X^-. If now $x(t)$ is a geodesic on the space X^-, the equation for it has the form

$$\frac{d}{dt}\left(\frac{x^{-1}\dot{x}+\dot{x}x^{-1}}{2}\right)=0. \tag{72}$$

Note that this equation can be obtained from the equation for geodesics for the two-sided invariant [under the action of the group $G=SL(n,C)$] metric $ds^2=\mathrm{Tr}(x^{-1}dxx^{-1}dx)$. Indeed, from the condition $\delta ds=0$ we obtain

$$\ddot{x}-\dot{x}x^{-1}\dot{x}=0, \tag{73}$$

whence (72) follows directly.

It is obvious that the following curve is a geodesic on X^-:

$$x(t)=b\exp(2\hat{a}t)\,b^+;\qquad b\in SL(n,C),\quad \hat{a}^+=\hat{a},\quad \mathrm{Sp}\,\hat{a}=0. \tag{74}$$

We now represent the Hermitian, positive-definite matrix $x(t)$ in the form

$$x(t)=U(t)\exp\{2aQ(t)\}U^{-1}(t), \tag{75}$$

where $U(t)$ is a unitary matrix; $Q(t)=\mathrm{diag}[q_1(t),\ldots,q_n(t)]$ is a diagonal matrix with vanishing trace.

Using (75) to calculate $\dot{x}x^{-1}$ and $x^{-1}\dot{x}$, we obtain

$$1)\quad (\dot{x}x^{-1}+x^{-1}\dot{x})/2=2aU(t)L(t)U^{-1}(t), \tag{76}$$

where

$$L(t)=P+\frac{i}{4a}[\exp(-2aQ)M\exp(2aQ)-\exp(2aQ)M\exp(-2aQ)]; \tag{77}$$

$$M=-iU^{-1}(t)\dot{U}(t) \tag{78}$$

is the "angular velocity of rotation", $P=\dot{Q}$;

$$2)\quad (\dot{x}x^{-1}-x^{-1}\dot{x})/2i=U\widetilde{M}U^{-1}, \tag{79}$$

where

$$\widetilde{M}=M-[\exp(2aQ)M\exp(-2aQ)+\exp(-2aQ)M\exp(2aQ)]/2. \tag{80}$$

On the other hand, it follows from (74) that the matrices

$$(\dot{x}x^{-1}+x^{-1}\dot{x})/2=b\hat{a}b^{-1}+(b^+)^{-1}\hat{a}b^+; \tag{81}$$

$$(\dot{x}x^{-1}-x^{-1}\dot{x})/2i=[b\hat{a}b^{-1}-(b^+)^{-1}\hat{a}b^+]/i \tag{82}$$

do not depend on the time.

Differentiating Eqs. (76), (79), (81), and (82) with respect to the time, we obtain equations of Lax's type:

$$i\dot{L}=[M,L]; \tag{83}$$

$$i\dot{\widetilde{M}}=[M,\widetilde{M}], \tag{84}$$

where the matrices, L and $\widetilde{M}$, and M are given by Eqs. (77), (80), and (78). Note that besides the matrix L we have also obtained the matrix $\widetilde{M}$, which also is subjected to an isospectral deformation with the course of time.

We now take the matrices L and M in the form (15) and (16), where $x(q)=a\coth aq(q=1)$. Then, as is well known, Eq. (83) is satisfied. Substituting L and M in Eq. (77), we see that this equation too for L and M is satisfied. We can also verify that Eq. (84) is satisfied at the same time.

Thus, if L and M are chosen in the form (15) and (16) and $x(q)=a\coth aq$, all the consistency conditions are satisfied and we arrive at the final result: *the quantities* $\exp[2aq_j(t)]$, *where* $q_j(t)$ *are solutions of the equations of motion for the system* (II), *are eigenvalues of the matrix*

$$x(t)=b\exp(2\hat{a}t)\,b, \tag{85}$$

where

$$b=\exp[aQ(0)];\quad Q=\mathrm{diag}(q_1,\ldots,q_n), \tag{86}$$

and the matrix $\hat{a}$ is found from the condition

$$2aL(0)=b\hat{a}b^{-1}+b^{-1}\hat{a}b; \tag{87}$$

$$\hat{a}_{jk}=ap_j\delta_{jk}+ia^2(1-\delta_{jk})\,\mathrm{sh}^{-1}a(q_j-q_k). \tag{88}$$

In conclusion, we note that all results for the system (III) can be obtained from the corresponding results for the system (II) by the substitution $a\to ia$. In the limit $a\to 0$, we find the results for the system (I) [$V(q)=q^{-2}$].

2. THE QUANTUM CASE

Completely Integrability of the Considered Systems. We now turn to the generalization of the results obtained for the considered many-particle systems to the quantum case. We recall that we study systems described by the Hamiltonian

$$\hat{H}=\frac{1}{2}\sum_{j=1}^{n}\hat{p}_j^2+U(q_1,\ldots,q_n);\quad \hat{p}_j=-i\frac{\partial}{\partial q_j}\ (\hbar=1); \tag{89}$$

$$U(q_1,\ldots,q_n)=\sum_{j<k}V(q_j-q_k), \tag{90}$$

where the two-particle potential $V(q)$ is a function having one of the five forms (I)–(V).

We shall say that the quantum system (89) is completely integrable if there exist n independent differential operators that commute with the Hamiltonian $\hat{H}$ and with one another, i.e., a complete set of integrals of the motion. In the classical limit ($h\to 0$), the commutators go over into the Poisson brackets, and this definition coincides with the definition of complete integrability of classical mechanics (Liouville's theorem[34]).

We show that the systems studied for the classical case also remain completely integrable for the quantum case.[3)] We consider first systems with potentials $V(q)$ of the form (I)–(IV). For this, to construct integrals of the motion for arbitrary n, we use the results of Refs. 22 and 27. (For systems (I), one can also take the results of Ref. 45).

Following Ref. 22, it is convenient to consider the set of classical integrals of the motion $J_k (k=2,\ldots,n)$, which are the coefficients of the characteristic polynomial $\det(L-\lambda I)$, where the matrix L is determined by Eq. (15), and I is the unit matrix. Note first that in the quantum case we do not here encounter the problem of defining the corresponding operators $\hat{J}_k$, i.e., the problem of the correct arrangement of the momentum opertors $\hat{p}_j$ in the classical expressions, since $\hat{J}_k$ has the form of a sum of terms, each of which contains only commuting operators. Thus, we have a set of well-defined operators $\hat{J}_2,\ldots,\hat{J}_n=\det\hat{L}$.

