Theory of multiple scattering of pions by nuclei

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The general theory of multiple scattering of pions by nuclei is expounded on the basis of relativistic potential theory. Different formulations of this theory are considered. The connection between the t matrix for scattering of a pion by a nucleon in their center-of-mass frame and the same matrix in an arbitrary frame is considered in the framework of the general theory. On the basis of the Lippman-Schwinger equation for the T matrix for the scattering of a pion by a nucleus, a general expression is obtained (in the form of a series) for the amplitude of πA scattering in the fixed-scatterer approximation (FSA), and different solvable models are considered in this approximation. The currently available approximate methods of solution of the problem of πA scattering are discussed in detail. Allowance for various effects that enable one to go beyond the fixed-scatterer approximation is considered on the basis of the first-order optical potential. Consideration is also given to the results of the first attempts to construct the second-order optical potential and to take into account real absorption of pions by nuclei in πA scattering. The problem of pion–deuteron scattering is considered separately on the basis of the relativistic three-particle equations. The main attention is devoted to pion-nucleus scattering in the region of the (3,3) resonance.

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INTRODUCTION

Study of the interaction of pions with nuclei is one of the central problems of the physics of intermediate energies. The recently commissioned meson factories open up wide possibilities for comprehensive and detailed quantitative study of this interaction. At the present state of theory, there is no unified dynamical scheme for treating all the phenomena that are observed when pions interact with nuclei. Therefore, different theoretical approaches are used to investigate particular aspects of the problem. One such approach to pion–nucleus interactions is the general theory of multiple scattering of a particle by a bound system. This theory enables one to find the differential and integral cross sections of elastic scattering, σ_{el}, and the total cross section σ_{tot} for scattering of a pion by a nucleus as a function of the pion kinetic energy T_π.

The theory of multiple scattering of a pion by a nucleus relates the pion–nucleus collision matrix to the free-state pion–nucleon collision matrix, which is assumed known to some degree or other. In this respect, the theory is phenomenological, but it is nevertheless sufficiently general to be used as the basis for investigating various aspects of the pion–nucleus dynamics.

Aspects of the theory of multiple scattering relating to pion–nucleus scattering were discussed in Ref. 1 as part of the general problem of the pion–nucleus interaction. However, in Ref. 1 the aspects of the theory of multiple scattering were considered only schematically and, further, the results of investigations published only up to the middle of 1974 were discussed. In the last three or four years many papers have been published on both the fundamentals of the theory and its development and on the application of this theory to particular problems of πA scattering.

The present review is devoted to a more detailed analysis of the state of the theory of multiple scattering of pions by nuclei. The main attention is devoted to the formulation of the theory and its further development, the analysis of approximate solutions to the problem, and the main assumptions used at present in concrete calculations. We consider separately pion–deuteron scattering, for which (in a definite range of the energy T_π) it is possible to give a three-particle formulation of the problem on the basis of Faddeev's equations or the relativistic generalizations of these equations.

1. GENERAL THEORY OF MULTIPLE SCATTERING OF PIONS BY NUCLEI

The theory of multiple scattering is in essence a potential theory, and it presupposes that the interaction of the incident particle, in this case a pion, with nuclei is of a two-particle nature. As a result, the Hamiltonian of the system consisting of the pion and the nucleus (πA) is represented in the form

$$H = h_\pi + H_A + \sum_{i=1}^{A} v_i, \tag{1}$$

where h_π is the Hamiltonian of the free pion, H_A is the Hamiltonian of the nucleus, and v_i is the potential of the interaction of the pion with nucleon i of the nucleus. In the framework of relativistic potential theory,[2-6] it is found that even in the first order in the square of the ratio of the particle velocity to the velocity of light three-particle forces occur in the expansion of the interaction. Therefore, the Hamiltonian (1) is an approximation that does not take into account the many-particle forces in the pion–nucleus interaction. In addition, the theory based on the Hamiltonian (1) ignores an important physical process—real absorption of the pion by the nucleus, which proceeds basically through the two-nucleon mechanism (see, for example, Ref. 7). Below, we formulate the general theory of the multiple scattering of pions by nuclei, following Refs. 8–11.

The T matrix corresponding to the Hamiltonian (1.1) is determined by the solution of the Lippmann–Schwinger equation:

$$T(E) = \begin{cases} \sum_i v_i + \sum_i v_i G(E)\, T(E); & (2) \\ \sum_i v_i + T(E)\, G(E) \sum_i v_i, & (3) \end{cases}$$

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where

$$G(E)=[E+i0-h_\pi-H_A]^{-1}; \tag{4}$$

E is the energy of the system.

This form of expression for the T matrix presupposes that the iteration series of Eqs. (2) and (3) converge or that the corresponding inverse operators exist.

It is important that the state vectors of the nucleus between which the matrix elements of the operator T and, generally, physical operators are taken must be antisymmetric under permutation of the nucleons. We denote by $\mathscr{A}=a^2$ the operator that projects the Hilbert space of the nucleus onto antisymmetrized state vectors, which form a subspace orthogonal to the remaining part of the Hilbert space. Because of the symmetry of the operator $\sum_i v_i$ under permutation of the nucleons, its matrix elements between the state vectors of these two different parts of the Hilbert space are zero. Therefore, Eqs. (2) and (3) can be written without loss of generality in the form

$$T=\begin{cases} Av+AvG\mathscr{A}T; & (5)\\ Av+T\mathscr{A}GvA, & (6)\end{cases}$$

where

$$\mathscr{A}v=\sum_i v_i. \tag{7}$$

The main physical quantity determined in the theory of multiple scattering is the submatrix of the total T matrix corresponding to elastic scattering. It is frequently convenient to reduce the problem of finding this matrix to that of finding some effective potential, which is called the optical potential. It is desirable and convenient to express this potential in terms of the pion–nucleon scattering matrix, which in principle is experimentally observable.

We represent the projection operator a as the sum of an operator p that projects the space of antisymmetric state vectors of the nucleus onto the ground state, and the operator Q, which separates all possible excited antisymmetric state vectors:

$$\left.\begin{aligned} &\mathscr{P}+Q=\mathscr{A}\\ &\mathscr{A}^2=\mathscr{A},\quad \mathscr{P}^2=\mathscr{P},\quad Q^2=Q,\quad \mathscr{P}Q=Q\mathscr{P}=0.\end{aligned}\right\} \tag{8}$$

It is obvious that the equations $\mathscr{A}G=G\mathscr{A}$; $\mathscr{P}G=G\mathscr{P}$; $QG=GQ$, which will be frequently used below, hold.

By definition, the optical potential is an operator by means of which the elastic-scattering matrix can be determined from a Lippmann–Schwinger equation of the following form:

$$T(E)=\begin{cases} U(E)+U(E)\mathscr{P}G(E)T(E); & (9)\\ U(E)+T(E)G(E)\mathscr{P}U(E). & (10)\end{cases}$$

We introduce the auxiliary operators

$$\tau(E)=\begin{cases} v+vG(E)\mathscr{A}\tau(E); & (11)\\ v+\tau(E)\mathscr{A}G(E)v; & (12)\end{cases}$$

$$\hat{\tau}(E)=\begin{cases} v+vG(E)Q\hat{\tau}(E); & (13)\\ v+\hat{\tau}(E)QG(E)v. & (14)\end{cases}$$

Determining the operator v from Eq. (12) and substituting in (5), we obtain

$$T=[1+\tau\mathscr{A}G]^{-1}A\tau+[1+\tau\mathscr{A}G]^{-1}A\tau G\mathscr{A}T,$$

which is equivalent to the equation

$$T=A\tau+(A-1)\tau G\mathscr{A}T. \tag{15}$$

Similarly, we can show that

$$T=A\hat{\tau}+[A\hat{\tau}\mathscr{A}G-\hat{\tau}QG]T. \tag{16}$$

Introducing the new operator

$$T'=T(A-1)/A, \tag{17}$$

we can reduce Eq. (15) to the form

$$T'=(A-1)\tau+(A-1)\tau G\mathscr{A}T'. \tag{18}$$

We determine an operator U' in such a way that the equation for the matrix T' takes the form (10), i.e.,

$$T'(E)=\begin{cases} U'(E)+U'(E)G(E)\mathscr{P}T'(E); & (19)\\ U'(E)+T'(E)\mathscr{P}G(E)U'(E). & (20)\end{cases}$$

Solving formally Eq. (18) for T', substituting the obtained expression for T' in (20), and making algebraic transformations similar to the ones used in the derivation of (15), we obtain the equation for the optical potential:

$$U'(E)=(A-1)\tau(E)+(A-1)\tau(E)G(E)QU'(E). \tag{21}$$

On the basis of (16), we can similarly obtain

$$T(E)=\begin{cases} \hat{U}(E)+\hat{U}(E)G(E)\mathscr{P}T(E); & (22)\\ \hat{U}(E)+T(E)\mathscr{P}G(E)\hat{U}(E), & (23)\end{cases}$$

where U is determined from the equation

$$\hat{U}(E)=A\hat{\tau}(E)+(A-1)\hat{\tau}(E)G(E)Q\hat{U}(E). \tag{24}$$

Equations (13), (14), and (22)–(23) are the basic relations of the theory of multiple scattering in Watson's formulation,[8] and Eqs. (11), (12), (17), and (19)–(21) are the basis of the Kerman–McManus–Thaler (KMT) formulation.[9] It is evident from the derivation that the two formulations are equivalent in an exact solution of the problem. They differ in an approximate solution. This difference is due to the difference between the auxiliary operators τ [(11), (12)] and $\hat{\tau}$ [(13), (14)], which represent the t matrix of pion scattering by a nucleon in a nucleus. It can be seen from the definition that τ and $\hat{\tau}$ are many-particle operators that take into account in different ways the influence of the nuclear medium on the distinguished nucleon that interacts with the pion. Determining v from Eq. (14), substituting the result in (11), and using the algebraic transformations performed above, we can readily find the connection between τ and $\hat{\tau}$:

$$\tau(E)=\begin{cases} \hat{\tau}(E)+\hat{\tau}(E)G(E)\mathscr{P}\tau(E); & (25)\\ \hat{\tau}(E)+\tau(E)\mathscr{P}G(E)\hat{\tau}(E). & (26)\end{cases}$$

We now consider the fulfillment of the unitarity condition for the elastic scattering matrix

$$T_{el}=\mathscr{P}T\mathscr{P}. \tag{27}$$

For this, using the equality $\mathscr{P}^2=\mathscr{P}$, we obtain from Eqs. (19), (20), (22), and (23)

$$T'_{el}(E)=\begin{cases} V'_{opt}(E)+V'_{opt}(E)G_0(E)T'_{el}(E); & (28)\\ V'_{opt}(E)+T'_{el}(E)G_0(E)V'_{opt}(E), & (29)\end{cases}$$

where

$$T'_{el}=\mathscr{P}T'\mathscr{P};\quad V'_{opt}=\mathscr{P}U'\mathscr{P};\quad G_0=\mathscr{P}G\mathscr{P}, \tag{30}$$

and

$$T_{el}(E)=\begin{cases}\hat{V}_{opt}(E)+\hat{V}_{opt}(E)G_0(E)T_{el}(E); & (31)\\ \hat{V}_{opt}(E)+T_{el}(E)G_0(E)V'_{opt}(E), & (32)\end{cases}$$

where

$$\hat{V}_{opt}=\mathcal{P}\hat{U}\mathcal{P}. \quad (33)$$

Equations (28), (29) and (31), (32) are equations of the form

$$\mathcal{T}(E)=\begin{cases}\mathcal{V}(E)+\mathcal{V}(E)\mathcal{G}(E)\mathcal{T}(E);\\ \mathcal{V}(E)+\mathcal{T}(E)\mathcal{G}(E)\mathcal{V}(E),\end{cases}$$

where $\mathcal{V}(E)$ in the general case is a non-Hermitian operator; $\mathcal{G}(E)$ is the Green's operator. It is easy to show[12] that the matrix $\mathcal{T}(E)$ satisfies a unitary relation of the form

$$\mathcal{T}(E)-\mathcal{T}^+(E)=\mathcal{T}^+(E)[\mathcal{G}(E)-\mathcal{G}^+(E)]\mathcal{T}(E)$$
$$+[1+\mathcal{T}^+(E)\mathcal{G}^+(E)][\mathcal{V}(E)-\mathcal{V}^+(E)][1+\mathcal{G}(E)\mathcal{T}(E)].$$

We use this relation for the matrices T_{el} and T'_{el} determined by the relations (27), (31)–(33), (28)–(30). We also use a relation obtained from (4):

$$G(E)-G^+(E)=-2\pi i\delta(E-h_\pi-H_A).$$

As a result, we obtain the equation

$$T_{el}(E)-T^+_{el}(E)=-2\pi i T^+_{el}(E)\mathcal{P}\delta(E-h_\pi-H_A)\mathcal{P}T_{el}(E)$$
$$+[1+T^+_{el}(E)G^+_0(E)][\hat{V}_{opt}(E)-\hat{V}^+_{opt}(E)][1+G_0(E)T_{el}(E)] \quad (34)$$

for Watson's formulation, and

$$T_{el}(E)-T^+_{el}(E)=-2\pi i\frac{A-1}{A}T^+_{el}(E)\mathcal{P}\delta(E-h_\pi-H_A)$$
$$\times\mathcal{P}T_{el}(E)+\frac{A}{A-1}[1+T'^+_{el}(E)G^+_0(E)][V'_{opt}(E)-V'^+_{opt}(E)]$$
$$\times[1+G_0(E)T'_{el}(E)] \quad (35)$$

for the KMT formulation.

It can be seen that the first term on the right-hand side of (34) is proportional to the cross section of elastic scattering, while the second term corresponds to inelastic processes, which are directly related to the non-Hermitian part of the optical potential $\hat{V}_{opt}$. With regard to the first term on the right-hand side of (35), before the elastic scattering cross section it does not include the quantity

$$T^+_{el}(E)\mathcal{P}\delta(E-h_\pi-H_A)\mathcal{P}T_{el}(E)/A.$$

And since the relation (35) is exact and therefore equivalent to (34), the second term on the right-hand side of (35) must contain a contribution from elastic scattering, i.e., the first term does not completely exhaust the elastic channel. Thus, whereas in Watson's formulation the contribution to the unitarity condition from elastic scattering is rigorously separated from the contribution of the inelastic processes, in the KMT formulation these contributions are not completely separated. Therefore, in an approximate solution of the scattering problem in the KMT formulation it is not certain what inelastic channels have been ignored in this approximation. This shortcoming of the KMT formulation was pointed out in Ref. 13, and this question was discussed in Ref. 14.

In the approximate solution of the problem of pion scattering by nuclei, it is sometimes convenient to represent the T matrix as a series (Watson series). To obtain this, we note that in accordance with the definition (7) the matrix elements of v_i and v between antisymmetric wave functions are equal, and therefore so will be the matrix elements of the operator τ [(11), (12)] and the operator τ_i defined by the equations

$$\tau_i(E)=\begin{cases}v_i+v_iG(E)\mathcal{A}\tau_i(E); & (36)\\ v_i+\tau_i(E)\mathcal{A}G(E)v_i. & (37)\end{cases}$$

We define the operators

$$T_i=v_i+v_iG\mathcal{A}T. \quad (38)$$

Using Eqs. (5), we see that

$$T=\sum_i T_i. \quad (39)$$

If from Eq. (37) we find v_i and substitute in (38), then, using (39), we find

$$T_i(E)=\tau_i(E)+\tau_i(E)G(E)\mathcal{A}\sum_{j\neq i}T_j(E). \quad (40)$$

Solving this system of equations iteratively, we obtain for T (39) the required series

$$T(E)=\sum_i\tau_i(E)+\sum_i\sum_{j\neq i}\tau_i(E)G(E)\mathcal{A}\tau_j(E)$$
$$+\sum_i\sum_{j\neq i}\sum_{k\neq j}\tau_i(E)G(E)\mathcal{A}\tau_j(E)G(E)\mathcal{A}\tau_k(E)+\ldots. \quad (41)$$

Before we consider the different approximations used at present to investigate pion scattering by nuclei in the framework of the above thoery of multiple scattering, we must particularize the method used to describe the nucleus. Let H_A^{cm} be the Hamiltonian of the internal motion of the nucleus; then in the framework of relativistic potential theory H_A will have the form ($\hbar=c=1$)

$$H_A=\sqrt{\mathbf{K}_A^2+(H_A^{cm})^2}-M_A. \quad (42)$$

Here and in what follows, the energy is measured from the ground-state energy of the nucleus (M_A is the corresponding mass), and $\mathbf{K}_A$ is the total momentum of the nucleus.

We introduce antisymmetrized state vectors $|\mathbf{K}_A\alpha\rangle=|\mathbf{K}_A\rangle|\alpha\rangle$, of the nucleus, which satisfy the Schrödinger equation

$$H_A|\mathbf{K}_A\alpha\rangle=E_{A\alpha}|\mathbf{K}_A\alpha\rangle, \quad (43)$$

where

$$E_{A\alpha}=\sqrt{\mathbf{K}_A^2+\mathcal{E}_{A\alpha}^2}-M_A. \quad (44)$$

Here, $\mathcal{E}_{A\alpha}=MA+\mathcal{E}_\alpha$ is the total energy of the nucleus A in the center-of-mass system; $\mathcal{E}_\alpha$ is the energy of the internal motion of the nucleus, which is determined by the equation

$$H_A^{cm}|\alpha\rangle=(MA+\mathcal{E}_\alpha)|\alpha\rangle. \quad (45)$$

In modern theory, nuclei with $A\geq 3$ nucleons are treated on the basis of nonrelativistic quantum mecahnics (for the deuteron, a quasirelativistic treatment is possible). Accordingly, we must consider the nonrelativistic limit of the expression H_A (42):

$$H_A\approx H_A^{cm}-M_A+\mathcal{K}_A^c;\quad \mathcal{K}_A^c=\mathbf{K}_A^2/(2MA). \quad (46)$$

In this case, we obtain for the energy

$$E_{A\alpha}\approx E_A(K_A)+\mathcal{E}_\alpha+B_A, \quad (47)$$

where

$$B_A=MA-M_A;\quad E_A(K_A)=\mathbf{K}_A^2/(2MA)\approx\sqrt{\mathbf{K}_A^2+M_A^2}-M_A. \quad (48)$$

It is clear that the state vectors $|\mathbf{K}_A\alpha\rangle$ satisfy the com-

pleteness condition

$$\sum_{\alpha}\int |\mathbf{K}_A\alpha\rangle\, d\mathbf{K}_A\, \langle\alpha\mathbf{K}_A| = \mathcal{A}. \tag{49}$$

We shall assume that these state vectors are orthonormalized in accordance with the equation

$$\langle \mathbf{K}'_A\alpha'|\alpha\mathbf{K}_A\rangle = \delta(\mathbf{K}'_A - \mathbf{K}_A)\,\delta_{\alpha'\alpha}. \tag{50}$$

2. THE πN SCATTERING MATRIX

In the theory of multiple scattering it is desirable, as we pointed out in Sec. 1, to express the πA scattering matrix in terms of the πN scattering matrix. In the present section, we consider some models of the πN interaction and questions associated with the use of the πN scattering matrix in the theory of multiple scattering of pions by nuclei.

In potential theory, the point of departure is the Lippmann–Schwinger equation for the t matrix, which we initially write down in the pion–nucleon center-of-mass system:

$$t^{cm}(\omega_{cm}) = v + v g_{cm}(\omega_{cm})\, t^{cm}(\omega_{cm}), \tag{51}$$

where

$$g_{cm}(\omega_{cm}) = [\omega_{cm} + i0 - h_\pi^{cm} - h_n^{cm}]^{-1} \equiv [\omega_{cm} + i0 - h_0^{cm}]^{-1}; \tag{52}$$

h_π^{cm} and h_n^{cm} are the Hamiltonian operators of the free pion and nucleon in the πN center-of-mass system; v in the general case is the nonlocal operator of the πN interaction, which depends on the spins and isospins of the particles. It can be represented in the form

$$\langle \mathbf{p}'|v|\mathbf{p}\rangle = 4\pi \sum_{ljIm_jm_I} v_{ljI}(p',\ p)\, \langle \hat{\mathbf{p}}'|ljIm_jm_I\rangle\, \langle ljIm_jm_I|\hat{\mathbf{p}}\rangle, \tag{53}$$

where $\langle \hat{\mathbf{p}}|ljIm_jm_I\rangle$ are the eigenfunctions of the orbital and total angular momenta and the isospin moment of the πN system and satisfy the well-known conditions of orthonormality and completeness:

$$\int \langle l'j'I'm_{j'}m_{I'}|\hat{\mathbf{p}}\rangle\, d\hat{\mathbf{p}}\, \langle \hat{\mathbf{p}}|ljIm_jm_I\rangle = \delta_{l'l}\delta_{j'j}\delta_{I'I}\delta_{m_{j'}m_j}\delta_{m_{I'}m_I}; \tag{54}$$

$$\sum_{ljIm_jm_I} \langle \hat{\mathbf{p}}'|ljIm_jm_I\rangle\, \langle ljIm_jm_I|\hat{\mathbf{p}}\rangle = \delta(\hat{\mathbf{p}} - \hat{\mathbf{p}}). \tag{55}$$

In (53), $\mathbf{p}$ is the relative momentum of the πN system. It is clear that the t matrix can be written in a form analogous to (53):

$$\langle \mathbf{p}'|t^{cm}(\omega_{cm})|\mathbf{p}\rangle = 4\pi \sum_{ljIm_jm_I} t_{ljI}^{cm}(p',\ p;\ \omega_{cm}) \times \langle \hat{\mathbf{p}}'|ljIm_jm_I\rangle\, \langle ljIm_jm_I|\hat{\mathbf{p}}\rangle. \tag{56}$$

On the basis of Eqs. (51)–(56), we can readily obtain an equation for the partial-wave t matrices:

$$t_{ljI}^{cm}(p',\ p;\ \omega_{cm}) = v_{ljI}(p',\ p) + 4\pi \int \frac{v_{ljI}(p',\ p'')\, t_{ljI}^{cm}(p'',\ p;\ \omega_{cm})\, p''^2\, dp''}{\omega_{cm} + i0 - \omega_{cm}(p'')}, \tag{57}$$

where

$$\left.\begin{aligned} &\omega_{cm}(p) = \omega_\pi(p) + \omega_n(p);\\ &\omega_\pi(p) = \sqrt{\mathbf{p}^2 + m^2};\\ &\omega_n(p) = \sqrt{\mathbf{p}^2 + M^2}; \end{aligned}\right\} \tag{58}$$

m and M are the pion and nucleon masses. The partial-wave amplitude of πN scattering is related to t_{ljI}^{cm} by the well-known relation

$$f_{ljI}(k) = -(2\pi)^2 \mu_{\pi N}(k)\, t_{ljI}^{cm}(k,\ k;\ \omega_{cm}(k)) = \exp[i\delta_{ljI}(k)] \sin\delta_{ljI}(k)/k, \tag{59}$$

where δ_{ljI} are the phase shifts;

$$\mu_{\pi N}(k) = \omega_\pi(k)\,\omega_n(k)/[\omega_\pi(k) + \omega_n(k)] \tag{60}$$

is the reduced "mass" of the πN system. For the partial-wave t matrices, there follows from Eq. (57) a unitarity condition off the energy shell:

$$t_{ljI}^{cm}(p',\ p;\ \omega_{cm}(k)) - t_{ljI}^{cm*}(p,\ p';\ \omega_{cm}(k)) = -\frac{i\mu_{\pi N}(k)\, k}{2}\, t_{ljI}^{cm}(p',\ k;\ \omega_{cm}(k))\, t_{ljI}^{cm*}(p,\ k;\ \omega_{cm}(k)). \tag{61}$$

One of the models of the πN interaction that is frequently used in the theory of the scattering of pions by nuclei is the separable model, which in the region of the (3, 3) resonance ($l=1$, $j=I=\frac{3}{2}$) has a certain theoretical justification.[15] In the framework of potential theory, this model presupposes that $v_\gamma(p', p)$ (below, to shorten the expressions, we take $\gamma \equiv ljI$) can be represented in the form

$$v_\gamma(p',\ p) = \lambda_\gamma v_\gamma(p')\, v_\gamma(p), \tag{62}$$

where λ_γ is a constant and $v_\gamma(p)$ is a form factor. Then, solving Eq. (57), we obtain

$$t_\gamma^{cm}(p',\ p;\ \omega_{cm}(k)) = v_\gamma(p')\, D_\gamma^{-1}(\omega_{cm}(k))\, v_\gamma(p), \tag{63}$$

where

$$D_\gamma(\omega_{cm}(k)) = \lambda_\gamma^{-1} - 4\pi \int \frac{v_\gamma^2(p)\, p^2\, dp}{\omega_{cm}(k) + i0 - \omega_{cm}(p)}. \tag{64}$$

The form factors $v_\gamma(p)$ in (62) and (63) are usually determined by two methods. In the first, the functions $v_\gamma(p)$ are specified in the simplest possible form compatible with the correct behavior of the πN scattering amplitude at low energies ($f_\gamma(k) \sim k^{2l}$). The parameters of the functions $v_\gamma(p)$ are determined in such a way as to obtain the experimentally measured phase shifts in a definite range of energies. In the second method, the inverse problem of πN scattering is solved using all information on the phase shifts in the energy range $(0, +\infty)$. Such a problem was solved, for example, in Refs. 16 and 17. In Ref. 16, the potential was taken in the form (63), where λ_γ is assumed to be independent of the energy. It was found that the form factors $v_\gamma(p)$ determined in this manner are, first, complex even at momenta below the corresponding threshold of inelastic processes of the πN collision and, second, have undesirable oscillations in a wide range of momenta. In Ref. 17, all inelastic channels in πN collisions were taken into account effectively through an energy-dependent optical potential of the elastic channel with a separable form:

$$\mathcal{V}_\gamma(p',\ p;\ \omega_{cm}(k)) = \lambda_\gamma(\omega_{cm}(k))\, \mathcal{V}_\gamma(p')\, \mathcal{V}_\gamma(p). \tag{65}$$

Here, $\lambda_\gamma[\omega_{cm}(k)]$ is a function that is real below the threshold of the inelastic processes (thereafter it is complex); the form factors of the elastic channels are real. The t matrix corresponding to the potential (65) can be obtained by solving an equation of the type (57), in which $v_\gamma(p', p)$ is replaced by the potential (65). One obtains

$$t_\gamma^{cm}(p',\ p;\ \omega_{cm}(k)) = \mathcal{V}_\gamma(p')\, D_\gamma^{-1}(\omega_{cm}(k))\, \mathcal{V}_\gamma(p), \tag{66}$$

where

$$D_\gamma^{-1}(\omega_{cm}(k)) = \lambda_\gamma^{-1}(\omega_{cm}(k)) - 4\pi \int \frac{\mathcal{V}_\gamma^2(p)\, p^2\, dp}{\omega_{cm}(k) + i0 - \omega_{cm}(p)} \tag{67}$$

$$= \lambda_\gamma^{-1} - 4\pi \int \frac{g_\gamma^2(p)\, p^2\, dp}{\omega_{cm}(k) + i0 - \omega_{cm}(p)}. \tag{68}$$

The functions $\mathscr{V}_\gamma(p)$ and $g_\gamma(p)$ were determined in Ref. 17 in terms of the phase shifts. Note that to determine $D_\gamma^{-1}[\omega_{cm}(k)]$ it is not necessary to know the function $g_\gamma(p)$, and it can be determined in accordance with the formula

$$D_\gamma^{-1}(\omega_{cm}(k)) = t_\gamma^{cm}(k,\ k;\ \omega_{cm}(k))/\mathscr{V}_\gamma^2(k), \tag{69}$$

which follows directly from (66). The relation requires knowledge of the t matrix on the energy shell at all values of the energy.

