Nuclear structure investigated using the system of principal axes of the ellipsoid of inertia

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The main results obtained by investigating the structure of light nuclei by means of generalized hyperspherical functions are presented. Considerable attention is devoted to the advantages of this method in comparison with the traditional approaches to the study of nuclear shape.

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INTRODUCTION

Since the pioneering work of Bohr and Mottelson,[1] it has been customary to investigate the interaction between collective and intrinsic motions in nonspherical nuclei by writing the wave function of a stationary state of a system of nucleons in the form

$$\Psi_M = \sum_{\Omega, K} \varphi^{I\tau}_{\Omega K}(\beta, \gamma)\, \chi_\Omega D^I_{MK}(\theta_i), \qquad (1)$$

where $\varphi^{I\tau}_{\Omega K}$ are the components of the collective wave function, χ_Ω is the intrinsic function, I is the angular momentum of the system, M is the projection of the angular momentum onto a fixed (nonrotating) axis, K is the projection of the angular momentum onto an intrinsic axis, τ is an additional quantum number, and Ω is the projection of the angular momentum of the intrinsic motion onto the intrinsic axis. The collective degrees of freedom β, γ, θ_i in the expression (1) represent vibrations and rotation of the ellipsoidal surface of the nucleus. The expression (1) arose out of attempts to extend the collective model to odd nuclei, and also to take into account not only collective but other degrees of freedom.

In the earliest and simplest variants of the generalized Bohr-Mottelson model, the intrinsic function χ_Ω was associated with only the states of an individual nucleon, but following Nilsson's work[2] the intrinsic function χ_Ω came to be regarded as a Slater determinant constructed from the single-particle orbitals of the nucleons of the nucleus in the average nonspherical field. The equipotentials of the average field have the form of ellipsoidal surfaces whose principal axes are directed along the principal axes of the ellipsoidal surface of the nucleus. One can therefore say that the intrinsic axes of the nucleus are determined in such a model as the axes of the intrinsic average nonspherical field. As a rule, it is assumed that the intrinsic field has a symmetry axis and that Ω and K are integrals of the motion.

The Slater determinant of the intrinsic function of heavy nonspherical nuclei is constructed from the single-particle orbitals in an average field whose potential (Nilsson potential or deformed Woods-Saxon potential) is chosen to satisfy the condition that the successive population of the states of the potential leads to the observed sequence of spins of the odd nuclei and also the condition that all the other results obtained in calculations with such a single-particle potential agree with the experiments. For nuclei of the p shell and nuclei of the s–d shell, the parameters of the potential of the average field (or, which is the same thing, the parameters of the single-particle orbitals of the Slater determinant) are found by minimizing the energy of the system of mutually interacting nucleons of the nucleus.[3]

In any model with an intrinsic function, problems arise through the fact that in the intrinsic frame the coordinates of the individual nucleons are not independent but are constrained through subsidiary conditions—the intrinsic motion must not lead to a displacement of the center of mass of the system or of the orientation of the intrinsic axes of the system, the parameters β and γ describing the quadrupole shape of the nuclear surface must not change, and the same applies to the volume of the nucleus. To take into account all these constraints, it is necessary to give a microscopic meaning to the collective degrees of freedom β, γ, θ_i and the parameter that determines the volume of the nucleus by expressing these degrees of freedom in terms of the single-particle coordinates of the nucleons. In addition, it is necessary of specify the explicit dependence of the collective variables on the nucleon coordinates, since one cannot otherwise uniquely and consistently determine many operators used to calculate the spectra of nuclei, the angular momenta in different states, and the transition probabilities.

The above-mentioned difficulty still remains in the case when all the collective degrees of freedom are coupled to just the parameters of the nonspherical intrinsic field and one does not get involved with questions relating to the shape of the nuclear surface as, for example, in the investigation of light nuclei, which do not have a well-defined surface.

In none of the currently considered models of the intrinsic functions are couplings between the nucleon coordinates in the intrinsic frame taken into account, nor is any dependence of the collective variables on the single-particle variables introduced. In all cases, the arguments of the Slater determinant are single-particle coordinates of the nucleons that vary independently in their domain of definition. Essential use of this is made in the integration of the square of the Slater determinant when, for example, the intrinsic quadrupole moment is calculated or when the distribution function of the single-particle density of nucleons in the intrinsic coordinate system is found,[4,5] or, finally, when the projection algorithm is implemented. Therefore, the intrinsic functions are not what they purport to be and do not give direct information related to the intrinsic axes of the nucleus. The coordinate axes with respect to which they are defined are fixed axes, and the orbitals of the Slater determinant belong in reality to the fixed, and not intrinsic, field. The wave functions themselves contain packets of states with different values of the total orbital angular momentum of the nucleus and its projection onto the fixed axis, although these packets are constructed in such a way as to ensure that the axes of the intrinsic system are oriented along the axes of the fixed system. This conclusion is in complete accord with the Peierls–Yoccoz projection algorithm,[6] which is based on a packet of states that is noninvariant under rotation of the axes of the fixed frame. Note that the intrinsic function in the expression (1) must depend only on the intrinsic variables and, therefore, must be invariant under rotations of the fixed coordinate system.

All that remains therefore is to assume that under certain conditions it is possible to ignore the constraints that the nucleon coordinates should satisfy in the intrinsic frame and to identify the intrinsic frame with the fixed frame determined by the axes of the fixed average field. However reasonable this assumption may appear, it requires justification; the only question concerns the construction of the theoretical scheme which can provide this justification and yield the conditions under which the intrinsic and fixed frames can be identified.

1. CHOICE OF THE INTRINSIC COORDINATE SYSTEM

The simplest possibility for introducing an intrinsic coordinate system in order to construct the intrinsic wave function is to place its origin at the center of mass of the nucleus and direct the axes along the principal axes of the ellipsoid of inertia. Such an idea for selecting the intrinsic coordinate system was put forward long ago, soon after the appearance of the Bohr–Mottelson collective model[1,7,8] in connection with the need to justify this model.[8-10] However, the idea can be implemented only after we have found all the independent configuration variables of the wave function of A nucleons suitable for working in a coordinate system whose axes are directed along the principal axes of the ellipsoid of inertia.[11-13]

In order to take into account the couplings between the single-particle coordinates of the A nucleons in the system of the principal axes of the ellipsoid of inertia, it is expedient to introduce intrinsic variables in such a way that the variation of these variables does not change the position of the center of mass, the orientation of the axies of the ellipsoid of inertia, or the values of its semiaxes. Intrinsic variables with this property are $3A$-9 generalized Eulerian angles α_i in an $(A$-1)-dimensional space.[11] Together with the spin and isospin variables, these must be the arguments of the components χ_ν of the intrinsic function. If as arguments of the components $u_\nu^{I\tau}$ of the collective function we choose the semiaxes a, b, c of the ellipsoid of inertia or the related variables ρ, β, γ (β and γ are analogous to the corresponding Bohr–Mottelson collective variables),

$$\left.\begin{aligned}&\rho^2=a^2+b^2+c^2;\ \rho^2\beta\cos\gamma=c^2-(a^2+b^2)/2;\\&\rho^2\beta\sin\gamma=(\sqrt{3}/2)(a^2-b^2),\end{aligned}\right\}\qquad(2)$$

and as Eulerian angles φ, θ, ψ we choose the angles of orientation of the axes of the ellipsoid of inertia, then the expansion for the nuclear wave function takes the form

$$\Psi_M^{I\tau}=\sum_{\nu,K}u_{\nu K}^{I\tau}(abc)\,D_{MK}^{I}(\varphi\theta\psi)\,\chi_\nu(\{\alpha_i\},\{\tau_j\},\{\sigma_l\}),\qquad(3)$$

where σ_l are the spin variables of the A nucleons, τ_j are the isospin variables, and α_i are the $3A$-9 generalized Eulerian angles in the $(A$-1)-dimensional space chosen in accordance with a definite rule.[11-13]

The expansion (3) can be used irrespective of whether the nucleus is spherical or not. The index ν is related to the indices of the representations and the basis indices of the rotation group in the $(A-1)$-dimensional space, and the components χ_ν of the intrinsic function are superpositions of generalized hyperspherical functions in this space.[13,14] The series with respect to the index ν is infinite, and therefore the exact intrinsic function and the exact collective function contain infinitely many components. However, in approximate expressions for $\Psi_M^{I\tau}$, the series in ν can be truncated, which corresponds to a definite form of the model. Relations between the Jacobi vectors of the system of A nucleons and the three sets of fixed variables $(a, b, c; \varphi, \theta, \psi; \alpha_i)$ were established in Ref. 11. By comparing the collective variables a, b, c and φ, θ, ψ with the collective variables of the Bohr–Mottelson model one can establish the meaning of the former and demonstrate their usefulness for studying quadrupole deformations of nuclei.

In accordance with the ideas used in the Bohr–Mottelson collective model, a nucleus can be likened to a drop of incompressible nuclear matter. During small quadrupole vibrations of this drop, the drop preserves its ellipsoidal shape, but the relationships between the semiaxes of the ellipsoid change. In addition, the ellipsoidal drop rotates in space. The collective wave function of the Bohr–Mottelson model depends on the Eulerian angles φ, θ, ψ, which determine the orientation of the ellipsoidal drop in space, and also on the variables $\tilde{\beta}$ and $\tilde{\gamma}$,[1)] which characterize the relationships between the semiaxes of an ellipsoidal drop of fixed volume and determine the shape of the nucleus in the intrinsic coordinate system associated with the principal axes of the drop. If ellipsoidal deformations of the incompressible drop and its rotation in space are associated with the motion of the ellipsoid of inertia of the drop, the relationships between the semiaxes of the ellipsoid of inertia will be identical to those between the semiaxes of the ellipsoidal drop, and the Eulerian angles of the orientation of the axes of the ellipsoid of in-

[1)]To distinguish the variables β and γ introduced above [see Eq. (2)] from the Bohr–Mottelson variables, we shall place a tilde above the latter.

ertia will be equal to the Eulerian angles of rotation of the ellipsoidal drop, i.e., the collective variables associated with the ellipsoid of inertia of the system will in this case lead to nothing new and will literally reproduce the motion of the ellipsoidal drop.

However, if we describe the system of particles by means of all of its degrees of freedom and if the arguments of the wave function are the single-particle variables, then the choice of the three semiaxes of the ellipsoid of inertia and the three Eulerian angles of orientation of the ellipsoid of inertia as collective variables makes it possible to separate the quadrupole motions of the system (the motions of the ellipsoid of inertia) from the intrinsic motions, which leave the ellipsoid of inertia unchanged, and to find the components of the collective wave function corresponding to a particular microscopic model of the nucleus or a particular approximate solution of the many-nucleon Schrödinger equation; this choice therefore enables us to obtain information about the quadrupole deformations of the nucleus in the intrinsic coordinate system associated, as in the Bohr–Mottelson model, with the principal axes of the ellipsoid of inertia. Hitherto, such information has been extracted from model or approximate many-particle wave functions by methods which give the parameters of the quadrupole deformations (of the nuclear shape) referred to a coordinate system which is not attached to the intrinsic axes of the nucleus.

The configuration variables listed above $(a, b, c; \varphi, \theta, \psi; \alpha_i)$ were used to formulate the method of generalized hyperspherical functions in Refs. 13 and 15.

2. METHOD OF GENERALIZED HYPERSPHERICAL FUNCTIONS

The method of generalized hyperspherical functions is a method of solving the many-particle Schrödinger equation for a system of nucleons with given interaction potential. This method presupposes a representation of the nuclear wave function in the form of the series (3) with respect to a basis of generalized hyperspherical functions χ_ν. The expansion (3), constructed for known model and approximate functions, makes it possible to determine the nature of the collective and intrinsic motions, to establish the intrinsic shape of the nuclei in different states, and to obtain the distribution of the nucleon density in the intrinsic coordinate system.

The functions χ_ν, like the wave functions of the shell model, are antisymmetric under permutation of the nucleons, but, in contrast to the shell functions, are invariant under translations of the center of mass of the system and rotations of the coordinate axes. Moreover, the functions χ_ν are invariant under deformations of the ellipsoid of inertia of the system of A nucleons, i.e., under redistributions of the masses of the nucleons in space that change only the semiaxes of the ellipsoid of inertia and their orientation in space; this makes the functions χ_ν convenient for separating the intrinsic motion from the collective motions of monopole and quadrupole type.

The basis of the functions χ_ν is complete, and a function of the generalized Eulerian angles that has no singularities can always be represented as a series in χ_ν. However, in regions of the configuration space of the single-particle variables where the wave function depends on the generalized Eulerian angles in a δ-functional manner, the convergence of the expansion (3), like the analogous expansion of the wave function with respect to the hyperspherical basis,[16] may be poor. A δ-functional dependence on the generalized Eulerian angles is inherent in wave functions of noncompact configurations when isolated subsystems are distinguished. Such a dependence can also be observed in the case of compact configurations if short-range repulsive forces act between the particles of the system. Then if any two particles approach each other, the wave function tends rapidly to zero once distances equal to the range of the repulsive forces have been reached. Therefore, the first terms of the expansion (3) approximate the wave function well only under the condition that the regions of the configuration space where the convergence is poor make a small contribution to the total normalization integral.

The various terms in the expansion (3) can be interpreted from the point of view of the part they play in reproducing the details of the nucleon density distribution in the coordinate system associated with the ellipsoid of inertia of the nucleus and also in the description of the collective excitations. This interpretation may have heuristic value if in the expansion (3) it is necessary to take into account many terms and one therefore needs a principle for selecting the principal terms among them.

Like the minimal approximation in the method of hyperspherical functions,[17,18] the minimal approximation in the method of generalized hyperspherical functions requires the retention in (3) of only those functions χ_ν that are contained in the wave function of the oscillator shell model for the ground state of the considered nucleus with A nucleons. Of course, even the minimal approximation of the method of generalized hyperspherical functions provides a more accurate description of the properties of the ground state of the nucleus and some of its collective excitations than the oscillator shell model or the minimal approximation of the method of K harmonics. This is achieved by an optimal choice of the functions $u^{l}_{\nu K}$, for which one can find a system of equations, by the use of the variational principle, or by direct substitution of the expression (3) in the Schrödinger equation for the system of A nucleons.

The minimal approximation corresponds to choosing the simplest functions χ_ν of the intrinsic motion. It can be assumed that the spectrum of monopole and quadrupole excitations, which is determined merely by the averaged characteristics of the intrinsic motion, is satisfactorily reproduced even when the chosen functions of the intrinsic motion need to be made more exact.