However, the commutator $[\hat{J}_k,\hat{J}_l]$ of two operators is not a well-defined operator and, therefore, the vanishing of the operator $[\hat{J}_k,\hat{J}_l]$ does not follow from the vanishing of the Poisson brackets $\{\hat{J}_k,\hat{J}_l\}$ but requires a further investigation. In particular, one must prove that $[\hat{J}_2,\hat{J}_k]=[\hat{H},\hat{J}_k]=0$, i.e., that the operators $\hat{J}_k$ are integrals of the motion.

We note first that it is sufficient to show that the highest operator $\hat{J}_W$ is an integral of the motion, i.e., that $[\hat{H},\hat{J}_n]=0$.

For, as is readily seen,

$$\Big[\sum_{l=1}^{n} q_l,\hat{J}_k\Big]=\mathrm{i}\Big(\sum_l\frac{\partial}{\partial p_l}\Big)\hat{J}_k=\mathrm{i}\,(n-k+1)\,\hat{J}_{k-1}. \qquad (91)$$

But, using the Jacobi identity for the operators $\sum_{l=1}^{n} q_l$, $\hat{H}$, and $\hat{J}_k$, we see that if $\hat{J}_k$ is an integral of the motion, then so is $\hat{J}_{k-1}$.

We prove that the operator $\hat{J}_n=\det(\hat{L})$ is an integral of the motion. Following Ref. 22, we concentrate our attention on the dependence of $\hat{J}_n=\det(\hat{L})$ on $\hat{p}_1$, $\hat{p}_2$, and $x_{12}=x(q_1-q_2)$:

$$\hat{J}_n=\hat{A}_{12}(\hat{p}_1\hat{p}_2-x_{12}^2)+\hat{B}_1\hat{p}_1+\hat{B}_2\hat{p}_2+\hat{B}_{12}x_{12}+\hat{C}_{12}. \qquad (92)$$

Here, the coefficients $\hat{A}_{12},\hat{B}_1,\hat{B}_2,\hat{B}_{12}$ and $\hat{C}_{12}$ do not depend on x_{12}, p_1, and p_2.

It is easy to show that the commutator $[\hat{H},\hat{J}_n]$ depends linearly on x'_{kl}, and the contribution of x'_{12} is therefore given by the expression

$$\dot{\hat{J}}_n=\mathrm{i}\,[\hat{H},\hat{J}_n]=2(\hat{B}_2-\hat{B}_1)\,x_{12}x'_{12}+\hat{B}_{12}[(\hat{p}_1-\hat{p}_2)\,x'_{12}+x'_{12}(\hat{p}_1-\hat{p}_2)]/2+\ldots \qquad (93)$$

It can be seen from (93) that the first term in the sum is a well-defined operator, whereas the second is not. Therefore, after reduction of the terms of this type to normal form additional terms arise, and from the vanishing of the Poisson brackets $\{H,J_n\}$ there follows vanishing of the commutator only if these additional terms vanish. We show that this is indeed the case.

For convenience, we use a different expression for the operator $\hat{J}_n$, which was obtained in Ref. 27. Namely,

$$\hat{J}_n=\exp\Big\{-\frac{1}{2}\sum_{k,l}x^2(q_k-q_l)\frac{\partial}{\partial\hat{p}_k}\frac{\partial}{\partial\hat{p}_l}\Big\}(\hat{p}_1\ldots\hat{p}_n). \qquad (94)$$

It follows from this expression that the operator $\hat{J}_n$ contains only quadratic terms in $x_{kl}=x(q_k-q_l)$ and, in particular, quadratic terms in x_{12}. Therefore, the expression (92) for $\hat{J}_n$ can be represented in the form

$$\hat{J}_n=\hat{A}_{12}(\hat{p}_1\hat{p}_2-x_{12}^2)+\hat{B}_1\hat{p}_1+\hat{B}_2\hat{p}_2+\hat{C}'_{12}, \qquad (95)$$

where $\hat{A}_{12},\hat{B}_1,\hat{B}_2$, and $\hat{C}_{12}$ as before do not depend on $\hat{p}_1$, $\hat{p}_2$, or x_{12}. Now, after commutation of the operator J_n, written down in the form (95), with H there arise only well-defined operators and their cancellation follows from the vanishing of the Poisson brackets $\{H,J_n\}$. Thus, we have proved that the operators $\hat{J}_k$ are integrals of the motion.

In the classical case, the integrals of the motion J_k are functions of the eigenvalues of the matrix L. They are homogeneous polynomials of degree k in the variables $p_1,\ldots,p_n,x(q_k-q_l)$, and it follows from (94) that they depend only on $x^2(q_k-q_l)$. In addition, they are invariant under permutations,

$$J_k(s\mathbf{p},s\mathbf{q})=J_k(\mathbf{p},\mathbf{q}), \qquad (96)$$

and the term of the highest power in the momenta in J_k does not depend on the coordinates.

We now consider the operator $[\hat{J}_k,\hat{J}_l]$. It follows from the Jacobi identity that this operator is an integral of the motion, of homogeneity degree $k+l$ in the variables p_j and $x(q_k-q_l)$ and invariant under permutations. However, the coefficients of the highest terms in the momenta are not constant. Under fulfillment of those conditions, one can show[32] that such an integral of the motion is identically equal to zero. This completes the proof of the complete integrability of the systems (I)–(IV).

Besides the integrals $\hat{J}_k$, which are the coefficients of the characteristic polynomial of the matrix $\hat{L}$, interest also attaches to the integrals $\hat{I}_k$ corresponding to the classical expressions $I_k=k^{-1}\,\mathrm{Tr}(L^k)$. To obtain these integrals, it is convenient to express I_k in terms of J_l by means of the well-known formulas for symmetric functions and then reduce the expression for I_k to normal form.

We give the explicit expressions for $\hat{I}_3$, $\hat{I}_4$, and $\hat{I}_5$:

$$3\hat{I}_3=\sum_k\hat{p}_k^3+3\sum_{k\neq l}V(q_k-q_l)\,\hat{p}_l; \qquad (97)$$

$$4\hat{I}_4=\sum_k\hat{p}_k^4+2\sum_{k\neq l}V(q_k-q_l)(2\hat{p}_l^2+\hat{p}_k\hat{q}_l)+\sum_{j\neq k}V^2(q_j-q_k)+2\sum_{j,k,l}V(q_j-q_k)\,V(q_j-q_l)+\sum_{k\neq l}\{2\mathrm{i}V'(q_k-q_l)\,p_l-V''(q_k-q_l)\}; \qquad (98)$$

[3)] In some special cases, a complete set of integrals of the motion was constructed earlier: for systems of three or four particles in Ref. 14, and for a system of five particles in Ref. 15.

$$5\hat{I}_5=\sum_k \hat{p}_k^5+5\sum_{k\neq l} V(q_k-q_l)(p_l^3+p_k^2 p_l)$$
$$+5\sum V^2(q_j-q_k)\,p_k+5\sum V(q_j-q_k)\,V(q_k-q_l)\,p_l$$
$$+5\sum V(q_j-q_k)\,V(q_l-q_k)\,p_k$$
$$+5\sum \{V'(q_k-q_l)\,\mathrm{i}p_l^2-V''(q_k-q_l)\,p_l\}\,. \quad (99)$$

As we have already noted, using the complete set of commuting operators $\hat{I}_2(\mathbf{p},\mathbf{q}),\ldots,I_n(\mathbf{p},\mathbf{q})$ for the system (I), we can construct a complete set of raising and lowering operators for the system (V). These operators satisfy the relations

$$[H,\,B_m^{\pm}]=\pm\, m\omega B_m^{\pm}. \quad (100)$$

Indeed, it follows from (100) that the operator B_m^+ (respectively, B_m^-) raises (respectively, lowers) the energy of the state by $m\omega$.