It is clear that these two methods of determining the t matrix lead to different behaviors off the energy shell. Therefore, if they are investigated when one is finding the T matrix of pion–nucleus scattering, one can in principle obtain information about the off-shell behavior of the πN scattering matrix. This is one of the aims of studying πA scattering.

For the state $\gamma \equiv ljI = 1\,\frac{3}{2}\,\frac{3}{2}$, the t matrix in the region of the resonance can be represented in the form

$$t^{cm}_{13/2\,3/2}(p',\ p;\ \omega_{cm}(k)) \equiv t^{cm}_{33}(p',\ p;\ \omega_{cm}(k)) = \frac{\lambda_{33} p' \tilde{v}_1(p')\,\tilde{v}_1(p)\,p}{\omega_{cm}(k) - \omega_R + i\Gamma(k)/2}, \tag{70}$$

where $\omega_R \equiv \omega_R(k_R)$ is the resonance energy, k_R is the corresponding momentum, $\Gamma(k)$ is the width of the resonance, and λ_{33} is a constant. It is possible to express $\Gamma(k)$ and λ_{33} in terms of a form factor, the values of the πN scattering amplitude, and the width of the resonance $\omega_{cm}(k) = \omega_R$ at the point of resonance by using the unitarity condition (61) and the optical theorem for the amplitude (59):

$$\lambda_{33} = \frac{\Gamma}{4}\,\frac{\sigma_{33}^{tot}(k_R)}{(2\pi)^3\,\mu_{\pi N}(\omega_R)\,k_R \tilde{v}_1^2(k_R)},\quad \Gamma \equiv \Gamma(k_R); \tag{71}$$

$$\Gamma(k) = \frac{\Gamma}{8}\,\frac{\mu_{\pi N}(\omega_{cm}(k))}{(2\pi)^3\,\mu_{\pi N}(\omega_R)}\,\frac{k\tilde{v}_1^2(k)}{k_R \tilde{v}_1^2(k_R)}\,k^2 \sigma_{33}^{tot}(k_R). \tag{72}$$

In the theory of multiple scattering of pions by nuclei in the region of the (3, 3) resonance, one also uses the Chew–Low model,[18] which describes the scattering of a p-wave pion by a fixed nucleon. The scattering matrix in this theory can be obtained from the expression (56) by substituting in it $l=1$, $jI \equiv \alpha = \frac{1}{2}\,\frac{1}{2};\ \frac{3}{2}\,\frac{1}{2};\ \frac{1}{2}\,\frac{3}{2};\ \frac{3}{2}\,\frac{3}{2}$ and

$$t^{cm}_{1\alpha}(p',\ p;\ \omega_{cm}(k)) = -\frac{p'v(p')}{\sqrt{\omega_\pi(p')}}\,\frac{h_\alpha(\omega_\pi(k))}{4\pi}\,\frac{pv(p)}{\sqrt{\omega_\pi(p)}}, \tag{73}$$

where $v(p)$ is the nucleon form factor; $h_\alpha(\omega_\pi)$ has the form

$$= \frac{h_\alpha(\omega_\pi)}{\lambda_\alpha/\omega_\pi} \Big/ \left(1 + \lambda_\alpha \omega_\pi \int \frac{p^3 v^2(p)\,d\omega_\pi(p)}{\omega_\pi^2(p)\,(\omega_\pi + i0 - \omega_\pi(p))} - \omega_\pi \int \frac{p^3 v^2(p)\,B_\alpha(\omega_\pi(p))\,d\omega_\pi(p)}{\omega_\pi^2(p)\,(\omega_\pi + \omega_\pi(p))}\right). \tag{74}$$

Here, $B_\alpha(\omega_\pi)$ is a function determined from the crossing-symmetry relation; λ_α is expressed in terms of the πN coupling constant f ($f^2 = 0.08$) in accordance with

$$\lambda_\alpha = \frac{2}{3}\left(\frac{f}{m}\right)^2 \begin{pmatrix} -4, & \alpha = {}^1/_2\,{}^1/_2 \\ -1, & \alpha = {}^3/_2\,{}^1/_2 \\ -1, & \alpha = {}^1/_2\,{}^3/_2 \\ 2, & \alpha = {}^3/_2\,{}^3/_2 \end{pmatrix}. \tag{75}$$

In the theory of multiple scattering of pions by nuclei, the corresponding T matrix can be expressed in terms of the t matrix of the πN collision in the pion–nucleus center-of-mass system. We therefore face the problem of relating this matrix to the matrix in the pion–nucleon center-of-mass system, i.e., t^{cm}. This question was discussed in a number of papers.[18-22] In our view, the natural and most convenient solution of this problem, corresponding to the common formulation of the problem of pion–nucleus scattering in which the interaction of the pion with the nucleus is treated phenomenologically, was given in Ref. 23 on the bases of relativistic potential theory. In the case of pion scattering by the deuteron, a different formulation is possible, which is more general and has a field-theoretical basis. It will be considered separately in the final section of this paper.

In relativistic potential theory, the interaction between the particles (in our case, between the pion and the nucleon) does not depend on the energy and in the general case is nonlocal and is introduced phenomenologically. Thus, in the πN center-of-mass system we have Eqs. (51) and (52), where $v = v(\mathbf{r}, \mathbf{p})$. Here, $\mathbf{r}$ and $\mathbf{p}$ are the relative radius vector and the momentum of the πN system. In an arbitrary system in which the pion and nucleon momenta are $\mathbf{p}_\pi$ and $\mathbf{p}_n$, respectively,

$$t(\omega) = \mathrm{v} + \mathrm{v} g(\omega)\, t(\omega), \tag{76}$$

where

$$g(\omega) = [\omega + i0 - h_0]^{-1} = [\omega + i0 - h_\pi - h_n]^{-1}; \tag{77}$$

$$h_0 = \sqrt{(h_0^{cm})^2 + \mathbf{P}^2};\quad \omega(p, P) = \sqrt{\omega_{cm}^2(p) + \mathbf{P}^2};\quad \mathbf{P} = \mathbf{p}_\pi + \mathbf{p}_n; \tag{78}$$

$$h^{cm} \equiv h_0^{cm} + v(\mathbf{r}, \mathbf{p});\ h = \sqrt{(h^{cm})^2 + \mathbf{P}^2},\quad \mathrm{v} = h - h_0. \tag{79}$$

It should be emphasized that in this (potential!) theory the particles are always considered on the mass shell.

The relative momentum in the expressions (51)–(69) and (76)–(79) is defined in accordance with

$$\mathbf{p} = \mathbf{p}_{\pi n} + \mathbf{P}\,\frac{\mathbf{p}_{\pi n}\mathbf{P}}{\omega_{cm}(p)\,[\omega_{cm}(p) + \omega(p, P)]} \equiv \mathbf{p}(\mathbf{p}_\pi, \mathbf{p}_n); \tag{80}$$

$$\mathbf{p}_{\pi n} = \frac{\omega_n(p_n)\,\mathbf{p}_\pi - \omega_\pi(p_\pi)\,\mathbf{p}_n}{\omega(p,\ P)}. \tag{81}$$

The expression (80) can be obtained by going over from the four–momenta defined in an arbitrary system,

$$P = (P_0,\ \mathbf{P}),\ P_0 = \omega_\pi(p_\pi) + \omega_n(p_n),\ \mathbf{P} = \mathbf{p}_\pi + \mathbf{p}_n;\quad p_{\pi n} = p_n^0 p_\pi - p_\pi^0 p_n / P_0 = (0,\ \mathbf{p}_{\pi n}), \tag{82}$$

to the four-momenta in the πN center-of-mass system ($P_{cm} = (\sqrt{P^2}, 0)$) by means of a Lorentz transformation:

$$\mathscr{L}_{i\mu}(P)\,(p_\pi + p_n)^\mu = 0,\quad \mu = 0, 1, 2, 3, \tag{83}$$

where

$$\mathscr{L}_{00}(P) = P_0/\sqrt{P^2};\ \mathscr{L}_{0i}(P) = -\mathscr{L}_{i0}(P) = P_i/\sqrt{P^2},\quad \mathscr{L}_{ij} = \delta_{ij} - P_i P_j/\sqrt{P^2}\,(P_0 + \sqrt{P^2}),\quad i, j = 1, 2, 3. \tag{84}$$

It is readily shown that the Jacobian of this transformation is

$$J(\mathbf{p}_\pi \mathbf{p}_n;\ \mathbf{pP}) = |D(\mathbf{p}_\pi,\ \mathbf{p}_n)/D(\mathbf{p},\ \mathbf{P})| = \frac{\omega_\pi(p_\pi)\,\omega_n(p_n)}{\omega_\pi(p)\,\omega_n(p)}\,\frac{\omega_\pi(p) + \omega_n(p)}{\omega_\pi(p_\pi) + \omega_n(p_n)}. \tag{85}$$

We introduce the state vectors $|\mathbf{p}_\pi \mathbf{p}_n\rangle$ and $|\mathbf{pP}\rangle$, which satisfy the following conditions of orthonormality and completeness:

$$\langle \mathbf{p}_\pi' \mathbf{p}_n' | \mathbf{p}_\pi \mathbf{p}_n \rangle = \delta(\mathbf{p}_n' - \mathbf{p}_n)\,\delta(\mathbf{p}_\pi' - \mathbf{p}_\pi); \tag{86}$$

$$\int |\mathbf{p}_\pi \mathbf{p}_n\rangle\, d\mathbf{p}_\pi\, d\mathbf{p}_n\, \langle \mathbf{p}_n \mathbf{p}_\pi| = 1; \tag{87}$$

$$\langle \mathbf{P}'\mathbf{p}' | \mathbf{pP} \rangle = \delta(\mathbf{P}' - \mathbf{P})\,\delta(\mathbf{p}' - \mathbf{p}); \tag{88}$$

$$\int |\mathbf{pP}\rangle\, d\mathbf{p}\, d\mathbf{P}\, \langle \mathbf{Pp}| = 1. \tag{89}$$

Using the equation $d\mathbf{p}_\pi d\mathbf{p}_n = J(\mathbf{p}_\pi \mathbf{p}_n; \mathbf{pP}) d\mathbf{p}\, d\mathbf{P}$, we can readily see that the matrix relating these two sets of state vectors $|\mathbf{p}_\pi \mathbf{p}_n\rangle$ and $|\mathbf{pP}\rangle$ has the form

$$\langle \mathbf{Pp}|\mathbf{p}_\pi \mathbf{p}_n\rangle = \delta(\mathbf{P}-\mathbf{p}_\pi-\mathbf{p}_n)\,\delta[\mathbf{p}-\mathbf{p}(\mathbf{p}_\pi, \mathbf{p}_n)]\, J^{-1/2}(\mathbf{p}_\pi \mathbf{p}_n; \mathbf{pP}). \tag{90}$$

From Eqs. (51) and (76) there follows directly the equation

$$\langle \mathbf{P'p'}|t(\omega)|\mathbf{pP}\rangle = \delta(\mathbf{P'}-\mathbf{P})\, t(\mathbf{p'}, \mathbf{p}; \omega, \mathbf{P}). \tag{91}$$

Using the well-known connection between the t matrix and the wave function and the circumstance that, because the potential $v(\mathbf{r}, \mathbf{p})$ is independent of the energy the center-of-mass variables and the variables of the relative motion separate in the wave function in the momentum representation, we can show[23] that

$$t(\mathbf{p'}, \mathbf{p}; \omega(p, P), \mathbf{P}) = F(p', p; P)\, t^{cm}(\mathbf{p'}, \mathbf{p}; \omega(p)), \tag{92}$$

where

$$F'(p', p; P) = [\omega_{cm}(p') + \omega_{cm}(p)]/[\omega(p', P) + \omega(p, P)]. \tag{93}$$

Equation (92) relates the half-off-shell t matrix in an arbitrary frame to the same t matrix in the center-of-mass system.

On the basis of the relations (86)–(91), we obtain

$$\langle \mathbf{p}'_\pi \mathbf{p}'_n|t(\omega)|\mathbf{p}_\pi \mathbf{p}_n\rangle = \delta(\mathbf{P'}-\mathbf{P})\, t(\mathbf{p}'_\pi \mathbf{p}'_n, \mathbf{p}_\pi \mathbf{p}_n; \omega), \tag{94}$$

where

$$t(\mathbf{p}'_\pi \mathbf{p}'_n, \mathbf{p}_\pi \mathbf{p}_n; \omega) = t(\mathbf{p'}, \mathbf{p}; \omega, \mathbf{P}) \times [J(\mathbf{p}'_\pi \mathbf{p}'_n; \mathbf{p'P})\, J(\mathbf{p}_\pi \mathbf{p}_n; \mathbf{pP})]^{-1/2}. \tag{95}$$

Using Eqs. (92) and (93), we obtain from (95)

$$\sqrt{\omega_\pi(p'_\pi)\,\omega_n(p'_n)}\, t(\mathbf{p}'_\pi \mathbf{p}'_n, \mathbf{p}_\pi \mathbf{p}_n; \omega_\pi(p_\pi)+\omega_n(p_n))\sqrt{\omega_\pi(p_\pi)\,\omega_n(p_n)} = \sqrt{\omega_\pi(p')\,\omega_n(p')}\, t^{cm}(\mathbf{p'}, \mathbf{p}; \omega(p))\sqrt{\omega_\pi(p)\,\omega_n(p)}\,\Phi(p', p; P), \tag{96}$$

where

$$\Phi(p', p; P) = \sqrt{\frac{\omega(p', P)}{\omega(p')}}\,\frac{\omega(p')+\omega(p)}{\omega(p', P)+\omega(p, P)}\sqrt{\frac{\omega(p, P)}{\omega(p)}}. \tag{97}$$

It is clear from the definition that $\Phi(p, p; P) = 1$, and then Eq. (96) reduces to the well-known[8] connection between the on-shell t matrix in an arbitrary frame and the same t matrix in the center-of-mass system:

$$\sqrt{\omega_\pi(p'_\pi)\,\omega_n(p'_n)}\, t(\mathbf{p}'_\pi \mathbf{p}'_n, \mathbf{p}_\pi \mathbf{p}_n; \omega_\pi(p_\pi)+\omega_n(p_n))\sqrt{\omega_\pi(p_\pi)\,\omega_n(p_n)} = \sqrt{\omega_\pi(p')\,\omega_n(p')}\, t^{cm}(\mathbf{p'}, \mathbf{p}; \omega(p))\sqrt{\omega_\pi(p)\,\omega_n(p)},$$
$$\omega_\pi(p_\pi)+\omega_n(p_n) = \omega(p). \tag{98}$$

In Ref. 19 and all subsequent investigations[20-22] into the problem of finding the connection between the t matrix of πN scattering in the pion–nucleus center-of-mass system and the t matrix in the pion–nucleon center-of-mass system it was assumed that Eq. (98) holds for the t matrices off the energy shell as well. As can be seen from the above, this assumption is not justified.

Note that if the nucleon is considered in the nonrelativistic limit (which means going to the limit $m/M \to 0$ in the calculation of the kinematic factors), the function $\Phi(p', p; P)$ will again be equal to unity:

$$\Phi(p', p; P) \xrightarrow[\frac{m}{M}\to 0]{} 1. \tag{99}$$

Bearing in mind that in the nonrelativistic limit the total energy $\omega_i(p_i)$ is

$$\omega_i(p_i) \approx m_i + \mathbf{p}_i^2/2m_i \equiv m_i + E_i(p_i), \tag{100}$$

from (96) in the nonrelativistic limit with respect to the nucleon we obtain

$$\sqrt{\omega_\pi(p'_\pi)}\, t(\mathbf{p}'_\pi \mathbf{p}'_n, \mathbf{p}_\pi \mathbf{p}_n; \omega_\pi(p_\pi)+M+E_n(p_n))\sqrt{\omega_\pi(p_\pi)} = \sqrt{\omega_\pi(p')}\, t^{cm}(\mathbf{p'}, \mathbf{p}; \omega_\pi(p)+M+E_n(p))\sqrt{\omega_\pi(p)}. \tag{101}$$

If we consider the pion in the nonrelativistic limit, we obtain

$$t(\mathbf{p}'_\pi \mathbf{p}'_n, \mathbf{p}_\pi \mathbf{p}_n; E(p, P)) = t^{cm}(\mathbf{p'}, \mathbf{p}; E_r(p)), \tag{102}$$

where

$$E(p, P) = \mathbf{P}^2/[2(m+M)] + \mathbf{p}^2/(2\mu_{\pi N}) \equiv E_c(P) + E_r(p), \tag{103}$$

which expresses Galilean invariance of the t matrix and follows directly from (51) and (76).

The connection between the off-shell πN collision matrices in an arbitrary frame and in the πN center-of-mass system can be readily found[23] from Eq. (76) if it is assumed that the πN system does not have bound states (so that the state vectors of the continuum form a complete system) and the relation (92) is used. Then

$$t(\mathbf{p'}, \mathbf{p}; \omega, P) = F(p', p; P)\, t^{cm}(\mathbf{p'}, \mathbf{p}; \omega(p)) + \int d\mathbf{p}'' F(p', p''; P)\, F(p'', p; P) \times t^{cm}(\mathbf{p'}, \mathbf{p}''; \omega(p''))\, t^{cm*}(\mathbf{p}, \mathbf{p}''; \omega(p'')) \times \left[\frac{F^{-1}(k, p''; P)}{\omega(k)+i0-\omega(p'')} - \frac{F^{-1}(p, p''; P)}{\omega(p)+i0-\omega(p'')}\right], \tag{104}$$

where k is determined from $\omega(k) = \sqrt{\omega^2 - \mathbf{P}^2}$. The expression (104) in conjunction with (94) and (95) determines the required connection between t and t^{cm}. Note that in Ref. 24 this connection was obtained on the basis of a formulation of relativistic potential theory which differs from the one assumed by Bakamjian and Thomas[2] and has the form

$$\langle \mathbf{p}'_\pi \mathbf{p}'_n|t(\omega)|\mathbf{p}_\pi \mathbf{p}_n\rangle = [J(\mathbf{p}'_\pi \mathbf{p}'_n; \mathbf{p'P})\, J(\mathbf{p}_\pi \mathbf{p}_n; \mathbf{pP})]^{-1/2} \times \frac{\delta(\mathbf{p'}/\omega(p') - \mathbf{P}/\omega(p))}{[\omega(p')\,\omega(p)]^{3/2}}\,\frac{\omega(p, P)}{\omega(p)}\, t^{cm}\left(\mathbf{p'}, \mathbf{p}; \frac{\omega(p)}{\omega(p, P)}\omega\right). \tag{105}$$

It should be emphasized that the on-shell matrix (105) coincides with the matrix determined by Eqs. (94), (95), and (104).

In Ref. 23, it was suggested that the expression (104) should be expanded in powers of $\mathbf{P}^2/(m+M)^2$. In our opinion, rather than doing this, it would be better to make an expansion in k/M (k is the momentum of the nucleon in the nucleus) where this ratio really is small, which, in its turn, is the main argument for using the nonrelativistic nuclear wave functions in the calculations.

3. FIXED-SCATTERER APPROXIMATION

We introduce state vectors $|\mathbf{k}_0\rangle$ for free motion of a pion with momentum $\mathbf{k}_0$ (for simplicity, we ignore the pion isospin), which are determined by the equations

$$\left.\begin{aligned} &h_\pi|\mathbf{k}_0\rangle = \omega_\pi(k_0)|\mathbf{k}_0\rangle, \\ &\int |\mathbf{k}_0\rangle\, d\mathbf{k}_0\, \langle \mathbf{k}_0| = 1, \\ &\langle \mathbf{k}'_0|\mathbf{k}_0\rangle = \delta(\mathbf{k}'_0 - \mathbf{k}_0). \end{aligned}\right\} \tag{106}$$

Then on the basis of (4) and (45)–(49) we obtain for the Green's operator

$$G(E)\mathcal{A}=\sum_{\alpha}\int\frac{|\mathbf{k}''\mathbf{K}_A''\alpha\rangle\, d\mathbf{k}_0''\, d\mathbf{K}_A''\,\langle\alpha\mathbf{K}_A''\mathbf{k}_0''|}{\omega_\pi(k_0)+E_A(K_A)+i0-\omega_\pi(k_0'')-E_A(K_A'')-\mathscr{E}_\alpha^*}. \quad (107)$$

In this expression, ignoring the excitation energy $\mathscr{E}_\alpha^*$ of the nucleus, and also the recoil energy of the nucleus compared with the kinetic energy of the incident pion, i.e., setting

$$E_A(k_A'')-E_A(K_A)+\mathscr{E}_\alpha^*\ll\omega_\pi(k_0)-m \quad (108)$$

and using the completeness conditions (49) and (106), we obtain

$$G(E)\mathcal{A}\approx G_\pi(\omega_\pi(k_0))\mathcal{A}, \quad (109)$$

where

$$G_\pi(\omega_\pi(k_0))=\int\frac{|\mathbf{k}_0''\rangle\, d\mathbf{k}_0''\,\langle\mathbf{k}_0''|}{\omega_\pi(k_0)+i0-\omega_\pi(k_0'')}=[\omega_\pi(k_0)+i0-h_\pi]^{-1} \quad (110)$$

is the Green's operator for free motion of the pion.