The wave function of the oscillator shell model for the ground state of magic nuclei contains just a single function $\chi_\nu = \chi_0$, which considerably simplifies the problem. The spectrum of monopole and quadrupole vibrations of magic nuclei in the minimal approximation was calculated in Refs. 19–21. For nuclei with oscillator shells that are not closed, even the minimal approximation involves several functions χ_ν.

3. BASIS FUNCTIONS

The structure of the basis functions χ_ν was investigated in Refs. 22–24, 14. These functions can be represented as superpositions of generalized hyperspherical functions $D^{[f]}_{[\nu][\mu]}$ of an $(A-1)$-dimensional space:

$$\chi^{[f_1f_2f_3]}_{[\nu]} = \sum_{[\mu]} D^{[f_1f_2f_3]}_{[\nu][\mu]}(\alpha_1, \alpha_2, \ldots, \alpha_{3A-9}) F_{[\mu]}(\sigma_1\tau_1; \sigma_2\tau_2; \ldots; \sigma_A\tau_A). \qquad (4)$$

The dependence of the superposition coefficients $F_{[\mu]}$ on the spin-isospin variables is chosen to make the function $\chi^{[f_1f_2f_3]}_{[\nu]}$ antisymmetric under transposition of the spatial and spin-isospin coordinates of any pair of nucleons.

The generalized hyperspherical functions $D^{[f]}_{[\mu][\nu]}$ are matrix elements of the matrices of the irreducible representations of the rotation group in the $(A-1)$-dimensional space. The superscript $[f_1f_2f_3]$ of the D functions gives the representation to which the functions belong, while the subscripts $[\nu]$ and $[\mu]$ label the rows and columns of the matrix of the irreducible representation.

In Eq. (4), the arguments of the functions D are the generalized Eulerian angles that specify the orientation of three mutually orthogonal vectors in the $(A-1)$-dimensional space with respect to some original coordinate system, which we call the *fixed* $(A-1)$-dimensional coordinate system. In addition, we shall consider a frame, called the *privileged* frame, which we attach to three distinguished vectors (the remaining $A-4$ unit vectors of the privileged frame can be specified arbitrarily).

The right-hand subscripts $[\mu]$ of a D function characterize its transformation properties under rotations made in the coordinate planes of the fixed system, while the left-hand subscripts characterize the transformation properties of the D function under rotations in planes of the privileged frame.

If $A>7$, the subscript $[\mu]$ of the function D of the representation $[f_1f_2f_3]$ contains $3A-15$ numbers, which are determined by the canonical Gel'fand—Tseĭtlin reduction. These numbers are conveniently distributed among three sets of $A-3$, $A-5$, and $A-7$ elements, respectively. The elements of the first set we denote by μ_{i1}, where $i=1,2,\ldots,A-3$; those of the second set, by μ_{j2}, where $j=1,2,\ldots,A-5$; and those of the third set, by μ_{k3}, where $k=1,2,\ldots,A-7$. In row $[\mu]$, these sets are listed successively one after the other, beginning with the first.

To each permutation of the nucleons, there corresponds a definite rotation in the $(A-1)$-dimensional space of the fixed coordinate system and, therefore, a definite set of generalized Eulerian angles β_l specifying this rotation. The transformation of the generalized hyperspherical functions under permutation of the nucleons is expressed by the equation

$$D^{[f_1f_2f_3]}_{[\nu][\mu]}(\{\alpha_i\}) = \sum_{[\mu']} D^{[f_1f_2f_3]}_{[\mu'][\mu]}(\{\beta_k\}) D^{[f_1f_2f_3]}_{[\nu][\mu']}(\{\alpha_i'\}), \qquad (5)$$

where α_i are the generalized Eulerian angles in the original coordinate system, and α_i' are the generalized Eulerian angles in the system that is rotated with respect to the first.

Reduction of the group $O(A-1)$ with respect to the group of permutations of the coordinates of the A nucleons means that, given functions $D^{[f_1f_2f_3]}_{[\nu][\mu]}$ of the representation $[f_1f_2f_3]$ with fixed index $[\nu]$, one can construct basis vectors corresponding to particular irreducible representations of the symmetric group.

The invariance group of the Hamiltonian of the system of A nucleons is the group of 24 different transformations of the privileged frame. Besides the four transformations of the group D_2 (the identity transformation and rotations through the angle π in the three planes of the privileged frame that each pass through two principal unit vectors), this group contains the six elements of the privileged frame. The Jacobi vectors are invariant under transformations of this group. The wave function of the system of A nucleons must therefore also be an invariant of the group.

The basis functions of the irreducible representations of the symmetry group of the privileged frame can be constructed from linear combinations of the generalized hyperspherical functions $D^{[f_1f_2f_3]}_{[\nu][\mu]}$ with fixed indices $[f_1f_2f_3]$ and $[\mu]$.

The functions $F_{[\mu]}$ in (4) belong to a state with a definite value of the total spin S and the total isospin T of the system of A nucleons. In addition, they are basis vectors of a definite irreducible representation of the group of permutations of A nucleons. Summation over the index $[\mu]$ ensures antisymmetry of the function $\chi^{[f_1f_2f_3]}_{[\nu]}$ under permutation of the nucleon coordinates.

The representations $[f_1f_2f_3]$ are only some of the total number of representations of the group $SO(A-1)$. The basis functions of all the remaining representations $[f_1f_2f_3, \ldots, f_{[(A-1)/2]}]$ have more than three nonzero indices f_i and depend not only on the generalized Eulerian angles $\alpha_1\alpha_2, \ldots, \alpha_{3A-9}$ but also on the other Eulerian angles [in the $(A-1)$-dimensional space there are altogether $(A-1)(A-2)/2$ of them], on which the wave function of the system of particles cannot depend because of its invariance under all possible rotations in planes of the privileged coordinate system. The planes pass through unit vectors perpendicular to the three principal unit vectors of the system. Therefore, all such representations must be discarded. For the same reason, among all the basis functions $D^{[f_1f_2f_3]}_{[\nu][\mu]}$ we must retain only the ones for which in the first and second column of indices $[\nu]$ of the Gel'fand—Tseĭtlin reduction only two quantum numbers and one quantum number, respectively, are nonzero, i.e., $[\nu] \to [\nu_{11}, \nu_{21}; \nu_{12}]$. Instead of using the quantum numbers $\nu_{11}, \nu_{21}, \nu_{12}$, we can classify the basis functions of the representation $[f_1f_2f_3]$ by using the quasiangular momentum j, its projection j_ζ, and the additional quantum number s: $[\nu] \to \{jj_\zeta s\}$. The quasiangular momentum and its projection are the total orbital angular momentum of the intrinsic subsystem determined by the generalized Eulerian angles and its projection onto one of the principal axes of the ellipsoid of inertia.

We have already pointed out above that the motion of the nucleons in the system of the principal axes of the ellipsoid of inertia cannot be described by independently varying single-particle coordinates, since nine additional constraints satisfied by the nucleon coordinates in this system must be taken into account. The presence of the constraints reflects the fact that in this system we have already eliminated nine degrees of freedom, or distinguished a subsystem—the ellipsoid of inertia. Moreover, the second subsystem (the intrinsic subsystem), whose degrees of freedom are the generalized Eulerian angles, executes its motion with respect to the principal axes of the ellipsoid of inertia. Therefore, in the kinetic-energy Hamiltonian $\hat{T}$ of the system (see Ref. 13) two angular momentum operators occur. One of them is the operator $\hat{I}$ of the total angular momentum, while the second is the operator $\hat{J}$ of the quasiangular momentum. They can both be expressed in terms of projections onto the principal axes of the ellipsoid of inertia:

$$\hat{\mathbf{I}}=\{\hat{I}_\xi,\ \hat{I}_\eta,\ \hat{I}_\zeta\};\ \hat{\mathbf{J}}=\{\hat{J}_\xi,\ \hat{J}_\eta,\ \hat{J}_\zeta\};$$

$$\hat{T}=\hat{T}_{abc}+\frac{\hbar^2}{2m}\Big\{\frac{b^2+c^2}{(b^2-c^2)^2}(\hat{I}_\xi^2+\hat{J}_\xi^2)+\frac{c^2+a^2}{(c^2-a^2)^2}(\hat{I}_\eta^2+\hat{J}_\eta^2)$$
$$+\frac{a^2+b^2}{(a^2-b^2)^2}(\hat{I}_\zeta^2+\hat{J}_\zeta^2)+\frac{4bc}{(b^2-c^2)^2}\hat{I}_\xi\hat{J}_\xi+\frac{4ca}{(c^2-a^2)^2}I_\eta J_\eta$$
$$+\frac{4ab}{(a^2-b^2)^2}\hat{I}_\zeta\hat{J}_\zeta+\frac{1}{3}\Big(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\Big)\hat{L}_0$$
$$+\frac{2}{3}\Big(\frac{1}{c^2}-\frac{1}{2a^2}-\frac{1}{2b^2}\Big)\hat{L}_{20}+\frac{2}{3}\frac{\sqrt{3}}{2}\Big(\frac{1}{a^2}-\frac{1}{b^2}\Big)L_{22+}\Big\}.$$

All first and second derivatives with respect to a, b, c are included in the term $\hat{T}_{abc}$, and the operators $\hat{L}_0, \hat{L}_{20}, \hat{L}_{22+}$ are constructed from three expressions $\hat{L}_\xi, \hat{L}_\eta, \hat{L}_\zeta$ that are quadratic in the generators $\hat{J}_{ik}$ of the group $SO(A-1)$:

$$\hat{L}_0=\hat{L}_\xi+\hat{L}_\eta+\hat{L}_\zeta;\ \hat{L}_{20}=\hat{L}_\zeta-(\hat{L}_\xi+\hat{L}_\eta)/2;\ \hat{L}_{22+}=(\sqrt{3}/2)(\hat{L}_\xi-\hat{L}_\eta),$$

where

$$\hat{L}_\xi=\sum_{i=1}^{A-4}J_{i\xi}^2;\quad \hat{L}_\eta=\sum_{i=1}^{A-4}J_{i\eta}^2;\quad \hat{L}_\zeta=\sum_{i=1}^{A-4}J_{i\zeta}^2.$$

The generalized hyperspherical function $D^{[f_1f_2f_3]}_{\{jj_\zeta s\}[\mu]}$ is an eigenfunction of the operator $\hat{L}_0$;

$$\hat{L}_0D^{[f_1f_2f_3]}_{\{jj_\zeta s\}[\mu]}=[f_1(f_1+A-3)+f_2(f_2+A-5)$$
$$+f_3(f_3+A-7)-j(j+1)]\,D^{[f_1f_2f_3]}_{\{jj_\zeta s\}[\mu]}.$$

When the operators $\hat{L}_{20}$ and $\hat{L}_{22+}$ are applied to the functions D, they do not change their superscripts $[f]$ or the right-hand subscripts $[\mu]$, but they can change the left-hand subscripts. In particular, the operator $\hat{L}_{20}$ conserves the projection of the quasiangular momentum, while the operator $\hat{L}_{22+}$ increases or decreases it by two. Both of these operators can increase or decrease (by not more than two) the value of the quasiangular momentum.

The action of the operators $\hat{J}_\xi, \hat{J}_\eta, \hat{J}_\zeta$ of the quasiangular momentum on the generalized hyperspherical function $D^{[f_1f_2f_3]}_{\{jj_\zeta s\}[\mu]}$ is subject to the same rules as the action of the operators of the angular-momentum projections on a state with definite angular momentum and its projection.

The Cartesian projections of the Jacobi vectors of the system of A nucleons can be expressed in terms of the generalized Eulerian angles by means of generalized hyperspherical functions belonging to the representation [1]:

$$\left.\begin{aligned}
q_{ml}=&-\frac{ia}{\sqrt{2}}(D^1_{1m}+D^1_{-1m})\,D^{[1]}_{[11][11\ldots1]}\\
&+\frac{b}{\sqrt{2}}(D^1_{1m}-D^1_{-1m})\,D^{[1]}_{[1][11\ldots1]}+cD^1_{0m}D^{[1]}_{[0][11\ldots1]},\\
&m=0,\ \pm1;\\
D^1_{km}=&D^1_{km}(\varphi\theta\psi);\quad D^{[1]}_{[\nu][\mu]}=D^{[1]}_{[\nu][\mu]}(\{\alpha_i\}),
\end{aligned}\right\}\qquad(6)$$

where D^1_{km} are spherical Wigner functions; a, b, c and φ, θ, ψ are collective variables; m is the index of the Cartesian coordinates; and l is the number of the Jacobi vector. The right-hand subscript of the function $D^{[1]}_{[\nu][11\ldots1]}$ contains $l-1$ unities.

The following equations illustrate the connection between the basis classified in accordance with the quasiangular momentum and its projection and the basis classified in accordance with the quantum numbers ν_{11}, ν_{21}, ν_{12}:

$$D^{[1]}_{\{10\}[\mu]}=D^{[1]}_{[0][\mu]};\quad D^{[1]}_{\{1\pm\}[\mu]}=(\pm1/\sqrt{2})(D^{[1]}_{[1][\mu]}\pm D^{[1]}_{[11][\mu]}).\qquad(7)$$

Among the simplest of the generalized hyperspherical functions

$$D^{[f_1f_2f_3]}_{[\nu_{11}\nu_{21}\nu_{12}][\mu_{11}\mu_{21}\ldots\mu_{A-3,1};\,\mu_{12}\mu_{22}\ldots\mu_{A-5,2};\,\mu_{13}\mu_{23}\ldots\mu_{A-7,3}]}$$

are the functions of the zeroth row of the representation $[f00]$. If the notation adopted in the monograph of Ref. 26 is used for the generalized Eulerian angles α_i, the functions of this row can be represented in the form

$$D^{[1]}_{[0][0]}=\cos\theta^{n-1}_{n-1};\quad D^{[1]}_{[1][0]}=\sin\theta^{n-1}_{n-1}\cos\theta^{n-2}_{n-2};$$
$$D^{[1]}_{[11][0]}=\sin\theta^{n-1}_{n-1}\sin\theta^{n-2}_{n-2}\cos\theta^{n-3}_{n-3}.$$

The volume element for the angles $\theta^{n-1}_{n-1}, \theta^{n-2}_{n-2}, \theta^{n-3}_{n-3}$ is

$$\sin^{n-2}\theta^{n-1}_{n-1}d\theta^{n-1}_{n-1}\sin^{n-3}\theta^{n-2}_{n-2}d\theta^{n-2}_{n-2}\sin^{n-4}\theta^{n-3}_{n-3}d\theta^{n-3}_{n-3},$$
$$0\leqslant\theta^{n-1}_{n-1},\ \theta^{n-2}_{n-2};\ \theta^{n-3}_{n-3}\leqslant\pi,\ n=A-1>5.$$

We have the formula

$$D^{[f]}_{[k_1k_2][0]}\sim2^{(3n-9)/2+k_1+k_2}\Gamma\Big(\frac{n-2}{2}+k_1\Big)\Gamma\Big(\frac{n-3}{2}+k_2\Big)\Gamma\Big(\frac{n-4}{2}\Big)$$
$$\times\Big[\frac{(f-k_1)!\,(k_1-k_2)!\,k_2!}{(2\pi)^3}\,\frac{[(n-2)/2+f]\,[(n-3)/2+k_1]\,[(n-4)/2+k_2]}{\Gamma(n-2+f+k_1)\,\Gamma(n-3+k_1)\,\Gamma(n-4+k_2)}\Big]^{1/2}$$
$$\times C^{(n-2)/2+k_1}_{f-k_1}(\cos\theta^{n-1}_{n-1})\,C^{(n-3)/2+k_2}_{k_1-k_2}(\cos\theta^{n-2}_{n-2})\,C^{(n-4)/2}_{k_2}(\cos\theta^{n-3}_{n-3})$$
$$\times\sin^{k_1}\theta^{n-1}_{n-1}\sin^{k_2}\theta^{n-2}_{n-2}.$$

This establishes the connection between generalized hyperspherical functions of the zeroth row and the Gegenbauer polynomials $C^r_m(t)$.

