# New approach in quantum theory and its application to low and high energy nuclear physics

D. A. Kirzhnits

*P. N. Lebedev Physics Institute, USSR Academy of Sciences, Moscow*

G. Yu. Kryuchkov

*Physics Research Institute, Armenian Academy of Sciences, Erevan*

N. Zh. Takibaev

*Institute of High Energy Physics, Kazakh Academy of Sciences, Alma Ata*

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A method is proposed for the description of a quantum system on the basis of the law of evolution of the system, not with the time, as is customary, but with the coupling constant. Applied to the few-body problem in low-energy nuclear physics, the method leads to simple analytic expressions for the phase shifts of neutron scattering by the deuteron, the triton binding energy, and the virial coefficients. In problems of quantum field theory the method, being a realization of the axiomatic program, leads to exceptional solutions that do not have a "ghost" pole (in quantum electrodynamics), nonrenormalizable divergences (in the theory of the $V - A$ interaction), and so forth.

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## INTRODUCTION

*Formulation of the Problem and Basic Equations.* Many problems in theoretical nuclear physics reduce to the solution of the following general question: How do the characteristics of a complex system of particles or fields depend on the parameters of the elementary interaction between them? At lower energies, the elementary interaction is given by the two-body internucleon forces; at high energies, it is given by the initial interaction between the fields specified by the corresponding Lagrangian.

The standard dynamical schemes usually employed to solve these problems (the Hamiltonian or Lagrangian formalism, the path-integral method, etc.) are based ultimately on the law of time evolution

$$i\,\partial\Psi/\partial t = H\Psi; \quad \partial O/\partial t = i\,[H, O] \tag{1}$$

for the wave function $\Psi$ and the operator $O$. Here, the Hamiltonian $H$ appears as an operator of displacement in time.

Writing the Hamiltonian of the system in the form

$$H = H_0 + gV, \tag{2}$$

where $H_0$ is the free Hamiltonian, $gV$ is the interaction Hamiltonian, and $g$ is the coupling constant, and solving the corresponding dynamical equations, we also obtain the possibility of determining the characteristics of the system of particles or fields as functions of the parameters occurring in the interaction Hamiltonian $gV$. These include, above all, the coupling constant $g$, which either appears in the Hamiltonian itself (in the case of the Coulomb interaction, it is the product of the charges of the interacting particles) or may be introduced formally and taken equal to unity when the calculations have been completed.

In the solution of the problem posed above, there is a nontrivial possibility of completely avoiding the dynamical stage in the calculation based on the description of the time evolution of the system. One can directly pose the problem of describing the evolution of the system, not with the course of time, but with respect to the coupling constant $g$ itself: from the value $g = 0$, when the interaction is switched off and the state of the system is known, to the actual value of $g$.[1)] It is obvious that we then obtain a complete description of the system.

The corresponding evolution equations are obtained in complete analogy with (1) (Ref. 1):

$$i\,\partial\Psi/\partial g = \Gamma\Psi; \quad \partial O/\partial g = i\,[\Gamma, O], \tag{3}$$

where $\Gamma$ is the operator of displacement with respect to the coupling constant. This operator can readily be found by differentiating the first equations in (1) and (3) with respect to $g$ and $t$, respectively, and equating the results:

$$\frac{\partial\Gamma_H(t)}{\partial t} = \frac{\partial H}{\partial g}, \quad \Gamma_H(t) = \int_{-\infty}^{t} dt' V_H(t'), \tag{4}$$

where the subscript $H$ denotes the Heisenberg representation. It remains to find the equation for the operator $\Gamma$ itself, or, which is the same thing, for the operator $V$. Using the second equation in (3), we obtain

$$\frac{dV_H(t)}{dg} = i\int_{-\infty}^{t} dt'\,[V_H(t'), V_H(t)]. \tag{5}$$

Equations (3)–(5) are a complete system of equations that describe an arbitrary quantum-mechanical system

[1)] Of course, instead of the coupling constant one could take any other parameter $\lambda$ characterizing the system. Then $V$ in the following expressions must be replaced by $\partial H/\partial\lambda$.

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of particles or fields. We distinguish the most important quantity characterizing such a system, the scattering matrix (the evolution matrix) $S(t) \equiv S(t, -\infty)$:

$$S(t) = T \exp\left[ -ig \int_{-\infty}^{t} dt' V_{in}(t') \right],$$

which acts on the space of "in" states (see, for example, Ref. 2). Differentiating the $S$ matrix with respect to $g$ and using the relation $V_H(t) = S^+ T(V_{in}(t) S)$, we obtain

$$i \frac{dS(t)}{dg} = S(t) \Gamma(t) = S(t) \int_{-\infty}^{t} dt V_H(t). \tag{6}$$

For the "complete" $S$ matrix $S \equiv S(\infty)$, we have

$$i \frac{dS}{dg} = S \int_{-\infty}^{\infty} dt V_H(t). \tag{7}$$

*Aspects of the Method.* The approach presented in this paper, which is based on description of the evolution with varying coupling constant—we shall call it the coupling-constant evolution method—could replace the ordinary dynamical methods, but it is by no means equivalent to them. The method has numerous features which give it particular advantages over the standard method in the solution of many problems of quantum mechanics and quantum field theory.

The most important feature of the method is that its equations correspond to a formulation of the quantum-mechanical problem which is more general than the usual one. It has long been felt that the well-known difficulites of quantum field theory arise from the excessively detailed description in relativistic quantum physics of the process of interaction of fields "from point to point," for which the measuring possibilities are inadequate. In accordance with this point of view, the conviction that only the states of the system before and after the scattering event, i.e., as $t \to \pm\infty$, are physically real must lead to a modification of the equations of quantum field theory such that new solutions, free of the difficulties, are obtained besides (or instead of) the usual "bad" solutions. Although the situation in quantum field theory has significantly improved in recent years, this problem has by no means lost its significance.

Following the book of Bogolyubov and Shirkov,[3] it is convenient to introduce into the expression for the action $g \int d^4x L(x) = -g \int dt V_H(t)$ the "interaction switching on" function $g(x)$ in place of the coupling constant $g$, the action then being written in the form $\int d^4x g(x) L(x)$; here, $L(x)$ is the interaction Lagrangian density in the Heisenberg representation divided by $g$. In the usual formulation of the problem, the axioms of quantum field theory (unitarity of the scattering matrix $S\{g(x)\}$, causality, invariance, the correspondence principle, etc.) are satisfied by the scattering matrix for an arbitrary function $g(x)$. In particular, the choice $g(x) = g\theta(t - x_0)$ yields the evolution matrix $S(t) \equiv S(t, -\infty)$, which corresponds to the usual Hamiltonian formulation of the problem.

The point of view put forward above (it is sometimes called the axiomatic approach) corresponds to less stringent requirements: The axioms of quantum field theory are satisfied only by the scattering matrix $S\{g(x)\}$ with a function $g(x)$ infinitely close to the constant $g$. Introducing the operator

$$L(x) = -i (\delta S / \delta g(x)) S^+ \tag{8}$$

and making it satisfy the causality condition, we find

$$\delta L(x) / \delta g(y) = 0, \quad x \leqslant y,$$

and combining this last relation with the formula

$$\delta L(x) / \delta g(y) - \delta L(y) / \delta g(x) = i [L(y), L(x)],$$

which follows from the unitarity condition $S^+ S = 1$, we obtain

$$i \delta L(x) / \delta g(y) = \theta(x - y) [L(x), L(y)]. \tag{9}$$

Expansion of the relations (8) and (9) with allowance for the correspondence principle in a functional series in $g(x)$ leads to axiomatic perturbation theory, which was considered in detail in Ref. 3.

But if in (8) and (9) we let the function $g(x)$ tend to the coupling constant, making the substitution

$$\left[ \frac{\delta A}{\delta g(x)} = B(x) \right] \to \left[ \frac{\partial A}{\partial g} = \int d^4x B(x) \right],$$

we return to Eqs. (7) and (5). Thus, the coupling-constant evolution method is a realization (and, moreover, a very simple and effective one) of the axiomatic program of quantum field theory. Below, we shall really see that the corresponding equations, when applied to quantum electrodynamics and the $V - A$ theory of the weak interaction, have unusual solutions free of the standard difficulties. These solutions cannot be obtained in the framework of the usual Hamiltonian approach, since they correspond to an evolution matrix $S(t)$ which is nonunitary for $t \neq \infty$.

Another feature of the coupling-constant evolution method is particularly helpful for field-theory models with a nonstandard interaction (nonlocal field theory, a theory with higher derivatives, essentially nonlinear interactions). In these cases, the usual representation of the $S$ matrix in the form of a time-ordered exponential is unsuitable: Either the very concepts of "earlier" and "later" lose their meaning or the $S$ matrix loses its unitarity, relativistic invariance, etc. The use of Eq. (7) (to which there corresponds and $S$ matrix in the form of an exponential ordered with respect to the coupling constant) eliminates these difficulties, if, as is usually the case, the interaction Lagrangian $gV_H$ is Hermitian, relativistically invariant, and so forth. It is also possible that this approach will be helpful in gauge theories, in which the unitarity of the $S$ matrix also presents tricky problems, for the solution of which it proves necessary to introduce additional unphysical particles.

One further feature of the coupling-constant evolution method is important for nonrelativistic problems of low-energy nuclear physics. In this case, as will be shown below, the solution of Eq. (7) for the $S$ matrix does not present difficulties and, for example, for two-particle scattering of elementary or composite particles this equation directly determines the phase shift in terms of the matrix element of the operator $V$. The main difficulty of the calculation is in the solution of Eq. (5) for

the operator $V$.

It is important that even the simplest iterative methods of solution of (5) leave the operator $V$ at each stage of the iterations Hermitian and time-retarded. This leads to the fulfillment of a very important condition: At each stage in the successive approximations the $S$ matrix remains unitary (and causal). And this property, which is lost in ordinary perturbation theory, in its turn ensures a very rapid convergence of the iteration series. In the very simple problems of three-body theory considered below we shall see that the lowest iteration is already sufficient.

What we have said makes it possible to avoid the cumbersome numerical solution of the Faddeev equations and to obtain solutions to many problems in few-body theory in explicit analytic form. We have in mind scattering problems, determination of the energy of a bound state, and the calculation of higher virial coefficients. We shall not get involved here in more particular features and advantages of the coupling-constant evolution method. They can be seen in the body of the text.

*Brief Historical Review.* The coupling-constant evolution method, as an independent quantum-mechanical approach, was formulated by one of the authors in 1965 in Ref. 4. However, from the earliest years of quantum mechanics different authors had proposed relations of a special or ancillary nature, having a differential form in the coupling constant. We shall give here a brief review of the corresponding results.

The best known relation, which is called the Hellman–Feynman formula,[5] gives an expression for the derivative with respect to the coupling constant of the energy $E_n$ of a stationary state[2)]:

$$dE_n/dg = V_{nn}. \tag{10}$$

Actually, this formula had already been obtained in 1926 by Born and Fermi in their justification of Ehrenfest's adiabatic principle (see the exposition of their work in Pauli's book.[6] Formula (10) and its thermodynamic analog $dF/dg = \langle V \rangle$ ($F$ is the free energy and the angular brackets denote thermodynamic averaging) are used in the solution of many problems in quantum mechanics and statistical physics.

Another formula of the same kind is the Gell-Mann-Low formula,[8] which determines the shift $\Delta E$ of the ground-state energy of a system in terms of the mean value $\langle S \rangle$ of the scattering matrix over this state:

$$\Delta E = -(i\delta/2)\, d \ln \langle S \rangle / dg,$$

where $\delta$ is the infinitesimally small parameter of adiabatic switching on of the interaction.

Equations determining the derivative with respect to the coupling constant of the state vector or of the matrix element of an operator were studied in many papers, beginning with the above-mentioned papers of Born and Fermi.[6,9] However, these equations were not used with a view to finding the state vector or the matrix element themselves, but rather to establish how these quantities change under the influence of various perturbations.

Among the relations that have become widely used in recent years we must include the differential equation of the renormalization group—the Callan–Symanzik equation[10] (see also Ref. 11).

A large group of studies have been made into relationships that are differential in the coupling constant and connect Green's functions (or coefficient functions in the expansion of the scattering matrix with respect to normal products of field operators[12] of different order). Such relations can be obtained from Eq. (7) above. We must mention here the investigations of Caianiello's group[13] into renormalization problems in different models of quantum field theory, and also the work of Petrina and his collaborators in constructive field theory.[14] We can also include here the investigations of Terent'ev[15] and one of the present authors,[16] which had the aim of using recursion relations between matrix elements of different structure to derive the doubly logarithmic and infrared asymptotic behaviors of different amplitudes.