In the Hilbert space of the nucleus, we introduce the basis vectors $|\mathbf{r}_1\dots\mathbf{r}_A\rangle$ and $|\mathbf{k}_1\dots\mathbf{k}_A\rangle$ (for the **r** and **p** representations, respectively), which satisfy the completeness and orthonomality relations

$$\left.\begin{aligned}&\int|\mathbf{r}_1\dots\mathbf{r}_A\rangle\prod_{i=1}^{A}d\mathbf{r}_i\,\langle\mathbf{r}_A\dots\mathbf{r}_1|=1;\\&\langle\mathbf{r}_1'\dots\mathbf{r}_A'|\mathbf{r}_1\dots\mathbf{r}_A\rangle=\prod_{i=1}^{A}\delta(\mathbf{r}_i'-\mathbf{r}_i).\end{aligned}\right\} \quad (111)$$

Similar relations hold for $|\mathbf{k}_1\dots\mathbf{k}_A\rangle$. Then in the **r** representation on the basis of (109) and (110), we obtain

$$\begin{aligned}&\langle\mathbf{r}_A'\dots\mathbf{r}_1'\mathbf{r}_0'|G(E)\mathcal{A}|\mathbf{r}_0\mathbf{r}_1\dots\mathbf{r}_A\rangle\\&\approx\prod_{i=1}\delta(\mathbf{r}_i'-\mathbf{r}_i)\,\langle\mathbf{r}_0'|G_\pi(\omega_\pi(k_0))|\mathbf{r}_0\rangle\,\mathcal{A}.\end{aligned} \quad (112)$$

We assume that the operator $Av=\sum_i v_i$ in the **r** representation has the form

$$\begin{aligned}&\langle\mathbf{r}_A'\dots\mathbf{r}_1'\mathbf{r}_0'|\sum_i v_i|\mathbf{r}_0\mathbf{r}_1\dots\mathbf{r}_A\rangle\\&=\prod_i\delta(\mathbf{r}_i'-\mathbf{r}_i)\sum_i\langle\mathbf{r}_0'-\mathbf{r}_i|v_i|\mathbf{r}_0-\mathbf{r}_i\rangle.\end{aligned} \quad (113)$$

Then, substituting the expressions (112) and (113) in Eq. (5), we can readily verify that the T matrix satisfies the approximate equation

$$\begin{aligned}&\langle\mathbf{r}_A'\dots\mathbf{r}_1'\mathbf{r}_0'|T(E)|\mathbf{r}_0\mathbf{r}_1\dots\mathbf{r}_A\rangle\\&\approx\prod_i\delta(\mathbf{r}_i'-\mathbf{r}_i)\,\langle\mathbf{r}_0'|T^{\mathrm{fs}}(\omega_\pi(k_0);\,\mathbf{r}_1\dots\mathbf{r}_A)|\mathbf{r}_0\rangle,\end{aligned} \quad (114)$$

where T^{fs} is determined by the equation

$$\begin{aligned}&T^{\mathrm{fs}}(\omega_\pi(k_0);\mathbf{r}_1\dots\mathbf{r}_A)=\sum_i\langle\mathbf{r}_0'-\mathbf{r}_i|v_i|\mathbf{r}_0-\mathbf{r}_i\rangle\\&+\sum_i\int\langle\mathbf{r}_0'-\mathbf{r}_i|v_i|\mathbf{r}_0''-\mathbf{r}_i\rangle\, d\mathbf{r}_0''\,\langle\mathbf{r}_0''|G_\pi(\omega_\pi(k_0))\mathcal{A}|\mathbf{r}_0'''\rangle\, d\mathbf{r}_0'''\\&\times\langle\mathbf{r}_0'''|T^{\mathrm{fs}}(\omega_\pi(k_0));\,\mathbf{r}_1\dots\mathbf{r}_A|\mathbf{r}_0'''\rangle.\end{aligned} \quad (115)$$

It is clear that $T^{\mathrm{fs}}(\omega_\pi(k_0);\mathbf{r}_1\dots\mathbf{r}_A)$ is the matrix for scattering of a pion with momentum $\mathbf{k}_0$ on fixed scattering centers $\mathbf{r}_1\dots\mathbf{r}_A$.

Note that the expression (113) for the potential of the πA interaction is the limiting case $(m/M\to 0)$ of the Galilean-invariant expression

$$\begin{aligned}&\langle\mathbf{r}_A'\dots\mathbf{r}_1'\mathbf{r}_0'|\sum_i v_i|\mathbf{r}_0\mathbf{r}_1\dots\mathbf{r}_A\rangle=\sum_i\langle\mathbf{r}_0'-\mathbf{r}_i'|v_i|\mathbf{r}_0-\mathbf{r}_i\rangle\\&\times\delta\left[\frac{m\mathbf{r}_0'+M\mathbf{r}_i'}{m+M}-\frac{m\mathbf{r}_0+M\mathbf{r}_i}{m+M}\right]\prod_{j\neq i}\delta(\mathbf{r}_j'-\mathbf{r}_i).\end{aligned} \quad (116)$$

Thus, the potential (113) is not Galilean invariant but it is translationally invariant. Therefore, the t matrix is not Galilian invariant in the fixed-scatterer approximation (FSA). The circumstance that the Galilean-invariant two-particle potential (116) together with the approximation (109) does not in the general case lead to the FSA was noted in Ref. 25. The effects associated with the violation of Galilean invariance in the FSA were investigated in Ref. 26 for proton scattering by the deuteron at energies ~100 MeV. It was shown that these effects play an important part. With regard to pion scattering, the effects of the violation of Galilean invariance are, so far as we know, unknown, having not yet been investigated, though we can expect that the effects for pions will not be large because of the small ratio $m/M\approx\frac{1}{7}$.

For the fulfillment of the relation (108), i.e., for the validity of the FSA, it is necessary (but not sufficient) that the recoil energy of the target particles (nucleons) be much less than the energy of the incident particle (pion)[27]:

$$(\mathbf{k}_0'-\mathbf{k}_0)^2/(2M)\ll\omega_\pi(k_0)-m, \quad (117)$$

where $\mathbf{k}_0$ and $\mathbf{k}_0'$ are the pion momenta before and after scattering.

The matrix of elastic scattering of a pion by a nucleus in the FSA is determined on the basis of the relations (114) and (115):

$$\begin{aligned}&\langle 0\mathbf{K}_A'\mathbf{k}_0'|T^{\mathrm{fs}}(E)|\mathbf{k}_0\mathbf{K}_A 0\rangle=\int\langle 0\mathbf{K}_A'|\mathbf{r}_1\dots\mathbf{r}_A\rangle\\&\times\langle\mathbf{k}_0'|T^{\mathrm{fs}}(\omega_\pi(k_0);\,\mathbf{r}_1\dots\mathbf{r}_A)|\mathbf{k}_0\rangle\prod_i d\mathbf{r}_i\,\langle\mathbf{r}_1\dots\mathbf{r}_A|\mathbf{K}_A 0\rangle,\end{aligned} \quad (118)$$

where T^{fs} satisfies the equation

$$\begin{aligned}&\langle\mathbf{k}_0'|T^{\mathrm{fs}}(\omega_\pi;\,\mathbf{r}_1\dots\mathbf{r}_A)|\mathbf{k}_0\rangle=\sum_i\langle\mathbf{k}_0'|v_i(\mathbf{r}_i)|\mathbf{k}_0\rangle\\&+\sum_i\langle\mathbf{k}_0'|v_i(\mathbf{r}_i)|\mathbf{k}_0''\rangle\frac{d\mathbf{k}_0''a}{\omega_\pi+i0-\omega_\pi(k_0'')}\langle\mathbf{k}_0|T^{\mathrm{fs}}(\omega_\pi;\,\mathbf{r}_1\dots\mathbf{r}_A)|\mathbf{k}_0\rangle;\end{aligned} \quad (119)$$

$$\langle\mathbf{k}_0'|v_i(\mathbf{r}_i)|\mathbf{k}_0\rangle=\exp(-i\mathbf{k}_0'\mathbf{r}_i)\,\langle\mathbf{k}_0'|v_i|\mathbf{k}_0\rangle\exp(i\mathbf{k}_0\mathbf{r}_i). \quad (120)$$

Introducing the radius vector $\mathbf{R}_A$ of the center of mass of the nucleus and the radius vectors $\mathbf{r}_i^0$ of the nucleons relative to $\mathbf{R}_A$:

$$\begin{aligned}&\mathbf{r}_i=\mathbf{r}_i^0+\mathbf{R}_A;\quad \mathbf{R}_A=\sum_i\mathbf{r}_i/A;\\&\sum_i\mathbf{r}_i^0=0\end{aligned} \quad (121)$$

and using the obvious equation

$$\prod_i d\mathbf{r}_i=\delta\Big(\sum_i\mathbf{r}_i^0\Big)\, d\mathbf{R}_A\prod_i d\mathbf{r}_i^0, \quad (122)$$

we can obtain from (118)

$$\begin{aligned}&\langle 0\mathbf{K}_A'\mathbf{k}_0'|T^{\mathrm{fsa}}(E)|\mathbf{k}_0\mathbf{K}_A 0\rangle\\&=\delta(\mathbf{K}_A'+\mathbf{k}_0'-\mathbf{K}_A-\mathbf{k}_0)\,T_{00}^{\mathrm{fsa}}(\mathbf{k}_0',\mathbf{k}_0;\omega_\pi(k_0)),\end{aligned} \quad (123)$$

where

$$\begin{aligned}&T_{00}^{\mathrm{fsa}}(\mathbf{k}_0',\mathbf{k}_0;\omega_\pi(k_0))\\&=\int\rho_{00}(\mathbf{r}_1^0\dots\mathbf{r}_A^0)\,\delta\Big(\sum_i\mathbf{r}_i^0\Big)\\&\times\prod_i d\mathbf{r}_i^0\,\langle\mathbf{k}_0'|T^{\mathrm{fs}}(\omega_\pi(k_0);\,\mathbf{r}_1^0\dots\mathbf{r}_A^0)|\mathbf{k}_0\rangle;\end{aligned} \quad (124)$$

$$\rho_{00}(\mathbf{r}_1^0\dots\mathbf{r}_A^0)=\langle 0|\mathbf{r}_1^0\dots\mathbf{r}_A^0\rangle\langle\mathbf{r}_1^0\dots\mathbf{r}_A^0|0\rangle. \quad (125)$$

Here, $\langle\mathbf{r}_1^0\dots\mathbf{r}_A^0|0\rangle$ is the wave function of the internal

motion of the nucleus in the ground state, ρ_{00} is the corresponding density, and $T^{fs}(\omega_\pi; \mathbf{r}_1^0 \dots \mathbf{r}_A^0)$ satisfies the operator equation

$$T^{fs}(\omega_\pi; \mathbf{r}_1^0 \dots \mathbf{r}_A^0) = \sum_i v_i(\mathbf{r}_i^0) + \sum_i v_i(\mathbf{r}_i^0) G_\pi(\omega_\pi) \mathcal{A} T^{fs}(\omega_\pi; \mathbf{r}_1^0 \dots \mathbf{r}_A^0), \quad (126)$$

where $v_i(\mathbf{r}_i^0)$ is determined by the equation

$$\langle \mathbf{k}_0' | v_i(\mathbf{r}_i^0) | \mathbf{k}_0 \rangle = \exp(-i\mathbf{k}_0'\mathbf{r}_i^0) \langle \mathbf{k}_0' | v_i | \mathbf{k}_0 \rangle \exp(i\mathbf{k}_0\mathbf{r}_i^0). \quad (127)$$

We introduce the operator $\tau_i(\omega_\pi; \mathbf{r}_i^0)$, which satisfies the equation

$$\tau_i(\omega_\pi; \mathbf{r}_i^0) = v_i(\mathbf{r}_i^0) + v_i(\mathbf{r}_i^0) G_\pi(\omega_\pi) \mathcal{A} \tau_i(\omega_\pi; \mathbf{r}_i^0). \quad (128)$$

Equations (126) and (128) are analogous to Eqs. (2) and (38), respectively, and therefore by analogy with (41) we can immediately write

$$T^{fs}(\omega_\pi; \mathbf{r}_1^0 \dots \mathbf{r}_A^0) = \sum_i \tau_i(\omega_\pi; \mathbf{r}_i^0) + \sum_i \sum_{j \neq i} \tau_i(\omega_\pi; \mathbf{r}_i^0) G_\pi(\omega_\pi) \mathcal{A} \tau_j(\omega_\pi; \mathbf{r}_j^0) + \sum_i \sum_{j \neq i} \sum_{k \neq j} \tau_i(\omega_\pi; \mathbf{r}_i^0) G_\pi(\omega_\pi) \mathcal{A} \tau_j(\omega_\pi; \mathbf{r}_j^0) \times G_\pi(\omega_\pi) \mathcal{A} \tau_k(\omega_\pi; \mathbf{r}_k^0) + \dots. \quad (129)$$

It is easy to see that

$$\langle \mathbf{k}_0' | \tau_i(\omega_\pi; \mathbf{r}_i^0) | \mathbf{k}_0 \rangle = \exp(-i\mathbf{k}_0'\mathbf{r}_i^0) \langle \mathbf{k}_0' | t_i(\omega_\pi) | \mathbf{k}_0 \rangle \exp(-i\mathbf{k}_0\mathbf{r}_i^0), \quad (130)$$

where

$$\langle \mathbf{k}_0' | t_i(\omega_\pi) | \mathbf{k}_0 \rangle = \langle \mathbf{k}_0' | v_i | \mathbf{k}_0 \rangle + \int \frac{\langle \mathbf{k}_0' | v_i | \mathbf{k}_0'' \rangle d\mathbf{k}_0'' \langle \mathbf{k}_0'' | t_i(\omega_\pi) | \mathbf{k}_0 \rangle}{\omega_\pi + i0 - \omega_\pi(k_0'')}. \quad (131)$$

It is evident from the derivation of the expression (129) that it could be obtained directly from (41) and (36), using the completeness approximation (109). Without loss of generality, the operator A can be omitted in (126) and (127).

We consider the case when the potential of the πA interaction is local, i.e., when

$$\langle \mathbf{r}_A' \dots \mathbf{r}_1' \mathbf{r}_0' | \sum_i v_i | \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_A \rangle = \prod_i \delta(\mathbf{r}_i' - \mathbf{r}_i) \sum_i \delta(\mathbf{r}_0' - \mathbf{r}_0) v_i(\mathbf{r}_0 - \mathbf{r}_i). \quad (132)$$

This expression is not only translationally invariant but also Galilean invariant, since it is a special case of the expression (116) when $\langle \mathbf{r}_0' - \mathbf{r}_i | v_i | \mathbf{r}_0 - \mathbf{r}_i \rangle = \delta(\mathbf{r}_0' - \mathbf{r}_0) v_i(\mathbf{r}_0 - \mathbf{r}_i)$. Introducing the new variables

$$\mathbf{R} = (m\mathbf{r}_0 + MA\mathbf{R}_A)/(m + MA); \quad \mathbf{r} = \mathbf{r}_0 - \mathbf{R}_A \quad (133)$$

and taking into account the relation (121), we can represent the potential (132) in the form

$$\langle \mathbf{r}_A' \dots \mathbf{r}_1' \mathbf{r}_0' | \sum_i v_i | \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_A \rangle = \delta(\mathbf{R}' - \mathbf{R}) \delta(\mathbf{r}' - \mathbf{r}) \prod_i \delta(\mathbf{r}_i^{0\prime} - \mathbf{r}_i^0) \sum_i v_i(\mathbf{r} - \mathbf{r}_i^0). \quad (134)$$

Note that in the expression (116) $\mathbf{r}_0' \neq \mathbf{r}_0$ and therefore $\mathbf{R}' \neq \mathbf{R}$, as in the case in the expression (134). Because of this, the results obtained below on the basis of (134) will not be valid for the nonlocal potential (116).

If we use nonrelativistic kinematics, the expression (107) for the Green's operator can be reduced to the form

$$G(E)\mathcal{A} = \sum_\alpha \int \frac{|\mathbf{k}''\mathbf{K}''\rangle d\mathbf{k}'' d\mathbf{K}'' \langle \mathbf{K}''\mathbf{k}''|}{E(k) + i0 - E(k'') - \mathscr{E}_\alpha^*} |\alpha\rangle\langle\alpha|, \quad (135)$$

where

$$\mathbf{k} = (MA\mathbf{k}_0 - m\mathbf{K}_A)/(m + MA); \quad \mathbf{K} = \mathbf{k}_0 + \mathbf{K}_A; \quad (136)$$

$$E(k) = \mathbf{k}^2/(2M_{\pi A}); \quad \mu_{\pi A} = mMA/(m + MA). \quad (137)$$

If in the expression (135) we ignore the excitation energy $\mathscr{E}_\alpha^*$ of the internal states of the nucleus compared with the energy of the relative motion of the pion and the nucleus, i.e., we set

$$\mathscr{E}_\alpha^* \ll E(k), \quad (138)$$

then, using the completeness conditions (49), we obtain from (135)

$$G(E)\mathcal{A} \approx G_\pi(E(k))\mathcal{A}, \quad (139)$$

where

$$G_\pi(E(k)) = \int \frac{|\mathbf{k}''\rangle d\mathbf{k}'' \langle \mathbf{k}''|}{E(k) + i0 - E(k'')} = [E(k) + i0 - \mathscr{H}_r]^{-1}. \quad (140)$$

Here, $\mathscr{H}_r$ is the operator of the kinetic energy of the relative motion of the pion and the nucleus. Therefore, in this case we have

$$\langle \mathbf{r}_A' \dots \mathbf{r}_1' \mathbf{r}_0' | G(E)\mathcal{A} | \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_A \rangle \approx \delta(\mathbf{R}' - \mathbf{R}) \prod_i \delta(\mathbf{r}_i^{0\prime} - \mathbf{r}_i^0) \langle \mathbf{r}' | G_\pi(E(k)) | \mathbf{r} \rangle \mathcal{A}. \quad (141)$$

In this approximation, using the expression (140), we readily see that the solution of Eq. (5) is

$$\langle \mathbf{r}_A' \dots \mathbf{r}_1' \mathbf{r}_0' | T(E) | \mathbf{r}_0 \mathbf{r}_1 \dots \mathbf{r}_A \rangle \approx \delta(\mathbf{R}' - \mathbf{R}) \prod_i \delta(\mathbf{r}_i^{0\prime} - \mathbf{r}_i^0) \langle \mathbf{r}' | T^{fs}(E(k); \mathbf{r}_1^0 \dots \mathbf{r}_A^0) | \mathbf{r} \rangle, \quad (142)$$

where $T^{fs}(E(k); \mathbf{r}_1^0 \dots \mathbf{r}_A)$ is the matrix for scattering of a pion with momentum $\mathbf{k}$ by fixed scatterers at $\mathbf{r}_1^0 \dots \mathbf{r}_A^0$, for which the relations (123)–(125) and (129), in which $\omega_\pi(k_0)$ must be replaced by $E(k)$, hold.

Thus, if the interaction of the incident pion with the nucleus is local, the reduction of the problem of πA scattering to the problem of scattering by fixed scatterers does not impose restrictions on the recoil energy of the nucleus, i.e., it is sufficient to require that the excitation energy of the internal state of the nucleus in the Green's function be negligible. This does not lead to a violation of Galilean invariance, in contrast to the case of a nonlocal interaction.

There exist three cases when the problem of the scattering of a pion (or an arbitrary particle) by a nucleus can be solved in a closed form in the FSA: 1) when the eikonal approximation is used for the πN collision matrix; 2) when the πN interaction is separable; and 3) when the potentials of the interaction of the pion with the different nucleons of the nucleus do not overlap. The first case corresponds to the Glauber[28] theory of multiple scattering (in this connection, see also the review of Ref. 29). The derivation of the expression for the πA scattering matrix in Glauber's theory from the Watson series (41) in the framework of the (local) potential model of the interaction was considered in a number of papers.[30-33] In Refs. 30 and 31, this question was investigated for the example of the deuteron, and in Refs. 32 and 33 in the general case for any nucleus. An elegant, in our opinion, derivation of the Glauber amplitude is given in Ref. 33, in which an important result is also proved. To obtain the Glauber result from summation of the series (129), it is necessary to take into account as well the terms corresponding to πN backward scattering [for example, the terms

with $k=j$ in the third term of the series (129)]. Thus, one can show the incorrectness of the assertion that Glauber theory in the elementary event takes into account only forward scattering.

The solution of the scattering problem in the FSA for a separable potential was considered in Ref. 27, and for nonoverlapping potentials in Refs. 34 and 35. These solutions can be directly obtained by summing the series (129) for the T matrix in the FSA. For a separable potential, we use the expressions (53) and (56). Omitting the spin–isospin dependence, we find

$$\langle \mathbf{k}' \mid v \mid \mathbf{k}\rangle = 4\pi \sum_{lm_l} \lambda_l v^*_{lm_l}(\mathbf{k}')\, v_{lm_l}(\mathbf{k}); \tag{143}$$

$$\langle \mathbf{k}' \mid t(\omega_\pi(k_0)) \mid \mathbf{k}\rangle = 4\pi \sum_{lm_l} v^*_{lm_l}(\mathbf{k}')\, D_l^{-1}(\omega_\pi(k_0))\, v_{lm_l}(\mathbf{k}), \tag{144}$$

where

$$v_{lm_l}(\mathbf{k}) = v_l(k)\langle \hat{\mathbf{k}} \mid lm_l\rangle; \quad \hat{\mathbf{k}} = \mathbf{k}/k; \quad D_l(\omega_\pi(k_0)) = \lambda_l^{-1} \tag{145}$$

$$-4\pi \int \frac{v^*_{lm_l}(\mathbf{k})\, v_{lm_l}(\mathbf{k})\, d\mathbf{k}}{\omega_\pi(k_0) + i0 - \omega_\pi(k)} = -\frac{v_l^2(k_0)\,\mu_{\pi N}(2\pi)^2}{4\pi A_l(k_0)}. \tag{146}$$

We introduce the notation

$$\left.\begin{aligned} &e_i v_{l'm_{l'}} \equiv \exp(i\mathbf{k}'\mathbf{r}_i^0)\, v_{l'm_{l'}}(\mathbf{k}');\\ &v_{lm_l} e_j \equiv v_{lm_l}(\mathbf{k}) \exp(i\mathbf{k}\mathbf{r}_j^0);\\ &V_{lm_l} e_j \equiv D_l^{-1}(\omega_\pi)\, v_{lm_l} e_j. \end{aligned}\right\} \tag{147}$$

Then on the basis of (130), we obtain

$$\langle \mathbf{k}' \mid \tau_i(\omega_\pi; \mathbf{r}_i^0) \mid \mathbf{k}\rangle = \sum_{l'm_{l'}} \sum_{lm_l} (e_i v_{l'm_{l'}})^*\, \delta_{l'l}\delta_{m_{l'}m_l} D_l^{-1}(\omega_\pi)\, (v_{lm_l} e_i). \tag{148}$$

Further, for the pion Green's function (110) we have

$$\langle \mathbf{k}' \mid G_\pi(\omega_\pi) \mid \mathbf{k}\rangle = \delta(\mathbf{k}'-\mathbf{k})/[\omega_\pi + i0 - \omega_\pi(k)] \equiv \delta(\mathbf{k}'-\mathbf{k})\, G_\pi(\omega_\pi; k). \tag{149}$$

We introduce

$$\left.\begin{aligned} &\mathcal{G}^{ij}_{l'm_{l'};\, lm_l} = -D_l^{-1}(\omega_\pi(k_0))\, G^{ij}_{l'm_{l'};\, lm_l}(\omega_\pi(k_0))\, \mathcal{A};\\ &\mathcal{G}^{i=j}_{l'm_{l'};\, lm_l} = 0, \end{aligned}\right\} \tag{150}$$

where

$$G^{ij}_{l'm_{l'};\, lm_l}(\omega_\pi) = \int (e_i v_{l'm_{l'}})\, G_\pi(\omega_\pi; k)\, (e_j v_{lm_l})^*\, d\mathbf{k}. \tag{151}$$

In the special case when $i=j$,

$$G^{i=j}_{l'm_{l'};\, lm_l}(\omega_\pi) = \delta_{l'l}\delta_{m_{l'}m_l}\, 4\pi \int v_l^2(k)\, k^2\, dk/[\omega_\pi + i0 - \omega_\pi(k)]. \tag{152}$$

These quantities form a $(2l'+1)A \times (2l+1)A$ matrix $\mathcal{G}$. In this notation, the series (129) can be represented in the form

$$\begin{aligned} &\langle \mathbf{k}' \mid T^{\mathrm{fs}}(\omega_\pi; \mathbf{r}_1^0 \ldots \mathbf{r}_A^0) \mid \mathbf{k}\rangle\\ &= \sum_{il'm_{l'}lm_l} (e_i' v_{l'm_{l'}})^*\, \delta_{l'l}\delta_{m_{l'}m_l}\, (e_i v_{lm_l})\\ &+ \sum_{il'm_{l'}jlm_l} (e_i' v_{l'm_{l'}})^*\, \mathcal{G}^{ij}_{l'm_{l'};\, lm_l} e_j V_{lm_l}\\ &+ \sum_{il'm_{l'}jl''m_{l''}k} (e_i' v_{l'm_{l'}})^*\, \mathcal{G}^{ij}_{l'm_{l'};\, l''m_{l''}}\, \mathcal{G}^{jk}_{l''m_{l''};\, lm_l} + \ldots. \end{aligned} \tag{153}$$

We represent the quantities $e_i v_{lm_l}(e_i V_{lm_l})$ in the form of a column, whose elements we label by the indices $i=1,2,\ldots,A$; $-l \leqslant m_l \leqslant l$, $l=1,2,\ldots$ We denote these sets by $(\underline{ev})$ and $(\underline{Ve})$, respectively. Then the set of quantities $(e_i v_{lm_l})$ can be represented in the form of the row $(\underline{ev})^+$. In the new notation, the series (153) takes the form[27]

$$\langle \mathbf{k}' \mid T^{\mathrm{fs}}(\omega_\pi; \mathbf{r}_1^0 \ldots \mathbf{r}_A^0) \mid \mathbf{k}\rangle = (\underline{e'v'})^+ [1 - \mathcal{G} + \mathcal{G}\mathcal{G} - \ldots]\, (\underline{Ve}); \tag{154}$$

$$\langle \mathbf{k}' \mid T^{\mathrm{fs}}(\omega_\pi; \mathbf{r}_1^0 \ldots \mathbf{r}_A^0) \mid \mathbf{k}\rangle = (\underline{e'v'})^+ [1 + \mathcal{G}]^{-1}\, (\underline{Ve}). \tag{155}$$

Thus, the problem of finding the matrix for scattering of a particle (in our case a pion) by a nucleus in the FSA in the model of a separable interaction reduces to the construction of the matrix $\mathcal{G}$ and the inversion of the matrix $1+\mathcal{G}$. It can be seen from the definition that the matrix $\mathcal{G}$ depends on the coordinates $\mathbf{r}_1^0 \ldots \mathbf{r}_A^0$ of the nucleons of the nucleus as parameters, with respect to which the expression (155) should be averaged over the wave function of the nucleus in accordance with (124). The solution of the problem in this form is extremely difficult even for the simplest nucleus—the deuteron. With increasing mass number A of the nucleus, the dimension of the matrix $\mathcal{G}$ rapidly increases, and with it the difficulties of solving the problem. This is one of the main reasons why (155) has not, so far as we know, been used by anyone for nuclei with $A \geqslant 3$. Naturally, one must ask whether the iterative expansion (154) could not be used to concrete calculations. This question was investigated in Ref. 36 for the example of pion scattering by the deuteron in the region of the (3, 3) resonance, and it was shown that for physically reasonable values of the parameters (in particular, for the radius of the repulsive core of the *NN* interaction) the series (154) does not converge, although the expression (155) exists.