The generalized hyperspherical functions of the representation [1] with maximal value of the right-hand subscript have the form

$$D^{[1]}_{[0][11\ldots1]}=(-1)^n\frac{1}{\sqrt{2}}\sin\theta^{n-1}_{n-1}\sin\theta^{n-1}_{n-2}\ldots\sin\theta^{n-1}_2\exp(i\theta^{n-1}_1);$$
$$D^{[1]}_{[1][11\ldots1]}=(-1)^n\frac{1}{\sqrt{2}}\Big\{\cos\theta^{n-1}_{n-1}\sin\theta^{n-1}_{n-2}\ldots\sin\theta^{n-1}_2\cos^{n-2}_{n-2}$$
$$+\sum_{s=1}^{n-3}\cos\theta^{n-1}_{n-1-s}\sin^{n-1}_{n-2-s}\ldots\sin\theta^{n-1}_2\sin\theta^{n-2}_{n-2}\ldots\cos\theta^{n-2}_{n-2-s}$$
$$+i\sin\theta^{n-2}_{n-2}\sin\theta^{n-2}_{n-3}\ldots\sin\theta^{n-2}_1\Big\}\exp(i\theta^{n-1}_1).$$

The limit formula is

$$D^{[f]}_{[k_1k][0]} \sim \text{const}\left(\frac{n}{2}\right)^{3/2} \frac{H_{f-k_1}(z)\, H_{k_1-k_2}(y)\, H_{k_2}(x)}{\sqrt{2^f (f-k_1)!\,(k_1-k_2)!\,k_2!}}$$

for

$$n \to \infty;\ f/n,\ k_1/n,\ k_2/n \ll 1,$$

where

$$\sqrt{2/n}\, z = \cos\theta^{n-1}_{n-1};\quad \sqrt{2/n}\, y = \cos\theta^{n-2}_{n-2};\quad \sqrt{2/n}\, x = \cos\theta^{n-3}_{n-3};$$
$$(1-2x^2/n)^{n/2}(1-2y^2/n)^{n/2}(1-2z^2/n)^{n/2}\, dx\, dy\, dz \to \exp(-x^2-y^2-z^2)\, dx\, dy\, dz.$$

4. EXPANSION IN GENERALIZED HYPERSPHERICAL FUNCTIONS OF THE WAVE FUNCTIONS OF THE TRANSLATIONALLY INVARIANT SHELL MODEL

Having gone over from the single-particle vectors $\mathbf{r}_i$ of the system of A nucleons to the center-of-mass coordinates R_x, R_y, R_z, the collective variables a, b, c and φ, θ, ψ, and the generalized Eulerian angles $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{3A-9}$, we can represent the wave functions of the oscillator shell model as superpositions of the functions $\chi^{[f]}_{[\nu]}$. Let $\tilde{\Psi}(\mathbf{r}_1\sigma_1\tau_1; \mathbf{r}_2, \sigma_2\tau_2 \ldots; \mathbf{r}_A\sigma_A\tau_A)$ be one such shell function belonging to a state whose orbital angular momentum is L with projection M of the angular momentum onto the fixed axis. Suppose, in addition, that on the transition to the center-of-mass system this wave function factorizes into the form

$$\tilde{\Psi} = \Psi F(R), \tag{8}$$

where R is the vector of the center of mass of the system, and $F(R)$ is the wave function of the zero-point vibrations of the center of mass. Then Ψ, the wave function of the oscillator shell model with the center-of-mass motion separated (the wave function of the translationally invariant shell model[27]), has the expansion

$$\Psi = \Psi_{LM} = \sum_{[f][\nu]k} \Phi^{[f]L}_{[\nu]K}(abc)\, D^L_{KM}(\varphi\theta\psi)\, \chi^{[f]}_{[\nu]}. \tag{9}$$

By choosing an appropriate linear combination of Slater determinants, one can construct Ψ_{LM} which are such that they contain $\chi^{[f]}_{[\nu]}$ with only definite values of the representation indices $[f]$.

Now the functions Ψ_{LM} are eigenfunctions of the Hamiltonian of a system of A particles with the two-body potential

$$\text{const}\,(\mathbf{r}_i - \mathbf{r}_j)^2.$$

The representation indices $[f]$ are integrals of the motion of such a system. Together with the right-hand indices $[\mu]$ of the generalized hyperspherical functions, the quantities

$$D^{[f]}_{[\nu][\mu]}(\{\alpha_i\})$$

are such integrals of the motion.

However, wave functions of definite symmetry with respect to permutation of the spatial coordinates, i.e., basis functions of irreducible representations of the group of permutations, are linear combinations of functions with different values of $[\mu]$.

When the group of $(A-1)$-dimensional rotations is reduced with respect to the permutation group of the A particles, an additional degeneracy arises if there is a multiplicity greater than unity for the occurrence in a reducible representation of the permutation group determined by the basis functions $D^{[f]}_{[\nu][\mu]}$ with fixed indices $[f]$ and $[\nu]$ of any of the irreducible representations of the permutation group that are conjugate with respect to the symmetry of the isospin function.

Thus, the representation indices $[f]$, the Young pattern of the permutation group, and the repetition indices that arise on the reduction of the group of $(A-1)$-dimensional rotations with respect to the permutation group are, together with the orbital angular momentum L, the spin, and the isospin, are quantum numbers needed to identify such states of the system of nucleons as are represented in the form of a superposition of determinants of the oscillator shell model. If the set of quantum numbers is to be made complete, one must find a further seven integrals of the motion. These integrals can be established by solving the system of equations for the collective functions.

In the method of generalized hyperspherical functions, the representations $[f]$ of the group $SO(A-1)$ have the same significance as the configurations of single-particle states in the shell model. But, in contrast to the latter, every representation $[f]$ together with its corresponding basis can be associated not with one state of the nucleus, but with a complete spectrum of states with several branches of collective excitations.

In the expansion (9), the coefficients $\Phi^{[f]}_{[f]K}$ are fixed in accordance with the requirement that Ψ be one of the functions of the translationally invariant shell model and, therefore, one of the eigenfunctions of the Hamiltonian of A nucleons with an oscillator two-body interaction. In this case, when one solves the problem of the motion of nucleons with an arbitrary interaction, the coefficients $u^{[f]L}_{[\nu]k}$ must be found from the system of equations of the method of generalized hyperspherical functions.

To derive equations for the coefficients $u^{[f]}_{[\nu]K}$, it is first of all necessary to list the quantum numbers of the basis functions $\chi^{[f]}_{[\nu]}$ that are used to construct the expansion of the wave function of the A-nucleon system. We can do this readily if we restrict ourselves to the minimal approximation and construct the expansion using only basis functions $\chi^{[f]}_{[\nu]}$ in the set of major configurations of the oscillator shell model. It is also easy to find the coefficients $\Phi^{[f]}_{[\nu]K}$ of the major configurations of the shell model. These are useful not only in an investigation of the shape of the nucleon density distribution given by the shell model but also as structural elements of trial functions for direct variational calculations.

Below, we give simple examples that illustrate the quantum numbers of the minimal basis and the oscillator coefficients for systems of six and eight nucleons.

The wave functions of the oscillator shell model (without allowance for the factor which contains the dependence on the center-of-mass coordinates) corresponding to the ground state of the ^{6}He nucleus and zero total orbital angular momentum of the system has the form

$$\Psi_0^{[2]}=\Phi_{\{0\}0}^{[2]0}\chi_{\{0\}}^{[2]}+\Phi_{\{20\}0}^{[2]0}\chi_{\{20\}}^{[2]}+\Phi_{\{22+\}0}^{[2]}\chi_{\{22+\}}^{[2]}, \tag{10}$$

where

$$\left.\begin{aligned}&\Phi_{\{\nu\}0}^{[2]0}=N_n\tilde{\Phi}_{\{\nu\}}^{[2]}\exp[-(a^2+b^2+c^2)/2];\\ &\tilde{\Phi}_{\{0\}}^{[2]}=1\,(a^2+b^2+c^2)/\sqrt{5};\ \tilde{\Phi}_{\{20\}}^{[2]}=c^2-(a^2+b^2)/2;\\ &\tilde{\Phi}_{\{22+\}}^{[2]}=(\sqrt{3}/2)\,(a^2-b^2);\\ &\chi_{\{22+\}}^{[2]}=[\chi_{\{22\}}^{[2]}+\chi_{\{2-2\}}^{[2]}]/\sqrt{2};\end{aligned}\right\} \tag{11}$$

N_n is a normalization constant.

The function $\Psi_{2M}^{[2]}$ belonging the same shell of the ^{6}He nucleus but to the state with angular momentum $L=2$ is determined by the coefficients

$$\Phi_{\{\nu\}K}^{[2]2}=N_n\tilde{\Phi}_{\{\nu\}K}^{[2]2}\exp[-(a^2+b^2+c^2)/2];$$

$$\tilde{\Phi}_{\{0\}0}^{[2]2}=[c^2-(a^2+b^2)/2]/\sqrt{5};\quad \tilde{\Phi}_{\{0\}2+}^{[2]2}=\sqrt{3/5}\,(a^2-b^2)/2;$$

$$\tilde{\Phi}_{\{20\}0}^{[2]2}=c^2+(a^2+b^2)/4;\quad \tilde{\Phi}_{\{20\}2+}^{[2]2}=(-\sqrt{3}/4)\,(a^2-b^2);$$

$$\tilde{\Phi}_{\{22+\}0}^{[2]2}=(-\sqrt{3}/4)\,(a^2-b^2);\quad \Phi_{\{22+\}2+}^{[2]2}=(3/4)\,(a^2+b^2);$$

$$\tilde{\Phi}_{\{22-\}2-}^{[2]2}=(-3/2)\,ab;\quad \tilde{\Phi}_{\{21+\}1+}^{[2]2}=(-3/2)\,bc;\quad \Phi_{\{21-\}1-}^{[2]2}=(3/2)\,ca,$$

in which $\tilde{\Phi}_{\{\nu\}K+}^{[f]L}$ is, by definition, the coefficient of $(D_{KM}^L+D_{-KM}^L)\chi_{[\nu]}^{[t]}/\sqrt{2}$; $\tilde{\Phi}_{\{\nu\}K}^{[f]L}$ is the coefficient of $(D_{KM}^L-D_{-KM}^L)\chi_{[\nu]}^{[f]}/\sqrt{2}$.

The oscillator function of the ground state of the ^{8}Be nucleus has the form[28]

$$\begin{aligned}\Psi_0^{[4]}=&\Phi_{\{0\}0}^{[4]0}\chi_{\{0\}}^{[4]}+\Phi_{\{20\}0}^{[4]0}\chi_{\{20\}}^{[4]}+\Phi_{\{22+\}}^{[4]0}\chi_{\{22+\}}^{[4]0}+\Phi_{\{40\}0}^{[4]0}\chi_{\{40\}}^{[4]}\\&+\Phi_{\{42+\}0}^{[4]0}\chi_{\{42+\}}^{[4]}+\Phi_{\{44+\}0}^{[4]0}\chi_{\{44+\}}^{[4]},\end{aligned} \tag{12}$$

where

$$\left.\begin{aligned}&\Phi_{\{\nu\}0}^{[4]0}=N_n\tilde{\Phi}_{\{\nu\}}^{[4]}\exp[-(a^2+b^2+c^2)/2];\\ &\tilde{\Phi}_{\{0\}}^{[4]}=\sqrt{21/10}\,[a^4+b^4+c^4+(2/3)\,(a^2b^2+b^2c^2+c^2a^2)];\\ &\tilde{\Phi}_{\{20\}}^{[4]}=3\{c^4-(a^4+b^4)/2-[a^2b^2-(b^2c^2+c^2a^2)/2]/3\};\\ &\tilde{\Phi}_{\{22+\}}^{[4]}=3[(\sqrt{3}/2)\,(a^4-b^4)-(b^2c^2-c^2a^2)/2\sqrt{3}];\\ &\tilde{\Phi}_{\{40\}}^{[4]}=(3/8)\sqrt{11/10}\,\{8[c^2-(a^2+b^2)/2]^2+(a^2-b^2)^2\};\\ &\tilde{\Phi}_{\{42+\}}^{[4]}=(3/8)\sqrt{11/10}\,4\sqrt{5}\,(a^2-b^2)\left[c^2-\frac{(a^2+b^2)}{2}\right];\\ &\tilde{\Phi}_{\{44+\}}^{[4]}=(3/8)\sqrt{11/10}\sqrt{35}\,(a^2-b^2)^2.\end{aligned}\right\} \tag{13}$$

Within the p shell, the major oscillator configurations in each case contain basis functions of only one representation $[f]$. But for the $s-d$ shell there is degeneracy, and several representations of the group $SO(A-1)$ correspond to the major configurations. For example, to the ground state of ^{20}Ne there correspond the representations [12, 4, 4] and [8, 8, 4], and to the ^{24}Mg ground state there correspond [16, 8, 4], [12, 12, 4], and [12, 8, 8]. Therefore, in accordance with the definition given above, the minimal basis for nuclei of the $s-d$ shell must include functions $\chi_{[\nu]}^{[f]}$ of different representations $[f]$ of the group $SO(A-1)$. However, if one of these representations is dominant in the expansion of the ground-state wave function of the given nucleus, the minimal approximation can be defined differently, with inclusion of only the generalized hyperspherical functions of the dominant representation in the minimal basis.

