The Newman-Penrose formalism and its application in the general theory of relativity

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In this review, we give a systematic exposition of the Newman-Penrose formalism in the general theory of relativity and we use this formalism to describe some general properties of gravitational fields that are intimately related to the behavior of null curves and null surfaces in Riemannian space-time. Actual methods of integrating the complete system of Newman-Penrose equations are considered in detail for gravitational fields of some algebraically special types in the Petrov classification in vacuum and in radiation-filled space-time. A separate section is devoted to applications of the spin coefficient formalism to the description of the propagation of gravitational and electromagnetic waves of small amplitude in a given external gravitational field, in particular, in the field of a black hole.

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INTRODUCTION

In the general theory of relativity formulated by Einstein, the gravitational field is a manifestation of the curvature of four-dimensional Riemannian space-time. In the general case of curved space-time, there does not exist a distinguished, globally defined class of equivalent frames of reference analogous to the inertial frames in flat space-time that can be conveniently used for formulating all the laws of nature. Therefore, one of the fundamental principles of general relativity is the principle of general covariance, according to which all physical laws are formulated in a form that is formally suitable in any frame of reference and in any four-dimensional coordinate system.

However, the actual form of the physical laws and the form of the various properties of a given phenomenon depend strongly on the method by which the phenomenon is described (i.e., on the choice of the space-time coordinates, the vectors of the four-dimensional bases, the field functions, etc.) and they will take their simplest form if the description is chosen to reflect the specific features of the considered problem (or class of problems), i.e., if the choice takes into account the symmetry, the existence of distinguished directions, and other properties which are known *a priori*.

It sometimes happens that such a choice of the method of description (which is not complete but leaves some arbitrariness), made in a universal manner, can lead to modifications of the mathematical formalism of the complete theory that are helpful in many fields of application.

"Sometimes, by reformulating an old theory in an unusual (though mathematically equivalent) way, previously unexpected possibilities for modifying the theory may appear as mathematically natural."[1]

One such modification of the mathematical formalism of general relativity is the Newman–Penrose formalism (the spin coefficient method), which is based on the introduction in space-time of null complex tetrads and a special choice of the field variables (which leads to a change in the field equations themselves). This formalism makes it possible to exploit naturally the geometrical properties of null curves and null surfaces and is therefore helpful in large classes of problems in general relativity; for example, in investigating the structure and the most general properties of gravitational fields that are associated with the behavior of congruences of null geodesics and null surfaces, which are characteristic surfaces for the gravitational field equations: for the study of fields of algebraically special types in the Petrov classification, whose definition entails the presence in space-time of null directions that have definite properties and are distinguished by the curvature of space-time itself; and, finally, in various problems relating to the propagation of gravitational, electromagnetic, and other waves, in which, of course, families of null geodesics (rays) and null surfaces (wave fronts) of the waves play an important part.[1)]

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1)After this review had been written, an extensive review entitled "The Newman–Penrose method in the general theory of relativity" by V. P. Frolov was published [Tr. Fiz. Inst. Akad. Nauk SSSR **96**, 72 (1977)]. Both reviews contain a detailed exposition of the fundamentals of the Newman–Penrose formalism, but differ strongly in the description of its various applications. For example, in Sec. 3 of the present review we consider in more detail the process of integration of the complete system of field equations and we give some new exact solutions. The material of Sec. 4 (application of the formalism to the propagation of waves in external fields) has no counterpart in Frolov's review. On the other hand, Frolov devotes considerable attention to results relating to asymptotically flat gravitational fields, whereas our review contains a description of only a few of these results.

Professor J. D. Zund (University of New Mexico, USA) has kindly sent us a reprint of his paper "Notes on the projective geometry of spinors" [J. D. Zund, Annali di Matematica Pura ed Applicata (IV) **110**, 29 (1976)], which is not included in the bibliography of the present review. In particular, Zund considers in detail the spinor aspects of the Newman–Penrose formalism, the algebraic classification of electromagnetic and gravitational fields, and some other questions.

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1. THE NEWMAN-PENROSE FORMALISM

The Newman–Penrose formalism can be constructed in the usual "tensor" (tetrad) manner by projection onto vectors of a complex null tetrad and by using special notation for various tensor relations. However, these constructions take their most natural form in a different approach that is based on the introduction of spinors and leads to the same system of field equations.

In this section, we describe the tensor and spinor variants of the Newman–Penrose formalism. The main equations are collected together for convenience in the Appendix. In writing the present section, we have mainly used Refs. 1–5.

Tensor Formulation. Complex Null Tetrads and the Newman–Penrose Scalars. The Newman–Penrose formalism presupposes the choice at every point of space-time (a four-dimensional pseudo-Riemannian space R^4 with metric signature −2) of a basis tetrad of four null vectors: two real vectors l^j and n^j and two complex vectors m^j and $\overline{m}^j$, which are related to the vectors of an ordinary orthonormal basis $\{e^j_{(0)}, e^j_{(1)}, e^j_{(2)}, e^j_{(3)}\}$ by[2)]

$$\left.\begin{array}{ll} l^j = e^j_{(0)} + e^j_{(1)}; & m^j = (1/\sqrt{2})\{e^j_{(2)} + ie^j_{(3)}\}; \\ n^j = (1/2)\{e^j_{(0)} - e^j_{(1)}\}; & \overline{m}^j = (1/\sqrt{2})\{e^j_{(2)} - ie^j_{(3)}\} \end{array}\right\} \quad (1)$$

(where $e^j_{(0)}$ is a timelike vector and $e^j_{(1)}, e^j_{(2)}, e^j_{(3)}$ are spacelike vectors). The Newman–Penrose vectors satisfy the orthogonality and normalization conditions

$$l_j l^j = n_j n^j = m_j m^j = l_j m^j = n_j m^j = 0; \; l_j n^j = -m_j \overline{m}^j = 1. \quad (2)$$

The components of the metric tensor g_{ij} can be expressed in terms of the components of the vectors (1) in accordance with

$$g_{ij} = l_i n_j + n_i l_j - m_i \overline{m}_j - \overline{m}_i m_j. \quad (3)$$

For the covariant derivatives along the directions of the vectors (1) the notation

$$D = l^i \nabla_i; \quad \Delta = n^i \nabla_i; \quad \delta = m^i \nabla_i; \quad \overline{\delta} = \overline{m}^i \nabla_i, \quad (4)$$

where ∇_i is the operator of covariant differentiation, is used.

By analogy with the Ricci rotation coefficients, for the null complex tetrad (1) rotation coefficients are introduced; these are also called spin coefficients, and they are denoted by

$$\left.\begin{array}{lll} \kappa = m^i D l_i; & \pi = -\overline{m}^i D n_i; & \varepsilon = (n^i D l_i - \overline{m}^i D m_i)/2; \\ \rho = m^i \overline{\delta} l_i; & \lambda = -\overline{m}^i \overline{\delta} n_i; & \alpha = (n^i \overline{\delta} l_i - \overline{m}^i \overline{\delta} m_i)/2; \\ \sigma = m^i \delta l_i; & \mu = -\overline{m}^i \delta n_i; & \beta = (n^i \delta l_i - \overline{m}^i \delta m_i)/2; \\ \tau = m^i \Delta l_i; & \nu = -\overline{m}^i \Delta n_i; & \gamma = (n^i \Delta l_i - \overline{m}^i \Delta m_i)/2. \end{array}\right\} \quad (5)$$

The commutators of the operators (4), applied to a scalar function, have the form

$$\left.\begin{array}{l} \Delta D - D\Delta = (\gamma + \overline{\gamma}) D + (\varepsilon + \overline{\varepsilon}) \Delta - (\tau + \overline{\pi}) \overline{\delta} - (\overline{\tau} + \pi) \delta; \\ \delta D - D\delta = (\overline{\alpha} + \beta - \overline{\pi}) D + \kappa \Delta - \sigma \overline{\delta} - (\overline{\rho} + \varepsilon - \overline{\varepsilon}) \delta; \\ \delta\Delta - \Delta\delta = -\overline{\nu} D + (\tau - \overline{\alpha} - \beta) \Delta + \overline{\lambda} \overline{\delta} + (\mu - \gamma + \overline{\gamma}) \delta; \\ \overline{\delta}\delta - \delta\overline{\delta} = (\overline{\mu} - \mu) D + (\overline{\rho} - \rho) \Delta - (\overline{\alpha} - \beta) \overline{\delta} - (\overline{\beta} - \alpha) \delta. \end{array}\right\} \quad (6)$$

2) In what follows, lower-case letters of the Latin alphabet $i, j, k, \ldots$ denote four-dimensional tensor indices, which take the values 0, 1, 2, 3. The bar denotes the complex conjugate.

When tensor relations are expressed in this formalism, the tensors in them must be projected onto the vectors of the null tetrad. We shall consider in what follows the projections onto the vectors (1) of the Maxwell tensor, as a real bivector, and of the Riemann curvature tensor. A complete set of independent variables, which are called Newman–Penrose scalars and for which a special notation is adopted in the Newman–Penrose formalism, can be separated as follows from these projections.

Real bivectors in four-dimensional space form a six-dimensional linear space, and therefore the six linearly independent bivectors

$$U_{ij} = l_{[i} m_{j]}; \; V_{ij} = n_{[i} \overline{m}_{j]}; \; M_{ij} = l_{[i} n_{j]} - m_{[i} \overline{m}_{j]}; \quad (7)$$

$$\overline{U}_{ij} = l_{[i} \overline{m}_{j]}; \; \overline{V}_{ij} = n_{[i} m_{j]}; \; \overline{M}_{ij} = l_{[i} n_{j]} - \overline{m}_{[i} m_{j]} \quad (8)$$

form a basis in this space.

The dual transformation of the bivector F_{ij} is defined as the transformation

$$F_{ij} \to i\overset{*}{F}_{ij}; \; \overset{*}{F}_{ij} = \epsilon_{ijkl} F^{kl}/2, \quad (9)$$

where ε_{ijkl} is the Levi-Civita tensor ($\varepsilon_{ijkl} = \varepsilon_{[ijkl]}$, $\varepsilon_{0123} = -\sqrt{-g}$). Every real bivector F_{ij} can be associated in a one-to-one manner with a complex bivector $\overset{+}{F}_{ij}$ which is self-dual, i.e., invariant under the transformation (9):

$$\overset{+}{F}_{ij} = (F_{ij} + i\overset{*}{F}_{ij})/2; \; F_{ij} = \overset{+}{F}_{ij} + \overset{\mp}{F}_{ij}. \quad (10)$$

It is readily seen that the bivectors (7) are self-dual, whereas the bivectors (8) are anti-self-dual [i.e., under the dual transformation (9) they are multiplied by −1]. Therefore, the decomposition of the self-dual bivector $\overset{+}{F}_{ij}$ with respect to the basis (7)–(8) contains only the bivectors (7) but not the bivectors (8) [the bivectors (7) form a basis in the three-dimensional complex space of self-dual bivectors]:

$$\overset{+}{F}_{ij} = -2\varnothing_0 V_{ij} - 2\varnothing_1 M_{ij} + 2\varnothing_2 U_{ij}. \quad (11)$$

Here, the numerical factors in the coefficients of the decomposition are introduced to simplify the subsequent expressions. From the definitions (7), the conditions (2), and the expressions (10), we obtain expressions for calculating the coefficients ϕ_0, ϕ_1, ϕ_2:

$$\left.\begin{array}{l} \varnothing_0 = \overset{+}{F}_{ij} U^{ij} = F_{ij} l^i m^j; \\ \varnothing_1 = \overset{+}{F}_{ij} M^{ij}/2 = F_{ij} (l^i n^j - m^i \overline{m}^j) \, 2; \\ \varnothing_2 = -\overset{+}{F}_{ij} V^{ij} = -F_{ij} n^i \overline{m}^j. \end{array}\right\} \quad (12)$$

Equations (10)–(12) make it possible to express all projections of the real bivector F_{ij} onto the null tetrad (1) in terms of three complex scalars.

We now consider the Riemann tensor. This tensor can be decomposed into "irreducible" parts, which are determined by the traceless four-valent tensor C_{ijkl} (Weyl tensor) and the traceless two-valent tensor Φ_{ij} and the scalar Λ:

$$R_{ijkl} = C_{ijkl} + \Phi_{ik} g_{jl} - \Phi_{il} g_{jk} + \Phi_{jl} g_{ik} - \Phi_{jk} g_{il} + 2\Lambda (g_{il} g_{jk} - g_{ik} g_{jl}), \quad (13)$$

where $2\Phi_{ij} = R_{ij} - R g_{ij}/4$ is the traceless part (deviator) of the Ricci tensor R_{ij} ($R_{ij} = R^k{}_{ikj}$), R is the scalar curvature ($R = R^k{}_k$), and $R = -24\Lambda$.

The Weyl tensor, which is defined by (13), has the algebraic properties

$$C_{ijkl}=-C_{jikl}=-C_{ijlk}=C_{klij}; \quad (14)$$

$$C_{ijkl}+C_{iljk}+C_{iklj}=0;\ C^{k}{}_{ikj}=0. \quad (15)$$

The Weyl tensor C_{ijkl} can also be associated in a one-to-one manner with the "self-dual Weyl tensor" $\overset{+}{C}_{ijkl}$ in accordance with

$$\overset{+}{C}_{ijkl}=(C_{ijkl}+i\overset{*}{C}_{ijkl})/2;\ C_{ijkl}=\overset{+}{C}_{ijkl}+\overline{\overset{+}{C}}_{ijkl}, \quad (16)$$

where $\overset{*}{C}_{ijkl}=\frac{1}{2}\varepsilon_{klmn}C^{mn}{}_{ij}=\varepsilon_{ijmn}C^{mn}{}_{kl}/2$ [this last equation holds by virtue of (15)], so that the tensor $\overset{+}{C}_{ijkl}$ is self-dual with respect to any (either the first or the second) antisymmetric pair of indices. Therefore, the tensor $\overset{+}{C}_{ijkl}$ can be decomposed with respect to binary products of the self-dual bivectors (7). From the symmetry of C_{ijkl} (and, therefore, of $\overset{+}{C}_{ijkl}$ as well) under the interchange of antisymmetric pairs of indices, it follows that this decomposition must be symmetric under the interchange of any pair of bivectors. Equations (15) are equivalent to the two equations $C^{k}{}_{ikj}=0$, $\overset{*}{C}{}^{k}{}_{ikj}=0$ or the single complex equation $\overset{+}{C}{}^{k}{}_{ikj}=0$. This last condition has the consequence that in the decomposition of $\overset{+}{C}_{ijkl}$ with respect to products of the bivectors (7) the coefficient of the product $U_{ij}V_{kl}$ (which is equal to the coefficient of $V_{ij}U_{kl}$) differs only in sign from the coefficient of $M_{ij}M_{kl}$. Thus, the decomposition for $\overset{+}{C}_{ijkl}$ has the form

$$\begin{aligned}\overset{+}{C}_{ijkl}=&-4\Psi_0V_{ij}V_{kl}-4\Psi_1(V_{ij}M_{kl}+M_{ij}V_{kl})\\&-4\Psi_2(M_{ij}M_{kl}-U_{ij}V_{kl}-V_{ij}U_{kl})\\&+4\Psi_3(U_{ij}M_{kl}+M_{ij}U_{kl})-4\Psi_4U_{ij}U_{kl}\end{aligned} \quad (17)$$

[as in (11), we have here introduced numerical coefficients for convenience]. The coefficients $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ are calculated in accordance with the formulas

$$\left.\begin{aligned}\Psi_0&=-\overset{+}{C}_{ijkl}U^{ij}U^{kl}=-C_{ijkl}l^im^jl^km^l;\\ \Psi_1&=-\overset{+}{C}_{ijkl}U^{ij}M^{kl}/2=-C_{ijkl}l^in^jl^km^l;\\ \Psi_2&=-\overset{+}{C}_{ijkl}M^{ij}M^{kl}/4=-C_{ijkl}(l^in^jl^kn^l\\&\qquad-l^in^jm^k\overline{m}^l)/2;\\ \Psi_3&=\overset{+}{C}_{ijkl}V^{ij}M^{kl}/2=-C_{ijkl}n^il^jn^k\overline{m}^l;\\ \Psi_4&=-\overset{+}{C}_{ijkl}V^{ij}V^{kl}=-C_{ijkl}n^i\overline{m}^jn^k\overline{m}^l.\end{aligned}\right\} \quad (18)$$

The Weyl tensor C_{ijkl}, which has ten independent components, is completely determined in accordance with Eqs. (16)–(18) in the null tetrad by the five complex coefficients $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$.

For the projections of the symmetric tensor Φ_{ij}, which determine by virtue of (13) the deviator of the Ricci tensor, the following notation is adopted:

$$\left.\begin{aligned}&\Phi_{00}=\Phi_{ij}l^il^j; && \Phi_{10}=\Phi_{ij}l^i\overline{m}^j; && \Phi_{20}=\Phi_{ij}\overline{m}^i\overline{m}^j;\\ &\Phi_{01}=\Phi_{ij}l^im^j; && \Phi_{11}=\Phi_{ij}l^in^j; && \Phi_{21}=\Phi_{ij}n^i\overline{m}^j;\\ &\Phi_{02}=\Phi_{ij}m^im^j; && \Phi_{12}=\Phi_{ij}n^im^j; && \Phi_{22}=\Phi_{ij}n^in^j.\end{aligned}\right\} \quad (19)$$

By virtue of the symmetry and the vanishing of the trace of the tensor Φ_{ij}, the projections (19) of this tensor satisfy the relations $\Phi_{mn}=\overline{\Phi}_{nm}$, where $m, n=0, 1, 2$, and therefore only six of the quantities in (19) are independent (three are real and three are complex).

The five complex scalars (18), the six independent scalars (19), and the scalar Λ determine the complete set of projections of the Riemann tensor R_{ijkl} onto the null tetrad (1).

Gravitational field equations in the Newman–Penrose formalism. In the usual formulation of the general theory of relativity given by Einstein, the gravitational field is described by the components of the metric g_{ij}, which are the potentials of this field and satisfy the system of Einstein equations, which form a nonlinear hyperbolic system of second-order equations:

$$R_{ij}-Rg_{ij}/2=\varkappa T_{ij}, \quad (20)$$

where the Ricci tensor R_{ij} and the scalar curvature R are expressed in terms of the metric tensor and its derivatives, T_{ij} is the energy-momentum tensor of matter, $\varkappa=8\pi G/c^4$ is Einstein's gravitational constant, and G is Newton's gravitational constant.

In the Newman–Penrose formalism, three groups of quantities become the variables of the gravitational field:

1) the components of the vectors $l^i, n^i, m^i, \overline{m}^i$ of the null tetrad (1) (which in accordance with (3) uniquely determine the components of the metric);

2) the spin coefficients (5);

3) the complete set of independent projections of the Riemann tensor, i.e., $\Psi_0, \Psi_1, \ldots, \Psi_4; \Phi_{mn}, \Lambda$.

The equations for these variables, which are the gravitational field equations in the Newman–Penrose formalism, can be divided into three groups on the basis of their "origin."

The first group of field equations are linear combinations of the projections of the Bianchi identities onto the vectors of the null tetrad:

$$\nabla_mR_{ijkl}+\nabla_kR_{ijlm}+\nabla_lR_{ijmk}=0; \quad (21)$$

for the choice of the variables made above, these are no longer identities, and all the terms of the projected equations (21) must be expressed [by means of (13), (16)–(19) and the definitions (5)] in terms of $\Psi_0, \Psi_1, \ldots, \Psi_4, \Phi_{mn}, \Lambda$, the spin coefficients (5), and the operators (4), i.e., solely in terms of the field variables that belong to the three listed groups. The independent linear combinations of the equations then obtained are given in Appendix A.

Similarly, projecting onto the vectors of the null tetrad (1) the equation that defines the Riemann tensor,

$$(\nabla_i\nabla_j-\nabla_j\nabla_i)z^k=R^h{}_{eij}z^l, \quad (22)$$

in which z^i is taken to be successively all the basis vectors $l^i, n^i, m^i, \overline{m}^i$, we obtain the second group of field equations, which is given in Appendix B.

The third group of equations which must be satisfied by the variables characterizing the gravitational field is formed by the so-called coordinate equations, which can be obtained as follows. On the space-time manifold we consider four independent[3)] scalar functions u, r, θ, φ. Applying the operators (4) to these functions, we obtain new functions, for which we introduce the notation

[3)]The independence of the functions u, r, θ, φ means that their gradients are linearly independent at every point.

$$\left.\begin{array}{llll} Y^0 = Du; & X^0 = \Delta u; & \xi^0 = \delta u; \\ V = Dr; & U = \Delta r; & \omega = \delta r; \\ Y^2 = D\theta; & X^2 = \Delta\theta; & \xi^2 = \delta\theta; \\ Y^3 = D\varphi; & X^3 = \Delta\varphi & \xi^3 = \delta\varphi. \end{array}\right\} \quad (23)$$

It is readily seen that if these functions u, r, θ, φ are taken as new coordinates, the functions (23) are the components of the vectors l^i, n^i, m^i in the new coordinate system,

$$l^i = \{Y^0, V, Y^2, Y^3\};\; n^i = \{X^0, U, X^2, X^3\};\; m^i = \{\xi^0, \omega, \xi^2, \xi^3\}, \quad (24)$$

and, therefore, the set of these functions coincides with the first group of field variables. It follows from the definitions (23) that these variables cannot be arbitrary but must satisfy certain constraint equations; these can be most readily obtained from the commutation relations (6) by applying them successively to the functions u, r, θ, φ and using (23). These equations are given in Appendix C.

The three groups (A.1)–(A.11), (A.12)–(A.29), and (A.30)–(A.37) of Newman–Penrose equations for the gravitational field are the purely "kinematic" constraints on the field variables. The coupling to matter appears only when, in accordance with the Einstein equations (20), the deviator Φ_{ij} of the Ricci tensor and the scalar Λ are expressed in terms of the energy-momentum tensor T_{ij} and its trace $T^l{}_l$ in accordance with the expressions[4)]

$$2\Phi_{ij} = \varkappa(T_{ij} - 1/4\, T^l{}_l g_{ij});\; \Lambda = \varkappa T^l_l/24. \quad (25)$$

In particular, in the regions of space where there is no matter, so that $T_{ij} = 0$, we obtain from (25) the result $\Phi_{ij} = 0$ (i.e., all $\Phi_{mn} = 0$) and $\Lambda = 0$.

But if there are in space not only the gravitational field but also other forms of matter, such as nongravitational fields, then to obtain the complete system of equations that determine the behavior of not only the gravitational field but also the matter generating it the three groups of Newman–Penrose equations must be augmented by, besides (25), the equations satisfied by the matter which generates the gravitational field (for example, the Maxwell equations for the electromagnetic field, the Weyl equations for the neutrino, etc.). All these equations must also be expressed in terms of the formalism.

Let us consider, for example, the case when there is also an electromagnetic field in space.

Maxwell equations in the Newman–Penrose formalism. The Maxwell equations in tensor form are

$$\nabla_k F^{ik} = -(4\pi/c)\, J^i;\; \nabla_i F_{jk} + \nabla_j F_{ki} + \nabla_k F_{ij} = 0, \quad (26)$$

where F_{ij} is the Maxwell tensor and J^i is the four-current vector. Equations (26) are equivalent to a single equation for the self-dual bivector $\overset{+}{F}_{ij}$:

$$\nabla_j \overset{+}{F}{}^{ij} = -(4\pi/c)\, J^i. \quad (27)$$

Using the decomposition (11) for $\overset{+}{F}{}^{ij}$ and projecting (27) onto the vectors of the null tetrad (1), we obtain equations that are the Maxwell equations in the Newman–Penrose formalism. These equations are given in Appendix D.

The projections of the energy-momentum tensor for the electromagnetic field can be readily calculated if in the usual expression for this tensor in terms of the Maxwell tensor[6] the expression for the latter in terms of $\overset{+}{F}_{ij}$ (10) is substituted; for this, the decomposition (11) is used. For $\Phi_{\mathfrak{mn}}$ and Λ we then have by virtue of (12) and (19)

$$\Phi_{\mathfrak{mn}} = (\varkappa/4\pi)\, \varnothing_{\mathfrak{m}} \overline{\varnothing}_{\mathfrak{n}};\; \Lambda = 0, \quad (28)$$

where $\mathfrak{m}, \mathfrak{n} = 0, 1, 2$ and $\phi_{\mathfrak{mn}}$ are determined from F_{ij} in accordance with (12).

Spinor Formulation. Spinors in Riemannian space. Spinors are introduced into a Riemannian space by constructing at every point of space-time, in addition to the tangent vector and tensor spaces, the two-dimensional complex linear space S of "spin vectors", and also all possible "spin-vector" (spinor) spaces that are tensor products of the space S, its dual space S_*, and their complex conjugate spaces $\overline{S}$ and $\overline{S}_*$.