We now consider the solution of the scattering problem in the FSA for nonoverlapping potentials. We assume that the potential v_i is nonlocal, with limited range (d_i), and with center at the point $\mathbf{r}_i$ $(i=1,2,\ldots,A)$. It is clear that the two potentials v_i and v_j will not overlap if

$$\langle \mathbf{r}_0' - \mathbf{r}_i \mid v_i \mid \mathbf{r}_0 - \mathbf{r}_i\rangle \neq 0$$

for

$$\left.\begin{aligned} &|\mathbf{r}_0 - \mathbf{r}_i| < d_i; \quad |\mathbf{r}_0' - \mathbf{r}_i| < d_i;\\ &r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = |\mathbf{r}_i^0 - \mathbf{r}_j^0| \geqslant d_i + d_j. \end{aligned}\right\} \tag{156}$$

We show that in this case the series (129) can be summed in a closed form. For this, we write the pion Green's function (110) in the coordinate representation:

$$\langle \mathbf{r}_0' \mid G_\pi(\omega_\pi(k_0)) \mid \mathbf{r}_0\rangle = -\frac{2\omega_\pi(k_0)}{4\pi} \frac{\exp[ik_0 \mid \mathbf{r}_0' - \mathbf{r}_0 \mid]}{\mid \mathbf{r}_0' - \mathbf{r}_0 \mid}. \tag{157}$$

We introduce the new variables

$$\boldsymbol{\rho}_i' = \mathbf{r}_0 - \mathbf{r}_i^0; \quad \boldsymbol{\rho}_j = \mathbf{r}_0 - \mathbf{r}_j^0, \tag{158}$$

which in accordance with the definition of the potential (156) satisfy the conditions

$$\rho_i' \leqslant d_i; \quad \rho_j \leqslant d_j; \quad |\boldsymbol{\rho}_i' - \boldsymbol{\rho}_j| < r_{ij}. \tag{159}$$

Then the expression (157) can be represented in the form

$$\begin{aligned} \langle \mathbf{r}_0' \mid G_\pi(\omega_\pi(k_0)) \mid \mathbf{r}_0\rangle &= -\frac{2\omega_\pi(k_0)}{4\pi} \frac{\exp[ik_0 \mid (\boldsymbol{\rho}_i' - \boldsymbol{\rho}_j) - \mathbf{r}_{ij} \mid]}{\mid (\boldsymbol{\rho}_i' - \boldsymbol{\rho}_j) - \mathbf{r}_{ij} \mid}\\ &= -2\omega_\pi(k_0) \sum_{l_0 m_{l_0}} k_0 j_{l_0}(k_0 \mid \boldsymbol{\rho}_i' - \boldsymbol{\rho}_j \mid)\, h_{l_0}^{(+)}(k_0 r_{ij})\\ &\times \langle \widehat{\boldsymbol{\rho}_i' - \boldsymbol{\rho}_j} \mid l_0 m_{l_0}\rangle \langle l_0 m_{l_0} \mid \hat{\mathbf{r}}_{ij}\rangle, \end{aligned} \tag{160}$$

where $j_l(x)$ and $h_l^{(+)}(x)$ are known spherical Bessel and Hankel functions, respectively. Determining

$$j_{l_0}(k_0 \mid \boldsymbol{\rho}_i' - \boldsymbol{\rho}_j \mid) \langle \boldsymbol{\rho}_i' - \boldsymbol{\rho}_j \mid l_0 m_{l_0}\rangle$$

from the plane-wave expansion

$$\sum_{l_0 m_{l_0}} i^{l_0} \sqrt{\frac{2}{\pi}}\, j_{l_0}(k_0 |\rho_i' - \rho_j|)\, \widehat{\langle \rho_i' - \rho_j |}\, l_0 m_{l_0}\rangle \langle l_0 m l_0 | \hat{\mathbf{k}}_0\rangle$$
$$= \langle \rho_i' - \rho_j | \mathbf{k}_0\rangle = (2\pi)^{3/2} \langle \rho_i' | \mathbf{k}_0\rangle \langle \mathbf{k}_0 | \rho_j\rangle$$
$$= \sum_{l' m_{l'}} i^{l'} \sqrt{\frac{2}{\pi}}\, j_{l'}(\rho_i' k_0)\, \langle \hat{\rho}_i' | l' m_{l'}\rangle \langle l' m_{l'} | \hat{\mathbf{k}}_0\rangle$$
$$\times \sum_{l m_l} i^l \sqrt{\frac{2}{\pi}}\, j_l(\rho_j k_0)\, \langle \hat{\mathbf{k}}_0 | l m_l\rangle \langle l m_l | \hat{\rho}_j\rangle, \tag{161}$$

we can reduce the expression (160) to

$$\langle \mathbf{r}_0' | G_\pi(\omega_\pi(k)) | \mathbf{r}_0\rangle \equiv \langle \rho_i' | G_\pi(\omega_\pi(k_0); \mathbf{r}_{ij}) | \rho_j\rangle$$
$$= -\frac{\omega_\pi(k_0) k_0}{2\pi} \sum_{l' m_{l'} l m_l} i^{l'-l} j_{l'}(k_0 \rho_i')\, j_l(k_0 \rho_j)$$
$$\times \langle \hat{\rho}_i' | l' m_{l'}\rangle \langle l m_l | \hat{\rho}_j\rangle\, F^{ij}_{l' m_{l'};\, l m_l}(k_0; \mathbf{r}_{ij}), \tag{162}$$

where

$$F^{ij}_{l' m_{l'};\, l m_l}(k_0; \mathbf{r}_{ij}) = \sum_{l_0 m_{l_0}} \sqrt{4\pi(2l'+1)(2l+1)}$$
$$\times \begin{pmatrix} l' & l & l_0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l' & l & l_0 \\ -m_{l'} & m_l & -m_{l_0} \end{pmatrix} (-)^{m_{l'}+m_{l_0}}$$
$$\times \langle \hat{\mathbf{r}}_{ij} | l_0 m_{l_0}\rangle\, h^{(+)}_{l_0}(k_0 r_{ij}). \tag{163}$$

On the basis of (130), (144), and (145),

$$\langle \mathbf{k}_0' | \tau_i(\omega_\pi(k_0); \mathbf{r}_i^0) | \mathbf{k}_0\rangle = \exp(-i\mathbf{k}_0' \mathbf{r}_i^0)$$
$$\times [4\pi \sum_{l m_l} \langle k_0' | t_l^i(\omega_\pi(k_0)) | k_0\rangle \langle \hat{\mathbf{k}}_0' | l m_l\rangle \langle l m_l | \hat{\mathbf{k}}_0\rangle] \exp(i\mathbf{k}_0 \mathbf{r}_i^0), \tag{164}$$

where

$$\langle k_0' | t_l^i(\omega_\pi(k_0)) | k_0\rangle = v_l(k_0') v_l(k_0)/D_l(\omega_\pi(k_0)). \tag{165}$$

Using (162) and (164), we can readily show that

$$\langle \mathbf{k}_0' | \tau_i(\omega_\pi(k_0); \mathbf{r}_i^0)\, G_\pi(\omega_\pi(k_0))\, \mathscr{A} \tau_i(\omega_\pi(k_0); \mathbf{r}_i^0) | \mathbf{k}_0\rangle$$
$$= \exp(-i\mathbf{k}_0' \mathbf{r}_i^0) [4\pi \sum_{l' l m_{l'} m_l} \langle k_0' | t_{l'}^i(\omega_\pi(k_0)) | k_0\rangle$$
$$\times k_0 F^{ij}_{l' m_{l'};\, l m_l}(k_0; \mathbf{r}_{ij})\, \mathscr{A} \langle k_0 | f_l^j(\omega_\pi(k_0)) | k_0\rangle \langle \hat{\mathbf{k}}_0' | l' m_{l'}\rangle$$
$$\times \langle l' m_{l'} | \hat{\mathbf{k}}_0\rangle \exp(i\mathbf{k}_0 \mathbf{r}_j^0), \tag{166}$$

where

$$f_l(\omega) = -(2\pi)^2 \omega t_l(\omega). \tag{167}$$

Similarly,

$$\langle \mathbf{k}_0' | \tau_i(\omega_\pi(k_0); \mathbf{r}_i^0)\, G_\pi(\omega_\pi(k_0))$$
$$\times \mathscr{A} \tau_i(\omega_\pi(k_0); \mathbf{r}_j^0)\, G_\pi(\omega_\pi(k_0))\, \mathscr{A} \tau_k(\omega_\pi(k_0); \mathbf{r}_k^0) | \mathbf{k}_0\rangle$$
$$= \exp(-i\mathbf{k}_0' \mathbf{r}_i^0) [4\pi \sum_{l' l'' m_{l'} m_{l''} m_l} \langle k_0' | t_{l''}^i(\omega_\pi(k_0)) | k_0\rangle$$
$$\times k_0 F^{ij}_{l' m_{l'};\, l'' m_{l''}}(k_0; \mathbf{r}_i)\, \mathscr{A} \langle k_0 | f_{l''}^j(\omega_\pi(k_0)) | k_0\rangle$$
$$\times k_0 F^{jk}_{l'' m_{l''};\, l m_l}(k_0, \mathbf{r}_{jk}^0)\, \mathscr{A} \langle k_0 | f_l^k(\omega_\pi(k_0)) | k_0\rangle$$
$$\times \langle \hat{\mathbf{k}}_0' | l' m_{l'}\rangle \langle l m_l | \hat{\mathbf{k}}_0\rangle \exp(i\mathbf{k}_0 \mathbf{r}_k^0). \tag{168}$$

From (164), (166), and (168) we can see the structure of an arbitrary term of the series (129). It is important to emphasize that for $k_0' = k_0$ each term of this series depends on the t matrix of πN scattering on the energy shell. This is a consequence of the use of the expressions (162) and (163) for the Green's functions, which, in their turn, are related to the choice of the potential (156). The assertion that for nonoverlapping potentials of the interaction of the incident particle with the different nucleons of the nucleus the amplitude of elastic scattering of the particle by the nucleus depends on the t matrix of the scattering of the particle by a nucleon only on the energy shell was first formulated in Ref. 37 and called Bég's theorem. This theorem is also proved in Refs. 27, 34, 35, and 38.

We introduce the matrix F, whose elements $F^{ij}_{l' m_l;\, l m_l'}(k_0; \mathbf{r}_{ij})$ are labeled by the indices $i = 1, 2, \ldots, A$; $-l \leq m_l \leq l = 1, 2, \ldots$. At the same time, we require that $F^{ii} = 0$. In this representation, it is clear that

$$f_l^j(\omega_\pi(k_0)) \equiv \langle k_0 | f_l^j(\omega_\pi(k_0)) | k_0\rangle = \exp[i\delta_l^j(k_0)] \sin\delta_l^j(k_0)/k_0 \tag{169}$$

are the elements of a diagonal matrix f. If we now introduce a column $C(\mathbf{k}_0)$ with elements $\langle l m_l | \hat{\mathbf{k}}_0\rangle \times \exp(i\mathbf{k}_0 \mathbf{r}_i^0)$, then on the basis of Eqs. (164), (166), (168), and (129) we can write[34]

$$\langle \mathbf{k}_0' | T^{fs}(\omega_\pi(k_0); \mathbf{r}_1^0 \ldots \mathbf{r}_A^0) | \mathbf{k}\rangle \tag{170}$$
$$= \begin{cases} C^+(\mathbf{k}_0') f [1 + k_0 F \mathscr{A} f + k_0 F \mathscr{A} f k_0 F \mathscr{A} f + \ldots] C(\mathbf{k}_0); \\ C^+(\mathbf{k}_0') f [1 - k_0 F \mathscr{A} f]^{-1} C(\mathbf{k}_0). \end{cases} \tag{171}$$

Thus, the problem of finding the matrix for scattering of a pion by a nucleus in the FSA for nonoverlapping potentials has been reduced to the construction of the matrix F, whose dimension depends on the number of partial waves which determine the πN interaction and on the mass number of the nucleus. Further, it is necessary to invert the matrix $[1 - k_0 F \mathscr{A} f]$, which depends on the nucleon coordinates $\mathbf{r}_i^0$ as on parameters, and it is then necessary to average (171) over the wave function of the ground state of the nucleus in accordance with (124). It is clear that the difficulties associated with the practical realization of the scheme for calculating the πA scattering matrix on the basis of (171) are of the same order as the difficulties associated with the use of (155). Note that in the derivation of (170) and (171), fewer restrictions were imposed on the πN interaction potential than in the derivation of (154) and (155). The first expression of (170) and (171) was obtained without particularization of the form of the potential and a restriction was imposed only on the range $r_{\pi N}$ of the πN interaction, namely, $2r_{\pi N}$ must be much less than the mean distance r_0 between the nucleons in the nucleus. Bearing in mind that $r_{\pi N} \approx 0.5$ F and $r_0 \approx 1.2$ F, we may conclude that good quantitative results are not to be expected from the use of (170) and (171). This is also confirmed by the calculations of Ref. 38 made for pion scattering on the ^{4}He nucleus in the region of the (3, 3) resonance, for which the first two terms in the expansion (170) were taken into account.

So far as we know, the convergence of the iterative series (170) has not been investigated. However, in our opinion the same result for (170) is to be expected as for the series (154), since the convergence of the series (129) should not essentially depend on the model in which the matrix $\tau_i(\omega_\pi)$ is calculated. The poor convergence of the series (129) is evidently due to the approximations in which it is derived.

4. FIRST-ORDER OPTICAL POTENTIAL

Experimental and theoretical investigations of pion scattering by nuclei are at present mainly made in the region of energies of the (3, 3) resonance. The great interest in this region is due to the hope of obtaining information about the behavior of the Δ resonance in nuclear matter and, in particular, the hope of elucidating the part played by the Δ resonance in the determination of various properties of the nucleus, i.e., in determining the extent to which the dynamics of the nucleus is

determined by the Δ resonance, which is one of the manifestations of the meson degrees of freedom of the nucleus.

In the FSA the production and propagation of the Δ resonance in the nucleus is precluded for pion scattering by nuclei in the region of the (3, 3) resonance. This is readily seen from the expression (130) for the matrix for collision of a pion with a nucleon in the nucleus, which corresponds to scattering of a pion by a nucleon fixed at the point $\mathbf{r}_i^0$. This aspect of the use of the FSA was pointed out in Ref. 41. A simple estimate of the mean free path of the Δ resonance shows that it is of order 1 F and comparable with the mean internucleon distance in the nucleus. It follows that the production and propagation of the Δ resonance in the nucleus must be taken into account when one is considering the scattering of pions by nuclei in the region of the (3, 3) resonance. And, since these effects are not taken into account in the FSA, it is necessary to go beyond the framework of this approximation. That the use of the FSA in the region of the (3, 3) resonance is suspect can also be seen from the fact that in the expression (130) the t matrix of free πN scattering does not depend on the energy and momentum of the nucleon in the nucleus. Such an approximation is hardly justified if one bears in mind the strong energy dependence of the πN collision matrix in the region of the (3, 3) resonance.

We consider various approximations that go beyond the framework of the FSA to greater or lesser extents. In the general case, the solution of this problem reduces to solving the equation

$$T(E)=T^{\mathrm{fs}}(E_0)+T^{\mathrm{fs}}(E_0)[G(E)-G_0(E_0)]\mathscr{A}T(E), \tag{172}$$

which is obtained from the exact equation for the T matrix (2) and the equation for T^{fs} with the Green's function $G_0(E_0)$ (110) if the potential is nonlocal or with $G_0(E_0)$ (140) if the potential is local. One of the variants of (172) was considered in Ref. 39 for π^3He(^{3}H) scattering.

Coherent Approximation. As we have seen in the preceding section, the FSA can be obtained if in the Watson series (41) and in the Green's function $G(E)\mathscr{A}$ (107) the excitation energies of the nucleus in all states are ignored but the states themselves are taken into account through the completeness condition (49). Instead of this, in $G(E)\mathscr{A}$ we shall now ignore all excited (internal) states of the nucleus with the corresponding energies, retaining only the ground state, i.e., we set

$$G(E)\mathscr{A}\approx G_0(E_0)\mathscr{P}=\int\frac{|\mathbf{k}_0''\mathbf{K}_A''\rangle\, d\mathbf{k}_0''\, d\mathbf{K}_A''\,\langle\mathbf{K}_A''\mathbf{k}_0''|}{\omega_\pi(k_0)+E_A(K_A)+i0-\omega_\pi(k_0'')-E_A(K_A'')}\,|0\rangle\langle 0|. \tag{173}$$

We then obtain

$$T(E)\approx\sum_i\tau_i(E)+\sum_i\sum_{j\neq i}\tau_i(E)G_0(E_0)\mathscr{P}\tau_j(E)+\sum_i\sum_{j\neq i}\sum_{k\neq j}\tau_i(E)G_0(E_0)\mathscr{P}\tau_j(E)G_0(E_0)\mathscr{P}\tau_k(E)+\dots. \tag{174}$$

It can be seen from this expression that the considered approximation reduces to neglecting the excitation of the nucleus in all intermediate scatterings of the pion by different nucleons of the nucleus (hence the epithet *coherent* for the approximation). At the same time, the excitation of the nucleus is assumed to be completely included in the operator $\tau_i(E)$ in (36) and (37) through the Green's function $G(E)$ (107).

From the relation (174) for the matrix elements with respect to the ground state, we obtain

$$\begin{aligned}T_{00}(E)&\approx A\tau_{00}(E)+A\tau_{00}(E)G_0(E_0)(A-1)\tau_{00}(E)\\&+A\tau_{00}(E)G_0(E_0)(A-1)\tau_{00}(E)G_0(E_0)(A-1)\tau_{00}(E)+\dots\\&=A\tau_{00}(E)+A\tau_{00}(E)G_0(E_0)\left[\frac{A-1}{A}T_{00}(E)\right],\end{aligned} \tag{175}$$

where

$$T_{00}(E)\equiv\langle 0|T(E)|0\rangle; \tag{176}$$

$$\tau_{00}(E)\equiv\langle 0|\tau(E)|0\rangle=\langle 0|\tau_i(E)|0\rangle. \tag{177}$$

From Eq. (175), we have

$$T'^{\mathrm{coh}}_{00}(E)=U^{\mathrm{coh}}_{00}(E)+U^{\mathrm{coh}}_{00}(E)G_0(E_0)T'^{\mathrm{coh}}_{00}(E), \tag{178}$$

where

$$T'^{\mathrm{coh}}_{00}(E)=\frac{A-1}{A}T^{\mathrm{coh}}_{00}(E);\quad U^{\mathrm{coh}}_{00}(E)=(A-1)\tau_{00}(E). \tag{179}$$

Note that (178) could be obtained directly from (19) and (20) if in (21) for the optical potential the term taking in to account the excited states of the nucleus is ignored. The optical potential obtained in such an approximation is called the *first-order optical potential* in the KMT formulation, i.e.,

$$U'^{(1)}(E)=U^{\mathrm{coh}}(E)=(A-1)\tau(E)=[(A-1)/A]\sum_i\tau_i(E). \tag{180}$$

Note that this last equation holds only in the space of antisymmetric states of the nucleus.

We now show that in the first (in the optical potential) approximation Watson's formulation of the theory of multiple scattering is identical with the KMT formulation. For this we note that from (24)

$$\hat{U}^{(1)}(E)=A\hat{\tau}(E)=\sum_i\hat{\tau}_i(E), \tag{181}$$

where $\hat{\tau}_i(E)$ satisfies the equation

$$\hat{\tau}_i(E)=\begin{cases}v_i+v_iG(E)Q\hat{\tau}_i(E); & (182)\\ v_i+\hat{\tau}_i(E)QG(E)v_i. & (183)\end{cases}$$

On the basis of Eq. (181), we obtain from (22)

$$T^{(1)}(E)=\sum_i\hat{\tau}_i(E)+\sum_i\hat{\tau}_i(E)G(E)\mathscr{P}T^{(1)}(E). \tag{184}$$

It is clear that between the operators τ_i and $\hat{\tau}_i$ there is a connection analogous to (25) and (26), i.e.,

$$\tau_i(E)=\begin{cases}\hat{\tau}_i(E)+\hat{\tau}_i(E)G(E)\mathscr{P}\tau_i(E); & (185)\\ \hat{\tau}_i(E)+\tau_i(E)\mathscr{P}G(E)\hat{\tau}_i(E). & (186)\end{cases}$$

Note that (184) and (185), (186) can be obtained from (5), (7), and (36), (37), respectively, by the replacement of v_i by $\hat{\tau}_i$ and $G\mathscr{A}$ by $G\mathscr{P}$, and therefore for $T^{(1)}(E)$ we can write down directly a series analogous to the series (41) obtained from (5), (7) and (36), (37):

$$T^{(1)}(E)=\sum_i\tau_i(E)+\sum_i\sum_{j\neq i}\tau_i(E)G(E)\mathscr{P}\tau_j(E)+\sum_i\sum_{j\neq i}\sum_{k\neq j}\tau_i(E)G(E)\mathscr{P}\tau_j(E)\mathscr{P}\tau_k(E)+\dots. \tag{187}$$

From this there follows directly the expression (175) and, therefore, Eq. (178). Thus, we have shown that in the first approximation in the optical potential, i.e., when the second terms on the right-hand sides are ig-

noted in the equations for the optical potential (21) and (24), the matrix for pion scattering by a nucleus is the same in both the Watson and KMT formulations of the theory of multiple scattering.[40] It is clear that a difference between these formulations appears when τ_i and $\hat{\tau}_i$ are taken in some approximation instead of the exact solutions of Eqs. (36), (37) and (182), (183). Therefore, the difference between the matrices $T^{(1)}(E)$ obtained in the two different formulations will to some degree determine the accuracy of the given approximation.

Impulse Approximation and the Role of Fermi Motion. One of the frequently used approximations for $\tau_i(E)$ or $\hat{\tau}_i(E)$ is the impulse approximation, which amounts to the replacement of τ_i or $\hat{\tau}_i$ by the free πN matrix $t_i(\omega)$ (76). Using the well-known operator algebra, we can readily show that

$$\tau_i(E) = t_i(\omega) + t_i(\omega)\,[G(E)\,\mathcal{A} - g(\omega)]\,\tau_i(E); \tag{188}$$

$$\hat{\tau}_i(E) = t_i(\omega) + t_i(\omega)\,[G(E)\,Q - g(\omega)]\,\hat{\tau}_i(E). \tag{189}$$

Here, ω is a parameter that is usually chosen in one of two forms:

$$\omega = \omega_\pi(k_0) + \omega_n(p_i), \tag{190}$$

where $\mathbf{p}_i$ is the momentum of nucleon i in the nucleus, and

$$\omega = E + M = \omega_\pi(k_0) + E_A(K_A) + M. \tag{191}$$

The variant (190), which has a transparent physical meaning,[8] leads in the nonrelativistic limit to a Galilean-invariant matrix $t_i(\omega)$, and to the variant (191) there corresponds a Galilean non-invariant matrix $t_i(\omega)$. The choice of ω in the form (190) has the further advantage over the choice of ω in the form (191) that in the first case the optical potential depends on the half-off-shell $t_i(\omega)$ matrix, as will be seen below. Therefore, the connection between the $t_i(\omega)$ matrices in the pion–nucleus and pion–nucleon center-of-mass systems will be determined by the simple formula (92). When ω is chosen in the form (191), the optical potential depends on the off-shell $t_i(\omega)$ matrix, and the connection is given by the more complicated (104). But the variant (190) has the shortcoming that in this case the connection between the matrices τ_i and $\hat{\tau}_i$ and the matrix $t_i(\omega)$ is not given by a simple relationship of the form (188) and (189). It is therefore hard to find the corrections to the impulse approximation. In the variant (191), the correction entails allowance of the second term on the right-hand sides of Eqs. (188) and (189).