One can show that operators of this kind are

$$\left.\begin{aligned}B_k^+&=\hat{I}_k(\hat{p}+\mathrm{i}\omega q,\,q)\quad\text{or}\quad D_k^+=\hat{J}_k(\hat{p}+\mathrm{i}\omega q,\,q);\\ B_k&=\hat{I}_k(\hat{p}-\mathrm{i}\omega q,\,q)\quad\text{or}\quad D_k=J_k(\hat{p}-\mathrm{i}\omega q,\,q).\end{aligned}\right\} \quad (101)$$

Using these operators, we can readily construct integrals of the motion and obtain the energy spectrum and the corresponding wave functions for the system (V).

Note that the wave function of the ground state is annihilated by all operators B_k (or D_k):

$$B_k\Psi_0=0\quad\text{or}\quad D_k\Psi_0=0. \quad (102)$$

Note also that the operators B_3, B_4 and B_3^+, B_4^+ were constructed in Ref. 14, and the operators B_5 and B_5^+ in Ref. 15.

Systems (I). Such systems are characterized by the Hamiltonian

$$H=\frac{1}{2}\sum_{j=1}^{n} p_j^2+g^2\sum_{j<k}(q_j-q_k)^{-2};\quad p_j=-\mathrm{i}\partial/\partial q_j. \quad (103)$$

We are interested in the properties of the solutions of the Schrödinger equation

$$H\Psi_k=E_k\Psi_k;$$
$$\Psi_k(\mathbf{q})=0\quad\text{for}\quad q_j=q_{j+1},\ \mathbf{q}=(q_1,\ldots,q_n)\,. \quad (104)$$

We list these properties (some of them are obvious).

1. For $g^2>-\frac{1}{4}$, the operator H is self-adjoint (in other words, there is no "fall toward the center"), and therefore its eigenvalues E_k are real.

2. The energy spectrum is continuous and fills the half-axis $0\leqslant E_k<\infty$.

3. The wave function of a state with zero energy has the form

$$\Psi_0(\mathbf{q})=\prod_{j<k}(q_j-q_k)^{\mu},\quad \mu(\mu-1)=g^2. \quad (105)$$

To see this, substitute $\Psi_0(\mathbf{q})$ in the Schrödinger equation (104) for $E_k=0$ and use the identity

$$\sum_{k,\,l;\,k\neq l}(q_j-q_k)^{-1}(q_j-q_l)^{-1}\equiv 0. \quad (106)$$

4. For $E_k>0$, the solution of the Schrödinger equation (104) can be conveniently sought in the form

$$\Psi_k=\Phi_k\Psi_0. \quad (107)$$

For the function $\Phi_k(\mathbf{q})$, we then obtain the equation

$$-\Big(\Delta+2\mu\sum_{j<k}(q_j-q_k)^{-1}(\partial_j-\partial_k)\Big)\,\Phi_k(\mathbf{q})=k^2\Phi_k(\mathbf{q}), \quad (108)$$

where

$$\Delta=\sum_j \partial_j^2;\quad \partial_j=\partial/\partial q_j;\quad k^2=2E_k. \quad (109)$$

The function $\Phi_k(\mathbf{q})$ is at the same time normalized by the condition $\Phi_k(0)=1$.

5. In the cases $\mu=0$ and $\mu=1$ ($g=0$), explicit expressions can be given for $\Phi_k(\mathbf{q})$:

$$\Phi_k^{(0)}(\mathbf{q})=\frac{1}{n!}\sum_s \exp[\mathrm{i}(s\mathbf{k}\mathbf{q})], \quad (110)$$

where the summation is over all permutations;

$$\Phi_k^{(1)}(\mathbf{q})=C\sum_s\{\varepsilon(s)\exp[\mathrm{i}(s\mathbf{k}\mathbf{q})]\}\Big/\prod_{j<l}(q_j-q_l), \quad (111)$$

where $\varepsilon(s)=+1$ for an even and $\varepsilon(s)=-1$ for an odd permutation.

We also give the expression for $\Phi_k^1(\mathbf{q})$:

$$\Phi_k^{(1)}(\mathbf{q})=\prod_{j<l}\frac{\sin(k_j-k_l)(q_j-q_l)}{(k_j-k_l)(q_j-q_l)}. \quad (112)$$

6. For three values of g^2, namely $g^2=-\frac{1}{4}$, $g^2=2$, and $g^2=12$ for $n=3$, and integral representation for the function $\Phi_k(\mathbf{q})$ is known.[33] Using it, one can calculate the S matrix.

7. The functions $\Phi_k(\mathbf{q})$ are analytic in q_j and invariant under permutations. They can therefore be expanded in series in invariants:

$$S_l=\sum_{j=1}^{n} q_j^l,\quad l=2,\,3,\,\ldots,\,n; \quad (113)$$

$$\Phi_k(\mathbf{q})=\sum C_{m_2,\,\ldots,\,m_n}S_2^{m_2}\ldots S_n^{m_n}. \quad (114)$$

Equation (108), rewritten for this case in the variables $S_2,\ldots,S_n$, has the form[14]

$$\sum_{l,\,m=2}^{n} lm\Big(S_{m+l-2}-\frac{1}{n}S_{l-1}S_{m-1}\Big)\frac{\partial^2\Phi_k}{\partial S_l\,\partial S_m}$$
$$+\sum_{l=2}^{n} l(l-1)\Big(1-\frac{1}{n}\Big)S_{l-2}\frac{\partial\Phi_k}{\partial S_l}+\mu\sum_{l=2}^{n}[S_0S_{l-2}+S_2S_{l-4}+\ldots$$
$$\ldots+S_{l-2}S_0-(l-1)\,S_{l-2}]\,l\frac{\partial\Phi_k}{\partial S_l}=-k^2\Phi_k(\mathbf{q}). \quad (115)$$

From Eq. (115), we can readily obtain the first few terms in the expansion of $\Phi_n(\mathbf{q})$ as $q\to 0$.