5. MATRIX ELEMENTS OF THE POTENTIAL-ENERGY OPERATOR

Once the quantum numbers of the basis functions χ_ν have been determined, to derive the system of equations for the coefficient functions $u_{\nu K}^L(abc)$ it is necessary to find the matrix elements of the Hamiltonian $\hat{H}=\hat{T}+\hat{V}$ between the functions χ_ν. The matrix elements of the kinetic-energy operator $\langle\nu|\hat{T}|\nu'\rangle$ reduce to the known matrix elements of the generators of the groups $SO(A-1)$. The main difficulty therefore resides in calculating the matrix elements of the potential-energy operator of the system of nucleons:

$$\hat{V}=\sum_{i<j}^{A}v(r_{ij},\,\sigma_i\sigma_j,\,\tau_i\tau_j).$$

To analyze the kinetic-energy operator, it is convenient to use wave functions of the translationally invariant shell model. Let Ψ and $\overline{\Psi}$ be two such functions:

$$\Psi=\Psi_{LM}=\sum_{\nu K}\Phi_{\nu K}^L(abc)\,D_{KM}^L(\varphi\theta\psi)\,\chi_\nu;$$

$$\overline{\Psi}=\overline{\Psi}_{LM}=\sum_{\nu' K'}\overline{\Phi}_{\nu' K'}^L(abc)D_{K'M}^L(\varphi\theta\psi)\,\chi_{\nu'}.$$

In a special case, the function $\overline{\Psi}_{LM}$ may be identically equal to the function Ψ_{LM}. If $\overline{\Psi}_{LM}\neq\Psi_{LM}$, then the former will be of interest only under the condition that it includes functions χ_ν which do not occur in Ψ_{LM}.

If the matrix element

$$U=\sum_{M=-L}^{L}\langle\overline{\Psi}_{LM}^*|\hat{V}|\Psi_{LM}\rangle$$

is defined without integration with respect to the collective variables a, b, c and φ, θ, ψ, then U is a function of the variables a, b, c, and it can be represented as the sum

$$U(abc)=\sum_{\nu\nu' K}\overline{\Phi}_{\nu' K}^{*L}(abc)\,\Phi_{\nu K}^L(abc)\,\langle\chi_{\nu'}^*|\,\hat{V}\,|\chi_\nu\rangle,$$

the derivation of which is based on the identity

$$\sum_{M=-L}^{L}D_{K'M}^{*L}(\varphi\theta\psi)\,D_{KM}^L(\varphi\theta\psi)\equiv\delta_{KK'}.$$

The required matrix elements

$$U_{\nu\nu'}(abc)=\langle\chi_{\nu'}^*|\hat{V}|\chi_\nu\rangle$$

are contained in $U(abc)$, and to extract them we must calculate not only the function $U(abc)$ but also $\Phi_{\nu K}^L(abc)$.

Thus, consider the integral

$$U(abc)=\sum_{M=-L}^{L}\int d\alpha_1\,d\alpha_2\,\ldots\,d\alpha_{3A-9}\overline{\Psi}_{LM}^*\hat{V}\Psi_{LM},$$

in which the spin-isospin Pauli matrices of the potential-energy operator $\hat{V}$ are averaged with respect to the spin—isospin functions occurring in Ψ_{LM} and $\overline{\Psi}_{LM}^*$. The integration is performed over the $3A-9$ generalized Eulerian angles chosen specially among the $(A-1)(A-2)/2$ generalized Eulerian angles of the $(A-1)$-dimensional space.

Let q_{kx}, q_{ky}, q_{kz} be the Cartesian components of the $(A-1)$ Jacobi vectors in an arbitrary coordinate system and $q_{k\xi}, q_{k\eta}, q_{k\zeta}$ be the Cartesian components of the same vectors but in a coordinate system whose axes are directed along the principal axes of the ellipsoid of inertia, with $k=1, 2, \ldots, A-1$. In addition, suppose that

$$\sum_{k=1}^{A-1}q_{k\xi}^2=a_0^2,\ \sum_{k=1}^{A-1}q_{k\eta}^2=b_0^2,\ \sum_{k=1}^{A-1}q_{k\zeta}^2=c_0^2.$$

Using the two identities[13]

$$\int \delta(a^2-a_0^2)\,\delta(b^2-b_0^2)\,\delta(c^2-c_0^2)\,da_0^2\,db_0^2\,dc_0^2 \equiv 1;$$

$$|a_0^2-b_0^2|\,|b_0^2-c_0^2|\,|c_0^2-a_0^2| \int \delta\Big(\sum_{l=1}^{A-1} q_{lx}q_{ly}\Big) \times \delta\Big(\sum_{l=1}^{A-1} q_{ly}q_{lz}\Big)\,\delta\Big(\sum_{l=1}^{A-1} q_{lz}q_{lx}\Big)\,d\varphi\sin\theta\,d\theta\,d\psi \equiv 1;$$

where φ, θ, ψ are the Eulerian angles specifying the orientation in space of the principal axes of the ellipsoid of inertia, we transform $U(abc)$ to the form

$$\left.\begin{aligned} U(abc) &= \frac{8}{(abc)^{A-5}} \int \delta\Big(\sum_{l=1}^{A-1} q_{lx}q_{ly}\Big) \\ &\times \delta\Big(\sum_{l=1}^{A-1} q_{ly}q_{lz}\Big)\,\delta\Big(\sum_{l=1}^{A-1} q_{lz}q_{lx}\Big) \\ &\times \delta\Big(a^2-\sum_{l=1}^{A-1} q_{lx}^2\Big)\,\delta\Big(b^2-\sum_{l=1}^{A-1} q_{ly}^2\Big)\,\delta\Big(c^2-\sum_{l=1}^{A-1} q_{lz}^2\Big) \\ &\times \sum_{M=-L}^{L} \overline{\Psi}_{LM}^{*}\hat{V}\Psi_{LM}\,d\tau_{3A-3}; \\ d\tau_{3A-3} &\equiv d\mathbf{q}_1\,d\mathbf{q}_2 \ldots d\mathbf{q}_{A-1} = (a_0b_0c_0)^{A-4} \\ &\times |a_0^2-b_0^2|\,|b_0^2-c_0^2|\,|c_0^2-a_0^2|\,da_0\,db_0\,dc_0 \\ &\times d\varphi\sin\theta\,d\theta\,d\psi\,d\alpha_1\,d\alpha_2 \ldots d\alpha_{3A-9}. \end{aligned}\right\} \quad (14)$$

We now introduce integral representations for the δ functions:

$$\delta(a^2-a_0^2)\,\delta(b^2-b_0^2)\,\delta(c^2-c_0^2) = \frac{1}{(2\pi)^3}\int\!\!\int\!\!\int_{-\infty}^{\infty} dk_1\,dk_2\,dk_3 \exp[ik_1(a^2-a_0^2)+ik_2(b^2-b_0^2)+ik_3(c^2-c_0^2)].$$

The first three δ functions in the integrand of (14) are nonzero only when the Cartesian coordinate axes are directed along the principal axes of the ellipsoid of inertia. In just this case, the sums

$$\sum_{l=1}^{A-1} q_{lx}^2,\quad \sum_{l=1}^{A-1} q_{ly}^2,\quad \sum_{l=1}^{A-1} q_{lz}^2$$

in the arguments of the three following δ functions are equal to a_0^2, b_0^2, c_0^2, respectively, which makes it possible to replace these quantities by sums:

$$\delta\Big(\sum_{l=1}^{A-1} q_{lx}q_{ly}\Big)\,\delta\Big(\sum_{l=1}^{A-1} q_{ly}q_{lz}\Big)\,\delta\Big(\sum_{l=1}^{A-1} q_{lz}q_{lx}\Big) = \frac{1}{(2\pi)^3}\int\!\!\int\!\!\int_{-\infty}^{\infty} dp_1\,dp_2\,dp_3 \exp\Big[-i\sum_{l=1}^{A-1}(p_1q_{ly}q_{lz}+p_2q_{lz}q_{lx}+p_3q_{lx}q_{ly})\Big].$$

The quadratic form in the Cartesian components of the Jacobi vectors that occurs in the exponential in the integral representations of all the δ functions can be reduced to diagonal form by a rotation of the coordinate axes of the three-dimensional space in which the Jacobi vectors are defined.[13] To implement this reduction, we set

$$t = (k_1+k_2+k_3)/2\sqrt{3};$$
$$\widetilde{\alpha}_{\pm 2} = (k_1-k_2 \mp ip_3);\quad \widetilde{\alpha}_{\pm 1} = (p_2 \mp ip_1);$$
$$\widetilde{\alpha}_0 = (2/\sqrt{3})\,[(k_3-(k_1+k_2)/2)]$$

and we then introduce new variables for integral representations of the δ functions:

$$\widetilde{\lambda}\cos\widetilde{\delta} = \sum_{\mu} \alpha_\mu D_{0\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi});$$
$$\widetilde{\lambda}\sin\widetilde{\delta} = \frac{1}{\sqrt{2}}\sum_{\mu} \alpha_\mu\,[D_{2\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi}) + D_{-2\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi})].$$

The orientation of the new system with respect to the original system is determined by the Eulerian angles $\widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}$ which, in turn, satisfy the three conditions

$$\sum_{\mu} \alpha_\mu D_{1\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi}) = 0;\quad \sum_{\mu} \alpha_\mu D_{-1\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi}) = 0;$$
$$\sum_{\mu} \alpha_\mu D_{2\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi}) = \sum_{\mu} \alpha_\mu D_{-2\mu}^2(\widetilde{\varphi}\widetilde{\theta}\widetilde{\psi}).$$

Then, as in Ref. 13, we can represent the expression for $U(abc)$ as an integral over the complete configuration space A of the single-particle vectors $\mathbf{r}_l$ with volume element $d\tau_{3A}$:

$$\begin{aligned} U(abc) &= \frac{1}{2}\int \ldots \int dt\,\widetilde{\lambda}^4\,d\widetilde{\lambda}\sin 3\widetilde{\delta}\,d\widetilde{\delta}\,d\widetilde{\varphi}\sin\widetilde{\theta}\,d\widetilde{\theta}\,d\widetilde{\psi} \\ &\times D(t, \widetilde{\lambda}, \widetilde{\delta}, \widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}; a, b, c) \int d\tau_{3A} \\ &\times \exp\Big\{\sum_{l=1}^{A} i(-s_1r_{lx'}^2 - s_2r_{ly'}^2 - s_3r_{lz'}^2)\Big\} \sum_{M=-L}^{L} \overline{\Psi}_{LM}^{*}\hat{V}\Psi_{LM}, \end{aligned} \quad (15)$$

where

$$s_1 = (2/\sqrt{3})\,[t+(\widetilde{\lambda}/2)\cos(\widetilde{\delta}-2\pi/3)];$$
$$s_2 = (2/\sqrt{3})\,[t+(\widetilde{\lambda}/2)\cos(\widetilde{\delta}+2\pi/3)];$$
$$s_3 = (2/\sqrt{3})\,[t+(\widetilde{\lambda}/2)\cos\widetilde{\delta}];$$
$$D(t, \widetilde{\lambda}, \widetilde{\delta}, \widetilde{\varphi}, \widetilde{\theta}, \widetilde{\psi}; a, b, c) = \frac{\sqrt{(1+is_1)(1+is_2)(1-is_3)}}{8\pi^{15/2}(abc)^{A-5}} \exp(ik_1a^2+ik_2b^2+ik_3c^2);$$
$$d\tau_{3A} = d\mathbf{r}_1\,d\mathbf{r}_2 \ldots d\mathbf{r}_A.$$

We assume that the Jacobi vectors $\mathbf{q}_l$ are normalized in such a way that

$$\sum_{l=1}^{A-1} q_l^2 + \frac{1}{\sqrt{A}}\Big(\sum_{l=1}^{A}\mathbf{r}_l\Big)^2 = \sum_{l=1}^{A} r_l^2.$$

We express the shell-model wave functions Ψ and $\overline{\Psi}$, which are chosen in the form of combinations of Slater determinants that are homogeneous in the nucleon coordinates, in the Cartesian coordinate system. Then the exponential with the sum in its argument can be included in the Slater determinants if the single-particle states are modified by replacing the isotropic three-dimensional oscillator by an anisotropic oscillator whose vibration frequencies along the x', y', z' axes differ from the corresponding frequencies of the isotropic oscillator by the factors $1+is_1$, $1+is_2$, $1+is_3$. At the same time, a compensating factor must be added in front of the Slater determinants to preserve their correct normalization.

Thus, we have shown that to calculate $U(abc)$ it is sufficient to integrate the expressions containing the Slater determinants. It follows that the matrix elements of the potential-energy operator between the functions χ_ν can be found without our having to construct explicitly functions χ_ν satisfying the Pauli principle (these functions are contained implicitly in the Slater determinants), and the operation of averaging the spin-isospin matrices of the potential-energy operator of the system can also be performed by means of the Slater determinants.

The combination of the simplest Slater determinants of the oscillator shell model for the ground state of the ^{6}He nucleus (total orbital angular momentum $L=0$, total spin $S=0$, total isospin $T=1$) has the form

$$(\overline{\Psi}_{xx}+\overline{\Psi}_{yy}+\overline{\Psi}_{zz})/\sqrt{3}. \quad (16)$$

Each of the terms of this combination corresponds to a definite filling by two neutrons of the p shell of excited states of the three-dimensional oscillator. The subscripts indicate the quantum numbers of the filled excited states of the three-dimensional oscillator (thus, x corresponds to single excitation of the oscillator which vibrates along the x axis and zero-point vibrations of the other two oscillators) and the number of filled states (it is equal to the number of subscripts). The serial number of the subscript indicates a definite combination of single-particle spin-isospin quantum numbers.

For the ground state of the ^{8}Be nucleus, the combination of the simplest Slater determinants which corresponds to $L=0, S=0, T=0$ has the somewhat different form

$$\frac{1}{\sqrt{45}}\sum_{i,j=1}^{3}(\widetilde{\Psi}_{x_ix_ix_jx_j}+\widetilde{\Psi}_{x_ix_jx_ix_j}+\widetilde{\Psi}_{x_ix_jx_jx_i})$$
$$\equiv\frac{1}{\sqrt{45}}[\widetilde{\Psi}_{(\mathbf{r}_1\mathbf{r}_2)(\mathbf{r}_3\mathbf{r}_4)}+\widetilde{\Psi}_{(\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_2\mathbf{r}_4)}+\widetilde{\Psi}_{(\mathbf{r}_1\mathbf{r}_4)(\mathbf{r}_2\mathbf{r}_3)}], \qquad (17)$$

where $x_1=x, x_2=y, x_3=z$.

In a second form of expression, the summation sign is replaced by symbols of scalar products, by means of which the subscripts are combined (sums of their pairwise products). To a definite combination of spin-isospin quantum numbers, there no longer corresponds the position of the subscript but only its serial number, since the position and serial number no longer coincide in all cases.