To determine the first-order optical potential in the impulse approximation, it is necessary to find the matrix element of the operator $t_i(\omega)$. In the normalization of the basis state vectors adopted in the previous sections,

$$\langle \mathbf{k}_0'\mathbf{K}_A'\alpha' \mid t_i(\omega) \mid \alpha\mathbf{K}_A\mathbf{k}_0\rangle = \int \langle \mathbf{K}_A'\alpha' \mid \mathbf{p}_1 \ldots \mathbf{p}_i' \ldots \mathbf{p}_A\rangle\, d\mathbf{p}_i' \,\langle \mathbf{p}_i'\mathbf{k}_0' \mid t_i(\omega) \mid \mathbf{k}_0\mathbf{p}_i\rangle \times d\mathbf{p}_i \prod_{j\neq i} d\mathbf{p}_j \langle \mathbf{p}_1 \ldots \mathbf{p}_i \ldots \mathbf{p}_A \mid \alpha\mathbf{K}_A\rangle, \tag{192}$$

where the matrix $\langle \mathbf{p}_i'\mathbf{k}_0' \mid t_i(\omega) \mid \mathbf{k}_0\mathbf{p}_i\rangle$ in the general case (when relativistic kinematics is used for the pion and the nucleon) is determined by Eqs. (94), (95), and (104).

But if we take into account the fact that the wave functions of nuclei with $A \geq 3$ are usually considered in the nonrelativistic approximation, it would be more consistent to use for $\langle \mathbf{p}_i'\mathbf{k}_0' \mid t_i(\omega) \mid \mathbf{k}_0\mathbf{p}_i\rangle$ the expression obtained from (94), (95), and (104) in the nonrelativistic limit with respect to the nucleon. In this case, on the basis of Eqs. (80), (81), (85)–(95), (100), and (104), we obtain

$$\langle \mathbf{p}_i'\mathbf{k}_0' \mid t_i(\omega) \mid \mathbf{k}_0\mathbf{p}_i\rangle = \delta(\mathbf{K}_{0i}' - \mathbf{K}_{0i})\, N(\mathbf{f}'\mathbf{k}_0';\ \mathbf{f}\mathbf{k}_0)\, t(\mathbf{f}',\ \mathbf{f};\ \omega,\ \mathbf{K}_{0i}), \tag{193}$$

where

$$N(\mathbf{f}'\mathbf{k}_0';\ \mathbf{k}\mathbf{k}_0) = \sqrt{\frac{\omega_\pi(\mathbf{f}')\,\omega_\pi(\mathbf{f})\,(M+\omega_\pi(k_0'))\,(M+\omega_\pi(k_0))}{\omega_\pi(k_0')\,\omega_\pi(k_0)\,(M+\omega_\pi(\mathbf{f}'))\,(M+\omega_\pi(\mathbf{f}))}}; \tag{194}$$

$$\mathbf{f} = \frac{M\mathbf{k}_0 - \omega_\pi(k_0)\,\mathbf{p}_i}{M+\omega_\pi(k_0)} \equiv b\mathbf{k}_0 - a\mathbf{p}_i; \quad a+b=1; \quad \mathbf{K}_{0i} = \mathbf{k}_0 + \mathbf{p}_i; \tag{195}$$

$$t(\mathbf{f}',\ \mathbf{f};\ \omega,\ \mathbf{K}_{0i}) = t^{cm}(\mathbf{f}',\ \mathbf{f};\ \omega(\mathbf{f})) + \int d\mathbf{k}''\, t^{cm}(\mathbf{f}',\ \mathbf{f}'',\ \omega(\mathbf{f}'')) \times t^{cm*}(\mathbf{f},\mathbf{f}'';\ \omega(\mathbf{f}'')) \left[\frac{1}{\sqrt{\omega^2 - \mathbf{K}_{0i}^2} + i0 - \omega(\mathbf{f}'')} - \frac{1}{\omega(\mathbf{f}) + i0 - \omega(\mathbf{f}'')}\right]. \tag{196}$$

Introducing new variables in accordance with

$$\left.\begin{aligned} &\mathbf{k} = \frac{MA\mathbf{k}_0 - \omega_\pi(\mathbf{k}_0)\,\mathbf{K}_A}{MA + \omega_\pi(k_0)} \equiv b_A\mathbf{k}_0 \\ &- a_A\mathbf{K}_A, \quad a_A - b_A = 1, \quad \mathbf{K} = \mathbf{k}_0 + \mathbf{K}_A; \\ &\mathbf{p}_i = \mathbf{p}_i^0 + \mathbf{K}_A/A, \end{aligned}\right\} \tag{197}$$

we obtain on the basis of (180), (181), and (192)

$$\langle \mathbf{k}_0'\mathbf{K}_A'0 \mid \begin{bmatrix} U_{\text{imp}}'^{(1)}(E) \\ \hat{U}_{\text{imp}}^{(1)}(E) \end{bmatrix} \mid 0\mathbf{K}_A\mathbf{k}_0\rangle = \delta(\mathbf{K}' - \mathbf{K})\, V_{00}^{(1)}(\mathbf{k}',\ \mathbf{k};\ \omega,\ \mathbf{K}), \tag{198}$$

where

$$V_{\alpha'\alpha}^{(1)}(\mathbf{k},\ \mathbf{k};\ \omega,\ \mathbf{K}) = \begin{bmatrix} A-1 \\ A \end{bmatrix} N(\mathbf{f}'\mathbf{k}_0';\ \mathbf{f}\mathbf{k}_0) \times \int \rho_{\alpha'\alpha}\left(\mathbf{p}^0 + \frac{A-1}{A}(\mathbf{k}-\mathbf{k}'),\ \mathbf{p}^0\right) d\mathbf{p}^0 \times t(\mathbf{f}',\ \mathbf{f};\ \omega,\ \mathbf{K}_{0i}) \equiv \begin{bmatrix} A-1 \\ A \end{bmatrix} N t_{\alpha'\alpha}(\mathbf{f}',\ \mathbf{f};\ \omega,\ \mathbf{K}). \tag{199}$$

Here, the factor $(A-1)$ corresponds to the KMT formulation and the factor A to Watson's formulation; the vectors $\mathbf{f}$ and $\mathbf{K}_{0i}$ are defined in accordance with

$$\mathbf{f} = \frac{MA + \omega_\pi(k_0)}{A\,(M + \omega_\pi(k_0))}\,\mathbf{k} - \frac{\omega_\pi(k_0)}{M + \omega_\pi(k_0)}\,\mathbf{p}_i^0; \tag{200}$$

$$\mathbf{K}_{0i} = \frac{A-1}{A}\,\mathbf{k} + \frac{M + \omega_\pi(k_0)}{MA + \omega_\pi(k_0)}\,\mathbf{K}. \tag{201}$$

The function $\rho_{\alpha'\alpha}(\mathbf{p}', \mathbf{p})$ in the expression (199) is determined by

$$\rho_{\alpha'\alpha}(\mathbf{p}',\ \mathbf{p}) = \int \langle \mathbf{r}' \mid \mathbf{p}'\rangle \langle \alpha' \mid \mathbf{x}_{A-2} \ldots \mathbf{x}_1\mathbf{r}'\rangle \times d\mathbf{r}'\, d\mathbf{x}_1 \ldots d\mathbf{x}_{A-2}\, d\mathbf{r}\, \langle \mathbf{r}\mathbf{x}_1 \ldots \mathbf{x}_{A-2} \mid \alpha\rangle \langle \mathbf{p} \mid \mathbf{r}\rangle, \tag{202}$$

where $\mathbf{r} = \mathbf{r}_i - \mathbf{R}_{A-1}$ and $\mathbf{x}_1 \ldots \mathbf{x}_{A-2}$ are Jacobi coordinates for the internal motion of the nucleus. Using the single-particle fractional-parentage expansion of the internal state vectors of the nucleus,

$$|\alpha\rangle = \sum_{\alpha_b\alpha_r} C_{\alpha_b\alpha_r}^{\alpha} |\alpha_b\rangle\, |\alpha_r\rangle, \tag{203}$$

we obtain from (202) (schematically)

$$\rho_{\alpha'\alpha}(\mathbf{p}',\ \mathbf{p}) = \sum_{\alpha_b'\alpha_b} a_{\alpha_b'\alpha_b}^{\alpha'\alpha} \rho_{\alpha_b'\alpha_b}(\mathbf{p}',\ \mathbf{p}), \tag{204}$$

where

$$\rho_{\alpha_b'\alpha_b}(\mathbf{p}',\ \mathbf{p}) = \langle \alpha_b' \mid \mathbf{p}'\rangle \langle \mathbf{p} \mid \alpha_b\rangle; \tag{205}$$

$$a_{\alpha_b'\alpha_b}^{\alpha'\alpha} = \sum_{\alpha_r'\alpha_r} C_{\alpha_b'\alpha_r'}^{\alpha'} C_{\alpha_b\alpha_r}^{\alpha} \langle \alpha_r' \mid \alpha_r\rangle. \tag{206}$$

In the expressions (203) and (205), $|\alpha_b\rangle$ are vectors of single-particle states in the nucleus.

In the new variables $(\mathbf{f}, \mathbf{K}_{0i})$ and $(\mathbf{k}, \mathbf{K})$, the expressions (190) and (191) take the form

$$\omega = \sqrt{\omega^2(\mathbf{f}) + \mathbf{K}_{0i}^2}; \quad \omega(\mathbf{f}) = \omega_\pi(\mathbf{f}) + \omega_n(\mathbf{f}) \approx \omega_\pi(\mathbf{f}) + M; \tag{207}$$

$$\omega = \sqrt{\omega_{\pi A}^2(k) + \mathbf{K}^2} M_A + M; \omega_{\pi A}(k) = \omega_\pi(k) + \omega_A(k) \approx \omega_\pi(k) + M. \tag{208}$$

It can be seen that when ω is chosen in the form (190) the integral term in the expression (104) becomes equal to zero. In addition, the matrix $t(\mathbf{f}', \mathbf{f}; \omega, \mathbf{K}_{0i})$ ceases to depend on $\mathbf{K}_{0i}$ and, therefore, on $\mathbf{K}$. As a result, for the optical potential we obtain a comparatively simple expression. This is one of the important advantages of choosing ω in the form (190) compared with the choice of ω in the form (191).

If in the integrand of (199) we ignore the dependence of the t matrix on the momentum of a nucleon in the nucleus relative to the center-of-mass $\mathbf{p}^0$ [this dependence arises from (200)], then, integrating $\rho_{00}[\mathbf{p}^0 + [(A-1)/A](\mathbf{k}-\mathbf{k}')]$ over $\mathbf{p}_0$, we obtain

$$V_{00}^{(1)}(\mathbf{k}', \mathbf{k}; \omega, \mathbf{K}) \approx \begin{bmatrix} A-1 \\ A \end{bmatrix} N(\mathbf{f}_0' \ \mathbf{k}_0'; \ \mathbf{f}_0 \mathbf{k}_0) \times t(\mathbf{f}_0', \mathbf{f}_0; \omega, \mathbf{K}_{0i}) \rho_{00}\left[\frac{A-1}{A}(\mathbf{k}-\mathbf{k}')\right], \tag{209}$$

where

$$\rho_{\alpha'\alpha}\left(\frac{A-1}{A}\mathbf{q}\right) = \int \exp\left(i\frac{A-1}{A}\mathbf{q}\mathbf{r}\right) \langle\alpha'|\mathbf{x}_{A-2} \ldots \mathbf{x}_1\mathbf{r}\rangle \times d\mathbf{x}_1 \ldots d\mathbf{x}_{A-2}\, d\mathbf{r} \langle \mathbf{r}\mathbf{x}_1 \ldots \mathbf{x}_{A-2}|\alpha\rangle; \tag{210}$$

$$\mathbf{f}_0 = \frac{MA + \omega_\pi(k_0)}{A(M + \omega_\pi(k_0))}\mathbf{k}. \tag{211}$$

In the limit of an infinitely heavy nucleon $(m/M \to 0)$ and, therefore, an infinitely heavy nucleus $[(A-1)/A \to 1]$, we obtain from (209), using (194), (195), (197), (201), (207), (208), (210), and (211), $\mathbf{f} \approx k_0 \approx k$, $\mathbf{K}_{0i} \approx \mathbf{k}$ and

$$V_{00}^{(1)}(\mathbf{k}', \mathbf{k}; \omega, \mathbf{K}) \approx (A-1)\, t(\mathbf{k}_0', \mathbf{k}_0; \omega_\pi(k_0))\, \rho_{00}(\mathbf{k}_0 - \mathbf{k}_0'). \tag{212}$$

This expression determines the first-order optical potential in the FSA. Indeed, as we have seen, the solutions of Eqs. (36) and (37) in the FSA have the form (124), and the corresponding series (129) for the T matrix reduces to Eq. (178), where U_{00}^{coh} is determined from (198) and (212) if in (129) the projection operator $\mathscr{A}$ is replaced by the operator $\mathscr{P} = |0\rangle\langle 0|$, i.e., in the intermediate states all excited states of the nucleus are ignored. Note that to improve the approximation of "freezing" the nucleons in the nucleus, which reduces the optical potential (199) to the factorized form (209), the nucleon momentum in the nucleus $\mathbf{p}_i = -\mathbf{k}/A + \mathbf{p}^0$ (in the pion–nucleus center-of-mass system) in the t matrix is sometimes taken to be equal not to $-\mathbf{k}/A$, but rather to some mean value $\mathbf{p}_0$, and the t matrix is again taken in front of the integral. In Ref. 42, it was suggested that $\mathbf{p}^0$ should be chosen in the form $\mathbf{p}_0 = -\mathbf{k}/A + [(A-1)/2A](\mathbf{k}' - \mathbf{k})$, which arises naturally[43] if the wave function of the nucleus has a Gaussian form (the main component of the wave function of a nucleus with $A = 3$ or 4 in the oscillator model).

Thus, in contrast to the FSA the first-order optical potential (198) takes into account the motions of the nucleons in the nucleus both kinematically and dynamically (Fermi motion), this being achieved by averaging of the free πN collision matrix over the momenta of the nucleon in the nucleus. The problem of allowing for the Fermi motion of the nucleons in the nucleus in πA scattering in the region of the (3, 3) resonance was considered in Refs. 41 and 44–46. In Refs. 41 and 46, this question was investigated in the approximation in which the intermediate states are taken into account in the framework of the model of nuclear matter, while in Refs. 44 and 45 this question was considered from the state in the shell model with an oscillator potential. In both cases, it was shown that allowance for the Fermi motion has a strong influence on the cross section in the region of the (3, 3) resonance as well as on the energy dependence of this cross section.

Let us now consider the applicability of the impulse approximation. When the energy parameter ω is chosen in the form (191), the condition of applicability of the impulse approximation can be obtained by estimating the second term on the right-hand sides of Eqs. (188) and (189). So far as I know, such an estimate has not hitherto been made. With regard to the choice (190), in this case, as we have noted above, Eqs. (188) and (189) do not hold. A condition of applicability of the impulse approximation for this case was obtained in Ref. 8 by estimating the second term of a certain expansion for $\tau_i(E)$ with respect to the t matrix for collision of the incident particle and the nucleon. In the presence of a resonance in the scattering (in our case, πN scattering) the condition of applicability of the impulse approximation takes the form[8]

$$\frac{f_{\pi N}}{\lambda\!\!\!^{-}} \frac{2\varepsilon_N}{\Gamma} \ll 1, \tag{213}$$

where $f_{\pi N}$ is the πN scattering amplitude, $\lambda\!\!\!^{-} = 1/k_0$ is the de Broglie wavelength of the pion, Γ is the width of the resonance and ε_N is the energy of stripping of a nucleon from the nucleus. The estimate shows[8] that in the region of the (3, 3) resonance the condition (213) is not satisfied, since the parameter $2f_{\pi N}\varepsilon_N/(\lambda\!\!\!^{-}\Gamma)$ is of order unity. The condition (213), which is sufficient but not necessary, is too stringent. Investigations into the scattering of a pion by the deuteron made in the framework of the three-particle equations showed[47-50] that the iteration series for the T matrix converges fairly well, which is due to mutual partial cancellation of the higher terms (beginning with the second) of the series.

For scattering of a pion by a nucleus more complicated than the deuteron, a numerical estimate of the accuracy of the impulse approximation can, in our opinion, be obtained by using the theorem proved above on the equivalence of Watson's formulation and the KMT formulation for the T matrix in the first approximation in the optical potential. In this case, the difference between these formulations arises because of the impulse approximation for τ_i and $\hat{\tau}_i$. The difference can be seen in the equation for the T_{00} matrix:

$$T_{00}(E) = At_{00}(\omega) + \begin{bmatrix} A-1 \\ A \end{bmatrix} t_{00}(\omega) G_0(E) T_{00}(E), \tag{214}$$

which is obtained from Eqs. (19), (20) and (22), (23) by

using (180), (192), and (199). In (214), the factor $(A-1)$ corresponds to the KMT formulation, and the factor A to Watson's formulation.

Allowance for Binding, Recoil, and the Pauli Principle for Nucleons in the Nucleus. We now consider the possibility of improving the impulse approximation. Such an improvement must always take into account the fact that the nucleon with which the incident pion collides is in the environment of the remaining nucleons of the nucleus. The excitation of the nucleus resulting from the collision of the pion with the distinguished nucleon, which is not taken into account in the FSA, is an important aspect of pion–nucleus dynamics, and this must be taken into account in the first place.

To investigate this question, one can proceed from Eqs. (36) and (37) for the matrix of pion collision with a nucleon of the nucleus. Using the expression (107) for the Green's function and the Galilian invariance of the πN interaction potential v_i, we readily obtain, using (197),

$$\langle \alpha' \mathbf{K}'_A \mathbf{k}'_0 | v_i | \mathbf{k}_0 \mathbf{K}_A \alpha \rangle = \delta(\mathbf{K}' - \mathbf{K}) [\langle \mathbf{k}' | v_{\alpha'\alpha} | \mathbf{k} \rangle = \langle \alpha' \mathbf{k}' | v | \mathbf{k} \alpha \rangle]; \quad (215)$$

$$\langle \alpha' \mathbf{K}'_A \mathbf{k}'_0 | \tau_i(E) | \mathbf{k}_0 \mathbf{K}_A \alpha \rangle = \delta(\mathbf{K}' - \mathbf{K}) [\langle \mathbf{k}' | \tau_{\alpha'\alpha}(E) | \mathbf{k} \rangle = \langle \alpha' \mathbf{k}' | \tau_i(E) | \mathbf{k} \alpha \rangle], \quad (216)$$

where

$$\langle \mathbf{k}' | \tau_{\alpha'\alpha}(E) | \mathbf{k} \rangle = \langle \mathbf{k}' | v_{\alpha'\alpha} | \mathbf{k} \rangle + \sum_{\alpha''} \int \frac{\langle \mathbf{k}' | v_{\alpha'\alpha''} | \mathbf{k}'' \rangle d\mathbf{k}'' \langle \mathbf{k}'' | \tau_{\alpha''\alpha}(E) | \mathbf{k} \rangle}{\omega_\pi(|\mathbf{k} + a_A \mathbf{K}|) + E_A(|-\mathbf{k} + b_A \mathbf{K}|) + i0 - \omega_\pi(|\mathbf{k}'' + a_A \mathbf{K}|) - E_A(|-\mathbf{k}'' + b_A \mathbf{K}|) - \mathscr{E}^*_{\alpha''}}; \quad (217)$$

$$\langle \mathbf{k}' | v_{\alpha'\alpha} | \mathbf{k} \rangle = \int \rho_{\alpha'\alpha} \left[\mathbf{p}^0 + \frac{A-1}{A} (\mathbf{k} - \mathbf{k}'); \ \mathbf{p}^0 \right] \times d\mathbf{p}^0 \left\langle \mathbf{k}' - \mathbf{k} + \frac{b}{b_A} \mathbf{k} - a\mathbf{p}^0 | v | \frac{b}{b_A} \mathbf{k} - a\mathbf{p}^0 \right\rangle. \quad (218)$$

Note that the denominator of the integrand in (217) for relativistic kinematics depends on the total momentum of the system and, therefore, $\langle \mathbf{k}' | \tau_{\alpha'\alpha}(E) | \mathbf{k} \rangle$ also depends on the total momentum. In the nonrelativistic limit, the dependence disappears, since

$$\omega_\pi(|\mathbf{k} + a_A \mathbf{K}|) + E_A(|-\mathbf{k} + b_A \mathbf{K}|) - \omega_\pi(|\mathbf{k}'' + a_A \mathbf{K}|) - E_A(|-\mathbf{k}'' + b_A \mathbf{K}''|) \approx E_{\pi A}(E) - E_{\pi A}(k''); \ E_{\pi A}(k) = \mathbf{k}^2/(2\mu_{\pi A}). \quad (219)$$

Usually, Eq. (217) is considered in the πA center-of-mass system. The question arises of the connection between the τ matrix in this system and the τ matrix in an arbitrary system, in which Eq. (217) was written down; we have already considered this question in Sec. 2.

Equation (217), written down in the πA center-of-mass system, was used in Ref. 51 to estimate the correction of the impulse approximation resulting from allowance for the binding of the nucleon in the nucleus and the excitation of the nucleus. In Ref. 51, the shell model with an oscillator potential was used. It was assumed that in the region of the (3, 3) resonance the πA interaction as well as the πN interaction factorizes. In addition, an approximation was made that in some manner is equivalent to a completeness approximation with respect to the vectors of the internal state of the nucleus, namely, it was assumed that

$$\langle \mathbf{k}' | v_{\alpha'\alpha} | \mathbf{k} \rangle \approx \lambda \rho^*_{\alpha'\alpha}(\mathbf{k}') v^*(\mathbf{k}') v(\mathbf{k}) \rho_{\alpha'\alpha}(\mathbf{k}) \approx \lambda \rho^*_{\alpha'0}(\mathbf{k}') v^*(\mathbf{k}') v(\mathbf{k}) \rho_{\alpha 0}(\mathbf{k}), \quad (220)$$

where $\rho_{\alpha'\alpha}(\mathbf{k})$ is determined by (210).

The solution of Eq. (217) with the potential (220) and with allowance for (218) has the form

$$\langle \mathbf{k}' | \tau_{00}(E) | \mathbf{k} \rangle = \rho^*_{00}(\mathbf{k}') v^*(\mathbf{k}') v(\mathbf{k}) \rho_{00}(\mathbf{k}) \Big/ \left[\lambda^{-1} - \sum_\alpha \int \frac{|v(\mathbf{q})|^2 \rho_{\alpha 0}(\mathbf{q}) d\mathbf{q}}{E_{\pi A}(k) + i0 - E_{\pi A}(q) - \mathscr{E}^*_\alpha} \right]. \quad (221)$$

Concrete calculations made for the nucleus ^{12}C showed[51] that allowance in this manner for the binding of the nucleon in the nucleus and for the excitation of the nucleus in the intermediate states shifts the position of the resonance to higher energies by ≈12% and leads to a shrinking of the resonance by an amount of the same order.

The excitations of the nucleus in the intermediate states in the τ matrix can be taken into account through Eqs. (217) "quasiclassically" if the excitation energy $\mathscr{E}^*_\alpha$ is replaced by some mean excitation energy $\overline{\mathscr{E}}$ and the completeness condition $\sum |\alpha\rangle \langle \alpha| = 1$ is then used. The result is

$$\langle \mathbf{k}' | \tau_{00}(E) | \mathbf{k} \rangle \approx \langle 0\mathbf{k}' | t(E - \overline{\mathscr{E}}) | \mathbf{k}0 \rangle, \quad (222)$$

where $\tau(E - \overline{\mathscr{E}})$ is the matrix of free πN scattering with energy shifted by $\overline{\mathscr{E}}$. Binding of the nucleon in the nucleus was taken into account in this manner in Refs. 44 and 52, in which the inadequacy of such an approximation in the region of the (3, 3) resonance was demonstrated; this is due to the strong energy dependence of the πN t matrix in this region.