In the same direction, particularly worthy of mention is the work of Šoln[17] (model of bremsstrahlung at high energies) and Mal'tsev *et al.*[18] (S-matrix model of multiparticle hadron production), in which a new method is developed for analyzing inelastic hadron collisions in the presence of leading particles which carry away an appreciable fraction of the energy and the momentum. Here, it is possible to draw certain conclusions about the mechanisms of multiparticle production, about the multiplicity distributions, and so forth.

In conclusion, let us briefly mention papers in which differential relations in the coupling constant were allotted a much more constructive part than in the investigations listed above. We are referring to applications to theories (nonlocal field theory, theories with higher derivatives, essentially nonlinear theories) in which the standard formalism of quantum field theory encounters difficulties. In Ref. 19 (see also Ref. 20) and in Šoln's papers[21] an equation of the type (7) was used systematically to describe such interactions and it led to the automatic fulfillment of the unitarity condition for the $S$ matrix.

Finally, in this list of studies with something in common with the coupling-constant evolution method we must, of course, mention the investigations in axiomatic field theory using the interaction switching-on function[3] (see also Ref. 22).

*Outline of the Exposition.* The material presented in the review is arranged as follows. Section 1 (there are four sections in the paper) contains a general formulation of the coupling-constant evolution method as applied to nonrelativistic quantum-mechanical problems and to quantum field theory.

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[2)] It is interesting to note that this formula can be generalized to the case of an external perturbation that is periodic in the time, the quasienergy appearing on its left-hand side instead of $E_n$ and an averaging over the period of the external perturbation appearing on the right-hand side.[7]

In Sec. 2, the theory of the coupling-constant evolution method in application to few-body problems in nonrelativistic quantum mechanics and in quantum field theory is presented. We consider in detail the scattering problem, making considerable use of the formalism of the Jost function. In Sec. 3, we formulate the problem of three or more bodies with applications to problems of scattering by a composite particle, the calculation of the energy of bound three-particle states, the derivation of higher virial coefficients in the thermodynamics of the hadron medium, and so forth.

Finally, in Sec. 4 the method is applied to quantum electrodynamics. As in Sec. 2, particular attention is devoted to the possiblility of obtaining new solutions of the coupling-constant evolution equations that do not contain the standard difficulties.

In the paper we use a system of units in which $\hbar = 1$; in addition, in relativistic problems the velocity of light is taken equal to unity; in nonrelativistic problems the mass of the elementary particle (nucleon) is taken equal to unity.

## 1. FORMULATION OF THE SCATTERING PROBLEM

The aim of this section is to give a systematic and consistent derivation of the basic relationships of the coupling-constant evolution method as applied to the scattering problem; in nonrelativistic theory, we include here as a composite part the problem of determining bound states. We consider the nonrelativistic scattering problem, an entity which will subsequently be important-the evolution matrix at finite $t$-, and the quantum-field scattering formalism, including the renormalization procedure.

The derivation given in the Introduction of the basic equations (5) and (7) was essentially heuristic and neither rigorous nor complete. We did not consider the boundary conditions for $\Gamma(t)$ [see (4)], the inclusion of bound states (composite particles) in the scheme, or quasilocal terms in Eq. (9). The present section will put Eqs. (5) and (7) on a firmer footing and make them more precise in a form convenient for applications.

*Nonrelativistic Scattering Problem.* Our treatment[23] will be based on an approach that was already outlined in the papers of Born and Fermi referred to earlier. We consider an arbitrary nonrelativistic quantum-mechanical system under stationary external conditions. The system may contain both elementary particles and complexes made up from them. The corresponding Schrödinger equation is [see (2)]

$$(H - E_\nu)\,\Psi_\nu^\pm = 0. \tag{11}$$

The state vectors $\Psi_\nu^\pm$, which form a complete system normalized to unit volume, correspond to outgoing and incoming waves for the continuum. As is well known, the corresponding perturbation series contain additions $\pm i\delta$ in the rules for avoiding the singularities. In the analogous expressions for the discrete spectrum, for which the vectors $\Psi^\pm$ coincide (standing wave), the singularities are understood in the principal-value sense. To unify the formulas and simplify the form of the equations in this method, it is convenient to make the substitution

$$\Psi_\nu^\pm = \exp(\pm iE_\nu/\delta)\,|\,\nu\rangle_\pm.$$

We introduce the notation

$$\langle\Psi_\mu^+\,|\,O\,|\,\Psi_\nu^+\rangle = \exp[i(E_\nu - E_\mu)/\delta]\,O_{\mu\nu};\quad O_{\mu\nu} \equiv {}_+\langle\mu\,|\,O\,|\,\nu\rangle_+$$

for the matrix element of an arbitrary operator $O$ and

$$\langle\Psi_\mu^-\,|\,\Psi_\nu^+\rangle = \exp[i(E_\mu + E_\nu)/\delta]\,S_{\mu\nu};\quad S_{\mu\nu} \equiv {}_-\langle\mu\,|\,\nu\rangle_+$$

for the scattering matrix for transition from the state $\mu$ into the state $\nu$.[2] Note that this last quantity is nonzero only on the isoenergy surface $E_\mu = E_\nu$: multiplying (11) from the left by ${}_-\langle\mu\,|$, we obtain $(E_\mu - E_\nu)S_{\mu\nu} = 0$.

We now differentiate (11) with respect to $g$:

$$(H - E_\nu)\,d\Psi_\nu^+/dg = (dE_\nu/dg - V)\,\Psi_\nu^+.$$

Multiplication of this equation from the left by $\bar{\Psi}_\nu^+$ gives the Hellman–Feynman relation

$$dE_\nu/dg = V_{\nu\nu}. \tag{12}$$

Multiplying (11) from the left by $\bar{\Psi}_\sigma^+$ with arbitrary $\sigma$, we obtain an equation for the state vector:

$$d\,|\,\nu\rangle_+/dg = \sum_\sigma V_{\sigma\nu}\,|\,\sigma\rangle_+/(E_\nu - E_\sigma + i\delta). \tag{13}$$

Noting further that by definition of the $S$ matrix ${}_-\langle\mu\,| = \sum_\sigma S_{\mu\sigma} + \langle\sigma\,|$, we obtain

$$\frac{d\,{}_-\langle\mu\,|}{dg} = \sum_{\sigma,\tau} S_{\mu\sigma}V_{\sigma\tau}\,{}_+\langle\tau\,|/(E_\mu - E_\tau + i\delta). \tag{14}$$

Using (13), we can readily find the equation for the matrix element of any operator $O$ which does not depend explicitly on $g$:

$$\frac{dO_{\mu\nu}}{dg} = \sum_\sigma [V_{\mu\sigma}O_{\sigma\nu}/(E_\mu - E_\sigma - i\delta) + O_{\mu\sigma}V_{\sigma\nu}/(E_\nu - E_\sigma + i\delta)].$$

In particular, for the operator $V$ itself we obtain the closed equation

$$\frac{dV_{\mu\nu}}{dg} = \sum_\sigma V_{\mu\sigma}V_{\sigma\nu}\left(\frac{1}{E_\mu - E_\sigma - i\delta} + \frac{1}{E_\nu - E_\sigma + i\delta}\right). \tag{15}$$

Finally, using (13), (14), and the definition given above for the $S$ matrix, we find for it the equation

$$\frac{dS_{\mu\nu}}{dg} = -2\pi i \sum_\sigma S_{\mu\sigma}V_{\sigma\nu}\delta(E_\sigma - E_\nu), \tag{16}$$

in which, as we have already emphasized, the values of $E_\mu$ and $E_\nu$ are equal.

The set of equations (12, (15), and (16) gives a complete discription of the scattering process and the bound states. We must add the initial conditions for $g$, which are discussed in the following section. It is important to note that Eqs. (15) and (16) are identical with Eqs. (5) and (7) obtained in the Introduction. To see this, it is sufficient to transform the last pair of equations to the energy representation.

In many cases, it is more convenient to describe scattering processes in the language of the Jost function[24] rather than the $S$ matrix. With a view to introducing it, we shall regard the energy $E_\nu$ in Eq. (16) as a free parameter capable of taking all values, including complex values; accordingly, the $S$ matrix itself becomes a func-

tion of $E$. We now define the Jost matrices $F_\pm(E)$ by

$$S_{\mu\nu} = \sum (F_+^{-1})_{\mu\sigma} (F_-)_{\sigma\nu}, \tag{17}$$

the matrix $F_+$ being regarded as an analytic function of on the first (physical) sheet of the energy and tending to unity as $E \to \infty$ on a large circle.

Substituting (17) in (16), where we take

$$-2\pi i\delta(E_\sigma - E) = 1/(E_\sigma - E + i\delta) - 1/(E_\sigma - E - i\delta),$$

identifying the term containing the derivative $dF_+/dg$ with the term containing $E + i\delta$ (see the above condition of analyticity of $F_+$), and taking the trace, we obtain

$$\frac{d}{dg} \mathrm{Sp}\left(F_\pm^{-1} \frac{dF_\pm}{dg}\right) = -\sum_\nu \frac{V_{\nu\nu}}{E - E_\nu \pm i\delta}.$$

Finally, using the well-known formula for the derivative of a determinant, we obtain the final equation

$$\frac{d}{dg} \ln \det(F_\pm) = -\sum_\nu \frac{V_{\nu\nu}}{E - E_\nu \pm i\delta}. \tag{18}$$

The Jost determinant, defined by this formula, can completely replace the $S$ matrix in the description of the scattering process itself or of bound states. There are simple methods, especially convenient for general multichannel scattering, which enable one to extract the appropriate physical information from the multisheeted function $\det[F_\pm(E)]$.[24]

To conclude this subsection, we give a simple general formula for Jost determinants:

$$\det F_\pm(E) = \prod_\nu (E - E_\nu \pm i\delta) / \prod_\nu (E - E_\nu^0 \pm i\delta). \tag{19}$$

The denominator in this expression-it corresponds to noninteracting particles-ensures the correst normalization to unity as $E \to \infty$. To derive (19), it is simplest to use (18) or (12). In what follows, we shall consider the numerous consequences of Eq. (19); here, we restrict ourselves to giving a simple relation that makes it possible to express the sum over the energy levels in terms of the Jost determinant:

$$\sum_\nu f(E_\nu) = \sum_\nu f(E_\nu^0) + \frac{1}{\pi} \int_{-\infty}^{\infty} d\,Ef(E) \frac{d}{dE} \mathrm{Arg} \det F_+(E). \tag{20}$$

For the derivation of (20) it is sufficient to note that

$$\sum_\nu [\delta(E - E_\nu) - \delta(E - E_\nu^0)] = \frac{1}{\pi} \frac{d}{dE} \mathrm{Im} \ln \det F_+ = \frac{1}{\pi} \frac{d}{dE} \mathrm{Arg} \det F_+.$$

Sums of the type (20) are encountered, for example, in quantum statistics (the partition function).

*Evolution Matrix.* In the Introduction we have already pointed out that the equations of the coupling-constant evolution method are more comprehensive than the ordinary dynamical equations and that under certain conditions they have additional solutions for which the evolution matrix $S(t)$ is nonunitary at finite $t$. In the present section we obtain the conditions under which this matrix ceases to be unitary. The material presented here applies not only to nonrelativistic mechanics but also to quantum field theory.

We consider Eq. (16), which determines the evolution of the scattering matrix. We introduce the generalized scattering amplitude $f_{\mu\nu}$, which is defined on the isoenergy surface $E_\mu = E_\nu$:

$$S_{\mu\nu} = \delta_{\mu\nu} + 2\pi i f_{\mu\nu} \delta(E_\mu - E_\nu). \tag{21}$$

Substitution of this expression in (16) gives an equation for the scattering amplitude:

$$\frac{df_{\mu\nu}}{dg} = -V_{\mu\nu} - 2\pi i \sum_\sigma f_{\mu\sigma} V_{\sigma\nu} \delta(E_\sigma - E_\nu). \tag{22}$$

We shall use this equation in the following section.