8. As is well known, these systems are completely integrable. A complete set of integrals of the motion $\hat{J}_k$ (or $\hat{I}_l$) is given by Eqs. (94).

9. From the explicit form of the integrals of the motion (constancy of the coefficients of the highest powers of the momenta) it follows that the behavior of the function $\Psi_k(\mathbf{q})$ in the asymptotic limit $|\mathbf{q}|\to\infty$ is

$$\Psi_k(\mathbf{q})\sim\sum_s c(s\mathbf{k})\exp[\mathrm{i}(s\mathbf{k}\cdot\mathbf{q})], \quad (116)$$

where the summation is over all permutations of s. For the values of the coupling constants g given in property 6, we have the factorization

$$c(\mathbf{k})=\prod_{j<l}c(k_j-k_l)=c\prod_{j<l}(k_j-k_l)^{-\mu}, \quad (117)$$

which was obtained in Refs. 46 and 47. The problem of factorization was also considered in Ref. 38. There is no doubt that factorization of $c(\mathbf{k})$ also holds for arbitrary value of g, but a proof of this fact is not known to the author.

10. We now consider a different class of solutions of

the Schrödinger equation (104); namely, following Refs. 12 and 13, we seek a solution of the form

$$\tilde{\Phi}_{kl}(\mathbf{q})=R_{kl}(r)P_l(\mathbf{q}),\quad r=|\mathbf{q}|, \tag{118}$$

where $P_l(\mathbf{q})$ is a homogeneous function of degree l satisfying the equation

$$(\Delta+2\mu\sum_{j<k}(q_j-q_k)^{-1}(\partial_j-\partial_k))P_l=0. \tag{119}$$

For the function $R_{kl}(r)$, we obtain the equation

$$-\left(\frac{d}{dr^2}+\frac{2}{r}\left(\frac{n-1}{2}+l+\mu\frac{n(n-1)}{2}\right)\frac{d}{dr}\right)R_{kl}=k^2R_{kl}. \tag{120}$$

The solution of this equation, normalized by the condition $R_{kl}(0)=1$, is

$$R_{kl}(r)=2^{\tilde{\mu}-1/2}\Gamma(\tilde{\mu}+1/2)(kr)^{-(\tilde{\mu}-1/2)}J_{\tilde{\mu}-1/2}(kr);$$

$$\tilde{\mu}=(n-1)/2+l+\mu n(n-1)/2, \tag{121}$$

where $J_\mu(x)$ is the Bessel function of order μ.

11. Equation (119) has polynomial solutions, which may naturally be called *generalized harmonic polynomials*. In contrast to the ordinary Laplace operator ($\mu=0$), all the polynomial solutions of this equation are invariant under permutations.[13]

12. We denote by $g_n(l)$ the number of solutions of degree l of Eq. (119), i.e., the dimensionality of the space of generalized harmonic polynomials of degree l. The operator on the left-hand side of Eq. (119) maps the space of polynomials of degree l that are invariant under permutations—this space has dimension $f_n(l)$—onto the space of those of degree $l-2$, which has dimension $f_n(l-2)$. Generalized harmonic polynomials are the kernel of this mapping, and therefore

$$g_n(l)=f_n(l)-f_n(l-2). \tag{122}$$

The dimension of $g_n(l)$ is equal to the number of solutions of the equation

$$l=3l_3+\ldots+nl_n \tag{123}$$

in non-negative integers. Hence, we obtain an expression for the generating function

$$G_n(z)=\sum_{l=0}^{\infty}g_n(l)z^l; \tag{124}$$

$$G_n(z)=[(1-z^3)(1-z^4)\ldots(1-z^n)]^{-1}. \tag{125}$$

To find $g_n(l)$ for small values of n, we consider the equation

$$l=l_1+2l_2+\ldots+nl_n \tag{126}$$

and the generating function

$$H(z)=[(1-z)(1-z^2)\ldots(1-z^n)]^{-1} \tag{127}$$

for the number of solutions $h_n(l)$ of this equation. We give the quantities $h_n(l)$, which are calculated in Ref. 48:

$$h_2(l)=[l/2]+1; \tag{128}$$

$$h_3(l)=(l+2)(l+4)/12-1/72+(-1)^l/8+(2/9)\cos(2\pi l/3); \tag{129}$$

or

$$h_3(l)=\left\{\frac{(l+2)(l+4)}{12}\right\}; \tag{130}$$

$$h_4(l)=\left\{\frac{(l+2)}{144}\left(l^2+13l+37+9\,\frac{1\pm(-1)^l}{2}\right)\right\}; \tag{131}$$

$$h_5(l)=\left[\frac{(l+1)(l+2)(l+3)(l+24)+155l^2+15l(67+3(-1)^l)}{2880}\right]. \tag{132}$$

In these formulas, $[x]$ and $\{x\}$ denote the integral part of x and the number nearest x. The asymptotic behavior of $h_n(l)$ for $l\gg n$ is determined by the strongest singularity of the function $H_n(z)$, which is at the point $z=1$. From this, we readily obtain

$$h_n(l)\sim l^{n-1}/[(n-1)!\,n!]. \tag{133}$$

13. Comparing (125) and (127), we find

$$g_n(l)=h_n(l)-h_n(l-1)-h_n(l-2)+h_n(l-3). \tag{134}$$

14. We give explicit expressions for the simplest generalized harmonic polynomials[14]:

$$\begin{aligned}&P_3=S_3;\\&P_4=(n+1+n(n-1)\mu)S_4-(3(1-1/n)+(2n-3)\mu)S_2^2;\\&P_5=(n+5+n(n-1)\mu)S_5-5(2(1-1/n)+(n-2)\mu)S_3S_2.\end{aligned} \tag{135}$$

An explicit expression for P_l for arbitrary l is unknown.

Systems (V). These systems are described by the Hamiltonian

$$H=\frac{1}{2}\sum_j p_j^2+g^2\sum_{j<k}(q_j-q_k)^{-2}+\frac{\omega^2}{2}\sum_j q_j^2;\quad p_j=-i\partial/\partial q_j. \tag{136}$$

We are interested in the spectrum and eigenfunctions of the Schrödinger equation

$$\left.\begin{aligned}&H\Psi_l=E_l\Psi_l\quad\text{for } q_1\leqslant q_2\leqslant\ldots\leqslant q_n;\\&\Psi_l(\mathbf{q})=0\quad\text{for } q_j=q_{j+1};\ \int|\Psi_l(\mathbf{q})|^2d^nq<\infty.\end{aligned}\right\} \tag{137}$$

We list some of their properties.

1. As before, H for $g^2>-\frac{1}{4}$ is a self-adjoint operator and its eigenvalues E_l are real.

2. The energy spectrum is discrete and $E_l>0$. The energy E_l is characterized by n quantum numbers $l=(l_1,\ldots,l_n)$.