The problem of allowing for the binding of the nucleon in the nucleus and for the Pauli principle can be investigated for the example of the scattering of a pion by a nucleon moving in a given external field that simulates the field produced by the core (the system of $A-1$ nucleons in the nucleus) in the limit of infinite mass of the core ($A - 1 \to \infty$). Such a model was considered in Refs. 53 and 54. It can be applied to real nuclei when it is possible to ignore the center-of-mass motion of the nucleus in the scattering process. If in (215)–(218) and (194), (197), (202)–(206) we go to the limit ($A \to \infty$, $a_A \to 0$, $b_A \to 1$), we obtain the basic equation of the model:

$$\langle \mathbf{k}'_0 | \tau_{\alpha_b^0 \alpha_b^0}(E) | \mathbf{k}_0 \rangle = \langle \mathbf{k}'_0 | v_{\alpha_b^0 \alpha_b^0} | \mathbf{k}_0 \rangle + \sum_{\alpha_b} \int \frac{\langle \mathbf{k}'_0 | v_{\alpha_b^0 \alpha_b^0} | \mathbf{k}''_0 \rangle d\mathbf{k}''_0 \langle \mathbf{k}''_0 | \tau_{\alpha_b^0 \alpha_b^0}(E) | \mathbf{k}_0 \rangle}{E + i0 - \omega_\pi(k''_0) - \mathscr{E}^*_{\alpha_b}}, \quad (223)$$

where α_b are the quantum numbers of the nucleon in the external field, α_b^0 corresponding to the ground state; $\mathscr{E}^*_{\alpha b}$ is the excitation energy of the nucleon in this field, and

$$\langle \mathbf{k}'_0 | \tau_{\alpha'_b \alpha_b}(E) | \mathbf{k}_0 \rangle = \int \langle \alpha'_b | \mathbf{p}' \rangle d\mathbf{p}' \langle \mathbf{p}' \mathbf{k}'_0 | \tau(E) | \mathbf{k}_0 \mathbf{p} \rangle d\mathbf{p} \langle \mathbf{p} | \alpha_b \rangle; \quad (224)$$

$$\langle \mathbf{k}'_0 | v_{\alpha'_b \alpha_b} | \mathbf{k}_0 \rangle = \int \rho_{\alpha'_b \alpha_b}(\mathbf{p}'; \ \mathbf{p}' + \mathbf{k}'_0 - \mathbf{k}_0) d\mathbf{p} \langle \mathbf{k}'_0 - a(\mathbf{k}'_0 + \mathbf{p}') | v | \mathbf{k}_0 - a(\mathbf{k}'_0 + \mathbf{p}') \rangle, \quad (225)$$

where the function $\rho_{\alpha'_b \alpha_b}(\mathbf{p}', \mathbf{p})$ is determined by the expression (205).

The main approximation made in Ref. 53 in solving Eq. (223) consists of replacing the state vectors $|\alpha_b\rangle$ of the bound nucleon by the state vectors of a free nu-

cleon. In other words, in the intermediate states the model of nuclear matter is used for the nucleus. Accordingly, E and $\mathscr{E}^*_{\alpha_b}$ are taken in the form (the kinematics is nonrelativistic!)

$$E=E_\pi(k_0);\quad \mathscr{E}^*_{\alpha_b}=E_N(p)+\langle V_N\rangle-\langle h_N-M\rangle_0-\langle V_N\rangle_0, \tag{226}$$

where h_N is the Hamiltonian operator for the free nucleon; V_N is the potential of the external field; $\langle h_N-M\rangle_0$, $\langle V_N\rangle_0$, and $\langle V_N\rangle$ are the corresponding expectation values with respect to the ground state and the excited states, $\langle V_N\rangle$ being assumed to be independent of $\mathbf{p}$. In this model, we readily obtain from (223) and (225) (ground state)

$$\begin{aligned}&\langle \mathbf{k}_0'|\tau_{00}(E)|\mathbf{k}_0\rangle\\ &=\int\langle 0|\mathbf{p}'\rangle\langle \mathbf{p}'+\mathbf{k}_0'-\mathbf{k}_0|0\rangle\langle b_0\mathbf{k}_0'-a_0\mathbf{p}'|\tau(E)|\mathbf{k}_0-a_0(\mathbf{k}_0'+\mathbf{p}')\rangle\,d\mathbf{p}',\end{aligned} \tag{227}$$

where

$$\begin{aligned}&\langle b_0\mathbf{k}_0'-a_0\mathbf{p}'|\tau(E)|\mathbf{k}-a_0(\mathbf{k}_0'+\mathbf{p}')\rangle\\ &=\langle b_0\mathbf{k}_0'-a_0\mathbf{p}'|v|\mathbf{k}_0-a_0(\mathbf{k}_0'+\mathbf{p}')\rangle\\ &+\int\frac{\langle b_0\mathbf{k}_0'-a_0\mathbf{p}'|v|\mathbf{y}\rangle\,d\mathbf{y}\,\langle\mathbf{y}|\tau(E)|\mathbf{k}_0-a_0(\mathbf{k}_0'+\mathbf{p}')\rangle}{E_r(|b_0\mathbf{k}_0-a_0\mathbf{p}'|)+i0-E_r(y)+B};\end{aligned} \tag{228}$$

$$B=\langle h_N-M\rangle_0-E_N(p')+\langle V_N\rangle_0-\langle V_N\rangle;\quad E_r(y)=\mathbf{y}^2/2\mu_{\pi N}; \tag{229}$$

$$a_0=m/(m+M);\quad b_0=M/(m+M). \tag{230}$$

If the potential v is chosen in the form (143) and (145), then the corresponding solution of Eq. (228) has the form

$$\begin{aligned}&\langle b_0\mathbf{k}_0'-a_0\mathbf{p}'|\tau(E)|\mathbf{k}_0-a_0(\mathbf{k}_0'+\boldsymbol{\rho}')\rangle\\ &=4\pi\sum_{lm_l}\frac{v^*_{lm_l}(b_0\mathbf{k}_0'-a_0\mathbf{p}')\,v_{lm_l}(\mathbf{k}_0-a_0(\mathbf{k}_0'+\mathbf{p}'))}{D_l(\mathbf{k}_0',\ \mathbf{p}';\ B)},\end{aligned} \tag{231}$$

where

$$D_l(\mathbf{k}_0',\ \mathbf{p}';\ B)=\lambda_l^{-1}-\int\frac{d\mathbf{y}\,|v_l(\mathbf{y})|^2}{E_r(|b_a\mathbf{k}_0'-a_0\mathbf{p}'|)+i0-E_r(y)+B}. \tag{232}$$

Substituting (231) and (232), we finally obtain

$$\begin{aligned}&\langle \mathbf{k}_0'|\tau_{00}(E)|\mathbf{k}_0\rangle=\int\langle 0|\mathbf{p}'\rangle\,d\mathbf{p}'\,\langle\mathbf{p}'+\mathbf{k}_0'-\mathbf{k}_0|0\rangle\\ &\times\sum_{lm_l}\frac{4\pi v^*_{lm_l}(b_0\mathbf{k}_0'-a_0\mathbf{p}')\,v_{lm_l}(\mathbf{k}_0-a_0(\mathbf{k}_0'+\mathbf{p}'))}{D_l(\mathbf{k}_0',\ \mathbf{p}';\ B)}.\end{aligned} \tag{233}$$

The quantity B (229), which enters (231) and (233) through the function D_l (232), takes into account the excitation of the bound nucleon (imitation of single-particle excitations in real nuclei) in intermediate states in the case of elastic scattering of the pion. If the dependence on B is ignored, then (231) is the free πN matrix in a coordinate system with origin at the center of the field. Thus, in this case the expression (233) will be the matrix of pion scattering by a bound nucleon in the impulse approximation, this taking into account the effect of recoil of the nucleon and averaging over the momenta of the bound nucleon (Fermi motion) in accordance with the general expression (199). If we consider the limit of infinite nucleon mass ($m/M\to 0$ and, therefore, $a_0\to 0, b_0\to 1$), we return to the FAS. The analysis made for π^{16}O scattering in the region of the (3, 3) resonance shows[53] that allowance for the excitation of the "nucleus" ($B\neq 0$) increases the real part of the optical potential and decreases its imaginary part. As a result, there is a decrease in the cross section of forward scattering and, therefore, of the total cross section, while the cross section at large (backward) angles increases. These effects depend on the chosen value of $A'\equiv\langle h_N-M\rangle_0+\langle V_N\rangle_0-\langle V_N\rangle$. It was also shown that for scattering through large angles allowance for the Fermi motion is also important,[53] as is allowance for the excitation of the nucleus in the intermediate states.

Closer to the real scattering of a pion by light nuclei is the model of an oscillator potential for the field in which the nucleon moves. The basic equation of this model can be obtained from (223) by setting in this equation $E=\omega_\pi(k_0)$, $\alpha_b=n_bl_bm_b$ and $E_{\alpha b}=E_{Nb}=\eta_bN_b=\eta_b\times(2n_b+l_b)$, where $n_b=0,1,\ldots$ is the radial quantum number, l_b is the orbital angular momentum, m_b is its projection (for simplicity, we here ignore the spin–orbit interaction), and η_b is the frequency of the oscillator field. As a result, we obtain

$$\begin{aligned}&\langle \mathbf{k}_0'|\tau_{n_b^0l_b^0m_b^0;\ n_b^0l_b^0m_b^0}(\omega_\pi(k_0))|\mathbf{k}_0\rangle=\langle \mathbf{k}_0'|v_{n_b^0l_b^0m_b^0;\ n_b^0l_b^0m_b^0}|\mathbf{k}_0\rangle\\ &+\sum_{\substack{n_bl_bm_b\\(N_b=2n_b+l_b)}}\int\frac{\langle \mathbf{k}_0'|v_{n_b^0l_b^0m_b^0;\ n_bl_bm_b}|\mathbf{k}_0''\rangle\,d\mathbf{k}_0''\,\langle \mathbf{k}_0''|\tau_{n_bl_bm_b;\ n_b^0l_b^0m_b^0}(\omega_\pi(k_0))|\mathbf{k}_0\rangle}{\omega_\pi(k_0)+i0-\omega_\pi(k_0'')-\eta_bN_b},\end{aligned} \tag{234}$$

where

$$\begin{aligned}&\langle \mathbf{k}_0'|v_{n_b'l_b'm_b';\ n_bl_bm_b}|\mathbf{k}_0\rangle\\ &=\int\rho_{n_b'l_b'm_b';\ n_bl_bm_b}(\mathbf{p}';\ \mathbf{p}'+\mathbf{k}_0'-\mathbf{k}_0)\,\langle \mathbf{k}_0'-a_0(\mathbf{k}_0'+\mathbf{p}')\\ &\times|v|\mathbf{k}_0-a_0(\mathbf{k}_0'+\mathbf{p}')\rangle\,d\mathbf{p}'.\end{aligned} \tag{235}$$

In Ref. 54, a study was made of a special case of this model, namely, the limiting case when $m/M\to 0$ and, therefore, $a_0\to 0$ and $b_0\to 1$ in the expression (235). Conceptually, this model is similar to Chew and Low's model.[18] For it to be close to this model, the radial part of the πN interaction potential in the state with $l=1$ was chosen in accordance with (73) in for form

$$\langle p'|v_1|p\rangle=\lambda_1\frac{p'v(p')}{\sqrt{\omega_\pi(p')}}\frac{p\vartheta(p)}{\sqrt{\omega_\pi(p)}}. \tag{236}$$

In this case, the expression (235) takes the form

$$\langle \mathbf{k}_0'|v_{n_b'l_b'm_b';\ n_bl_bm_b}|\mathbf{k}_0\rangle=\langle \mathbf{k}_0'|v_1|\mathbf{k}_0\rangle\,\rho_{n_b'l_b'm_b';\ n_bl_bm_b}(\mathbf{k}_0-\mathbf{k}_0'), \tag{237}$$

where

$$\rho_{n_b'l_b'm_b';\ n_bl_bm_b}(\mathbf{q})=\int\exp(i\mathbf{q}\mathbf{r}_b)\,\langle n_b'l_b'm_b'|\,\mathbf{r}_b\rangle\,d\mathbf{r}_b\,\langle \mathbf{r}_b|n_bl_bm_b\rangle. \tag{238}$$

In Ref. 54, Eq. (234) with the kernel (237) was solved by the method of Padé approximation, a partial expansion having first been made. In particular, the second Padé approximant

$$\tau_{l_\pm}=\tau^{(1)}_{l_\pm}[1-\tau^{(2)}_{l_\pm}/\tau^{(1)}_{l_\pm}] \tag{239}$$

was used; here $\tau^{(1)}_{l_\pm}$ and $\tau^{(2)}_{l_\pm}$ are the first and second iteration of the equation for $\tau_{l_\pm}$ ($l_\pm$ corresponds to $j=l\pm\frac{1}{2}$; here, j is the total angular momentum). The expression (239) gives the exact solution of the Lippmann–Schwinger equation for the free πN collision matrix, and it is to be expected that it will be a good approximation to the exact solution of Eq. (234). Concrete calculations made for the case that simulated π^{16}O scattering without allowance for the Pauli principle showed[34] that allowance for the binding of the nucleon in the nucleus shifts the resonance energy downward compared with the resonance energy of free πN scattering. But if allowance is made for the Pauli principle, which reduces to

the elimination of corresponding terms in the summation in (234), the resonance energy of $\pi^{16}O$ scattering approaches the resonance energy of free πN scattering. Thus, the effect of the Pauli principle and the binding of the nucleon in the nucleus virtually compensate each other for the resonance energy.

It should be emphasized that the results discussed above were obtained[54] under a number of simplifying assumptions, among which the main one is the approximation of "freezing" of the nucleon during the time of its interaction with the pion; this reduces (235) to the expression (237). It would be interesting to solve Eq. (234) using (235) without additional assumptions. At the present time, such a problem is more realistic, even for a p-wave πN (separable!) interaction, which is the main interaction for πA scattering in the region of the resonance. This problem was considered for an s-wave interaction in Refs. 52 and 55 and was solved under some further simplifying assumptions. It was shown that to obtain the cross section in the region of the resonance neither the impulse approximation, nor the approximation of a "frozen" nucleon, nor the "quasiclassical" approximation is really suitable.

The problem of finding the corrections to the impulse approximation due to allowance for binding of the nucleon in the nucleus and the Pauli principle can be investigated on the basis of Eq. (188). In this equation, the corrections are associated with the second term on the right-hand side. Choosing ω in the form $\omega = E + M$, we obtain

$$\tau_i(E) = t_i(E+M) + t_i(E+M)\,[G(E)\mathcal{A} - g(E+M)]\,\tau_i(E), \quad (240)$$

where $G(E)$ and $g(E)$ are determined from (4) and (77), an $t(E+M)$ by Eq. (76). In the nonrelativistic limit, the Hamiltonian H_A (42) of the nucleus can be represented in the form

$$H_A = H_{A-1} + \mathcal{K}_N^i + V_i + B_i = \mathcal{K}_{A-1}^c + H_{A-1}^{cm} + \mathcal{K}_N^i + V_i + B_i, \quad (241)$$

where H_{A-1}^{cm} is the Hamiltonian of the internal excitation of nucleus $(A-1)$; $\mathcal{K}_{A-1}^c$ is the operator of the kinetic energy of the center-of-mass motion; $\mathcal{K}_N^i$ is the operator of the kinetic energy of the distinguished nucleon i; V_i is the operator of the potential energy of the interaction of the "residual nucleus" $(A-1)$; B_i is the energy needed to separate nucleus A into nucleon i and the "residual nucleus" $(A-1)$. If the operator $\mathcal{A}$ in (240) is omitted, i.e., the Pauli principle is ignored, then under the assumption that the residual nucleus $(A-1)$ is not excited in the considered process ("inert" core), Eq. (240) takes the form

$$\tau_i(E) = t_i(E+M) + t_i(E)\,[E + i0 - h_\pi - \mathcal{K}_N^i]^{-1} \times V_i\,[E - B_i + i0 - \mathcal{K}_{A-1}^c - \mathcal{K}_N^i - V_i]^{-1}\,\tau_i(E). \quad (242)$$

Using the relation

$$V_i\,[E - B_i + i0 - h_\pi - \mathcal{K}_N^i - V_i]^{-1} = t_{ic}\,(E - B_i - h_\pi)\,[E - B_i + i0 - \mathcal{K}_N^i - \mathcal{K}_{A-1}^c]^{-1},$$

where t_{ic} is the matrix of the collision of the distinguished nucleon i with the inert core, we reduce Eq. (242) to the form

$$\tau_i(E) = t_i(E+M) + t_i(E+M)\,[E + i0 - h_\pi - \mathcal{K}_N^i]^{-1} \times t_{ic}\,(E - B_i - h_\pi)\,[E - B_i + i0 - \mathcal{K}_N^i - \mathcal{K}_{A-1}^c]^{-1}\,\tau_i(E). \quad (243)$$

In Ref. 56, in which the correction associated with allowing for the binding of the nucleon in the nucleus was calculated on the basis of Eq. (243), the second term was taken into account in the first approximation, i.e., the matrix

$$\tau_i^{(2)}(E) = t_i(E+M) + t_i(E+M)\,[E + i0 - h_\pi - \mathcal{K}_N^i]^{-1} \times t_{ic}\,(E - B_i - h_\pi)\,[E - B_i + i0 - \mathcal{K}_N^i - \mathcal{K}_{A-1}^c]^{-1}\,t_i(E+M) \quad (244)$$

was considered. In Ref. 56, further simplifying assumptions were made in order to calculate this matrix: 1) for the nucleus, the independent-particle model was used, and accordingly the operator of the kinetic energy of the core $\mathcal{K}_{A-1}^c$ was omitted for nuclei with filled single-particle states (244); 2) the Green's functions were replaced by δ functions, with the result that the matrix of scattering of a nucleon by the core, t_{ic}, was on the energy shell, and it was calculated in a model of an optical potential having the form of a rectangular well with spin–orbit interaction; 3) the first term of (244) was calculated in the approximation of a nucleon frozen in the nucleus ("factorization" approximation); 4) in the second term of (244) the πN scattering matrix was taken on the energy shell; 5) for the averaged (with respect to the wave functions of the single-particle states) matrix t_{ic} in the **r** representation the local approximation was used.

For the example of $\pi^{12}C$ scattering in this model it was shown[56] that allowance for the effect of the binding of the nucleon in the nucleus leads to an important correction (50–100%) to the optical potential of lowest order at energies $120 < T < 420$ MeV, but for the total and differential cross sections this correction is only 10–20%. The large value of the correction to the optical potential is evidently due to the use of the factorization approximation in the optical potential of lowest order [see (202)], which, as we have noted many times above, is a bad approximation in the region of the (3, 3) resonance.

For the model just considered, an important element was the neglect of both the excitation of the internal motion of the residual nucleus $(A-1)$ and the motion of it in the center-of-mass system. This approximation corresponds to the independent-particle model used in Ref. 56 for nuclei with filled single-particle states. One can consider a model in which the core is again assumed to be "inert," i.e., in H_A (241) the Hamiltonian H_{A-1}^{cm} is ignored, but allowance is made for the recoil of the core through the kinetic energy operator $\mathcal{K}_{A-1}^c$. With regard to the operator V_i, it is replaced by the single-particle operator V_{ic}. This model, which is called the three-particle model, was proposed by Révai,[57] and it was then considered in Refs. 40 and 58–60. We shall show that the equation for the matrix of pion scattering by a bound nucleon in this model has the form[40,60]

$$\tau_i(E) = \hat{t}_i(E - B_i) + t_i\,(E - B_i)\,\hat{G}_{0i}\,(E - B_i)\,\hat{t}_{ic}\,(E - B_i)\,\hat{G}_{0i}\,(E - B_i)\,\tau_i(E). \quad (245)$$

Here

$$E = \omega_\pi(k_0) + E_A(K_A);$$

$$\hat{t}_i(E) = v_i + v_i\hat{G}_{0i}\,(E - B_i)\,\hat{t}_i\,(E - B_i); \quad (246)$$

$$\hat{t}_{ic}\,(E - B_i) = V_{ic} + V_{ic}\hat{G}_{0i}\,(E - B_i)\,\hat{t}_{ic}\,(E - B_i), \quad (247)$$

where

$$\hat{G}_{0i}(E-B_i)=[E-B_i+i0+h_\pi-\mathscr{H}_N^i-\mathscr{H}_{A-1}^c]^{-1}. \quad (248)$$

Note that the expression (245) can be obtained from Eq. (243) if obvious substitutions are made in it. The expression (245) is too complicated for practical use, and therefore further simplifying approximations were made in Ref. 60: 1) the second term in the expression (245) was completely ignored, so that the τ_i matrix could be obtained in the impulse approximation, but with allowance for the three-particle kinematics; 2) the operator $h_\pi - m + \mathscr{H}_N^i$ was represented as a sum of the operators of the kinetic energies of the relative motion of the pion and the nucleon, $\mathscr{H}_{\pi N}^r$, and the motion of their center of mass, $\mathscr{H}_{\pi N}^c$. This, strictly speaking, is valid when not only the nucleon but also the pion is considered in the nonrelativistic limit. As a result of these simplifications, we obtain

$$\tau_i(E)\approx\tau_i^{\text{imp}}(E)=t_i^r(E-m-B_i-\mathscr{H}_{\pi N}^c-\mathscr{H}_{A-1}^c). \quad (249)$$

If we again use the nonrelativistic kinematics and ignore the transformation of the t matrix from the pion–nucleon center-of-mass system to the pion–nucleus center-of-mass system, we obtain[60]

$$\langle \mathbf{k}'\mathbf{p}_i'|\tau_i(E)|\mathbf{k}\mathbf{p}_i\rangle\approx\delta(\mathbf{k}'+\mathbf{p}_i'-\mathbf{k}-\mathbf{p}_i)\,t_i^r(\mathbf{f}',\mathbf{f};E_{\pi A}^r(k) - B_i-(\mathbf{k}+\mathbf{p}_i)^2/(2\mu_{\pi N,A-1})), \quad (250)$$

where $\mathbf{k}$ and $\mathbf{f}(\mathbf{k}',\mathbf{f}')$ are defined in accordance with (194), (196), and (197) in the nonrelativistic limit with respect to the nucleon, and

$$E_{\pi A}^r(k)=\mathbf{k}^2/2\mu_{\pi A}=E_{\pi A}(k);$$
$$\mu_{\pi N,A-1}=(m+M)M(A-1)/(m+MA). \quad (251)$$

Note that at energies $E_{\pi A}(k)$ less than the energy of breakup of the nucleus into the nucleon and the residual nucleus B_i the t matrix (249) and, therefore, the optical potential $U_{00}'^{(1)}$ (180) are real, so that the elastic unitarity (35) is ensured (to accuracy $1/A$). Such unitarity is not satisfied, for example, for the ordinary impulse approximation, since for either choice (190) or (191) of ω we obtain complex $t_i(\omega)$. Although the impulse approximation in the three-particle model has in this respect an advantage over the usual impulse approximation, it is less justified and consistent that the model in which (in accordance with the independent-particle model) one ignores not only the excitations of the residual nucleus $(A-1)$ but also the motion of its center of mass.[41,52,55] The point is that for ^{4}He, ^{12}C, and ^{16}O nuclei, for which πA scattering has been investigated, the energy of the binding of the nucleon to the core is greater than (for the ^{4}He nucleus considerably greater) or of the order of the mean binding energy of a nucleon in the core, which therefore cannot be regarded as an inert structureless "particle" that recoils as a whole, as is assumed in the three-particle model.

In Refs. 54, 56, and 60, the results of which were discussed above, the Pauli principle for the nucleons in the nucleus is taken into account very approximately, which is motivated by the difficulties that arise in the solution of Eqs. (36), (37), or (182), (183) for finite nuclei. These difficulties partly derive from the need for simultaneous allowance of several interconnected effects: the binding of the nucleon in the nucleus, the Fermi motion of the nucleon, and the Pauli principle. The last two effects are more readily investigated in the model of nuclear matter for πA scattering, in which there is no problem of allowing for the binding of the nucleon. This was done in Refs. 61–64. In Refs. 61–63, only the effect of the Pauli principle was considered. This was due to the use of the Chew–Low model for the πN interaction (static model with respect to the nucleon!). In Ref. 64, the effect of the Pauli principle was investigated together with the effect of the Fermi motion.

Bearing in mind that the nuclear-matter model is a special case of the independent-particle model, in which the single-particle states are states of free motion of the nucleons, in the expression H_A (241) we should ignore H_{A-1} and V_i, and instead of B_i we should take $-\varepsilon_n$, where ε_n is the single-particle energy. Further, bearing in mind that the model of nuclear matter corresponds to the limit of an infinitely heavy nucleus $(A\to\infty)$, in (197) we should set $\mathbf{k}=0$, which immediately leads to $\mathbf{k}=\mathbf{k}_0$ and $\mathbf{p}_i=\mathbf{p}_i^0$. Then, using Watson's formulation of the theory of multiple scattering (here, we follow Ref. 64), on the basis of Eqs. (181)–(183) and (4) we obtain

$$\langle \mathbf{k}_0'|\hat{U}_{00}(E)|\mathbf{k}_0\rangle=\sum_i\langle 0\mathbf{k}_0'|\hat{\tau}_i(E)|\mathbf{k}_0 0\rangle = \sum_{n\neq 0}\langle n\mathbf{k}_0'|\hat{\tau}(E+\varepsilon_n)|\mathbf{k}_0 n\rangle, \quad (252)$$

where

$$\langle n'\mathbf{k}_0'|\hat{\tau}(E+\varepsilon_n)|\mathbf{k}_0 n\rangle=\langle n'\mathbf{k}_0'|v|\mathbf{k}_0 n\rangle + \sum_{n''\neq 0}\int\frac{\langle n'\mathbf{k}_0'|v|\mathbf{k}_0''n''\rangle\,d\mathbf{k}_0''\langle n''\mathbf{k}_0''|\hat{\tau}(E+\varepsilon_n)|\mathbf{k}_0 n\rangle}{E+\varepsilon_n+i0-\omega_\pi(k_0'')-\varepsilon_{n''}}. \quad (253)$$

Remembering that for nuclear matter $\varepsilon_n=E_N(p_n)$, and introducing the matrix $t(\omega)$ of free πN scattering, on the basis of Eqs. (189), (253), and (77) we obtain

$$\langle \mathbf{p}'\mathbf{k}_0'|\tau(\omega_\pi(k_0)+E_N(p))|\mathbf{k}_0\mathbf{p}\rangle=\langle \mathbf{p}'\mathbf{k}_0'|t(\omega)|\mathbf{k}_0\mathbf{p}\rangle + \int\langle \mathbf{p}'\mathbf{k}_0'|t(\omega)|\mathbf{k}_0''\mathbf{p}''\rangle\,d\mathbf{p}''d\mathbf{k}_0''\Big[\frac{\Theta(p''-p_F)}{\omega_\pi(k_0)+E_N(p)+i0-\omega_\pi(k_0'')-E_N(p'')} - \frac{1}{\omega+i0-\omega_\pi(k_0'')-E_N(p'')}\Big]\langle \mathbf{p}''\mathbf{k}_0|\tau(\omega_\pi(k_0)+E_N(p))|\mathbf{k}_0\mathbf{p}\rangle, \quad (254)$$

where

$$\Theta(x)=\begin{cases}1, x>0\\ 0, x<0\end{cases};$$

and p_F is the Fermi momentum.

