Theory of elastic scattering and bremsstrahlung of fast charged particles in crystals

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A theory of elastic scattering and bremsstrahlung of fast particles in periodic structures is presented. Particular attention is devoted to interference and coherence effects. The scattering and bremsstrahlung are investigated in the framework of the Born approximation and also in the framework of the quasiclassical approximation and classical electrodynamics. The bremsstrahlung of channeled particles is considered specially.

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INTRODUCTION

The problem of the interaction of fast charged particles with matter has always attracted much attention of both theoreticians and experimentalists. Of particular imporantance in this problem is the interaction of fast charged particles with crystals, since the periodic structure of the crystals can lead to the occurrence of interference and coherence effects. Because of these effects, the bremsstrahlung photons emitted when particles pass through a crystal have a high degree of monochromaticity and are linearly polarized. This accounts for the practical importance of the interaction of relativistic electrons with crystals for high-energy physics. There is however a further, purely theoretical aspect to the question. The point is that although the interaction of a fast electron with the individual atoms of the crystal is determined by the parameter $Ze^2/\hbar v$ ($Z|e|$ is the charge of the nucleus and v the velocity of the particle), so that it would seem permissible to use the first Born approximation of quantum perturbation theory at small Z, the influence of the large number of atoms of the crystal with which the electron interacts has the consequence that the first Born approximation rapidly becomes invalid. Thus, we need to make an investigation more accurate than the Born approximation of the interaction of a fast charged particle with a crystal lattice. An important fact then is that the fast particle interacts effectively with all the atoms that are in the zone in which the radiation is formed. The length of this zone is determined by $l \sim 2\varepsilon\varepsilon'/m^2\omega$, where ε is the energy of the incident particle, m is its mass, ω is the frequency of the photon, and $\varepsilon' = \varepsilon - \hbar\omega$. (Here and in what follows, we use a system of units in which the velocity of light is unity, $c=1$.) If there are N_l atoms in this zone, then the effective interaction parameter is not $Ze^2/\hbar v$, but $N_l Ze^2/\hbar v$, and this quantity can, in general, take arbitrary values, so that the interaction of the particle with the crystal can be either very weak or very strong. The nature of the elastic scattering and bremsstrahlung is accordingly very different.

In the present paper, we present the theory of elastic and inelastic interactions of fast charged particles with a crystalline medium and we establish how the scattering and bremsstrahlung change as a function of the effective interaction parameter. Simultaneously, we elucidate the status of the results obtained in the framework of the Born approximation in the general picture of scattering and bremsstrahlung of relativistic particles.

Since the particles are assumed to be fast, we can use the quasiclassical approximation, and this enables us to obtain the results of both the Born approximation and the classical theory of scattering and bremstrahlung.

First, we present a theory of elastic scattering of fast particles in the quasiclassical approximation and then the classical theory of radiation and quasiclassical theory of bremsstrahlung of an electron on the simplest periodic structure consisting of a linear chain of atoms. This makes it possible to establish all the characteristic features in the dependence of the scattering and bremsstrahlung cross sections on the effective interaction parameter and the energy of the particle. We then describe the theory of coherent bremsstrahlung of relativistic electrons (and positrons) in single crystals. The paper ends with a consideration of the bremsstrahlung of a fast particle under channeling conditions.

1. ELASTIC SCATTERING OF FAST PARTICLES IN THE QUASICLASSICAL APPROXIMATION

The wave function of a particle in the quasiclassical approximation in an external field $U(\mathbf{r})$ has the form[1]

$$\Psi(\mathbf{r}) = f(\mathbf{r}) \exp(iS(\mathbf{r})/\hbar), \tag{1}$$

where $S(\mathbf{r})$ is the classical action determined by the equation

$$(\nabla S)^2 = [\varepsilon - U(\mathbf{r})]^2 - m^2, \tag{2}$$

and $f(\mathbf{r})$ is a bispinor satisfying the equation

$$2(\nabla S)(\nabla f) + (\nabla^2 S) f + \gamma_0 \boldsymbol{\gamma} (\nabla U) f = i\hbar \nabla^2 f. \tag{3}$$

For fast particles, S can be written in the form $S = \mathbf{pr} + \chi(\mathbf{r})$, where $\mathbf{p}$ is the momentum of the incident particle, and one can seek χ and f as series in inverse powers of the momentum[2]:

$$\left.\begin{aligned} \chi(\mathbf{r}) &= \chi_0(\mathbf{r}) + \chi_1(\mathbf{r}) + \ldots; \\ f(\mathbf{r}) &= f_0 + f_1(\mathbf{r}) + \ldots \end{aligned}\right\} \tag{4}$$

Solving Eqs. (2) and (3), we find

$$
\left.\begin{aligned}
\chi_0(\mathbf{r}) &= \frac{(-1)}{v}\int_{-\infty}^{z} U(\boldsymbol{\rho}, z')\,dz';\\
\chi_1(\mathbf{r}) &= -\frac{1}{2p}\int_{-\infty}^{z} [(\nabla\chi_0)^2 - U^2]\,dz';\\
f_0 &= u;\\
f_1(\mathbf{r}) &= -\frac{1}{2p}\int_{-\infty}^{z} [\nabla^2\chi_0 + \gamma_0\boldsymbol{\gamma}(\nabla U)]\,u\,dz',
\end{aligned}\right\} \tag{5}
$$

where ρ is the impact parameter (the z axis is parallel to the momentum $\mathbf{p}$ of the particle), $v = p/\varepsilon$, u is the bispinor of a free electron, and γ_0 and $\boldsymbol{\gamma}$ are Dirac matrices satisfying $\gamma_0^+ = \gamma_0$ and $\boldsymbol{\gamma}^+ = -\boldsymbol{\gamma}$. [It is assumed that in the limit $z \to -\infty$ the wave function has the plane-wave form: $\psi(\mathbf{r}) \to u\exp(i\mathbf{pr}/\hbar)$.]

Knowing the wave function $\psi(\mathbf{r}) = \varphi(\mathbf{r})\exp(i\mathbf{pr}/\hbar)$ of the particle, we can find the amplitude $a(\mathbf{q})$ of elastic scattering of the particle in the field $U(\mathbf{r})$:

$$a(\mathbf{q}) = \frac{-1}{4\pi\hbar^2}\int d\mathbf{r}\exp(i\mathbf{qr}/\hbar)\,U(\mathbf{r})\,\overline{u}'\gamma_0\varphi(\mathbf{r}), \tag{6}$$

where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ is the momentum transfer and u' is the bispinor of the scattered particle ($\overline{u}' = u'^*\gamma_0$). The differential cross section $d\sigma_e(\mathbf{q})$ of elastic scattering of the particle into the element dO' of solid angle is related to $a(\mathbf{q})$ by

$$d\sigma_e(\mathbf{q}) = |a(\mathbf{q})|^2\,dO'. \tag{7}$$

The integrated cross section of elastic scattering is, in accordance with the optical theorem, equal to

$$\sigma_e = 4\pi(\hbar/p)\,\mathrm{Im}\,a(\mathbf{q})|_{\mathbf{q}=0}. \tag{8}$$

Using the quasiclassical wave function, we can obtain the following expression for the amplitude of elastic scattering[2,3]:

$$
\begin{aligned}
a(\mathbf{q}_\perp) = {} & i\overline{u}'\gamma_0 u\frac{v}{4\pi\hbar}\int d\boldsymbol{\rho}\exp(i\mathbf{q}_\perp\boldsymbol{\rho}/\hbar)\\
& \times\{1 - \exp[i(\chi_0(\rho) + \chi_1(\rho) + \ldots)/\hbar]\},
\end{aligned} \tag{9}
$$

where

$$
\left.\begin{aligned}
\chi_0(\rho) &= -\frac{1}{v}\int_{-\infty}^{\infty} U(\boldsymbol{\rho}, z)\,dz;\\
\chi_1(\rho) &= \frac{1}{pv^2}\int_{-\infty}^{\infty}\left[z(\nabla_\perp U)\left(\nabla_\perp\int_{-\infty}^{z} U(\boldsymbol{\rho}, z')\,dz'\right) - \frac{1-v^2}{2}U^2\right]dz
\end{aligned}\right\} \tag{10}
$$

($\mathbf{q}_\perp$ is the projection of the vector $\mathbf{q}$ onto the plane perpendicular to the z axis, and $\nabla_\perp = \partial/\partial\boldsymbol{\rho}$).

For an axially symmetric potential, the elastic scattering amplitude (9) and the total cross section (8) are equal to

$$a(\mathbf{q}_\perp) = i\overline{u}'\gamma_0 u\frac{v}{2\hbar}\int_0^{\infty}\rho\,d\rho\,J_0(q_\perp\rho/\hbar)\{1 - \exp[i(\chi_0 + \chi_1 + \ldots)/\hbar]\}; \tag{11}$$

$$\sigma_e = 4\pi\,\mathrm{Re}\int_0^{\infty}\rho\,d\rho\{1 - \exp[i(\chi_0 + \chi_1 + \ldots)/\hbar]\}. \tag{12}$$

Equation (9) for the scattering amplitude is valid under the conditions

$$|\overline{U}|/pv \ll 1;\; z_{\mathrm{eff}}|\chi_0|/p\rho^2 \ll 1;\; \hbar z_{\mathrm{eff}}/p\rho^2 \ll 1, \tag{13}$$

where z_{eff} is the range of the coordinate z in which the external force acts on the particle. For example, in the case of scattering on an individual atom z_{eff} is in order of magnitude equal to the screening length R.[1)]

We emphasize that the conditions (13) are satisfied with greater accuracy, the higher is the energy of the particle.

The quantity χ_0 in (9) can be arbitrary, but two limiting cases are of particular interest: $|\chi_0| \ll \hbar$ and $|\chi_0| \gg \hbar$; the first corresponds to the Born approximation and the second to classical mechanics. In the first case, in accordance with (11),

$$d\sigma_e(\mathbf{q}_\perp) = \frac{2\pi}{\hbar^4}\left|\int_0^{\infty}\rho\,d\rho\,J_0(q_\perp\rho/\hbar)(\chi_0(\rho) + \chi_1(\rho) + \ldots)\right|^2 q_\perp\,dq_\perp. \tag{14}$$

Ignoring here χ_1, we obtain the elastic scattering cross section in the first Born approximation. If we take into account χ_1 but ignore χ_1^2, we obtain the elastic scattering cross section with allowance for the second Born approximation.

In the limiting case $|\chi_0| \gg \hbar$ and $|\chi_1| \gg \hbar$ (but $|\chi_0| \gg |\chi_1|$), the differential cross section of elastic scattering (7) through angles ϑ_e satisfying the condition $p\vartheta_e\rho(\vartheta_e) \gg \hbar$ (i.e., for $q_\perp\rho \gg \hbar$) agrees with the result of the classical theory of scattering,

$$d\sigma_e(\vartheta_e) = 2\pi\rho(\vartheta_e)\,|d\rho(\vartheta_e)/d\vartheta_e|\,d\vartheta_e, \tag{15}$$

where $\rho(\vartheta_e)$ is the classical impact parameter as a function of the scattering angle. Indeed, if $|\chi_0| \gg \hbar$, $|\chi_1| \gg \hbar$, and $q_\perp\rho \gg \hbar$, the integral that determines the elastic scattering amplitude (11) can be calculated by the method of steepest descent[5]:

$$
\begin{aligned}
& \int_0^{\infty}\rho\,d\rho\,J_0(q_\perp\rho)\{1 - \exp[i(\chi_0(\rho) + \chi_1(\rho) + \ldots)/\hbar]\}\\
& \approx \hbar\sqrt{\frac{\tilde{\rho}}{q_\perp}\left|\frac{d\tilde{\rho}}{dq_\perp}\right|}\exp[i(-q_\perp\tilde{\rho} + \chi_0(\tilde{\rho}) + \chi_1(\tilde{\rho}) + \ldots)/\hbar],
\end{aligned}
$$

where the function $\tilde{\rho} = \rho(q_\perp)$ is determined by the relation

$$q_\perp = \left|\frac{d}{d\rho}(\chi_0(p) + \chi_1(\rho) + \ldots)\right|. \tag{16}$$

Substituting this expression in (7) and averaging over the spin states of the electron, we obtain (15).

Thus, the quasiclassical scattering amplitude (11) admits the passage to the limit of the region of applicability of the Born approximation and the region of applicability of classical mechanics.

If in (9) we retain $\chi_0(\rho)$ but ignore $\chi_1(\rho)$, we can obtain the amplitude of elastic scattering in the eikonal approximation[4,6,7]:

$$a(\mathbf{q}_\perp) = i\overline{u}'\gamma_0 u\frac{v}{4\pi\hbar}\int d\boldsymbol{\rho}\exp(i\mathbf{q}\boldsymbol{\rho}/\hbar)\{1 - \exp(i\chi_0(\rho)/\hbar)\}. \tag{17}$$

We see that $\chi_1(\rho)$ determines the correction to the eikonal approximation proportional to p^{-1}.