Finally, we give the combination of Slater determinants for the nucleus ^{12}C, using not only the scalar-product symbol but also the vector-product symbol:

$$\widetilde{\Psi}_{([\mathbf{r}_1\mathbf{r}_5][\mathbf{r}_2\mathbf{r}_6])([\mathbf{r}_3\mathbf{r}_7][\mathbf{r}_4\mathbf{r}_8])}+\widetilde{\Psi}_{([\mathbf{r}_1\mathbf{r}_5][\mathbf{r}_3\mathbf{r}_7])([\mathbf{r}_2\mathbf{r}_6][\mathbf{r}_4\mathbf{r}_8])}$$
$$+\widetilde{\Psi}_{([\mathbf{r}_1\mathbf{r}_5][\mathbf{r}_4\mathbf{r}_8])([\mathbf{r}_2\mathbf{r}_6][\mathbf{r}_3\mathbf{r}_7])}.$$

Note that the states 1 and 5, 2 and 6, 3 and 7, and 4 and 8 have the same spin-isospin quantum numbers.

We now consider again the expression (15). Before we integrate over $d\tau_A$, we expand the Slater determinants and represent the integrated function as a sum of products of single-particle functions. Because the latter are orthogonal, many terms of this sum vanish after integration over $d\tau_A$. We integrate each of the remaining terms over the three components of the vector of the particle whose coordinates occur only in the argument of the exponential in the given term of the sum. With regard to the remaining single-particle variables, even if they occur only in the exponential, it is expedient to integrate with respect to them after integration with respect to the parameters of the integral representations of the δ functions. However, in the three-dimensional space in which the single-particle vectors are defined, it is first necessary to make a rotation to ensure our return to the parameters p_i and k_i and then to the δ functions. In contrast to the arguments of the δ functions of the original expressions (14), the arguments of the new δ functions contain, not the Jacobi coordinates q_l, but the coordinates $\mathbf{r}_l$ of $A-1$ particles:

$$U(a, b, c)=\frac{8}{(abc)^{A-5}}\int\delta\left(\sum_{l=1}^{A-1}r_{lx}r_{ly}\right)\delta\left(\sum_{l=1}^{A-1}r_{ly}r_{lz}\right)$$
$$\times\delta\left(\sum_{l=1}^{A-1}r_{lz}r_{lx}\right)\delta\left(a^2-\sum_{l=1}^{A-1}r_{lx}^2\right)\delta\left(b^2-\sum_{l=1}^{A-1}r_{ly}^2\right)\delta\left(c^2-\sum_{l=1}^{A-1}r_{lz}\right)$$
$$\times\mathcal{G}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{A-1})\exp\left(-\sum_{l=1}^{A-1}r_l^2\right)d\tau'_{3A-3}, \qquad (18)$$

where $\mathcal{G}$ is the sum of the products of the interaction potential v_{st} of two nucleons in certain spin-isospin states and polynomials that are homogeneous in the single-particle coordinates and invariant under rotations in the three-dimensional space. Thus, for ^{6}He for the function (17)

$$\mathcal{G}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)=(4/3)\pi^{-15/2}\{3[v_{31}(r)+v_{13}(r)](\mathbf{r}_3\mathbf{r}_4)^2$$
$$+v_{13}(r)(\mathbf{r}_1\mathbf{r}_2)^2+[9v_{33}(r)+3v_{31}(r)+3v_{13}(r)+v_{11}(r)](\mathbf{r}_1\mathbf{r}_3)^2/2$$
$$-[9v_{33}(r)-3v_{31}(r)-3v_{13}(r)+v_{11}(r)](\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_2\mathbf{r}_3)/2\}, \qquad r=|\mathbf{r}_1-\mathbf{r}_2|.$$

For ^{8}Be for the functions (18)

$$\mathcal{G}(r_1, r_2, \ldots, r_7)=\frac{16}{15}\pi^{-15/2}\Big\{\frac{9v_{33}(r)+3v_{31}(r)+3v_{13}(r)+v_{11}(r)}{2}$$
$$\times([\mathbf{r}_3\mathbf{r}_4)(\mathbf{r}_5\mathbf{r}_6)+(\mathbf{r}_3\mathbf{r}_5)(\mathbf{r}_4\mathbf{r}_6)+(\mathbf{r}_3\mathbf{r}_6)(\mathbf{r}_4\mathbf{r}_5)]^2+[(\mathbf{r}_1\mathbf{r}_2)(\mathbf{r}_3\mathbf{r}_4)$$
$$+(\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_2\mathbf{r}_4)+(\mathbf{r}_1\mathbf{r}_4)(\mathbf{r}_2\mathbf{r}_3)]^2+[(\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_4\mathbf{r}_5)+(\mathbf{r}_1\mathbf{r}_4)(\mathbf{r}_3\mathbf{r}_5)$$
$$+(\mathbf{r}_1\mathbf{r}_5)(\mathbf{r}_3\mathbf{r}_4)]^2+[(\mathbf{r}_2\mathbf{r}_3)(\mathbf{r}_4\mathbf{r}_5)+(\mathbf{r}_2\mathbf{r}_4)(\mathbf{r}_3\mathbf{r}_5)+(\mathbf{r}_2\mathbf{r}_5)(\mathbf{r}_3\mathbf{r}_4)]^2)$$
$$-\frac{9v_{33}(r)-3v_{31}(r)-3v_{13}(r)+v_{11}(r)}{2}([(\mathbf{r}_3\mathbf{r}_4)(\mathbf{r}_5\mathbf{r}_6)+(\mathbf{r}_3\mathbf{r}_5)(\mathbf{r}_4\mathbf{r}_6)$$
$$+(\mathbf{r}_3\mathbf{r}_6)(\mathbf{r}_4\mathbf{r}_5)]^2+[(\mathbf{r}_1\mathbf{r}_2)(\mathbf{r}_3\mathbf{r}_4)+(\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_2\mathbf{r}_4)+(\mathbf{r}_1\mathbf{r}_4)(\mathbf{r}_3\mathbf{r}_2)]^2+2[(\mathbf{r}_1\mathbf{r}_3)(\mathbf{r}_4\mathbf{r}_5)$$
$$+(\mathbf{r}_1\mathbf{r}_4)(\mathbf{r}_3\mathbf{r}_5)+(\mathbf{r}_1\mathbf{r}_5)(\mathbf{r}_3\mathbf{r}_4)][(\mathbf{r}_3\mathbf{r}_2)(\mathbf{r}_4\mathbf{r}_5)+(\mathbf{r}_2\mathbf{r}_4)(\mathbf{r}_3\mathbf{r}_5)+(\mathbf{r}_2\mathbf{r}_5)(\mathbf{r}_3\mathbf{r}_4)])\Big\}.$$

If the coordinate axes are directed along the principal axes of the ellipsoid of inertia of the system of $A-1$ particles, and if the function $\mathcal{G}$, which is invariant under rotations of the coordinate axes, we go over from the single-particle vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{A-1}$ to the semiaxes a'_0, b'_0, c'_0 of the ellipsoid of inertia and the generalized Eulerian angles $\alpha'_1, \alpha'_2, \ldots, \alpha'_{3A-9}$, then in (18) we can integrate over a'_0, b'_0, c'_0 and the Eulerian angles φ', θ', ψ' of the orientation of the ellipsoid of inertia. At the same time, it should be borne in mind that in the system of the ellipsoid of inertia the single-particle vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{A-1}$ are related to the collective variables a'_0, b'_0, c'_0 and the generalized Eulerian angles by the same relations that connect the Jacobi vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{A-1}$ to the semiaxes a_0, b_0, c_0 of the ellipsoid of inertia of the system of A particles (with the center-of-mass motion separated) and the generalized Eulerian angles $\alpha_1, \alpha_2, \ldots, \alpha_{3A-9}$. The upshot is that U is given by the expression

$$U(a, b, c)=\int\mathcal{G}(a, b, c, \alpha'_1, \alpha'_2, \ldots, \alpha_{3A-9})$$
$$\times\exp(-a^2-b^2-c^2)\,d\alpha'_1\,d\alpha'_2, \ldots, d\alpha'_{3A-9},$$

from which it is now not difficult to extract the required matrix elements $U_{\nu\nu'}(a, b, c)$.

Performing all the necessary operations, we can represent the matrix elements for ^{6}He in the form[29]

$$U_{\nu\nu'}=\sum_{i=1}^{4}U_{\nu\nu'}^{(i)};$$
$$U_{\nu\nu'}^{(1)}=\int d\tau\,[v_{31}(p)+v_{13}(p)]\langle\,|\hat{A}_1|\nu'\rangle;$$
$$U_{\nu\nu'}^{(2)}=\int d\tau v_{13}(p)\langle\nu|\hat{A}_2|\nu'\rangle;$$
$$U_{\nu\nu'}^{(3)}=\int d\tau\frac{1}{3}[9v_{33}(p)+3v_{31}(p)+3v_{13}(p)+v_{11}(p)]\langle\nu|\hat{A}_3|\nu'\rangle;$$
$$U_{\nu\nu'}^{(4)}=\int d\tau\frac{1}{3}[9v_{33}(p)-3v_{33}(p)+v_{11}(p)]\langle\nu|\hat{A}_4|\nu'\rangle,$$

where

$$p=\sqrt{2}\sqrt{a^2r^2\sin^2\theta_2\cos^2\varphi_2+b^2r^2\sin^2\theta_2\sin^2\varphi_2+c^2r^2\cos^2\theta_2};$$
$$d\tau=r^2\,dr\sin\theta_2\,d\theta_2\,d\varphi_2;\quad 1\geqslant r\geqslant 0;\quad \pi\geqslant\theta_2\geqslant 0;\quad 2\pi\geqslant\varphi_2\geqslant 0;$$

$v_{31}, v_{33}, v_{13}, v_{11}$ are the usually defined components of the exchange potential and ν is a definite set of quantum numbers. For ^6He, if we restrict ourselves to the ν contained in the wave function (16), it is convenient to classify the basis functions by using the quasiangular momentum j, its projection k, and also the quantum number n, which is uniquely related to j by $2n+j=2$, $j=0,\ 2$. Then

$$\langle njk\,|\,A_i\,|\,n'j'k'\rangle=A_{i1}R_{nj}(r)\,Y_{jk}(\theta_2,\ \varphi_2)\,R_{n'j'}(r)\,Y_{j'k'}(\theta_2,\ \varphi_2)$$
$$+\sum_{n_1,\ j_1,\ k_1}A_{i2}C_{n_1j_1k_1}^{njk,\ n'j'k'}R_{n_1j_1}(r)\,Y_{j_1k_1}(\theta_2,\ \varphi_2)+A_{i3}\delta_{jj'}\delta_{kk'};$$
$$A_{11}=1/5;\quad A_{21}=13/20;\quad A_{31}=-1/4;\quad A_{41}=-7/20;$$
$$A_{12}=(2/15)\sqrt{7/\pi};\quad A_{22}=(1/60)\sqrt{7/\pi};\quad A_{32}=(-1/24)\sqrt{7/\pi};$$
$$A_{42}=(13/120)\sqrt{7/\pi};\quad A_{13}=21/10\pi;\quad A_{23}=21/80\pi;$$
$$A_{33}=21/16\pi;\quad A_{43}=21/80\pi;$$
$$C_{100}^{100;\ 100}=1;\quad C_{0,2k}^{100;\ 0,2k}=-2\,(20k0\,|\,2k);$$
$$C_{100}^{0,2k;\ 0,2k'}=-2\sqrt{5}\,(22kk'\,|\,00);\qquad C_{0,2k_1}^{0,2k;\ 0,2k'}=\sqrt{35/2}\,(22kk'\,|\,2k_1),$$

where $(jj'kk'\,|\,j_1k_1)$ are Clebsch-Gordan coefficients.

6. DISTRIBUTION FUNCTION OF THE SINGLE-PARTICLE DENSITY OF NUCLEI IN THE INTRINSIC COORDINATE SYSTEM[30,31]

To study the shape of light nuelci, one frequenctly considers the shape of the surfaces of equal density of nucleons in the intrinsic coordinate system of the nucleus. However, since the intrinsic system is not defined, the shape of the nucleon density distribution in the intrinsic system is identified with the shape of the density distribution in the fixed system, this identification being justified by the validity of the adiabatic approximation.[32,33] The limitations of such an approach are obvious. The shape of the density distribution is found for a superposition of states with different angular momenta (this superposition is nominally called the *intrinsic function*) but not for a state with definite value of the angular momentum. The problem of calculating the shape of the equal-density surfaces in states with definite value of the angular momentum is not even posed. It is even doubtful whether the adiabatic approximation is valid, since even in the most favorable situation there is no safe margin for the conditions of applicability of the adiabatic approximation for nuclei of the p and s-d shells.

If the intrinsic coordinate system of the nucleus is tied to the principal axes of the ellipsoid of inertia, one can construct the distribution function of the single-particle density of nucleons in this system, and with it the constant-density surfaces. The latter give a direct representation of the intrinsic shape of the nuclei.

The distribution function $f_M^{I\tau}(\xi,\eta,\zeta)$ of the single-particle density of the nucleus in the intrinsic system can be expressed in terms of an integral of the product of the square of the modulus of the wave function and three δ functions over the complete space of variation of the collective and intrinsic variables[34]:

$$f^{I\tau}(\xi,\ \eta,\ \zeta)=\int|\Psi_M^{I\tau}|^2\,\delta\left(\xi-\sqrt{\frac{A-1}{A}}\,ar\sin\theta'\sin\varphi'\right)$$
$$\times\,\delta\left(\eta-\sqrt{\frac{A-1}{A}}\,br\sin\theta'\cos\varphi'\right)\delta\left(\zeta-\sqrt{\frac{A-1}{A}}\,cr\cos\theta'\right)$$
$$\times\,d\tau_{abc}\,d\tau_{\Omega}\,d\tau_{\alpha_i};$$
$$\mathbf{q}=\mathbf{r}_1-(\mathbf{r}_1+\mathbf{r}+\ldots+\mathbf{r}_A)/A=\{\xi,\ \eta,\ \zeta\},\tag{19}$$

where $d\tau_{abc}, d\tau_{\Omega}, d\tau_{\alpha_i}$ are the volume elements in the space of the variables $a,\ b,\ c,\ \varphi,\ \theta,\ \psi$, and α_i, respectively; r, θ', φ' are the intrinsic variables related by simple equations to three of the $3A-9$ generalized Eulerian angles:

$$r\cos\theta'=\cos\theta_{n-1}^{n-1};\qquad r\sin\theta'\cos\varphi'=\sin\theta_{n-1}^{n-1}\cos\theta_{n-2}^{n-2};$$
$$r\sin\theta'\sin\varphi'=\sin\theta_{n-1}^{n-1}\sin\theta_{n-2}^{n-2}\cos\theta_{n-3}^{n-3},$$

with $1\geqslant r\geqslant 0, \pi\geqslant\theta'\geqslant 0, 2\pi>\varphi'\geqslant 0$, and the volume element in the space of these three independent intrinsic variables has the form

$$d\tau_{r,\ \theta',\ \varphi'}=(1-r^2)^{A-6/2}\,r^2\,dr\sin\theta'\,d\theta'\,d\varphi'.$$

If we restrict ourselves to an integration over $a,\ b,\ c$ and α_i on the right-hand side of (19), we obtain the function $F_M^{I\tau}(\xi,\eta,\zeta;\ \varphi,\theta,\psi)$, which is related to $f^{I\tau}(\xi,\eta,\zeta)$ by

$$f^{I\tau}(\xi,\ \eta,\ \zeta)=\int F_M^{I\tau}(\xi,\ \eta,\ \zeta;\ \varphi,\ \theta,\ \psi)\,d\Omega.$$

This new function can be of interest in an investigation of the shape of nuclei in states with nonzero angular momentum. In a state with zero angular momentum $(I=0)$, the function $F_0^{I\tau}$ does not depend on the Eulerian angles and differs by only a constant factor from the moment $f^{0\tau}$.