In the space S, an antisymmetric bilinear form $\Omega(u, v)$ is introduced by specifying in a certain basis in S the coefficients ε_{AB}[5)]:

$$\Omega(u, v) = \varepsilon_{AB} u^A v^B, \text{where } \varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

this form being invariant under all linear unimodular transformations on S:

$$u^{A'} = \Lambda^A{}_{.B} u^B, \text{ where } \det \|\Lambda^A{}_{.B}\| = 1.$$

To this transformation in the space $\overline{S}$ there corresponds the transformation with the complex-conjugate matrix $\overline{\Lambda}$ (spinor indices that transform by means of $\overline{\Lambda}$ are indicated by a dot): $\overline{u}^{\dot{A}'} = \overline{\Lambda}^{\dot{A}}{}_{\dot{B}} \overline{u}^{\dot{B}}$.

The matrix ε_{AB}, which is invariant under unimodular transformations, can be used to raise and lower spinor indices:

$$u_A = \varepsilon_{BA} u^B;\quad u^A = \varepsilon^{AB} u_B;\quad u_{\dot{A}} = \varepsilon_{\dot{B}\dot{A}} u^{\dot{B}};\quad u^{\dot{A}} = \varepsilon^{\dot{A}\dot{B}} u_{\dot{B}},$$

where

$$\varepsilon_{AB} = \varepsilon^{AB} = \varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Because S has two dimensions, for an arbitrary spinor ξ we have the important identity

$$\xi_{\ldots A \ldots B \ldots} - \xi_{\ldots B \ldots A \ldots} = \varepsilon_{AB} \xi_{\ldots C \ldots}{}^{C}{}_{\ldots}, \quad (29)$$

which indicates that only symmetric spinors are irreducible, i.e., cannot be expressed in terms of symmetric spinors of lower valence.

Because the group of unimodular transformations in S is locally isomorphic to the proper Lorentz group, a one-to-one correspondence can be established between spinors of even rank and four-tensors. This correspondence can be represented explicitly as follows. If the matrix of a two-valent spinor $u^{A\dot{B}}$ in some basis in

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[4)] If the Einstein equations with cosmological term λ are considered, $R_{ij} - Rg_{ij}/2 + \lambda g_{ij} = \varkappa T_{ij}$, then in Eq. (25) one must take Λ to be $(\varkappa T^l{}_l - 4\lambda)/24$, where $T = T_l{}^l$ is the trace of the energy-momentum tensor.

[5)] Here and in what follows, upper-case Latin letters $A, B, C, \ldots$ denote the spinor indices and take the values 0 and 1.

S is Hermitian, it can be written in the form

$$u^{A\dot{B}} = u^i \sigma_i^{B\dot{A}} = \frac{1}{\sqrt{2}} \begin{pmatrix} u^0 + u^1 & u^2 + iu^3 \\ u^2 - iu^3 & u^0 - u^1 \end{pmatrix}, \tag{30}$$

where the u^i are real, the matrices $\sigma_i^{A\dot{B}}$ are Hermitian, and $\sigma_1^{A\dot{B}}, \sigma_2^{A\dot{B}}, \sigma_3^{A\dot{B}}$ differ only by a factor $1/\sqrt{2}$ from the Pauli matrices:

$$\left.\begin{aligned} \sigma_0^{A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_1^{A\dot{B}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\ \sigma_2^{A\dot{B}} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_3^{A\dot{B}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \end{aligned}\right\} \tag{31}$$

The Hermiticity of $u^{A\dot{B}}$ as well as the value of its determinant $2\det\|u^{A\dot{B}}\| = (u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2$ are conserved under an arbitrary unimodular transformation in S. Therefore, under this transformation the u^i remain real and undergo a Lorentz transformation. Thus, the u_i can be regarded as the components of some real vector in an orthonormal basis of Minkowski space.

The $\sigma^i{}_{A\dot{B}}$ can be used to establish a one-to-one correspondence between tensors and even-valence spinors, i.e., given the components of tensors, to calculate the components of the spinors corresponding to them and vice versa. Since $\sigma^i{}_{A\dot{B}}$ satisfy the relations

$$g_{ij}\sigma^i_{A\dot{B}}\sigma^j_{C\dot{D}} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}; \quad \varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}\sigma_i^{A\dot{C}}\sigma_j^{B\dot{D}} = g_{ij}, \tag{32}$$

which follow from (30) or (31), for a third-rank tensor $X^k{}_{ij}$, for example, we have

$$X^{E\dot{F}}{}_{A\dot{B}C\dot{D}} = \sigma^i_{A\dot{B}}\sigma^j_{C\dot{D}}\sigma_k^{E\dot{F}}X^k{}_{ij}; \quad X^k{}_{ij} = \sigma_i^{A\dot{B}}\sigma_j^{C\dot{D}}\sigma^k_{E\dot{F}}X^{E\dot{F}}{}_{A\dot{B}C\dot{D}}.$$

If the considered tensor has certain symmetry properties, its spinor equivalent may have a fairly simple form and may be expressible in terms of spinors of lower valence. In what follows, we shall consider some tensors and their spinor equivalents.

It follows from (32) that the metric tensor g_{ij} corresponds to the spinor $\varepsilon_{AB}\varepsilon_{\dot{C}\dot{D}}$. For the Levi-Civita discriminant tensor, we have

$$\varepsilon_{ijkl} \leftrightarrow \varepsilon_{A\dot{B}C\dot{D}E\dot{F}G\dot{H}} = i(\varepsilon_{AE}\varepsilon_{CG}\varepsilon_{\dot{B}\dot{H}}\varepsilon_{\dot{D}\dot{F}} - \varepsilon_{AG}\varepsilon_{CE}\varepsilon_{\dot{B}\dot{F}}\varepsilon_{\dot{D}\dot{H}}). \tag{33}$$

For the spinor equivalent of the Maxwell tensor F_{ij}, i.e., for the spinor $F_{A\dot{B}C\dot{D}} = \sigma^i{}_{A\dot{B}}\sigma^j{}_{C\dot{D}}F_{ij}$, we can write by virtue of (29)

$$\begin{aligned} F_{A\dot{B}C\dot{D}} \equiv F_{AC\dot{B}\dot{D}} &= \varepsilon_{AC}F_E{}^E{}_{(\dot{B}\dot{D})}/2 + \varepsilon_{\dot{B}\dot{D}}F_{(AC)\dot{H}}{}^{\dot{H}}/2 \\ &+ F_{(AC)(\dot{B}\dot{D})} + \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}F_{E\dot{H}}{}^{E\dot{H}}/4. \end{aligned}$$

Since F_{ij} is antisymmetric, $F_{(AC)(\dot{B}\dot{D})} = 0$ and $F_{E\dot{H}}{}^{E\dot{H}} = 0$. In addition, since F_{ij} is a real tensor, the spinors $F_{(AC)\dot{H}}{}^{\dot{H}}$ and $F_E{}^E{}_{(\dot{B}\dot{D})}$ are complex conjugates. Therefore, the spinor $F_{A\dot{B}C\dot{D}}$ corresponding to the real bivector F_{ij} can be represented in the form

$$F_{A\dot{B}C\dot{D}} = (\varepsilon_{AC}\bar{\varphi}_{\dot{B}\dot{D}} + \varepsilon_{\dot{B}\dot{D}}\varphi_{AC})/2, \tag{34}$$

where $\varphi_{AC} \equiv F_{(AB)\dot{H}}{}^{\dot{H}}$ is a symmetric second-rank spinor. Using (33), we can show that for the self-dual bivector $\overset{+}{F}_{ij}$ (10) the correspondence $\overset{+}{F}_{ij} \leftrightarrow \varepsilon_{\dot{B}\dot{D}}\varphi_{AC}/2$ holds.

Using the algebraic properties of the curvature tensor R_{ijkl}, we can obtain an expression analogous to (34) for the spinor corresponding to it:

$$\begin{aligned} -R_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} &= \Psi_{ABCD}\varepsilon_{\dot{E}\dot{F}}\varepsilon_{\dot{G}\dot{H}} + \bar{\Psi}_{\dot{E}\dot{F}\dot{G}\dot{H}}\varepsilon_{AB}\varepsilon_{CD} + 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{E}\dot{F}}\varepsilon_{\dot{G}\dot{H}} \\ &- \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{E}\dot{H}}\varepsilon_{\dot{F}\dot{G}}) + \varepsilon_{AB}\Phi_{CD\dot{E}\dot{F}}\varepsilon_{\dot{G}\dot{H}} + \varepsilon_{CD}\Phi_{AB\dot{G}\dot{H}}\varepsilon_{\dot{E}\dot{F}}, \end{aligned} \tag{35}$$

where $\Psi_{ABCD}, \Phi_{AB\dot{G}\dot{H}}, \Lambda$ are the so-called curvature spinors, which have the properties

$$\Psi_{ABCD} = \Psi_{(ABCD)}; \quad \Phi_{AC\dot{B}\dot{D}} = \Phi_{(AC)(\dot{B}\dot{D})} = \bar{\Phi}_{AC\dot{B}\dot{D}}; \quad \Lambda = \bar{\Lambda}.$$

The decomposition (35) is intimately related to the decomposition (13), since the relations

$$\begin{aligned} C_{ijkl} \leftrightarrow C_{A\dot{E}B\dot{F}C\dot{G}D\dot{H}} &= -\Psi_{ABCD}\varepsilon_{\dot{E}\dot{F}}\varepsilon_{\dot{G}\dot{H}} - \bar{\Psi}_{\dot{E}\dot{F}\dot{G}\dot{H}}\varepsilon_{AB}\varepsilon_{CD}; \\ \Phi_{ij} \leftrightarrow \Phi_{AB\dot{C}\dot{D}}; &\quad \Lambda = -R/24, \end{aligned}$$

where $\overset{+}{C}_{ijkl} \leftrightarrow -\Psi_{ABCD}\varepsilon_{\dot{E}\dot{F}}\varepsilon_{\dot{G}\dot{H}}$, hold.

So far, all our constructions have been made at a single point of Riemannian space. To describe the structure of spinor fields in space-time, it is necessary to define the operation of covariant differentiation for spinors:

$$\nabla_i\xi^A = (\partial/\partial x^i)\,\xi^A + \Gamma_{iB}^A\xi^B.$$

The coefficients $\Gamma_i{}^A{}_B$ are defined in such a way that the spinor connection they define is, by virtue of the correspondence between tensors and spinors, compatible with the affine connection of the Riemannian space. To determine the coefficients, it is sufficient to require fulfillment of the conditions

$$\nabla_i\sigma^j_{A\dot{B}} = 0; \quad \nabla_i\varepsilon_{AB} = 0.$$

Like Eq. (22), which expresses the commutator of two covariant derivatives of a vector in terms of the Riemann tensor and serves as the definition of this tensor, the second-order covariant derivatives of a spinor satisfy relations which contain the curvature spinors and can also serve as the definition of these spinors:

$$\left.\begin{aligned} &\nabla_{(A}{}^{\dot{E}}\nabla_{B)\dot{E}}\xi_C = -\Psi_{ABCD}\xi^D - 2\Lambda\varepsilon_{C(B}\xi_{A)}; \\ &\nabla_{E(\dot{C}}\nabla^E{}_{\dot{D})}\xi_A = \Phi_{AB\dot{C}\dot{D}}\xi^B, \end{aligned}\right\} \tag{36}$$

where $\nabla_{A\dot{B}} = \sigma^i_{A\dot{B}}\nabla_i$.

Further, from the Bianchi identities (21) and the expression (35), which relates the Riemann tensor to the curvature spinors, there follow equations for these spinors:

$$\nabla^D{}_{\dot{E}}\Psi_{ABCD} = \nabla_{(C}{}^{\dot{H}}\Phi_{AB)\dot{E}\dot{H}}; \quad \nabla^{A\dot{E}}\Phi_{AB\dot{E}\dot{H}} = -3\nabla_{B\dot{H}}\Lambda. \tag{37}$$

Equations (36) and (37) are the basic spinor equations that describe the gravitational field. To obtain the Newman–Penrose equations, these equations must be projected onto a spinor basis (basis dyad).

"Dyad" components of spinors and the Newman–Penrose equations for the gravitational field. The basis dyad $\zeta_a{}^A$ ($a = 0, 1$ labels the spinors of the basis[6]) must be chosen in such a way that the relations

$$\varepsilon_{AB}\zeta_a{}^A\zeta_b{}^B = \varepsilon_{ab}; \quad \varepsilon^{ab}\zeta_a{}^A\zeta_b{}^B = \varepsilon^{AB}, \text{ where } \varepsilon_{ab} = \varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{38}$$

[6]Here, we introduce one further type of index: lower case initial letters of the Latin alphabet $a, b, c, \ldots = 0, 1$ label the spinors of the basis dyad. The letters $i, j, k, \ldots$ of the same alphabet will also be used in what follows to denote four-dimensional tensor indices.

TABLE I. Spin coefficients.

$\Gamma_{abc\dot{d}} =$

$c\dot{d}$ \ ab	00	01 or 10	11
$0\dot{0}$	κ	ε	π
$1\dot{0}$	ρ	α	λ
$0\dot{1}$	σ	β	μ
$1\dot{1}$	τ	γ	ν

are satisfied. Spinors are then determined by the set of their projections onto the spinors of this basis; for example, for the spinor $X^A{}_{B\dot{C}}$ we have $X^a{}_{b\dot{c}} = X^A{}_{B\dot{C}}\zeta^a{}_A\zeta_b{}^B\zeta^{\dot{C}}_{\dot{c}}$, and $X^A_{B\dot{C}} = \zeta_a{}^A\zeta^b{}_B\bar{\zeta}^{\dot{c}}{}_{\dot{C}}X^a{}_{b\dot{c}}$, where $\zeta^a{}_A = \varepsilon^{ab}\varepsilon_{BA}\zeta_b{}^B$.

The dyad components of $\sigma^i{}_{A\dot{B}}$, i.e., $\sigma^i{}_{a\dot{b}} = \zeta_a{}^A\zeta_{\dot{b}}{}^B\sigma^i{}_{A\dot{B}}$, are linearly independent four-vectors that satisfy in accordance with (32) and (38) normalization and orthogonality conditions that are identical with (2), and they can therefore be taken as the vectors of the Newman–Penrose null tetrad (1):

$$l^i = \sigma^i_{0\dot{0}};\quad n^i = \sigma^i_{1\dot{1}};\quad m^i = \sigma^i_{0\dot{1}};\quad \overline{m}^i = \sigma^i_{1\dot{0}}. \tag{39}$$

The analogs of the Ricci rotation coefficients for the basis spinor dyad $\zeta_a{}^A$ are the quantities

$$\Gamma_{abc\dot{d}} = \zeta_a{}^A\nabla_{c\dot{d}}\zeta_{bA},\ \text{where}\ \nabla_{c\dot{d}} = \zeta_c{}^C\bar{\zeta}_{\dot{d}}{}^{\dot{D}}\nabla_{C\dot{D}} = \sigma^i_{c\dot{d}}\nabla_i. \tag{40}$$

Using (39), we can readily show that $\Gamma_{abc\dot{d}}$ are equal to the quantities defined in (5) (Table I). For this reason, the quantities (5) were called spin coefficients.

The dyad components of the differentiation operators $\nabla_{a\dot{b}}$ are the same as the operators (4) defined for the tetrad (39):

$$D = \nabla_{0\dot{0}};\quad \Delta = \nabla_{1\dot{1}};\quad \delta = \nabla_{0\dot{1}};\quad \overline{\delta} = \nabla_{1\dot{0}}. \tag{41}$$

The dyad components of the curvature spinors are directly related to the projections of the Weyl tensor and the deviator of the Ricci tensor onto the tetrad vectors (39). The connection is given in Table II and by the formulas ($o^A = \zeta_0{}^A$, $\iota^A = \zeta_1{}^A$):

$$\left.\begin{aligned}
\Psi_0 &= \Psi_{0000} = \Psi_{ABCD}o^Ao^Bo^Co^D;\\
\Psi_1 &= \Psi_{0001} = \Psi_{ABCD}o^Ao^Bo^C\iota^D;\\
\Psi_2 &= \Psi_{0011} = \Psi_{ABCD}o^Ao^B\iota^C\iota^D;\\
\Psi_3 &= \Psi_{0111} = \Psi_{ABCD}o^A\iota^B\iota^C\iota^D;\\
\Psi_4 &= \Psi_{1111} = \Psi_{ABCD}\iota^A\iota^B\iota^C\iota^D.
\end{aligned}\right\} \tag{42}$$

TABLE II. Dyad components of $\Phi_{AB\dot{C}\dot{D}}$ and their connection with the tetrad components of the deviator of the Ricci tensor.

$\Phi_{ab\dot{c}\dot{d}} =$

ab \ $\dot{c}\dot{d}$	$\dot{0}\dot{0}$	$\dot{0}\dot{1}$	$\dot{1}\dot{1}$
00	Φ_{00}	Φ_{01}	Φ_{02}
01	Φ_{10}	Φ_{11}	Φ_{12}
11	Φ_{20}	Φ_{21}	Φ_{22}

Projecting Eqs. (36), in which the spinors of the basis dyad are taken as the ξ^A, onto these spinors, and using the notation (41)–(42) and Tables I and II, we obtain equations that are identical with the second group (A.12)–(A.29) of Newman–Penrose equations. Similarly, Eqs. (37) give the first group (A.1)–(A.11) of Newman–Penrose equations.

If there are any forms of matter present in space, for example, nongravitational fields, the equations that govern them must also be expressed in spinor form and then projected onto the spinors of the basis dyad using the notation given above.

Equations of a free massless spinor field. As an example, we consider the equations that govern a free (i.e., sourceless) massless spinor field of spin s. It is well known that such a field is described by a symmetric spinor of rank $2s$ and in the simplest case satisfies the equations[7)]

$$\nabla^{A\dot{B}}\varphi_{ABC\ldots} = 0. \tag{43}$$

It is noteworthy that for $s = 2$ these equations are identical with Eqs. (37) in vacuum, where $\Phi_{mn} = 0$, and $\Lambda = 0$ by virtue of the Einstein equations (20).

For $s = 1$, Eqs. (43) are the equations for the spinor φ_{AB} determined by the Maxwell tensor in accordance with (34); for it follows from the Maxwell equations (26) in the absence of sources ($J^i = 0$) in conjunction with (34) that $\nabla^{A\dot{B}}\varphi_{AC} = 0$. Projecting these equations onto the basis dyad and setting

$$\varnothing_0 = \varphi_{AB}o^Ao^B;\quad \varnothing_1 = \varphi_{AB}o^A\iota^B;\quad \varnothing_2 = \varphi_{AB}\iota^A\iota^B,$$

we obtain the Maxwell equations (A.38)–(A.41) (see Appendix D).

For $s = \frac{1}{2}$, Eqs. (43) go over into equations of the form

$$\nabla^{A\dot{B}}\xi_A = 0, \tag{44}$$

which are called the Weyl equations and which describe the behavior of neutrinos. Projecting Eqs. (44) onto the basis dyad and introducing the notation $\chi_{+1} = \xi_A o^A$, $\chi_{-1} = \xi_A\iota^A$, we obtain the Weyl equations in the Newman–Penrose formalism derived in Ref. 9.

Dirac equations, i.e., equations for a massive charged field with spin $s = \frac{1}{2}$, provide an example of equations for wave fields of a different type. These equations are given in terms of the Newman–Penrose formalism in Ref. 10.

Thus, the tensor and spinor approaches described in this section are equivalent, since they lead to the same

7) This generalization to the case of curved space of the usual equations $\partial^{A\dot{B}}\varphi_{ABC\ldots} = 0$ is the simplest and most natural, but it should be borne in mind that s cannot be large, since Eqs. (43) become inconsistent when $s \geqslant 3/2$. If they are to be consistent, conditions must be satisfied that relate $\varphi_{ABC\ldots}$ to the curvature spinors[7,8] (the case $s = 2$ is an exception, since these consistency conditions are satisfied automatically for the free gravitational field with $\varphi_{ABCD} = \Psi_{ABCD}$).

system of equations for the gravitational field and the matter producing it.

Rotations of the Null Tetrads and the Spinor Bases and Transformation of the Field Variables. Null Geodesic Coordinate System. The variables that in the Newman–Penrose formalism characterize the gravitational field, i.e., the components of the null tetrad $l^i, n^i, m^i, \overline{m}^i$, the quantities $\Psi_0, \Psi_1, \ldots, \Psi_4, \Phi_{mn}, \Lambda$, and the spin coefficients are not determined uniquely in a given gravitational field, since $l^i, n^i, m^i, \overline{m}^i$ are the components of vectors that depend on the choice of the coordinate system, $\Psi_0, \Psi_1, \ldots, \Psi_4, \Phi_{mn}$ and the spin coefficients, albeit scalars, transform under rotations of the null tetrad (or the spinor basis dyad), and only Λ is an invariant under transformations of the coordinates and rotation of the bases.

We consider first rotations of the spinor bases and the null tetrads. An arbitrary unimodular transformation that preserves the normalization and orthogonality conditions (38) of the spinor basis $\{o^A, \iota^A\}(o^A = \zeta_0^A, \iota^A = \zeta_1^A)$ can be represented as the product of elements of three mutually noncommuting Abelian groups of transformations:

$$\mathrm{I}\left\{\begin{array}{l}\tilde{o}^A = p o^A;\\ \tilde{\iota}^A = p^{-1}\iota^A;\end{array}\right.\quad \mathrm{II}\left\{\begin{array}{l}\tilde{o}^A = o^A;\\ \tilde{\iota}^A = \iota^A + a o^A;\end{array}\right.\quad \mathrm{III}\left\{\begin{array}{l}\tilde{o}^A = o^A + b\iota^A;\\ \tilde{\iota}^A = \iota^A,\end{array}\right. \tag{45}$$

where p, a, and b are arbitrary complex numbers.

Since the group of unimodular transformations is locally isomorphic to the proper Lorentz group, the transformations (45) of the spinor dyad generate in accordance with (39) proper Lorentz rotations of the null tetrads that preserve the conditions (2) or, which is equivalent, the form of Eq. (3). These rotations are products of elements of the following four groups of transformations[4,5]:

$$\left.\begin{array}{ll}\tilde{G}\left\{\begin{array}{l}\tilde{l}^i = G l^i;\\ \tilde{n}^i = G^{-1} n^i;\\ \tilde{m}^i = m^i;\end{array}\right. & \tilde{A}\left\{\begin{array}{l}\tilde{l}^i = l^i;\\ \tilde{n}^i = n^i + A\bar{A} l^i + \bar{A} m^i + A\overline{m}^i;\\ \tilde{m}^i = m^i + A l^i;\end{array}\right.\\ \tilde{H}\left\{\begin{array}{l}\tilde{l}^i = l^i;\\ \tilde{n}^i = n^i;\\ \tilde{m}^i = \exp(\mathrm{i}H) m^i;\end{array}\right. & \tilde{B}\left\{\begin{array}{l}\tilde{l}^i = l^i + B\bar{B} n^i + \bar{B} m^i + B\overline{m}^i;\\ \tilde{n}^i = n^i;\\ \tilde{m}^i = m^i + B n^i.\end{array}\right.\end{array}\right\} \tag{46}$$

The two real parameters G and H, and also the real and imaginary parts of the complex parameters A and B, which can be expressed in terms of the parameters p, a, and b in (45) in accordance with

$$G = p\bar{p};\quad H = 2\arg p;\quad A = \bar{a},\quad B = b,$$

are the six independent parameters of the proper Lorentz group.

The transformation laws for the field variables under rotations of the null tetrads (46) [or the spinor bases (45)] are found from the definitions of these quantities. In Appendix E, we have written out the transformation laws for $\Psi_0, \Psi_1, \ldots, \Psi_4, \phi_m$ under transformations of the groups $\tilde{G}$, $\tilde{H}$, and $\tilde{A}$ (or under the corresponding transformations of the spinor dyads of groups I and II). The analogous rules for transformations of the group $\tilde{B}$ or its corresponding group III for the spinors can be readily obtained from the rules of Appendix E by making the parameter substitution $A \to \bar{B}(a \to b)$ and also the following transformation of the basis spinors and corresponding transformation of the null basis vectors:

$$o^A \to \mathrm{i}\iota^A;\quad \iota^A \to \mathrm{i}o^A;\quad l^i \leftrightarrow n^i;\quad m^i \leftrightarrow \overline{m}^i, \tag{47}$$

which leads to the transformation of the field variables indicated in Table III.