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1) For scattering on an individual atom in the case $|\chi_0| \lesssim \hbar$ Schiff[4] obtained a restriction on the scattering angles ϑ_e for which an expansion in inverse powers of the momentum of the particle is valid, namely, $\vartheta_e < (pR)^{-1/2}$. It is easy to see that this restriction is contained in the inequalities (13), since in the considered case $z_{\mathrm{eff}} \sim R$ and $\rho \sim \hbar/p\vartheta_e$.

To conclude this section, we note that Eq. (9) for the elastic scattering amplitude obtained using the quasiclassical wave functions encompasses the well-known results of other authors who obtained the scattering amplitude by other methods.[8-14,16] For example, in the nonrelativistic approximation for $|\chi_1| \ll \hbar$ the correction to the eikonal scattering amplitude of a spinless particle was obtained in Ref. 8 by Wu using the method proposed by Schiff[4] for summing the Born series for the scattering amplitude. Wu's result agrees with Eq. (9) if in it we expand the exponential $\exp[i\chi_1(\rho)/\hbar]$, ignoring in $\chi_1(\rho)$ the terms of order v/c and replacing $\bar{u}'\gamma_0 u$ by 2ε (this substitution corresponds to the transition to a spinless particle).

In Refs. 9 and 10, a method of eikonal expansion of the elastic scattering amplitude was developed. This method was then used by Wallace[11-13] to find the scattering amplitude of a fast spinless particle. To terms of order p^{-2}, Wallace's result agrees with Eq. (9) if $\bar{u}'\gamma_0 u$ in it is replaced by 2ε.

Baker[14,15] found the eikonal wave function of a fast nonrelativistic spinless particle in an external centrally symmetric field with correction of order p^{-1} by solving an integral equation. Using this wave function, he obtained the elastic scattering amplitude. Baker's result agrees with Eq. (9) if in the latter we ignore in the expression for $\chi_1(\boldsymbol{\rho})$ the terms v/c, use the central symmetry of the potential, and replace $\bar{u}'\gamma_0 u$ by 2ε.

The elastic scattering amplitude (9) with allowance for $\chi_1(\boldsymbol{\rho})$ was obtained recently by the method of the operator quasiclassical approximation developed in the investigations of Baĭer, Katkov, and Fadin.[16,17]

2. CLASSICAL THEORY OF THE RADIATION OF A FAST PARTICLE

Coherent effects accompanying bremsstrahlung in periodic structures can already be described by classical radiation theory, and we therefore present here the results of the classical theory of the radiation of a fast charged particle moving in an arbitrary field.

The energy emitted by a fast particle moving along a trajectory $\mathbf{r} = \mathbf{r}(t)$ in the frequency interval $d\omega$ into the element dO of solid angle is determined in classical electrodynamics by the formula[18]

$$d\mathscr{E}_{\mathbf{k}} = \frac{e^2}{(2\pi)^2} |\mathbf{k} \times \mathbf{l_k}|^2 \, dO \, d\omega, \tag{18}$$

where

$$\mathbf{l_k} = \int_{-\infty}^{\infty} dt \, \mathbf{v}(t) \exp[i(\omega t - \mathbf{kr}(t))]; \tag{19}$$

$\mathbf{v}(t) = \dot{\mathbf{r}}(t)$, and $\mathbf{k}$ is the wave vector of the electromagnetic wave.

If an external electric field $\mathbf{E}(\mathbf{r})$ acts on the particle, the particle velocity $\mathbf{v}(t)$ is determined by the equation

$$\frac{d}{dt} \frac{m\mathbf{v}(t)}{\sqrt{1 - \mathbf{v}^2(t)}} = e\mathbf{E}(\mathbf{r}). \tag{20}$$

In the case of high energies in which we are interested, we can seek a solution of this equation as an expansion in inverse powers of the energy ε of the incident particle[19,20]:

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{v}_1(t) + \mathbf{v}_2(t) + \ldots, \tag{21}$$

where $\mathbf{v}_0$ is the velocity of the incident particle (i.e., the velocity of the particle as $t \to -\infty$) and $v_n(t) \sim \varepsilon^{-}$. To find the individual terms of this expansion, it is convenient to proceed from the formal solution of Eq. (20):

$$\mathbf{v}(t) = \left(\mathbf{v}_0 + \varepsilon^{-1} \int_{-\infty}^{t} e\mathbf{E}(\mathbf{r}')\, dt'\right) \left(1 + \varepsilon^{-1} \int_{-\infty}^{t} e\mathbf{v}(t')\, \mathbf{E}(\mathbf{r}')\, dt'\right)^{-1}, \tag{22}$$

where $\mathbf{r}' = \mathbf{r}(t')$ (in deriving (22), we have used the relation $d[m(1-v^2)^{-1/2}]/dt = e\mathbf{Ev}$). To the expansion (21) for the velocity there corresponds the expansion of the particle's radius vector

$$\mathbf{r}(t) = \boldsymbol{\rho} + \mathbf{v}_0 t + \int_{-\infty}^{t} \mathbf{v}_1(t')\, dt' + \ldots,$$

where ρ is the impact parameter. It is readily seen that the first terms of the expansion are

$$\left.\begin{aligned} \mathbf{v}_1(t) &= \varepsilon^{-1} \int_{-\infty}^{t} e\,(\mathbf{E}' - \mathbf{v}_0(\mathbf{v}_0\mathbf{E}'))\, dt'; \\ \mathbf{v}_2(t) &= -\frac{1}{2}\mathbf{v}_0(\mathbf{v}_{1\perp})^2 - \mathbf{v}_{1\perp}\varepsilon^{-1} \int_{-\infty}^{t} e\mathbf{v}_0\mathbf{E}'\, dt' \\ &+ \int_{-\infty}^{t} dt' \int_{-\infty}^{t'} dt''\, (\mathbf{v}_{1\perp}(t'')\nabla)\, \mathbf{v}_{1\perp}(t'), \end{aligned}\right\} \tag{23}$$

where $\mathbf{E}' = \mathbf{E}(\boldsymbol{\rho} + \mathbf{v}_0 t)$ and $\mathbf{v}_{1\perp}$ is the component of the velocity $\mathbf{v}_1(t)$ perpendicular to $\mathbf{v}_0$.

Determining the terms of the expansion (21), we can calculate $\mathbf{l_k}$, representing it first in the form

$$\mathbf{l_k} = i \int_{-\infty}^{\infty} dt \exp[i(\omega t - \mathbf{kr}(t))] \frac{d}{dt}\left(\frac{\mathbf{v}(t)}{\omega - \mathbf{kv}(t)}\right). \tag{24}$$

This form of expression for $\mathbf{l_k}$ has the advantage that the integrand contains the acceleration $\dot{\mathbf{v}}$, which is directly determined by the electric field $\mathbf{E}(\mathbf{r})$. Substituting (21) in (24), we find

$$\mathbf{l_k} = i\exp(-i\mathbf{k}\boldsymbol{\rho}) \int_{-\infty}^{\infty} dt \exp\left[i(\omega - \mathbf{kv}_0)t - i\mathbf{k}\int_{-\infty}^{t} \mathbf{v}_1\, dt' - \ldots\right] \times \frac{d}{dt}\left(\frac{\mathbf{v}_0 + \mathbf{v}_1(t) + \ldots}{\omega - \mathbf{kv}_0 - \mathbf{kv}_1(t) - \ldots}\right). \tag{25}$$

If the particle has a high energy, then the emission and scattering occur at small angles. The scattering angle is determined by the expression $\sin\vartheta_e = |\mathbf{v}_\perp(\infty)|/v$. For small angles ($\vartheta_e \ll 1$), using (22) we readily obtain the expression for the scattering angle:

$$\vartheta_e = \vartheta_e(\rho) = p^{-1} | \nabla_\perp (\chi_0(\rho) + \chi_1(\rho) + \ldots)|, \tag{26}$$

where the functions $\chi_0(\rho)$ and $\chi_1(\rho)$ are determined by Eqs. (10), $z = v_0 t$, $\nabla_\perp = \partial/\partial\rho$, $e\mathbf{E} = \nabla U$. With regard to the emission angle ϑ_k, in order of magnitude $\vartheta_k \sim m/\varepsilon$.

If the scattering angle is small compared with the emission angle ($\vartheta_e \ll \vartheta_k$), then $|\mathbf{kv}_1| \ll (\omega - \mathbf{kv}_0)$, and

$$\mathbf{l_k} = \frac{i}{\omega - \mathbf{kv}_0} \exp(-i\mathbf{k}\boldsymbol{\rho})\left(\mathbf{W} + \mathbf{v}_0 \frac{\mathbf{kW}}{\omega - \mathbf{kv}_0}\right)\left[1 + O\left(\frac{\vartheta_e}{\vartheta_k}\right)\right], \tag{27}$$

where

$$\mathbf{W} = \int_{-\infty}^{\infty} dt \exp[i(\omega - \mathbf{kv}_0)t]\, (\dot{\mathbf{v}}_1(t) + \dot{\mathbf{v}}_2(t) + \ldots).$$

Substitution of this expression in (18) gives

$$d\mathscr{E}_{\mathbf{k}}=\frac{e^2}{(2\pi)}\frac{\omega^2}{(\omega-\mathbf{kv})^2}\left[\mathbf{W}_\perp^2-(1-\mathbf{v}_0^2)\frac{(\mathbf{kW}_\perp)^2}{(\omega-\mathbf{kv}_0)^2}\right]dO\,d\omega. \quad (28)$$

We now introduce the effective radiation $dI_{\mathbf{k}}$ (Ref. 18):

$$\frac{dI_{\mathbf{k}}}{d\omega}=\int\frac{d\mathscr{E}_{\mathbf{k}}}{d\omega}\,d\boldsymbol{\rho}. \quad (29)$$

Expanding the potential $U(\mathbf{r})$ in the Fourier integral

$$U(\mathbf{r})=(2\pi)^{-3}\int U(\mathbf{q})\exp(i\mathbf{qr})\,d\mathbf{q} \quad (30)$$

and ignoring the term $\dot{\mathbf{v}}_2$, we obtain

$$\frac{dI_{\mathbf{k}}}{d\omega}=\frac{e^2}{4\pi^3}\frac{\delta'}{m^2}\int dq_\parallel\,d\mathbf{q}_\perp\frac{q_\perp^2}{q_\parallel^2}\left[1-2\frac{\delta'}{q_\parallel}\left(1-\frac{\delta'}{q_\parallel}\right)\right]\times|U(\mathbf{q})|^2[1+O(v_e/v_k,\ v_2/v_1)], \quad (31)$$

where $\delta'=\omega m^2/2\varepsilon^2$, and $q_\parallel$ and $q_\perp$ are the components of the vector $\mathbf{q}$ parallel to and perpendicular to $\mathbf{v}_0$, with $q_\parallel=(\omega-\mathbf{kv}_0)\geqslant\delta'$.

Note that δ' in (31) has a deep physical meaning, namely, it characterizes the size of the region in which the radiation is formed; for since the electron velocity is close to the velocity of light, the photon emitted by the electron when it collides with an atom moves for a certain time together with the electron, and these two particles cannot be regarded as free particles.[21] The length l of the spatial region traversed before the electron "lags" behind the photon and the two particles behave as free particles can be called the zone of formation of the radiation. But the length of this zone is of order $(\delta')^{-1}$. To see this, we note that the longitudinal "dimension" of the electron is of order $r_0(m/\varepsilon)$ $(r_0\sim e^2/m)$, while the "size" of the photon is k^{-1}, and therefore the electron lags behind the photon enough for the spatial regions $r_0(m/\varepsilon)$ and k^{-1} to cease to overlap after a time $\Delta t=(r_0m/\varepsilon+k^{-1})/(c-v)$. During this time, the electron traverses the path $l=v\Delta t$. Thus, we really do have $l\sim(\delta')^{-1}$.

The effective radiation can also be determined in the case when the relationship between ϑ_e and ϑ_k is arbitrary but

$$z_{\rm eff}\ll(\delta')^{-1}, \quad (32)$$

where $z_{\rm eff}$ is the effective distance along $\mathbf{v}_0$ over which the particle is subject to the external field. If the inequality (25) is satisfied, the exponential in the integrand of (25) can be replaced by unity, and we obtain

$$\mathbf{I}_{\mathbf{k}}=i\exp(-i\mathbf{k}\boldsymbol{\rho})\left(\frac{\mathbf{v}(\infty)}{\omega-\mathbf{kv}(\infty)}-\frac{\mathbf{v}_0}{\omega-\mathbf{kv}_0}\right)[1+O(z_{\rm eff}\delta')], \quad (33)$$

and therefore

$$d\mathscr{E}_{\mathbf{k}}=\frac{e^2}{(2\pi)^2}\left|\mathbf{k}\times\left(\frac{\mathbf{v}(\infty)}{\omega-\mathbf{kv}(\infty)}-\frac{\mathbf{v}_0}{\omega-\mathbf{kv}_0}\right)\right|^2dO\,d\omega. \quad (34)$$

after integration over dO, the effective radiation takes the form

$$\frac{dI_\omega}{d\omega}=\omega\int\frac{dW^{cl}(\rho)}{d\omega}\,d\boldsymbol{\rho}, \quad (35)$$

where

$$\frac{dW^{cl}(\rho)}{d\omega}=\frac{2e^2}{\pi\omega}\left[\frac{2\xi^2+1}{\xi\sqrt{\xi^2+1}}\ln\left(\xi+\sqrt{\xi^2+1}\right)-1\right];$$

$\xi=\varepsilon\vartheta_e(\rho)/2m$, where $\vartheta_e(\rho)$ is determined by (26).