3. The wave function of the ground state factorizes:

$$\Psi_0(\mathbf{q})=N_0(\mu)\prod_{j<k}(q_j-q_k)^\mu\exp(-\omega\mathbf{q}^2/2). \tag{138}$$

Here $g^2=\mu(\mu-1)$; $\mathbf{q}^2=\sum_j q_j^2$.

The ground-state energy is

$$E_0=\omega(n/2+\mu n(n-1)/2). \tag{139}$$

Formulas (138) and (139) can be readily obtained using the identity (106).

4. As above, it is convenient to seek a solution in the form

$$\Psi_l=\Phi_l\Psi_0. \tag{140}$$

For the function Φ_l we then obtain the equation

$$\left[-\frac{1}{2}\left(\Delta+2\mu\sum_{j<k}(q_j-q_k)^{-1}(\partial_j-\partial_k)\right)+\omega\sum_j q_j\partial_j\right]\Phi_l=(E_l-E_0)\Phi_l. \tag{141}$$

5. The operators

$$\hat{B}_m^{\pm}(\mathbf{p},\mathbf{q})=\hat{I}_m(\mathbf{p}\pm i\omega\mathbf{q},\mathbf{q}),\quad m=1,2,\ldots,n \tag{142}$$

or

$$\hat{D}_m^{\pm}(\mathbf{p},\mathbf{q})=\hat{J}_m(\mathbf{p}\pm i\omega\mathbf{q}) \tag{143}$$

satisfy the commutation relations

$$[H,B_m^{\pm}]=\pm m\omega B_m^{\pm} \tag{144}$$

and, therefore, raise (or lower) the energy of a state by $m\omega$.

Here, the operators $\hat{I}_m(\mathbf{p},\mathbf{q})$ and $\hat{J}_m(\mathbf{p},\mathbf{q})$ are integrals of the motion for the system (I), as defined above. Using them, one can readily construct integrals of the motion.

6. The wave function of the ground state is annihilated by all lowering operators:

$$B_m\Psi_0=0;\quad D_m\Psi_0=0. \tag{145}$$

7. Using the raising operators $B_1^+, B_2^+, \ldots, B_n^+$ (or $D_1^+, D_2^+, \ldots, D_n^+$), we can readily construct the eigenfunctions of $\hat{H}$. Namely,

$$\Psi_{l_1l_2,\ldots,l_n}=N_l\,(B_1^+)^{l_1}(B_2^+)^{l_2}\ldots(B_n^+)^{l_n}\,\Psi_0. \tag{146}$$

Note that these functions form a nonorthogonal basis, since the operators B_j^+ and B_m do not commute.

8. Hence, we find the energy spectrum

$$E_{l_1l_2,\ldots,l_n}=E_0+\omega\,(l_1+2l_2+\ldots+nl_n). \tag{147}$$

9. The multiplicity of degeneracy of $h_n(l)$ is determined by the number of solutions of Eq. (126) in nonnegative numbers. The generating function for $h_n(l)$ has the form (127).

10. For $\mu=1$, $g^2=0$ and the considered system (136) is an oscillator in the polyhedral corner

$$\Lambda=\{\mathbf{q}\mid q_1\leqslant q_2\leqslant\ldots\leqslant q_n\}.$$

11. Knowing the energy spectrum and the degeneracy, we can readily calculate the quantum partition function of the operator $\hat{H}$ for $\hbar\neq0$, $p_j=i\hbar\partial/\partial q_j$:

$$Z(\beta)=\mathrm{Sp}\exp(-\beta\hat{H})=\sum h_n(l)\exp(-\beta E_l). \tag{148}$$

From (147), we find

$$Z(\beta)=\exp(-\beta E_0)\,[(1-\exp(-\beta\hbar\omega))\,[1-\exp(-2\beta\hbar\omega)]\ldots[1-\exp(-n\beta\hbar\omega)]]^{-1}. \tag{149}$$

12. Letting $\hbar$ in this expression tend to zero, and μ to infinity in such a way that $\hbar\mu=\mathrm{const}$, we obtain the classical limit of the partition function[49]:

$$Z_{\mathrm{cl}}(\beta)=\frac{1}{n!\,(\hbar\omega\beta)^n}\exp(-\beta E^0_{\mathrm{cl}}). \tag{150}$$

13. On the other hand,

$$Z_{\mathrm{cl}}(\beta)=\frac{1}{(2\pi\hbar)^n}\int\exp(-\beta H(\mathbf{p},\mathbf{q}))\,d^npd^nq. \tag{151}$$

Integrating over the momenta and using the explicit form of $H=p^2/2+U(\mathbf{q})$, we obtain

$$Z_{\mathrm{cl}}(\beta)=\frac{1}{(2\pi\hbar)^n}\left(\frac{2\pi}{\beta}\right)^{n/2}\int\limits_{q_1\leqslant q_2\leqslant\ldots\leqslant q_n}\exp[-\beta U(\mathbf{q})]\,d^n\mathbf{q}. \tag{152}$$

Comparing (150) and (152), we obtain an expression for the multiple integral[49]

$$\int\limits_{-\infty}^{+\infty}\ldots\int\exp\left(-\left[\frac{\omega^2}{2}\sum_{j=1}^n q_j^2+g^2\sum_{j<k}(q_j-q_k)^{-2}\right]\right)d^nq=\left(\frac{2\pi}{\omega^2}\right)^{n/2}\exp\left\{-\omega g\,\frac{n(n-1)}{2}\right\}. \tag{153}$$

14. We now turn to the calculation of the normalization factor for the ground-state wave function. For this, we calculate the integral

$$I(\mu,\omega)=\int\left\{\prod_{j<k}(q_j-q_k)\right\}^{2\mu}\exp(-\omega\Sigma q_j^2)\,d^nq. \tag{154}$$

We note first that the function

$$\prod_{j<k}(q_j-q_k)^{2\mu}\exp\left(-\omega\sum_j q_j^2\right) \tag{155}$$

determines the probability distribution in the n-dimensional space of $\mathbf{q}$. For $\mu=\frac{1}{2}$, this distribution was considered by Wigner,[50] and for $\mu=1$ and 2 in Dyson's well-known papers[51,52] on the statistical theory of the levels of complex nuclei. This distribution is called a Gaussian ensemble (E_1, E_2, and E_4, respectively). Note that

$$I(\mu,\omega)=I(\mu,1)\,\omega^{-[n/2+\mu n(n-1)/2]} \tag{156}$$

and it is therefore sufficient to consider the integral (154) only for $\omega=1$ (in what follows, $I(\mu,1)$ will be denoted simply by $I(\mu)$).