TABLE III. Transformation of the Newman–Penrose scalars under the substitution $l^i \longleftrightarrow n^i$, $m^i \longleftrightarrow \overline{m}^i$ (47) of the basis vectors.

o^A	ι^A	l^i	n^i	m^i	$\overline{m}^i$	χ
$\mathrm{i}\iota^A$	$\mathrm{i}o^A$	n^i	l^i	$\overline{m}^i$	m^i	χ'

χ	D	Δ	δ	$\bar{\delta}$
χ'	Δ	D	$\bar{\delta}$	δ

χ	κ	ρ	σ	τ	ε	α	β	γ	π	λ	μ	ν
χ'	$-\nu$	$-\mu$	$-\lambda$	$-\pi$	$-\gamma$	$-\beta$	$-\alpha$	$-\varepsilon$	$-\tau$	$-\sigma$	$-\rho$	$-\kappa$

χ	Ψ_0	Ψ_1	Ψ_2	Ψ_3	Ψ_4
χ'	Ψ_4	Ψ_3	Ψ_2	Ψ_1	Ψ_0

χ	Φ_{00}	Φ_{01}	Φ_{02}	Φ_{10}	Φ_{11}	Φ_{12}	Φ_{20}	Φ_{21}	Φ_{22}	Λ	ϕ_0	ϕ_1	ϕ_2
χ'	Φ_{22}	Φ_{21}	Φ_{20}	Φ_{12}	Φ_{11}	Φ_{10}	Φ_{02}	Φ_{01}	Φ_{00}	Λ	$-\phi_2$	$-\phi_1$	$-\phi_0$

Note that the complete system of field equations is symmetric, i.e., it preserves its form under the transformation (47) and the corresponding transformation of the field variables.[8)]

Choosing special values of the parameters in the rotation of the null tetrads (or spinor dyads) in accordance with the transformation rules for the field variables, one can make the latter satisfy additional restrictions that simplify the integration of the field equations or the investigation of particular solutions. A special choice of a coordinate system related in a definite manner to a given field of null tetrads strongly influences the form of the coordinate equations and can simplify their integration as well as the calculation of the components of different tensors for the given gravitational field. Sometimes, some details of this choice need not be specified in advance but can be determined during the solution of the problem.

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8)This symmetry of the field equations was the basis of one further formulation of general relativity – the Geroch–Held–Penrose formalism.[11] This formalism is based on the specification at each point of space-time of, not a complete null tetrad, but only two null directions. As field variables, one chooses quantities that have homogeneous transformation properties under rotations of the null tetrads that do not change the given null directions, i.e., transformations of the groups $\tilde{G}$ and $\tilde{H}$ or group I. It is asserted by the creators of this formalism that it is in some cases preferable and can lead to significant simplifications compared with the ordinary spin coefficient method. For applications of it, see, for example, Ref. 12.

We consider some helpful and frequently used methods for choosing the tetrads and the coordinate system. In a number of cases, in particular in the description of wave processes, when concepts such as wave fronts, rays, and so forth, are encountered, a certain family of null hypersurfaces $u(x^i) = \mathrm{const}$ is defined in space. Every such family uniquely determines a congruence of null geodesics, the generators of each hypersurface of the family, and the vector gradient of these surfaces is tangent to the generators and is parallelly transported along them: $dx^i/dr = g^{ij}u_{,j}$, where r is an affine parameter on the geodesics. In this case, it is convenient to take the null tetrads and the coordinates associated with the null hypersurfaces (the wave fronts) and the null geodesics (rays), their generators. The vector l^i can be taken to coincide with the gradient $g^{ij}u_{,j}$, with the remaining tetrad vectors parallelly transported along the rays. Then for the spin coefficients, we have

$$\kappa = \varepsilon = \pi = 0; \quad \rho = \bar{\rho}; \quad \tau = \bar{\alpha} + \beta;$$

(Some of these conditions have a simple geometrical meaning: $\kappa = 0$ if and only if the integral curves of the vector l^i are geodesics, and if also $\varepsilon + \bar{\varepsilon} = 0$, then l^i is transported parallelly along its direction; $\rho = \bar{\rho}$ is the necessary and sufficient condition for this vector to be proportional to a gradient, and if also $\tau = \bar{\alpha} + \beta$, then l^i is a gradient vector.) For this choice of the tetrad, the vectors m^i and $\bar{m}^i$ are tangent to the hypersurfaces $u(x^i) = \mathrm{const}$, since $\delta u = m^i l_i = 0$, $\bar{\delta} u = \bar{m}^i l_i = 0$.

As coordinate x^0, we choose the function $u(x^i)$ that defines the family of null hypersurfaces; as the first coordinate, we choose the affine parameter r on the rays, and for the two remaining coordinates we choose θ and φ, which "label" the rays on each hypersurface and are constant along the rays. (Such coordinates are called null geodesic coordinates.[2,13]) Then the quantities (23) satisfy the equations

$$Y^0 = Y^2 = Y^3 = \xi^0 = 0; \quad X^0 = V = 1$$

and the vectors of the null tetrad in these coordinates have the components

$$l^i = \{0,\ 1,\ 0,\ 0\}; \quad n^i = \{1,\ U,\ X^2,\ X^3\}; \quad m^i = \{0,\ \omega,\ \xi^2,\ \xi^3\}. \tag{48}$$

The operator $D = \partial/\partial r$ is applied to a scalar.

Note that if we give up parallel transport of the vectors n^i, m^i, and $\bar{m}^i$ along l^i and make a tetrad rotation (46) with the parameter $A = -\omega$ we could make ω in (48) vanish. The components of the metric in the constructed coordinates have by virtue of (48) when $\omega = 0$ the form

$$g^{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2U & X^2 & X^3 \\ 0 & X^2 & \multicolumn{2}{c}{\multirow{2}{*}{$-\hat{g}^{ab}$}} \\ 0 & X^3 & & \end{pmatrix}, \quad g_{ij} = \begin{pmatrix} -2U - X_a X^a & 1 & X_2 & X_3 \\ 1 & 0 & 0 & 0 \\ X_2 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$-\hat{g}_{ab}$}} \\ X_3 & 0 & & \end{pmatrix},$$

where $\hat{g}^{ab} = \xi^a \bar{\xi}^b + \bar{\xi}^a \xi^b$, $\hat{g}^{ab}\hat{g}_{ac} = \delta^b_c$, $X_a = g_{ab}X^b$, and $a, b, c = 2, 3$. The tensor $\hat{g}_{ab}$ is the metric tensor on the coordinate surfaces $\{u = \mathrm{const},\ r = \mathrm{const}\}$, and $\hat{g} = \det\|\hat{g}_{ab}\| = -\det\|g_{ij}\|$. The quantity $d\sigma = \sqrt{\hat{g}}\, d\theta d\varphi$ is the element of area of these two-dimensional surfaces which, if the need arises, can be interpreted as the instantaneous positions of two-dimensional fronts in the three-dimensional physical space. (For a description of the geometry of such surfaces, see Ref. 11.) The two-dimensional vector X^a can be used to describe the polarization properties of waves.[12]

2. SOME GENERAL PROPERTIES OF GRAVITATIONAL FIELDS

In this section, we shall use the Newman–Penrose formalism to consider the focusing effect of a gravitational field on a congruence of null geodesics (an effect that is intimately related to the formation of singularities in space-time), the Petrov classification of all gravitational fields, and also some properties of asymptotically flat gravitational fields, in particular Sachs's peeling-off theorem.

Optical Scalars and Focusing Properties of the Gravitational Field. We consider a family of null geodesics in a small neighborhood of a given null geodesic Γ. We choose the null tetrads in such a way that the vector l^i is tangent to each geodesic of the family[9)] with the complete tetrad parallelly transported along each geodesic. Then for these tetrads,

$$\kappa = \varepsilon = \pi = 0. \tag{49}$$

We consider null geodesics in the neighborhood of Γ that "pierce" small two-dimensional areas w spanned by the vectors m^i and $\bar{m}^i$ on Γ, which are therefore orthogonal to Γ. (We shall not consider other null geodesics in the neighborhood of Γ.)

The point at which a given geodesic γ intersects the area w can be specified by a small vector η^i drawn to this point from the point P at which w intersects Γ. This vector can be represented in the form

$$\eta^i = (z\bar{m}^i + \bar{z}m^i)/2$$

and is therefore uniquely determined by the complex quantity $z = x + iy$, where x and y are Cartesian coordinates on w with the origin at P.[10)] Let us consider how z changes for each individual geodesic γ as we move from area to area along Γ. As we do this, the point z corresponding to the geodesic γ is displaced in the complex plane, so that every small neighborhood of any point of this plane undergoes an affine transformation. This transformation can be determined as follows. The considered family of geodesics is obviously two-dimensional, and all its elements can be labeled by specifying a certain complex function φ which is constant along each geodesic and therefore satisfies the condition $D\varphi = 0$. Let φ and φ' be two values of the function φ corresponding to two neighboring geodesics γ and γ'. Then to small quantities of second-order, $\varphi' \approx \varphi + \eta^i \nabla_i \varphi = \varphi + (\bar{z}\delta\varphi + z\bar{\delta}\varphi)/2$. Since $D\varphi = 0$ and $D\varphi' = 0$, using the commutation relations (6) for the operators D and δ, and also the freedom in the choice of φ, we obtain

9)It is assumed that within the considered neighborhood the geodesics do not cross.

10)Since the vectors m^i and $\bar{m}^i$ can, while remaining orthogonal to Γ, be subjected to rotations of the form $m^i \to m^i + Al^i$, $\bar{m}^i \to \bar{m}^i + \bar{A}l^i$, the area w spanned by these vectors is not determined uniquely. However, the quantity z does not depend on the actual choice at a given point on Γ of the area w orthogonal to Γ.

$$Dz = -\rho z - \sigma\bar{z}. \tag{50}$$

It follows that as the point P and the area w corresponding to it are displaced along Γ through dr, where dr is the increment of the affine parameter along Γ, the points of w with coordinates (x, y) undergo a small affine transformation with matrix $\delta_{ab}+A_{ab}\,dr\,(a, b=1, 2)$, where

$$A_{ab} = \Omega_{ab} + D_{ab} + \Theta\delta_{ab}$$

and

$$\Omega_{ab} = \begin{pmatrix} 0 & \operatorname{Im}\rho \\ -\operatorname{Im}\rho & 0 \end{pmatrix};$$

$$D_{ab} = \begin{pmatrix} -\operatorname{Re}\sigma & -\operatorname{Im}\sigma \\ -\operatorname{Im}\sigma & \operatorname{Re}\sigma \end{pmatrix}; \quad \Theta = -\operatorname{Re}\rho.$$

These formulas show that $\operatorname{Re}\rho$ determines the small expansion, $\operatorname{Im}\rho$ the small rotation, and σ in D_{ab} the shear, where the magnitudes of the shears that take place along the principal axes of the matrix $\delta_{ab}+D_{ab}\,dr$ are equal to $1\pm|\sigma|dr$, while the principal axes are rotated about the coordinate axes (and therefore about the tetrad parallelly transported along Γ) through the angle $\frac{1}{2}\arg\sigma$.[11)] [14–16]

The quantities $\omega=\operatorname{Im}\rho$, $\Theta=-\operatorname{Re}\rho$, and $|\sigma|$ are called *optical scalars*. The optical scalars can be calculated in accordance with the formulas[4]

$$|\omega| = \sqrt{\Omega_{ab}\Omega^{ab}/2} = \sqrt{\nabla_{[i}l_{j]}\nabla^i l^j/2}; \quad \Theta = A_c^c/2 = \nabla_i l^i/2;$$

$$|\sigma| = \sqrt{D_{ab}D^{ab}/2} = \sqrt{[\nabla_{(i}l_{j)}\nabla^i l^j - (\nabla_i l^i)^2/2]/2}.$$

Figure 1 shows the relative change in the shape of an initially circular narrow pencil of rays as they are displaced along Γ. This change is made up of an expansion (Fig. 1a), a rotation (Fig. 1b), and a pure shear (Fig. 1c).

The presence of curvature of space-time leads to the appearance of nonzero ρ and σ and, therefore, to expansion (in reality, negative, i.e., to contraction), rotation, and shear (distortion) of the pencil of rays even if at some moment $\rho=\sigma=0$ for this pencil. Indeed, under the condition (49) we obtain from (A.12) and (A.13) the equations that describe the change of ρ and σ along the rays:

$$\left.\begin{aligned} D\rho &= \rho^2 + \sigma\bar{\sigma} + \Phi_{00}; \\ D\sigma &= (\rho+\bar{\rho})\sigma + \Psi_0. \end{aligned}\right\} \tag{51}$$

(These equations, which in fact determine the behavior of the optical scalars, are called the Sachs equations.) If it is assumed that the local matter density in space is non-negative, i.e., that $T_{ij}u^iu^j \geq 0$ for any timelike vector u^i, it then follows by continuity that $\Phi_{00}=(\frac{1}{2})\times\varkappa T_{ij}l^il^j \geq 0$ for any null vector. Thus, if $\rho=\bar{\rho}$, i.e., if the pencil of rays is part of some null surface, then the right-hand side of the first equation in (51) is negative definite, which leads to focusing of the pencil of rays $(D\rho \geq 0)$.

If $\Phi_{00}=0$ (for example, when $T_{ij}=0$, i.e., when matter is not present), focusing can occur in this case because of a nonvanishing Ψ_0, which generates a value $\sigma\neq 0$, which, in turn, makes a positive contribution to $D\rho$ (see Ref. 1 for more detail on this).

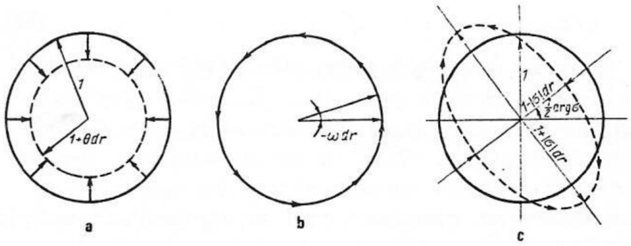

FIG. 1. Ehlers-Sachs diagram: a) expansion, b) rotation, c) shear.

As follows from Eq. (50) and the Sachs equations (51), the focusing has the consequence that the rays of a pencil that initially converge at some point must necessarily begin to intersect, i.e., reach a caustic, over a finite range of variation of their affine parameters.

Thus, one of the very general and important properties of curvature is a positive focusing along any null geodesic under the condition that the local matter density is positive. In particular, this focusing of the rays can result in the formation of true singularities in space-time. For example, numerous exact solutions of the field equations are known that describe the collision of plane waves of various nature (gravitational, electromagnetic, neutrino) with the formation of a true spacelike singularity in space-time as a result of the interaction of the waves and their mutual focusing.[17–20]

Petrov Classification of Gravitational Fields. In Refs. 21–24, Petrov established an algebraic classification of gravitational fields using Jordan normal forms for matrices in a six-dimensional bivector space. He found that in the most general case the gravitational field at a given point of space belongs to one of three types, which were subsequently called the Petrov types.

This classification acquired a definite geometrical interpretation after the concept of a principal null direction for the gravitational field had been introduced.[12)] According to Debever's theorem[25] in Sachs's formulation,[26] at each point of space-time there exists at least one and not more than four distinct null directions (called principal null directions) such that the null vector l^i corresponding to each of these directions satisfies the algebraic equation[13)]

[11)]The real and imaginary parts of μ and the modulus and argument of λ have a similar geometrical meaning for the congruence of integral curves of the vector field n^i (provided they are geodesics).

[12)]Physically, principal null directions for the gravitational field are distinguished by the property that, locally, along these directions there is no astigmatic focusing of an initially parallel pencil of rays. In vacuum, such a pencil of rays, moving along any such direction, is not focused locally at all.

Note also that by virtue of the Goldberg–Sachs theorem in vacuum[2] degenerate principal null directions for the gravitational field, which are also called propagation directions, have shear-free geodesics as integral curves.

[13)]The results described below are also presented in the reviews of Refs. 4 and 27.

$$l_{[i}C_{j]kl[m}l_{n]}l^k l^l = 0. \qquad (52)$$

The types of algebraic structure of the Weyl tensor, which differ in the number of noncoincident principal null directions, coincide with definite types in the Petrov classification. Thus, to the algebraically general Petrov type I there correspond four distinct principal null directions, whereas for all the algebraically special (degenerate) Petrov types some of these directions coincide: In type II there are three distinct directions (one of them is doubled); in type D, there are two directions, each of which is doubled; in type III, there are two directions, one of which is threefold degenerate; finally, in type N all four principal null directions coincide. Figure 2 is the so-called Penrose diagram for the gravitational field types in the Petrov classification. In the square brackets, the degeneracy of the principal null directions corresponding to the type is indicated. Type O denotes a vanishing Weyl tensor, i.e., it contains only conformally flat spaces.

The null vectors that in the algebraically special fields determine the multiple principal null directions also satisfy [in addition to (52)] equations that become progressively more stringent with increasing multiplicity of the degeneracy:

$$\left.\begin{array}{ll} \text{II}, D: & C_{ijk[l}l_{m]}l^j l^k = 0; \\ \text{III}: & C_{ijk[l}l_{m]}l^k = 0; \\ N: & C_{ijkl}l^l = 0. \end{array}\right\} \qquad (53)$$

The algebraic structure of the Weyl tensor belongs to a particular Petrov degenerate type if and only if the equation in the series (53) corresponding to this type has a solution which does not satisfy the next and more stringent condition.

We now consider the representation of this classification in terms of the Newman–Penrose formalism. Taking the vector l^i of the null basis tetrad along the principal null direction of highest multiplicity for the given type, and using the decomposition (17) of the Weyl tensor with respect to the bivector basis (7), we find that in this tetrad the following conditions hold for the various Petrov types:

$$\left.\begin{array}{ll} \text{I}: & \Psi_0 = 0, \quad \Psi_1 \neq 0; \\ \text{II}, D: & \Psi_0 = \Psi_1 = 0, \quad \Psi_2 \neq 0; \\ \text{III}: & \Psi_0 = \Psi_1 = \Psi_2 = 0, \quad \Psi_3 \neq 0; \\ N: & \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \quad \Psi_4 \neq 0. \end{array}\right\} \qquad (54)$$

In the case of type D, choosing in addition the tetrad vector n^i along the second degenerate null direction, we obtain $\Psi_4 = \Psi_3 = 0, \Psi_2 \neq 0$.

The converse is also true, namely, if $\Psi_0, \Psi_1, \ldots, \Psi_4$ in some tetrad belong to one of the types listed in (54), then [as can be readily shown using the decomposition (17)] the vector l^i is a principal null direction with the maximal multiplicity possible for the given type. Thus, the possibilities in a given field of choosing a null tetrad in which the quantities $\Psi_0, \Psi_1, \ldots, \Psi_4$ correspond to one of the forms listed in (54) stand in a one-to-one correspondence [as indicated in (54)] with the Petrov type of this field.[2,5]

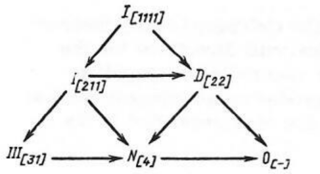

FIG. 2. Penrose diagram. The arrows indicate the directions of increasing degeneracy.

Regarding the $\Psi_0, \Psi_1, \ldots, \Psi_4$ as the dyad components of the curvature spinor Ψ_{ABCD} corresponding to the Weyl tensor in accordance with (35), the Petrov classification of gravitational fields can be formulated as a classification according to the types of algebraic structure of the spinor Ψ_{ABCD}.

The analog of Eq. (52), which determines the principal null directions, is an equation for the spinor ξ^A (called in what follows a fundamental spinor)

$$\Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0. \qquad (55)$$

The null direction determined by the spinor ξ^A in accordance with

$$l^i = \sigma^i_{A\dot{B}}\xi^A\bar{\xi}^{\dot{B}},$$

is a principal null direction. Equation (55) always has at least one and not more than four distinct (i.e., differing by not merely a numerical factor) solutions. The multiplicity of the degenerate fundamental spinors for each Petrov type is equal to the multiplicity of the principal null directions corresponding to them for this type as indicated in the Penrose diagram (see Fig. 2). The analogs of Eqs. (53) are the equations satisfied by the multiple solutions (55):

$$\begin{array}{lll} \text{I}: & \Psi_{ABCD}\xi^A\xi^B\xi^C\xi^D = 0; & \Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}; \\ \text{II}, D: & \Psi_{ABCD}\xi^A\xi^B\xi^C = 0; & \Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}; \\ \text{III}: & \Psi_{ABCD}\xi^A\xi^B = 0; & \Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}; \\ N: & \Psi_{ABCD}\xi^A = 0; & \Psi_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D \end{array}$$

(in type I there are no multiple solutions). On the right, we give the algebraic structure of the spinor Ψ_{ABCD}, expressed in terms of the fundamental spinors, corresponding to the field type. (Different letters are used to denote fundamental spinors that determine distinct principal null directions.) This structure can be readily obtained by factorizing the fourth-degree polynomial (55). The properties (54) of the dyad components of Ψ_{ABCD} hold if the fundamental spinor α^A, which has the highest multiplicity for the given type, is chosen as the first spinor of the basis dyad $\{o^A, l^A\}$.

The correspondence (54) between the Petrov type of the field and the algebraic structure of the $\Psi_0, \Psi_1, \ldots, \Psi_4$ (as tetrad components of the Weyl tensor or dyad components of the curvature spinor Ψ_{ABCD}) leads to a comparatively simple method for determining the field type.

For suppose that in some arbitrary vector or spinor basis the $\Psi_0, \Psi_1, \ldots, \Psi_4$ are known. We then seek rotation parameters for these bases such that the transformed tetrad vector l^i or the basis spinor o^A coincide, respectively, with the principal null direction and the fundamental spinor. [It is necessary to use the B rotations (46) or the b transformations (45), since l^i and o^A determine an unchanged null direction for the other rotations.] After these transformations, we must have $\tilde{\Psi}_0 = 0$ in the new basis in accordance with (52) or (55), i.e., the parameter b (or B) must satisfy the fourth-degree algebraic equation

$$\Psi_4 b^4+4\Psi_3 b^3+6\Psi_2 b^2+4\Psi_1 b+\Psi_0=0. \quad (56)$$

The four roots of this equation correspond to the four fundamental spinors and determine the four principal null directions, and multiple roots of (56) correspond to fundamental spinors and principal null directions of the same degeneracy. This follows from the fact that under the b rotations (45) or the B rotations (46) with $b=B$ the remaining quantities $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ transform in accordance with

$$\left.\begin{aligned}\tilde{\Psi}_1&=\Psi_1+3b\Psi_2+3b^2\Psi_3+b^3\Psi_4;\\ \tilde{\Psi}_2&=\Psi_2+2b\Psi_3+b^2\Psi_4;\\ \tilde{\Psi}_3&=\Psi_3+b\Psi_4;\\ \tilde{\Psi}_4&=\Psi_4.\end{aligned}\right\} \quad (57)$$

It can be seen from (56) and (57) that the polynomials in b on the right-hand sides of (57) are, except for a numerical coefficient, obtained by successive differentiation of (56) with respect to b for constant $\Psi_0, \Psi_1, \ldots, \Psi_4$. Therefore, if b is a multiple root of (56), then in the transformed basis not only Ψ_0 but also the quantities which follow it in the series $\Psi_0, \Psi_1, \ldots, \Psi_4$ vanish, so that the total number of vanishing tetrad components of the Weyl tensor is equal to the multiplicity of the root. Thus, in accordance with (54) the determination of the Petrov field type from the values of $\Psi_0, \Psi_1, \ldots, \Psi_4$ given in a certain basis reduces to the problem of determining the multiplicity of the roots of the fourth-degree polynomial (56).

By analogy with the above Petrov classification for gravitational fields according to the types of algebraic structure of the Weyl tensor, one can classify electromagnetic fields in accordance with the algebraic types of the Maxwell tensor or the self-dual bivector $\overset{+}{F}_{ij}$ (10) associated with it. Essentially, such a classification[14)] was already given by Ruse[29] and Synge.[28]

The principal null directions l^i for $\overset{+}{F}_{ij}$ and the fundamental spinors $\underset{+}{\xi}^A$ for the symmetric spinor φ_{AB} corresponding to $\overset{+}{F}_{ij}$ are determined in accordance with (34) by the equations

$$l_{[i}\overset{+}{F}_{j]k}l^k=0;\quad \varphi_{AB}\xi^A\xi^B=0.$$

Each of these equations has at least one and not more than two distinct solutions. Therefore, by analogy with the Petrov types we have the following types of electromagnetic field, which differ by the degeneracy of the fundamental spinors and the principal null directions: III_F [11], N_F [2]. (We can also add the type O_F for the case $\overset{+}{F}_{ij}=0$.) The index F denotes the type for the electromagnetic field.

The principal null directions and the fundamental spinors for these types satisfy equations analogous to (53):

$$\left.\begin{aligned}\mathrm{III}_F:&\quad l_{[i}\overset{+}{F}_{j]k}l^k=0;\quad \varphi_{AB}\xi^A\xi^B=0;\quad \varphi_{AB}=\alpha_{(A}\beta_{B)};\\ N_F:&\quad \overset{+}{F}_{jk}l^k=0;\quad \varphi_{AB}\xi^A=0;\quad \varphi_{AB}=\alpha_A\alpha_B.\end{aligned}\right\} \quad (58)$$

The algebraic structure of φ_{AB} is given on the right-hand sides in (58) in terms of the fundamental spinors.