Equation (35) is valid if the condition (32) is satisfied. In Addition, the expansion (21) must hold, and for this we require fulfillment of the inequality

$$\overline{U}/pv\ll1;\quad\frac{z_{\rm eff}\chi_0}{p\rho^2}\ll1, \quad (36)$$

where $\overline{U}$ is the mean value of the potential energy.

The second of these inequalities can be represented in the form

$$\Delta\rho\ll\rho,$$

where $\Delta\rho\sim z_{\rm eff}\vartheta_e$ and $\vartheta_e\sim\chi_0/p\rho$. Thus, the second of the inequalities means that the impact parameter must change little within the range of the forces.

3. BREMSSTRAHLUNG OF A FAST PARTICLE IN THE QUASICLASSICAL APPROXIMATION

The effective cross section for the emission of a photon by an electron in an external field in the lowest perturbation order in the interaction with the radiation is determined by the expression[1]

$$d\sigma=\frac{e^2}{4(2\pi)^4}\frac{p'}{p}|\mathbf{eJ}|^2\,\omega\,d\omega\,dO\,dO', \quad (37)$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the initial and final momenta of the electron far from the scatterer, $\mathbf{e}$ is the polarization vector of the photon, and $\mathbf{J}$ is the transition current. If the wave functions of the electron in the initial and final states in the field $U(\mathbf{r})$ are represented in the form

$$\Psi(\mathbf{r})=\varphi(\mathbf{r})\exp(i\mathbf{pr}/\hbar);$$
$$\Psi'(\mathbf{r})=\varphi'(\mathbf{r})\exp(i\mathbf{p'r}/\hbar),$$

then

$$\mathbf{J}=\mathbf{J}(\mathbf{q})=\int d\mathbf{r}\,\varphi'(\mathbf{r})\,\gamma\varphi(\mathbf{r})\exp(i\mathbf{qr}/\hbar) \quad (38)$$

and $\mathbf{q}=\mathbf{p}-\mathbf{p}'-\mathbf{k}$ is the momentum transferred to the external field.

The differential cross section (37) for bremsstrahlung of relativistic particles can be expressed[22] in terms of the components of $\mathbf{q}$:

$$d\sigma(\mathbf{q},y)=\frac{e^2}{(2\pi)^4}\frac{\varepsilon'}{\varepsilon}|\mathbf{eJ}(\mathbf{q})|^2\frac{\delta}{m^2}\frac{dy}{\sqrt{1-y^2}}\frac{d\omega}{\omega}\,dq_z'\,d\mathbf{q}_\perp, \quad (39)$$

where $q_z'=(q_z-q_\perp^2/2\varepsilon)\geqslant\delta$, $\delta=\omega m^2/2\varepsilon\varepsilon'$, and we have replaced the variable ϑ_k by the variable y, $1\geqslant y\geqslant-1$, which is defined by

$$\left(\frac{\varepsilon\vartheta_k}{m}\right)^2=\left(\frac{q_z'}{\delta}-1+\frac{q_\perp^2}{m^2}\right)+y\frac{2q_\perp}{m}\sqrt{\frac{q_z'}{\delta}-1}. \quad (40)$$

Using the wave functions (1) and (4) as $\Psi(\mathbf{r})$ and $\Psi'(\mathbf{r})$, we can find the bremsstrahlung cross section in the quasiclassical approximation. For the investigation of bremsstrahlung in crystals, particular interest attaches to the two cases when $\vartheta_e\ll\vartheta_k$ and $z_{\rm eff}$ and δ^{-1} bear an arbitrary relation to one another, and when $z_{\rm eff}\ll\delta^{-1}$ and the relationship between ϑ_e and ϑ_k is arbitrary. It should be borne in mind that the Born approximation may or may not be satisfied.

We consider first the case when $\vartheta_e\ll\vartheta_k$ and the relationship between $z_{\rm eff}$ and δ^{-1} is arbitrary.[3,22] Then in the first Born approximation, the bremsstrahlung cross section is

$$d\sigma=\frac{e^2}{4\pi^3}\frac{\delta}{m^2}\frac{\varepsilon'}{\varepsilon}\int dq'_z\,d\mathbf{q}_\perp\frac{q_\perp^2}{q_z'^2}\times\left[1+\frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'}-2\frac{\delta}{q'_z}\left(1-\frac{\delta}{q'_z}\right)\right]\times|U(\mathbf{q})|^2\frac{d\omega}{\omega}\left[1+O\left(\frac{\vartheta_e}{\vartheta_k},\ \frac{\chi_1}{\chi_0},\ \frac{\chi_0}{\hbar}\right)\right]. \tag{41}$$

If the condition of applicability of the Born approximation is not satisfied (but $\vartheta_e \ll \vartheta_k$), i.e., χ_0 is comparable with $\hbar$, then the bremsstrahlung cross section is given by

$$d\sigma=\frac{e^2}{\pi}\frac{\delta}{m^2}\frac{\varepsilon'}{\varepsilon}\int dq'_z\,d\mathbf{q}_\perp\frac{q_\perp^2}{q_z'^2}\times\left[1+\frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'}-2\frac{\delta}{q'_z}\left(1-\frac{\delta}{q'_z}\right)\right]\times|A(\mathbf{q})|^2\frac{d\omega}{\omega}\left[1+O\left(\frac{\vartheta_e}{\vartheta_k},\ \frac{\chi_1}{\chi_0}\right)\right], \tag{42}$$

where

$$\left.\begin{aligned}A(\mathbf{q})&=\varepsilon_0 q_\perp^{-1}\int_0^\infty \rho\,d\rho\,J_1(q_\perp\rho/\hbar)\exp[i(\chi_0+\chi_1+\ldots)/\hbar]\,W_\perp(\rho);\\ W_\perp(\rho)&=\varepsilon^{-1}\int_{-\infty}^{\infty}dz\exp(iq'_z z)\left(\frac{\partial}{\partial\rho}U(\rho,z)+\ldots\right).\end{aligned}\right\} \tag{43}$$

[In deriving Eq. (42), we have assumed that the potential of the external field is axially symmetric.]

If the range of the external forces is such that not only the condition $\vartheta_e \ll \vartheta_k$ but also the condition $z_{eff} \ll \delta^{-1}$ is satisfied, then Eq. (42) simplifies appreciably. Setting $\exp(iq'_z z)=1$ in (43), we obtain

$$d\sigma(\mathbf{q}_\perp)=dw(\mathbf{q}_\perp)\,d\sigma_e(\mathbf{q}_\perp)\,[1+O(z_{\text{eff}}\,\delta)], \tag{44}$$

where $d\sigma_e(\mathbf{q}_\perp)$ is the elastic scattering cross section determined by Eqs. (7) and (11), in which it is necessary to replace $\overline{u}'\gamma_0 u$ by 2ε, and

$$dw(\mathbf{q}_\perp)=\frac{2e^2}{3\pi\hbar}\frac{\varepsilon'}{\varepsilon}\left(1+\frac{3}{4}\frac{(\hbar\omega)^2}{\varepsilon\varepsilon'}\right)\frac{q_\perp^2}{m^2}\frac{d\omega}{\omega}. \tag{45}$$

We see that if the condition $z_{eff}\delta \ll 1$ is satisfied, the bremsstrahlung cross section factorizes at both low frequencies and in the region of frequencies $\hbar\omega \sim \varepsilon$.

But if $q_{eff} \ll m$, where q_{eff} determines the order of magnitude of the characteristic momentum transfers, then in (44) we can integrate with respect to $q_\perp$; as a result, using (13), we obtain

$$d\sigma=\frac{4e^2}{3m^2\hbar}\frac{\varepsilon'}{\varepsilon}\left(1+\frac{3(\hbar\omega)^2}{4\varepsilon\varepsilon'}\right)\int_0^\infty \rho\,d\rho\,|\nabla_\perp(\chi_0+\chi_1+\ldots)|^2\frac{d\omega}{\omega}. \tag{46}$$

In this case, the expression for the bremsstrahlung cross section has the same form irrespective of the value of the parameter $\chi_0/\hbar$, i.e., both in the Born approximation as well as in the quasiclassical approximation and in the classical limit (for $\hbar\omega \ll \varepsilon$).[19]

Hitherto we have assumed that $\vartheta_e \ll \vartheta_k$. We now drop this assumption, but we shall assume that $z_{eff}\delta \ll 1$. As before, the cross section factorizes[16,22]:

$$d\sigma(q_\perp)=dw(q_\perp)\,d\sigma_e(q_\perp)\,[1+O(z_{\text{eff}}\delta)], \tag{47}$$

where $d\sigma_e(q_\perp)$, as in (44), is the elastic scattering cross section. However, now

$$\left.\begin{aligned}dw(q_\perp)&=\frac{2e^2}{\pi\hbar}\frac{d\omega}{\omega}\frac{\varepsilon'}{\varepsilon}K(q_\perp/2m);\\ K(\xi)&=\frac{2\xi^2[1+(\hbar\omega)^2/2\varepsilon\varepsilon']+1}{\xi\sqrt{\xi^2+1}}\ln\left(\xi+\sqrt{\xi^2+1}\right)-1.\end{aligned}\right. \tag{48}$$

Note that

$$K(\xi)=(4\xi^2/3)[1+3(\hbar\omega)^2/4\varepsilon\varepsilon'],\qquad \xi\ll 1; \tag{49}$$

$$K(\xi)=2\ln(2\xi)(1+(\hbar\omega)^2/2\varepsilon\varepsilon'),\qquad \xi\gg 1. \tag{50}$$

For factorization of the bremsstrahlung cross section, the relationship between the angles ϑ_e and ϑ_k is not important; it is only important; it is only important that the condition $z_{eff} \ll \delta^{-1}$ be satisfied. It should be emphasized that in Eq. (47) the ratio $\chi_0/\hbar$ can be arbitrary.

If χ_0 is very large, i.e., if $\chi_0 \gg \hbar$ ($\chi_1 \gg \hbar$), then the bremsstrahlung cross section takes the form

$$d\sigma=4e^2\frac{d\omega}{\hbar\omega}\frac{\varepsilon'}{\varepsilon}\int_0^\infty \rho\,d\rho\,K(\xi)\big|_{\xi=(2m)^{-1}|\nabla_\perp(\chi_0+\chi_1+\ldots)|}.$$

Multiplying this by $\hbar\omega$ and assuming that $\hbar\omega \ll \varepsilon$, we obtain the classical formula (35).

4. ELASTIC SCATTERING OF FAST ELECTRONS BY A CHAIN OF ATOMS

In the preceding sections, we have presented the general theory of elastic scattering and bremsstrahlung of fast particles in an arbitrary external field. We now study the scattering and bremsstrahlung produced by a collection of atoms. Of particular interest is the case when the atoms form a periodic structure. The existence of such a structure may have an important influence on the nature of the scattering and the radiation.

We consider first the elastic scattering of fast electrons by a linear chain of N atoms placed along the z axis at distances a from each other. If $U(\rho, z)$ is the potential energy of the interaction of the electron with one atom, the potential energy of the interaction of the electron with the chain of N atoms will have the form

$$U^{(N)}(\rho,z)=\sum_{n=1}^{N}U(\rho,z-na). \tag{51}$$

Substituting (51) in the expressions (10) for χ_0 and χ_1, we obtain for $N \gg 1$

$$\left.\begin{aligned}&\chi_0^{(N)}=N\chi_0;\qquad \chi_1^{(N)}=\frac{N^3a}{12p}(\nabla\chi_0)^2;\\ &\chi_0\equiv\chi_0(\rho)=-\frac{1}{v}\int_{-\infty}^{\infty}U(\rho,z)\,dz.\end{aligned}\right\} \tag{52}$$

(The direction of motion of the particle incident on the chain coincides with the axis of the chain.) Hence, using (11), we can find the amplitude for elastic scattering of electrons by the chain of atoms[2]:

$$a^{(N)}(q_\perp)=i\,(\overline{u}'\gamma_0 uv/\hbar)\times\int_0^\infty \rho\,d\rho\,J_0(q_\perp\rho/\hbar)\left\{1-\exp\left[i\frac{N}{\hbar}\left(\chi_0+\frac{N^2a}{12p}(\nabla_\perp\chi_0)^2+\ldots\right)\right]\right\}. \tag{53}$$

This expression is valid if the inequalities (13) are satisfied. For a chain of atoms, $z_{eff} \sim Na$ and $\rho_{eff} \sim R$, and therefore the conditions (13) take the form

$$\left.\begin{aligned}&\overline{U}/pv\ll 1;\quad N^2a\chi_0/pR^2\ll 1;\\ &\hbar Na/pR^2\ll 1,\end{aligned}\right\} \tag{54}$$

where R is the screening length.