Let $a_{ij}(\varphi,\theta,\psi)$ be the matrix elements of the rotation matrix in three-dimensional space, and let φ,θ,ψ, as above, be the Eulerian angles of orientation of the principal axes of the ellipsoid of inertia with respect to the fixed frame; it is then easy to establish a correspondence between the coordinates ξ,η,ζ of a point that is fixed with respect to the intrinsic frame of the nucleus and the coordinates x,y,z of a point that is fixed with respect to the fixed frame:

$$\left.\begin{array}{l}\xi=a_{11}x+a_{12}y+a_{13}z;\\ \eta=a_{21}x+a_{22}y+a_{23}z;\\ \zeta=a_{31}x+a_{32}y+a_{33}z.\end{array}\right\}\tag{20}$$

Substitution of the relations (20) in the expression for the function $F_M^{I\tau}$ and subsequent integration of it over the Eulerian angles φ,θ,ψ lead to the function $\bar{f}_M^{I\tau}(x,y,z)$ of the single-particle density of nucleons in the fixed frame. Of course, the density $\bar{f}_M^{I\tau}$ can also be obtained by direct integration of the square of the modulus of the wave function $\Psi_M^{I\tau}$ over all the variables for fixed values of the coordinates of one of the nucleons in a center-of-mass frame whose axes are not tied to the intrinsic axes of the nucleus.

In states with zero angular momentum, the density $\bar{f}_0^{0\tau}$ in the fixed coordinate system depends only on

$$q=\sqrt{x^2+y^2+z^2},$$

and if

$$\xi=q\sin\varepsilon\cos\delta,\quad \eta=q\sin\varepsilon\sin\delta,\quad \zeta=q\cos\varepsilon,$$

then in this special case

$$\bar{f}_0^{0\tau}(q) = \frac{1}{4\pi} \int f^{0\tau}(\xi, \eta, \zeta) \sin\varepsilon \, d\varepsilon \, d\delta. \quad (21)$$

The intrinsic variable r appears in (19) only when $A > 4$. For a system of four nucleons, this variable does not exist (there remain only θ' and φ'), and the expression for the density therefore has the simpler form

$$f^{I\tau}(\xi, \eta, \zeta) = \int |\Psi_M^{I\tau}|^2 \delta\left(\xi - \sqrt{\frac{3}{4}} a \sin\theta' \cos\varphi'\right)$$
$$\times \delta\left(\eta - \sqrt{\frac{3}{4}} b \sin\theta' \sin\varphi'\right) \delta\left(\zeta - \sqrt{\frac{3}{4}} c \cos\theta'\right) d\tau_{abc} \, d\tau_{\alpha_i} d_\Omega. \quad (22)$$

Of the three Eulerian angles α_i that are intrinsic variables, two are the angles θ' and φ', and the third is the angle ψ', which varies from 0 to 2π. To the intrinsic variables there corresponds the volume element $d\alpha_i = \sin\theta' d\theta' d\varphi' d\psi'$. As regards the collective variables a, b, c, it is convenient for the following calculations to go over in accordance with Eqs. (2) from them to the variables ρ, β, γ, to which there corresponds the volume element

$$d\tau_{\rho\beta\gamma} = \rho^8 d\rho \beta^4 d\beta \, |\sin 3\gamma| \, d\gamma / \sqrt{1 - 3\beta^2 + 2\beta^3 \cos 3\gamma}. \quad (23)$$

The ranges of the new variables are determined by the inequalities

$$\infty > \rho > 0; \quad \pi > \gamma > 2\pi/3; \quad -1/2 \cos\gamma > \beta > 0. \quad (24)$$

Integration over the region

$$2\pi/3 > \gamma > \pi/3; \quad -1/2 \cos(\gamma + 2\pi/3) > \beta > 0 \quad (25)$$

leads to the same result but with ζ and ξ interchanged, whereas integration over the region

$$\pi/3 > \gamma > 0; \quad -1/2 \cos(\gamma + 2\pi/3) > \beta > 0 \quad (26)$$

corresponds to the replacement of ζ by ξ, ξ by η, and η by ζ. The Jacobian of the transformation from the arguments of the three δ functions to the variables ρ, φ', θ' (the Jacobian must be known for the integration of the δ functions) is equal to

$$(3/4)^{3/2} \rho^2 \sqrt{1 - 3\beta^2 + 2\beta^3 \cos 3\gamma} \sin\theta'. \quad (27)$$

One of the simplest wave functions of the system of four nucleons is the function of the translationally invariant shell model for the ground state of ^{4}He. By considering this function, we can understand at least the qualitative nature of the single-particle density function of ^{4}He in the intrinsic coordinate system. After the transition to the collective and internal variables, it takes the form

$$\Psi_0^0(^4\text{He}) = \Phi_0^0(\rho) \chi_{[0]}^{[0]}(\sigma_i, \tau_i); \quad \Phi_0^0 = \text{const} \exp(-\rho^2/2), \quad (28)$$

where the intrinsic function $\chi_{[0]}^{[0]}$, which is represented by a single component, depends only on the spin-isospin variables, while the collective function Φ_0^0 depends only on the global radius ρ. Substitution of (28) in (22) makes it possible to integrate over ρ and the angles φ', θ', ψ' and to reduce the problem of calculating the single-particle density to a two-dimensional integration:

$$f^0(\xi, \eta, \zeta) = \frac{64}{(3\pi)^{3/2}} \iint D^3 \exp(-D) \frac{\beta^4 d\beta \sin 3\gamma \, d\gamma}{1 - 3\beta^2 + 2\beta^3 \cos 3\gamma};$$
$$D = 4\left[\frac{\xi^2}{1 + 2\beta\cos(\gamma + 2\pi/3)} + \frac{\eta^2}{1 + 2\beta\cos(\gamma - 2\pi/3)} + \frac{\zeta^2}{1 + 2\beta\cos\gamma}\right]. \quad (29)$$

The single-particle density (29) is normalized in such a way that

$$\iiint_{-\infty}^{\infty} f^0(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta = 1.$$

The distribution of the single-particle density in the fixed coordinate system for the wave function of the translationally invariant shell model is well known:

$$\bar{f}_0^0(q) = (4/3\pi)^{3/2} \exp(4q^2/3). \quad (30)$$

The connection between (28) and (30) is determined by the integral (21).

We begin our investigation of the function $f^0(\xi, \eta, \zeta)$ by establishing the limiting value of $f^0(0, 0, \zeta)$ when ζ tends to zero. We set $\beta\cos\gamma = s$ and $\beta\sin\gamma = t$, and then

$$\left.\begin{aligned} &|\sin 3\gamma| \, d\gamma \, \beta^4 d\beta = |3s^2 - t^2| \, t \, dt \, ds; \\ &1 - 3\beta^2 + 2\beta^3 \cos 3\lambda = (1 + 2s)[(1 - s)^2 - 3t^2]; \\ &-\sqrt{3} s > t > 0; \quad 0 > s > -1/2. \end{aligned}\right\} \quad (31)$$

Therefore

$$f^0(0, 0, \zeta) = \frac{64}{(3\pi)^{3/2}} \int_{-1/2}^{0} ds \int_0^{-\sqrt{3}s} dt \left(\frac{4\zeta^2}{1 + 2s}\right)$$
$$\times \exp\left(-\frac{4\zeta}{1 + 2s}\right) \frac{(3s^2 - t^2)\, t}{(1 + 2s)[(1 - s)^2 - 3t^2]}.$$

The integration with respect to t is readily performed. If we then set $u = 4\zeta^2/(1 + 2s)$, the density along the ζ axis takes the form

$$f^0(0, 0, \zeta) = \frac{32}{(3\pi)^{3/2}} \int_{4\zeta^2}^{\infty} u^3 \exp(-u)$$
$$\times \left\{\frac{1}{8}\left(1 - \frac{4\zeta^2}{u}\right)^2 + \frac{2\zeta^2}{9} \frac{3u - 8\zeta^2}{u^2} \ln\left[16\zeta^2 \frac{3u^2 - 8\zeta^2}{(3u - 4\zeta^2)^2}\right]\right\} \frac{du}{u}. \quad (32)$$

It follows immediately from (32) that $\lim_{\zeta \to 0} f(0, 0, \zeta) = (4/3\pi)^{3/2}$ in accordance with the expression (30).

But if the plane $\zeta = 0$ is approached with $\xi^2 + \eta \neq 0$, the density increases as $\ln(1/|\zeta|)$, and the limit expression for the density can be represented in the form

$$f^0(\xi, \eta, \zeta \ll 1) \approx (\ln(1/4\zeta^2) - C)$$
$$\times [16/(3\pi)^{3/2}] \int_0^1 [d(\xi, \eta, \lambda)]^3 \exp[-d(\xi, \eta, \lambda)] \lambda \, d\lambda, \quad (33)$$

where

$$d(\xi, \eta, \lambda) = (8/3)[\xi^2/(1 + \lambda)];$$

C is Euler's constant. It follows from (33) that as we leave the plane $\zeta = 0$ along the normal to this plane the density decreases rapidly—much more rapidly than when we move in the direction perpendicular to the ζ axis. We can therefore say that in the intrinsic coordinate system the nucleons are localized near the plane $\zeta = 0$. Note that more than half of the mass of ^{4}He is in a planar layer between the planes $\zeta_0 = \pm 0.1$. Note that for ξ, η, ζ we have chosen units in which the oscillator radius is $r_0 = 1$.

The factor accompanying $\ln(1/4\zeta^2) - C$, which depends

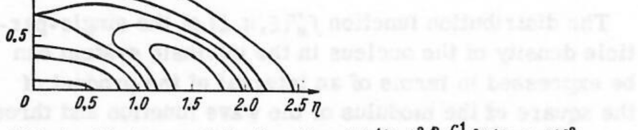

FIG. 1. Contours of the function $16/(3\pi)^{3/2} \int_0^1 [d(\xi, \eta; \lambda)]^3 \times \exp[-d(\xi, \eta; \lambda)] \lambda \, d\lambda$ [see Eq. (33)].

on ξ and η, determines the structure of the single-particle density function in the direction perpendicular to the ζ axis. In Fig. 1 we have plotted the contour in the (ξ, η) plane, each of which corresponds to a definite value of this factor.

Thus, if the ranges of the variables β and γ are determined by the conditions (24), the plane $\zeta = 0$ is distinguished. This distinguishing of one of the coordinate planes of the system of the principal axes of the ellipsoid of inertia (the one distinguished depends on the choice of the ranges of the collective variables β and γ) is an effect with a kinematic origin due to the fact that the volume element (23) in the space of collective variables of the system of four particles contains the factor $(1 - 3\beta^2 + 2\beta^3 \cos 3\gamma)^{-1/2}$, which tends to infinity when one of the three collective variables a, b, or c tends to zero. Vanishing of a, b, or c means that all four particles are in one plane. Therefore, the probability density for planar configurations of the four-particle system associated with the element of phase volume in the space of collective variables is equal to infinity, and this leads to the distinguishing of one of the planes, on which the distribution function of the single-particle density has a logarithmic singularity.

This distinguishing of one of the intrinsic planes occurs not only for the four-particle system but also, and to a greater extent, for the three-particle system, whose single-particle density

$$f^{I\tau}(\xi, \eta, \zeta) = \delta(\zeta)\, Q^{I\tau}(\xi, \eta) \tag{34}$$

contains a δ function of ζ, since the three particles always remain in one plane during their motion.

It follows from the expression (29) that the equation for the family of equal-density surfaces ($f^0 = \text{const}$) can be represented in the form

$$\zeta^2 = \mathcal{G}(f_0^\circ;\ \xi,\ \eta^2). \tag{35}$$

The ζ axis is the normal for the surfaces of this family. The surfaces (35) squeeze up to the plane $\zeta = 0$ with increasing distance of their points from the ζ axis: $|\zeta| \to 0$ if $\sqrt{\xi^2 + \eta^2} \to \infty$. For fixed value of f^0, Eq. (35) gives two surfaces symmetric about the plane $\zeta = 0$.

Now let $\Phi_0^0(A)$ be the ground-state wave function of a doubly magic nucleus with A nucleons (equal numbers of neutrons and protons) in the approximation of the translationally invariant shell model with isotropic oscillator basis. Its arguments are the spin-isospin variables and the Jacobi vectors $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_{A-1}$, and

$$\mathbf{q}_1 = \sqrt{\frac{A-1}{A}}\left(\mathbf{r}_1 - \frac{\mathbf{r}_2,\ \mathbf{r}_3 + \ldots + \mathbf{r}_A}{A-1}\right) = \sqrt{\frac{A}{A-1}}\,\mathbf{q}$$
$$= \sqrt{\frac{A}{A-1}}\left(\mathbf{r}_1 - \frac{1}{\sqrt{A}}\,\tilde{\mathbf{R}}\right), \tag{36}$$

where $\tilde{\mathbf{R}}$ is the specially normalized center-of-mass coordinate: $\tilde{\mathbf{R}} = (\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r})\sqrt{A}$.