In terms of the Newman–Pentose formalism, taking the vector l^i as the first vector of the null basis tetrad or the fundamental spinor α^A as the basis spinor o^A, we obtain in accordance with (58) for the scalars ϕ_0, ϕ_1, ϕ_2 the relations

$$\mathrm{III}_F:\ \varnothing_0=0,\quad \varnothing_1\neq 0;$$
$$N_F:\ \varnothing_0=\varnothing_1=0,\quad \varnothing_2\neq 0,$$

and if also, in type III_F, the vector n^i is taken to be the other principal null direction of the tensor F_{ij} corresponding to the fundamental spinor β^A, then for this type we have

$$\mathrm{III}_F:\ \varnothing_0=\varnothing_2=0,\quad \varnothing_1\neq 0.$$

In accordance with the transformation rules of the ϕ_0, ϕ_1, ϕ_2 under b rotations of the spinor dyads or B rotations of the tetrad (with $b=B$), the determination of the electromagnetic field type reduces to the determination of the multiplicity of the roots of an equation of second [and not fourth, like (56)] degree:

$$\varnothing_2 b^2+2\varnothing_1 b+\varnothing_0=0.$$

Asymptotically Flat Gravitational Fields and Sach's Peeling-Off Theorem. If it is assumed that a gravitational field is produced by a system of sources in a bounded region of space, outside which there are no sources, it is natural to expect that with increasing distance from the sources the field that they produce becomes "weaker" and that space-time approaches in its properties Minkowski space, i.e., is asymptotically flat.

Asymptotically flat gravitational fields represent a very important and interesting class of fields. There are two main reasons for this[1]: first, in the overwhelming majority of physical situations in which significant general relativistic effects can be expected (except for cosmological problems), the curvatures involved in a local process are many orders of magnitude greater than the characteristic curvature of the general cosmological background, i.e., in these cases it is a good approximation to assume that the field is asymptotically Euclidean; second, it is in asymptotically flat cases that the general theory of relativity "begins to resemble much of the rest of physics,"[1] namely, in these fields one can define the energy and momentum of a system of sources and the radiative energy loss by a system, consider incoming and outgoing waves and their scattering, and study the multipole structure of sources and also many other questions.[31-38]

A conformal mapping of space-time onto a compact manifold with boundary[1,4]; the construction of the so-called information function, which determines the radiation emitted by the system; and also the use of the complete group of asymptotic symmetries (the Bondi–Metzner–Sachs group)[33,34] in conjunction with the Newman–Penrose formalism or some of its elements—these constitute the elegant and powerful formalism that played a decisive role in the investigation of asymptotically flat fields.

Contenting ourselves here with these few references,

14) This classification is given in terms of the Newman–Penrose formalism in, for example, Ref. 30.

we consider in what follows one important property of asymptotically flat fields that is intimately related to the Petrov classification—Sach's peeling-off theorem.[31-32]

In accordance with this theorem, in asymptotically flat space at large distances from a localized system the gravitational field has a specific behavior, namely, along every null geodesic γ with affine parameter r which goes to infinity the tetrad components of the Weyl tensor in the leading approximation in $1/r$ have the form (Ref. 2)[15)]

$$\Psi_N = O(r^{-5+N}), \quad N = 0, 1, 2, 3, 4.$$

This means that far from the sources the complete space is divided asymptotically into five zones. The outermost zone is the wave zone, in which the only important quantity among the $\Psi_0, \Psi_1, \ldots, \Psi_4$ is $\Psi_4 = O(1/r)$, and the remainder are negligibly small, so that in the principal terms in $1/r$ in the wave zone the gravitational field has type N, and the tangent vector to γ coincides with the multiple principal null direction. The leading term of Ψ_4 in the wave zone can be directly expressed in terms of the information function,[4] which characterizes its connection with the asymptotic radiation field.

As we approach the sources from infinity, coming in along the null geodesic γ, we pass successively through the remaining four zones, in which the quantities $\Psi_3, \Psi_2, \Psi_1, \Psi_0$ are successively "switched on" and make an important contribution to the curvature, the degeneracy of the Petrov field type decreasing correspondingly as the sources are approached. The general asymptotic behavior of the curvature of space (determined by the Weyl tensor) along γ can be expressed schematically by the formula

$$C = N/r + \mathrm{III}/r^2 + \mathrm{II}/r^3 + \mathrm{I}/r^4 + \mathrm{I}'/r^5 + O(1/r^6).$$

Here, N, III, II, I are the Petrov field types in the corresponding asymptotic zones, and r is the affine parameter on γ.

In the first three zones, the field belongs asymptotically to the degenerate Petrov types, and the geodesic γ determines the principal multiple null direction of the field. In the fourth and fifth zones the field has an algebraically general type, but in the fourth zone the geodesic is, as before, along a principal direction, whereas in the closest, fifth zone the field is completely decoupled from γ.

The successive peeling off of the principal null directions from γ as one moves in from the most distant zone, the wave zone, to the nearer zones is shown in

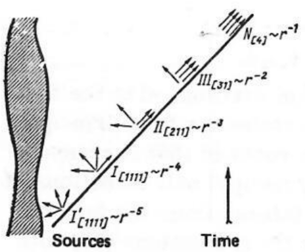

FIG. 3. Sachs diagram (peeling off and asymptotic behavior of the gravitational fields of island systems).

Fig. 3.

Thus, Sachs's peeling-off theorem gives a clear geometrical meaning to the types of gravitational fields and principal null directions. A similar peeling-off theorem can be formulated for electromagnetic fields in flat space generated by a localized system of charges, and also for asymptotically flat electrovacuum gravitational fields.[35]

3. THE NEWMAN-PENROSE FORMALISM AS A METHOD FOR INTEGRATING EINSTEIN'S EQUATIONS

The technique for integrating Einstein's equations in the Newman–Penrose formalism was developed mainly in Refs. 38–41. In many problems, this technique has definite advantages over a direct attempt to integrate the equations of the gravitational field. Let us list some of these advantages.

1. One of the most important points is the transition to "canonical" variables of the gravitational field that is made in the spin coefficient method. All three groups of field variables are appropriately transformed under Lorentz rotations of the null tetrad (see Sec. 1), whereas the complete system of Newman–Penrose equations remains invariant under such transformations (the variables of the first group depend in addition on the choice of the coordinate system). Such transformations can be used to make the transformed tetrad reflect the actual symmetry of the investigated spaces; if such a choice of the basis is made, the complete system of Newman–Penrose equations decomposes in many cases into a system of ordinary differential equations with separable variables.

2. The use of the Newman–Penrose formalism reduces the order of the gravitational field equations and simultaneously increases their number; for the Einstein equations are (quasilinear) equations of *second* order, whereas the complete system of Newman–Penrose equations contains only invariant differential operators of *first* order.

3. Some of the spin coefficients (such as ρ, σ, etc) used in the Newman–Penrose formalism have a simple geometrical meaning, so that it is not necessary to make an additional geometrical investigation of the space-time described by the particular solution of the Einstein equations. Moreover, from the given tetrad components of the Weyl tensor one can uniquely establish the Petrov type of the considered gravitational field (see Sec. 2).

It should also be noted that if there are other physi-

15)A detailed and rigorous definition of asymptotically flat space-time was given by Penrose.[1] The most important assumed properties of these fields, on which the proof of Sachs's theorem is based, are analyticity of the quantities characterizing the field and their derivatives in inverse powers of the affine parameter r on γ, and also specific behavior of one of the tetrad components of the Weyl tensor, namely Ψ_0 (the vector l^i of the null tetrad is directed along γ). It is assumed that $\Psi_0 = O(1/r^5)$. This assumption is satisfied in many special cases, for example, for retarded solutions in a weak field or even when radiation is incident on the system for a finite duration.[34]

cal fields, for example, the electromagnetic field, in the space-time as well as the gravitational field, knowledge of the complete set of projections onto the null tetrad of the corresponding tensors describing these fields gives information about the algebraic structure and asymptotic properties of these fields, the behavior of their associated principal null congruences, etc.

In a detailed investigation of the structure of the space-time generated by a given gravitational field, it frequently happens that not only the optical scalars but also some other spin coefficients are intimately related to the physical and geometrical characteristics of the processes taking place in the considered gravitational field. For example, if we have a plane gravitational wave propagating along some degenerate ($\rho=\sigma=0$) congruence of geodesic rays, the spin coefficient τ characterizes the rate of change of the direction of propagation of the wave; if $\tau=0$, then the family of wave fronts is plane-parallel. Below, in the present section, we shall consider the gravitational field generated by a body that radiates anisotropically high-frequency massless fields or ultrarelativistic particles and, because the radiation is anisotropic, suffers a recoil. The motion of such a body, which Kinnersley[41,104] has called a photon rocket, will obviously be accelerated. For the choice of the Newman–Penrose null tetrad that we make in this section, the spin coefficient ν characterizes the magnitude of this acceleration with respect to a locally geodesic coordinate system, while the imaginary part of γ characterizes the rate of change of its direction.

4. In the integration of the Einstein equations in the Newman–Penrose formalism a number of intermediate calculations can be automated. Thus, to obtain the transversal equations (see below), Kinnersley[41] used the language FORMAC in a computer. For the same purpose, one can use the language ANALITIK (an algorithmic language for describing computational processes using analytic transformations[42]) adapted to a MIR-2 computer[16)]; one can also use FORTRAN. [17)] In a number of problems in general relativity, one needs to know in analytic form (at least in the form of a functional series) solutions of the Einstein equations corresponding to a particular energy-momentum tensor; in such problems, the use of the Newman–Penrose formalism with further use of computational techniques can appreciably shorten the computing time.

As examples, which, however, have a very general nature, we consider below the integration of the complete system of Newman–Penrose equations for Petrov type D gravitational fields in vacuum and in radiation-filled space.

Type D Vacuum Metrics. In the fundamental investigation of Ref. 41, Kinnersley succeeded in finding explicitly all vacuum solutions of the Einstein equations corresponding to type D gravitational fields. We shall consider the main stages of this work and indicate some generalizations of it.

For type D gravitational fields, the only nonvanishing tetrad component of the Weyl tensor is Ψ_2 if the two real tetrad vectors l_i and n_i of the null tetrad are taken to coincide with the propagation directions (see Sec. 2). By the Goldberg–Sachs theorem,[2,45–47] the degenerate eigenvectors of the Weyl tensor in vacuum form a congruence of shear-free geodesics (and conversely). Therefore, as usual, we choose the coordinate system such that $l^i=\delta^i_1$ and the coordinate $x^1\equiv r$ is an affine parameter along the geodesic l^i. It is then found that the spin coefficient ε is purely imaginary and can be made to vanish by the tetrad rotation $m^i\to\exp(iH)m^i$. We can then write

$$\left.\begin{array}{l}\Psi_0=\Psi_1=\Psi_3=\Psi_4=0;\ \Psi_2\neq0;\ \sigma=\kappa=\nu=\lambda=\varepsilon=0;\\ l^i=\{0,1,0,0\};\ n^i\equiv\{X^0,U,X^2,X^3\};\ m^i\equiv\{\xi^0,\omega,\xi^2,\xi^3\}.\end{array}\right\}\quad(59)$$

Note that the tetrad vectors and the coordinate system are not yet uniquely determined but admit the following transformations, which preserve the relations (59):

$$x^{1\prime}=x^1+R^*(x^0,x^2,x^3);\quad(60)$$

$$x^{\alpha\prime}=x^{\alpha\prime}(x^0,x^2,x^3);\quad(61)$$

$$m^{i\prime}=m^i;\ n^{i\prime}=A^*n^i;\ l^{i\prime}=(A^*)^{-1}l^i;\ r'=A^*r;\quad(62)$$

$$n^{i\prime}=n^i;\ l^{i\prime}=l^i;\ m^{i\prime}=\exp(iH^*)m^i\quad(63)$$

(here and in all that follows, Greek indices $\alpha,\beta,\ldots$ take the values $0,2,3$, and quantities with an asterisk do not depend on r).

The tetrad components Φ_{mn} of the Ricci tensor and the scalar curvature Λ for vacuum gravitational fields vanish (the cosmological term is not present in the Einstein equations).

Restricting ourselves to the case $\rho\neq0$,[18)] we write down the general solution of Eq. (A.12) under the assumptions we have made:

$$\rho=-[r+R^*(x^0,x^2,x^3)+i\rho^*]^{-1};\quad R^*=\bar{R}^*;\quad \rho^*=\bar{\rho}^*,\quad(64)$$

where the quantities with an asterisk appear as constants of integration. It is readily seen that this solution is reduced by the transformation (60) to the form

$$\rho=-(r+i\rho^*)^{-1}.\quad(65)$$

The following integration of the radial equations containing the operator $D=\partial/\partial r$ requires some additional work. Namely, we apply the commutators (6) of the invariant derivatives to the scalars Ψ_2 and ρ; the resulting equations are nontrivial consequences of the Newman–Penrose system and are helpful in the integration of the

[16)]An example of a separate program in ANALITIK for calculating the commutators of the invariant derivatives (one of the stages in the integration of the Newman–Penrose system; see below) is given, for example, in Ref. 43.

[17)]For writing long programs, the advantages of FORTRAN over ANALITIK are fairly obvious, since the corresponding computer (BÉSM-6) is much faster than the MIR-2 and has larger memory. It should, however, be noted that the standard possibilities of FORTRAN do not permit one to carry out transformations on analytic expressions. For other languages used in automatic analytic transformations in general relativity, see, for example, Ref. 44.

[18)]The case $\rho=0$, which was considered in Ref. 41, leads to the so-called B metrics of Ehlers and Kundt,[48] the physical interpretation of which is not clear.

system:

$$\Delta\rho + D\mu = \rho(\gamma + \bar{\gamma}) + \pi\bar{\pi} - \tau\bar{\tau}; \quad (66)$$

$$\bar{\delta}\rho + D\pi = \rho(\alpha + \bar{\beta}); \quad (67)$$

$$\delta\mu + \Delta\tau = -\mu(\alpha + \bar{\beta}) + \tau(\gamma - \bar{\gamma}); \quad (68)$$

$$D(\bar{\delta}\rho) = 3\rho(\bar{\delta}\rho) + \rho^2(\pi - \alpha - \bar{\beta}); \quad (69)$$

$$2\rho\Delta\rho - D(\Delta\rho) = \rho^2(\gamma + \bar{\gamma}) - (\tau + \bar{\pi})\bar{\delta}\rho - (\bar{\tau} + \pi)\delta\rho. \quad (70)$$

Equation (69) with allowance for (A.12), (A.15), and (A.16) can be readily integrated:

$$\bar{\delta}\rho = \rho(\alpha + \bar{\beta}) - 2\bar{\tau}^*\rho^3. \quad (71)$$

Combining (71) and (67), we obtain the equation

$$D\pi = 2\bar{\tau}^*\rho^3. \quad (72)$$

Finally, successive integration of (72), (A.16), (A.15), (A.14), (A.30)–(A.32), (A.2), (A.17), (A.33), (70), and (66) leads to the following dependence of the gravitational field variables on the radial coordinate:

$$\left.\begin{aligned}
&\pi = \pi^* + \bar{\tau}^*\rho^2;\\
&\beta = \bar{\rho}\beta^*;\\
&\alpha = \rho\alpha^* - \pi^* + \rho^2\bar{\tau}^*;\\
&\tau = \rho\eta^* + \rho\bar{\rho}\tau^* - \bar{\pi}^*;\\
&\omega = \bar{\rho}\omega^* + \bar{\alpha}^* + \beta^* - \bar{\pi}^*/\bar{\rho};\\
&\xi^\alpha = \bar{\rho}\xi^{*\alpha};\\
&X^\alpha = X^{*\alpha} + \rho\bar{\rho}(\bar{\tau}^*\xi^{*\alpha} + \tau^*\bar{\xi}^{*\alpha}) + \rho\eta^*\bar{\xi}^{*\alpha} + \bar{\rho}\bar{\eta}^*\xi^{*\alpha};\\
&\Psi_2 = \rho^3\Psi^*;\\
&\gamma = \gamma^* + \rho(\eta^*\alpha^* - \bar{\tau}^*\bar{\pi}^*) + \bar{\rho}(\bar{\eta}^*\beta^* - \tau^*\pi^*)\\
&\quad + \rho^2(\Psi^*/2 + \bar{\tau}^*\eta^*) + \rho\bar{\rho}(\tau^*\alpha^* + \bar{\tau}^*\beta^*) + \rho^2\bar{\rho}\tau^*\bar{\tau}^* - r\pi^*\bar{\pi}^*;\\
&U = U^* - r(\gamma^* + \bar{\gamma}^* + \eta^*\pi^* + \bar{\eta}^*\bar{\pi}^*) + r^2\pi^*\bar{\pi}^*\\
&\quad + \rho[\bar{\tau}^*(\bar{\alpha}^* + \beta^*) - \bar{\tau}^*\eta^* + \eta^*\bar{\omega}^* - \Psi^*/2]\\
&\quad + \bar{\rho}[\tau^*(\alpha^* + \bar{\beta}^*) - \tau^*\bar{\eta}^* + \bar{\eta}^*\omega^* - \bar{\Psi}^*/2]\\
&\quad + \rho\bar{\rho}(\bar{\tau}^*\omega^* + \tau^*\bar{\omega}^* - \tau^*\bar{\tau}^*) - (\rho/\bar{\rho})\bar{\tau}^*\bar{\pi}^* - (\bar{\rho}/\rho)\tau^*\pi^*;\\
&\Delta\rho = -\rho^2 M^* + \rho^2\eta^*(\alpha^* + \bar{\beta}^*) + \rho(\gamma^* + \bar{\gamma}^* + \eta^*\pi^*)\\
&\quad + \rho\bar{\eta}^*\bar{\pi}^* + \rho\bar{\rho}[\eta^*(\bar{\alpha}^* + \beta^*) - \tau^*\pi^* + \bar{\tau}^*\bar{\pi}^* - \eta^*\bar{\eta}^*]\\
&\quad - \rho^3(\Psi^*/2 + \bar{\tau}^*\eta^*) - \rho^2\bar{\rho}[\bar{\Psi}^*/2 + \bar{\tau}^*\eta^* - \tau^*(\alpha^* + \bar{\beta}^*)\\
&\quad - \bar{\tau}^*(\bar{\alpha}^* + \beta^*)] - \rho^3\bar{\rho}\tau^*\bar{\tau}^* + r^2\rho^2\pi^*\bar{\pi}^*;\\
&\mu = \mu^* + \rho(M^* - \bar{\tau}^*\bar{\pi}^*) + \bar{\rho}\tau^*\pi^* + \rho^2(\Psi^*/2 + \bar{\tau}^*\eta^*)\\
&\quad + \rho\bar{\rho}\bar{\Psi}^*/2 + \rho^2\bar{\rho}\tau^*\bar{\tau}^* - r^2\rho\pi^*\bar{\pi}^*.
\end{aligned}\right\} \quad (73)$$

In the next stage, the relations (73) are substituted into the equations of the Newman–Penrose system that have not yet been used, and these remaining equations are then reduced to a polynomial form with respect to ρ (for brevity, we give the result of this procedure under the simplifying assumption $\pi^* = 0$). In the resulting equations, we equate the terms of equal powers of ρ on the right- and left-hand sides and, after fairly lengthy transformations, we obtain a number of algebraic consequences.

$$\eta^* = \mu^* = U^* + M^* = 0 \text{ and } \omega^* = -i\rho^*(\bar{\alpha}^* + \bar{\beta}^*),$$

and a system of transversal equations of the form

$$\xi^{*\alpha}(\Psi^*)_{,\alpha} = -3\Psi^*(\bar{\alpha}^* + \beta^*); \quad \bar{\xi}^{*\alpha}(\Psi^*)_{,\alpha} = -3\Psi^*(\alpha^* + \bar{\beta}^*); \quad (74)$$

$$X^{*\alpha}(\Psi^*)_{,\alpha} = -3\Psi^*(\gamma^* + \bar{\gamma}^*); \quad (75)$$

$$\xi^{*\alpha}(\rho^*)_{,\alpha} = -\rho^*(\bar{\alpha}^* + \beta^*) - i\tau^*; \quad (76)$$

$$X^{*\alpha}(\rho^*)_{,\alpha} = -\rho^*(\gamma^* + \bar{\gamma}^*); \quad (77)$$

$$\xi^{*\alpha}(\tau^*)_{,\alpha} = -\tau^*(3\bar{\alpha}^* + \beta^*); \quad (78)$$

$$\bar{\xi}^{*\alpha}(\tau^*)_{,\alpha} = -\tau^*(\alpha^* + 3\bar{\beta}^*) + 2i\rho^* U^* + (\Psi^* - \bar{\Psi}^*)/2; \quad (79)$$

$$X^{*\alpha}(\tau^*)_{,\alpha} = -\tau^*(\gamma^* + 3\bar{\gamma}^*); \quad (80)$$

$$\xi^{*\alpha}(U^*)_{,\alpha} = -2U^*(\bar{\alpha}^* + \beta^*); \quad (81)$$

$$X^{*\alpha}(U^*)_{,\alpha} = -2U^*(\gamma^* + \bar{\gamma}^*); \quad (82)$$

$$X^{*\alpha}(\alpha^*)_{,\alpha} - \bar{\xi}^{*\alpha}(\gamma^*)_{,\alpha} = \gamma^*(\bar{\beta}^* - \alpha^*); \quad (83)$$

$$X^{*\alpha}(\beta^*)_{,\alpha} - \xi^{*\beta}(\gamma^*)_{,\alpha} = \gamma^*(\bar{\alpha}^* + \beta^*) - 2\bar{\gamma}^*\beta^*; \quad (84)$$

$$\xi^{*\alpha}(\alpha^*)_{,\alpha} - \bar{\xi}^{*\alpha}(\beta^*)_{,\alpha} = 2\beta^*(\bar{\beta}^* - \alpha^*) + 2i\rho^*\gamma^* - U^*; \quad (85)$$

$$\xi^{*\alpha}(X^{*\beta})_{,\alpha} - X^{*\alpha}(\xi^{*\beta})_{,\alpha} = 2\bar{\gamma}^*\xi^{*\alpha} - (\bar{\alpha}^* + \beta^*)X^{*\beta}; \quad (86)$$

$$\bar{\xi}^{*\alpha}(\xi^{*\beta})_{,\alpha} - \xi^{*\alpha}(\bar{\xi}^{*\beta})_{,\alpha} = (\alpha^* - \bar{\beta}^*)\xi^{*\beta} + (\beta^* - \bar{\alpha}^*)\bar{\xi}^{*\beta} - 2i\rho^* X^{*\beta}. \quad (87)$$

The remainder of the problem reduces to finding all solutions of this system and determining the coordinate dependence of the components of the vectors of the null tetrad.

We begin by making coordinate transformations (61) in order to achieve $X^{*\alpha} = \delta_0^\alpha$. The freedom in the choice of the coordinate system that then remains consists of the transformations

$$x^{0'} = x^0 + f(x^2, x^3); \quad (88)$$

$$x^{2'} = g(x^2, x^3); \quad (89)$$

$$x^{3'} = h(x^2, x^3). \quad (90)$$

We then use a transformation (62) to achieve fulfillment of the condition $\Psi^*\bar{\Psi}^* = \text{const}$. After this, Eqs. (74), (75) and (81), (82) give, respectively

$$\alpha^* + \bar{\beta}^* = \gamma^* + \bar{\gamma}^* = \omega^* = 0; \quad U^* = \text{const}, \quad (91)$$

and we can again use the allowed tetrad A^* transformations (62) to reduce U^* to the form

$$(U^*)' = (A^*)^2 U^* = -\varepsilon^0 \equiv \begin{cases} -1/2, & U^* < 0;\\ 0, & U^* = 0;\\ +1/2, & U^* > 0. \end{cases} \quad (92)$$

The conditions (91) and (92) appreciably simplify the system of transversal equations; when integrating this system, it is convenient to consider separately the cases $\tau^* = 0$ and $\tau^* \neq 0$. We begin with the first case.