The second and third inequalities restrict the number of atoms in the chain. As N increases, the first inequality to be violated is obviously $N^2 a\chi_0/pR^2 \ll 1$. Rewriting it in the form $N\chi_0/pR \ll R/Na$, and noting that the mean scattering angle of the particle is in order of magnitude $\overline{\vartheta}_e \sim N\chi_0/pR$ [see (67)],we conclude that the scattering angle must be small compared with R/Na (this quantity can be called the opening angle of the chain). In other words, the expression (53) for the scattering amplitude is valid if the scattered particle remains within the limits of the chain over its whole length. Since the means scattering angle decreases with increasing energy, the allowed length of the chain for which (53) is valid increases with increasing energy.

Note that the condition of applicability of the Born approximation for the chain of atoms has the form $N\chi_0 \ll \hbar$ ($\chi_0^{(N)} \ll \hbar$), which does not depend on ε in the limit $v \to C$. Therefore, for large ε the region of applicability of (53) is considerably larger than that of the Born scattering amplitude.

We now elucidate the characteristic features of the behavior of the elastic scattering cross section as a function of the number of atoms in the chain.

If the number of atoms in the chain is such that the condition of applicability of the Born approximation is satisfied, $N\chi_0 \ll \hbar$, then the exponential in (53) can be expanded in a series. We then obtain the following expression for the differential cross section of elastic scattering of unpolarized particles

$$d\sigma_e^{(N)}(q_\perp) = 2\pi (N/\hbar)^2 \left\{ \int_0^\infty \rho\, d\rho\, J_0(q_\perp \rho/\hbar) \right.$$
$$\left. \times [\chi_0(\rho) + (N^2 a/12p)(\nabla_\perp \chi_0(\rho))^2 + \ldots] \right\}^2 (q_\perp/\hbar)\, d(q_\perp/\hbar). \tag{55}$$

Since $N^2 a(\nabla\chi_0)^2/12p \ll \chi_0$, in the first Born approximation the scattering cross section is proportional to the square of the number of atoms in the chain.[2,23-25] In other words, in the Born approximation the particle interacts coherently with the chain, i.e., it is not the cross sections but the amplitudes of scattering on each of the scattering centers that are added.

The second term in the square brackets in Eq. (55) corresponds to the contribution to the scattering cross section from the second Born approximation. We see that the relative contribution of the second Born approximation is also proportional to the square of the number of atoms in the chain.

When the second Born approximation is taken into account, the cross section depends on the sign of the charge of the particle. For the scattering of an electron and positron on an individual atom, the difference between their elastic scattering cross sections is slight. For example, for scattering in the Coulomb field of an individual atom, we obtain from (14) the following expression for the small-angle scattering cross section:

$$d\sigma_e(q_\perp) = \frac{8\pi (Ze^2)^2}{v^2 q_\perp^3} \left(1 - \frac{e}{|e|}\frac{Ze^2}{\hbar v}\frac{\pi}{2}\frac{q_\perp}{p}\frac{v^2}{c^2} + \ldots\right) dq_\perp, \tag{56}$$

i.e., for large p the correction that takes into account the sign of the charge is vanishingly small. But if the scattering takes place on a chain of atoms, the difference between the electron and positron scattering cross sections can be large, since it is proportional to N^2.

We now consider large N, when $N\chi_0 \gg \hbar$. In this case, the Born approximation does not hold. If $N\chi_0 \gg \hbar$, then for all scattering angles except a very small region near zero (where $q_\perp \rho \sim p\vartheta_e \rho(\vartheta_e) \le \hbar$), the method of stationary phase can be used to calculate the scattering amplitude. In this manner we can find the momentum transfer as a function of the impact parameter:

$$q_\perp = N |\nabla_\perp \chi_0| [1 + (N^2 a/6p) \nabla_\perp^2 \chi_0 + \ldots] \tag{57}$$

and obtain the differential scattering cross section [see (15)]

$$d\sigma_e^{(N)}(q_\perp) = 2\pi\rho(q_\perp) \left| \frac{d\rho(q_\perp)}{dq_\perp} \right| dq_\perp. \tag{58}$$

For small scattering angles ($p\vartheta_e \rho(\vartheta_e) \lesssim \hbar$) we cannot use the method of stationary phase, since there is no stationary point near zero angle. In this case, the scattering amplitude must be calculated directly in accordance with (53).

For a chain of atoms with screened Coulomb potential

$$U(r) = \frac{Ze|e|}{r} \exp(-r/R) \tag{59}$$

we can obtain an explicit expression for the scattering cross section for $NZe^2/\hbar v \gg 1$. It is necessary to distinguish three characteristic ranges of values of the momentum transfer $q_\perp$. For small $q_\perp$, namely $q_\perp \lesssim \hbar/(R \ln(NZe^2/\hbar v))$, to calculate the cross section we must use the expression (53) for the scattering amplitude. In this case,[24]

$$d\sigma_e^{(N)}/dq_\perp = (2\pi R^2/q_\perp) \ln^2 (NZe^2\gamma \sqrt{\pi}/\hbar v)\, J_1$$
$$\times [(q_\perp R/\hbar) \ln (NZe^2\gamma \sqrt{\pi}/\hbar v)], \tag{60}$$

where $\gamma = \exp(-G)$, in which $G = 0.577\ldots$ is Euler's constant.

For momentum transfers $q_\perp \le \hbar/R \ln(NZe^2/hv)$, the differential scattering cross section can be calculated by the method of stationary phase in accordance with Eqs. (57) and (58). If at the same time $\hbar/(R\ln(NZe^2/\hbar v) \lesssim q_\perp \lesssim 2NZe^2/vR$, then[2]

$$d\sigma_e/dq_\perp = (2\pi R^2/q_\perp) \ln (2NZe^2/q_\perp vR; \tag{61}$$

but if $q_\perp \gtrsim 2NZe^2/vR$, then

$$d\sigma_e^{(N)}/dq_\perp = 8\pi (NZe^2)^2/v^2 q_\perp^3. \tag{62}$$

Integrating in the differential cross section of elastic scattering with respect to the momentum transfer, we obtain the total scattering cross section $\sigma_e^{(N)}$. In the Born approximation,

$$\sigma_e^{(N)} = 2\pi \left(\frac{N}{\hbar}\right)^2 \int_0^\infty \rho\, d\rho \left(\chi_0 + \frac{N^2 a}{12} (\nabla_\perp \chi_0)^2 + \ldots\right)^2. \tag{63}$$

It follows from (63) that for $N\chi_0 \ll \hbar$ the total scattering cross section is appreciably smaller than the geometrical cross section of an atom: $\sigma_e^{(N)} \approx (N\chi_0/\hbar)^2 R^2 \ll \pi R^2$. This means that the chain of atoms is "transparent" for the beam of incident particles.

The total elastic scattering cross section in the Born approximation increases in proportion to N^2 with increasing N. For $N\chi_0 \gg \hbar$, this relationship between $\sigma_e^{(N)}$

and N breaks down. An estimate of the total scattering cross section for $N\chi_0 \gg \hbar$ shows that in order of magnitude it is equal to

$$\sigma_e^{(N)} \approx \bar{\rho}^2, \tag{64}$$

where $\bar{\rho}$ is the effective value of the impact parameter in the integral (12); ρ is determined by the relation $N\chi_0(\bar{\rho}) \sim \hbar$.

For a chain of atoms whose potential is represented by the screened Coulomb potential, $\chi_0^{(N)} = -2NZe|e|K_0(\rho/R)$, where $K_0(\rho/R)$ is a Bessel function.[26] From this, we can readily find the order of magnitude of $\bar{\rho}$: $\bar{\rho} \sim R\ln(2NZe^2/\hbar v)$. Therefore, the total elastic scattering cross section for $NZe^2/\hbar v \gg 1$ has the form[24–25]

$$\sigma_e^{(N)} \sim R^2 \ln^2(2NZe^2/\hbar v). \tag{65}$$

Thus, for $NZe^2/\hbar v \gg 1$ the total elastic scattering cross section is not proportional to the square of the number of atoms in the chain, in contrast to the Born approximation. The quadratic dependence on N is replaced by a weaker, logarithmic dependence. In other words, for $NZe^2/\hbar v \gg 1$ the total scattering cross section is almost independent of N. The total cross section is now equal in order of magnitude to the geometrical cross section of an atom: $\sigma_e^{(N)} \sim \pi R^2$. This means that the chain of atoms ceases to be transparent, i.e., all particles that strike the "impact" area of order πR^2 are scattered.

The correction due to $\chi_1(\rho)$ in (57), as in (55), is proportional to N^2 and depends on the sign of the charge of the incident particle.

Note that in the region of applicability of the Born approximation the mean square of the momentum transfer does not depend on the number of atoms N but is determined basically by the transverse dimensions of the interaction region:

$$\overline{q_\perp^2} = \frac{\int q_\perp^2 \, (d\sigma_e^{(N)}/dq_\perp)\, dq_\perp}{\sigma_e^{(N)}} \sim \left(\frac{\hbar}{R}\right)^2, \tag{66}$$

whereas for $NZe^2/\hbar v \gg 1$, in accordance with (57)–(65),

$$\overline{q_\perp^2} \sim (NZe^2/R)^2. \tag{67}$$

Summarizing, we can say that the nature of the elastic scattering of fast particles by a chain of atoms is considerably altered when one goes beyond the framework of the Born approximation. In the range of applicability of the Born approximation ($N\chi_0 \ll \hbar$), the differential and the total cross section of elastic scattering are proportional to the square of the number of atoms in the chain, and the mean scattering angle does not depend on N. If the condition of quasiclassical motion of the particle in the field of the chain of atoms ($N\chi_0 \ll \hbar$) is satisfied, the total scattering cross section is hardly dependent on N, while the mean scattering angle is proportional to N.

5. BREMSSTRAHLUNG OF RELATIVISTIC PARTICLES ON A CHAIN OF ATOMS WHEN THE RADIATION IS FORMED IN A LARGE ZONE

We now turn to the investigation of the bremsstrahlung resulting from the passage of particles through a chain of atoms. This will be coherent or incoherent, depending on the number of atoms in the chain, the energy of the particle, and the photon frequency. Our task will be to establish the dependence of the radiation intensity of these quantities. For this, we use the general theory developed in Secs. 2 and 3.

An important influence on the nature of the bremsstrahlung of relativistic particles is to be found in the ratio of the chain length Na to the coherence length, which, as we shall now show, coincides with the length δ^{-1} of the zone in which the radiation is formed. Suppose that a particle, moving with velocity v, emits two waves $\exp[i(\omega t - \mathbf{k}\mathbf{r})]$ at the beginning and end of a path section $l = v\Delta t$. Then the phase difference between the two waves will be $\Delta\varphi = (\omega\Delta t - kl) = (\omega l/v)(1 - v\cos\vartheta_k)$, where ϑ_k is the angle between $\mathbf{l}$ and $\mathbf{k}$. The relativistic particle radiates in a narrow cone along the direction of its motion with opening angle of order m/ε, and therefore $(1 - v\cos\vartheta_k) \sim m/2\varepsilon$. If $\Delta\varphi \lesssim \pi/2$, the amplitudes of the waves will add, i.e., the radiation will be coherent in nature. We see that the condition of coherence has the form $l(\omega m^2/2\varepsilon) \lesssim \pi/2$. The length l which is determined by this condition is called the coherence length.[27–30] Thus, $l \sim \delta^{-1}$, i.e., the coherence length is in order of magnitude equal to the length of the zone in which the radiation is formed.

We consider first the simplest case when this zone is appreciably longer than the length of the chain, $\delta^{-1} \gg Na$ (Figs. 1a and 1b); at a sufficiently high energy of the particle or sufficiently low frequency of the photon, the inequality $\delta^{-1} \gg Na$ can always be satisfied. Since the particle is acted on by the field in the spatial region $z_{eff} \sim Na$, the coherence length will be appreciably greater than z_{eff}. Therefore, the intensity of the bremsstrahlung of the relativistic particles on the chain of atoms, $dI^{(N)}/d\omega$, will be determined by Eq. (47):

$$\frac{dI^{(N)}}{d\omega} = \hbar\omega \int_0^\infty \left[\frac{dw(q_\perp)}{d\omega}\,\frac{d\sigma_e^{(N)}(q_\perp)}{dq_\perp}\right] dq_\perp. \tag{68}$$

Since the bremsstrahlung probability $dw(q)/d\omega$ varies slowly with $q_\perp$, and the differential scattering cross section decreases rapidly with increasing $q_\perp$, the main contribution to the integral (68) is made by the same values of the momentum transfer as to the total cross section of inelastic scattering of fast particles by the chain of atoms. Therefore, if q_{eff} determines the order of magnitude of the characteristic momentum transfers

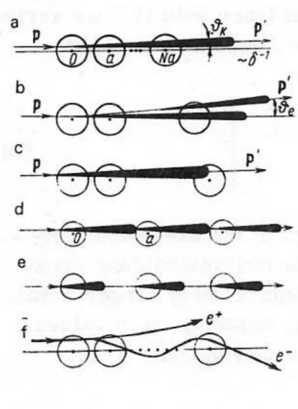

FIG. 1.

in elastic scattering, the bremsstrahlung intensity will in order of magnitude be given by

$$dI^{(N)}/d\omega \approx \hbar\omega\,(dw\,(q_{\rm eff})/d\omega)\,\sigma_e^{(N)}, \qquad (69)$$

where $\sigma_e^{(N)}$ is the total cross section for elastic scattering of the particles by the chain of atoms. Using (49) and (50), which determine the asymptotic behaviors of the function $d\omega(q_\perp)/d\omega$, we obtain the following estimates for the bremsstrahlung intensity at small and large q_{eff} values[31]:

$$\frac{dI^{(N)}}{d\omega} \sim \begin{cases} e^2\,(q_{\rm eff}/m)^2\,\sigma_e^{(N)}, & q_{\rm eff} \ll m; \qquad (70) \\ e^2 \ln\,(q_{\rm eff}/m)\,\sigma_e^{(N)}, & q_{\rm eff} \gg m. \qquad (71) \end{cases}$$

Thus, the dependence of the bremsstralung intensity on the number of atoms in the chain is determined by the N dependence of both the total elastic scattering cross section and the effective momentum transfer.