The matrix element $\langle \Psi_0^0 | f(q) | \Psi_0^0 \rangle$ of the operator $f(\mathbf{q})$, which acts only on q, can be represented in the form

$$\langle \Psi_0^0 | f(\mathbf{q}) | \Psi_0^0 \rangle = \int d\mathbf{k}\, C(\mathbf{k}) \langle \Psi_0^0 | \exp(i\mathbf{k}\mathbf{q}) | \Psi_0^0 \rangle, \tag{37}$$

where

$$C(\mathbf{k}) = \frac{1}{(2\pi)^3} \int d\mathbf{q}\, f(\mathbf{q}) \exp(i\mathbf{k}\mathbf{q}),$$

but

$$\langle \Psi_0^0 | \exp(i\mathbf{k}\mathbf{q}) | \Psi_0^0 \rangle = \langle \Psi_0^0 | \exp(i\mathbf{k}\mathbf{r}_1 - i\mathbf{k}\tilde{R}/\sqrt{A}) \cdot$$
$$\times | \Psi_0^0 \rangle \int d\tilde{\mathbf{R}} \exp(-\tilde{R}^2 + i\mathbf{k}\tilde{R}/\sqrt{A})\, \pi^{-3/2} \exp(k^2/4A) \cdot$$
$$= \exp(k^2/4A) \langle \tilde{\Psi} | \exp(i\mathbf{k}\mathbf{r}_1) | \tilde{\Psi} \rangle, \tag{38}$$

where

$$\langle \tilde{\Psi} | \exp(i\mathbf{k}\mathbf{r}_1) | \tilde{\Psi} \rangle = \int d\tau_A |\tilde{\Psi}|^2 \exp(i\mathbf{k}\mathbf{r}_1);$$
$$\tilde{\Psi} = \pi^{-3/4}(-\tilde{R}^2/2)$$

is the ground-state wave function of the doubly magic nucleus in the oscillator shell model: $d\tau_A = d\mathbf{r}_1 d\mathbf{r}_2 \ldots d\mathbf{r}_A$.

We denote by m the number of the last closed shell of the magic nucleus; then

$$A = (2/3)(m+1)(m+2)(m+3), \quad m = 0, 1, 2, \ldots;$$
$$\langle \tilde{\Psi} | \exp(i\mathbf{k}\mathbf{r}_1) | \tilde{\Psi} \rangle = (4/3)\, L_m^{(3)}(k^2/2) \exp(-k^2/4), \tag{39}$$

where $L_m^\alpha(x)$ is the generalization of the Laguerre polynomial. Therefore

$$\langle \Psi_0^0 | \exp(i\mathbf{k}\mathbf{q}) | \Psi_0^0 \rangle = \frac{4}{A} L_m^{(3)}(k^2/2) \exp\left(-\frac{A-1}{A}\frac{k^2}{4}\right). \tag{40}$$

As a result, the single-particle density of the magic nucleus in which m shells are closed can be expressed by the following formula in the center-of-mass system of the nucleus:

$$P_m(q^2) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \frac{4}{A} L_m^{(3)}\left(\frac{k^2}{2}\right) \exp\left(-\frac{A-1}{A}\frac{k^2}{4} - i\mathbf{k}\mathbf{q}\right). \tag{41}$$

In particular, for $^{16}O (m=1)$

$$P_1(q^2) = \frac{1}{5\pi^{3/2}}\left(1 + \frac{128}{45}q^2\right) \exp\left(-\frac{16}{15}q^2\right). \tag{42}$$

In the general case,

$$P_m(q^2) = \overline{P}_m(q^2) \exp\left(-\frac{A}{A-1}q^2\right), \tag{43}$$

where $\overline{P}_m(q^2)$ is a polynomial of degree m in q^2. The density $P_m(q^2)$ is normalized in such a way that the integral over the complete space of variation of the vector $\mathbf{q}$ is equal to unity when it is multiplied by $[A/(A-1)]$. The function $\Psi_0^0(A)$ in the system of the principal axes of the ellipsoid of inertia can be represented as the product of two factors—the collective function and the intrinsic function $\chi_{\{0\}}^{[kkk]}(\alpha_i, \sigma_k, \tau_k)$:

$$\Psi_0^0(A) = \Phi_0^0 \chi_{\{0\}}^{[hkk]};\ \Phi_0^0 = \text{const}\,(abc)^k \exp[-(a^2+b^2+c^2)/2];$$
$$k = mA/4. \tag{44}$$

Using the results of Ref. 13, we can express the matrix element of $f(\xi, \eta, \zeta)$ between the intrinsic functions χ_0 of the ground states $\psi_0^0(A)$ of magic nuclei in terms of the operator $\overline{P}_m(-\partial/\partial\lambda)$ and the function $I_m(A, \lambda; a, b, c)$:

$$\langle \chi_{\{0\}}^{[hkk]} | f(\xi, \eta, \zeta) | \chi_{\{0\}}^{[hkk]} \rangle = \overline{P}_m\left(-\frac{\partial}{\partial\lambda}\right) I_m(A, \lambda; a, b, c)|_{\lambda=1}; \tag{45}$$

$$I_m(A, \lambda;\ a, b, c) = \frac{1}{\lambda^{3/2}} \frac{\Gamma(m_1+2)}{\Gamma(m_1+1/2)} \int_0^1 (1-r^2)^{m_1-1/2} r^2\, dr$$
$$\times \int_0^\pi \sin\theta'\, d\theta' \int_0^{2\pi} d\varphi' f\left(\sqrt{\frac{A-1}{A}} \frac{r}{\sqrt{\lambda}} a \sin\theta' \cos\varphi',\ \sqrt{\frac{A-1}{A}}\right.$$
$$\left. \times \frac{r}{\sqrt{\lambda}} b \sin\theta' \sin\varphi',\ \sqrt{\frac{A-1}{A}} \frac{r}{\sqrt{\lambda}} c \cos\theta'\right);$$
$$m_1 = (A+2k-5)/2. \tag{46}$$

Introducing the differentiation $\overline{P}_m(-\partial/\partial\lambda)$ in the integrand and making some simple transformations, we obtain

$$\langle\chi_{\{0\}}^{[kkk]}|f(\xi,\eta,\zeta)|\chi_{\{0\}}^{[kkk]}\rangle = \frac{\Gamma(m_1+2)}{\Gamma(m_1+1/2)}\int_0^1 (1-r^2)^{m_1-m-1/2} Q_m(r^2)\,r^2\,dr \times \int_0^{\pi}\sin\theta'\,d\theta' \int_0^{2\pi} d\varphi' f\Big(\sqrt{\frac{A-1}{A}}\,ra\sin\theta'\cos\varphi',\ \sqrt{\frac{A-1}{A}}\,rb \times \sin\theta'\sin\varphi',\ \sqrt{\frac{A-1}{A}}\,rc\cos\theta'\Big), \quad (47)$$

where $Q_m(r^2)$ is a polynominal of degree m in r^2.

It is convenient to perform the subsequent calculations in the collective variables ρ, β, γ. The collective function Φ_0^0 in these variables has the form

$$\Phi_0^0(\rho,\beta,\gamma) = \frac{8}{\sqrt{27}}\left(\frac{4}{27}\right)^{m/2}\frac{1}{\sqrt{\Gamma(m_1+1)\,\Gamma(2m_1+3)}} \times \rho^{3k}\exp(-\rho^2/2)\,(1-3\beta^2+2\beta^3\cos 3\gamma)^{k/2}. \quad (48)$$

It is normalized in such a way that unity is the result of integrating $\Psi_0^0(\rho,\beta,\gamma)$ with the volume element

$$d\tau_{\rho\beta\gamma} = \rho^{3A-4}\,d\rho\,(1-3\beta^2+2\beta^3\cos 3\gamma)^{A-5/2}\beta^4\,d\beta\,\sin 3\gamma\,d\gamma \quad (49)$$

over the region

$$0<\rho<\infty;\ 0<\gamma<\pi/3;\ 0<\beta<1/[2\cos(\gamma-2\pi/3)].$$

To find $f_0^0(\xi,\eta,\zeta)$, the distribution function of the single-particle density in the system of principal axes of the ellipsoid of inertia, we set

$$f(\xi,\eta,\zeta) = \delta\Big(\xi-\sqrt{\frac{A-1}{A}}\,ar\sin\theta'\cos\varphi'\Big) \times \delta\Big(\eta-\sqrt{\frac{A-1}{A}}\,br\sin\theta'\sin\varphi'\Big)\,\delta\Big(\zeta-\sqrt{\frac{A-1}{A}}\,cr\cos\theta'\Big) \quad (50)$$

and we integrate the matrix element (47) over the collective variables ρ, β, γ with weight $[\Phi_0^0(\rho,\beta,\gamma)]^2$. The three δ functions in the integrand make it possible to reduce the six-fold integral to a triple integral:

$$f_0^0(\xi,\eta,\zeta) = \left(\frac{4}{27}\right)^{m_1+1/2}\frac{16}{\Gamma(2m_1+2)\,\Gamma(m_1+1/2)}\int_0^1 (1-r^2)^{m_1-m-1/2} \times Q_m(r^2)\,r^2\,dr\int_0^{\pi/3}\sin 3\gamma\,d\gamma\int_0^{1/-[2\cos(\gamma-2\pi/3)]}(1-3\beta^2 + 2\beta^3\cos 3\gamma)^{m_1-1/2}\beta^4\,d\beta\,[D_A]^{3m_1+9/2}\exp[-D_A], \quad (51)$$

where

$$D_A = \frac{3A}{A-1}\Big[\frac{\xi^2}{1+2\beta\cos(\gamma+2\pi/3)} + \frac{\eta^2}{1+2\beta\cos(\gamma-2\pi/3)} + \frac{\zeta^2}{1+2\beta\cos\gamma}\Big]\frac{1}{r^2}. \quad (52)$$

In the particular case of oxygen $^{16}O(m=1)$, the polynomial $Q_m(r^2)$ has the form

$$Q_1(r^2) = (1+23r^2)/5\pi^{3/2}. \quad (53)$$

The integration in (51) with this polynomial for $A=16$ was performed in Ref. 31. It was found that the constant-density surfaces of ^{16}O in the intrinsic coordinate system of this nucleus resemble ellipsoidal surfaces, the semiaxes of the ellipsoids being in the ratios 0.65:0.85:1.10.

7. EFFECTIVE NONSPHERICITY AND NONAXIALITY OF NUCLEI[34-37]

The quantitative criteria generally used to estimate the nonsphericity of nuclei—the quadrupole moment in some state and the probability of $E2$ transition between different states of one rotational band—give only indirect information about nuclear nonsphericity. The parameters of the anisotropic oscillator basis used for Hartree-Fock calculations are also not directly related to the shape of the nucleus.

The most natural and direct way of estimating the nonsphericity of nuclei and the values of their quadrupole deformations is based on an investigation of the components $u_{\nu K}^{I\tau}(\rho,\beta,\gamma)$ of the collective function [see (3)], since it is in terms of the collective variables ρ, β, γ that one determines the nonzero components of the tensor Q_{2k} of the mass quadrupole moment in the system of the principal axes of the ellipsoid of inertia, or, which is the same thing, in the system of the principal axes of the tensor of the quadrupole moment. To within a factor equal to the nucleon mass,

$$Q_{20} = 2\rho^2\beta\cos\gamma;\quad Q_{22+} = 2\rho^2\beta\sin\gamma = (Q_{22}+Q_{2-2})/\sqrt{2}; \quad Q_{21} = Q_{2-1} = (Q_{22}-Q_{2-2})\sqrt{2} = 0. \quad (54)$$

The collective components $u_{\nu K}^{I\tau}$ give the most complete picture of the quadrupole deformations of the nucleus. In particular, they determine the function

$$W^{I\tau}(\rho,\beta,\gamma) = \sum_{\nu K}|u_{\nu K}^{I\tau}(\rho,\beta,\gamma)|^2, \quad (55)$$

in terms of which the probability density of different values of ρ, β, γ is expressed. To estimate the extent to which the ellipsoid of inertia of a given state differs on the average from a sphere, we can consider the average value $\overline{\beta}^2$. It characterizes the effective nonsphericity of the nucleus. As regards the effective nonaxiality of the ellipsoid of inertia of the nucleus, it can be characterized by the mean value $\overline{\beta^3\cos 3\gamma}$. Both $\overline{\beta^2}$ and $\overline{\beta^3\cos 3\gamma}$ can be readily calculated if the function $W^{I\tau}$ is known. For example,

$$\left.\begin{array}{c}\overline{\beta}^2 = \int W^{I\tau}\beta^2\,d\tau_\rho\,d\tau_{\beta\gamma};\\ d\tau_\rho = \rho^{3A-4}\,d\rho;\ d\tau_{\beta\gamma} = (1-3\beta^2+2\beta^3\cos 3\gamma)^{(A-5)/4} \\ \times\beta^4\,d\beta\,\sin 3\gamma\,d\gamma.\end{array}\right\} \quad (56)$$

The integration with respect to ρ is from zero to infinity, and the integration with respect to β and γ is over the area of the triangle formed by the two rays that emanate from the origin at the angles $\gamma=0$ and $\gamma=\pi/3$ and the straight line for which $1+2\beta\cos(\gamma+2\pi/3)=0$.

Let us now consider particular nuclei. To describe them, we shall use the wave functions of the translationally invariant oscillator shell model, which are a reasonable zeroth approximation for light nuclei. We begin by analyzing the shape of the ellipsoid of inertia of light magic nuclei.

The single component of the collective wave function (28) of the ^4He ground state does not depend on β or γ:

$$u_{0,0}^0(^4He) = \Phi_0^0(^4He) \sim \exp(-\rho^2/2). \quad (57)$$

Therefore, the probability density of different values of

β and γ is determined solely by the dependence of the element of phase volume on these variables:

$$d\tau_{\beta\gamma}(^4\mathrm{He}) = \beta^4\, d\beta \sin 3\gamma\, d\gamma / \sqrt{1 - 3\beta^2 + 2\beta^3 \cos 3\gamma}. \tag{58}$$

For fixed values of γ, the probability density increases with increasing β and becomes infinite at the maximal (allowed) values of β.

The mean value of β^2, calculated in accordance with the function (57) with allowance for (58), is

$$\overline{\beta^2}(^4\mathrm{He}) = 0.455. \tag{59}$$

To get a perspicuous scale for this value, we must somewhat modify the definition of β and introduce the new variable

$$\tilde{\beta} = \sqrt{4\pi/5}\,\beta, \tag{60}$$

whose values can be directly compared with the values taken by the deformation parameter $\tilde{\beta}$ used by Bohr and Mottelson.[1] The appearance of the factor $\sqrt{4\pi/5}$ is a result of the following comparison.