Equation (16) follows from Eq. (7) transformed to the energy representation. The analogous equation (6) for the evolution matrix, written in matrix form, becomes

$$\frac{dS_{\mu\nu}(t)}{dg} = -i \sum_\sigma S_{\mu\sigma}(t) \int_{-\infty}^{t} dt' V_{\sigma\nu}(t').$$

Introducing the corresponding analog of (21),

$$S_{\mu\nu}(t) = \delta_{\mu\nu} + \frac{\exp[i(E_\mu - E_\nu)t]}{E_\mu - E_\nu - i\delta} F_{\mu\nu}, \tag{23}$$

where the "scattering amplitude" $F_{\mu\nu}$ on the isoenergy surface coincides with $f_{\mu\nu}$, we obtain the equation

$$\frac{dF_{\mu\nu}}{dg} = -V_{\mu\nu} - \sum_\sigma F_{\mu\sigma} V_{\sigma\nu} \left(\frac{1}{E_\mu - E_\sigma - i\delta} - \frac{1}{E_\nu - E_\sigma + i\delta}\right). \tag{24}$$

We now write down the condition of unitarity $S^+(t)S(t) = 1$ of the evolution matrix and differentiate it with respect to $g$. The unitarity condition becomes

$$S^+(t) \frac{dS(t)}{dg} = \left\{S^+(t) \frac{dS(t)}{dg} S^+(t)\right\} S(t).$$

Using (6), we can cast it in the form of an "associator"[3)]

$$\{V(t')\{S^+(t)S(t)\}\} - \{\{V(t')S^+(t)\}S(t)\} = 0,$$

whence, using (23), we find

$$(E_\mu - E_\nu)\left(\sum_\sigma \sum_\tau - \sum_\tau \sum_\sigma\right) \frac{F_{\mu\tau}^+ F_{\tau\sigma} V_{\sigma\nu} + V_{\mu\tau} F_{\tau\sigma}^+ F_{\sigma\nu}}{(E_\mu - E_\sigma - i\delta)(E_\sigma - E_\tau)(E_\tau - E_\nu - i\delta)} = 0. \tag{25}$$

This condition is automatically satisfied for the total scattering matrix $S = S(\infty)$ because of the necessary fulfillment of the equation $E_\mu = E_\nu$. However, at finite $t$, for which the energy conservation law $E_\mu = E_\nu$ is not satisfied, the condition (25) requires a rearrangement of the order of summation over the states. It is well known that this requires uniform convergence of the infinite sums, which, in its turn, requires that the expression under the summation sign decrease sufficiently rapidly with the energy. Later, we shall really see that the equations of the coupling-constant evolution method have solutions that decrease at high energies insufficiently fast to ensure unitarity of the evolution matrix. We shall see that for such solutions the phase shift increases with the energy, the scattering cross section oscillates, vanishing infinitely many times, etc.

If the evolution matrix really is nonunitary, it can be associated with an interaction $V^{\mathrm{eff}}$ which is non-Hermitian at finite $t$. This interaction is most readily determined by means of the ordinary equation for the $S$ matrix:

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[3)] By analogy with the commutator, we define the associator as $\{a\{bc\}\} - \{\{ab\}c\}$, where the brackets determine the order of the operations.

$$i\,dS(t)/dt = gV^{\text{eff}}(t)\,S(t).$$

Using (23), we find

$$gV^{\text{eff}}_{\mu\nu} = -F_{\mu\nu} - \sum_{\sigma} F^{+}_{\mu\sigma}F_{\sigma\nu}/(E_\sigma - E_\mu + i\delta). \tag{26}$$

In the following section, taking concrete examples, we shall illustrate our relations.

*Scattering Problem in Quantum Field Theory.* We now turn to the formulation of the scattering problem in relativistic theory, having in mind local renormalizable field theory without subsidiary conditions. We shall not consider the difficult problem of composite particles, although it is precisely in the framework of the coupling-constant evolution method that we have confidence that it can be solved basically in the same manner as in nonrelativistic theory. We can assume that composite particles are included in the complete system of "in" states of the system.

We determine more precisely the relations given in the Introduction. With regard to Eq. (8) for the scattering matrix, it has the standard form (7). This can be seen after transition to the derivative with respect to $g$:

$$\frac{dS}{dg} = iS\int d^4x L(x). \tag{27}$$

The equation for the Lagrangian itself is a more complicated matter.[3,4,25] From the point of view of the renormalization procedure, it is important that the right-hand side of (9) contains an indeterminate form at actually coincident points $x=y$; this can be eliminated by adding to this right-hand side a quasilocal (nonzero only when $x=y$) term $\Lambda(x,y)$. This does not contradict axiomatic considerations, since the quasilocal term does not not affect the causality and unitarity conditions. For the operator $\Lambda$ we can also readily write down an equation similar to (9), the right-hand side of which contains a new quasilocal term. For this, in turn, we can write down an equation with a new quasilocal term, and so forth. In the general case, we then obtain an infinite chain of equations with infinitely many quasilocal terms.

However, for a renormalizable theory such as quantum electrodynamics the situation simplifies considerably. In this case, the number of quasilocal terms is finite and greater by unity than the number of particle species (renormalization of the particle masses and the coupling constant). The coupling constant is renormalized by means of a quasilocal term for which the quantity $\int d^4y\Lambda(x,y)$, which arises on the transition from (9) to the equation differential in $g$, is simply proportional to $L(x)$. With regard to the renormalization of the particle masses, labeling the particle species by the subscript $j$, we denote $\int d^4y\Lambda_j(x,y)=\Lambda_j(x)$. We arrive at the system of renormalized equations

$$\left.\begin{aligned}\frac{dL(x)}{dg} &= i\int d^4y\theta(x-y)[L(x),\ L(y)] + \alpha L(x) + \sum_j \beta_j\Lambda_j(x);\\ \frac{d\Lambda_j(x)}{dg} &= i\int d^4y\theta(x-y)[\Lambda_j(x),\ L(y)],\end{aligned}\right\} \tag{28}$$

where the constants $\beta_j$ are determined from the conditions of stability of the single-particle states $|1_j\rangle$,[4)]

$$S|1_j\rangle = |1_j\rangle, \tag{29}$$

and the constant $\alpha$ is determined by the condition of charge renormalization (the scattering amplitude at threshold or the residue of the amplitude at the corresponding pole must take a given, experimentally determined value).

To Eqs. (27) and (28) we must also add the boundary conditions at $g=0$:

$$S=1;\quad L(x)=L_{\text{in}}(x);\quad \Lambda_j(x)=\Lambda_{j,\text{in}}(x). \tag{30}$$

Here, $L_{\text{in}}$ is the given interaction Lagrangian in the "in" representation divided by $g$ (for example, in quantum electrodynamics it is $:\psi_{\text{in}}\hat{A}_{\text{in}}\psi_{\text{in}}:$); $\Lambda_{j,\text{in}}$ is the mass renormalization counter term (for bosons it is $:\varphi^2_{j,\text{in}}:$, and for fermions $:\bar{\psi}_{j,\text{in}}\psi_{j,\text{in}}:$).

It is important to emphasize[1] that, in contrast to the usual formalism, we have not had to introduce a special counter term for the wave-function renormalization. Since the introduction of Green's functions, vertex parts, etc., is redundant in the coupling-constant evolution method, the charge is here renormalized as a whole without separation of individual contributions of the Green's function, vertices, etc.

To confirm the correctness of the counter-term structure in Eq. (28) and elucidate the physical meaning of $\alpha$ and $\beta_j$, we consider the usual expression for the $S$ matrix:

$$S = T\exp\left[i\int d^4x\left(g_0(g)L_{\text{in}}(x) + \sum_j \delta M_j(g)\Lambda_{j,\text{in}}(x)\right)\right].$$

Here, $g_0$ and $g$ are, respectively, the bare and renormalized coupling constants; $\delta M_j$ are the shifts of the masses (for Fermi particles) or the squares of the masses (for Bose particles).

Differentiating formally the expression obtained for $g$, we can readily see that it completely corresponds to Eqs. (27) and (28) given above if we take

$$\left.\begin{aligned}&L(x) = g_0'S^+T(L_{\text{in}}(x)S) + \sum_j \delta M_j'S^+T(\Lambda_{\text{in}}(x)S);\\ &\alpha = g_0''/g_0';\quad \beta_j = \delta M_j'' - \delta M_j'g_0''/g_0'.\end{aligned}\right\} \tag{31}$$

where the prime denotes the derivative with respect to $g$.

As in nonrelativistic scattering, it is convenient to transform (27) and (28) to the energy representation, going over to matrix elements in the system of "in" states, which from the very start are assumed to be independent of $g$. Using the translational invariance, we obtain

$$\langle\mu|L(x)|\nu\rangle = \exp[i(p_\mu - p_\nu)x]L_{\mu\nu},\quad L_{\mu\nu} = \langle\mu|L(0)|\nu\rangle$$

and similarly for $\Lambda_j$. In addition, we introduce the relativistic scattering amplitude $f$:

$$S_{\mu\nu} = \delta_{\mu\nu} + (2\pi)^4 i\delta(p_\mu - p_\nu)f_{\mu\nu}. \tag{32}$$

This gives the required equations:

[4)] In addition, we must of course also have fulfillment of the vacuum stability condition $S|0\rangle = |0\rangle$ [or $L_{00}=0$; see (35)].

$$\left.\begin{aligned} df_{\mu\nu}/dg &= L_{\mu\nu} + (2\pi)^4 i \sum_\sigma f_{\mu\sigma} L_{\sigma\nu} \delta^4(p_\sigma - p_\nu); \\ dL_{\mu\nu}/dg &= -\sum_\sigma L_{\mu\sigma} L_{\sigma\nu} (A^-_{\mu\sigma} + A^+_{\nu\sigma}) + \alpha L_{\mu\nu} + \sum_j \beta_j \Lambda^j_{\mu\nu}; \\ d\Lambda^j_{\mu\nu}/dg &= -\sum_\sigma (\Lambda^j_{\mu\sigma} L_{\sigma\nu} A^-_{\mu\sigma} + L_{\mu\sigma} \Lambda^j_{\sigma\nu} A^+_{\nu\sigma}), \end{aligned}\right\} \quad (33)$$

where we have introduced the notation

$$A^\pm_{\mu\nu} = \delta^3(\mathbf{p}_\mu - \mathbf{p}_\nu)/(E_\mu - E_\nu \pm i\delta).$$

The boundary conditions are ($g = 0$)

$$f_{\mu\nu} = gL^{in}_{\mu\nu}; \quad L_{\mu\nu} = L^{in}_{\mu\nu}; \quad \Lambda_{j,\mu\nu} = \Lambda^{in}_{j,\mu\nu}, \quad (34)$$

and the stability conditions (29) can be written in the form

$$L_{ij,ij} = 0. \quad (35)$$

## 2. TWO-PARTICLE SCATTERING

In this section, we consider the application of the coupling-constant evolution method to problems of elastic scattering of two particles. More precisely, in nonrelativistic problems we study the scattering of two elementary particles at arbitrary energies or the scattering of an elementary particle by a composite particle (or the scattering of a composite particle by a composite particle) below the threshold for the breakup of the composite particle. In relativistic problems, we consider scattering with allowance for not more than two particles in the intermediate states, which corresponds to the first stage in the expansion in the number of intermediate particles. Already in this simplest approximation we encounter numerous nontrivial and instructive results.

The treatment is given in the center-of-mass system of the colliding particles, the arguments being the energy $E$ in this system and the corresponding momentum. In the case of scattering with the participation of composite particles, $E$ must be understood as the kinetic energy-like the momentum $k$, it does not depend on the coupling constant $g$. Assuming an expansion with respect to spherical functions, we shall give only the relations for the corresponding spherical harmonics, omitting the index $l$.