In Sec. 4, we have seen that important changes in the nature of the dependence of q_{eff} and $\sigma_e^{(N)}$ on N arise if the condition of applicability of the Born approximation is not satisfied, and we therefore consider separately two possibilities: when the Born approximation holds and when it does not.

In the first case ($N\chi_0 \ll \hbar$), the total cross section of elastic scattering is proportional to the square of the number of atoms in the chain [see Eq. (63)], and the effective value of the momentum transfer does not depend on N [$q_{eff} \sim \hbar/R$, Eq. (66)]. Since $\hbar/R \ll m$, it follows in accordance with (70) that the bremsstrahlung intensity in the Born approximation is given by

$$dI^{(N)}/d\omega \sim e^2\,(NZe^2/m)^2;\ NZe^2/\hbar v \ll 1;\ Na \ll \delta^{-1}, \qquad (72)$$

i.e., the bremsstrahlung intensity preserves the quadratic dependence on N of the cross section characteristic of elastic scattering.

In the second case, $(N\chi_0 \gg \hbar)\,\sigma_e^{(N)} \sim R^2$ and $q_{eff} \sim NZe^2/R$. Since now the momentum q_{eff} is proportional to N, depending on the number of atoms in the chain its value may be either less than or greater than m. If $q_{eff} \sim NZe^2/R \ll m$, then in accordance with (70) we again arrive at a quadratic dependence of the bremsstrahlung intensity on N:

$$\frac{dI^{(N)}}{d\omega} \sim e^2\left(\frac{NZe^2}{m}\right)^2;\ \ \frac{NZe^2}{mR} \ll 1;\ \ \frac{NZe^2}{\hbar v} \gg 1;\ \ Na \ll \delta^{-1}. \qquad (73)$$

However, in contrast to the case of the Born approximation, the quadratic dependence of the intensity on the number of atoms in the chain is now due to the N dependence of the momentum transfer.

If $NZe^2/R \gg m$, then in accordance with (71) we arrive at a much weaker, logarithmic, dependence of the bremsstrahlung intensity on N:

$$\left.\begin{array}{l} \dfrac{dI^{(N)}}{d\omega} \sim e^2R^2 \ln \dfrac{NZe^2}{mR};\ \ \dfrac{NZe^2}{mR} \gg 1; \\ \dfrac{NZe^2}{\hbar v} \gg 1;\ \ Na \ll \delta^{-1}. \end{array}\right\} \qquad (74)$$

Note that the coherent nature of the bremstrahlung—the quadratic dependence of the bremsstrahlung cross section on N—is retained for appreciably larger N values than for elastic scattering, namely, up to values $N \sim mR/Ze^2$. It is readily seen that $q_{eff}/m \sim \vartheta_e\ \vartheta_k$, where $\vartheta_e \sim q_{eff}/p$ are the characteristic scattering angles and $\vartheta_e \sim m/\varepsilon$ are the characteristic angles of emission of the relativistic particles. One can therefore say that the coherent nature of the radiation persists until the particles leaves the radiation cone (the cone with opening angle $\vartheta_k \sim m/\varepsilon$ around the momentum p) due to its interaction with the atoms of the chain.

Note that in accordance with (74) the bremsstrahlung intensity on a chain of N^* atoms for which $N^*Ze^2/mR \sim 1$ and on a longer chain $N > N^*$ is virtually the same (because of the weak, logarithmic, dependence of $dI^{(N)}/d\omega$ on N). However, this does not mean that a scattered particle leaves the chain at length N^*a; over this length, it merely leaves the radiation cone (see Fig. 1b). The condition for the particle to traverse the complete chain of N atoms, $NZe^2/\varepsilon R < R/Na$, and the condition for it to go outside the radiation cone, $NZe^2/R > m$, do not contradict one another.

Since $dI^{(N)}/d\omega$ hardly depends on N when the condition for coherence is not satisfied, the bremsstrahlung intensity $N^{-1}dI^{(N)}/d\omega$ per atom tends to zero as $N \to \infty$. (However, the condition $Na \ll \delta^{-1}$ must be satisfied.) Therefore, at large N one can say that the bremsstrahlung is suppressed.

We recall that the bremsstrahlung is also suppressed in an amorphous medium. This is the so-called Landau–Pomeranchuk effect[32] due to multiple scattering of the fast particle within the zone in which the radiation is formed.[27,28,33,34]

An important feature of the suppression of bremsstrahlung in the case of interaction of fast particles with a chain of atoms (and also in the case of interaction with a crystal) is the circumstance that the suppression begins at much lower energies than in the case of an amporphous medium.[35]

In an amorphous medium, the mean-square angle of scattering of the particle within the radiation formation zone $l \sim \delta^{-1}$ is[27,32]

$$\overline{\vartheta_e^2} \sim (\varepsilon_s^2/\varepsilon^2)\,(\delta^{-1}/L),$$

where L is the radiation length and $\varepsilon_s^2 = 4\pi \cdot 137 m^2$. The radiation is suppressed for $\overline{\vartheta}_e > \vartheta_k$. Therefore, in an amorphous medium the suppression occurs when

$$\varepsilon \gtrsim m\,(\omega/\varepsilon)\,(m^2/\varepsilon_s^2)\,mL. \qquad (75)$$

For a chain of atoms, the suppression occurs when the conditions $N \gtrsim mR/Ze^2$ and $Na \lesssim \delta^{-1}$ are satisfied. It follows that this occurs when

$$\varepsilon \gtrsim m\,(\omega/\varepsilon)\,m^2aR/Ze^2. \qquad (76)$$

This energy is appreciably lower than the energy determined by Eq. (75). For example, for silicon the energy (75) is two orders of magnitude greater than the energy (76).

All the above estimates hold independently of the form of the potential of an individual atom. However, they are essentially qualitative in nature. It is therefore of interest to obtain quantitative relationships that simultaneously illustrate the qualitative picture of the radiation. This can be done by assuming that the potential of an individual atom has the form of the screened Coulomb potential $U(r)=(Ze|e|/r)\exp(-r/R)$. For such a potential, one can directly calculate the bremsstrahlung intensity $dI^{(N)}/d\omega$ in accordance with Eq. (68).[31] (Recall that this expression is valid if $Na \ll \delta^{-1}$.)

If $NZe^2/mR \ll 1$, then integration of the expression (68) leads to the result

$$\frac{dI^{(N)}}{d\omega}=\frac{16e^2}{3}\left(\frac{NZe^2}{m}\right)^2\frac{\varepsilon'}{\varepsilon}\left\{\left(1+\frac{3}{4}\frac{(\hbar\omega)^2}{\varepsilon\varepsilon'}\right)\times\left[\ln(mR\hbar)+\frac{1}{2}-\left(\frac{NZe^2}{\hbar v}\right)^2\times\sum_{n=1}^{\infty}\frac{1}{n[n^2-(NZe^2/\hbar v)^2]}\right]+\frac{1}{12}\right\}. \quad (77)$$

Note that this expression is essentially the well-known Bethe–Maximon expression[36,37] for the bremsstrahlung intensity of a fast electron on a single Coulomb center obtained in the Furry–Sommerfeld–Maue approximation,[38] it being merely necessary to replace the charge $Z|e|$ of the nucleus in the Bethe–Maximon expression by $NZ|e|$. In other words, in the case of bremsstrahlung when $Na \ll \delta^{-1}$ the chain of atoms behaves like a single center with charge $NZ|e|$.

If $NZe^2/\hbar v \ll 1$, the expression (77) goes over into the result of the Born approximation. In this case, as one would expect, the radiation is coherent in nature. However, as we have pointed out above, coherence is also obtained when the Born approximation is invalid. Namely, if $NZe^2/\hbar \gg 1$, but, as before, $NZe^2/mR \ll 1$, then (77) becomes

$$\frac{dI^{(N)}}{d\omega}=\frac{16e^2}{3}\left(\frac{NZe^2}{m}\right)^2\frac{\varepsilon'}{\varepsilon}\left\{\left(1+\frac{3}{4}\frac{(\hbar\omega)^2}{\varepsilon\varepsilon'}\right)\times\left[\ln\frac{mR}{NZe^2}-G+\frac{1}{2}\right]+\frac{1}{12}\right\}. \quad (78)$$

Thus, in this case too the bremsstrahlung intensity is proportional to N^2, though there is now also an additional, weak logarithmic dependence on N.

For $NZe^2/mR \gg 1$, Eqs. (77) and (78) do not hold and integration of the general expression (68) leads to

$$\frac{dI^{(N)}}{d\omega}=\frac{4e^2}{3}R^2\frac{\varepsilon'}{\varepsilon}\left(1+\frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'}\right)\ln^3\frac{NZe^2}{mR}. \quad (79)$$

Equations (77)–(79) show that the coherent nature of the bremsstrahlung of fast particles on the chain of atoms is lost only when $NZe^2/mR \sim 1$.

6. BREMSSTRAHLUNG OF RELATIVISTIC PARTICLES ON A CHAIN OF ATOMS WHEN THE RADIATION IS FORMED IN A SMALL ZONE

In the preceding section, we assumed that $Na \ll \delta^{-1}$. We now consider the case when the size of the radiation formation zone is small or comparable with the chain length: $Na \gtrsim \delta^{-1}$. For this, we use Eq. (42), which determines the bremsstrahlung intensity for arbitrary Na and δ^{-1}, provided $\vartheta_e \ll \vartheta_k$. When $Na \sim \delta^{-1}$, this last condition leads to a restriction on the energy:

$$\varepsilon \ll m(\omega/\varepsilon)m^2aR/Ze^2. \quad (80)$$

This inequality is satisfied in a wide range of particle energies [for example, for silicon it is satisfied up to energies $\varepsilon \sim 10^6(\hbar\omega/\varepsilon)m$].

For a chain of N atoms, the function $W_{\perp}^{(N)}(\rho)$ has in accordance with (43) the form

$$W_{\perp}^{(N)}(\rho)=\frac{\sin(Nq'_z a/2)}{\sin(q'_z a/2)}W_{\perp}(\rho)[1+O(\varepsilon^{-1})],$$

where $W_{\perp}(\rho)=\varepsilon^{-1}\nabla_{\perp}\int_{-\infty}^{\infty}U(\rho,z)dz$, and the bremsstrahlung intensity is

$$\frac{dI^{(N)}}{d\omega}=\int_{\delta}^{\infty}\frac{\sin^2(Nq'_z a/2)}{\sin^2(q'_z a/2)}\frac{d^2I}{dq'_z\,d\omega}dq'_z. \quad (81)$$

Here, $d^2I/dq'_z d\omega$ is the intensity of bremsstrahlung on an individual atom in the frequency interval $d\omega$ and in the interval of momentum transfers dq'_z:

$$\frac{d^2I}{dq'_z\,d\omega}=\frac{e^2}{\pi}\frac{\delta}{m^2}\frac{1}{q_z'^2}\int dq_{\perp}q_{\perp}^2|A(q)|^2\frac{\varepsilon'}{\varepsilon}\times\left[1+\frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'}-2\frac{\delta}{q'_z}\left(1-\frac{\delta}{q'_z}\right)\right].$$

When $\delta^{-1} \gg Na$, the ratio of the squares of the sines in the integrand is equal to N^2 (since $q'_z \sim \delta$), and, therefore, in this case the radiation is coherent. Note that when $Na \ll \delta^{-1}$ the position of the atoms in the chain does not influence the coherent nature of the radiation—the intensity if proportional to N^2 both when the atoms are arranged at equal distances from one another and at unequal distances.

But if $Na \gtrsim \delta^{-1}$, the ratio of the squares of the sines does not lead to the appearance of the factor N^2 and the coherence is lost (for $Na \ll \delta^{-1}$, the phase difference between successively emitted waves is small compared with $\pi/2$, and therefore the wave amplitudes are added and the bremsstrahlung intensity is proportional to N^2, but when $Na \gtrsim \delta^{-1}$ the emitted waves can partly cancel each other).