For $\mu=\frac{1}{2}$, 1, and 2 this integral has been calculated (see Ref. 52 and the references given there). In Ref. 52 it was conjectured that for arbitrary μ the result is

$$I(\mu)=\frac{(2\pi)^{n/2}}{n!}\,\frac{\prod_{k=1}^n\Gamma(1+\mu k)}{[\Gamma(1+\mu)]^n}. \tag{157}$$

The author of the present paper does not know whether this hypothesis has now been proved.

15. Here, we can also consider a different class of solutions of the Schrödinger equation (141), namely solutions of the form

$$\tilde{\Phi}_{lm}(\mathbf{q})=R_{lm}(r)\,P_m(\mathbf{q});\quad r=|\mathbf{q}|, \tag{158}$$

where $P_m(\mathbf{q})$ is a homogeneous polynomial of degree m satisfying Eq. (119).

For the function $R_{lm}(r)$, we obtain the equation

$$-\left(\frac{d^2}{dr^2}+\frac{2}{r}\left(\frac{n-1}{2}+m+\mu\,\frac{n(n-1)}{2}\right)\frac{d}{dr}\right)R_{lm}+2\omega r\,\frac{d}{dr}R_{lm}=2(E_l-E_0-m\omega)\,R_{lm}, \tag{159}$$

whose solution is

$$R_{lm}(r)=C_{lm}L_l^{\tilde{\mu}}(\omega r^2); \tag{160}$$

$\tilde{\mu}=m+(n-3)/2+n(n-1)\mu/2$, and $L_l^{\tilde{\mu}}(x)$ is a Laguerre polynomial of degree l.

Systems (II). Such systems are characterized by the Hamiltonian

$$H=\frac{1}{2}\sum_{j=1}^n p_j^2+g^2a^2\sum_{j<k}\mathrm{sh}^{-2}\,a\,(q_j-q_k),\quad p_j=-i\partial/\partial q_j. \tag{161}$$

We are interested in the properties of solutions of the Schrödinger equation

$$H\Psi_k=E_k\Psi_k,\ q_1\leqslant q_2\leqslant\ldots\leqslant q_n;\quad \Psi_k(\mathbf{q})=0\ \text{ for }\ q_j=q_{j+1}. \tag{162}$$

We list some of these properties.

We note that the system has the properties 1 and 2 of the systems (I).

3. A wave function of the form

$$\Psi^-=\left\{\prod_{j>k}\mathrm{sh}\,(q_j-q_k)\right\}^{\mu}\quad(a=1) \tag{163}$$

is an eigenfunction of the operator H with eigenvalue

$$E^-=-2\rho^2\mu^2=-\frac{\mu^2}{6}\,n\,(n^2-1);\quad \rho=\left(\frac{n-1}{2},\frac{n-3}{2},\ldots,-\frac{n-1}{2}\right). \tag{164}$$

To prove this, we use the identity

$$\sum_{k=l} \operatorname{cth}(q_j-q_k)\operatorname{cth}(q_j-q_l)=0. \tag{165}$$

Note, however, that the function $\Psi^-(q)$ increases exponentially as $|\mathbf{q}|\to\infty$, and therefore E^- does not belong to the spectrum of H.

4. We seek eigenfunctions of the Schrödinger equation in the form

$$\Psi_{\mathbf{k}}=\Phi_{\mathbf{k}}\Psi^- \tag{166}$$

For the function $\Phi_{\mathbf{k}}^{\mu}(\mathbf{q})$, we then obtain the equation

$$\{\Delta+2\mu\sum_{j<k}\operatorname{cth}(q_j-q_k)(\partial_j-\partial_k)\}\,\Phi_{\mathbf{k}}(\mathbf{q})=-(\mathbf{k}^2+4\rho^2\mu^2)\,\Phi_{\mathbf{k}}(\mathbf{q}). \tag{167}$$

5. For the special values $\mu=0$ and $\mu=1$ of the constants corresponding to $g^2=0$, we can write down explicit formulas for $\Phi_{\mathbf{k}}(\mathbf{q})$:

$$\Phi_{\mathbf{k}}^{(0)}(\mathbf{q})=\frac{1}{n!}\sum_s \exp\{i(s\mathbf{k},\mathbf{q})\}; \tag{168}$$

$$\Phi_{\mathbf{k}}^{(1)}(\mathbf{q})=c\,\frac{\sum_s \varepsilon(s)\exp\{i(s\mathbf{k},\mathbf{q})\}}{\prod_{j<l}\operatorname{sh}(q_j-q_l)}, \tag{169}$$

where the summation is over all permutations of s, and $\varepsilon(s)=\pm1$ for even and odd permutations, respectively.

6. The asymptotic behavior of the wave functions $\Psi_{\mathbf{k}}(\mathbf{q})$ as $|\mathbf{q}|\to\infty$ is

$$\Psi_{\mathbf{k}}(\mathbf{q})\sim\sum_s c(s\mathbf{k})\exp\{i(s\mathbf{k},\mathbf{q})\}. \tag{170}$$

For the special cases $\mu=\frac{1}{2}$, 2, and 4 (for $n=3$) the function $c(k)$ factorizes[46,47] and, therefore, so does the S matrix. This factorization evidently also holds for all values of μ (and accordingly g^2).

7. In these distinguished cases [$\mu=\frac{1}{2}$, 2 and $\mu=4$ (for $n=3$)], an integral representation for the function $\Phi_{\mathbf{k}}(\mathbf{q})$ is known,[46,47] and from this the factorization assertion can be obtained. Thus, for $\mu=\frac{1}{2}$ we have[46]

$$\Phi_{\mathbf{k}}(\mathbf{q})=c_0\exp\{(i\mathbf{k}-\rho,\mathbf{q})\}$$
$$\times\int_{-\infty}^{\infty}\!\!\int\frac{\Delta_1^{i(k_1-k_2)/2-1/2}\Delta_2^{i(k_2-k_3)/2-1/2}\dots\Delta_{n-1}^{i(k_{n-1}-k_n)/2-1/2}}{D_1^{i(k_1-k_2)/2+1/2}D_2^{i(k_2-k_3)/2+1/2}\dots D_{n-1}^{i(k_{n-1}-k_n)/2+1/2}}\prod_{j>k}dx_{jk}. \tag{171}$$

Here, the integration is over the subgroup of real lower-triangular matrices X_- with units on the diagonal. In the numerator we have the upper corner minors Δ_j of the matrix $[\exp(Q)X\exp(-Q)]'[\exp(Q)X\exp(-Q)]$, where $Q=\operatorname{diag}(q_1,\dots,q_n)$, and in the denominator the upper corner minors of the matrix $(X'X)$; the prime denotes the transpose. The components of the vector ρ are $\rho_j=(n+1)/2-j$.