*Scattering of Two Nonrelativistic Particles.* In the considered case $k = (2m^*E)^{1/2}$, and the level density is $\rho = d^3k/dE = m^*k/2\pi^2$, where $m^*$ is the reduced mass of the colliding particles. Assuming for simplicity that they have no spin, we use for the scattering amplitude in (21) the well-known expression in terms of the phase shift $\delta$:

$$f = [\exp(2i\delta) - 1]/2ik. \quad (36)$$

The transition to spherical harmonics diagonalizes the $S$ matrix and the scattering amplitude, the Jost matrices, etc. This makes it possible to solve Eq. (22) and, using (36), we obtain the important relation

$$d\delta/dg = -\pi\rho V_{kk'}, \quad k = k', \quad (37)$$

where the spherical harmonic in the angle between $\mathbf{k}$ and $\mathbf{k}'$ is understood. In conjunction with (12), this relation shows that the matrix elements of the interaction potential $V$ diagonal in the energy have a direct physical meaning, determining the derivative with respect to the coupling constant of the energy of a level (in the discrete spectrum) or the phase shift (in the continuum). Using (12) and (37), we directly point out the helpful sum rule

$$\sum_n E_n = \frac{1}{\pi}\int_0^\infty dE\,(\delta - \delta^B), \quad (38)$$

where $E_n$ is the energy of the bound state with given $l$; the superscript $B$ indicates the Born approximation. The derivation of (38) is based on the obvious equation $d\,\mathrm{Tr}(V)/dg = 0$, where the trace is over the states with a given value of $l$. The sum rule (38) was recently obtained by Puff[26] by a considerably more complicated method (see also the earlier papers of Refs. 27).[5)]

The above-mentioned diagonalization of the Jost matrices also makes it possible to solve Eq. (18). Denoting the Jost function $F_+$ by $u$, we obtain for it the well-known expression[24]

$$u = \prod_n \left(1 - \frac{E_n}{E}\right) \exp\left\{\frac{1}{\pi}\int_0^\infty \frac{dE'\delta(E')}{E - E' + i\delta}\right\}. \quad (39)$$

The representation (19), which is valid for all (discrete and continuous) levels with given $l$, can be used to give a simple expression for the phase shift in terms of the number of levels

$$\Delta N(E) = \sum_\nu [\theta(E - E_\nu) - \theta(E - E^0_\nu)]$$

that intersect the energy $E$ in the process of switching on the interaction:

$$\delta(E) = -\mathrm{Im}\ln u = -\pi\Delta N(E). \quad (40)$$

From this Levinson's well-known theorem[30] follows directly:

$$\delta(0) - \delta(\infty) = \pi N, \quad (41)$$

where $N$ is the number of bound states, since the change in the total number of levels is zero by virtue of the completeness theorem. The expression (40) is related to the trace formula (see, for example, Ref. 31).[6)]

In some cases (for example, for the point interaction considered below) the integral over the energy in (39) diverges. To eliminate this difficulty, it is convenient to introduce a renormalized Jost function $u$ normalized to unity at $E = 0$:

$$\tilde{u} = \prod_n \left(1 - \frac{E}{E_n}\right) \exp\left\{\frac{E}{\pi}\int_0^\infty \frac{dE'\delta(E')}{E'(E - E' + i\delta)}\right\}. \quad (42)$$

It is easy to show that such a transition is in complete agreement with the general renormalization procedure given earlier.

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[5)] One can also derive sum rules containing negative powers of the energy (in the two-body problem see Ref. 28, in the three-body problem Ref. 29).

[6)] The expression (40) cannot be understood literally, since otherwise the phase shift would always be a multiple of $\pi$. In fact, one must understand an averaging (smoothing) with respect to $E$ over an interval greater than the spacing between the levels. This is due to the fact that in the scattering process the collimator is small compared with the system itself, i.e., the spacing between levels is small compared with the energy spread in the initial wave packet.

*Separable Interaction.* Using the representation (19) for the Jost function and Eq. (12), we can readily obtain for this function the equation

$$\frac{d^2u}{dg^2}=Au;$$
$$A=\sum_{\mu\nu}(V_{\mu\mu}V_{\nu\nu}-|V_{\mu\nu}|^2)/(E-E_\mu+i\delta)(E-E_\nu+i\delta). \tag{43}$$

It can be seen from this that for an interaction $V_{\mu\nu}$ which can be represented as a product of two factors depending on $\mu$ and $\nu$, respectively, $A=0$ and the Jost function is linear in the coupling constant. Using the condition $u=1$ at $g=0$ and (40), we find

$$u=1+\frac{1}{\pi}\int_0^\infty \frac{dE'\delta^B(E')}{E-E'+i\delta}, \tag{44}$$

where, in accordance with (37), $\delta^B=-\pi g\rho V^0_{kk}$; $V^0$ is the potential $V$ at $g=0$.

An interaction having the above property and leading to Eq. (44) is said to be separable. A very important example of such an interaction is the Yamaguchi potential,[32] which is successfully used in low-energy nuclear physics and which will be employed in the following calculations. For this potential, $V^0_{kk}$, has the form

$$V^0_{kk'}=-4\pi\gamma^3/[(k^2+\gamma^2)(k'^2+\gamma^2)], \tag{45}$$

where $\gamma=1.44F^{-1}$ is the reciprocal range of the forces. The corresponding phase shift in the Born approximation is

$$\delta^B=gk\gamma^4/(k^2+\gamma^2)^2$$

and nonzero only in the $s$ state. It can readily be shown on the basis of (15) that $V_{kk}$, separates because $V^0_{kk}$, does [cf. (48) below].

The potential (45) describes the proton-neutron interaction in a triplet spin state. The corresponding Jost function has the form [see (3.8)]:

$$u=1-g\gamma^2/(\gamma-ik)^2.$$

From this, we can readily find an expression for the energy of the bound state (deuteron) by determining the zero of the Jost function [see (3.4)]:

$$E_0\equiv-\varkappa^2=-\gamma^2(g^{1/2}-1)^2, \tag{46}$$

where $\varkappa=0.23F^{-1}$ is the reciprocal of the scattering length. The dimensionless coupling constant $g$ in (45) is normalized in such a way that $g=1$ corresponds to the threshold of deuteron formation.

It remains to find the scattering characteristics. Using (40), we obtain[7)]

$$\operatorname{tg}\delta=2kg\gamma^3/[(k^2+\gamma^2)^2-g\gamma^2(\gamma^2-k^2)],$$

whence, using Levinson's theorem (41), we obtain

$$\delta=\pi-\operatorname{arctg}\left(\frac{k}{\varkappa}\right)-\operatorname{arctg}\left(\frac{k}{2\gamma+\varkappa}\right)-2\operatorname{arctg}(k/\gamma). \tag{47}$$

We also give an expression for the derivative:

$$d\delta/dg=2k\gamma^3/(k^2+\varkappa^2)(k^2+(2\gamma+\varkappa)^2),$$

in terms of which the exact matrix element of the potential can be expressed,

$$V_{kk'}=-4\pi\left(\frac{d\delta(k)}{k\,dg}\frac{d\delta(k')}{k'\,dg}\right)^{1/2}\exp[i(\delta(k)-\delta(k'))]. \tag{48}$$

Letting $\gamma$ tend to infinity in the above formulas, we arrive at the limit of a point interaction. In this limit, $g\to1$ and we must regard as fixed the quantity $\varkappa$ defined by Eq. (46). For a point interaction, we must go over to the renormalized Jost function, which has the form

$$\tilde{u}=1-ik/\varkappa.$$

The phase shift is now determined by

$$\delta=\pi-\operatorname{arctg}(k/\varkappa). \tag{49}$$

*Scattering of Two Relativistic Particles.* Turning to the relativistic problem of two-particle scattering,[33] we encounter many complicating circumstances. The most fundamental of them is that now the number of particles in the intermediate states is in no way limited and the problem is essentially from the very beginning no longer restricted to two bodies. As we have already said, this difficulty can be avoided by selecting only those scattering diagrams that contain not more than two particles in the intermediate states.

More precisely, this selection must be relativistic in nature, i.e., we must take into account both types of motion of particles-forward and backward in time (see the diagrams of Fig. 1, which correspond to the four-fermion interaction considered below). This means that in the corresponding equations the Green's function $(E-E_\mu\pm i\delta)^{-1}$ is replaced by the expression

$$(E-E_\mu\pm i\delta)^{-1}+(E+E_\mu)^{-1}=2E/(E^2-E_\mu^2\pm i\delta),$$

and for the energy integral we have

$$\int dE'(E-E'\pm i\delta)^{-1}(\ldots)\to\int ds'(s-s'\pm i\delta)^{-1}(\ldots),$$

where $s=E^2$ is the Mandelstam variable.

This makes clear the approximate nature of the treatment based on the choice of the diagrams in Fig. 1. Instead of the crossing-symmetric expressions that depend on all the Mandelstam variables, we are dealing with a dependence on only the variable $s$. In the following approximations, which take into account a larger number of intermediate particles, a dependence on the second Mandelstam variable also appears.

We restrict the treatment to the relativistic scattering problem corresponding to the four-fermion interaction

$$L_{in}=:(\bar{\psi}_{in}O\psi_{in})(\bar{\psi}_{in}O\psi_{in}):$$

Here, the matrices $O$ are either 1 (scalar variant), or $\gamma_5$ (pseudoscalar variant), or $\gamma_\alpha(1+\gamma_5)/4\sqrt{2}$ $(V-A)$. In the considered approximation there is no mass renormalization of the particles, and this allows us to set $\Lambda$

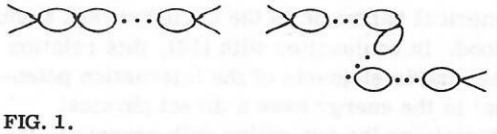

FIG. 1.

[7)] *Translator's Note.* The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

$=0$ in (33). For the listed interactions, we have the important relation[33]

$$\sum_{\sigma}{}' L^{\text{in}}_{\mu\sigma}(0)\, L^{\text{in}}_{\sigma\nu}(0) = \rho(s_\sigma)\, L^{\text{in}}_{\mu\nu}(0), \tag{50}$$

where the prime on the summation sign means that the summation is over all the variables except the energy. The function $\rho$ in (50) is the relativistic generalization of the level density, which was denoted above by the same letter[8]:

$$\rho(s) = \sum_{\nu}{}' L^{\text{in}}_{\nu\nu} = \frac{1}{2\pi} \sum_{q_1, q_2} \delta^4(q_1 + q_2 - p)\, \mathrm{Sp}\,[O(\hat{q_1} \pm m_1)\, O(\hat{q_2} \pm m_2)], \tag{51}$$

where $s = p^2$, $m_1$ and $m_2$ are the particle masses in the intermediate state, and $\delta^4(p) = (2\pi)^4 \delta(p)$. In particular, for the $V-A$ interaction when $s \gg m^2$ we have $\rho(s) = -s/4\sqrt{2}\pi^2$.

Equation (50) shows first of all that the spinor and vector structures of $L_{\text{in}}$ are reproduced on the transition to quadratic, cubic, etc., combinations of this quantity, which makes it possible to separate directly the corresponding structure, setting

$$f_{\mu\nu} = L^{\text{in}}_{\mu\nu}\varphi(s); \quad L_{\mu\nu} = L^{\text{in}}_{\mu\nu}\lambda(s_\mu, s_\nu), \tag{52}$$

and in what follows we shall deal with only scalar quantities. Substitution of (52) in Eq. (33) gives a representation of the amplitude in terms of the phase shifts,

$$\varphi(s) = [\exp(2i\delta(s)) - 1]/2\pi i\rho(s), \tag{53}$$

and the relation between the Lagrangian and the phase shift:

$$d\delta(s)/dg = \pi\rho(s)\,\lambda(s, s). \tag{54}$$

The intimate analogy with the nonrelativistic case is obvious, and the entire difference resides in the transition to the argument $s$ and the different expression for the function $\rho(s)$.

Another important consequence of (50) is the possibility of regarding the considered interaction as separable. Using (50), (52), and (33), it is easy to show that $\lambda(s, s')$ in (52) really does decompose into the product of two factors that depend on $s$ and $s'$, respectively. This is not so remarkable, since the four-fermion local interaction is the relativistic generalization of the separable point interaction.

What we have said enables us to expect that the relativistic Jost function corresponding to (53),[9]

$$u(s) = \exp\left\{\frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds'\delta(s')}{s - s' + i\delta}\right\}, \quad s_0 = (m_1 + m_2)^2$$

will, as in nonrelativistic theory, be a linear function of the coupling constant. This is in fact readily seen from (54) and (33). Introducing the Born phase shift $\delta^B = \pi g\rho(s)$, we obtain

$$u = 1 + g\int_{s_0}^{\infty} \frac{ds'\rho(s')}{s - s' + i\delta}.$$

It is more convenient to go over to the renormalized Jost function, normalizing it at the threshold $s_0$ (this means that the renormalized charge is determined by the value of the scattering amplitude at the threshold[1]):

$$\tilde{u} = 1 + g(s - s_0)\int_{s_0}^{\infty} \frac{ds'\rho(s')}{(s - s' + i\delta)(s' - s_0)}. \tag{55}$$

From this, in particular, we obtain an expression for the phase shift:

$$\mathrm{tg}\,\delta = -\frac{\mathrm{Im}\,u}{\mathrm{Re}\,u} = \pi g\rho(s)\left[1 + g(s - s_0)\int_{s_0}^{\infty} \frac{ds'\rho(s')}{(s - s')(s' - s_0)}\right]^{-1}. \tag{56}$$

The integrals in the last two expressions are divergent for the $V-A$ interaction. This is a manifestation of the nonrenormalizability of such an interaction.