I. *The Newman–Unti–Tamburino (NUT) family* ($\tau^* = 0$).[40] It follows from Eqs. (74)–(79) that

$$\Psi^* \equiv m + 2ib\varepsilon^0 = \text{const}; \quad \rho^* \equiv b = \text{const}. \quad (93)$$

We now use a transformation (63) to achieve

$$\gamma^* = 0. \quad (94)$$

This is possible because γ^* goes over under the transformation (63) into $\gamma^{*\prime} = \gamma^* + (1/2)i\partial H^*/\partial x^0$. The condition (94) is not violated under subsequent transformations (63) with the function $H^* = H^*(x^2, x^3)$. Using, finally, the coordinate transformations (89)–(90) and the method that was first applied in Ref. 40 and then developed further in Refs. 49 and 50, we reduce ξ^{*2} and ξ^{*3} to the form

$$\xi^{*2'} = p(x^2, x^3); \quad \xi^{*3'} = ip(x^2, x^3), \quad (95)$$

where $p(x^2, x^3)$ is, in general, a complex function whose imaginary part can be made to vanish by means of (63). The freedom in the transformations of the coordinates x^2 and x^3 that remains after this reduces to a conformal transformation $\zeta' = \zeta'(\zeta)$, where $\zeta \equiv x^2 + ix^3$ and ζ' is an arbitrary analytic function. Using (92)–(96), we can rewrite Eqs. (83)–(87) in the form

$$2\alpha^* = \bar{\nabla}p; \quad (96)$$

$$2ib = p^2[\nabla(\bar{\xi}^{*0}/p) - \bar{\nabla}(\xi^{*0}/p)]; \quad (97)$$

$$2\varepsilon^0 = (\sqrt{2}p)^2\nabla\bar{\nabla}\ln(\sqrt{2}p), \quad (98)$$

where $\nabla \equiv \partial/\partial x^2 + i\partial/\partial x^3$. Equation (98) is the equation for a flat (conformally flat) surface of constant curvature with the metric $(\sqrt{2}p)^2\delta^{\mu\nu}$, $\mu, \nu = 2, 3$. Any such surface can be reduced by a conformal transformation to a pseudosphere ($2\varepsilon^0 = -1$), sphere ($2\varepsilon^0 = +1$), or plane ($2\varepsilon^0 = 0$), for which it is known that

$$\sqrt{2}\, p(x^2, x^3) = 1 + \varepsilon^0 \zeta\bar{\zeta}/2.$$

Substituting this expression for p in the remaining equation (97), we note that

$$\xi^{*0} = -ib\zeta/\sqrt{2} \qquad (99)$$

is a solution (at least a particular solution) of the inhomogeneous equation. Since, however, the coordinate transformation (88) induces $\xi^{*0\prime} = \xi^{*0} + p\nabla f$, and the condition of integrability of the equation $\nabla f = -\xi^{*0}/p$, written in the form $\nabla\bar{\nabla} f = \bar{\nabla}\nabla f$, is equivalent to the homogeneous equation (97), the general solution of (97) can be reduced by an appropriate transformation (88) to the form (99).

This completes the integration of the Newman–Penrose equations in the considered case. Combining the above results, we obtain

$$\left.\begin{array}{l} l^i = \{0, 1, 0, 0\};\ n^i = \{1, U, 0, 0\};\ m^i = \bar{\rho}\{-ib\zeta/\sqrt{2}, 0, p, ip\}; \\ l_i = \{1, 0, -A^0, -B^0\};\ n_i = \{-U, 1, A^0U, B^0U\}; \\ m_i = (-1/2\rho p)\{0, 0, 1, i\}, \end{array}\right\} \qquad (100)$$

where $U = -\varepsilon^0 + \rho\bar{\rho}(mr + 2b^2\varepsilon^0)$, $A^0 \equiv bx^3/\sqrt{2}p$, and $B^0 \equiv -bx^2\sqrt{2}\,p$.

Metric I.A ($\varepsilon^0 = +\frac{1}{2}$). Applying to the general solution (100) the coordinate transformation[19])

$$t = x^0 - \int (dr/2U);\quad \sin\theta = (\zeta\bar{\zeta})^{1/2}(1 + \zeta\bar{\zeta}/4)^{-1};\quad \operatorname{tg}\varphi = x^3/x^2,$$

we obtain the metric in the standard coordinates

$$ds^2 = \Phi[dt + 4b\sin^2(\theta/2)\,d\varphi]^2 - \Phi^{-1}dr^2 - (r^2 + b^2)(d\theta^2 + \sin^2\theta\,d\varphi^2);$$
$$\Phi \equiv 1 - (2mr + 2b^2)/(r^2 + b^2).$$

Metric I. B ($\varepsilon^0 = -\frac{1}{2}$):

$$ds^2 = \Phi[dt + 4b\,\mathrm{sh}^2(\theta/2)\,d\varphi]^2 - \Phi^{-1}dr^2 - (r^2 + b^2)(d\theta^2 + \mathrm{sh}^2\theta\,d\varphi^2);$$
$$\Phi \equiv -1 - (2mr - 2b^2)/(r^2 + b^2).$$

Metric I. C ($\varepsilon^0 = 0$):

$$ds^2 = \Phi(dt + b\theta^2 d\varphi)^2 - \Phi^{-1}dr^2 - (r^2 + b^2)(d\theta^2 + \theta^2 d\varphi^2);$$
$$\Phi \equiv -2mr/(r^2 + b^2).$$

II. *The Kerr–NUT family* ($\tau^* \neq 0$).[41] We use the transformation (63) to make $i\tau^*$ a real function [this is possible because under this transformation τ^* goes over into $\tau^{*\prime} = \tau^*\exp(iH^*)$]. From Eqs. (74), (75), (78), (79), and (85), we obtain

$$\Psi^* \equiv m - i[4\beta^*\tau^* - 2\rho^*\varepsilon^0] = \text{const},\quad \beta^* = \bar{\beta}^*;$$
$$\xi^{*\alpha}(\tau^*)_{,\alpha} = 2\beta^*\tau^*;\quad \xi^{*\alpha}(\beta^*)_{,\alpha} = -\varepsilon^0/2 - 2(\beta^*)^2.$$

Further, we choose a new coordinate $x^{2\prime}$ such that $\rho^* = \rho^*(x^{2\prime})$ ($\rho^* \neq \text{const}$). By virtue of (77), $\xi^{*\alpha}(\rho^*)_{,\alpha}$ is real, and therefore ξ^{*2} must also be real. Making then allowed coordinate transformations, we can achieve

$$\xi^{*2} = -1/\sqrt{2};\quad \mathrm{Re}(\xi^{*0}) = \mathrm{Re}(\xi^{*3}) = 0.$$

Using these relations, we deduce from the equations

$$\xi^{*\alpha}(\xi^{*\beta})_{,\alpha} = \bar{\xi}^{*\alpha}(\xi^{*\beta})_{,\alpha};\quad \xi^{*\alpha}(\tau^*)_{,\alpha} = \bar{\xi}^{*\alpha}(\tau^*)_{,\alpha};$$
$$\xi^{*\alpha}(\beta^*)_{,\alpha} = \bar{\xi}^{*\alpha}(\beta^*)_{,\alpha};$$
$$X^{*\alpha}(\xi^{*\beta})_{,\alpha} = X^{*\alpha}(\tau^*)_{,\alpha} = X^{*\alpha}(\beta^*)_{,\alpha} = 0$$

that $\xi^{*\alpha}$, τ^*, and β^* depend only on x^2, so that the transversal partial differential equations reduce to ordinary differential equations with separable variables. We represent the result of integration of these equations for various special cases in the following form ($x^0 = u$; $x^1 = r$; $x^2 = \theta$; $x^3 = \varphi$; $a = \text{const}$; $b = \text{const}$).

Case II.A ($\varepsilon^0 = +\frac{1}{2}$);

$$\rho = -(r + ib - ia\cos\theta)^{-1}.$$
$$ds^2 = \rho\bar{\rho}(r^2 - 2mr - b^2 + a^2\cos^2\theta)\,du^2 + 2du\,dr$$
$$-4\rho\bar{\rho}[b\cos\theta(r^2 - 2mr - b^2 + a^2) - a\sin^2\theta(mr + b^2)]\,du\,d\varphi$$
$$-2(a\sin^2\theta + 2b\cos\theta)\,dr\,d\varphi - [r^2 + (b - a\cos\theta)^2]\,d\theta^2$$
$$+\rho\bar{\rho}[(r^2 - 2mr - b^2 + a^2)(a\sin^2\theta + 2b\cos\theta)^2 - \sin^2\theta(r^2 + b^2 + a^2)^2]\,d\varphi^2.$$

Case II. B ($\varepsilon^0 = -\frac{1}{2}$, $|\beta^*| > \frac{1}{2}/\sqrt{2}$) and $\rho = -(r - ib + ia\,\mathrm{ch}\theta)^{-1}$;

$$ds^2 = -\rho\bar{\rho}(r^2 + 2mr - b^2 + a^2\mathrm{ch}^2\theta)\,du^2 + 2du\,dr$$
$$-4\rho\bar{\rho}[b\,\mathrm{ch}\,\theta(r^2 + 2mr - b^2 + a^2) - a\,\mathrm{sh}^2\theta(mr - b^2)]\,du\,d\varphi$$
$$+2(a\,\mathrm{sh}^2\theta - 2b\,\mathrm{ch}\,\theta)\,dr\,d\varphi - [r^2 + (b - a\,\mathrm{ch}\,\theta)^2]\,d\theta^2$$
$$+\rho\bar{\rho}[(r^2 + 2mr - b^2 + a^2)(a\,\mathrm{sh}^2\theta - 2b\,\mathrm{ch}\,\theta)^2 - \mathrm{sh}^2\theta(r^2 + b^2 + a^2)^2]\,d\varphi^2.$$

Case II. C ($\varepsilon^0 = -\frac{1}{2}$, $|\beta^*| < \frac{1}{2}\sqrt{2}$) and $\rho = -(r - ib + ia\,\mathrm{sh}\theta)^{-1}$.

$$ds^2 = -\rho\bar{\rho}(r^2 + 2mr - b^2 + a^2\mathrm{sh}^2\theta)\,du^2 + 2du\,dr$$
$$+4\rho\bar{\rho}[b\,\mathrm{sh}\,\theta(r^2 + 2mr - b^2 - a^2) - a\,\mathrm{ch}^2\theta(mr - b^2)]\,du\,d\varphi$$
$$+2(a\,\mathrm{ch}^2\theta - 2b\,\mathrm{sh}\,\theta)\,dr\,d\varphi - [r^2 + (b - a\,\mathrm{sh}\,\theta)^2]\,d\theta^2$$
$$-\rho\bar{\rho}[(r^2 + 2mr - b^2 - a^2)(a\,\mathrm{ch}^2\theta - 2b\,\mathrm{sh}\,\theta)^2 + \mathrm{ch}^2\theta(r^2 + b^2 - a^2)^2]\,d\varphi^2.$$

Case II. D ($\varepsilon^0 = -\frac{1}{2}$, $|\beta^*| = \frac{1}{2}\sqrt{2}$);

$$\rho = -[r - ib + ia\exp(\theta)]^{-1}.$$
$$ds^2 = -\rho\bar{\rho}[r^2 + 2mr - b^2 + a^2\exp(2\theta)]\,du^2$$
$$+2du\,dr + 4\rho\bar{\rho}\{b\exp(\theta)[r^2 + 2mr - b^2]$$
$$-a\exp(2\theta)[mr - b^2]\}\,du\,d\varphi + 2[a\exp(2\theta)$$
$$-2b\exp(\theta)]\,dr\,d\varphi - \{r^2 + [b - a\exp(\theta)]^2\}\,d\theta^2$$
$$-\rho\bar{\rho}\{(r^2 + 2mr - b^2)[a\exp(2\theta) - 2b\exp(\theta)]^2$$
$$+(r^2 + b^2)^2\exp(2\theta)\}\,d\varphi^2.$$

Case II. E ($\varepsilon^0 = 0$, $\beta^* \neq 0$) and $\rho = -(r + ib + i\theta^2/2)^{-1}$;

$$ds^2 = -\rho\bar{\rho}(2mr + 2b + \theta^2)\,du^2 + 2du\,dr$$
$$+2\rho\bar{\rho}\theta^2(r^2 - 2mbr - mr\theta^2/2 - b^2 - b\theta^2/2)\,du\,d\varphi$$
$$+(b\theta^2 + \theta^4/4)\,dr\,d\varphi - [r^2 + (b + \theta^2/2)^2]\,d\theta^2$$
$$-\rho\bar{\rho}[(2mr + 2b)(b\theta^2 + \theta^4/4)^2 + \theta^2(r^2 + b^2)^2]\,d\varphi^2.$$

Case II. F ($\varepsilon^0 = 0$, $\beta^* = 0$) and $\rho = -(r + i\theta)^{-1}$;

$$ds^2 = -2\rho\bar{\rho}mr\,du^2 + 2du\,dr + 2\rho\bar{\rho}(r^2 - 2mr\theta^2 + \theta^2)\,du\,d\varphi$$
$$+2\theta^2\,dr\,d\varphi - (r^2 + \theta^2)\,d\theta^2 - \rho\bar{\rho}(r^4 + 2mr\theta^4 - \theta^4)\,d\varphi^2.$$

Note that in the limit $a \to 0$ the II metrics go over into the I metrics as follows: II.A into I.A; II.B, II.C, II.D into I.B; and II.E, II.F into I.C. The solution of greatest interest is II.A, which was first obtained in Ref. 51 and was analyzed in detail in Ref. 52. Here and in the remainder of the present section, we shall use a system of units in which the velocity of light in vacuum c and the Newtonian gravitational constant G are equal to unity. In these units, m is identified with the total energy (mass) of the source and a (the Kerr parameter) with the specific angular momentum; the parameter b,

[19])*Translator's note*. The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

TABLE IV. Properties of the vacuum solutions of the Einstein equations of Kerr–Newman–Unti–Tamburino type (v is the velocity, $\mathfrak{M}$ is the angular momentum of the source, and t, s, and l are, respectively, timelike, spacelike, and null directions.

$\mathfrak{M}$ \ v	t	s	l
t	—	II.B	—
s	II. A	II.C	II.F
l	—	II.D	II.E

which introduces an additional rotation into the congruence of geodesic rays, is called the NUT parameter ("imaginary mass," "mass of magnetic monopole type"[51] etc.).

The solution II.A goes over as $a, b \to 0$ into the Schwarzschild metric; it is therefore natural to assume that the gravitational field described by the metric II.A is produced by a source moving with velocity less than the velocity of light. With regard to the remaining solutions in the family given above, it may be remarked[41] that the gravitating source corresponding to these solutions moves along a spacelike ($\varepsilon^0 = -\frac{1}{2}$) or null ($\varepsilon^0 = 0$) world line. The additional subdivision in these cases corresponds to different spatial disposition of the four-vector of the angular momentum of the source, which is orthogonal to the velocity four-vector (Table IV). Thus, the Newman–Penrose formalism enables one to obtain in a unified manner complete families of solutions corresponding to a given set of parameters and describing sources of tardyon, tachyon,[20)] and luxon types.

Finally, we consider the case $\pi^* \neq 0$ (Ref. 41); the integration then leads to the Ehlers–Kundt C metric ($\pi^* \neq 0, \rho^* = 0$),[48] and also to a solution that is the generalization of the C metric to the case of rotation ($\pi^* \neq 0, \rho^* \neq 0$). In accordance with the interpretation of Ref. 57, the C metric describes the gravitational field of a uniformly accelerated massive body that emits gravitational waves; it is remarkable that in this case the space-time is asymptotically flat. It is noteworthy that the most general "particle-like" solution of the vacuum Einstein equations of type D ($\pi^* \neq 0, \rho^* \neq 0$) was obtained for the first time by the spin coefficient method.

[20)]On the basis of the tachyon solutions obtained here, one cannot, without additional experimental investigations or theoretical restrictions derived from causality, conclude that superluminal motion is "real" or "physically unacceptable." We intentionally do not consider here the (very large) group of problems associated with superluminal motion and superluminal frames of reference, referring the interested reader to the literature.[53-55] In connection with tachyons in general relativity, see Ref. 56.

Kinnersley's investigation in Ref. 41 has fundamental importance. The point is that the type D includes, in particular, an important class of vacuum gravitational fields, namely, stationary asymptotically flat gravitational fields that have a nonsingular simply connected event horizon (gravitational fields of black holes). A simple analysis of all solutions of type D shows that the most general solution having these properties is the Kerr metric[61] (the solution II.A for $b = 0$). From this important physical conclusions can be drawn.

Suppose that as a result of the collapse of a massive star a black hole is formed, which, after rapid emission of gravitational waves, goes over into a stationary ground state. In accordance with what we have said above, this state is entirely described by the Kerr metric, and it is therefore uniquely determined by the specification of the mass m of the black hole and its specific angular momentum a. "Black holes have no hair"—this is how Wheeler picturesquely characterized the property of stationary black holes that for their description one requires no other additional characteristics such as multipole electromagnetic and gravitational moments, baryon charge, etc. (see, for example, Refs. 58–60).[21)]

Some generalizations of Kinnersley's solutions[41] (to the case of gravitational theory with a cosmological term, the presence in space-time of an electromagnetic field, etc.) belonging to the type D are given in Refs. 68–72. Note that the most general "particle-like" solution of the Einstein–Maxwell equations with a cosmological term of type D, which is given in Ref. 72 and which was obtained by a method similar to the Newman–Penrose method, contains seven arbitrary parameters; the "genealogical tree" of this solution (according to Plebański and Demianski[72]) is shown in Fig. 4.

We now consider some other investigations devoted to solutions of the Einstein and Einstein–Maxwell equations in the Newman–Penrose formalism in which methods analogous to those in Refs. 38–41 were used to solve the complete system of Newman–Penrose equations.

It should be noted first that type II metrics in the Petrov classification are far less amenable to detailed investigation by the spin coefficient method than the type D metrics. Historically, the first type II vacuum solutions of the Einstein equations containing a nondegenerate, shear-free, nonrotating (or twist-free) congruence of geodesic rays were investigated by Robinson and Trautman[77] before the creation of the Newman–Penrose formalism, and therefore these metrics are called the Robinson–Trautman metrics. The Robinson–Trautman metrics and also their Einstein–Maxwell generalizations are represented in the Newman–Penrose formalism in Refs. 39 and 143, respectively. The much larger class of solutions of vacuum Einstein equations that generalize the Robinson–Trautman solutions to the case

[21)]The properties of stationary black holes are described, for example, in Ref. 62, which also contains further references.

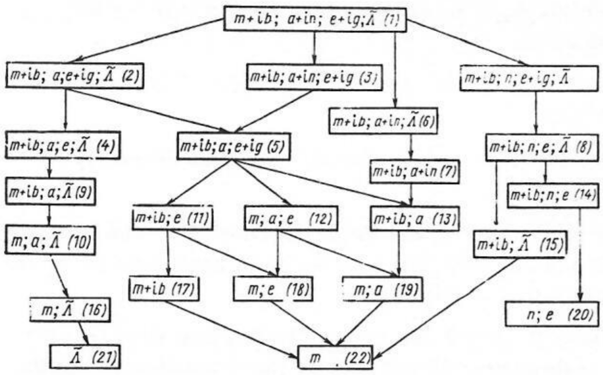

FIG. 4. "Genealogical tree" of the general Petrov type D "particle-like" solution of the Einstein-Maxwell equations with cosmological term.[72] Here m is the mass, b is the NUT parameter, a is the rotation, n is the acceleration, e and g are the electric and magnetic charges of the source, and $\tilde{\Lambda}$ is the cosmological term; 1) Plebański, Demianski[72]; 2) Plebański[73]; 3) Kinnersley[74]; 4) Carter[75]; 5) Demianski and Newman[51]; 6) Carter[75]; 7) Kinnersley[41]; 8) Carter[75]; 9) Frolov[69]; 10) Demianski[78]; 11) Brill[85]; 12) Newman *et al.*,[79] Perjés,[80] Ernst[81]; 13) Demianski,[82] Kramer and Neugebauer,[83] I. Robinson, J. Robinson and Zund[84]; 14) Levi-Civita,[76] Newman and Tamburino,[135] I. Robinson and Trautman,[77] Ehlers and Kundt[48]; 15) Demanski,[86] Frolov[69]; 16) Kottler[87]; 17) Newman, Tamburino and Unti,[40] Taub[89]; 18) Reissner and Nordström[88]; 19) Kerr[61]; 20) Bertotti,[90] I. Robinson[91]; 21) De Sitter[92]; 22) Schwarzschild[93].

when the congruence of geodesic rays has rotation (twisting congruence) was considered by Talbot[49] by the spin coefficient method; in the general case, the metric form was found to depend on three arbitrary functions of the angular coordinates and the time that satisfy a system of three nonlinear partial differential equations which cannot be reduced to quadratures. Finally, electrovac, algebraically special gravitational fields containing a shear-free congruence of null geodesics with expansion were investigated by Lind[50]; as in Ref. 49, the problem of integrating the complete system of Newman–Penrose equations was reduced to that of solving a system of fewer (in this case, five) nonlinear partial differential equations for functions that do not depend on the affine parameter r along the rays. Some nontrivial particular solutions of the Einstein and Einstein–Maxwell equations in the classes of metrics described in Refs. 49 and 50 were obtained (by other methods) in Refs. 84 and 144–146. An interesting interpretation of the solutions of Ref. 50 was proposed in Ref. 148.

Similar in approach to Refs. 49 and 50 are Refs. 63 and 64, in which Trim and Wainwright investigated algebraically special solutions of the equations of the gravitational field with an energy-momentum tensor describing electromagnetic and neutrino fields.

We consider briefly the results of Refs. 67 and 147, in which the spin coefficient method was used to consider regular (i.e., without angular singularities) electrovac gravitational fields for which a multiple eigenvector of the Weyl tensor coincides at each point of space-time with one of the principal null directions of the Maxwell tensor and is tangent to a diverging shear-free geodesic congruence (for such gravitational fields, the name "Kerr–Maxwell spaces" was used in Refs. 67 and 147). It was shown, first, that the class of Kerr–Maxwell spaces of the Petrov types III and N is empty and, second, that the Kerr–Newman metric[79] is the only solution of the field equations in Kerr–Maxwell spaces without radiation characterized by a compact surface $r = \text{const}$, $u = \text{const}$ (for the proof of the uniqueness theorem for the Kerr–Newman metric, see also Ref. 50).

Conformally flat solutions of the Einstein–Maxwell equations with non-null ($\phi_0 = \phi_2 = 0$, $\phi_1 \neq 0$) and null ($\phi_0 = \phi_1 = 0$, $\phi_2 \neq 0$) electromagnetic field were studied by the spin coefficient method in Refs. 149 and 150, respectively. It was found that all such solutions reduce, on the one hand, to the so-called Bertotti–Robinson electromagnetic universe,[90,91,149] and, on the other, to the solution of Ref. 150, which describes a family of plane electromagnetic waves ($\varkappa = \rho = \sigma = \tau = 0$). Generalizations of these solutions to the case of the Einstein–Maxwell equations with the energy-momentum tensor of a "null fluid" (see below) are given in Refs. 151 and 152.

Finally, with regard to algebraically general solutions of the gravitational field equations (Petrov type I), which have hitherto been little investigated in the Newman–Penrose formalism, we merely restrict ourselves to a reference to the literature[39,65,66,153] without making any additional comments.

Some "Radiating" Solutions of Einstein's Equations. In the Newman–Penrose formalism, we now consider type D solutions of Einstein's equations with the energy-momentum tensor of a "null fluid":

$$8\pi T_{ij} = 2Sl_i l_j, \; l_i l^i = 0, \tag{101}$$

where l_i is the degenerate eigenvector of the Weyl tensor ($\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$; $\Psi_2 \neq 0$), which coincides with one of the real vectors of the null tetrad. As is shown in Refs. 94–96, the tensor (101) can be regarded as the energy-momentum tensor of high-frequency monochromatic radiation averaged over a region of space containing a large number of wavelengths. It can be shown that the high-frequency (electromagnetic, gravitational, etc.) waves propagate along null geodesics of a background space with slowly varying metric g_{ij} produced by the averaged effective energy and momentum distribution (101).

Thus, the Einstein equations with right-hand side of the form (101) describe the (nonlinear) reaction of the waves on the background, or, as one sometimes says, the "warping" of the background by the radiation. The first solution of these equations was apparently obtained by Vaidya[97–99] and was investigated in detail in Ref. 100. This solution describes the gravitational field of a spherically symmetric isotropically radiating body; the generalization of this solution to the case when the radiation can carry away charge was given in Ref. 101. Later, Kinnersley[102] found a solution with radiation distributed in accordance with the law $S = S_0(u) + S_1(u) \times \cos\theta(u)$, where u is a retarded time. Because of the

[22] We recall that by a nondegenerate congruence of geodesic rays we understand a congruence for which at least one of the optical scalars is nonzero.

anisotropy of the radiation, the source for the Kinnersley metric must suffer recoil, and therefore must move with acceleration in any inertial frame of reference. Therefore, the Kinnersley solution explicitly contains three functions, namely $a(u)$, $b(u)$, and $c(u)$, which are the independent components of the acceleration four-vector.