8. Going in (171) to the limit $q_j\to\infty$ in such a way that $q_1-q_2\to\infty,\dots,q_{n-1}-q_n\to\infty$, we obtain an expression for $c(\mathbf{k})$ in (170) which determines the behavior of the function $\Phi_{\mathbf{k}}(\mathbf{q})$ in the limit $|\mathbf{q}|\to\infty$:

$$c(\mathbf{k})=c_0\int_{-\infty}^{+\infty}\!\!\int\!\!\int D_1^{-i(k_1-k_2)/2-1/2}\dots D_{n-1}^{-i(k_{n-1}-k_n)/2-1/2}\prod_{j>k}dx_{jk}. \tag{172}$$

This integral was calculated in Ref. 46, and it was found that $c(\mathbf{k})$ factorizes:

$$\left.\begin{aligned}c(\mathbf{k})&=c_0\prod_{j<l}c(k_j-k_l);\\ c(k_j)&=\frac{\Gamma(\mu+1/2)}{\sqrt{\pi}}\,\frac{\Gamma(ik_j)}{\Gamma(ik_j+\mu)}.\end{aligned}\right\} \tag{173}$$

Systems (III). Such systems are characterized by the Hamiltonian

$$H=\frac{1}{2}\sum_j p_j^2+g^2a^2\sum_{j<k}\sin^{-2}a(q_j-q_k). \tag{174}$$

We consider the eigenvalue problem

$$H\Psi_l(\mathbf{q})=E_l\Psi_l(\mathbf{q}); \tag{175}$$

$$\Psi_l(\mathbf{q})=0\quad\text{for }q_j=q_{j+1}; \tag{176}$$

$$\int|\Psi_l(\mathbf{q})|^2d^nq<\infty;$$
$$0\leqslant q_1\leqslant q_2\leqslant\dots\leqslant q_n\dots\leqslant a^{-1}\pi. \tag{177}$$

Such systems were considered for the first time in Ref. 18 and describe a system of n particles on a circle which repel each other.

We list the properties of such systems.

1. As before, the operator H is self-adjoint for $g^2>-\frac{1}{4}$.

2. The energy spectrum is discrete, $E_l>0$. The energy is characterized by n quantum numbers $l=(l_1,\dots,l_n)$, and

$$E_l=(l+2\mu\rho)^2/2. \tag{178}$$

3. In accordance with (105) and (163) there exists the solution

$$\Psi^+(\mathbf{q})=N(\mu)\prod_{j<k}(\sin(q_j-q_k))^{\mu};\quad \mu(\mu-1)=g^2, \tag{179}$$

which corresponds to the energy

$$E^+=2\mu^2\rho^2=\mu^2n(n^2-1)/6. \tag{180}$$

4. We seek a solution of Eq. (175) in the form

$$\Psi_l(\mathbf{q})=\Phi_l(\mathbf{q})\,\Psi^+(\mathbf{q}). \tag{181}$$

For the function $\Phi_l(\mathbf{q})$ we obtain the equation

$$B\Phi_l=[\Delta+2\mu\sum_{j<k}\operatorname{ctg}(q_j-q_k)(\partial_j-\partial_k)]\,\Phi_l=-2(E_l-E^+)\,\Phi_l. \tag{182}$$

5. All the formulas for $\Phi_l(\mathbf{q})$ can be obtained from the formulas for the systems (II) by including in them the parameter a and then replacing it by ia. We give the formulas for $\Phi_l(\mathbf{q})$ when $\mu=1$ [cf. (169)]:

$$\Phi_l(\mathbf{q})=\prod_{j<k}\left(\frac{\rho_j-\rho_k}{l_j-l_k}\right)\frac{\sum_s\varepsilon(s)\exp[i(sk\mathbf{q})]}{\prod_{j<k}\sin(q_j-q_k)}. \tag{183}$$

This function is proportional to the character of the irreducible $SU(n)$ representation with highest weight l. Similarly, replacing a by ia in (171), we obtain an integral representation for $\Phi_l(\mathbf{q})$.

6. One can show[18] that the operator B in (182) in the basis of the exponentials $\exp[i(l,\mathbf{q})]$ has triangular form. From this we obtain an expression for the energy spectrum:

$$(E_l-E_0)=[(l+2\mu\rho)^2-4\mu^2\rho^2]/2, \tag{184}$$

where

$$\rho=\left(\frac{\rho-1}{2},\frac{\rho-3}{2},\dots,-\frac{\rho-1}{2}\right).$$

7. We now calculate the normalization factor $N(\mu)$ of the solution $\Psi^+(\mathbf{q})$, namely $N(\mu)=I^{-1}(\mu)$, where

$$I(\mu)=\int\Big\{\prod_{j<k}\sin(q_k-q_j)\Big\}^{2\mu}d^nq,$$
$$0\leqslant q_1\leqslant q_2\leqslant\dots\leqslant q_n\leqslant\pi. \tag{185}$$

Up to a factor, this integral is equal to the partition function, which determines the distribution function of the eigenvalues of the Hamiltonian describing n successive levels of heavy nuclei.[51] Its value was predicted for all values of μ in Ref. 51[4)]:

$$I(\mu)=C\frac{\Gamma(1+n\mu)}{[\Gamma(1+\mu)]^n};\quad C=\frac{\pi^n 2^{-\mu n(n-1)}}{n!}. \tag{186}$$

This formula is proved in Refs. 53 and 54.

APPENDIX

Solution of the functional equation (18)

Suppose the odd function $x(\xi)$ satisfies the functional equation

$$x(\xi)x'(\eta)-x(\eta)x'(\xi)=x(\xi+\eta)[z(\xi)-z(\eta)]. \tag{A.1}$$

It is easy to see that if $x(\xi)$ is regular at $\xi=0$, then $x(\xi)\equiv 0$ and this function must have as $\xi\to 0$ the behavior

$$x(\xi)\sim g(\xi^{-1}+\gamma\xi)\quad (g\neq 0). \tag{A.2}$$

To find the general solution of (A.1), we must, following Ref. 24, let η tend to zero. Equating the coefficients of different powers of η, we obtain

$$\left.\begin{aligned}&z(\xi)\sim g(\xi^{-2}-\delta);\\ &z(\xi)=g\left(\frac{1}{2}\frac{x''(\xi)}{x(\xi)}+\gamma-\delta\right).\end{aligned}\right\} \tag{A.3}$$

However, since the function $z(\xi)$ is determined up to a constant, setting $\delta=\gamma$, we can assume that

$$z(\xi)=gx''(\xi)/[2x(\xi)]. \tag{A.4}$$

Thus, any solution of Eq. (A.1) must also satisfy the equation

$$x(\xi)x'(\eta)-x(\eta)x'(\xi)=\frac{g}{2}\left(\frac{x''(\xi)}{x(\xi)}-\frac{x''(\eta)}{x(\eta)}\right)x(\xi+\eta). \tag{A.5}$$