It is important to note that (55) is not always a solution of the original equations (33). From these, we can obtain an equation directly for the phase shift[33, 34]:

$$\delta''(s)/\delta'(s) = \frac{2}{\pi}(s - s_0)\int_{s_0}^{\infty} \frac{ds'\delta'(s')}{(s' - s_0)(s' - s)}.$$

It follows that the Jost function can have neither singularities nor zeros (in the absence of bound states) off the cut. At the same time, the expression (55) for $\delta^B < 0$, like the identical result of ordinary perturbation theory, can have a zero for $s < 0$ ("ghost" pole; see Sec. 3 below). These "ghost" solutions of the equation for the Jost function, which arise from the additional differentiation of the original equations with respect to the coupling constant and which contradict the initial system of axioms, must be rejected. It is important that (as will be shown below) there are, besides these solutions, exceptional solutions in the corresponding cases which do not contain such difficulties.

*Exceptional Solution.* As we have already said in the introduction, the equations of the coupling-constant evolution method correspond to a more flexible formulation of the problem than the usual approach, and one can therefore expect the occurrence in it of exceptional solutions that are not found in the standard method. This is indeed the case, and the exceptional solutions are in many cases free of the difficulties that are inherent in the usual solutions.

We begin by considering nonrelativistic scattering on the basis of a separable interaction potential for which $A$ in (43) is zero. We repeat the derivation of (43) without assuming that the expression under the summation sign decreases sufficiently rapidly (and one can therefore change the order of summation). Simple calculations give

$$d^2u/dg^2 = \left(\sum_{\mu}\sum_{\nu} - \sum_{\nu}\sum_{\mu}\right) V_{\mu\mu}V_{\nu\nu}/(E_\nu - E_\mu)(E - E_\nu + i\delta). \tag{57}$$

Going over directly to the renormalized Jost function (which corresponds to subtraction from the right-hand side of the last equation of its value at $E=0$), introducing the phase shift by means of (37), and assuming that the phase shift at large momenta has the asymptotic behavior

$$\lim_{k\to\infty}(d\delta(k)/dg)/k = C(g), \tag{58}$$

[8] The signs ± in (51) correspond to a particle and antiparticle, respectively, in the intermediate state.

[9] Bound states are assumed to be absent [see (67)].

we finally obtain the equation

$$d^2\tilde{u}/dg^2 + C^2k^2\tilde{u} = 0. \tag{59}$$

This is indeed a complicated equation, since its coefficient must be consistent with the asymptotic behavior of its solution [see (40)].

Equation (59) has a formal solution corresponding to $C=0$. We have investigated it earlier. Here we shall show that $C\neq 0$ corresponds to a nonunitary evolution matrix and, for this reason, the solutions considered above with $C=0$ correspond to the usual Hamiltonian formulation of the problem. With regard to the solutions with $C\neq 0$ to be considered below, they correspond to the axiomatic formulation of the problem discussed in the introduction.

First, we sketch the proof of the fact that the sum in the condition (25), which is proportional to $d[S^*(t)S(t)]/dg$, really does vanish together with $C^2$ (see the detailed proof in Ref. 35). This reduces to substituting in the indicated sum the representation (48) and solving the equation for the amplitude (24), which for a separable interaction has the form

$$F_{kk'} = \left(\frac{d\delta(k)}{dg}\Big/\frac{d\delta(k')}{dg}\right)^{1/2}\frac{\exp[i\delta(k)]\sin\delta(k)}{\sqrt{kk'}}.$$

It is easy to see that the sum under consideration reduces to the sum on the right-hand side of (57). Completing the discussion of this question, we give the expression for the effective Hamiltonian corresponding to the exceptional solution (13) [see (26)]:

$$V^{\text{eff}}_{kk'} \sim \left(\frac{k}{k'}\frac{d\delta^B(k')}{dg}\Big/\frac{d\delta(k)}{dg}\right)^{1/2}\frac{\sin\delta(k)}{k}.$$

This expression really is non-Hermitian, but becomes Hermitian for $C=0$.

Note that the exceptional solution is associated with a cross section that oscillates with increasing energy and vanishes infinitely many times. This can be seen directly from (58):

$$\sigma \sim \sin^2\delta(k) \sim \sin^2\left(k\int_0^g dgC(g)\right). \tag{60}$$

All the relations we have given can be readily generalized to relativistic scattering in the approximation that was considered in the previous subsection. We give directly the relations corresponding to (58) and (59):

$$\frac{d^2\tilde{u}}{dg^2} + (s-s_0)C^2u = 0; \quad C = \lim_{s\to\infty}\frac{d\delta(s)}{dg}\Big/\sqrt{s}. \tag{61}$$

*Examples of Exceptional Solutions.* We now turn to concrete applications of Eq. (59). We shall give a general method for determining $C$ based on solution of the inverse scattering problem. Here, we shall restrict ourselves to cases when the form of this quantity can be guessed on the basis of simple dimensional arguments.

We shall consider a situation in which there is scaling and the theory contains only a single dimensionless parameter $\xi = gk^{2n}$, with $n>0$. Accordingly, the Born phase shift can be written in the form $\delta^B = A\xi$, the number $A$ being assumed positive for the sake of definiteness. In this case, from dimensional considerations

$$C(g) = \sigma g^{1/2n-1},$$

where $\sigma$ is a new numerical constant. Equation (59) takes the form

$$\partial^2\tilde{u}/\partial\xi^2 + \sigma^2\xi^{1/n-2}\tilde{u} = 0,$$

and its solution is

$$u(\xi) = \frac{i\,\text{sign}(g)\,\pi}{\Gamma(n)}\left(\frac{\chi}{2}\right)^n H_n^{(i)}(\chi), \tag{62}$$

where $i=1$ for $g<0$, $i=2$ for $g<0$, and

$$\chi = 2\left[\frac{An\Gamma^2(n)}{\pi}|\xi|\right]^{1/2n}; \quad \sigma = \frac{\text{sign}(g)}{n}\left(\frac{An\Gamma^2(n)}{\pi}\right)^{1/2n}.$$

In deriving (62) we have used the boundary conditions $u=1$ at $g=0$ and an asymptotic condition that follows from (59) in the limit $k\to\infty$:

$$\tilde{u} \approx (\delta(k)/kC)^{1/2}\exp\left(-ik\int_0^g dgC(g)\right).$$

We also give the expression for the phase shift:

$$\text{tg}\,\delta = -\text{sign}(g)\,J_n(\chi)/N_n(\chi), \tag{63}$$

Here, $H_n^{(i)}$, $J_n$, and $N_n$ are Bessel functions.

Referring to the original papers in Refs. 33–35 for a discussion of the ghost solutions, we turn to concrete examples.

We begin with the case $n=\frac{1}{2}$, which corresponds to a point interaction, it being convenient to take $1/\varkappa$ (the scattering length) as the coupling constant.[10] Then

$$\tilde{u} = \exp(-ik/\varkappa),$$

and the phase shift is $k/\varkappa$. One can show[4] that for the considered model all solutions of the coupling-constant evolution equations are exhausted by this solution and the ordinary solution [see (49)].

The following example corresponds to $n=3/2$; this is the nonrelativistic analog of the nonrenormalizable theory with $V^0_{kk'} \sim (kk')$ (interaction in the $p$ state). The ordinary solution for this model contains divergent integrals. Setting $\delta^B = gk^3$, we have for the phase shifts

$$\delta = (3g)^{1/3}k - \text{arctg}\,[(3g)^{1/3}k].$$

This is the first example we have encountered when the transition to the axiomatic formulation of the problem is rewarded by the appearance of a "good" (in this case, renormalizable) solution.

Finally, we consider the more realistic (although now less directly significant after the creation of the unified Weinberg–Salam theory) example of electron–neutrino scattering in the framework of the theory of the $V-A$ interaction.[33,35] In the asymptotic region $s \gg m^2$ ($m$ is the electron mass) we have a situation with scaling with $n=2$ corresponding to $\delta^B = Gs/2\sqrt{2}\pi$, where $G$ is the Fermi constant [cf. (59) and (61)]. We give the expressions for the phase shift,

$$\text{tg}\,\delta = J_2(\chi)/N_2(\chi); \quad \chi = 2(Gs/\sqrt{2}\pi)^{1/4}$$

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[10] Note that for nonpoint, "smoothed" interactions that decrease with increasing energy there are no exceptional solutions at all.[33]

and the scattering cross section

$$\sigma_{ev}=(8\pi/s)\,J_2^2(\chi)/[J_2^2(\chi)+N_2^2(\chi)].$$

At low energies, the last expression is equal to its Born expression, and at high energies it oscillates, as we have already noted. Here too we have obtained a finite result for the scattering characteristics in a model for which the ordinary method gives a divergent result.

Note that in the last two cases the result is nonanalytic in the coupling constant at the point $g=0$, which means that there is no hope of obtaining it perturbatively.

## 3. THREE-BODY PROBLEM

The method presented here can also be used effectively in the nonrelativistic few-body problem (three or more bodies) to describe many-particle collisions and many-particle bound states. The method usually employed, which is based on the Faddeev equations,[36-38] necessitates cumbersome numerical calculations. Attempts to use the first iterations of these equations (the first Born terms) for rough semiquantitative estimates do not lead to success because of the strong violation of unitarity and the divergence of the Born series. This last circumstance gave rise to the attempt to attack the problem, not on the basis of the dynamical equations, but directly using the conditions of causality (analyticity) and unitarity of the $S$ matrix.[39] In this sense, the coupling-constant evolution method is intermediate in nature: Having the advantages of the dynamical approach (above all, uniqueness), it guarantees fulfillment of causality and unitarity at each step in the successive approximations. It is not surprising that the series of successive approximations converges rapidly in the framework of the coupling-constant evolution method and leads to simple analytic expressions for the quantities in which we are interested.

*Separation of Disconnected Diagrams.* If the number of particles in the system exceeds two (the number of particles directly coupled by the two-body interaction), the process of their scattering is described by not only connected diagrams, for which all particles are involved in the interaction, but also disconnected diagrams, which correspond to one or more particles passing freely through the interaction region. One of the difficulties in the theory of few-body scattering that was overcome in the Faddeev equations was related to the disconnected diagrams.

In the framework of the coupling-constant evolution method, the disconnected diagrams can not only be readily separated, but this method also leads naturally to a convergent iterative procedure.[23] We begin the discussion of this question with the most complicated equation (15) for the matrix elements of the interaction $V$ under the assumption that the many-particle interaction is based on the two-body potential $V^{(2)}$. We split $V_{\mu\nu}$ into connected and disconnected parts:

$$V_{\mu\nu}=V^c_{\mu\nu}+U_{\mu\nu}, \tag{64}$$

where $U=\sum I_i V_i^{(2)}$; $i=1,2,3$ is the number of the particle which does not undergo scattering (and it is simultaneously the number of the interacting pair), $I_i$ symbolizes the $\delta$ function of the momentum conservation law of particle $i$ (Fig. 2) (the open circle in Fig. 2 corresponds to the potential $V^{(2)}$; the black circle, to $V^c$). Substituting (64) in (15) and using the analogous equation for $V^{(2)}$, we obtain the following inhomogeneous equation for the connected part of $V$:

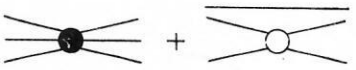

FIG. 2.

$$\frac{dV^c_{\mu\nu}}{dg}=\sum_\sigma\{(U_{\mu\sigma}U_{\sigma\nu})^c+U_{\mu\sigma}V^c_{\sigma\nu}+V^c_{\mu\sigma}U_{\sigma\nu}+V^c_{\mu\sigma}V^c_{\sigma\nu}\}\times\left[\frac{1}{E_\mu-E_\sigma-i\delta}+\frac{1}{E_V-E_\sigma+i\delta}\right], \tag{65}$$

where the superscript $c$ in the first term indicates omission of the term $(UU)^c=\sum_{i\neq j}I_iV_i^{(2)}I_jV_j^{(2)}$ diagonal in the particle number (Fig. 3).