If $Na \gg \delta^{-1}$, we can use the formula

$$\frac{\sin^2(Naq'_z/2)}{\sin^2(aq'_z/2)}=N\frac{2\pi}{a}\sum_{g_j}\delta(q'_z-g_j),$$

where $g_j=(2\pi/a)j$, $j=0,\pm1,\pm2,\ldots$, and we obtain

$$\frac{dI^{(N)}}{d\omega}=N\frac{2\pi}{a}\sum_{g_j\geq\delta}\frac{d^2I}{dg_j\,d\omega}. \quad (82)$$

We now determine the sum over g_j in (82). We consider first the case when $\delta^{-1} \ll a$ (see Fig. 1e). The summation then begins with the value $j=j_{\min}\approx a\delta/2\pi$, which is appreciably greater than unity. We are therefore concerned with summation of the series $\sum_{n=j}^{\infty}f(n)$, where $j \gg 1$ and the function $f(n)$ decreases rapidly with increasing n. Such a sum can be replaced by the integral[39]

$$\sum_{n=j}^{\infty}f(n)\approx\int_{j}^{\infty}f(n)dn+O(f(j)).$$

Thus, when $\delta^{-1} \ll a$ we arrive at the result

$$dI^{(N)}/d\omega=N\,dI/d\omega, \quad (83)$$

where $dI/d\omega$ is the bremsstrahlung intensity on an individual atom: $dI/d\omega \sim e^2(Ze^2/m)^2$. Thus, when $a\delta \gg 1$ the bremsstrahlung intensity on the chain of atoms does not differ from the intensity in an amorphous medium. This is so because the bremsstrahlung takes place independently on the individual atoms of the chain when $a \gg \delta^{-1}$.

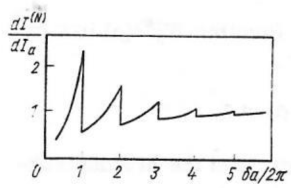

FIG. 2.

Now suppose $a \ll \delta^{-1} \ll Na$, i.e., the size of the radiation formation zone is appreciably greater than the distance between the atoms of the chain but less than the length of the chain. Then the summation in (82) should begin with $j_{min}=1$. Since $d^2I/dg_j d\omega \sim 2\pi\delta/ag_j^2 = (a\delta/2\pi)j^{-2}$, the main contribution to the sum over g_j is made by just the first few terms. Therefore, we now have

$$dI^{(N)}/d\omega \sim a\delta N\, dI/d\omega. \tag{84}$$

We see that when $a \ll \delta^{-1} \ll Na$ the bremsstrahlung of a particle moving along the chain of atoms contains the small factor $a\delta$, which is absent in the case of the bremsstrahlung of a fast particle in an amorphous medium. This is a manifestation of the periodic structure of the chain and the fact that its length is greater than the radiation formation zone.

We consider, finally, the case when the radiation formation zone is comparable in length with the distance between the atoms of the chain, $a \sim \delta^{-1} \ll Na$ (Fig. 1d). A characteristic result in this case is that for certain values of $a\delta$ there are additional changes in the radiation intensity. The point is that for $2\pi/a > \delta$ the summation in (82) should begin with $j_{min}=1$. But if $4\pi/a > \delta^{-1} > 2\pi a$, the summation should begin with $j_{min}=2$, etc. Therefore, at $2\pi/a = \delta$ there is a jump in the bremsstrahlung intensity, the contribution of the terms with $j=1$ dropping out of the sum over j. This intensity jump can be readily found for $\hbar\omega \ll \varepsilon$:

$$\frac{dI^{(N)}(2\pi/a \geqslant \delta)\, d\omega}{dI^{(N)}\left(\frac{2\pi}{a} \leqslant \delta < \frac{4\pi}{a}\right)\Big/ d\omega} = \frac{\sum\limits_{n=1}^{\infty} n^{-2}(1-2n^{-1}+2n^{-2})}{\sum\limits_{n=2}^{\infty} n^{-2}(1-2n^{-1}+2n^{-2})} \approx 3.5\,.$$

Similarly, at $4\pi/a = \delta$ the bremsstrahlung intensity changes by a factor 2.02; at $6\pi/a = \delta$, by a factor 1.35, etc. The graphical dependence of $dI^{(N)}/d\omega$ on the parameter $a\delta/2\pi$ is shown in Fig. 2.

TABLE I.

Bremsstrahlung intensity $dI^{(N)}/d\omega$	Conditions of applicability	Features of the bremsstrahlung
$N^2 I'_\omega$	$\delta^{-1} \gg Na$ $\vartheta_e \ll \vartheta_\kappa$	Coherence
$\sim e^2R^2$	$\delta^{-1} \gg Na$ $\vartheta_e \gg \vartheta_\kappa$	Suppression of bremsstrahlung (analog of the Landau-Pomeranchuk effect)
$\sim a\delta N I'_\omega$	$Na \gg \delta^{-1} \gg a$	Suppression of bremsstrahlung due to interference
$\sim N I'_\omega$	$Na \gg \delta^{-1} \sim a$	Sharp interference peaks in the bremsstrahlung spectrum
$N I'_\omega$	$a \gg \delta^{-1} \gg R$	Bremsstrahlung as in an amorphous medium
(See Sec. 8 below)	$Na \geqslant R\, \vartheta_{ch}$ $N^2aZe^2/\varepsilon R^2 \sim 1$	Bremsstrahlung under conditions of the channeling regime

Note that all the results obtained are valid for both relativistic electrons and relativistic positrons. The difference between the bremsstrahlung of these two kinds of particle, $dI_-^{(N)}$ and $dI_+^{(N)}$, appears only in small corrections[2,40]:

$$dI_{\mp}^{(N)}/d\omega = (dI^{(N)}/d\omega)\,[1 \pm O(N^2aZe^2/\varepsilon R^2)], \tag{85}$$

which, however, increase with increasing N and N^2.

In Table I we give the main results obtained in this and the preceding sections (I'_ω is the bremsstrahlung intensity on an individual atom in the frequency interval $d\omega$).

7. BREMSSTRAHLUNG OF FAST ELECTRONS IN A CRYSTAL

We now show that the bremsstrahlung of fast particles in crystals has the same features as the bremsstrahlung of fast particles established in the previous sections for the case when the particles pass along chains of atoms.[3]

We consider first the bremsstrahlung of electrons in a crystal in the first Born approximation. If $U(\mathbf{r}-\mathbf{r}_n)$ is the potential energy of the interaction of an electron with an individual atom, the potential energy of the interaction of the electron with the complete crystal is

$$U_c(\mathbf{r}) = \sum_n U(\mathbf{r}-\mathbf{r}_n), \tag{86}$$

and its Fourier transform is

$$U_c(\mathbf{q}) = \left(\frac{2\pi}{a}\right)^3 \sum_{\mathbf{g}_j} \delta(\mathbf{q}-\mathbf{g}_j)\, U(\mathbf{q}),$$

where $\mathbf{g}_j$ is a vector of the reciprocal lattice and a is the lattice constant (for simplicity, we consider a cubic lattice). It follows that

$$|U_c(\mathbf{q})|^2 = N_c \left(\frac{2\pi}{a}\right)^3 \sum_{\mathbf{g}_j} \delta(\mathbf{q}-\mathbf{g}_j)\, |U(\mathbf{q})|^2, \tag{87}$$

where N_c is the total number of atoms in the crystal. Substituting this expression in (41), we obtain the bremsstrahlung intensity in the first Born approximation[27,41–45]:

$$\frac{dI_c}{d\omega} = N_c \frac{e^2}{4\pi^3} \frac{\delta}{m^2} \left(\frac{2\pi}{a}\right)^3 \sum_{\mathbf{g}_j} |U(\mathbf{g}_j)|^2$$
$$\times \frac{\varepsilon'}{\varepsilon} \frac{g_\perp^2}{g^2} \left[1 + \frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'} - 2\frac{\delta}{g}\left(1-\frac{\delta}{q}\right)\right], \tag{88}$$

where $g_\parallel$ is the projection of the vector g_j onto the direction of motion of the electron, $g_\parallel \geqslant \delta$.

If $\delta^{-1} \ll a$, the summation over $\mathbf{g}_j$ can be replaced by an integration. In this case, the crystal structure will not affect the bremsstrahlung, and it will take place in the same way as in an amorphous medium.

The crystal structure affects the bremsstrahlung only when the radiation formation zone is greater than or comparable with the lattice constant, $\delta^{-1} \gtrsim a$; for one can then have interference of waves emitted by the electron on different atoms of the lattice, and this can lead to an appreciable change in the nature of the radiation. Formally, this finds its reflection in the fact that in

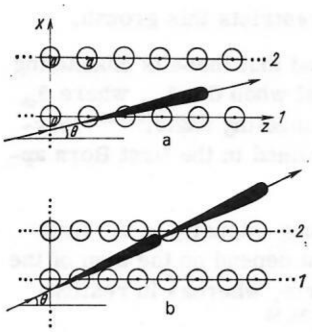

FIG. 3.

the expression for the bremsstrahlung intensity one can no longer replace the summation over the vectors of the reciprocal lattice by an integration.

The crystal lattice will have the greatest influence on the radiation when $\delta^{-1} \gg a$ and the lattice is oriented in a definite manner relative to the beam of the incident particles, namely, when there is a small angle Θ between the direction of motion of the particle and the crystallographic axis, $\Theta < R/a$. There is then a large number of atoms arranged along the crystallographic axis within the radiation formation zone. We must distinguish two possibilities: $R/\Theta \lesssim \delta^{-1} \ll a/\Theta$ and $\delta^{-1} \gtrsim a/\Theta$.

If $R/\Theta \lesssim \delta^{-1} \ll a/\Theta$, the interference between the waves emitted on different chains of atoms is slight (Fig. 3a; two parallel chains of atoms of the crystal are denoted by the numbers 1 and 2). But if $\delta^{-1} \gtrsim a/\Theta$ (see Fig. 3b), then the interference between the emissions on different chains will be important. In both cases, the coherent nature of the bremsstrahlung on the atoms of the same chain will be preserved, since $\delta^{-1} \gg a$.

We transform the general expression (88), remembering that $\delta^{-1} \gg a$ and $\Theta < R/a$. In this case, $g_\parallel = g_{jz} + \Theta(g_{jx}\cos\alpha + g_{jy}\sin\alpha)$, where α is the angle between the (z, p) and (x, z) planes, and the main contribution to the sum over g_{jz} is given by just the single term with $g_{jz} = 0$. Indeed, the contribution of this term to the bremsstrahlung intensity is

$$(dI_c/d\omega)_{g_{jz}=0} \sim N_c e^2 (Ze^2/m)^2 R/a\Theta, \tag{89}$$

while the contribution of the remaining terms with $g_{jz} \neq 0$ is

$$(dI_c/d\omega)_{g_{jz}\neq 0} = N_c e^2 (Ze^2/m)^2 a\delta,$$

and therefore

$$\frac{(dI_c/d\omega)\, g_{jz}\neq 0}{(dI_c/d\omega)\, g_{jz}=0} \sim \frac{a}{\delta^{-1}}\,\frac{a\Theta}{R}. \tag{90}$$

The relation (89) can also be represented in the form

$$dI_c/d\omega \sim \bar{N}\,(N_l)^2 e^2 (Ze^2/m)^2, \tag{91}$$

where $\bar{N} \sim N_c/N_l$ in which N_l is the number of atoms in the radiation formation zone ($N_l \sim R/a\Theta$). The quantity $\bar{N}$ is the number of groups of atoms of the crystal on which the bremsstrahlung occurs coherently.

Thus, for $\delta^{-1} \gg a$ and $\Theta < R/a$ the bremsstrahlung intensity is given by

$$\frac{dI_c}{d\omega} = N_c \frac{e^2}{4\pi^3}\frac{\delta}{m^2}\left(\frac{2\pi}{a}\right)^3 \sum_{g_{jx},\, g_{jy}} |U(\mathbf{g})|^2 \frac{\varepsilon'}{\varepsilon}\frac{g_\perp^2}{g_\parallel^2}\Big[1 + \frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'} - 2\frac{\delta}{g_\parallel}\Big(1 - \frac{\delta}{g_\parallel}\Big)\Big]\Big[1 + O\Big(a\delta\frac{a\Theta}{R}\Big)\Big], \tag{92}$$

where $g_\parallel = \Theta(g_{jx}\cos\alpha + g_{jy}\sin\alpha) \geqslant \delta$. The fact that the main contribution to the bremsstrahlung intensity is made by the single term with $g_{jz} = 0$ means that the details of the crystal structure along the z axis are unimportant. Therefore, when studying the bremsstrahlung of fast particles moving near the crystallographic z axis, one can use the potential averaged along the axis:

$$U(x, y) = \frac{1}{N_z a}\int_{-\infty}^{\infty} U(\mathbf{r})\,dz. \tag{93}$$

Let us now establish the dependence of the bremsstrahlung intensity (92) on the angle α between the (z, p) and (z, x) planes. If $\alpha = 0$ (i.e., the motion of the particle takes place in the (z, x) plane), the inequality $g_\parallel \geqslant \delta$ takes the form $\Theta g_{jx} \geqslant \delta$, and the summation over g_{jy} can be replaced by an integration. The remaining sum over g_{jx} depends strongly on the parameter $a\delta/2\pi\Theta$, and at integral values of this parameter the bremsstrahlung spectrum has pronounced peaks (Fig. 4). A similar situation also occurs when $\alpha = \pi/2$, except that the parts played by the x and y axes are interchanged.