The Vaidya and Kinnersley solutions are of Petrov type D. Here, we shall obtain all type D solutions of the Einstein equations with right-hand side of the form (101) ($\Phi_{22} \neq 0$, and the remaining Φ_{mn} and Λ are equal to zero) that contain a twist-free shear-free nondegenerate congruence of geodesic rays ($\varkappa = \sigma = \varepsilon = 0, \rho = \bar{\rho} \neq 0$).[103] Using the notation (introduced earlier) that has now become generally accepted, we write down the result of the radial integrations for the corresponding system of Newman–Penrose equations:

$$\rho = -1/r;\ \pi = \pi^* + \bar{\tau}^*/r^2;\ \beta = -\beta^*/r;\ \alpha = -\alpha^*/r - \pi^* + \bar{\tau}^*/r^2;$$
$$\tau = -\eta^*/r + \tau^*/r^2 - \bar{\pi}^*;\ \gamma = \gamma^* - (1/r)(\eta^*\alpha^* - \bar{\tau}^*\bar{\pi}^*)$$
$$-(1/r)(\bar{\eta}^*\beta^* - \tau^*\pi^*) + (1/r^2)(\Psi^*/2 + \bar{\tau}^*\eta^* + \tau^*\alpha^* + \bar{\tau}^*\beta^*)$$
$$-\tau^*\bar{\tau}^*/r^3 - r\pi^*\bar{\pi}^*;\ \Psi_2 = -\Psi^*/r^3;\ \Phi_{22} = \Phi^*/r^2\ \ \xi^\alpha = -\xi^{*\alpha}/r;$$
$$\omega = -\omega^*/r + (\bar{\alpha}^* + \beta^*) + r\bar{\pi}^*;\ X^\alpha = X^{*\alpha}$$
$$+(1/r^2)(\bar{\tau}^*\xi^{*\alpha} + \tau^*\bar{\xi}^{*\alpha}) - \eta^*\bar{\xi}^{*\alpha}/r - \bar{\eta}^*\xi^{*\alpha}/r;$$
$$U = U^* - r(\gamma^* + \bar{\gamma}^* + \eta^*\pi^* + \bar{\eta}^*\bar{\pi}^*) + r^2\pi^*\bar{\pi}^*$$
$$-(1/r)[(\bar{\alpha}^* + \beta^*)\bar{\tau}^* - \bar{\tau}^*\eta^* + \eta^*\bar{\omega}^* - \Psi^*/2]$$
$$-(1/r)[(\alpha^* + \bar{\beta}^*)\tau^* - \tau^*\bar{\eta}^* + \bar{\eta}^*\omega^* - \bar{\Psi}^*/2] + (1/r^2)[(\bar{\tau}^*\omega^*$$
$$+\tau^*\bar{\omega}^*) - \tau^*\bar{\tau}^*] - \bar{\tau}^*\bar{\pi}^* - \tau^*\pi^*;\ \mu = \mu^* - (1/r)(M^* - \bar{\tau}^*\bar{\pi}^*)$$
$$-\tau^*\pi^*/r + (1/r^2)(\Psi^*/2 + \bar{\Psi}^*/2 + \bar{\tau}^*\eta^*) - \tau^*\bar{\tau}^*/r^3 + r\pi^*\bar{\pi}^*;$$
$$\nu = (\Phi^*\pi^*/\Psi^*)r - (1/r)(\Phi^*\tau^*/3\Psi^*) + \nu^*;\ \lambda = 0.$$

Substituting these expressions in the as yet unused equations of the Newman–Penrose system and separating the variables, we obtain a number of algebraic relations,

$$\omega^* = \tau^* = \eta^* = 0;\ \mu^* = \bar{\mu}^*;\ \Psi^* = \bar{\Psi}^*;\ M^* + U^* = 0;\ \nu^*\pi^* = 0,$$

and the system of transversal equations

$$\xi^{*\alpha}(\Psi^*)_{,\alpha} = -3\Psi^*(\alpha^* + \beta^*); \qquad (102)$$
$$X^{*\alpha}(\Psi^*)_{,\alpha} = -3\Psi^*(\gamma^* + \bar{\gamma}^* + \mu^*) - \Phi^*; \qquad (103)$$
$$\xi^{*\alpha}(\Phi^*)_{,\alpha} = 3\bar{\nu}^*\Psi^* - 4\Phi^*(\bar{\alpha}^* + \beta^*)]; \qquad (104)$$
$$\bar{\xi}^{*\alpha}(\pi^*)_{,\alpha} = \pi^*(\bar{\beta}^* - \alpha^*); \qquad (105)$$
$$\xi^{*\alpha}(\pi^*)_{,\alpha} = \pi^*(\bar{\alpha}^* - \beta^*) - \mu^*; \qquad (106)$$
$$X^{*\alpha}(\pi^*)_{,\alpha} = -\pi^*(\gamma^* - \bar{\gamma}^*) + \pi^*\Phi^*/\Psi^*; \qquad (107)$$
$$\bar{\xi}^{*\alpha}(\mu^*)_{,\alpha} = -2\pi^*U^* - \mu^*(\alpha^* + \bar{\beta}^*); \qquad (108)$$
$$X^{*\alpha}(\mu^*)_{,\alpha} = -\mu^*(\mu^* + \gamma^* + \bar{\gamma}^*) + \mu^*\Phi^*/\Psi^*; \qquad (109)$$
$$\bar{\xi}^{*\alpha}(U^*)_{,\alpha} = -3\pi^*\Psi^* - 2U^*(\alpha^* + \bar{\beta}^*); \qquad (110)$$
$$X^{*\alpha}(U^*)_{,\alpha} + \xi^{*\alpha}(\nu^*)_{,\alpha} = -2U^*(\mu^* + \gamma^* + \gamma^*) - \nu^*(\bar{\alpha}^* + 3\beta^*); \qquad (111)$$
$$\bar{\xi}^{*\alpha}(\nu^*)_{,\alpha} = -\nu^*(\bar{\beta}^* + 3\alpha^*); \qquad (112)$$
$$X^{*\alpha}(\alpha^*)_{,\alpha} - \bar{\xi}^{*\alpha}(\gamma^*)_{,\alpha} = \nu^* + \gamma^*(\bar{\beta}^* - \alpha^*) - \alpha^*\mu^* - \pi^*U^*; \qquad (113)$$
$$X^{*\alpha}(\beta^*)_{,\alpha} - \xi^{*\alpha}(\gamma^*)_{,\alpha} = -\beta^*(\mu^* - \gamma^* + 2\bar{\gamma}^*) + \bar{\alpha}^*\gamma^* - \bar{\pi}^*U^*; \qquad (114)$$
$$\xi^{*\alpha}(\alpha^*)_{,\alpha} - \bar{\xi}^{*\alpha}(\beta^*)_{,\alpha} = -U^* + 2\beta^*(\bar{\beta}^* - \alpha^*)]; \qquad (115)$$
$$\xi^{*\alpha}(X^{*\beta})_{,\alpha} - X^{*\alpha}(\xi^{*\beta})_{,\alpha} = (\mu^* + 2\bar{\gamma}^*)\xi^{*\beta} - (\bar{\alpha}^* - \beta^*)X^{*\beta}; \qquad (116)$$
$$\bar{\xi}^{*\alpha}(\xi^{*\beta})_{,\alpha} - \xi^{*\alpha}(\bar{\xi}^{*\beta})_{,\alpha} = (\alpha^* - \bar{\beta}^*)\xi^{*\beta} + (\beta^* - \bar{\alpha}^*)\bar{\xi}^{*\beta}. \qquad (117)$$

The transversal equations are conveniently integrated separately for the three different cases corresponding to the condition $\nu^*\tau^* = 0$.

Case A. $\nu^* = 0$ and $\tau^* = 0$. The integration is performed as in Case I considered earlier. We obtain

Metric A.1: $ds^2 = [1 - 2m(u)/r]du^2 + 2du\,dr - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$.

Metric A.2: $ds^2 = [-1 - 2m(u)/r]du^2 + 2du\,dr - r^2(d\theta^2 + \mathrm{sh}^2\theta d\varphi^2)$.

Metric A.3: $ds^2 = [-2m(u)/r]du^2 + 2du\,dr - r^2(d\theta^2 + \theta^2 d\varphi^2)$.

The Metric A.1 is the Vaidya metric[97-100]; the interpretation of the remaining solutions is analogous to the interpretation of I.B and I.C.

Case B. $\nu^* \neq 0$ and $\pi^* = 0$. We assume first that $U^* < 0$ (sub-case C.1) and specify the quantities $\xi^{*\alpha}$ in the gauge

$$\xi^{*0} = \mathrm{Re}\,\xi^{*0};\ \xi^{*2} = \mathrm{Re}\,\xi^{*2};\ \xi^{*3} = \mathrm{Re}\,\xi^{*3} - i/\sqrt{2}; \qquad (118)$$

the remaining coordinate transformation freedom is

$$x^{0\prime} = x^{0\prime}(x^0, x^2);\ x^{2\prime} = x^{2\prime}(x^0, x^2); \qquad (119)$$
$$x^{3\prime} = x^3 + f(x^0, x^2). \qquad (120)$$

As a result of the tetrad A^* and H^* transformations, respectively, we obtain

$$U^* = -1/2;\ \nu^* = \bar{\nu}^*. \qquad (121)$$

Using these relations, we deduce from (111), (112), and (116) the equations $\beta^* = \bar{\beta}^*$ and $(\mathrm{Re}\,\xi^{*0})_{,3} = (\mathrm{Re}\,\xi^{*2})_{,3} = 0$, so that, using the coordinate transformations (119), we obtain

$$\mathrm{Re}\,\xi^{*0} = 0;\ \mathrm{Re}\,\xi^{*2} = -1/\sqrt{2}. \qquad (122)$$

It follows similarly from (116) that $(X^{*0})_{,2} = (X^{*0})_{,3} = 0$; then, after an allowed coordinate transformation, we arrive at

$$X^0 = 1. \qquad (123)$$

Finally, we write out the general solution of Eq. (115):

$$\beta^* = -(1/2\sqrt{2})\,\mathrm{ctg}\,[x^2 + \tilde{f}(x^0)]. \qquad (124)$$

If we now take the quantity in the square brackets in (124) as the new coordinate $x^{2\prime}$, Eq. (117) can be readily solved, and its solution can be reduced by means of a coordinate transformation (120) to the form

$$\mathrm{Re}\,\xi^{*3} = -(x^3/\sqrt{2})\,\mathrm{ctg}\,(x^2). \qquad (125)$$

The further integration of the system of transversal equations is simple, and we give the result directly:

$$\mathrm{Re}\,\gamma^* = -(1/2)a(u)\cos\theta;\ \mathrm{Im}\,\gamma^*$$
$$= (1/\sin\theta)[\tilde{A}(u)\cos(\varphi/\sin\theta) + \tilde{B}(u)\sin(\varphi/\sin\theta];$$
$$\nu^* = (1/\sqrt{2})a(u)\sin\theta;$$
$$X^{*2} = -a(u)\sin\theta + 2[\tilde{B}(u)\cos(\varphi/\sin\theta)$$
$$-\tilde{A}(u)\sin(\varphi/\sin\theta)];$$
$$X^{*3} = -2\tilde{A}(u)[(\varphi/\sin\theta)\sin(\varphi/\sin\theta) + \cos(\varphi/\sin\theta)]\cos\theta$$
$$-2\tilde{B}(u)[-(\varphi/\sin\theta)\cos(\varphi/\sin\theta)$$
$$+\sin(\varphi/\sin\theta)]\cos\theta - \varphi a(u)\cos\theta + g(u)\sin\theta,$$

where $x^0 \equiv u, x^2 \equiv \theta, x^3 \equiv \varphi$; $a(u), \tilde{A}(u), \tilde{B}(u)$, and $\tilde{g}(u)$ are constants of integration. The transition to the Kinnersley coordinates[102] is made by means of the transformation $\varphi' = (\varphi/\sin\theta) - \int_0^u \tilde{g}(\tau)d\tau$.

Metric B.I ($U^* < 0$):

$$ds^2 = [1 - 2ra(u)\cos\theta - r^2(f^2 + h^2\sin^2\theta) - 2m(u)/r]du^2$$
$$+ 2du\,dr + 2r^2 f\,du\,d\theta + 2r^2 h\sin^2\theta\,du\,d\varphi - r^2(d\theta^2 + \sin^2\theta\,d\varphi^2);$$
$$f \equiv -a(u)\sin\theta + b(u)\sin\varphi + c(u)\cos\varphi;$$
$$h \equiv \mathrm{ctg}\,\theta[b(u)\cos\varphi - c(u)\sin\varphi].$$

Metric B.II ($U^*>0;\ |\beta^*|>\frac{1}{2}\sqrt{2}$):

$$ds^2=[-1-2ra(u)\,\mathrm{ch}\,\theta-r^2(f^2+h^2\,\mathrm{sh}^2\,\theta)-2m(u)/r]\,du^2$$
$$+2du\,dr+2r^2f\,du\,d\theta+2r^2h\,\mathrm{sh}^2\,\theta\,du\,d\varphi-r^2(d\theta^2+\mathrm{sh}^2\,\theta\,d\varphi^2);$$
$$f\equiv-a(u)\,\mathrm{sh}\,\theta+b(u)\sin\varphi+c(u)\cos\varphi;$$
$$h\equiv\mathrm{cth}\,\theta\,[b(u)\cos\varphi-c(u)\sin\varphi].$$

Metric B.III ($U^*>0;\ |\beta^*|<\frac{1}{2}\sqrt{2}$):

$$ds^2=[-1-2ra(u)\,\mathrm{sh}\,\theta-r^2(f^2+h^2\,\mathrm{ch}^2\,\theta)$$
$$-2m(u)/r]\,du^2+2du\,dr+2r^2f\,du\,d\theta$$
$$+2r^2h\,\mathrm{ch}^2\,\theta\,du\,d\varphi-r^2(d\theta^2+\mathrm{ch}^2\,\theta\,d\varphi^2);$$
$$f\equiv-a(u)\,\mathrm{ch}\,\theta-b(u)\exp(\varphi)-c(u)\exp(-\varphi);$$
$$h\equiv\mathrm{th}\,\theta\,[b(u)\exp(\varphi)-c(u)\exp(-\varphi)].$$

Metric B.IV ($U^*>0;\ |\beta^*|=\frac{1}{2}\sqrt{2}$):

$$ds^2=\{-1+2ra(u)\exp(\theta)-r^2[f^2+h^2\exp(2\theta)]$$
$$-2m(u)/r\}\,du^2+2du\,dr+2r^2f\,du\,d\theta$$
$$+2r^2h\exp(2\theta)\,du\,d\varphi-r^2[d\theta^2+\exp(2\theta)\,d\varphi^2];$$
$$f\equiv a(u)\exp(\theta)-2b(u)\,\varphi;$$
$$h\equiv b(u)\,[\varphi^2-\exp(-2\theta)].$$

Metric B.V ($U^*=0$):

$$ds^2=[-2\theta r-r^2(f^2+h^2)-2m(u)/r]\,du^2$$
$$+2du\,dr+2r^2f\,du\,d\theta+2r^2h\,du\,d\varphi-r^2(d\theta^2+d\varphi^2);$$
$$f\equiv\frac{1}{2}(\varphi^2-\theta^2)+b(u);$$
$$h\equiv-\theta\varphi+c(u).$$

The Metric B.I is the Kinnersley metric.[102] The arbitrary (though nonincreasing because of the energy condition) function $m(u)$ describes the change in the mass of the source as a function of the retarded time; the function $a(u)$ characterizes the magnitude of the acceleration, and $b(u)$ and $c(u)$ characterize the rate of change of its direction. The comoving spherical coordinate system is oriented in such a way that the direction $\theta=0$ to the north pole always coincides with the direction of the acceleration. Note also that by virtue of the special choice of the tetrad gauge for the case B.V the function $a(u)$ for any value of the retarded time u is numerically equal to unity in our adopted system of units ($G=c=1$). For brevity, we have omitted this function in the expressions for the components of the metric tensor of the case B.V (the same applies to the gravitational field variables corresponding to the metric B*.V given below). This function can be readily reconstructed in the expressions by dimensional considerations. The metrics B.I, B.II–B.IV, and B.V in the limit $\nu^*\to 0$ go over into A.1, A.2, and A.3, respectively, and in the limit $\Phi^*\to 0$ into the metrics I.A, I.B, and I.C for $b=0$. In the three cases, the surface $r=\mathrm{const}$, $u=\mathrm{const}$ is a two-dimensional analytic manifold (Riemannian source) of elliptic, hyperbolic, or parabolic type, and the gravitating source moves along a timelike, spacelike, or null world line, respectively. The spatial disposition of the velocity and acceleration four-vectors of the source can be deduced from Table V.

Case C. $\nu^*=0$ and $\pi^*\neq 0$. We shall here omit the actual coordinate and tetrad transformations (which are similar to those made in Ref. 41) that make it possible to simplify and explicitly integrate the complete system of Newman–Penrose equations in this case. The

TABLE V. Family of "radiating" solutions of Kinnersley type to the Einstein equations (v is the velocity, a is the acceleration of the source, and t, s, and l denote timelike, spacelike, and null directions, respectively).

v \ a	t	s	l
t	—	—	B. I
s	B. II	B. III	B. IV
l	—	B. V	A. 3

resulting metric has the form ($x^0\equiv u;\ x^1\equiv r;\ x^2\equiv x;\ x^3\equiv y$):

$$ds^2=-2U\,du^2+2du\,dr+2c^2(u)\,r^2\,du\,dx$$
$$-\frac{r^2}{f^2(x)\,c^2(u)}[dx^2+f^4(x)\,dy^2];$$
$$U=-3mxc^2(u)-\frac{1}{2}c^4(u)\frac{d[f^2(x)]}{dx}r$$
$$+r\frac{1}{c(u)}\frac{dc(u)}{du}+\frac{m}{r}+\frac{1}{2}r^2f^2(x)\,c^6(u);$$
$$X^0=1,\ X^2=X^3=0;$$
$$\omega=\frac{1}{\sqrt{2}}c^3(u)\,f(x)\,r;$$
$$\xi^0=0,\ \xi^2=-\frac{f(x)\,c(u)i}{\sqrt{2}\,r},\ \xi^3=-\frac{ic(u)}{\sqrt{2}\,rf(x)};$$
$$\rho=-\frac{1}{r};$$
$$\pi=\frac{1}{\sqrt{2}}f(x)\,c^3(u);$$
$$\beta=-\frac{1}{r}\frac{c(u)}{4f(x)\sqrt{2}}\cdot(a-6mx^2);$$
$$\alpha=-\frac{1}{r}\frac{c(u)}{4f(x)\sqrt{2}}(6mx^2-a)-\frac{1}{\sqrt{2}}f(x)\,c^3(u);$$
$$\tau=-\frac{1}{\sqrt{2}}f(x)\,c^3(u);$$
$$\gamma=\frac{1}{4}c^4(u)\,(a-6mx^2)-\frac{1}{2c(u)}\cdot\frac{dc(u)}{du}$$
$$+\frac{m}{2r^2}-\frac{1}{2}rf^2(x)\,c^6(u);$$
$$\Psi_2=-\frac{m}{r^3};$$
$$\Phi_{22}=S=\frac{3m}{r^2c(u)}\cdot\frac{dc(u)}{du};$$
$$\mu=\frac{1}{2}c^4(u)\,(6mx^2-a)-\frac{1}{r}3mxc^2(u)$$
$$+\frac{m}{r^2}+\frac{1}{2}rf^2(x)\,c^6(u);$$
$$\nu=\frac{3r}{\sqrt{2}}f(x)\,c^2(u)\frac{dc(u)}{du},$$

where $f(x)\equiv\sqrt{-2mx^3+ax+b}$; a, b, m are arbitrary constants; $c(u)$ is an arbitrary positive function.

This solution in the limiting case $\Phi^*\equiv 0$ ($c(u)\equiv 1$) goes over into the Ehlers–Kundt C metric[48,41,57] and is a generalization of the latter to the case when high-frequency radiation with an energy-momentum tensor of the form (101) is present in the space-time.

Thus, the solutions A.1–A.3, B.I–B.V, and C exhaust the class of "radiating" nondegenerate type D twist-free solutions of the Einstein equations.

We now consider briefly the results of Ref. 118, in which the study of the gravitational field of radiating systems was continued. Frolov[118] found a general solution of Petrov type II of the Einstein equations with energy-momentum tensor (101), in which l_i represents a twist-free congruence of shear-free geodesics with expansion. As is shown in Ref. 64, the metric in this case has the form

$$ds^2 = 2du\,(dr - U\,du) - (r^2/2P^2)\,d\zeta\,d\bar{\zeta};\quad P = P(u, \zeta, \bar{\zeta}), \tag{126}$$

where U and P are real functions and ζ and $\bar{\zeta}$ are complex coordinates; $d\zeta = dx + i\,dy$ and $d\bar{\zeta} = dx - i\,dy$.

We now choose the Newman–Penrose null tetrad, making it satisfy the relations [the first vector of the tetrad is made coincident with the vector l_i in (101)]

$$l_i\,dx^i = du;\quad n_i\,dx^i = dr - U\,du;\quad m_i\,dx^i = (r/2P)\,d\zeta. \tag{127}$$

The spin coefficients corresponding to this tetrad are

$$\left.\begin{array}{c} \kappa = \varepsilon = \pi = \sigma = \tau = \lambda = 0;\quad \rho = \bar{\rho} = -1/r; \\ \alpha = -\bar{\beta} = -(1/r)\,\partial P/\partial\zeta; \\ \mu = U/r - \partial(\ln P)/\partial u;\quad \gamma = -(\partial U/\partial r)/2; \\ \nu = (2P/r)\,\partial U/\partial\zeta. \end{array}\right\} \tag{128}$$

In (118), Frolov then considered in detail the case when the surface r = const, u = const is compact (conformally equivalent to a complete sphere). In this case, it is convenient to represent the function P in the form $P = VP_0$, $P_0 = (1 + \zeta\bar{\zeta})/2$, where the stereographic coordinate ζ ranges over the complete complex plane, and $V = V(u, \zeta, \bar{\zeta})$ is a nonvanishing bounded function. The stereographic coordinates ζ and $\bar{\zeta}$ are related to the polar coordinates θ and φ on the unit sphere by the relation $\zeta = \exp(i\varphi)\cot(\theta/2)$; in these coordinates, the line element (126) is rewritten in the form

$$ds^2 = 2\,du\,(dr - U\,du) - (r^2/2V^2)\,(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{129}$$

Substituting (128) in the second group of Newman–Penrose equations [Eqs. (A.12)–(A.29) of Appendix B], we determine the tetrad components of the Weyl tensor (some of the equations are satisfied identically):

$$\left.\begin{array}{l} \Psi_0 = \Psi_1 = 0;\quad \Psi_2 = -m(u)/r^3;\quad \Psi_3 \\ = -(2P/r^2)\,\partial U^*/\partial\zeta = (V^3/r^2)\,\eth_0 R; \\ \Psi_4 = -\dfrac{4}{r}\dfrac{\partial}{\partial\zeta}\left[P^2\dfrac{\partial^2(\ln P)}{\partial\zeta\,\partial u}\right] - \dfrac{4}{r^2}\dfrac{\partial}{\partial\zeta}\left(P^2\dfrac{\partial U^*}{\partial\zeta}\right) \\ \qquad = -\dfrac{V^2}{r}\dfrac{\partial R}{\partial u} + \dfrac{1}{r^2}\bar{\eth}_0(V^4\eth_0 R), \\ \text{where} \\ U^* = -4P^2\partial^2(\ln P)/\partial\zeta\,\partial\bar{\zeta};\quad R = (1/V)\,\bar{\eth}_0^2 V. \end{array}\right\} \tag{130}$$

The differential operators $\eth_0$ and $\bar{\eth}_0$ (introduced in Refs. 136–138) used in (130) are applied to quantities with definite spin weight, raising ($\eth_0$) or lowering ($\bar{\eth}_0$) the spin weight by unity.

The remaining unused equations from the second group of Newman–Penrose equations lead to the following relations (to obtain them, one can also use the formulas of the Appendix Ref. 63):

$$S = (1/r^2)\,Q(u, \zeta, \bar{\zeta}); \tag{131}$$

$$U = -4P^2\partial^2(\ln P)/\partial\zeta\,\partial\bar{\zeta} + (r/P)\,\partial P/\partial u + m(u)/r; \tag{132}$$

$$\frac{1}{16}\frac{\partial}{\partial u}\left(\frac{m}{P^3}\right) - \frac{\partial^2}{\partial\zeta^2}\left(\frac{\partial^2 P}{\partial\bar{\zeta}^2}\right) + \frac{1}{P}\frac{\partial^2}{\partial\zeta^2}\left(P\frac{\partial^2 P}{\partial\bar{\zeta}^2}\right) = -\frac{Q(u, \zeta, \bar{\zeta})}{16P^3}. \tag{133}$$

The relation (131) shows that for different $r = r_0$ the same amount of energy flows through the surface r = const, u = const, and $Q = Q(u, \zeta, \bar{\zeta})$ characterizes the intensity of the radiation at a given time as a function of the direction. Equation (133), which goes over for $Q \equiv 0$ into the well-known Robinson–Trautman equation,[77] relates the metric to the radiation intensity. Conversely, specifying an arbitrary $P = P(u, \zeta, \bar{\zeta})$ and choosing a corresponding $m = m(u)$ to satisfy the condition $Q \geqslant 0$, we obtain a solution of the Einstein equations with the energy-momentum tensor (101).

Under the condition that there is no low-frequency (background) gravitational radiation and that the surface r = const, u = const is compact, the general solution of the Einstein equations with the radiation (101), where l_i is a twist-free shear-free nondegenerate congruence of geodesic rays, reduces to the Kinnersley solution[102,118] (see the metric B.I given above).