Equation (65), which of course cannot be solved exactly, by its very form suggests an iterative procedure for solution. To the zeroth iteration there corresponds the free term of (65), to the first iteration, the result of substituting the zeroth iteration in the term of (65) linear in $V^c$, etc. It is important to emphasize that at each stage of this iteration $V^c_{\mu\nu}$ is Hermitian (by its very construction) and causal (because of the correct conditions for avoiding the singularity). Therefore, the $S$ matrix is also unitary and causal at each stage of the successive approximations.[11)]

With regard to the separation of the disconnected part in the $S$ matrix, this can be done by representing the $S$ matrix in the form of a connected three-particle matrix and two-body $S$ matrices corresponding to all possible particle pairs. We shall not give the corresponding long expressions[41] but point out directly that for the simplest process of scattering of a particle on a bound state below the threshold of its breakup Eq. (37) can be used. For the process of neutron scattering by the deuteron considered below,

$$d\delta/dg=-(k/3\pi)\,V_{nd,\,nd}. \tag{66}$$

In the general case of multichannel scattering, it is more convenient to go over to the Jost matrix, in whose determinant, as we have already said, all the necessary information is contained. We give without derivation the simple equation for the determinant of the connected part of the Jost matrices [cf. (18)]:

$$\frac{d}{dg}\ln\det F_\pm=-\sum_\nu\frac{V^c_{\nu\nu}}{E-E_\nu+i\delta}. \tag{67}$$

*Scattering of the Neutron by the Deuteron.* We now turn to a definite calculation—the calculation of the phase shifts of neutron elastic scattering by the deuteron at energies below the threshold of deuteron disintegration.[23] We shall calcu-

[11)] Considering (37), we see that it is not the amplitude but the phase shift which is expanded; it it this that guarantees unitarity.

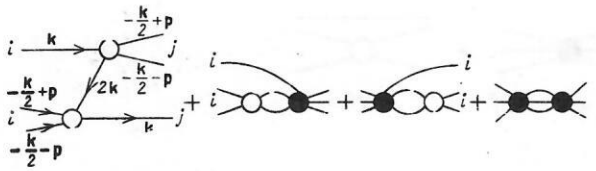

FIG. 3.

late the $s$- and $p$-wave phase shifts in the quartet state (total spin 3/2) and the $p$-wave phase shift in the doublet state (total spin 1/2). With regard to the $s$-wave phase shift of doublet scattering, its calculation is more difficult because of the presence of the bound state-the triton-and also because of the well-known anomalies at low energies (Ref. 40).[12)]

To the zeroth iteration of Eq. (65) there corresponds the exchange diagram in Fig. 4, in which the double line corresponds to the deuteron[13)]:

$$dV_{nd,\,nd}/dg = 2\,|V_{d,\,np}(q)|^2/(q^2+\varkappa^2). \tag{68}$$

Here, $V_{d,np}$ is the matrix element of deuteron disintegration, which depends on the momentum transfer $q = |\mathbf{k}+\mathbf{k}'/2| = |\mathbf{k}'+\mathbf{k}/2| = k(5/4-\cos\theta)^{1/2}$, in which $\theta$ is the angle between the vectors k and k'. It is convenient to renormalize the charge, taking $\varkappa^2$ as the new charge [see (46)]. Then in accordance with (28) we must have an additional term $\alpha V^c$ on the right-hand side of (68) with

$$\alpha = (d^2g/(d\varkappa^2)^2)/(dg/d\varkappa^2) \sim d^2\varkappa^2/dg^2.$$

But according to (46) and (15) $d^2\varkappa^2/dg^2 = -d^2E_d/dg^2$ is equal to $2\sum_\sigma |V_{d,\sigma}|^2/(E_\sigma - E_d)$ and the considered term itself corresponds to the diagram in Fig. 5. It can be seen that this term has the structure of a higher iteration, and it is therefore appropriate to take it into account at a later stage, where it compensates terms of similar structure arising from higher iterations of Eq. (65). The corresponding regrouping of the iteration series—it reduces to combining the renormalization terms with higher iterations-significantly improves the convergence of the iteration series.

Bearing in mind that the two-particle potential is separable (we take it in the Yamaguchi form), we can write

$$|V_{d,\,np}|^2 = V_{dd}V_{np,\,np} = (4\pi/q)\,d\delta^{(2)}(q)/d\varkappa^2; \quad V_{dd} = dE_d/d\varkappa^2 = -1,$$

where $\delta^{(2)}$ is the two-particle phase shift. When integrating the expressions (68) and (66), in which we must set $g = \varkappa^2$, we must bear in mind that null boundary conditions with respect to $\varkappa^2$ must be imposed as $\varkappa^2 \to \infty$,

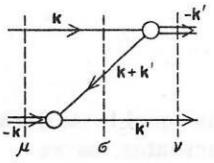

FIG. 4.

when the deuteron is bound infinitely strongly and does not participate in any processes requiring its dissociation.[23] This leads to the final expression for the phase shift of neutron scattering by the deuteron:

$$\delta(k) = -\frac{8k}{3q}(q^2+\varkappa^2)\int_{\varkappa^2}^{\infty}\frac{d\varkappa^2\delta^{(2)}(q)}{(q^2+\varkappa^2)^2}, \tag{69}$$

where we understand the taking of the corresponding spherical harmonic with respect to the angle $\theta$, which occurs in $q$.

Substituting the two-particle phase shift [see (47)] in (69) and noting that the parameter $\varkappa/\gamma$ is small, we obtain the following results. The quartet $s$-wave scattering length is

$$a_0 = -\frac{\delta(k)}{k}\Big|_{k=0} = \frac{16}{9\varkappa}\left(1-\frac{3\varkappa}{4\gamma}+\frac{3}{4}\left(\frac{\varkappa}{\gamma}\right)^2+\ldots\right) = 6.75\ \text{F}$$

(the experimental value is 6.35F). The principal term in the expansion in the parameter $\varkappa/\gamma$ of the expression for the quartet $s$-wave shift has the form

$$\delta(k) = -(8/3)\,[(3/4+1/3\xi^2)\,\mathrm{arctg}\,(3\xi/2) - (1/4+1/\xi^2)\,\mathrm{arctg}\,(\xi/2)],$$

where $\xi = k/\varkappa$. The phase shifts in other angular momentum and spin states have a similar form, which is shown in Fig. 6 by the continuous curves; the hatched corridors are the experimental errors (see Ref. 38). There is satisfactory agreement with experiment.

The convergence of the method can be gauged by an estimate of the following iteration, to which there corresponds the triangle diagram in Fig. 7 (for details, see Ref. 41). Its contribution to the quartet $s$-wave scattering length is

$$(128/27\varkappa)\,[\ln(9/8) - 1/12] \sim 0.1a_0.$$

The iteration series converges rapidly because the higher iterations have more distant singularities with smaller residues and discontinuities across the cuts; at the same time, each iteration meets the general requirements of unitarity and causality (analyticity).

We emphasize the differences between the diagrams used here and the usual diagrams of perturbation theory: a) in the first case the expansion is of the phase shift, but in the second of the scattering amplitude; b) in the first case the matrix element of the potential occurs in the vertex, in the second the two-particle scattering amplitude; c) in the first case we encounter integrals over $g$, in the second simply powers of $g$.

*Triton Binding Energy*. Among the problems assoc-

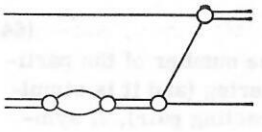

FIG. 5.

[12)] There are grounds for attributing these anomalies to virtual scattering of the neutron and proton in the singlet spin state, in which there is a virtual level near zero. Scattering of just this kind occurs in the triangle diagram considered below (see Fig. 7).

[13)] The change of sign in this equation is due to the exchange of a Fermi particle (for more detailed discussion of coefficients of this kind associated with the spin and isospin, see Ref. 37 and also Ref. 41).

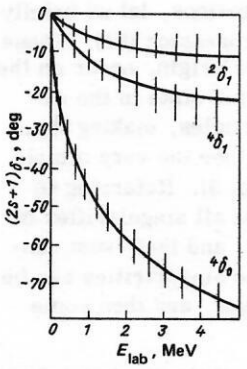


FIG. 6.

iated with doublet $s$-wave channel, we restrict ourselves to the solution of one of them-the calculation of the energy $E_t$ of the bound state of two neutrons and the proton (the triton).[42] The corresponding level, like the deuteron level, can be regarded as "shallow" compared with the depth of the potential well; the quantity

$$X^2=(3/4)(E_d-E_t), \tag{70}$$

which generalizes $\varkappa^2$ for the deuteron, is approximately 1/10 of $\gamma^2$. Using this circumstance, we shall work in the pole approximation, retaining in the corresponding equations only the terms that have the highest singularity in $(k^2+X^2)\to 0$. In particular, this enables us to ignore the contribution of intermediate states corresponding to three free nucleons, since the extra integration over the momentum "smears" the pole singularity. The absence of three-nucleon processes, in turn, makes it possible to ignore the separation of disconnected diagrams.

As before, we go over to a new coupling constant $\varkappa^2$, to which there corresponds the renormalization factor $\alpha=-\frac{1}{2}\varkappa^2$. In the pole approximation for the phase shift of $nd$ scattering, we can directly take an expression of the type (49):

$$\delta^{(2)}=\pi-\operatorname{arctg}(k/X),$$

whence, using (66), we obtain

$$V_{nd,\,nd}(k)=-3\pi\,(dX/d\varkappa^2)/(k^2+X^2). \tag{71}$$

The system of equations (15), which corresponds to Fig. 8 (the triple line represents the triton), takes the form

$$\left(\frac{d}{d\varkappa^2}+\frac{1}{2\varkappa^2}\right)V_{tt}=\frac{8}{3}\int\frac{d^3k\,|V_{t,\,nd}|^2}{k^2+X^2}; \tag{72}$$

$$\left(\frac{d}{d\varkappa^2}+\frac{1}{2\varkappa^2}\right)V_{t,\,nd}=\frac{4}{3}(V_{tt}-V_{dd})V_{t,\,nd}\left(\frac{i}{\delta}+\frac{1}{k^2+X^2}\right)$$
$$+\frac{4}{3}\int d^3k'V_{t,\,nd}V_{nd,\,nd}\left(\frac{1}{k^2-k'^2+i\delta}-\frac{1}{k'^2+X^2}\right); \tag{73}$$

$$\left(\frac{d}{d\varkappa^2}+\frac{1}{2\varkappa^2}\right)V_{nd,\,nd}=\frac{4}{3}V_{nd,\,t}V_{t,\,nd}\left(\frac{1}{k^2+X^2}+\frac{1}{k'^2+X^2}\right)$$
$$+\frac{4}{3}\int d^3k''V_{nd,\,nd}V_{nd,\,nd}\left(\frac{1}{k^2-k''^2-i\delta}+\frac{1}{k'^2-k''^2+i\delta}\right), \tag{74}$$

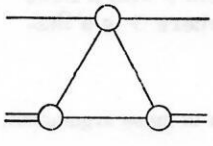

FIG. 7.

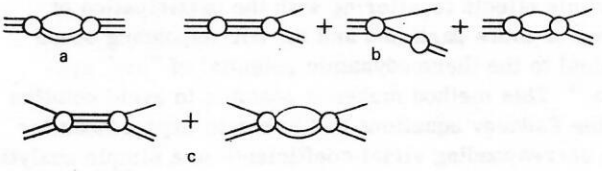


FIG. 8.

where $V_{tt}=dE_t/d\varkappa^2$.

It is easy to see that the matrix element factorizes near the pole:

$$V_{nd,\,nd}(k,\ k')=\chi^*(k')\,\chi(k);$$
$$V_{t,\,nd}=\chi^*(k)\,\beta,\quad |\beta|^2=V_{tt}-V_{dd}=-(3/4)\,dX^2/dg.$$

This makes it possible to avoid solutions of the complete system (72)–(73) and to restrict ourselves to the solution of Eqs. (71) and (72). We then obtain the final equation for $X$:

$$X\,d^2X/d\varkappa^2=-4/3. \tag{75}$$

This equation has a simple solution corresponding to the real case $\gamma/\varkappa\gg 1$:

$$X=2\left(\frac{2}{3}\ln\frac{\gamma}{\varkappa}\right)^{1/2}\varkappa.$$

Hence, using (70), we can readily find to the same accuracy an expression for the triton energy:

$$E_t=-\varkappa^2\ln(\gamma^2/\varkappa^2)=E_d\ln(\gamma^2/|E_d|). \tag{76}$$

Substitution of the empirical values $\gamma=1.44$ F$^{-1}$ and $E_d=-2.23$ MeV gives $|E_t|\approx 8$ MeV, which is fairly close to the experimental value 8.5 MeV.