The operator of the quadrupole moment of the mass distribution of the nucleons in the nucleus is related in a simple manner to the collective variables ρ, β, γ and the generalized spherical functions D^2_{KM} of the Eulerian angles φ, θ, ψ:

$$Q_{2M} = 2\rho^2\beta\,[\cos\gamma D^2_{0M} + \sin\gamma\,(D^2_{2M} + D^2_{2-M})/\sqrt{2}]. \tag{61}$$

If there are vibrations in the nucleus of the variable ρ about the equilibrium value ρ_0 and if the amplitude of these vibrations is small compared with ρ_0, then the mean square radius $\overline{r^2}$ of the nucleus satisfies the approximate relation $A\overline{r^2} \approx \rho_0^2$. In the same approximation, ρ^2 in (60) can be replaced by $A\overline{r^2}$. It follows from comparison of the resulting approximate expression for Q_{2M} and the same operator but in the Bohr–Mottelson model that

$$Q_{2M} = \sqrt{5/\pi}\, A\overline{r^2}\tilde{\beta}\,[\cos\tilde{\gamma} D^2_{0M} + \sin\tilde{\gamma}\,(D^2_{2M} + D^2_{-2M})\sqrt{2}];$$
$$\overline{r^2} = (3/5)\,R_0^2, \tag{62}$$

where R_0 is the radius of a sphere filled uniformly by the nuclear matter; there also follow (60) and the equation $\tilde{\gamma} = \gamma$.

It follows from (59) and (60) that $\sqrt{\overline{\tilde{\beta}^2}} = 1.07$ for ^{4}He, whereas the statistical value of $\tilde{\beta}$ in the Bohr–Mottelson model for heavy nonspherical nuclei (the equilibrium value about which the $\tilde{\beta}$ vibrations take place) is approximately equal to 0.3 – 0.35.

The collective component (57) corresponds to the wave function of a system in which oscillator forces of Wigner type act between the particles. If the oscillator forces are replaced by weak attractive forces, the mean value of β^2 is increased even more. But if on the background of the long-range attractive forces a short-range repulsion is introduced, $\overline{\beta^2}$ is somewhat decreased. The value of $\overline{\beta^2}$ is near zero if there are in the system forces (a combination of short-range repulsion and long-range attraction) which rigidly stabilize the motion of the four particles near the vertices of a tetrahedron.

The mean value of $\beta^3 \cos 3\gamma$ for the ground state of ^{4}He can also be readily found using (57) and (58) and is $\beta^3 \cos 3\gamma = 0.245$.

TABLE I. Parameters of effective deformation of light magic nuclei.

Nucleus	β_{eff}	γ_{eff}, deg	$c_{eff}:b_{eff}:a_{eff}$
^{4}He	0.67	12	1.52 : 0,77 : 0.31
^{16}O	0.27	23	1.22 : 0,97 : 0.76
^{40}Ca	0.144	27	1.12 : 0,99 : 0.87

If the relations

$$\beta^2_{eff} = \overline{\rho^4\beta^2}/\overline{\rho^4}; \quad \beta^3_{eff}\cos 3\gamma_{eff} = \overline{\rho^6\beta^3\cos 3\gamma}/\overline{\rho^6}$$

are used to determine β^2_{eff} and γ_{eff}, we then obtain[2)] for ^{4}He the result $\beta_{eff} = 0.672$ and $\gamma_{eff} = 12°25'$.

We recall that in the Bohr–Mottelson collective model $\overline{\tilde{\beta}^3 \cos 3\gamma} = 0$ for all states of a nucleus that executes harmonic vibrations about a spherically symmetric shape, and therefore in this case $\tilde{\gamma}_{eff} = 30°$.

Thus, on the average the ellipsoid of inertia of ^{4}He in the intrinsic coordinate system has an elongated nonaxial shape. Expressing in terms of β_{eff} and γ_{eff} the effective value of the three semiaxes of the ellipsoid of inertia:

$$c_{eff} \approx \sqrt{1 + 2\beta_{eff}\cos\gamma_{eff}}; \quad b_{eff} \approx \sqrt{1 + 2\beta_{eff}\cos(\gamma_{eff} - 2\pi/3)};$$
$$a_{eff} \approx \sqrt{1 + 2\beta_{eff}\cos(\gamma_{eff} + 2\pi/3)},$$

we find a ratio that also characterizes on the average the quadrupole moment of the mass distribution of the nucleons in the intrinsic coordinate system.

The values of β_{eff} and γ_{eff} and the ratios $c_{eff}:b_{eff}:a_{eff}$ for the ground states of the light magic nuclei are given in Table I. With increasing mass number A, the value of β_{eff} of the magic nuclei decreases (although even for ^{40}Ca it has a relatively large value), while γ_{eff} tends to $\pi/6$.

The mean values of $\rho^4\beta^2$ and $\rho^6\beta^3\cos 3\gamma$ can be calculated in any microscopic model without direct use of the functions $W^{I\tau}$ or the collective components $u^{I\tau}_{\nu K}$, provided the wave function of the model can be expressed in terms of the coordinates of the nucleons. Thus, in the same approximation of the translationally invariant shell model,

$$\beta^2_{eff} = \frac{3}{2}\frac{1}{E_0(E_0+1)}\left[g_2 - \frac{1}{2}L(L+1)\right] + \frac{5}{2}\frac{1}{E_0+1};$$
$$\beta^3_{eff}\cos 3\gamma_{eff} = \frac{1}{8E_0(E_0+1)(E_0+2)}\left\{36g_3 + 27\omega + 63\left[g_2 - \frac{1}{2}L(L+1)\right] + 70E_0\right\};$$
$$g_2 = (2/3)(\lambda^2 + \mu^2 + 2\lambda + 3\lambda + 3\mu);$$
$$g_3 = (\lambda - \mu)[(\lambda + 2\mu)(2\lambda + \mu) + 9(\lambda + \mu + 1)]/9;$$
$$E_0 = N_{min} + (3/2)(A - 1),$$

where L is the total orbital angular momentum of the state, λ and μ are the indices of the representation of $SU(3)$, ω is the eigenvalue of the Bargmann–Moshinsky operator, N_{min} is the smallest number of oscillator quanta of the state permitted by the Pauli principle for

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2) In the considered approximation, $\overline{\rho^4\beta^2} = \overline{\rho^4}\,\overline{\beta^2}$, $\overline{\rho^6\beta^3\cos 3\gamma} = \overline{\rho^6}\,\overline{\beta^3\cos 3\gamma}$.

TABLE II. Parameters of effective deformation of some nonmagic nuclei.

Nucleus	L	$(\lambda\mu)$	β_{eff}	γ_{eff}, deg	$c_{eff}:b_{eff}:a_{eff}$
^{6}He	0	(20)	0.58	14	1,46 : 0.82 : 0,44
	2		0.54	15	1.43 : 0.58 : 0.49
^{8}He	0	(40)	0,53	14	1.43 : 0.84 : 0,51
	2		0.52	15	1,41 : 0.85 : 0.52
	4		0.47	18	1.38 : 0.90 : 0.55
^{10}B	0	(22)	0.43	21	1.34 : 0.93 : 0.58
^{12}C	0	(04)	0.38	26	1.30 : 0.97 : 0,61
	2		0.37	25	1.29 : 0.97 : 0.63
	4		0.35	23	1.28 : 0.96 : 0.67
^{14}C	0	(02)	0.31	23	1.25 : 0.96 : 0.71
	2		0.30	23	1.24 : 0.96 : 0.73
^{18}O	0	(40)	0.27	22	1.23 : 0.96 : 0.76
	2		0.27	22	1.23 : 0.96 : 0.76
	4		0.26	23	1.21 : 0.97 : 0.77
^{20}Ne	0	(80)	0.29.	19	1.25 : 0.94 : 0.75
	2		0.29	19	1.25 : 0.94 : 0.75
	4		0.28	20	1.24 : 0.95 : 0.75
	6		0.27	21	1.23 : 0.96 : 0.75
	8		0.25	23	1.21 : 0.97 : 0.77
^{24}Mg	0	(84)	0.28	22	1.23 : 0.96 : 0.75
^{5}He	1	(10)	0.60	14	1,54 : 0.82 : 0.41
^{7}Li	1	(30)	0.55	14	1.43 : 0.84 : 0.49
	3		0.50	17	1.40 : 0.88 : 0.52

the representation $(\lambda\mu)$ of the $SU(3)$ scheme.

The effective deformation parameters of some nuclei of the p and $s-d$ shells are given in Table II.

Within the first oscillator shells, both the nonsphericity and the effective nonaxiality vary weakly on the transition from one nucleus to its neighbor, and the values of β_{eff} and γ_{eff} for the nonmagic nuclei are close to the corresponding values for the magic nuclei. The nonsphericity of the ground states of even-even nuclei ($L=0$) of the p shell is the greater, the larger is the number of nucleons, and ^{20}Ne has the greatest nonsphericity in the ground state for the nuclei of the $s-d$ shell.

As L varies from the minimal to the maximal value within the representation $(\lambda\mu)$ of $SU(3)$, the value of β_{eff} always only decreases, and γ_{eff} increases if the wave functions belong to a representation with $\lambda\neq 0$, $\mu=0$ and decreases if $\lambda=0, \mu\neq 0$. Thus, a nucleus that is prolate for $L=0(\lambda\neq 0, \mu=0)$ becomes oblate with increasing L, while a nucleus that is oblate for $L=0$ ($\lambda=0, \mu\neq 0$) becomes prolate with increasing L, but its nonsphericity decreases simultaneously in both the first and the second case.

Thus, the translationally invariant shell model using an isotropic oscillator basis predicts a large effective nonsphericity and an effective nonaxiality of the ellipsoid of inertia of nuclei of the p and $s-d$ shells, and within the $SU(3)$ group of rotational bands the degree of nonsphericity of the nuclear ellipsoid of inertia hardly changes as long as the orbital angular momentum L is small; but at the maximal L for the $(\lambda\mu)$ band the centrifugal stretching to which the rotating nucleus is subjected is appreciably weaker than at small L, and the degree of nonsphericity of the nucleus is accordingly smaller. This somewhat unexpected effect, which was noted earlier in a different form in Refs. 38–40 and called there anticentrifugal stretching, receives a simple explanation on the basis of the notion of a subsystem determined by the intrinsic degrees of freedom of the nucleus—the generalized Eulerian angles in the $(A-1)$-dimensional space.

Above, we have already noted that the quasiangular momentum j, the angular momentum of the intrinsic subsystem with respect to the principal axes of the ellipsoid of inertia, and its projection onto one of the axes of the ellipsoid of inertia can be conveniently used to classify the components χ_ν of the intrinsic function, which characterizes the motion of the subsystem. All the states of the nucleus belonging to the band $(\lambda\mu)$ have the same many-component intrinsic function, and each of the states is a superposition of components of the intrinsic function corresponding to different values of the quasiangular momentum and its projection. The presence of components with nonzero quasiangular momentum means that in all states of the rotational band there is intrinsic rotation. The intrinsic rotation is a result of the fact that for nonmagic nuclei the homogeneous polynomials of minimal degree satisfying the conditions of the Pauli principle and constructed from the single-particle nucleon coordinates must correspond to nonzero values of the quasiangular momentum.

The contribution of components with different values of the quasiangular momentum is determined by the component of the collective function. The possible values of the quasiangular momentum depend on the indices λ and μ. Suppose, for example, that the rotational band $(\lambda\mu)$ of an even-even nucleus corresponds to an even value $\lambda\neq 0$ and $\mu=0$. Then the quasiangular momentum takes all possible even values from 0 to λ, and even in the base state of the rotational band ($L=0$) components of the intrinsic function with nonzero quasiangular momentum are important. Moreover, in this state the intrinsic rotation favors the appearance of effective nonsphericity to a greater extent than in other states. Indeed, since the total angular momentum of the nucleus is zero, in order to compensate fully the angular momentum of the intrinsic rotation, the ellipsoid of inertia must rotate in the direction opposite to that of the intrinsic rotation, with the result that centrifugal forces distort the ellipsoid of inertia in such a way that the axis along which it is elongated is on the average perpendicular to the quasiangular momentum.

In the form in which it is expressed in Refs. 13 and 14, the Hamiltonian of the nucleus does not contain the angular momentum operators of the intrinsic rotation of the ellipsoid of inertia, but it does contain the operators of the projections $\hat{j}_\xi, \hat{j}_\eta, \hat{j}_\zeta$ of the quasiangular momentum and the operators of the projections L_ξ, L_η, L_ζ of the total orbital angular momentum of the nucleus onto the principal axes of the ellipsoid of inertia. The operators of the projections $\hat{R}_\xi, \hat{R}_\eta, \hat{R}_\zeta$ of the angular momentum of intrinsic rotation of the ellipsoid of inertia are related to the operators of the projections of the quasiangular momentum and the total angular momentum by simple equations: $\hat{R}_\xi=\hat{I}_\xi-\hat{j}_\xi, \hat{R}_\eta=\hat{I}_\eta-\hat{j}_\eta, \hat{R}_\zeta=\hat{I}_\zeta-\hat{j}_\zeta$. Therefore the part of the Hamiltonian of the system that in a state with zero total angular momentum is the kinetic-energy operator of the intrinsic rotation of the ellipsoid of inertia has the form

$$\frac{\hbar^2}{2m\rho^2\beta^2}\left[\frac{1-\beta\cos(\gamma-2\pi/3)}{\sin^2(\gamma-2\pi/3)}\,j_\xi^2+\frac{1-\beta\cos(\gamma+2\pi/3)}{\sin^2(\gamma+2\pi/3)}\,j_\eta^2+\frac{1-\beta\cos\gamma}{\sin^2\gamma}\,j_\zeta^2\right].$$

In the states of the rotational band in which the total angular momentum is nonzero, there is no need to compensate completely the angular momentum of the intrinsic rotation, and the intrinsic rotation of the ellipsoid of inertia is not so rapid. The greater is the total angular momentum of the state of the rotational band $(\lambda\mu)$, the slower the ellipsoid of inertia rotates. The slowing down of the intrinsic rotation of the ellipsoid of inertia somewhat reduces the effect of the centrifugal stretching of the nucleus and simultaneously reduces its degree of nonsphericity. However, as long as the total angular momentum of the nucleus is much smaller than the maximal value of the quasiangular momentum of the intrinsic function of the rotational band $(\lambda\mu)$, this decrease has virtually no influence on the effective nonsphericity and on the effective nonaxiality of the ellipsoid of inertia, and it is only in the states in which the total angular momentum of the nucleus has its maximal value that it can be appreciable. In these states, the angular momentum of the nucleus is directed on the average along the quasiangular momentum.

CONCLUSIONS

The method of generalized hyperspherical functions has passed through only the first stage of its development. Hitherto, it has been used only to investigate comparatively simple systems. The aim here has been to establish the potentialities of the method and its advantages over the traditional approaches. As regards the extension of the method to a greater number of nuclei and its development beyond the minimal approximation, this requires further development of the technique of computation of the matrix elements of various operators between generalized hyperspherical functions.

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