We again let η tend to zero. Then the coefficients of η^{-2}, η^{-1}, and 1 on the left- and right-hand sides of the equation are identically equal. Equating the coefficients of η, we find the equation

$$x(\xi)x'''(\xi)-3x'(\xi)x''(\xi)-12\gamma x(\xi)x'(\xi)=0. \tag{A.6}$$

Multiplying it by x^{-4} and integrating, we obtain

$$x^{-3}x''+6\gamma x^{-2}+c=0. \tag{A.7}$$

But $x(\xi)\sim g\xi^{-1}$ in the limit $\xi\to 0$. It follows that $c=-2g^{-2}$. Multiplying (A.7) by x^3x' and integrating, we find

$$(x')^2=g^{-2}x^4-2\mu x^2+\lambda\quad (\mu=3\gamma). \tag{A.8}$$

Note that from (A.7) there follows

$$z(\xi)=g\frac{x''(\xi)}{2x(\xi)}=g^{-1}x^2(\xi)+\mu g=g^{-1}V(\xi)+\text{const}. \tag{A.9}$$

Integrating (A.8) with the boundary condition $x(\xi)\sim g\xi^{-1}$ as $\xi\to 0$, we obtain an expression for the function that is the inverse of $x(\xi)$:

$$\xi(x)=\int_x^\infty dx/\sqrt{g^{-2}x^4-2\mu x^2+\lambda}. \tag{A.10}$$

This integral can be simplified in the following cases:

$$\begin{aligned}&1)\quad \mu=0,\quad \lambda=0,\quad x(\xi)=g\xi^{-1};\\ &2)\quad \mu=\pm a^2,\quad \lambda=g^2a^4,\quad x(\xi)=ga\,\mathrm{cth}\,a\xi,\quad ga\,\mathrm{ctg}\,a\xi;\\ &3)\quad \mu=\mp a^2/2,\quad \lambda=0,\quad x(\xi)=ga\,\mathrm{sh}^{-1}a\xi,\quad ga\sin^{-1}a\xi.\end{aligned} \tag{A.11}$$

In the remaining cases, the integral can be expressed in terms of elliptic functions. The explicit formulas for $x(\xi)$ depend on the positions of the roots z_1 and z_2 of the quadratic equation

$$z^2-2\mu g^2z+\lambda g^2=0. \tag{A.12}$$

I. Suppose $g^2\mu^2-\lambda>0$ and, therefore, z_1 and z_2 are real. We consider separately three cases:

1) $z_2<z_1<0$. We set $|z_1|=a^2$, $|z_2|=(1-k^2)a^2$. Then

$$\left.\begin{aligned}&x(\xi)=\frac{ga\,\mathrm{cn}(a\xi,k)}{\mathrm{sn}(a\xi,k)};\quad y(\xi)=ga^2\,\mathrm{dn}(a\xi,k)/\mathrm{sn}^2(a\xi,k);\\ &V(\xi)=g^2\,\mathrm{sn}^{-2}(a\xi,k)=g^2\mathscr{P}(a\xi)+\text{const}.\end{aligned}\right\} \tag{A.13}$$

2) $z_1<0, z_2>0$. We set $|z_1|=k^2a^2, z_2=(1-k^2)a^2$. Then

$$\left.\begin{aligned}&x(\xi)=ga\frac{\mathrm{dn}(a\xi,k)}{\mathrm{sn}(a\xi,k)};\quad y=g\frac{a^2\mathrm{cn}^2(a\xi,k)}{\mathrm{sn}^2(a\xi,k)};\\ &V(\xi)=g^2a^2\,\mathrm{sn}^{-2}(a\xi,k)=g^2a^2\mathscr{P}(a\xi)+\text{const}.\end{aligned}\right\} \tag{A.14}$$

3) $z_1>0, z_2>0$. Then

$$\left.\begin{aligned}&x(\xi)=ga\frac{1}{\mathrm{sn}(a\xi,k)};\quad y(\xi)=g\frac{a^2\,\mathrm{cn}(a\xi,k)\,\mathrm{dn}(a\xi,k)}{\mathrm{sn}^2(a\xi,k)};\\ &V(\xi)=g^2a^2\,\mathrm{sn}^{-2}(a\xi,k)=g^2a^2\mathscr{P}(a\xi)+\text{const}.\end{aligned}\right\} \tag{A.15}$$

II. Suppose $g\mu^2-\lambda<0$ and, therefore, z_1 and z_2 are complex. Then the expression $x^4-4\mu g^2x^2+\lambda g^2$ can be represented in the form

$$(x^2+2\nu x+g\sqrt{\lambda})(x^2-2\nu x+g\sqrt{\lambda}),\quad \nu=\sqrt{(\mu g^2+g\sqrt{\lambda})/2)}.$$

In the integral (A.10), we make the change of variables

$$x=(\lambda g^2)^{1/4}\frac{\tilde{x}-1}{\tilde{x}+1};\quad dx=(\nu g^2)^{1/4}\frac{2\,d\tilde{x}}{(\tilde{x}+1)^2}.$$

After this, the integral takes the form

$$\xi=\sqrt{\frac{2}{g\sqrt{\lambda}-\mu g^2}}\int_{-1}^{\tilde{x}}\frac{dx}{[(x^2+\tau^2)(x^2+\sigma^2)]^{1/2}}, \tag{A.16}$$

where

$$\tau^2=[(\lambda g^2)^{1/4}+\nu]/[(\lambda g^2)^{1/4}-\nu];\quad \sigma^2=\tau^{-2}. \tag{A.17}$$

Hence,

$$\tilde{x}(\xi)=a\,\mathrm{sn}^{-1}\left(a\sqrt{2/(g\sqrt{\lambda}-\mu g^2)}(\xi+\xi_0),k\right.$$

or

$$x(\xi)=(\lambda g^2)^{1/4}\frac{1-a^{-1}\mathrm{sn}\left(a\sqrt{2/(g\sqrt{\lambda}-\mu g^2)}(\xi-\xi_0),k\right)}{1+a^{-1}\mathrm{sn}\left(a\sqrt{2/(g\sqrt{\lambda}-\mu g^2)}(\xi-\xi_0),k\right)}. \tag{A.18}$$

If is easy to show that in all cases the potential has the form

$$V(\xi)=g^2a^2\mathscr{P}(a\xi)+\text{const}. \tag{A.19}$$

Indeed, it follows from Eq. (A.8) that

$$((x^2)')^2=4g^{-2}x^6-8\mu x^4+4\lambda x^2$$

or

$$(V')^2=(4g^{-2}V^2-8\mu V+4\lambda)V. \tag{A.20}$$

It remains to show that in all the considered cases the functions $x(\xi)$ and $z(\xi)$ satisfy the functional equation (A.1). This can be directly verified on (A.1) by means of the composition formula for elliptic functions (see Ref. 23).

[4)] For the special values $\mu=\frac{1}{2}$, 1, and 2, it was calculated in Ref. 51.

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