We now encounter the problem of transforming the matrix $t(\omega)$ on the transition from the pion–nucleus laboratory system to the pion–nucleon center-of-mass system. The solution of this problem is given by (193) and (196). They take a simple form in the approximations used in Ref. 64 to solve Eqs. (254): 1) the single-particle potential in nuclear matter is assumed to be local, i.e., the potential acting on a nucleon within the Fermi sea and outside it is assumed to be the same; 2) the parameter ω is chosen in the form (190), namely $\omega=\omega_\pi(k_0)+E_N(p)+M$, which leads to the vanishing of the integral term in (196); 3) nonrelativistic kinematics is used for the pion as well.

In the concrete calculations of Ref. 64, Schmit used a separable s-wave t matrix of πN scattering chosen to simulate the resonance behavior of this matrix. It was

shown that the influence of the Pauli principle on the optical potential strongly depends on the Fermi averaging; namely, when allowance is made for this motion, the effect of the Pauli principle in the region of the resonance is very small, whereas in the static model[61-63] this effect is important. At low energies, the effect of the Pauli principle is important,[64] but it is always appreciably smaller than in the static model.[61-63]

The model of nuclear matter for πA scattering also enables one to investigate comparatively easily important aspects of this problem in the region of the resonance associated with the production and propagation of the Δ resonance in nuclear matter. This question was considered in Refs. 64 and 41. In Ref. 41, in contrast to Ref. 64, the Pauli principle is ignored. Accordingly, it is more convenient to use the KMT formulation of the theory of multiple scattering [see (180) and (36), (37)], and therefore, instead of (252) and (253), remembering that

$$|n\rangle = |\mathbf{p}\rangle \text{ and } \varepsilon_n = E_N(p),$$

we obtain

$$\langle \mathbf{k}_0' | U_{00}^{(1)}(\omega_\pi) | \mathbf{k}_0 \rangle = \int\limits_{p \le p_F} \langle \mathbf{p}\mathbf{k}_0' | \tau(\omega_\pi + E_N(p)) | \mathbf{k}_0\mathbf{p} \rangle \, d\mathbf{p}, \tag{255}$$

where

$$\langle \mathbf{p}'\mathbf{k}_0' | \tau(\omega_\pi + E_N(p)) | \mathbf{k}_0\mathbf{p} \rangle = \langle \mathbf{p}'\mathbf{k}_0' | v | \mathbf{k}_0\mathbf{p} \rangle + \int \frac{\langle \mathbf{p}'\mathbf{k}_0' | v | \mathbf{k}_0''\mathbf{p}'' \rangle \, d\mathbf{k}_0'' d\mathbf{p}'' \langle \mathbf{p}''\mathbf{k}_0'' | \tau(\omega_\pi + E_N(p)) | \mathbf{k}_0\mathbf{p} \rangle}{\omega_\pi + E_N(p) + i0 - \omega_\pi(k_0'') - E_N(p'')}.$$

Note that in (255) we have omitted the factor $(A-1)/A \approx 1$ $(A \to \infty)$. We recall that in accordance with (78) and (195) in the nonrelativistic limit with respect to the nucleon we have

$$\begin{aligned} \omega_\pi(k_0) + \omega_n(p) &= \sqrt{[\omega_\pi(\mathbf{k}) + \omega_n(\mathbf{k})]^2 + (k_0 + \mathbf{p})^2} \\ &\approx \omega_\pi(\mathbf{k}) + \omega_n(\mathbf{k}) + \frac{(k_0+\mathbf{p})^2}{2(\omega_\pi(k_0) + M)} \\ &= h_0^{cm}(\mathbf{k}) + \frac{(k_0+\mathbf{p})^2}{2M^*}, \quad M^* = M + \omega_\pi(k_0). \end{aligned} \tag{256}$$

In accordance with Eqs. (193), (195), and (196), we obtain from (256)

$$\begin{aligned} &\langle \mathbf{p}'\mathbf{k}_0' | \tau(\omega_\pi(k_0) + E_N(p)) | \mathbf{k}_0\mathbf{p} \rangle \\ &= N\delta(\mathbf{k}_0' + \mathbf{p}' - \mathbf{k}_0 - \mathbf{p}) \, t^{cm}(b\mathbf{k}_0' - a\mathbf{p}'; \\ &b\mathbf{k}_0 - a\mathbf{p}; \ \omega_\pi(k_0) + E_N(p) - (\mathbf{k}_0 + \mathbf{p})^2/2M^*). \end{aligned} \tag{257}$$

Substituting this expression in (255), for the optical potential we obtain

$$\begin{aligned} &\langle \mathbf{k}_0' | U_{00}^{(1)}(\omega_\pi(k_0)) | \mathbf{k}_0 \rangle \equiv V_{opt}^{(1)}(\mathbf{k}_0', \mathbf{k}_0; \omega_\pi(k_0)) \\ &= \delta(\mathbf{k}_0' - \mathbf{k}_0) \int N t^{cm}(b\mathbf{k}_0 - a\mathbf{p}, b\mathbf{k}_0 - a\mathbf{p}; \\ &\omega_\pi(k_0) + E_N(p) - (\mathbf{k}_0 + \mathbf{p})^2/2M^*). \end{aligned} \tag{258}$$

It should be noted that in Ref. 41 the expression (258) was written down for arbitrary ω_π, i.e., for $\omega_\pi \neq \omega_\pi(k_0)$, which hold if the integral term in (196) is ignored.

For the investigation of the propagation of the pion in the nuclear medium with the optical potential (258) with $\omega_\pi \neq \omega_\pi(k_0)$, the resonance expression (70) was used in Ref. 41 for the πN scattering matrix t^{cm}. It was shown[64] that when a pion with energy in the region of the (3, 3) resonance propagates in nuclear matter two kinds of motion arise—genuine pion motion and $\Delta - h$ type of motion (h is a hole in the nuclear matter). These two motions are coupled, and go over continuously from one form to the other. They can be distinguished only at low densities of the nuclear matter ($\rho < \rho_c = 0.042\ F^{-3}$), where the pion form of motion is predominant. With increasing density, the relative weight of the $\Delta - h$ form of motion increases, and at densities close to and even lower than the density of real nuclei ($\rho \approx 0.15\ F^{-3}$) the $\Delta - h$ form of motion becomes as important as the pion form. This is a consequence of the allowance for the Fermi motion. It should be emphasized that these results were obtained without allowance for the Pauli principle in the τ matrix, but, as the model calculations of Ref. 64 show, they remain virtually unchanged when this effect is taken into account.

The picture of the production of the Δ isobar for pions with energies in the region of the (3, 3) resonance that we have in nuclear matter (for a review of the state of this problem, see Ref. 65) can also arise in finite nuclei if they are described by the independent-particle model and consist of shells with filled single-particle states. In contrast to nuclear matter, a hole in the independent-particle model corresponds to a single-particle state in some self-consistent field. It is clear that the $\Delta - h$ states will be degenerate if the interaction of the Δ isobar with the hole is not taken into account. If it is, the degeneracy will be lifted, and as a result we obtain an excitation spectrum of the nucleus of the type of particle–hole excitations in nuclei with filled single-particle states. Such allowance for the excitation of the nucleus in the construction of the first-order optical potential was considered in Refs. 66 and 67 for the example of π^4He. Here, we should point out that the applicability of this model to a nucleus as light as ^{4}He is doubtful because of the incorrect allowance for the center-of-mass motion of the nucleus in the independent-particle model. It is also difficult to justify the assumption that the potential of the interaction of nucleon i with the residual nucleus, V_i [see (241)], can be regarded as the potential of the interaction of the Δ isobar (pion + nucleon i in resonance) with the residual nucleus, as is done in Ref. 66 in the formulation of the problem in the framework of the theory of multiple scattering. In addition, this treatment introduces into the theory the additional phenomenological Δ–nucleus interaction, although allowance for this interaction is a necessary and important element of the Δ-isobar dynamics in the nucleus.

For π^4He scattering there is the possibility of allowing correctly for the center-of-mass motion of the nucleus and the Fermi motion by using the wave function obtained in the translationally invariant oscillator basis,[68] which makes it possible for this example to investigate various questions of the theory of multiple scattering of pions by nuclei.[69] The existence of fairly complete experimental data on π^4He scattering[70,71] makes such investigations desirable.

5. SECOND-ORDER OPTICAL POTENTIAL

We now consider the problem of finding the corrections to the first-order optical potential. These corrections, naturally, arise from the second term on the

right-hand side of Eq. (21) or (24). We consider first the approach to the solution of this problem proposed in Ref. 70. We proceed from Eq. (18) for the T matrix. Taking into account the equation $\mathscr{A}=\mathscr{P}+Q$ (8), from (18) we obtain

$$\mathscr{P}T'=\mathscr{P}\tau'+\mathscr{P}\tau'G\mathscr{P}T'+\mathscr{P}\tau'GQT'; \tag{259}$$

$$QT'=Q\tau'+Q\tau'G\mathscr{P}T'+Q\tau'GQT', \tag{260}$$

where T' is the matrix related to the T matrix by the equation $T'=(A-1)T/A$ (17), and τ' is the matrix

$$\tau'(E)=(A-1)\tau(E). \tag{261}$$

Solving Eq. (260) for QT' [taking into account the equations $Q^2=Q$ and $QG=GQ$ (8)],

$$QT'=\frac{1}{Q(1-G\tau')Q}[Q\tau'+Q\tau'G\mathscr{P}T']$$

$$=G^{-1}\frac{1}{Q(G^{-1}-\tau')}[Q\tau'+Q\tau'G\mathscr{P}T'] \tag{262}$$

and substituting it in (259), we obtain

$$\mathscr{P}T'=\mathscr{P}U'+\mathscr{P}U'G\mathscr{P}T', \tag{263}$$

where

$$U'=\tau'+\tau'\frac{1}{Q(E+i0-h_\pi-H_A-\tau')Q}Q\tau'$$

$$=\tau'+\tau'\frac{Q}{Q(E+i0-h_\pi-H_A)Q}U'. \tag{264}$$

It is clear that Eqs. (263) and (19), (20) coincide for elastic scattering and, therefore, the operator U' determined in accordance with (264) coincides with the optical potential (21) in the KMT formulation. Thus, for the optical potential $U'_{opt}(E)\equiv\langle 0|U'(E)|0\rangle$ we obtain from (264)

$$U'_{opt}(E)=\tau'_{00}(E)$$

$$+\sum_{\alpha',\alpha\neq 0}\tau'_{0\alpha'}(E)[E+i0-\mathscr{E}^*_\alpha-\mathscr{K}^c_A-h_\pi-\tau'_{\alpha'\alpha}(E)]^{-1}\tau'_{\alpha'0}(E), \tag{265}$$

where $E=\omega_\pi+E_A$. Equation (265) for the optical potential with the Green's operator

$$\tilde{G}(E)=[E+i0-h_\pi-H_A-\tau'(E)]^{-1} \tag{266}$$

is exact. The operator (266) in the general case is nondiagonal with respect to the excited internal states α of the nucleus because of the nondiagonality of the operator $\tau'(E)$. If in (265) we ignore the nondiagonal elements of the operator τ' and, therefore, of the operator $G(E)$, we obtain

$$U'_{opt}(E)\approx\tau'_{00}(E)$$

$$+\sum_{\alpha\neq 0}\tau'_{0\alpha}(E)[E+i0-\mathscr{E}^*_\alpha-\mathscr{K}^c_A-h_\pi-\tau'_{\alpha\alpha}(E)]^{-1}\tau_{\alpha 0}(E). \tag{267}$$

This expression was obtained in Ref. 72 and called the second-order optical potential.

As can be seen from the definition (11)–(12), the matrix $\tau(E)$ is a many-particle operator that takes into account the antisymmetrization of the state vectors of the nucleus and, therefore, takes into account the Pauli principle. In the determination of the optical potential (267) a significant simplification can be achieved if instead of the operator τ_i defined in accordance with (36) and (37) we use the operator $\bar{t}_i$, which is defined by the equation

$$\bar{t}_i(E)=\begin{cases} v_i+v_iG(E)t_i(E); & (268)\\ v_i+\bar{t}_i(E)G(E)v_i & (269)\end{cases}$$

and does not take into account antisymmetrization. Solving Eq. (269) for v_i and substituting the resulting expression in (36), we obtain

$$\tau_i(E)=\bar{t}_i(E)+\bar{t}_i(E)G(E)(\mathscr{A}-1)\tau_i(E). \tag{270}$$

As we have already said above, the operators τ (25), (26) and τ_i (36), (37) in the space of antisymmetric states of the nucleus are related by $\tau=\sum_i\tau_i/A$, and from (270) we therefore have

$$\tau(E)=\frac{1}{A}\sum_i\bar{t}_i(E)+\frac{1}{A}\sum_i\bar{t}_i(E)G(E)(\mathscr{A}-1)\tau_i(E). \tag{271}$$

On the basis of Hilbert's identity,

$$A^{-1}-B^{-1}=A^{-1}(B-A)B^{-1}=B^{-1}(B-A)A^{-1}.$$

From (4) and (266), we obtain

$$G(E)=\tilde{G}(E)-\tilde{G}(E)\tau'(E)G(E). \tag{272}$$

Taking into account the circumstance that the matrix elements of the operators τ and G between states of the spaces $\mathscr{A}$ and $1-\mathscr{A}$ are zero and, therefore, that these matrix elements are also zero for $\tilde{G}$, we see from (271) that the operator $G(\mathscr{A}-1)$ can be replaced by the operator $\tilde{G}(\mathscr{A}-1)$. As a result,

$$(A-1)\tau_{00}(E)=\frac{A-1}{A}\Big(\sum_i\bar{t}_i(E)\Big)_{00}$$

$$+\frac{A-1}{A}\Big(\sum_i\bar{t}_i(E)\tilde{G}(E)(\mathscr{A}-1)\tau_i(E)\Big)_{00}. \tag{273}$$

If in the second term on the right-hand side of this relation we replace the operator τ_i by the operator $\bar{t}_i$ and remember that the operator $\mathscr{A}\bar{t}_i$ has matrix elements only in the space of antisymmetric states, we can write

$$(A-1)\tau_{00}(E)\approx\frac{A-1}{A}\Big(\sum_i\bar{t}_i(E)\Big)_{00}$$

$$+\frac{A-1}{A}\Big(\sum_i\bar{t}_i(E)\tilde{G}(E)\sum_j\bar{t}_j(E)\Big)_{00}$$

$$-\frac{A-1}{A}\Big(\sum_i\bar{t}_i(E)\tilde{G}(E)\bar{t}_i(E)\Big)_{00}. \tag{274}$$

We now calculate the sum in the expression (267). For this, we make the following approximations: 1) we shall assume that the $\tau_{\alpha\alpha}$ in (266) are independent of the state (α) and replace them by the averaged quantity $\langle\tau'(E)\rangle=\bar{U}_0(E)$; 2) instead of $\mathscr{E}^*_\alpha$ we take some average excitation energy $\bar{\mathscr{E}}$ of the nucleus. Then, retaining in (271) only the first term and using the completeness of the states vectors $|\alpha\rangle$, we obtain

$$\sum_{\alpha\neq 0}\tau'_{0\alpha}(E)\tilde{G}_{\alpha\alpha}(E)\tau'_{\alpha 0}(E)$$

$$\approx\frac{(A-1)^2}{A^2}\Big[\Big(\sum_i\bar{t}_i(E)\bar{G}(E)\sum_j\bar{t}_j(E)\Big)_{00}$$

$$-\Big(\sum_i\bar{t}(E)\Big)_{00}\bar{G}(E)\Big(\sum_j\bar{t}_j(E)\Big)_{00}\Big], \tag{275}$$

where

$$\bar{G}(E)=[E+i0-\mathscr{K}^c_A-h_\pi-\bar{\mathscr{E}}-\bar{U}_0(E)]^{-1}. \tag{276}$$

Adding the expression (274) and (275) in accordance with (267) and (261), we obtain an approximate expression for the optical potential:

$$U'_{opt}(E)\approx U^{(1)}_{opt}(E)+U^{(2)}_{opt}(E), \tag{277}$$

where

$$U^{(1)}_{\text{opt}}(E)=(A-1)\langle 0|\frac{1}{A}\sum_i \bar{t}_i(E)|0\rangle, \quad (278)$$

$$U^{(2)}_{\text{opt}}(E)=(A-1)^2\Big[\frac{1}{A(A-1)}\sum_{i\neq j}\langle 0|\bar{t}_i(E)\bar{G}(E)\bar{t}_j(E)|0\rangle$$
$$-\langle 0|\frac{1}{A}\sum_i \bar{t}_i(E)|0\rangle\bar{G}(E)\langle 0|\frac{1}{A}\sum_j \bar{t}_j(E)|0\rangle\Big]. \quad (279)$$

This optical potential was used in Ref. 73 to investigate pion scattering by the ^{4}He nucleus. In the concrete calculations, further simplifying approximations were made: 1) the matrix $\bar{t}_i(E)$ was replaced by the free πN scattering matrix $t_i(\omega)$, in which the energy ω was chosen in accordance with the simplified three-particle model; 2) for the matrix elements $\langle 0|t_i(\omega)|\alpha\rangle$ the following approximate factorization was used:

$$\langle 0\mathbf{k}'|\sum_i t_i(\omega)|\mathbf{k}\alpha\rangle\approx\rho_{0\alpha}(\mathbf{q})\langle\mathbf{k}',\mathbf{p}_0-\mathbf{q}|t(\omega)|\mathbf{k}\mathbf{p}_0\rangle,$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the pion momenta before and after scattering in the pion–nucleus center-of-mass system, $\mathbf{q}=\mathbf{k}-\mathbf{k}'=\mathbf{k}_0-\mathbf{k}_0'$ is the momentum transfer, $\mathbf{k}_0$ is the momentum of the incident pion, $\rho_{0\alpha}(\mathbf{q})$ is determined by the expression (210), and $\mathbf{p}_0=-\mathbf{k}/A+(A-1)\mathbf{q}/2A$ in accordance with Refs. 41 and 42; 3) the averaged operator of the optical potential $\bar{U}_0(E)$ and $\bar{G}(E)$ (276) was replaced by the constant quantity $\langle\mathbf{k}_0|U^{(1)}_{\text{opt}}(E)|\mathbf{k}_0\rangle$. As a result of these approximations, the optical potential $U^{(2)}_{\text{opt}}$ can be expressed in terms of the correlation function

$$C(\mathbf{q}',\mathbf{q})=\rho_{00}(\mathbf{q}',\mathbf{q})-\rho_{00}(\mathbf{q}')\rho_{00}(\mathbf{q}), \quad (280)$$

where $\rho_{00}(\mathbf{q}',\mathbf{q})$ and $\rho_{00}(\mathbf{q})$ are defined in accordance with (202) and (210), respectively. In Ref. 73, the function $C(\mathbf{q}',\mathbf{q})$ was calculated in the framework of the shell model with phenomenological allowance for short-range pairing correlation. The correlation associated with correct allowance for the center-of-mass motion of the nucleus was also taken into account. It is clear that this correlation must be taken into account in the scattering of a pion by a light nucleus such as ^{4}He (as we have pointed out above). The problem of the connection between the πN scattering matrix in the pion–nucleon center-of-mass system and the same matrix in the pion–nucleus system was solved in Ref. 73 in the framework of the relativistic potential theory of Ref. 23. Use was made of an expansion in $\mathbf{Q}^2/(m+M)^2$, where $\mathbf{Q}$ is the total momentum of the πN system (for a discussion of this question, see below).

The second-order optical potential is usually investigated with a view to obtaining information about the pairing correlation of nucleons in the nucleus; this really is contained in the second-order potential. However, it must be borne in mind that in an exact solution of the problem information about the correlation can also be obtained from the first-order optical potential, since it is also included in the operators τ (11)–12) or $\hat{\tau}$ (25)–(26) through the Green's operator (4) and the projection operator $\mathscr{A}$. When the impulse approximation is used, the effect of the pairing correlation in τ or $\hat{\tau}$ is partly ignored.

A somewhat different representation than the one obtained above for the second-order optical potential was obtain in Ref. 11 on the basis of a correlation expansion of the total optical potential in the KMT formulation. This expansion, obtained directly on the basis of Eq. (2), has the form

$$U'(E)=U^{(1)}(E)+U^{(2)}(E)+\ldots, \quad (281)$$

where

$$U^{(1)}(E)=\frac{A-1}{A}\sum_i \bar{t}_i(E); \quad (282)$$

$$U^{(2)}(E)=\frac{A-1}{A}\sum_{i>j}(\bar{t}_{ij}(E)-\bar{t}_i(E)-\bar{t}_j(E)). \quad (283)$$

The matrix $\bar{t}_i$ ($\bar{t}_j$) in these relations is determined by (268) and (269), and $\bar{t}_{ij}=\Lambda_{ij}+\Lambda_{ji}$, where Λ_{ij} and Λ_{ji} satisfy the system of equations

$$\Lambda_{ij}=\bar{t}_i+\bar{t}_iG(1-\mathscr{P})\Lambda_{ji}; \quad (284)$$
$$\Lambda_{ji}=\bar{t}_j+\bar{t}_jG(1-\mathscr{P})\Lambda_{ij}, \quad (285)$$

or the two equivalent uncoupled equations

$$\Lambda_{ij}=\bar{t}_i+\bar{t}_iG(1-\mathscr{P})\bar{t}_j+\bar{t}_iG(1-\mathscr{P})\bar{t}_jG(1-\mathscr{P})\Lambda_{ij}; \quad (286)$$
$$\Lambda_{ji}=\bar{t}_j+\bar{t}_jG(1-\mathscr{P})\bar{t}_i+\bar{t}_jG(1-\mathscr{P})\bar{t}_iG(1-\mathscr{P})\Lambda_{ji}. \quad (287)$$

It is easy to see that, iterating these equations for $\bar{t}_{ij}=\Lambda_{ij}+\Lambda_{ji}$, we obtain the series

$$\bar{t}_{ij}(E)=\bar{t}_i(E)+\bar{t}_j(E)+\bar{t}_i(E)G(E)(1-\mathscr{P})\bar{t}_j(E)$$
$$+\bar{t}_j(E)G(1-\mathscr{P})\bar{t}_i(E)+\bar{t}_i(E)G(E)(1-\mathscr{P})\bar{t}_j(E)G(E)(1-\mathscr{P})$$
$$\times\bar{t}_i(E)+\bar{t}_j(E)G(E)(1-\mathscr{P})\bar{t}_i(E)G(E)(1-\mathscr{P})\bar{t}_j(E)+\ldots. \quad (288)$$

Note that (284)–(288) are also valid when G in them is replaced by $G\mathscr{A}$. Accordingly, $G(1-\mathscr{P})$ is replaced by GQ and the substitution of $G\mathscr{A}$ should also be made in Eqs. (268) and (269), which as a result reduce to (36) and (37) for τ_i. If in the expansion (288) we restrict ourselves to the single and double terms, then for the second-order optical potential we obtain

$$U^{(2)}(E)\approx\frac{A-1}{A}\sum_{i\neq j}\bar{t}_i(E)G(E)(1-\mathscr{P})\bar{t}_j(E). \quad (289)$$

The operator (289), like the corresponding part of the operator (267), is in the general case a many-particle operator. If for $\bar{t}_i$ we use the impulse approximation $\bar{t}_i(E)\approx t_i(\omega)$, then this operator is transformed into a two-particle operator. But the operator of real absorption of a pion by a nucleus[7] is also (basically) a two-particle operator. Therefore, when we consider the second-order optical potential we must take into account as well the real absorption of pions by nuclei. In Refs. 69 and 74, this effect was taken into account phenomenologically in the first-order optical potential. Such allowance was also made in Ref. 75 by introducing into the Hamiltonian of the πA interaction an operator of pair absorption or production R. Approximate allowance for this operator reduces to replacing the Green's function $G(E)$ in (177), (178) and (25), (26) by the function

$$\tilde{G}(E)=[G^{-1}(E)-\Delta]^{-1};\quad \Delta=RG(E)R.$$

Concrete calculations were made in Ref. 75 using instead of Δ its mean value $\langle\Delta\rangle=-iv_\pi\lambda_\alpha$,[76] in which v_a is the pion velocity, and λ_a is the mean free path until absorption in nuclear matter. The calculations made for π^4He scattering[60,75] showed that allowance for real absorption of the pion by the nucleus leads to an appreciable decrease of the cross section in the region of the (3, 3) resonance and that with decreasing energy

this effect increases relatively.