Thus, if the particles move in one of the crystallographic planes ($\alpha = 0$ or $\alpha = \pi/2$) and the angle Θ is small, there is in addition to the general increase in the bremsstrahlung intensity (due to the coherence) compared with the intensity in an amorphous medium an important effect due to the interference of waves emitted by a particle on different chains of the atoms of the crystal. This effect leads to the appearance of sharp peaks in the bremsstrahlung spectrum (cf. Figs. 2 and 4).

The characteristic dependences of the ratio of the bremsstrahlung intensity in the crystal to the intensity in an amorphous medium on the angle of entrance Θ for fixed values of ε and ω and on ω for fixed values of ε and Θ in accordance with (92) are shown in Fig. 4 (Refs. 45 and 46).

Hitherto, we have not assumed the existence of any inequality between δ^{-1} and a/Θ. Now suppose that the radiation formation zone is appreciably shorter than the

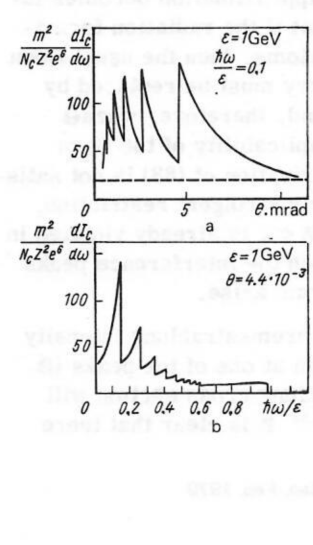

FIG. 4.

distance traversed by the particle between its sucessive collisions with different chains of atoms of the crystal (chains 1 and 2 in Fig. 3b). Then the periodicity of the crystal along the x and y axes is not important for the bremsstrahlung, since there is no interference of the waves emitted on different chains. Therefore, the summation over g_{jx} and g_{jy} in (92) can be replaced by an integration and the bremsstrahlung intensity can be expressed by

$$\frac{dI_c}{d\omega} = N_c 2e^2 \left(\frac{Ze^2}{m}\right)^2 \left[\frac{R}{a\Theta} F\left(\frac{\delta R}{\Theta}\right) + O(\delta a)\right],$$

where

$$F(\zeta) = \frac{\zeta}{a^2} \int_{\zeta}^{\infty} d\tilde{q}_x \int d\tilde{q}_y \, |\tilde{U}(\tilde{q})|^2$$

$$\times \frac{\varepsilon'}{\varepsilon} \frac{\tilde{g}^2}{\tilde{g}_x^2} \left[1 + \frac{(\hbar\omega)^2}{2\varepsilon\varepsilon'} - 2\frac{\zeta}{\tilde{g}_x}\left(1 - \frac{\zeta}{\tilde{g}_x}\right)\right] \quad (94)$$

and the tilde is used to designate the corresponding dimensionless quantities: $\mathbf{r} = \tilde{\mathbf{r}} R$, $\mathbf{g} = \tilde{\mathbf{g}}/R$, $U(\mathbf{r}) = (Ze|e|/R)\,\tilde{U}(\tilde{\mathbf{r}})$, and $U(\mathbf{q}) = Ze|e|R^2 U(\mathbf{g})$.

Note that this expression contains the two parameters $R/a\Theta$ and $\delta R/\Theta$. The first of them is the number of atoms "intersected" by the electron moving at angle Θ to an individual chain of atoms. The second characterizes the ratio between the length δ^{-1} of the radiation formation zone and the length $Na \sim R/\Theta$ of the part of the chain intersected by the electron.

Thus, Eq. (92) for the radiation intensity in the crystal leads to the following characteristic dependence of $dI_c/d\omega$ on δ, a, and Θ:

$$\frac{dI_c}{d\omega} \sim N_c \frac{dI}{d\omega}\left[\frac{R}{a\Theta}\mathcal{F}\left(\frac{\delta R}{\Theta}, \frac{\delta a}{2\pi\Theta}\right) + O(a\delta)\right], \quad (95)$$

where $dI/d\omega$ is the bremsstrahlung intensity on an individual atom and $\mathcal{F}(\delta R/\Theta, \delta a/2\pi\Theta)$ is a function of $\delta R/\Theta$ and $\delta a/2\pi\Theta$. For $\delta R/\Theta \sim 1$, this function is near unity, while as $\delta R/\Theta \to \infty$ it decreases rapidly and at integral values of $\delta a/2\pi\Theta$ has sharp peaks. Therefore if $\delta^{-1} \gtrsim R/\Theta$, then[47]

$$dI_c/d\omega \sim N_c (dI/d\omega) R/a\Theta.$$

If $\delta^{-1} \ll R/\Theta$, then

$$dI_c/d\omega \sim N_c (dI/d\omega)\,\delta a.$$

We have presented the main results of the theory of bremsstrahlung of relativistic particles in a crystal in the first Born approximation. However, in a number of important cases this approximation becomes invalid. We begin by noting that if the radiation formation zone contains $N \sim R/a\Theta$ atoms, then the usual Born parameter $Ze^2/\hbar v$ in the theory must be replaced by the parameter $RZe^2/a\Theta\hbar v$, and, therefore, when $\Theta < RZe^2/a\hbar$ the condition of applicability of the Born approximation used in the derivation of (88) is not satisfield.[27,31,48-50] This is a fairly stringent restriction, since the inequality $RZe^2/a\Theta\hbar \ll 1$ is already violated in the region of angles Θ in which the interference peaks in the bremsstrahlung spectrum arise.

We note further that if the bremsstrahlung intensity is expressed by Eq. (92), then at one of the peaks ($\Theta \sim R\delta$) as $\varepsilon \to \infty$ the bremsstrahlung cross section will increase unboundedly.[27,35,49,50] It is clear that there must be a mechanism that restricts this growth.

Recently, it has been found that there is channeling of fast particles in a crystal when $\Theta \lesssim \vartheta_{ch}$, where $\vartheta_{ch} \sim \sqrt{Ze^2/\varepsilon a}$ is the critical channeling angle.[51-53] However, this effect is not contained in the first Born approximation.

Finally, the bremsstrahlung intensity in the first Born approximation does not depend on the sign of the charge of the incident particle, whereas in reality there is such a dependence.[54,55]

Thus, the bremsstrahlung in crystals must be investigated more accurately than in the framework of the Born approximation. We establish first criteria for the applicability of the expression (92) for the bremsstrahlung intensity in a crystal. In Sec. 3 we have seen that the Born expression for the bremsstrahlung intensity (41) is valid if

$$\chi_0 \ll \hbar; \quad \vartheta_e \ll \vartheta_k; \quad \chi_1 \ll \chi_0. \quad (96)$$

Substituting in the expression for $\chi_0 = \int U dl$ the averaged potential $U(x, y)$ of the crystal and noting that $dl = dx/\Theta$, we obtain from the criterion $\chi_0 \ll \hbar$ a restriction on the angle of entrance Θ of the particle into the crystal: $RZe^2/a\Theta\hbar \ll 1$.

The condition $\vartheta_e \ll \vartheta_k$ means that $q_{eff} \ll m$. Since q_{eff} is in accordance with (92) of order $\hbar/R$, we arrive at the condition $\hbar/mR \ll 1$, which is always satisfied. Finally, the third of the inequalities (96) leads to a restriction on ε and Θ: $Ze^2/\varepsilon a\Theta^2 \ll 1$. This inequality is the condition whose fulfillment enables one to ignore the channeling effect.

Thus, the conditions of applicability of the first Born approximation for the investigation of the bremsstrahlung of electrons in a crystal have the form

$$RZe^2/a\Theta\hbar \ll 1; \quad \hbar/mR \ll 1; \quad Ze^2/\varepsilon a\Theta^2 \ll 1. \quad (97)$$

At high energies, the first of the conditions (97) to be violated when Θ decreases is the inequality $RZe^2/a\Theta\hbar \ll 1$.

We now investigate the bremsstrahlung intensity of fast particles in a crystal when the condition of applicability of the Born approximation is not satisfied. We consider the case when the conditions

$$\chi_0 \gg \hbar; \quad \vartheta_e \ll \vartheta_k; \quad \chi_1 \ll \chi_0, \quad (98)$$

are satisfied, i.e., when only the first of the inequalities (96) is not fulfilled. In this case, the bremsstrahlung spectrum can be found by means of the general expressions (14) and (42), which were obtained in the framework of classical electrodynamics and the quasiclassical approximation of quantum electrodynamics and which are valid if the conditions (98) are satisfied.

A striking feature of these expressions is that they have the same form as the expression (41) for the bremsstrahlung intensity obtained in the Born approximation. Therefore, if only the first of the conditions (97) is violated, the bremsstrahlung intensity of fast particles in the crystal will be determined by the expression (92), which is valid in the Born approximation.[2)]

Note, however, that the condition $\vartheta_e \ll \vartheta_k$ for $\chi_0 \gg \hbar$ is not equivalent to the condition $\hbar/mR \ll 1$, which is valid in the first Born approximation, but is equivalent to the inequality $Ze^2/ma\Theta \ll 1$, since for $\chi_0 \gg \hbar$ the momentum transfer is strictly related to the impact parameter, $q_\perp \approx \nabla\chi_0$. Therefore, the inequalities (98) lead to the following restrictions on the energy of the particle and the angle of its entrance into the crystal:

$$RZe^2/a\Theta\hbar \gg 1; \quad Ze^2/ma\Theta \ll 1; \quad Ze^2/\varepsilon a\Theta^2 \ll 1. \tag{99}$$

Since $Ze^2/ma\Theta \sim (\hbar/mR)\,(RZe^2/a\Theta\hbar)$ and $\hbar/mR \ll 1$, there exists a fairly large range of angles Θ in which the inequalities $RZe^2/a\Theta\hbar \gg 1$ and $Ze^2/ma\Theta \ll 1$ are satisfied. Note that it is precisely this range of Θ that is the most interesting for studying the bremsstrahlung of fast electrons in a crystal, since the bremsstrahlung intensity reaches its maximum in this region and there are sharp peaks in the bremsstrahlung spectrum.

Thus, the bremsstrahlung spectrum of relativistic particles in a crystal is determined by the expression (92) up to values of ε and Θ for which the following inequalities are satisfied:

$$Ze^2/ma\,\Theta \ll 1; \quad Ze^2/\varepsilon a\Theta^2 \ll 1. \tag{100}$$

The second of these inequalities contains the energy of the particle, and if the energy is sufficiently high this inequality can always be satisfied. The first inequality does not depend on ε and is violated when $\sim Ze^2/ma$.

We now obtain the bremsstrahlung intensity when the first of the inequalities (100) is not satisfied, i.e., when the inequality $Ze^2/ma\Theta \gtrsim 1$ is satisfied. For this, we use the expression (47) for the bremsstrahlung intensity, this holding under the conditions

$$z_{\rm eff} \ll \delta^{-1}; \quad \chi_1 \ll \chi_0 \tag{101}$$

(the relation between the angles ϑ_e and ϑ_k can be arbitrary). Since $z_{\rm eff} \sim R/\Theta$, the inequalities (101) take the form

$$\delta R/\Theta \ll 1; \quad Ze^2/\varepsilon a\Theta^2 \ll 1. \tag{102}$$

Using (47) and taking into account the first correction in the small parameter χ_1/χ_0, we obtain the following expression for the bremsstrahlung intensity:

$$\frac{dI_c}{d\omega} = N_c \frac{2e^2}{\pi} Ra\Theta \int_{-\infty}^{\infty} d\tilde{y} K(\xi(\tilde{y})), \tag{103}$$

where the function $K(\xi)$ is determined by Eq. (48),

$$\xi(\tilde{y}) = \frac{\varepsilon\vartheta_e(\tilde{y})}{2m} = \frac{Ze^2}{2ma\Theta}\Big\{\frac{\partial}{\partial\tilde{y}}\Big[\int_{-\infty}^{\infty}\tilde{U}(\tilde{x},\tilde{y})\,d\tilde{x} - \frac{e}{|e|}\frac{Ze^2}{\varepsilon a\Theta^2}$$
$$\times \int_{-\infty}^{\infty} d\tilde{x}\,\tilde{x}\Big(\frac{\partial}{\partial\tilde{y}}\tilde{U}(\tilde{x},\tilde{y})\Big)\Big(\frac{\partial}{\partial\tilde{y}}\int_{-\infty}^{\tilde{x}}\tilde{U}(\tilde{x}',\tilde{y})\,d\tilde{x}'\Big) + \ldots\Big]\Big\}$$

and the tilde, as before, is used to denote the corresponding dimensionless quantities. (In deriving (103) we have used the fact that the integral with respect to the variable x is equal to $N_z a\Theta$.)