In Refs. 105, 139, and 140, the Newman–Penrose formalism was used to investigate "radiating" solutions of the Einstein–Maxwell equations with a cosmological term of Petrov type II (in Refs. 104 and 141, a detailed investigation was made by the Kerr–Schild method[68,142]). The following system of equations was considered:

$$R_{ij} - Rg_{ij}/2 - \tilde{\Lambda}g_{ij} = 2(E_{ij} + \tilde{S}l_i l_j); \tag{134}$$

$$E_{ij} = -F_i{}^k F_{jk} + g_{ij}(F_{kl}F^{kl})/4; \tag{135}$$

$$F_{ij} = A_{j;i} - A_{i;j}; \tag{136}$$

$$F_{ij;k} + F_{ki;j} + F_{jk;i} = 0; \tag{137}$$

$$F^{ij}{}_{;j} = \tilde{J}^i;\quad \tilde{J}^i = 2Jl^i, \tag{138}$$

where l^i is the tangent vector to the twist-free shear-free null geodesic congruence with expansion:

$$(l^i{}_{;i})^2 = 2l_{(i;j)}l^{i;j};\quad l_{[i;j]}l^{i;j} = 0;$$
$$l_i l^i = 0,\quad l^j l_{i;j} = 0;\quad l^i{}_{;i} \neq 0.$$

Under the condition that there is no low-frequency gravitational radiation (understood as the vanishing of the information function[33,34] for the background metric) or angular singularities, a family of general solutions of this system was obtained corresponding to gravitating sources of, respectively, tardyon (B*.I), tachyon (B*.II–B*.IV), and luxon (B*.V) type. Below, for reference, we give the nonvanishing spin coefficients, the tetrad components of the Weyl and Maxwell tensors, the tetrad component of the four-current $\tilde{J}$, and the mean energy flux density $\tilde{S}$ of the high-frequency radiation, and also the components of the vectors of the null tetrad for solutions that directly generalize the metrics B.I–B.V to the case when the space-time contains a background (low-frequency) electromagnetic field. The functions $e(u)$ and $g(u)$ are, respectively, the electric and magnetic charges of the source, and the interpretation of the remaining arbitrary functions of the retarded time is analogous to the interpretation given above for the B metrics. Note that if we know the components of the vectors of the null tetrad, then, using Eqs. (3), we can also readily calculate the components of the metric; for this reason, we do not here write out g_{ij}, but merely mention that for $e(u) \equiv g(u) \equiv 0$ the corresponding metrics B* and B are identical.

The radial dependence of $\tilde{J}$ and $\tilde{S}$ for all the B* metrics is the same:

$$\tilde{J} = \frac{1}{r^2}J^* = \frac{1}{2r^2}\cdot\frac{d}{du}[e(u) + ig(u)];$$
$$\tilde{S} = \frac{1}{r^2}S^* + \frac{1}{2r^3}\cdot\frac{d}{du}[e^2(u) + g^2(u)].$$

Metric $B^*.I$ $(U^*<0)$:

$$\rho=-1/r;\ \beta=-\alpha=(1/2\sqrt{2}r)\operatorname{ctg}\theta;$$
$$\gamma=-(1/2)\,a(u)\cos\theta+(i/2\sin\theta)\,[c(u)\sin\varphi-b(u)\cos\varphi]$$
$$+(1/2r^2)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\cos\theta\}$$
$$-(1/2r^3)\,[e^2(u)+g^2(u)]-\tilde{\Lambda}r/6;\ \nu=(1/\sqrt{2})\,a(u)\sin\theta$$
$$+(\sqrt{2}/r^2)\,[e^2(u)+g^2(u)]\,a(u)\sin\theta;\ \mu=-1/2r$$
$$+(1/r^2)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\cos\theta\}$$
$$-(1/2r^3)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r/6;\ \Psi_2=-(1/r^3)\,\{m(u)$$
$$+2\,[e^2(u)+g^2(u)]\,a(u)\cos\theta\}+(1/r^4)\,[e^2(u)+g^2(u)];\ \Psi_3$$
$$=-(3/\sqrt{2}r^3)\,[e^2(u)+g^2(u)]\,a(u)\sin\theta;\ \varnothing_1=-(1/2r^2)\,[e(u)$$
$$+ig(u)];\ \varnothing_2=(1/\sqrt{2}r)\,[e(u)+ig(u)]\,a(u)\sin\theta;$$
$$S^*=-\frac{\partial}{\partial u}\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\cos\theta\}$$
$$+3a(u)\cos\theta\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\cos\theta\}$$
$$-3\,[e^2(u)+g^2(u)]\,a^2(u)\sin^2\theta+2\,[e^2(u)$$
$$+g^2(u)]\,a(u)\sin\theta\,[b(u)\sin\varphi+c(u)\cos\varphi];$$
$$\xi^0=0;\ \xi^2=1/\sqrt{2}r;\ \xi^3=i/\sqrt{2}r\sin\theta;\ \omega=0;$$
$$X^0=1;\ X^2=-a(u)\sin\theta+b(u)\sin\varphi$$
$$+c(u)\cos\varphi;\ X^3=\operatorname{ctg}\theta\,[b(u)\cos\varphi-c(u)\sin\varphi];$$
$$U=-1/2+(1/r)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\cos\theta\}$$
$$+ra(u)\cos\theta-(1/2r^2)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r^2/6.$$

Metric $B^*.II$ $(U^*>0;\ |\beta^*|>\frac{1}{2}\sqrt{2})$:

$$\rho=-1/r;\ \beta=-\alpha=(1/2\sqrt{2}r)\operatorname{cth}\theta;\ \gamma=-(1/2)\,a(u)\operatorname{ch}\theta$$
$$+(i/2\operatorname{sh}\theta)\,[c(u)\sin\varphi-b(u)\cos\varphi]+(1/2r^2)\,\{m(u)-2\,[e^2(u)$$
$$+g^2(u)]\,a(u)\operatorname{ch}\theta\}-(1/2r^3)\,[e^2(u)+g^2(u)]$$
$$-\tilde{\Lambda}r/6;\ \nu=-(1/\sqrt{2})\,a(u)\operatorname{sh}\theta+(\sqrt{2}/r^2)\,[e^2(u)$$
$$+g^2(u)]\,a(u)\operatorname{sh}\theta;\ \mu=1/2r+(1/r^2)\,\{m(u)$$
$$-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta\}-(1/2r^3)\,[e^2(u)$$
$$+g^2(u)]+\tilde{\Lambda}r/6;\ \Psi_2=-(1/r^3)\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta\}$$
$$+(1/r^4)\,[e^2(u)+g^2(u)];$$
$$\Psi_3=-(3/\sqrt{2}r^3)\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta;$$
$$\varnothing_1=-(1/2r^2)\,[e(u)+ig(u)];\ \varnothing_2=(1/\sqrt{2}r)$$
$$\times[e(u)+ig(u)]\,a(u)\operatorname{sh}\theta;\ S^*=-\frac{\partial}{\partial u}\,\{m(u)$$
$$-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta\}$$
$$+3a(u)\operatorname{ch}\theta\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta\}$$
$$-3\,[e^2(u)+g^2(u)]\,a^2(u)\operatorname{sh}^2\theta+2\,[e^2(u)$$
$$+g^2(u)]\,a(u)\operatorname{sh}\theta\,[b(u)\sin\varphi+c(u)\cos\varphi];$$
$$\xi^0=0;\ \xi^2=(1/\sqrt{2}r);\ \xi^3=i/\sqrt{2}r\operatorname{sh}\theta;$$
$$\omega=0;\ X^0=1;\ X^2=-a(u)\operatorname{sh}\theta+b(u)\sin\varphi$$
$$+c(u)\cos\varphi;\ X^3=\operatorname{cth}\theta\,[b(u)\cos\varphi-c(u)\sin\varphi];$$
$$U=1/2+(1/r)\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta\}$$
$$+ra(u)\operatorname{ch}\theta-(1/2r^2)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r^2/6.$$

Metric $B^*.III$ $(U^*>0;\ |\beta^*|<\frac{1}{2}\sqrt{2})$:

$$\rho=-1/r;\ \beta=-\alpha=(1/2\sqrt{2}r)\operatorname{th}\theta;$$
$$\gamma=-(1/2)\,a(u)\operatorname{sh}\theta+(i/2\operatorname{ch}\theta)\,[b(u)\exp(\varphi)-c(u)\exp(-\varphi)]$$
$$+(1/2r^2)\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta\}-(1/2r^3)[e^2(u)$$
$$+g^2(u)]-\tilde{\Lambda}r/6;$$
$$\nu=-(1/\sqrt{2})\,a(u)\operatorname{ch}\theta+(\sqrt{2}/r^2)\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta;$$
$$\mu=1/2r+(1/r^2)\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta\}$$
$$-(1/2r^3)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r/6;$$
$$\Psi_2=-(1/r^3)\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta\}+(1/r^4)$$
$$\times[e^2(u)+g^2(u)];$$
$$\Psi_3=-(3/\sqrt{2}r^3)\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta;$$
$$\varnothing_1=-(1/2r^2)\,[e(u)+ig(u)];$$
$$\varnothing_2=(1/r\sqrt{2})\,[e(u)+ig(u)]\,a(u)\operatorname{ch}\theta;$$
$$S^*=-\frac{\partial}{\partial u}\left\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta\right\}+3a(u)\operatorname{sh}\theta\,\{m(u)$$
$$-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta\}-3\,[e^2(u)+g^2(u)]\,a^2(u)\operatorname{ch}^2\theta$$
$$-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{ch}\theta\,[b(u)\exp(\varphi)+c(u)\exp(-\varphi)];$$
$$\xi^0=0;\ \xi^2=1/\sqrt{2}r;\ \xi^3=i/\sqrt{2}r\operatorname{ch}\theta;$$
$$\omega=0;\ X^0=1;\ X^2=-a(u)\operatorname{ch}\theta-b(u)\exp(\varphi)-c(u)\exp(-\varphi);$$
$$X^3=\operatorname{th}\theta\,[b(u)\exp(\varphi)-c(u)\exp(-\varphi)];$$
$$U=1/2+(1/r)\,\{m(u)-2\,[e^2(u)+g^2(u)]\,a(u)\operatorname{sh}\theta\}$$
$$+ra(u)\operatorname{sh}\theta-(1/2r^2)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r^2/6.$$

Metric $B^*.IV$ $(U^*>0;\ \beta^*=\pm\frac{1}{2}\sqrt{2})$:

$$\rho=-1/r;\ \beta=-\alpha=\mp\frac{1}{2\sqrt{2}r};\ \gamma=(1/2)\,a(u)\exp(\theta)$$
$$\mp ib(u)\exp(-\theta)+(1/2r^2)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta)\}$$
$$-(1/2r^3)\,[e^2(u)+g^2(u)]-\tilde{\Lambda}r/6;$$
$$\nu=\mp(1/\sqrt{2})\,a(u)\exp(\theta)\pm(\sqrt{2}/r^2)\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta);$$
$$\mu=1/2r+(1/r^2)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta)\}$$
$$-(1/2r^3)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r/6;$$
$$\Psi_2=-(1/r^3)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\exp\theta\}$$
$$+(1/r^4)\,[e^2(u)+g^2(u)];$$
$$\Psi_3=\mp(3/\sqrt{2}r^3)\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta);$$
$$\varnothing_1=-(1/2r^2)\,[e(u)+ig(u)];$$
$$\varnothing_2=\pm(1/\sqrt{2}r)\,[e(u)+ig(u)]\,a(u)\exp(\theta);$$
$$S^*=-\frac{\partial}{\partial u}\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta)\}-3a(u)\exp(\theta)$$
$$\times\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta)\}-3\,[e^2(u)$$
$$+g^2(u)]\,a^2(u)\exp(2\theta)+4\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta)\cdot b(u)\,\varphi;$$
$$\xi^0=0;\ \xi^2=\mp 1/\sqrt{2}r;\ \xi^3=i/\sqrt{2}r\exp(\theta);\ \omega=0;$$
$$X^0=1;\ X^2=a(u)\exp(\theta)-2b(u)\,\varphi;\ X^3=b(u)\,[\varphi^2-\exp(-2\theta)];$$
$$U=1/2+(1/r)\,\{m(u)+2\,[e^2(u)+g^2(u)]\,a(u)\exp(\theta)\}$$
$$-ra(u)\exp(\theta)-(1/2r^2)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r^2/6.$$

Metric $B^*.V$ $(U^*=0)$:

$$\rho=-1/r;\ \gamma=-(\theta+i\varphi)/2+(1/2r^2)\,\{m(u)-(2/3)\,[e^2(u)$$
$$+g^2(u)]\theta^3\}-(1/2r^3)\,[e^2(u)+g^2(u)]-\tilde{\Lambda}r/6;$$
$$\nu=\pm 1/\sqrt{2}\mp(\sqrt{2}/r^2)\,[e^2(u)+g^2(u)]\,\theta^2;$$
$$\mu=(1/r^2)\,\{m(u)-(2/3)\,[e^2(u)+g^2(u)]\,\theta^3\}-(1/2r^3)\,[e^2(u)$$
$$+g^2(u)]+\tilde{\Lambda}r/6;$$
$$\Psi_2=-(1/r^3)\,\{m(u)-(2/3)\,[e^2(u)+g^2(u)]\,\theta^3\}+(1/r^4)\,[e^2(u)+g^2(u)];$$
$$\Psi_3=\pm(3/\sqrt{2}r^3)\,[e^2(u)+g^2(u)]\,\theta^2;$$
$$\varnothing_1=-(1/2r^2)\,[e(u)+ig(u)]:$$
$$\varnothing_2=\mp(1/\sqrt{2}r)\,[e(u)+ig(u)]\,\theta^2;$$
$$S^*=-\frac{\partial}{\partial u}\,\{m(u)-(2,3)\,[e^2(u)+g^2(u)]\,\theta^3\}+3\theta\,\{m(u)$$
$$-(4/3)\,[e^2(u)+g^2(u)]\,\theta^3\}+2\theta^2\,[e^2(u)+g^2(u)]\,[b(u)+\varphi^2/2];$$
$$\xi^0=0;\ \xi^2=\mp 1/\sqrt{2}r;\ \xi^3=\mp i/\sqrt{2}r;$$
$$\omega=0;\ X^0=1;\ X^2=(\varphi^2-\theta^2)/2+b(u);\ X^3=-\theta\varphi+c(u);$$
$$U=\theta r+(1/r)\,\{m(u)-(2/3)[e^2(u)+g^2(u)]\,\theta^3\}$$
$$-(1/2r^2)\,[e^2(u)+g^2(u)]+\tilde{\Lambda}r^2/6.$$

As we have already mentioned above, the B* metrics describe the gravitational field of anisotropically radiating charged sources which differ by the spatial disposition of the velocity and accleration four-vectors (which are orthogonal to one another). Thus, for the metric B*.I the source velocity is timelike and, therefore, the acceleration is spacelike. In the cases B*.II, B*.III, and B*.IV, the velocity is spacelike, and the acceleration is timelike, spacelike, and null, respectively. Finally, case B*.V corresponds to null velocity and spacelike acceleration of the source (it is easy to show that if the velocity and acceleration are simultaneously null, the acceleration can be made to vanish by a reparametrization of the world line of the source[103]).

The examples of the use of the spin coefficient method for integragion and further investigation of definite solutions of the field equations in general relativity given in the present section may appear rather cumbersome on first acquaintance, and the formalism itself may seem inconvenient for this reason. However, this first impression deceives; for knowledge of all three groups of gravitational field variables in the Newman–Penrose formalism gives us essentially the most complete information about the structure of the considered space-time.

The large number of investigations of different solutions of the Einstein equations in the Newman–Penrose formalism during the last 10–15 years and the fundamental results obtained by means of this formalism clearly demonstrate the fruitfulness of this method for studying gravitational, electromagnetic, and other fields in the general theory of relativity.

4. PROPAGATION OF WAVES IN EXTERNAL VACUUM AND ELECTROVAC GRAVITATIONAL FIELDS

In this section, we apply the Newman–Penrose formalism to the propagation of gravitational and electromagnetic waves on the background of a vacuum or electrovac gravitational field.

We consider in detail the propagation of small-amplitude waves on a background of vacuum and electrovac fields of algebraically special Petrov types, in particular, in the exterior fields of rotating uncharged (Kerr field) or charged (Kerr–Newman field) black holes and the scattering of small-amplitude waves by a strong electromagnetic-gravitational wave.

In the approximation of geometrical optics, we consider the propagation of high-frequency waves of small amplitude in an arbitrary vacuum or electrovac gravitational field and we describe some properties of these processes.

Propagation of Small-Amplitude Waves in Gravitational Fields of Algebraically Special Types. In the linear approximation, we consider arbitrary small perturbations of the gravitational and electromagnetic fields on the background of a vacuum or electrovac field of algebraically special Petrov type.

Necessary properties of the background field. As background field, we can in what follows consider any vacuum or electrovac field for which in some null basis

$$\Psi_0 = \Psi_1 = 0; \quad \varnothing_0 = 0, \tag{139}$$

i.e., the gravitational field of the background is algebraically special, and we admit only background electromagnetic fields for which the degenerate principal null direction of the Weyl tensor is a principal null direction of the Maxwell tensor.

In the general case, from the conditions (139) and the Bianchi identities (A.1) and (A.4) we find that in this same basis

$$\kappa = \sigma = 0. \tag{140}$$

(The special cases $3\Psi_2 = \pm 2\Phi_{11}$ are an expection.) For the electromagnetic field of the background under the condition (139) one of the following three cases holds (the last two, in a specially chosen tetrad):

$$\left.\begin{array}{l} \text{a) } \Phi_{mn} = 0 - \text{no electromagnetic field,} \\ \text{b) } \Phi_{11} \neq 0,\ \Phi_{mn} = 0, \quad \text{if} \quad \neq 1 \ \text{ or } \ \neq 1, \\ \text{c) } \Phi_{22} \neq 0,\ \Phi_{mn} = 0, \quad \text{if} \ m \neq 2 \ \text{ or } \ \neq 2. \end{array}\right\} \tag{141}$$

Fields of this type include many well-known exact solutions of the Einstein or Einstein–Maxwell vacuum equations, in particular, the fields of static and stationary black holes (Schwarzschild, Kerr, Reissner–Nordström solutions), which are fields of type D, and also plane gravitational or gravitational-electromagnetic waves, which are of Petrov type N. We shall consider these solutions below.

Linearized equations for the perturbations. For arbitrary small perturbations of the background field, any component $\tilde{\chi}$ of the perturbed field can be represented in the form

$$\tilde{\chi} = \chi + \chi^B, \tag{142}$$

where χ is the corresponding component of the background field and χ^B is a small perturbation of it.[23] If the space also contains sources of electromagnetic and gravitational perturbations (particles or other fields) that contribute directly to the energy-momentum tensor, then for $\tilde{\Phi}_{mn}$ we shall have, instead of (142),

$$\tilde{\Phi}_{mn} = \Phi_{mn} + \Phi^B_{mn} + \Phi^M_{mn}, \tag{143}$$

where $\Phi^{(M)}_{mn}$ describe the contribution of the sources.

Retaining only the terms linear in the perturbations, we obtain from the Bianchi identities (A.1) and (A.4), and from (A.13), using (139)–(141) and (28), the following inhomogeneous equations for the perturbations of the field variables:

$$(D - 4\rho - 2\varepsilon)\Psi_1^B - (\bar{\delta} - 4\alpha + \pi)\Psi_0^B + (3\Psi_2 - 2\Phi_{11})\kappa^B - (D - 2\bar{\rho} - 2\varepsilon)\Phi^B_{01} = A; \tag{144}$$

$$(\delta - 4\tau - 2\beta)\Psi_1^B - (\Delta - 4\gamma + \mu)\Psi_0^B + (3\Psi_2 + 2\Phi_{11})\sigma^B + (\delta - 2\beta + 2\bar{\pi})\Phi^B_{01} - (D - 2\varepsilon + 2\bar{\varepsilon} - \bar{\rho})\Phi^B_{02} = B; \tag{145}$$

$$(D - \rho - \bar{\rho} - 3\varepsilon + \bar{\varepsilon})\sigma^B - (\delta - \tau - \bar{\alpha} + \bar{\pi} - 3\beta)\kappa^B - \Psi_0^B = 0, \tag{146}$$

where $A = (D - 2\bar{\rho} - 2\varepsilon)\Phi^M_{01} - (\delta - 2\beta - 2\bar{\alpha} + \bar{\pi})\Phi^{(M)}_{00}$;

$B = (D - 2\varepsilon + 2\bar{\varepsilon} - \bar{\rho})\Phi^{(M)}_{02} - (\delta - 2\beta + 2\bar{\pi})\Phi^{(M)}_{01}$.

We transform (144)–(146), using the commutation relations for the operators D and δ, which by virtue of (A.12)–(A.29) can be written in the form

$$[D - (p+1)\varepsilon + \bar{\varepsilon} + q\rho - \bar{\rho}](\delta - p\beta + q\tau) - [\delta - (p+1)\beta - \bar{\alpha} + \bar{\pi} + q\tau](D - p\varepsilon + q\rho) = \sigma\bar{\delta} - \kappa\Delta + (2q - p)\Psi_1 + q[(\Delta - 3\gamma - \bar{\gamma} - \mu + \bar{\mu})\kappa - (\bar{\delta} - 3\alpha + \bar{\beta} - \pi - \bar{\tau})\sigma] - p[(\alpha + \pi)\sigma - (\mu + \gamma)\kappa], \tag{147}$$

where p and q are arbitrary complex parameters, and the right-hand side of (147) vanishes for the background

[23] *Translator's note.* The superscript B derives from the first letter of the Russian word for "perturbation," and is retained to simplify composition.

field by virtue of the conditions (139) and (140).

Applying the operator $(\delta - 3\beta - \bar{\alpha} + \bar{\pi} - 4\tau)$ to (144) and subtracting from the resulting equation (145), to which we apply the operator $(D - 3\varepsilon + \bar{\varepsilon} - 4\rho - \bar{\rho})$, and using (147) with vanishing right-hand side, Eq. (146), and the Maxwell equations (A.38)–(A.41) for the background field, we obtain

$$\begin{aligned}&[(D-3\varepsilon+\bar{\varepsilon}-4\rho-\bar{\rho})(\Delta-4\gamma+\mu)-(\delta-3\beta-\bar{\alpha}\\&+\bar{\pi}-4\tau)(\bar{\delta}-4\alpha+\pi)-3\Psi_2]\Psi_0^B\\&-(\varkappa/4\pi)\bar{\varnothing}_1[(\delta-3\beta-\bar{\alpha}+\bar{\pi}-4\tau)(D-2\varepsilon)\\&+(D-3\varepsilon+\bar{\varepsilon}-4\rho+\bar{\rho})(\delta-2\beta)]\varnothing_0^B-[(\delta-3\beta\\&-\bar{\alpha}+\bar{\pi}-\tau)(2\Phi_{11}\varkappa^B)+(D-3\varepsilon+\bar{\varepsilon}-\rho-\bar{\rho})\\&\times(2\Phi_{11}\sigma^B)]+(\varkappa/4\pi)\bar{\varnothing}_2[(D-3\varepsilon-\bar{\varepsilon}\\&-4\rho)(D-2\varepsilon)]\varnothing_0^B=I_0,\end{aligned} \tag{148}$$

where

$$I_0=(\delta-3\beta-\bar{\alpha}+\bar{\pi}-4\tau)A-(D-3\varepsilon+\bar{\varepsilon}-4\rho-\bar{\rho})B.$$

We now consider the Maxwell equations (A.38)–(A.41). We apply to (A.38) the operator $(\delta - \beta - \bar{\alpha} + \bar{\pi} - 2\tau)$ and subtract from it (A.39), to which we apply the operator $(D - \varepsilon + \bar{\varepsilon} - 2\rho - \bar{\rho})$, after which we superimpose the perturbations. By virtue of (147), (139), (140), and (146) and the Maxwell equations for the background, we obtain

$$\begin{aligned}&[(D-\varepsilon+\bar{\varepsilon}-2\rho-\bar{\rho})(\Delta-2\gamma+\mu)-(\delta-\bar{\alpha}\\&-\beta+\bar{\pi}-2\tau)(\bar{\delta}-2\alpha+\pi)]\varnothing_0^B+4\varnothing_1\Psi_1^B\\&+2\varnothing_1[(\Delta-3\gamma-\bar{\gamma}-2\mu+\bar{\mu})\varkappa^B\\&-(\bar{\delta}-3\alpha+\bar{\beta}-2\pi-\bar{\tau})\sigma^B]-\varnothing_2\Psi_0^B=J_0,\end{aligned} \tag{149}$$

where

$$\begin{aligned}J_0=&(D-\varepsilon+\bar{\varepsilon}-2\rho-\bar{\rho})[(2\pi/c)J_{(m)}]\\&-(\delta-\beta-\bar{\alpha}+\bar{\pi}-2\tau)[(2\pi/c)J_{(l)}].\end{aligned}$$

In case a), when there is no background electromagnetic field, Eqs. (148) and (149) become independent and each of them contains only one unknown function:

$$\begin{aligned}&[(D-3\varepsilon+\bar{\varepsilon}-4\rho-\bar{\rho})(\Delta-4\gamma+\mu)-(\delta-\bar{\alpha}-3\beta+\bar{\pi}-4\tau)\\&\times(\bar{\delta}-4\alpha+\pi)-3\Psi_2]\Psi_0^B=I_0;\end{aligned} \tag{150}$$

$$\begin{aligned}&[D-\varepsilon+\bar{\varepsilon}-2\rho-\bar{\rho})(\Delta-2\gamma+\mu)-(\delta-\bar{\alpha}-\beta+\bar{\pi}-2\tau)\\&\times(\bar{\delta}-2\alpha+\pi)]\varnothing_0^B=J_0.\end{aligned} \tag{151}$$

Equations (150) and (151) describe the gravitational and electromagnetic perturbations, respectively. These equations were obtained by Teukolsky,[9] and were then used to describe perturbations on the background of the field of a rotating black hole—the Kerr field.[106–117,119,120]

In case b), i.e., when $\phi_2 = 0$, introducing the new variables

$$\left.\begin{aligned}\varnothing_{(\varkappa)}&=(\varkappa/4\pi)[(D-2\varepsilon-3\rho)\varnothing_0^B+2\varnothing_1\varkappa^B];\\ \varnothing_{(\sigma)}&=(\varkappa/4\pi)[(\delta-2\beta-3\tau)\varnothing_0^B+2\varnothing_1\sigma^B]\end{aligned}\right\} \tag{152}$$

and using (147), we can reduce Eqs. (148), (149), and (146) to the form

$$\begin{aligned}&[(D-3\varepsilon+\bar{\varepsilon}-4\rho-\bar{\rho})(\Delta-4\gamma+\mu)\\&-(\delta-\bar{\alpha}-3\beta+\bar{\pi}-4\tau)(\bar{\delta}-4\alpha+\pi)-3\Psi_2]\Psi_0^B\\&-\bar{\varnothing}_1[(\delta-\bar{\alpha}-3\beta-\bar{\pi}-\tau)\varnothing_{(\varkappa)}+(D-3\varepsilon+\bar{\varepsilon}-\rho+\bar{\rho})\varnothing_{(\sigma)}]=I_0;\end{aligned} \tag{153}$$

$$\begin{aligned}&[(D-6\rho-2\varepsilon)(\Delta-3\gamma-\bar{\gamma}+\bar{\mu})-2(3\Psi_2-2\Phi_{11})]\varnothing_{(\varkappa)}\\&-(D-6\rho-2\varepsilon)(\bar{\delta}-3\alpha+\bar{\beta}-\bar{\tau})\varnothing_{(\sigma)}+\frac{\varkappa}{\pi}\varnothing_1(\bar{\delta}-4\alpha+\pi)\Psi_0^B=J_1;\end{aligned} \tag{154}$$

$$\begin{aligned}&(D-3\varepsilon+\bar{\varepsilon}-3\rho-\bar{\rho})\varnothing_{(\sigma)}-(\delta-\bar{\alpha}-3\beta+\bar{\pi}-3\tau)\varnothing_{(\varkappa)}\\&-(\varkappa/2\pi)\varnothing_1\Psi_0^B=0,\end{aligned} \tag{155}$$

where $J_1 = (\varkappa/4\pi)[(D - 6\rho - 2\varepsilon)J_0 - 4\phi_1 A]$.