However, this agreement should probably not be given too great significance, if for no other reason than that (76) implies coincidence of the thresholds for the formation of the deuteron and the triton, which in fact differ slightly. Nevertheless, bearing in mind the limited accuracy of (76), the result must be regarded as reasonable. The simplicity of its form and method of derivation advantageously distinguishes the coupling-constant evolution method from the standard approach, in which $E_t$ is found after lengthy numerical solution of the Faddeev equations.

To conclude this subsection we point out that the case of a larger number of bound states can be treated similarly if they are all regarded as shallow. Denoting by $X_n$ ($n$ is the number of the level) the quantity analogous to (70), we obtain instead of (75)

$$X_n\left[\frac{d^2X_n}{d\varkappa^2}+\frac{dX_n}{d\varkappa}\sum_m{}'\frac{dX_m}{d\varkappa}\Big/(X_m-X_n)\right]=-\frac{4}{3}. \tag{77}$$

From this equation we obtain the useful sum rule

$$\frac{d^2}{d\varkappa^2}\sum_n X_n=-\frac{4}{3}\sum_n X_n^{-1}.$$

From (77) one can deduce a number of properties characteristic of the Efimov family of levels[43] that occur near the point $\varkappa=0.5$

*Virial Coefficients*. The coupling-constant evolution method also has important advantages in the solution of the problem of determining the contribution of many-

particle effects (scattering with the participation of three or more particles and the corresponding bound states) to the thermodynamic potential of "hot" system.[44] This method makes it possible to avoid solution of the Faddeev equations and to obtain expressions for the corresponding virial coefficients in a simple analytic form. In addition, it makes it possible to avoid the singularities for "forward" scattering that occur in the usual approach and whose elimination is not a simple matter (see Ref. 45).

The thermodynamic potential of a system of strongly interacting particles can be written in the form of the virial expansion[46]

$$\Omega=\Omega_0-\frac{1}{\beta}\sum_{N=2}^{\infty}(m_N/2\pi\beta)^{3/2}\exp(N\beta\mu)\,b_N, \tag{78}$$

where $\Omega_0$ is the thermodynamic potential of an ideal gas, $\beta=1/kT$ is the reciprocal temperature, $m_N$ is the total mass of the $N$ particles, and $\mu$ is the chemical potential.[14)] The quantity $b_N$, the virial coefficient, describes the effects of the interaction of the $N$ particles and can be expressed in the form

$$b_N=\mathrm{Sp}_N[\exp(-\beta H)]^c. \tag{79}$$

Here, the trace is taken over the system of appropriately symmetrized wave functions of the $N$ particles, and the superscript $c$, as above, indicates that only connected diagrams have been taken.

In the expression (79), we can immediately separate the contribution of the $N$-particle bound states, which can be conveniently written in the form

$$b_N^{\mathrm{discr}}=\sum_n(\exp(\beta|E_{nN}|)-1), \tag{80}$$

where $E_{nN}$ is the corresponding energy of the bound state. With regard to the contribution of the continuum, differentiating (79) with respect to $g$, using Eq. (12), and integrating over $g$, we obtain

$$b_N^{\mathrm{cont}}=-\beta\int_0^g dg\sum_\nu\exp(-\beta E_{\nu N})\,V^c_{\nu\nu}, \tag{81}$$

where $V^c$ has the same meaning as before. Accordingly, for its determination we can use the iteration procedure described above.

We show first that for the second virial coefficient the well-known Bethe–Uhlenbeck formula[46] is obtained. Using (37) and going over to an integration over the energy, we obtain

$$b_2=\sum_n[\exp(\beta|E_n|)-1]+\frac{\beta}{\pi}\sum_e(2l+1)\int_0^\infty dE\exp(-\beta E)\,\delta_l(E).$$

If we integrate here by parts and use Levinson's theorem (41), then we do indeed obtain the Bethe–Uhlenbeck formula

$$b_2=\sum_n\exp(\beta|E_n|)+\sum_l(2l+1)\frac{1}{\pi}\int_0^\infty dE\exp(-\beta E)\frac{d\delta_l(E)}{dE}. \tag{82}$$

[14)] The expression (78) is for a system of particles of one species or a system of different particles of equal mass represented with equal density.

Before we turn to concrete applications, let us briefly consider the singularities of the expression (81). These singularities, of a purely kinematic origin, occur on the coincidence of the initial and the final state in the expression $V^c_{\nu\nu}$ for three or more particles, making the energy denominator in (15) vanish (see the very simple example in the first diagram in Fig. 3). Referring to Ref. 44 for the details, we note that all singularities of this kind can be explicitly summed, and their sum vanishes identically. Therefore, these singularities can be deleted as they appear without danger, and then some finite contribution remains.

In conclusion, we calculate some of the first virial coefficients for a hot nonrelativistic nucleon gas in which the protons and neutrons have equal density. We consider the two regions $\xi\gg1$ and $\xi\ll1$, where $\xi=\beta\varkappa^2$; in the first of them, the nucleons are mainly combined in deuterons; in the second, however, there are few deuterons.[15)] Using (47) and (82), we obtain

$$b_{np}=\exp(\xi),\quad \xi\gg1;\quad b_{np}=1/2,\ \xi\ll1. \tag{83}$$

We now estimate the contribution of three-particle scattering of two neutrons and a proton in the quartet state. Using the first diagram in Fig. 3 and an unrenormalized coupling constant, we obtain

$$b_{nnp}=-\beta\int_0^g dg\int d^3k\,d^3p\exp\left(-\beta\left(\tfrac{3}{4}k^2+p^2\right)\right)V_{nnp,\,nnp},$$

where

$$V_{nnp,\,nnp}=\frac{64\pi^2}{3kp}\int_0^g dg\,\frac{d\delta^{(2)}(p)}{dg}\,\frac{d\delta^{(2)}(3k/2)}{dg}\Big/\left(p^2-\tfrac{9}{4}k^2\right).$$

This gives

$$\left.\begin{aligned} b_{nnp}&=-\mathrm{const}\,\xi^{1/2}, && \xi\gg1;\\ b_{nnp}&=\mathrm{const}, && \xi\ll1,\end{aligned}\right\} \tag{84}$$

where the values of constants not written out explicitly are of order unity.

Finally, we consider the contribution of neutron scattering by deuterons in the quartet state. The corresponding virial coefficient (see Fig. 4) is given by

$$b_{nd}=-\beta\int_0^{\varkappa^2}d\varkappa^2\int d^3k\exp\left(\beta\left(\varkappa^2-\tfrac{3^2}{4}\right)\right)V_{nd,\,nd},$$

where

$$V_{nd,\,nd}=-\frac{16\pi}{3}\int_{\varkappa^2}^\infty\frac{d\varkappa^2}{\frac{9}{4}k^2+\varkappa^2}\,\frac{d\delta^{(2)}(3k/2)}{d\varkappa^2}.$$

Finally,

$$b_{nd}=\tfrac{4}{27}(\exp\xi-1)\ln\xi. \tag{85}$$

## 4. QUANTUM ELECTRODYNAMICS AND THE PROBLEM OF THE GHOST POLE

In this final section, we apply the coupling-constant evolution method to quantum field theory models describing the interaction of a Fermi field $\psi$ with a Bose field $\varphi$ of the type $L^{in}=(\psi_{in}O\psi_{in})\varphi_{in}$, where $O$ is a ma-

[15)] For simplicity, we shall ignore the spin in these arguments.

trix. We shall primarily consider quantum electrodynamics, although we also give results corresponding to pseudoscalar theory and to the theory of interaction of fermions with a vector field.

These models admit application of the coupling-constant evolution method to the same extent as the relativistic models of the four-fermion interaction considered in Sec. 2. In particular, in the theory presented below we encounter exceptional solutions, which (in contrast to the usual solutions) do not have a ghost pole for spacelike momentum (the "zero-charge" problem[47]). The investigation of the exceptional solutions requires the formulation of a regular procedure for determining the function $C(g)$ (see Sec. 2), which describes the extent to which the evolution matrix at finite time is nonunitary. This procedure is identical with the one used to solve the inverse problem of scattering theory.

*Annihilation Scattering of a Particle by an Antiparticle.* For the considered models, the necessary selection of the diagrams corresponds to the annihilation scattering of a particle by an antiparticle when the intermediate states contain either the same objects as in the initial state or a boson (Fig. 9). This circumstance, which complicates the problem compared with the case of the four-fermion interaction, also leads to more complicated calculations, though the final result remains qualitatively the same.

We introduce the four-fermion Lagrangian

$$l^{in} = :(\bar{\psi}_{in} O \psi_{in})(\bar{\psi}_{in} O \psi_{in}): \tag{86}$$

(for quantum electrodynamics, it is also necessary to add the transverse projection operator $g_{\mu\nu} - k_\mu k_\nu / k^2$). Then in complete analogy with what we have said in Sec. 2, the condition (50) is satisfied for (86), and the function $\rho$ [see (51)] has for quantum electrodynamics ($O = \gamma_\mu$) the form ($m$ is the mass of the fermion)

$$\rho(s) = -(1/12\pi^2)(1-4m^2/s)^{1/2}(s+2m^2); \tag{87}$$

for pseudoscalar theory ($O = \gamma_5$) the coefficient $1/12$ is replaced by $1/8$, and the final factor in (87) is simply $s$.

In these models too one can reduce the treatment to scalar functions, assuming as in (52)

$$f_{\mu\nu} = l^{in}_{\mu\nu}\varphi(s); \quad L^{22}_{\mu\nu} = l^{in}_{\mu\nu}\lambda(s_\mu, s_\nu),$$

where $L^{22}$ is the matrix element of $L$ for the transition of two fermions into two fermions. The connection between $\varphi$ and $\lambda$ and the phase shift is given by the previous expressions (53) and (54). As in Sec. 2, we can introduce the Jost function $u(s)$, which contains complete information about the scattering process. Turning to the problem of the ghost pole of the Green's function $D = d(s)/(s - M^2)$ of a boson of mass $M$ (in quantum electrodynamics, it is here also necessary to add the transverse operator), we give the expression for the function $d$ in terms of $u$:

$$d(s, g) = \frac{2}{g}\int_0^g \frac{dg}{u^2(s, g)} \tag{88}$$

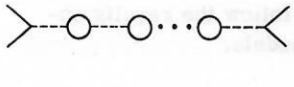

FIG. 9.

[see (53) and (54)].

We show (for details, see Refs. 34, 48, and 49) that the considered models of the interaction of Fermi and Bose fields can also be regarded as separable. This is understood in the sense that for them too the equation for the Jost function has the form

$$\left.\begin{aligned} &d^2\tilde{u}/d\alpha^2 + C^2(\alpha)(s-M)^2\tilde{u} = 0; \\ &C(\alpha) = \lim_{s\to\infty}(d\delta/d\alpha)/\sqrt{s}. \end{aligned}\right\} \tag{89}$$

Here we have gone over to the renormalized Jost function, which becomes equal to unity at $s = M^2$ (accordingly, the renormalized coupling constant refers to the same point), and as the coupling constant itself we have chosen $\alpha = g^2/4\pi$, on which (and not on $g$) the scattering characteristics depend.

Referring to the quoted papers, we merely sketch the proof of (89). For the selected set of diagrams, the following terms are important in the expansion of the operators $L$ and $\Lambda$ in normal products [see (28)]: in the operator $L$, the terms $l^{in}L^{22}$ and $L^{in}L^{12}$; in the operator $\Lambda$, in addition, the term $:\varphi_{in}\varphi_{in}:\Lambda^{11}$.[16] It can be shown that the system of equations for the functions $L^{12}, L^{22}, \Lambda^{11}, \Lambda^{12}, \Lambda^{22}$ that follows from (28) corresponds to the following structure of these functions:

$$L^{12} = \chi(s); \quad L^{22}(s_\mu, s_\nu) = \chi^*(s_\mu)\chi(s_\nu)[b(s_\mu) + b(s_\nu)];$$
$$\Lambda^{12} = \chi(s)\,b(s)\,\Lambda^{11}; \quad \Lambda^{22}(s_\mu, s_\nu) = \chi^*(s_\mu)\chi(s_\nu)\,b(s_\mu)\,b(s_\nu)\,\Lambda^{11},$$

where $b(s) \sim (s - M^2)^{-1}$. This is obviously a nearly separable structure. Using the results we have given, we can now show that the function $A$ in the equation analogous to (43) really is equal to $C^2(s - M^2)$.