In our opinion, the effect of the real absorption of the pion by the nucleus in the scattering process was taken into account more consistently in Ref. 77 for πd scattering. The results of Ref. 77 will be discussed in the following section.

6. πd SCATTERING

The general theory of multiple scattering of pions by nuclei presented in the foregoing sections on the basis of the Lippmann–Schwinger equation (1) can of course also be applied to the scattering of pions by the deuteron. But it is then necessary to remember that the deuteron is a weakly bound system that does not have bound excited states, so that the influence of the deuteron breakup channel on the elastic scattering of pions may be important. It is well known that the deuteron breakup channel is not correctly taken into account in the framework of the Lippmann–Schwinger equations, and it may therefore happen that Eq. (1) is not always a suitable equation for describing πd scattering even in the case of the elastic channel.

If the πd interaction is considered at pion kinetic energies $T_\pi \leq 300$ MeV, where one can ignore the production of new strongly interacting particles, then Faddeev's equations or their relativistic generalizations can be applied to this problem. It is clear that these equations directly describe the following channels:

$$\pi^\pm + d \Rightarrow \begin{cases} d + \pi^\pm; & (290) \\ np + \pi^\pm; & (291) \\ \binom{pp}{nn} + \pi^0. & (292) \end{cases}$$

The absorption channel $\pi d \to NN$ can be taken into account for πd scattering if the solution of the problem (290)–(292) is available (this question is discussed below).

The Faddeev equations for the processes (290)–(292) correctly take into account the breakup channels (291) and (292), i.e., they do not have the shortcoming inherent in the Lippmann–Schwinger equations (1). The use of the Faddeev equations or their relativistic generalizations has the advantage over the theory of multiple scattering presented in this review that they are directly expressed in terms of the matrices of πd collisions in the free state, i.e., there is no need to use the impulse approximation, whose applicability in the region $T_\pi \leq 300$ MeV is doubtful. In addition, since these equations can be solved by numerical methods in certain models of the two-body interactions without using, for example, the FSA, it is possible to investigate the accuracy of this approximation for the example of πd scattering. The convergence of the multiple scattering series can also be investigated for this example. Further, using the circumstance that the three-particle Faddeev equations contain the two-particle off-shell collision matrices, by studying πd scattering in the framework of these equations one can obtain information about the off-shell behavior of the πN and NN matrices. In this example, one can also investigate other questions of the dynamics of the pion–nucleus interaction. From what we have said the fundamental importance of studying πd scattering in the framework of the three-particle equations for pion–nuclear physics is evident.

Numerous investigations[47-50,78-86] have been made of πd scattering in the framework of the three-particle equations. Some of them[47,76-80,82,85] use the nonrelativistic equations (the Faddeev equations), while the remainder[48-50,81,82,84,86] use the relativistic equations. The effect of the relativistic kinematics is apparent even for the πd scattering length,[81] and therefore scattering of pions by the deuteron should be studied on the basis of the relativistic three-particle equations. The equations used in Refs. 82 and 86 were obtained in Ref. 87 by the Blankenbecler–Sugar method,[88] and in Refs. 48 and 49 equations obtained by this method in the isobar model[89] were used. With regard to Refs. 50, 81, and 84, the problem was investigated in these papers on the basis of the three-particle equations obtained in Ref. 90 in the framework of the Logunov–Tavkhelidze quasipotential approach.[91] We emphasize that in all these relativistic equations all particles are on the mass shell. The relativistic equations are potential equations, i.e., they are similar to the equations in relativistic potential theory discussed in Secs. 1 and 2 of this paper. But in contrast to this theory, in which all the propagators are linear in the energies, the equations used in Refs. 48, 49, 82, and 86 contain propagators quadratic in the energy and, as a consequence, the so-called cluster properties of the Faddeev equations are violated.

Note that this difficulty in equations of this kind can be avoided if one uses Wightman–Gårding relative coordinates[92] (so far as we know, this formalism has not hitherto been applied to πd scattering). With regard to the relativistic three-particle equations investigated in Refs. 50, 81, and 84, this difficulty does not arise for them because their propagators are linear in the energies. These equations are the closest to the Faddeev equations and admit a natural passage to the limit of them, and they are therefore close to the equations of relativistic potential theory. But there is a difference: in relativistic potential theory the potential does not depend on the energy [see (79)], but the quasipotential does.

The equations for the problem of scattering of a pion (particle 1) by a deuteron with allowance for the identity of the nucleons (particles 2 and 3) used in Refs. 50 and 84 have the form[81]

$$\mathcal{P}_{23}\tilde{u}_{11}(P_0)\,\mathcal{P}_{23} = \mathcal{P}_{23}\tilde{T}_2(P_0)\,\tilde{G}_0(P_0)\,\tilde{u}_{2+3,\,1}(P_0)\,\mathcal{P}_{23}; \quad (293)$$

$$\tilde{u}_{2+3,\,1}(P_0)\,\mathcal{P}_{23} = 2\tilde{G}_0^{-1}(P)\,\mathcal{P}_{23} + 2\tilde{T}_1(P_0)\,\tilde{G}_0(P_0)\,\mathcal{P}_{23}\tilde{u}_{11}(P_0)\,\mathcal{P}_{23} - P_{23}\tilde{T}_2(P_0)\,\tilde{G}_0(P_0)\,\tilde{u}_{2+3,\,1}(P_0)\,\mathcal{P}_{23}. \quad (294)$$

where P_{23} is the operator of transposition of the nucleons, P_0 is the total energy of the system, $G_0(P_0)$ is the Green's function, and

$$\mathcal{P}_{23} = \mathcal{P}_{23}^2 = \mathcal{P}_{23}^+ = (1 - P_{23})/2; \quad (295)$$

$$\tilde{u}_{2+3,\,1}(P_0) = \tilde{u}_{21}(P_0) - P_{23}\tilde{u}_{31}(P_0). \quad (296)$$

Here $\tilde{T}_1$ and $\tilde{T}_2$ are, respectively, the two-particle collision matrices for the pairs (3, 3) and (3, 1) in the

center-of-mass system of the three particles; $\tilde{u}_{11}$ is the operator of the transition $\pi d \to \pi d$; $\tilde{u}_{2+3,1}$ is an auxiliary operator. The operator of the transition $1+(2,3) \to 1+2+3$, i.e., the operator of pion scattering by the deuteron with breakup of the latter, can be expressed in terms of the solutions of the system of equations (293)–(294) in the form

$$\mathcal{P}_{23}\tilde{u}_{01}(P_0)\mathcal{P}_{23} = \tilde{G}_0^{-1}(P_0)\mathcal{P}_{23} + \mathcal{P}_{23}\tilde{T}_2(P_0)\tilde{G}_0(P_0)\tilde{u}_{2+3,1}(P_0)\mathcal{P}_{23} + \mathcal{P}_{23}\tilde{T}_1(P_0)\tilde{G}_0(P_0)\mathcal{P}_{23}\tilde{u}_{11}(P_0)\mathcal{P}_{23}. \quad (297)$$

The problem of finding the connection between the two-particle collision matrices in the center-of-mass system of the three particles and the same matrices in the center-of-mass systems of the corresponding pairs can be solved in the present approach much more simply than in the relativistic potential theory considered in the second section of this paper. To demonstrate this, we note that in the Jacobi coordinates

$$\mathbf{p}_{jk} = \frac{m_k\mathbf{p}_j - m_j\mathbf{p}_k}{m_j + m_k}; \quad p^0_{jk} = \frac{m_k p^0_j - m_j p^0_k}{m_j + m_k}, \quad ijk = 123, 231, 312; \quad (298)$$

$$\mathbf{q}_i = \frac{m_i(\mathbf{p}_j + \mathbf{p}_k) - (m_j + m_k)\mathbf{p}_i}{m_i + m_j + m_k} = \mathbf{P}_i = \mathbf{p}_j + \mathbf{p}_k = -\mathbf{p}_i, \quad (299)$$

where $\mathbf{p}_i$, $\mathbf{p}_j$, $\mathbf{p}_k$ and m_i, m_j, m_k are the momenta and masses of the particles, and p_i^0 is the fourth component of the four-momentum $p_i = (p_i^0, \mathbf{p}_i)$ in the center-of-mass system of the three particles. The two-particle matrix T_i in this system has the form[81]

$$\langle \mathbf{p}'_{jk}\mathbf{q}'_i \mid \tilde{T}_i(P_0) \mid \mathbf{p}_{jk}\mathbf{q}_i\rangle = \omega_i(\mathbf{q}_i)\,\delta(\mathbf{q}'_i - \mathbf{q}_i)\,\tilde{T}_i(\mathbf{p}'_{jk}\mathbf{p}_{jk}; P_{0i}\mathbf{P}_i), \quad (300)$$

where

$$\omega_i(p) = \sqrt{m_i^2 + \mathbf{p}^2}; \quad P_{0i} = P_0 - \omega_i(q_i); \quad (301)$$

$\tilde{T}_i(\mathbf{p}'_{jk}\mathbf{p}_{jk}; P_{0i}\mathbf{P}_i)$ satisfies the quasipotential equation[90]

$$\tilde{T}_i(\mathbf{p}'_{jk}\mathbf{p}_{jk}; P_{0i}\mathbf{P}_i) = \tilde{V}_i(\mathbf{p}'_{jk}\mathbf{p}_{jk}; P_{0i}\mathbf{P}_i) + \int \frac{\tilde{V}_i(\mathbf{p}'_{jk}\mathbf{p}''_{jk}; P_{0i}\mathbf{P}_i)\,\tilde{T}_i(\mathbf{p}''_{jk}\mathbf{p}_{jk}; P_{0i}\mathbf{P}_i)\,d\mathbf{p}''_{jk}}{2\omega_j(p''_j)\,\omega_k(p''_k)\,[P_{0i} + i0 - \omega_j(p''_j) - \omega_k(p''_k)]}. \quad (302)$$

The connection between this matrix and the same matrix in the center-of-mass system of pair jk has the form[81]

$$\tilde{T}_i(\mathbf{p}'_{jk}\mathbf{p}_{jk}; P_{0i}\mathbf{P}_i) = a_i(q'_{jk}q_jP_0)\sqrt{\omega_j(q'_{jk})\,\omega_k(q'_{jk})}\,T_i(\mathbf{q}'_{jk}\mathbf{q}_{jk}; P_i^2) \times \sqrt{\omega_j(q_{jk})\,\omega_k(q_{jk})}\,a_i(q_{jk}q_iP_0) \quad (303)$$

(similarly for $\tilde{V}_i$), where

$$a_i(q_{jk}q_iP_0) = \begin{cases} \left[\dfrac{[S_{jk}(q) + \mathbf{q}_i^2]^{1/2}[\sqrt{S_{jk}(q)} + \sqrt{P_i^2}]}{\sqrt{S_{jk}(q)}\,[P_{0i} + \sqrt{S_{jk}(q) + \mathbf{q}_i^2}]}\right]^{1/2}, & P_i^2 > 0; \\ \left[\dfrac{[S_{jk}(q) - \mathbf{q}_i]^{1/2}[\sqrt{S_{jk}(q) - \mathbf{q}_i^2} - P_{0i}]}{\sqrt{S_{jk}(q)}\,[\sqrt{S_{jk}(q) - \sqrt{-P_i^2}}]}\right]^{1/2}, & P_i^2 < 0; \end{cases} \quad (304)$$

$$P_i^2 = P_{0i}^2 - \mathbf{P}_i^2 = [P_0 - \omega_i(q_i)]^2 - \mathbf{q}_i^2;$$

$$S_{jk}(q) = [\omega_j(q_{jk}) + \omega_k(q_{jk})]^2 = [\omega_j(p_j) + \omega_k(p_k)]^2 - (\mathbf{p}_j + \mathbf{p}_k)^2, \quad (305)$$

and the matrix $T_i(q'_{jk}q_{jk}; P_i^2)$ is determined by an equation of Lippmann–Schwinger type:

$$T_i(\mathbf{q}'_{jk}\mathbf{q}_{jk}; P_i^2) = V_i(\mathbf{q}'_{jk}\mathbf{q}_{jk}; P_i^2) + \int \frac{V_i(\mathbf{q}'_{jk}\mathbf{q}''_{jk}; P_i^2)\,T_i(\mathbf{q}''_{jk}\mathbf{q}_{jk}; P_i^2)\,d\mathbf{q}''_{jk}}{\xi(P_i^2) + i0 - \omega_j(\mathbf{q}''_{jk}) - \omega_k(\mathbf{q}''_{jk})}; \quad (306)$$

$$\xi(P_i^2) = \begin{cases} \sqrt{P_i^2}, & P_i^2 > 0; \\ -\sqrt{-P_i^2}, & P_i^2 < 0. \end{cases} \quad (307)$$

The relative momentum of pair jk in (303)–(307) is determined by the expression[81]

$$\mathbf{q}_{jk} = -\mathbf{q}_j - \varphi_{jk}(\mathbf{q}_i\mathbf{q}_j)\,\mathbf{q}_i; \quad (308)$$

$$\varphi_{jk}(\mathbf{q}_i\mathbf{q}_j) = \frac{\omega_i(q_j)}{\sqrt{S_{jk}(q)}} + \frac{\mathbf{q}_i\mathbf{q}_j}{\sqrt{S_{jk}(q)}\,[\omega_j(q_j) + \omega_k(|\mathbf{q}_i + \mathbf{q}_j|) + \sqrt{S_{jk}(q)}]}, \quad (309)$$

which is obtained by a Lorentz transformation (84) on the momentum $\mathbf{p}_j = -\mathbf{q}_j$. Similarly, for the relative momentum of pair ki we have the expression

$$\mathbf{q}_{ki} = \mathbf{q}_i - \varphi_{ki}(\mathbf{q}_i\mathbf{q}_j)\,\mathbf{q}_j, \quad (310)$$

which is obtained from the expression for $\mathbf{q}_{ki}$ given in Ref. 81 by some simple transformations. Note that if the four-momentum $p_{jk} = (p^0_{jk}, \mathbf{p}_{jk})$ is defined, not by (298), but in accordance with

$$p_{jk} = \frac{\omega_k(|\mathbf{p}_k|)\,p_j - \omega_j(|\mathbf{p}_j|)\,p_k}{\omega_j(|\mathbf{p}_j|) + \omega_k(|\mathbf{p}_k|)}, \quad (311)$$

then as a result of a Lorentz transformation on this four-momentum we obtain for the relative momentum $\mathbf{q}_{jk}$ an expression of the form (80), (81). However, the form of the equations found in Ref. 81 remains unchanged.

Note that the expression (303), which holds for the off-shell two-particle collision matrix, is much simpler than the expression (104) obtained in Ref. 23 in relativistic potential theory. This is so because in the quasipotential approach used in the derivation of (303) the connection between $\tilde{V}_i$ and V_i, which depend on the energy, is given by an expression analogous to (303), whereas in the Lorentz-invariant potential theory the connection between $\mathbf{v}_i$ and v_i, which are assumed to be independent of the energy, is determined by the equation

$$v = \sqrt{(h_{0i}^{cm} - v_i)^2 + \mathbf{P}_i^2} - \sqrt{(h_{0i}^{cm})^2 - \mathbf{P}_i^2} \quad (312)$$

[see (78), (79)], where h_{0i}^{cm} is the Hamiltonian of the noninteracting particles j and k in the center-of-mass system of the pair.

Very important for applications is the fact that in (303) the matrix relating $\tilde{T}_i$ and T_i is not an integral but a simple function and that it is factorized. In this respect, an analogous connection between $\tilde{T}_i$ and T_i is obtained in Ref. 86,[1)] in which one of the variants (noncovariant) of the three-particle equations of Ref. 87 is used.

The problem of going over from the center-of-mass system of two particles to the center-of-mass system for three particles for the equations used in Refs. 48 and 49 reduces merely to the correct choice of the relative momenta of the pair of particles, this being due to the covariance of the corresponding equations in the isobar model of Ref. 87. One of the two variants of relative momenta used in Refs. 48 and 49 coincides with (310) and (311).

As we said above, one of the questions that can be studied by applying the three-particle equations to πd scattering is the question of the convergence of the

[1)] The quantity $\omega_0(q_i) = \sqrt{P_i^2}$ in (1.2) in Ref. 86 requires definition in accordance with (307).

multiple scattering series. In Ref. 48, an interesting result relating to πd scattering in the region of the (3, 3) resonance was obtained: the multiple scattering series converges much more rapidly in the three-particle approach than does the analogous series in the FSA. Thus, the reason for the poor convergence (divergence) of the series (154) is to be sought in the basic approximation (109) in which it is obtained. However, the multiple scattering series still converges slowly in the three-particle approach, this being due to NN rescattering in intermediate states. This effect plays an important part at large scattering angles. Attention was drawn to this circumstance for the first time in Ref. 47. Subsequently, this result was confirmed to a greater or lesser extent in subsequent calculations, and according to Ref. 50 the main contribution to the cross section from multiple scattering is due to NN rescattering; this means that in the region of the (3, 3) resonance the last term on the right-hand side of (294) can be ignored, and as a result we obtain from Eqs. (293) and (294) a separate equation for elastic scattering:

$$\mathcal{P}_{23}\tilde{u}_{11}\mathcal{P}_{23}=2\mathcal{P}_{23}\tilde{T}_2\mathcal{P}_{23}+2\mathcal{P}_{23}\tilde{T}_2\tilde{G}_0\tilde{T}_1\tilde{G}_0\mathcal{P}_{23}\tilde{u}_{11}\mathcal{P}_{23}. \quad (313)$$

Bearing in mind that $\tilde{G}_0\tilde{T}_1\tilde{G}_0=\tilde{G}_1-\tilde{G}_0$, where $\tilde{G}_1$ is the total Green's function of the first channel ($\pi d\to\pi d$), we obtain from (313)

$$\mathcal{P}_{23}\tilde{u}_{11}\mathcal{P}_{23}=2\mathcal{P}_{23}\tilde{T}_2\mathcal{P}_{23}+2\mathcal{P}_{23}\tilde{T}_2(\tilde{G}_1-\tilde{G}_0)\mathcal{P}_{23}\tilde{u}_{11}\mathcal{P}_{23}. \quad (314)$$

The operator $\tilde{G}_1$ coincides with the operator G (4) for the πd system, and therefore (314) is an equation equivalent to the Lippmann–Schwinger equations for πd scattering with allowance for the identity of the nucleons. It is clearly much simpler to solve Eqs. (314) or (313) than the system of equations (293) and (294). We emphasize once more that Eqs. (313) and (314) are approximate.

Let us briefly consider the part played by the relativistic kinematics in πd scattering in the region of the (3, 3) resonance. This question was considered in Refs. 49 and 50, and it was shown that for the πN system it is necessary to take into account the relativistic kinematics,[49] this leading to an appreciable effect even for forward scattering, and that the effect of complete allowance for the relativistic kinematics for the pion in the cross sections is of the same order as the effect of multiple scattering.[50]

One of the main aims of studying the pion–nucleus interaction is, as is well known, to obtain information about the off-shell behavior of the πN collision matrix. Application of the three-particle equations to πd scattering gives a unique possibility for extracting such information, since these equations directly contain the off-shell free πN collision matrix. This question was investigated for the scattering of pions with both zero energy[84] and energies in the region of the (3, 3) resonance.[50] For this purpose, two different πN collision matrices were used: one of them[83] corresponds to a potential of the form (62) with parameters chosen to describe the experimental data up to 300 MeV, and the other[17] is obtained by solving the inverse scattering problem and corresponds to the energy-dependent potential (65). The two matrices differ in their off-shell behavior even in the region of momenta $\leqslant 400$ MeV/c, which, as calculations show, is the main region in the integration when the integral equations (293) and (294) are solved.

It was shown in Ref. 50 that πd scattering in the region of the (3, 3) resonance is definitely sensitive to the off-shell behavior of the πN collision matrix, and the effect for the behavior of the πN collison matrix, and the effect for the πd scattering length was found to be more pronounced.[84] We mention here the weak sensitivity found in Ref. 86 of the πd scattering cross sections to the off-shell behavior of the πN collision matrix. This difference between the results is evidently due to the fact that in Ref. 86 the interaction parameters were chosen to give a good description of the πN scattering data only in the region of the (3, 3) resonance, the t matrix being taken in the form (70), and variation in the off-shell behavior was achieved only by changing the range of the πN interaction, i.e., by changing the parameter of the form factor $\tilde{v}_1(p)$. The construction of the t matrix used in Ref. 50 took into account the phase shifts in a much wider range of energies ($\leqslant 9$ GeV), and it is probable that in this case the effect of the off-shell behavior is not restricted to just the range of the πN interaction.

We now discuss the problem of allowing for the pion absorption channel $\pi d\to NN$ in πd scattering, which in Ref. 77 was considered on the basis of the channel-coupling method proposed in Ref. 93. Correct allowance for absorption of the pion by the system of nucleus requires that there should be no overcounting of the events of pion absorption and emission by the nucleon ($\pi N\rightleftarrows N$). The point is that, from the point of view of the field approach, the $\pi N\rightleftarrows N$ vertex participates in both the πN and NN interactions and in the real absorption of the pion by the system of nucleons. To avoid overcounting the $\pi N\rightleftarrows N$ vertices, Mizutani and Koltun[77] proceed from the Hamiltonian of a system with a fixed number of nucleons (in the considered case a system of two nucleons) and an arbitrary number of pions:

$$H=H_0+V_{\pi N}+V_{NN}+R+R^+,$$

where H_0 is the kinetic energy operator of the particles, V_{NN} is the operator of the interaction between the nucleons, not including pion exchange but including the exchange of heavy mesons (ρ, ω, etc.), R and R^+ are the operators of annihilation and creation of a pion by an individual nucleon, and $V_{\pi N}$ is the operator of the πN interaction that cannot be generated by any combination of the operators R and R^+. For example, $V_{\pi N}$ includes interaction through the exchange of heavy mesons. Two coupled channels were considered—the mesonless channel (with two nucleons and no pion) and the single-meson channel (with two nucleons and one pion); the remaining channels were taken into account effectively.

In Ref. 77, an expression was obtained for the T matrix of transitions between single-meson states ($\pi NN\rightleftarrows\pi NN$):

$$T_{11}(E)=T_{11}^F(E)+\langle\Psi_1^{(-)}\mid R_{10}[E+i0-h_0-W_{NN}(E)]^{-1}R_{01}\mid\Psi_1^{(-)}\rangle. \quad (315)$$

Here, T_{11}^F is the matrix for the $\pi NN\rightleftarrows\pi NN$ transitions obtained by solving the three-particle equations with

certain effective πN and NN interactions (in the general case, the three-particle πNN interaction is also taken into account), $|\Psi_i^{(-)}\rangle$ are the corresponding state vectors, h_0 is the effective kinetic energy operator of two nucleons, $W_{NN}(E)$ is the corresponding interaction operator, which, for example, reproduces the experimental NN phase shifts, and R_{01} and R_{10} are operators that couple the mesonless channel (NN) to the single-meson channel (πNN). The second term in (315) takes into account the contribution to the matrix of $\pi NN \rightleftarrows \pi NN$ transitions from transitions that proceed through the mesonless channel, i.e., it takes into account real absorption of the pion by the NN system.

Thus, Eq. (315) solves (in principle) the problem of taking into account absorption of a pion scattered by a deuteron. On the basis of this formula, the contribution from the second term of (315) to the πd scattering length was calculated in Ref. 77 in a definite model. The value of the correction was found to depend on the details of the model.

CONCLUSIONS

It is evident from the above analysis that the theory of multiple scattering based on relativistic potential theory has wide possibilities for studying pion-nucleus dynamics. From the analysis of the results obtained on the basis of the first-order optical potential, it can be seen that in the construction of the t matrix for the scattering of a pion by a nucleon bound in a nucleus it is necessary to make a further investigation of the part played by the excitation of the target nucleus and the Pauli principle. In this connection, considerable interest attaches to the exact solution of the model problem of the scattering of a pion by a nucleon bound in an external field. At the present time, it is possible to solve (numerically) this problem for an external oscillator field. The t matrix obtained in this manner can be applied to the real problem of pion scattering by light nuclei, for which most of the experimental investigations are made.

Allowance for the pair absorption of pions by nucleons of the target nucleus in the construction of the second-order optical potential is currently an important problem which has a bearing on the problem of obtaining reliable information about the pairing correlation of nucleons in the nucleus.

Calculations of πA scattering on the basis of the second-order optical potential will be informative with regard to the pairing correlation of nucleons if the part played by all effects associated with the first-order optical potential has first been elucidated; in this manner, it will be possible to separate the effects associated exclusively with the second-order optical potential.

Of great interest in our view is the study of the part played by the relativistic kinematics of the pion in πA ($A \geq 3$) scattering; that has proved to be significant for πd scattering.

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