Equation (103) is valid both when $Ze^2/ma\Theta \gg 1$ and when $Ze^2/ma\Theta \ll 1$. In the latter case, it takes the form

$$\frac{dI_c}{d\omega} = N_c \frac{2e^2}{3\pi}\Big(\frac{Ze^2}{m}\Big)^2 \frac{R}{a\Theta}\int_{-\infty}^{\infty} d\tilde{y}\Big[\frac{\partial}{\partial y}\int_{-\infty}^{\infty}\tilde{U}(\tilde{x},\tilde{y})\,d\tilde{x}$$
$$-\frac{e}{|e|}\frac{Ze^2}{\varepsilon a\Theta^2}\frac{\partial}{\partial\tilde{y}}\int_{-\infty}^{\infty} d\tilde{x}\,\tilde{x}\Big(\frac{\partial}{\partial\tilde{y}}\tilde{U}(\tilde{x},\tilde{y})\Big)$$
$$\times\Big(\frac{\partial}{\partial\tilde{y}}\int_{-\infty}^{\tilde{x}}\tilde{U}(\tilde{x}',\tilde{y})\,d\tilde{x}'\Big) + \ldots\Big]^2. \tag{104}$$

This expression differs from (94), which is of the Born kind, in that it contains a correction which determines the dependence of the bremsstrahlung intensity on the sign of the charge of the particle. Note that, depending on the form of the potential $U(x, y)$, this correction may either increase or decrease the bremsstrahlung of the electron (or positron); for the Gaussian potential $U(x, y) = U_0\exp[-(x^2+y^2)/R^2]$, the contribution of the correction is zero. With decreasing angle Θ, the difference between the bremsstrahlung of electrons and positrons rapidly increases. Under the conditions when the contribution of this correction becomes comparable with the contribution of the principal term ($Ze^2/\varepsilon a\Theta^2 \sim 1$), it is necessary to take into account the effect of channeling of the particles in the crystal, and, as will be shown in the following section, the difference between the bremsstrahlung of electrons and positrons will be large in this case.

Equations (103) and (104) show that when $Ze^2/ma\Theta > 1$ the coherent nature of the bremsstrahlung is lost and there is suppression of the bremsstrahlung (cf. Sec. 5). In particular, as $\varepsilon \to \infty$ the bremsstrahlung intensity at the maximum, i.e., at $\Theta \sim \delta R$, tends to zero.

Thus, the region of applicability of the expression for the bremsstrahlung intensity of fast particles in a crystal in the first Born approximation is appreciably larger than the region of applicability of the Born approximation itself. When $Ze^2/ma\Theta > 1$, the radiation is suppressed, and the bremsstrahlung intensity at the maximum does not increase unboundedly as $\varepsilon \to \infty$. Finally, the bremsstrahlung intensity depends on the sign of the charge of the particle, and this effect is determined by the parameter $Ze^2/\varepsilon a\Theta^2$. In the cases when channeling of the particles becomes important ($Ze^2/\varepsilon a\Theta^2 \sim 1$), the expressions given in this section cease to be valid.

8. BREMSSTRAHLUNG OF FAST ELECTRONS AND POSITRONS IN THE CASE OF CHANNELING

Finally, we consider the bremsstrahlung of relativistic electrons and positrons when the inequality $Ze^2/\varepsilon a\Theta^2 \ll 1$ is not satisfied in (100). Rewriting it in the form $Ze^2/\varepsilon a\Theta \ll \Theta$ and noting that in order of magnitude $Ze^2/\varepsilon a\Theta$ is the characteristic angle ϑ_e of scattering of a particle by a chain of atoms of the crystal, we see that this inequality requires the scattering angle to be small compared with the angle of entrance of the particles into the crystal: $\vartheta_e \ll \Theta$. But if $\vartheta_e \sim \Theta$, then we encoun-

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[2)] It was first suggested by Ter-Mikaelyan[27] that the expression for the bremsstrahlung cross section of relativistic electrons in a crystal obtained in the Born approximation could have a wider applicability than the Born approximation itself.

ter a new phenomenon—channeling, which results in the particles moving in the channels formed by the atoms of the crystal axes (or planes), deviating periodically from the direction of the channel axis through small angles $\vartheta_e \sim \vartheta_{ch}$, where ϑ_{ch} is the critical channeling angle.[51-53] In the case of axial channeling $\vartheta_{ch} \sim \sqrt{Ze^2/\varepsilon a}$, and in the case of planar channeling $\vartheta_{ch} \sim \sqrt{RZe^2/\varepsilon a^2}$. In the derivation of the expressions of Sec. 7, channeling was not taken into account, and we therefore now consider bremsstrahlung under channeling conditions.[56]

Since a channeled particle interacts with a large number of atoms, the effective constant of the coupling of the particle with the atoms will be large, and therefore one can use classical theory to describe the motion of the particle in the channel.

Assuming that the motion of the particles in the channel is nearly periodic in the plane perpendicular to the channel axis, so that $\mathbf{r}(t+T)=\mathbf{r}(t)+\mathbf{v}T$, where T is the period of this motion, we readily obtain from (18) the following expression for the spectral density:

$$\frac{d\mathscr{E}_{ch}}{d\omega}=\frac{e^2}{(2\pi)^2}\int\frac{\sin^2(MT(\omega-\mathbf{kv})/2)}{\sin^2(T(\omega-\mathbf{kv})/2)}|\mathbf{k}\times\mathbf{I}_T|^2\,dO, \tag{105}$$

where

$$\mathbf{I}_T=\int_0^T dt\,\dot{v}(t)\exp[i(\omega t-\mathbf{kr}(t))].$$

If the particle makes M periodic motions in the channel, with $M \gg 1$, and the critical channeling angle is appreciably smaller than the characteristic angle of emission by the relativistic particle, $\vartheta_{ch} \ll \vartheta_{ch}$, then, in accordance with (28), we deduce from (105) that

$$\frac{d\mathscr{E}_{ch}}{d\omega}=M\frac{e^2}{T}\sum_{g\geqslant\delta'}\frac{\omega W^2(g)}{g^2}\left[1-2\frac{\delta'}{g}\left(1-\frac{\delta'}{g}\right)\right], \tag{106}$$

where

$$\mathbf{W}(g)=\int_0^\infty dt\,\dot{\mathbf{v}}_\perp(t)\exp(igt);$$

$g(2\pi/T)n$, $n=1,2,\ldots$ $\delta'=\omega m^2/2\varepsilon^2$, and $\mathbf{v}(t)$ are the components of the particle velocity in the plane perpendicular to the channel axis. The total energy emitted by the particle in the channel under the same conditions is equal to[3)]

$$\Delta\mathscr{E}_{ch}=M\frac{2e^2}{3}\left(\frac{\varepsilon}{m}\right)^4\int_0^T\dot{\mathbf{v}}_\perp^2(t)\,dt\,(1+O(\vartheta_{ch}/\vartheta_h)). \tag{107}$$

In the special case of motion of a particle in an oscillator potential,

$$\mathbf{v}_\perp(t)=\mathbf{v}_\perp\sin(2\pi t/T),\quad v_z\neq v$$

and

$$\mathbf{W}(g)=\mathbf{v}_\perp\delta_{n1}$$

(δ_{n1} is the Kronecker delta). Equation (106) takes the form[59,60]

$$\frac{d\mathscr{E}_{ch}}{d\omega}=\begin{cases} M\frac{e^2}{4\pi}\mathbf{v}_\perp^2\omega T\left[1-2\frac{\delta' T}{2\pi}\left(1-\frac{\delta' T}{2\pi}\right)\right], & \delta'\leqslant\frac{2\pi}{T};\\ 0, & \delta'=\omega m^2/2\varepsilon^2>2\pi/T\end{cases} \tag{108}$$

Thus, in the case of oscillator motion of the channeled particle its bremsstrahlung spectrum has a sharp peak in the region of frequencies $\omega \sim 4\pi\varepsilon^2/m^2T$.

Multiplying $d\mathscr{E}_{ch}/d\omega$ by the sum of the cross sections of all channels S, we obtain the bremsstrahlung intensity of a beam of channeled particles (for unit flux density of the incident particles):

$$dI_{ch}/d\omega \approx d\mathscr{E}_{ch}/d\omega S. \tag{109}$$

Let us compare the total intensity and the bremsstrahlung spectrum of channeled particles with the corresponding quantities for an amorphous medium and for coherent bremsstrahlung in the absence of channeling.

It is easy to see that (106) has the same structure as (92) for the intensity of coherent bremsstrahlung in a crystal. In both cases, the spectrum has sharp peaks, but they correspond to different frequencies: in the case of channeling, the peaks occur at $\omega_{ch}=(4\pi\varepsilon^2/Tm^2)n$; for coherent bremsstrahlung, at $\omega_c=(4\pi\Theta\varepsilon^2/am^2)n$.

Note that the characteristic oscillation periods of channeled electrons and positrons, T_- and T_+, are different and are determined by the ratio of the width of the channel to the critical channeling angle. For electrons, the width of the channel is in order of magnitude equal to the screening length, while for positrons it is equal to the lattice constant, and therefore $T_- \sim R/\vartheta_{ch}$ and $T_+ \sim a/\vartheta_{ch}$. Therefore, in accordance with (106), the bremsstrahlung intensities of channeled electrons and positrons differ strongly.

The main contribution to the total bremsstrahlung intensity of channeled particles is made by the region of frequencies near $\omega \sim \varepsilon^2/m^2T$. Since $|\mathbf{W}(g)| \sim \vartheta_e \sim \vartheta_{ch}$ in this region of frequencies,

$$dI_{ch}/d\omega \sim Me^2(\vartheta_{ch}/\vartheta_h)^2 S. \tag{110}$$

In the case of axial channeling of electrons $S \sim N_\perp R^2$ and $M \sim N_z a/T_-$, where $N_\perp$ is the number of channels and N_z is the number of atoms along the direction of motion of the particles, and $N_z N_\perp = N_c$. For frequencies $\omega \sim \varepsilon^2\vartheta_c/m^2R$, we find in accordance with (110) that

$$dI_{ch}/d\omega \sim N_c e^2 aR\vartheta_{ch}(\vartheta_{ch}/\vartheta_h)^2.$$

In an amorphous medium

$$dI_a/d\omega \sim N_c e^2(Ze^2/m)^2$$

and therefore

$$(dI_{ch}/d\omega)/dI_a/d\omega \sim R/a\vartheta_{ch}. \tag{111}$$

It is readily seen that for relativistic electrons this ratio is always large; for example, for silicon at $\varepsilon \sim 10$ GeV we have $dI_{ch}/d\omega \sim 10^3 \cdot dI_a/d\omega$.

[3)] In deriving Eqs. (106) and (107), which describe the bremsstrahlung of a channeled particle, we have used the same assumptions as are usually made in the theory of ondulator radiation.[57,58] Therefore, Eqs. (106) and (107) describe the bremsstrahlung of not only a channeled particle but also the bremsstrahlung in an ondulator, the only difference being that the physical causes leading to the periodic motion of the particle in the crystal and in the ondulator are different.

Note that if the channeling condition is not satisfied, i.e., for $\Theta > \vartheta_{ch}$, we have in accordance with (92) when $\omega \sim \varepsilon^2\Theta/m^2R$ the relation $dI_c/d\omega \sim (R/a\Theta)(dI_a/d\omega$. But if $\Theta < \vartheta_{ch}$, we cannot use the expression (92). The bremsstrahlung intensity is determined in this case by (106). However, in the case of axial channeling of the electrons the expression (92) still leads in order of magnitude to the correct result (111) if we set $\Theta \sim \vartheta_{ch}$ in (92).

In accordance with (107), the total bremsstrahlung intensity of channeled particles is

$$I_{ch} \sim N_c e^2 a (\vartheta_{ch}/\vartheta_k)^2;$$

for an amorphous medium[1]

$$I_a \sim N_c \varepsilon e^2 (Ze^2/m)^2.$$

In the theory of coherent bremsstrahlung, in accordance with (95),

$$I_{ch} \sim (\varepsilon/m^2 a) I_a.$$

Therefore, in the case of axial channeling of electrons

$$I_{ch}/I_a \sim \varepsilon/ma^2, \quad I_{ch} \sim I_c. \tag{112}$$

For silicon at $\varepsilon \sim 10$ GeV we have $I_{ch} \sim 10 I_a$.

Thus, in the case of bremsstrahlung of electrons in an axial channel the bremsstrahlung spectrum has a distinguished range of frequencies in which the bremsstrahlung intensity appreciably exceeds the bremsstrahlung intensity in an amorphous medium.

If the electrons are incident at a small angle to a crystallographic plane, planar channeling occurs. The period of oscillatory motion in a planar channel is somewhat greater than in the case of axial channeling, since the critical angle ϑ_{ch} is somewhat reduced in the planar case. A large number of beam particles is then in the channeling regime, since we now have $S_{pl} \sim (a/R) S_{ax}$. Therefore, in contrast to axial channeling, the maximum of the bremsstrahlung spectrum in the case of planar channeling is shifted to lower frequencies. To obtain the spectrum and total bremsstrahlung intensity, we must multiply the expressions (111) and (112) by a/R and take ϑ_{ch} to be the critical angle of planar channeling.

The period T_+ of channeled positrons is a/R times greater than the period T_- of channeled electrons because of the greater width of the channel. Therefore, the maximum of the bremsstrahlung spectrum of positrons is shifted to lower frequencies than the maximum of the electron spectrum, and its value is a/R times smaller than the bremsstrahlung intensity of electrons.

Therefore, the bremsstrahlung spectrum in the case of channeling differs not only from the bremsstrahlung spectrum in the absence of channeling but also from the bremsstrahlung spectrum in an amorphous medium. This spectrum depnds on the nature of the channeling and on the sign of the charge of the channeled particle.

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