Turning to (89), we first give the ordinary solution corresponding to $C = 0$, Using the boundary condition

$$(d\delta/d\alpha)|_{\alpha=0} = 4\pi^2\rho(s)/(s - M^2),$$

we obtain the Jost function in the form of a linear function of $\alpha$:

$$\tilde{u}(s) = 1 + 4\pi\alpha(s - M^2)\int_{4m^2}^{\infty}\frac{ds'\rho(s')}{(s - s' + i\delta)(s' - M^2)^2}.$$

However, in accordance with (88),

$$d^{-1}(s) = 1 - (\alpha/3\pi)\ln(-s/m^2). \tag{90}$$

This expression has a ghost pole at

$$s = -m^2\exp(2\pi/\alpha).$$

We now turn to the determination of the exceptional solution. It is first of all necessary to establish the form of the function $C(\alpha)$. The problem will then be solved in general form, but we restrict ourselves to the asymptotic region $s \gg m^2$, where the Callan–Symanzik equation holds:

$$[m\partial/\partial m + \beta(\alpha)\alpha\partial/\partial\alpha]\alpha d(s, \alpha) = 0,$$

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[16] The conditions of stability of the vacuum and the single-particle space here eliminate the normal products 1, $:\bar{\psi}_{in}\bar{\psi}_{in}:$, and $:\varphi_{in}\varphi:$. In addition, the conditions of stability of the single-boson state fix the constant $\beta$ in (28) [see (35)].

where $\beta(\alpha)=\sigma\alpha+\ldots$ is the Callan–Symanzik function ($\sigma$ is a constant). Using the asymptotic behavior of the Jost function (see Sec. 2) and (88), we obtain

$$\alpha d(s)\sim\exp\left(2i\sqrt{s}\int_0^\alpha d\alpha C(\alpha)\right)-1.$$

From this we obtain the connection between the function $C(\alpha)$ and the Callan–Symanzik function[50]:

$$C(\alpha)\sim\frac{1}{m\alpha\beta(\alpha)}\exp\left(-\int_\alpha^\infty\frac{d\alpha}{\alpha\beta(\alpha)}\right)$$
$$=-\frac{1}{m\sigma\alpha^2}\exp\left(-\frac{1}{2\sigma\alpha}\right)+\ldots. \tag{91}$$

Using this formula, we can readily obtain a solution of Eq. (89) in the form

$$\tilde{u}(s)=i\pi\alpha\sigma H_0^{(1)}(x),$$

where

$$x=(\sqrt{s}/m\gamma)\exp(-1/2\alpha\sigma),$$

$\sigma=1/3\pi$ in quantum electrodynamics and $1/2\pi$ in the pseudoscalar theory; $\gamma=\exp C$, where $C$ is Euler's constant. Hence, using (88), we find an expression for the Green's function of the boson in the form

$$d(s,\alpha)=-(i/2\pi\alpha\sigma)[1+H_0^{(2)}(x)/H_0^{(1)}(x)]. \tag{92}$$

This expression does not have a ghost pole, which is not surprising, since its occurrence would contradict the original system of axioms.

The Green's function (92) can also be rewritten in the form

$$d^{-1}(s)=d_0^{-1}(s)+\alpha\pi\sigma\Psi(x), \tag{93}$$

where $d_0$ is the ordinary solution (90), and

$$\Psi(x)=(2/\pi)\ln(x\gamma/2)-N_0(x)/J_0(x)$$

is a meromorphic function of its argument. It plays the part of the well-known Castillejo–Dalitz–Dyson $R$ function,[51] and, thus, the coupling-constant evolution method can be regarded as a method of definite fixing of this function (see also Ref. 4).

Note that the series expansions in $\alpha$ of the exceptional and the ordinary solution are identical, which is due to the essential nonanalyticity of the function $\Psi$ in (93) [of the type $\exp(-1/\alpha)$] at the point $\alpha=0$.

*Determination of the Function $C(\alpha)$.* Hitherto, the form of the function $C$ has been found either on the basis of dimensional considerations or from relations that hold only in the asymptotic region. Here, we describe a general procedure for determining the function $C(\alpha)$ that, in particular, enables one to find an exceptional solution in models such as quantum electrodynamics outside the asymptotic region as well.[34,49]

We proceed from Eq. (89), which by the change of variables

$$\tilde{u}=z/\sqrt{C};\quad \xi=\int_0^\alpha d\alpha C;\quad k^2=s-M^2$$

can be reduced to the form of an ordinary Schrödinger equation

$$[d^2/d\xi^2+k^2-U(\xi)]z(k,\xi)=0 \tag{94}$$

with "potential"

$$U(\xi)=(1/\sqrt{C})\frac{d^2}{d\xi^2}\sqrt{C} \tag{95}$$

and boundary conditions

$$z|_{\xi\to0}\to\sqrt{C}\left(1-\frac{4i\pi\rho}{k^2}\int_0^\xi\frac{d\xi}{C}\right);$$
$$z|_{k\to\infty}=(k/\rho)^{1/2}\exp(-ik\xi). \tag{96}$$

These conditions are in reality superfluous and actually determine the potential (95). It can be found by using the well-known method for solving the inverse scattering problem.[24]

The solution is divided into two stages. In the first we solve the limit problem (the subscript 0 is appended to the corresponding quantities), to which there corresponds the asymptotic form of $\rho(k)$ as $k\to\infty$. In all interesting cases

$$\rho_0(k)=(\varkappa/4\pi^2)k^{2(n+1)},\quad n\geqslant0$$

and in the models considered here $n=0$. We assume further (and this will be confirmed) that $U_0(\xi)=(n^2-1/4)/\xi^2$. Substitution of $U_0$ in (94) gives the Bessel equation, and it can be seen from (96) that $z_0$ is proportional to $H_n^{(1)}(k\xi)$. Subjecting this quantity to the boundary conditions (96), we can uniquely determine the function $C_0(\xi)$:

$$C_0(\xi)=\frac{2\varkappa\Gamma^2(n+1)2^{2n}}{\pi(2n)^2}\xi^{1-2n},\quad n\neq0;$$
$$C_0(\xi)=\frac{2\varkappa}{\pi}\xi\ln^2\left(\frac{1}{\xi}\right),\qquad n=0. \tag{97}$$

For the function $z_0(k,\xi)$ itself, we obtain the simple expression

$$z_0(k,\xi)=(\xi k^2/8\pi)^{1/2}H_n^{(1)}(k\xi). \tag{98}$$

Referring to the quoted papers for the details, we give the results of the solution of the second stage of the problem, which refers to the nonasymptotic region. The principal part is played here by the Gel'fand-Levitan–Marchenko kernel $K(\xi,\xi')$, which is determined by the equation

$$K(\xi,\xi')=Q(\xi,\xi')-\int_0^\xi d\xi''K(\xi,\xi'')Q(\xi'',\xi'),$$

where

$$Q(\xi,\xi')=8\pi\int_{2m}^\infty\frac{dk}{k}\left(\sqrt{\rho(k)\rho_0(k)}-\rho_0(k)\right)z_0(k,\xi)z_0(k,\xi').$$

In terms of this kernel we can express the function $U(\xi)$ [and, therefore, the function $C(\alpha)$ in accordance with (95)]:

$$U(\xi)=U_0(\xi)-2dK(\xi,\xi)/d\xi.$$

In addition, the kernel $K$ determines the function $z(k,\xi)$ and thereby the Jost function of the problem:

$$z(k,\xi)=z_0(k,\xi)-\int_0^\xi d\xi'K(\xi,\xi')z_0(k,\xi').$$

These relations completely solve the problem, and from them, as a special case, there follow the results obtained above for the simplest models.

We apply the obtained relations to quantum electrodynamics with the aim of finding an exceptional solution without a ghost pole outside the asymptotic region. To logarithmic accuracy, the solution of the limit problem [see (97) and (98)] coincides with the results obtained above. With regard to the region $k\xi \sim k\varkappa \lesssim 1$, in it, as will be seen from what follows, the difference between the function $z$ and $z_0$ is small [of order $\exp(-1/\alpha)$]. When we carry out the second stage of this procedure, this enables us to regard the correction terms to the limit terms as small, which greatly facilitates the solution of the problem.

We restrict ourselves to giving the final expression for the photon Green's function [cf. (93)]:

$$d^{-1}(s) = d_0^{-1}(s) + \frac{\alpha}{3}\Psi(x) + \Delta(x), \tag{99}$$

where

$$d_0^{-1} = 1 - \frac{\alpha}{3\pi} R\left[\frac{3-R^2}{2}\ln\frac{R+1}{R-1} + R - \frac{8}{3R}\right], \quad R = \left(1 - \frac{4m^2}{s}\right)^{1/2}$$

is the nonasymptotic form of the Green's function corresponding to the usual dynamical solution, and

$$\Delta(x) = -\frac{8\alpha\tau}{3\pi\gamma^2 x^2}\left(1 - \frac{1-x/2}{J_0^2(x)}\right)\exp(-3\pi/\alpha - 5/6). \tag{100}$$

Here

$$x = (2\sqrt{s}/m\gamma)\exp(-3\pi/2\alpha - 5/6);$$

$$\tau = \frac{1}{2}\int_4^\infty dt\left[1 - \left(-1\frac{2}{t}\right)^{1/2}\left(1 - \frac{4}{t}\right)^{1/4}\right].$$

In the region $x \sim 1$, where the usual solution has a ghost pole, the second term on the right-hand side of (99), which "swamps" this pole, is important; in this region, the quantity (100) is exponentially small. But in the region $x \ll 1$ (in particular, in the currently experimentally accessible region $s \gtrsim m^2$) both the last terms on the right-hand side of (99) are exponentially small and the prinicpal part is played by the usual expression for the Green's function.

In conclusion, we make a remark that applies to all exceptional solutions of the equations of quantum field theory discussed above. All such solutions are characterized by the existence of directions in the complex $s$ plane along which the scattering amplitude increases exponentially. This could be regarded as a defect of such solutions if in the corresponding region for the solution of the problem there were sufficient selected diagrams whose sum corresponds to the discussed solution. But it is clear that at large $s$ a major part is played by many-particle diagrams with more than two particles in the intermediate states not taken into account above. Therefore, the question of the asymptotic behavior of the exact solutions of the equations in the coupling-constant evolution method (and, moreover, of the very existence of exceptional solutions) should, strictly speaking, be regarded as still open. A promising fact in this connection is the existence of exceptional solutions in nonrelativistic models (see Sec. 2) that are in complete correspondence with all the general principles of quantum field theory.

*Decay States.* In considering the models discussed above we have essentially assumed fulfillment of the inequality $M < 2m$, which ensures stability of the boson against decay into a fermion–antifermion pair. Violation of this inequality, which happens, for example, in the theory of the weak interaction, leads to instability of the boson, which, in its turn calls for a certain modification of the coupling-constant evolution formalism. We mention here some results of Ref. 52.

From the very fact of instability of the boson it follows that its states must be eliminated from the complete system of "in" states. Using this circumstance, one can show that everything said above concerning the Jost function remains true. This applies, in particular, to Eq. (89), which determines the Jost function. It is only necessary to remember that the coupling constant must now be normalized as the residue at the pole on the second sheet. In addition, it is of course necessary to assume that the width of the boson level is small compared with the boson mass.

The dynamical solution of this problem, as is well known, contains a Breit–Wigner resonance in the region $s \approx M^2$. As can be seen from Eq. (89), the ordinary and exceptional solutions of this equation almost coincide in this region. With regard to the asymptotic behavior of the scattering characteristics, they are determined by the same expressions, in which the fermion mass $m$ must be replaced by the largest of the existing masses-the mass $M$ of the boson.

We cannot list here all those people with whom we have had discussions which have helped to elucidate the questions touched upon here. An exception must be made only for Igor' Evgen'evich Tamm, whose approval, support, and constructive remarks played a decisive part in the formulation of the method presented here.

*Note added in proof.* After this paper had been sent to press, new results from the application of the coupling-constant evolution method to the three-body problem were published[53,54]; these relate to doublet scattering of the neutron by the deuteron, proton scattering by the deuteron, the binding energies of nuclei of tritium and helium-3, and the nuclear vertex constant $(t, nd)$. All these results, which were obtained in collaboration with the student F. M. Len'kov, are in good agreement with the experiments and the results of numerical calculations.

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