This system, which was obtained in a somewhat different form in Ref. 121, and also the equivalent systems in Refs. 122 and 123, is very complicated for analyzing the behavior of arbitrary small perturbations in the exterior field of a charged rotating black hole—the Kerr–Newman field. However, under some additional assumptions (see below), it can be significantly simplified, which means that it can be used, for example, to describe the propagation of electromagnetic and gravitational waves in the Kerr–Newman field or to study ultrarelativistic charged particles moving in the same field.[121,124]

In case c), setting $\phi_1 = 0$ in (148) and (149), we obtain

$$\begin{aligned}&[(D-3\varepsilon+\bar{\varepsilon}-4\rho-\bar{\rho})(\Delta-4\gamma+\mu)-(\delta-3\beta-\bar{\alpha}+\bar{\pi}-4\tau)\\&\times(\bar{\delta}-4\alpha+\pi)-3\Psi_2]\Psi_0^B\\&+(\varkappa/4\pi)\bar{\varnothing}_2(D-3\varepsilon-\bar{\varepsilon}-4\rho)(D-2\varepsilon)\varnothing_0^B=I_0;\end{aligned} \tag{156}$$

$$\begin{aligned}&[(D-\varepsilon+\bar{\varepsilon}-2\rho-\bar{\rho})(\Delta-2\gamma+\mu)\\&-(\delta-\beta-\bar{\alpha}+\bar{\pi}-2\tau)(\bar{\delta}-2\alpha+\pi)]\varnothing_0^B-\varnothing_2\Psi_0^B=J_0.\end{aligned} \tag{157}$$

This system of equations was obtained in Ref. 126 and was used there to analyze the scattering of small-amplitude waves by a strong electromagnetic-gravitational wave.[24)]

It is interesting to note that, although the null tetrads and the coordinate system in the perturbed space are chosen only to within arbitrary infinitesimal transformations, the functions Ψ_0, $\phi_{(k)}$, and $\phi_{(\sigma)}$ in (153)–(155) and also Ψ_0^B and ϕ_0^B in (150)–(151) and in (156)–(157) (ϕ_0^B by virtue of $\phi_1 = 0$) do not depend on this choice, since they are invariant under these transformations.

Each of the systems of equations (150)–(151), (153)–(155), and (156)–(157) in the presence of sources of perturbations—particles or other fields besides the electromagnetic and gravitational fields—is not yet closed and must be augmented by equations that determine the evolution of these sources. However, in many cases the motion of the sources can be calculated with neglect of the radiative reaction, i.e., under the assumption that the perturbations that are produced do not have an appreciable reaction back on the evolution of the sources. The perturbations of the electromagnetic and gravitational fields are then determined from the corresponding inhomogeneous equations (150)–(151), (153)–(155), or (156)–(157), whose right-hand sides are determined from the given motion of the sources.

Special Forms of the Background Field. We consider in more detail the use of Eqs. (150)–(151), (153)–(155), and (156)–(157) to describe the behavior of perturbations in the exterior fields of black holes.

Field of a rotating black hole—the Kerr field. The Kerr solution for a special choice of the null tetrad in Boyer–Lindquist coordinates (ct, r, θ, φ) is given by

24) These equations can also be used to analyze nonlinear effects if the nonlinear terms calculated in the linear approximation are taken as sources.

$$
\left.\begin{aligned}
&l^i=\{(r^2+a^2)/\Delta, 1, 0, a/\Delta\};\ n^i=(1/2)\,\rho\bar{\rho}\,\{r^2+a^2, -\Delta, 0, a\};\\
&m^i=-(1/\sqrt{2})\,\bar{\rho}\,\{ia\sin\theta, 0, 1, i/\sin\theta\};\ \Psi_2=m\rho^3;\\
&\rho=-(r-ia\cos\theta)^{-1}, \beta=-(1/2\sqrt{2})\,\bar{\rho}\,\mathrm{ctg}\,\theta;\\
&\pi=(1/\sqrt{2})\,i\,a\sin\theta\rho^2;\\
&\tau=-(1/\sqrt{2})\,ia\sin\theta\,\rho\bar{\rho};\\
&\mu=\Delta\rho^2\bar{\rho}/2;\ \gamma=\mu+(r-m)\,\rho\bar{\rho}/2;\ \alpha=\pi-\bar{\beta},
\end{aligned}\right\}\qquad(158)
$$

where $\Delta=r^2-2mr+a^2$, $m=GM/c^2$, $a=L/Mc$; M is the mass, and L is the angular momentum of the black hole. The remaining spin coefficients and the other Newman–Penrose scalars are zero.

On the background of the Kerr field, the equations (150) and (151) for the perturbations of the gravitational and electromagnetic fields, respectively, and also the equation for the scalar field $\varphi(\Box\varphi=-4\pi T)$ and the Weyl equations can be written in the form of a single "basis" equation with discrete parameter s ("spin weight") containing only one unknown function Y_s. (For gravitational perturbations, $s=2$, $Y_2=\Psi_0^B$; for electromagnetic perturbations, $s=1$, $Y_1=\phi_0^B$; for neutrinos, $s=\frac{1}{2}$, $Y_{1/2}=\chi_{+1}$; and for a scalar field, $s=0$, $Y_0=\varphi$.) Because a type D field has a second degenerate principal null direction, in the Kerr field we can obtain one further equation for another perturbed component of each of these fields, and all these equations reduce to the same basis equation for Y_s but for different values of the parameter s (for gravitational perturbations, $s=-2$, $Y_{-2}=\rho^{-4}\Psi_4^B$; for electromagnetic perturbations, $s=-1$, $Y_{-1}=\rho^{-2}\phi_2^B$; and for neutrinos, $s=-\frac{1}{2}$, $Y_{-1/2}=\rho^{-1}\chi_{-1}$). The basis equation for Y_s in Boyer–Lindquist coordinates admits complete separation of the variables, and after a Fourier expansion of the unknown function with respect to the azimuthal angular coordinate φ and a Fourier transformation with respect to the time, it reduces to the inhomogeneous radial equation[25) 9,107]:

$$
\begin{aligned}
&[\Delta d^2/dr^2+2(s+1)(r-m)\,d/dr+(K^2-2is(r-m)K)/\Delta\\
&\quad+4is\omega r-\lambda]\,{}_sR^k_{\omega n}(r)={}_sT^k_{\omega n},
\end{aligned}\qquad(159)
$$

where $K\equiv(r^2+a^2)\,\omega-an$; $\lambda={}_sA^k_{\omega n}-2a\omega n+a^2\omega^2-s(s+1)$;

ω is the wave frequency; n is the number of the azimuthal harmonic; ${}_sT^k_{\omega n}$ is the Fourier component, multiplied by $-2\rho^{s-|s|-1}\bar{\rho}^{-1}$, of the sources in Eqs. (150) and (151) for Ψ_0^B and ϕ_0^B or in the analogous equations for $\rho^{-4}\Psi_4^B$ and $\rho^{-2}\phi_0^B$. The eigenvalue problem that arises for the equation for the function of the polar angle θ, namely

$$
\begin{aligned}
&\Big[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+a^2\omega^2\cos^2\theta-\frac{n^2}{\sin^2\theta}\\
&\quad-2a\omega s\cos\theta-\frac{2sn\cos\theta}{\sin^2\theta}-s^2\,\mathrm{ctg}^2\,\theta\\
&\quad-s^2+{}_sA^k_{\omega n}\Big]\,{}_sS^k_{\omega n}(\theta)=0
\end{aligned}\qquad(160)
$$

leads under the additional condition that the solutions are regular at $\theta=0$ and $\theta=\pi$ to a complete orthonormal system of spin spheroidal harmonics ${}_sS^k_{\omega n}(\theta)$[26)] (the superscript $k=0,1,2,\ldots$ labels different harmonics).[9,107] In the general case, the unknown function $Y_s(t,r,\theta,\varphi)$ is determined from the solutions (159) and (160) in the form

$$
\begin{aligned}
&Y_s(t,r,\theta,\varphi)=\int\sum_{n=-\infty}^{n=+\infty}\sum_{k=0}^{k=+\infty}{}_sY^k_{\omega n}\frac{d\omega}{2\pi};\\
&{}_sY^k_{\omega n}={}_sR^k_{\omega n}(r)\,{}_sS^k_{\omega n}(0)\exp(-i\omega t+in\varphi).
\end{aligned}
$$

Thus, each field of perturbations (except the scalar field) is characterized for given ω, n, k by two functions ${}_sY^k_{\omega n}(t,r,\theta,\varphi)$, though it can be shown that these two functions are connected by simple local relations[107] that make it possible to express each of these functions uniquely in terms of the other. In addition, for each value of s the radial basis equation (159) can be expressed in six different forms corresponding to two further possible ways of choosing the time coordinate ("retarded" or "advanced" time, which are regular on the past and future horizon, respectively) and to a different choice of the null tetrad, which is regular, not on the past horizon like (158), but on the future horizon.

The 12 equations then obtained have the property that any one of the field functions in them uniquely determines all the remaining field functions, but the different forms of these equations and the different asymptotic behaviors of the solutions make it preferable to use one or other of the equations, depending on the actual conditions of the problem.[107] As is shown in Ref. 113, each of the functions Y_s for given field carries complete information about the perturbed field, determining it except for trivial terms corresponding to small mass and angular momentum additions and the appearance of a small charge of the black hole.

A further great convenience of using the Teukolsky equations to describe wave processes in the Kerr field is that there is no need to use the complete system of equations for the perturbations, since many of the most important characteristics of waves propagating in this field—the energy flux of waves coming in from or going out to infinity, the polarization of the waves, and also the spectral and angular distribution—can be determined by using only the function $\Psi_0^B(\Psi_4^B)$ for the gravitational perturbations and $\phi_0^B(\phi_2^B)$ for the electromagnetic perturbations.[9,107] Thus, for the angular distribution of the energy flux of the waves we have

$$
\left.\begin{aligned}
&\frac{d^2}{dt\,d\Omega}E^{\rm out}_{\rm el}=\lim_{r\to\infty}\frac{c}{2\pi}r^2|\phi^B_2|^2;\\
&\frac{d^2}{dt\,d\Omega}E^{\rm out}_{\rm gr}=\lim_{r\to\infty}\frac{c^5}{4\pi G}\frac{r^2}{\omega^2}|\Psi^B_4|^2;\\
&\frac{d^2}{dt\,d\Omega}E^{\rm in}_{\rm el}=\lim_{r\to\infty}\frac{c}{8\pi}r^2|\phi^B_0|^2;\\
&\frac{d^2}{dt\,d\Omega}E^{\rm in}_{\rm gr}=\lim_{r\to\infty}\frac{c^5}{64\pi G}\frac{r^2}{\omega^2}|\Psi^B_0|^2,
\end{aligned}\right\}\qquad(161)
$$

and the real and imaginary parts of $\Psi_0^B(\Psi_4^B)$ and $\phi_0^B(\phi_2^B)$ correspond to the two different polarization states of the waves.

25)The properties of radial functions satisfying the homogeneous equations (159), in particular their asymptotic behavior, are described in Ref. 107; for the construction of solutions of the radial equations in different cases, see Refs. 12, 106, 107, 112, 119, 120. A solution of the inhomogeneous radial equations can be constructed by the Green's function method.[12,119]

26)The properties of spheroidal harmonics and the approximate calculation of the eigenvalues corresponding to them are described for different special cases in Refs. 12, 106, 107, 112, 119, and 120.

It is somewhat more complicated to calculate the energy flux of the waves that fall into the horizon and are absorbed by the black hole:

$$\left.\begin{aligned} \frac{d^2}{dt\,d\Omega}E^H_{\text{el}} &= \lim_{r\to r_+}\frac{c}{2\pi}\frac{\omega\Delta^2}{8mr_+k}\left|\varnothing^{\text{B}}_0\right|^2;\\ \frac{d^2}{dt\,d\Omega}E^H_{\text{gr}} &= \lim_{r\to r_+}\frac{c^5}{64\pi G}\frac{\omega\Delta^4}{k(k^2+4\epsilon^2)(2mr_+)^3}\left|\Psi^{\text{B}}_0\right|^2, \end{aligned}\right\} \tag{162}$$

where $\Delta = r^2 - 2mr + a^2$, $r_+ = m + \sqrt{m^2 - a^2}$ is the radius of the horizon,

$$\epsilon = \sqrt{m^2 - a^2}/4mr_+;\ k = \omega - n\omega_+;\ \omega_+ = a/2mr_+.$$

Using the basis equation, one can show that for waves of each type (gravitational, electromagnetic, and others) the energy conservation law holds[107]:

$$\frac{d}{dt}E^{\text{in}} - \frac{d}{dt}E^{\text{out}} = \frac{d}{dt}E^H. \tag{163}$$

A remarkable fact follows from (162) and (163): If $k/\omega < 0$, then the energy carried away by the waves to infinity is greater than the energy brought into the black hole by the incoming waves; in other words, there is an amplification of waves of a definite kind when they are scattered in the field of a rotating black hole.[27)] [107,108,112,120] This process, which is absent if the black hole has zero angular momentum, is irreversible and is accompanied by an increase in the area of the horizon.[107,112]

The results described above make it possible to study in detail the interaction of a black hole with external fields, i.e., the absorption of waves by a black hole and their scattering in the exterior field of the hole, which is accompanied by amplification of certain waves and a change in the polarization of the waves, and also to consider various processes in the presence of sources of perturbations—the occurrence of "floating" orbits, tidal friction reducing the black hole angular momentum,[59,108–111] the tidal effect of the motion of a satellite on a black hole,[114,115] and the emission of particles in nonrelativistic[116] and unstable, relativistic, orbits, motion in these orbits being accompanied by emission of synchrotron nature.[12,117,119]

Field of a charged rotating black hole—the Kerr–Newman field. In the Kerr–Newman field, the components of the vectors of the null tetrad and the spin coefficients have expressions that are the same as the corresponding expressions for the Kerr field (158), in which, however, $\Delta = r^2 - 2mr + a^2 + q^2$, where $q^2 = GQ^2/c^4$ and Q is the charge of the black hole, and for the other nonvanishing Newman–Penrose scalars we have

$$\Psi_2 = m\rho^3 + q^2\bar{\rho}\rho^3;\ \varnothing^1 = -Q\rho^2/2;\ \Phi_{11} = q^2\rho^2\bar{\rho}^2/2.$$

If a black hole has an electric charge, this leads to qualitatively new features in the various phenomena that occur in its exterior field. In the first place, we have the occurrence of perturbations of the electromagnetic field and the emission of electromagnetic waves when even neutral matter moves in this field,[127] and there is also a mutual transformation of electromagnetic and gravitational waves, which occurs with appreciable intensity if these waves propagate in the immediate neighborhood of the charged black hole.[121,125]

Many details of these phenomena can be described on the basis of Eqs. (153)–(155) when their high-frequency solutions are considered.[121,124] For high-frequency perturbations, i.e., perturbations that oscillate rapidly compared with the more smoothly varying background, the terms of these equations containing derivatives of the components of the perturbed field are much larger in magnitude than the terms that do not contain derivatives or contain derivatives of lower order. Omitting in Eqs. (153)–(155) the terms of lowest order and introducing the variables

$$\Psi = \Psi^{\text{B}}_0;\ \Phi = \bar{\rho}^{-1}\delta\varnothing_{(k)},$$

we can reduce these equations in the considered approximation to two second-order equations[121]:

$$L\Psi - 2\bar{\rho}\bar{\varnothing}_1\Phi = I_0;\ L\Phi + \frac{\varkappa}{2\pi}\bar{\rho}^{-1}\varnothing_1\delta\bar{\delta}\Psi = J_2 \equiv \bar{\rho}^{-1}\delta J_1, \tag{164}$$

where $L = D\Delta - \delta\bar{\delta} + (-4\gamma + \mu)D + (-3\varepsilon + \bar{\varepsilon} - 4\rho - \bar{\rho})\Delta + (4\alpha - \pi)\delta + (\bar{\alpha} + 3\beta - \bar{\pi} + 4\tau)\bar{\delta}$. Note that the operator L in the principal terms is the same as the operator in the Teukolsky equation (150) for the Kerr field, and therefore admits separation of the variables. This leads to the same equation for the eigenvalues (160) for the angular harmonics. The operator $\delta\bar{\delta}$ in the highest terms (those containing second derivatives) is the same as the angular part of the operator L taken with opposite sign, and therefore $\delta\bar{\delta}$ in the second equation of (164) can be replaced by the angular part of the operator L taken with a minus sign. After expansion of the unknown functions with respect to the spheroidal harmonics ${}_sS^k_{\omega n}(\theta)$ and Fourier transformations with respect to φ and t, the angular part of the operator $\delta\bar{\delta}$ reduces to multiplication by $\rho\bar{\rho}\lambda/2$, where $\lambda = {}_sA^k_{\omega n} - 2a\omega n + a^2\omega^2 - s(s+1)$, and ${}_sA^k_{\omega n}$ is the eigenvalue for the harmonic ${}_sS^k_{\omega n}(\theta)$.[28)]

After these transformations, the variables in (164) still do not separate because of the presence of the factors $\rho\phi_1$ and $\bar{\rho}\bar{\phi}_1$, but if we restrict the treatment to a region of space around the equator near the black hole and extending up to the polar axis far from the black hole and determined by the condition

$$a\cos\theta| \ll r \tag{165}$$

(for small a, this condition is obviously satisfied identically), the dependence on θ in these factors can be ignored and the variables separate completely. In addition, under the condition (165) the system (164) after

---

[27)]The results of a numerical analysis of this phenomenon are given in Ref. 107.

[28)]For high-frequency waves, $|n| \gg 1$, $|\omega m| \gg 1$. In addition, under the conditions

$$n^2 - a^2\omega^2 \gg 1;\ k \ll |n| \tag{166}$$

the angular harmonics are concentrated near the equatorial plane and decrease rapidly as one moves away from it. (Precisely these conditions are satisfied by the frequencies emitted by, for example, ultrarelativistic particles in circular orbits near the black hole, which results in the synchrotron nature of their emission.) At the same time,

$${}_sA^h_{n\omega} = n^2 + \sqrt{n^2 - a^2\omega^2}(2k+1) + O(n^0, \omega^0) = n^2 + O(n, \omega)$$

and therefore $\lambda \approx (n - a\omega^2)$.

the introduction of the new variables

$$Y_{\pm} = \Psi \mp (c^2/\sqrt{G\lambda})\Phi$$

decomposes into two independent equations for Y_+ and Y_-:

$$(L \pm (q/r^3)\sqrt{\lambda})Y_{\pm} = I_{\pm},$$

where

$$I_{\pm} = I_0 \mp (c^2/\sqrt{G\lambda})J_2. \quad (167)$$

Under the same assumptions in a Kerr–Newman field, analogous equations can be obtained for the two other functions $\tilde{Y}_{\pm} = \tilde{\Psi} \mp (c^2/\sqrt{G\lambda})\tilde{\Phi}$, where

$$\tilde{\Psi} = \rho^{-4}\Psi_4^B;\ \tilde{\Phi} = \rho^{-4}\bar{\delta}\varnothing_{(v)};$$
$$\varnothing_{(v)} = (\varkappa/4\pi)[(\Delta + 2\gamma + 3\mu)\varnothing_2^B + 2\varnothing_1\nu^B].$$

The radial equations obtained from the separation of the variables can be conveniently described by introducing the parameter s [$s = 2$ for $Y_{\pm}$ in (167) and $s = -2$ for $\tilde{Y}_{\pm}$]:

$$(\mathscr{L} \mp 2q\sqrt{\lambda}/r)Y_{\pm}^s = (T_{\pm}^s)_{\omega n}^k, \quad (168)$$

where $\mathscr{L}$ has the same form as the operator in the Teukolsky equations (159) for the Kerr field, differing only in the expression for $\Delta(\Delta = r^2 - 2mr + a^2 + q^2)$.

The energy flux transported by waves that go out to infinity or come into the black hole from infinity is equal to the sum of the corresponding fluxes calculated in accordance with Eqs. (161), since far from the charged black hole these waves do not interact but propagate independently of one another. For these fluxes, we can obtain

$$\frac{d^2}{dtd\Omega}E^{\text{in}} = \lim_{r\to\infty}\frac{c^5}{G}\frac{r^2}{64\pi\omega^2}(|Y_+^{s=2}|^2 + |Y_-^{s=2}|^2);$$
$$\frac{d^2}{dtd\Omega}E^{\text{out}} = \lim_{r\to\infty}\frac{c^5}{G}\frac{r^{-6}}{4\pi\omega^2}(|Y_+^{s=-2}|^2 + |Y_-^{s=-2}|^2).$$

Everywhere except in small neighborhoods of the turning points of the potential barrier in (168), more precisely for $|r - r_{\text{t.p.}}| \lesssim |\omega|^{-1/2}$, the solutions of the homogeneous radial equations (168) can be constructed by the WKB method; in the small neighborhoods, they must be matched to the Airy or parabolic cylindrical functions that approximate the solution in these regions. The solutions of the inhomogeneous equations (167) are constructed in the usual manner by means of Green's functions. Under the conditions (166), the angular harmonics can be expressed in terms of Jacobi polynomials.[121]

Equations (168) can be used to analyze the propagation of high-frequency electromagnetic and gravitational radiation and the mutual transformation of these waves near the equatorial plane of a charged rotating black hole,[29)] and also to investigate the properties of the synchrotron emission of ultrarelativistic particles moving in the equatorial plane of this black hole.[121 30)]

Propagation of High-Frequency Waves in an Arbitrary External Electrovac Gravitational Field. We consider first, in the approximation of geometrical optics, the behavior of high-frequency perturbations of the electromagnetic and gravitational fields on the background of an arbitrary electrovac gravitational field. By high-frequency waves, one usually means in this connection small perturbations of the background field that vary appreciably over distances much shorter than the characteristic inhomogeneity distances of the background field. Thus, in the case of high-frequency waves we have a small parameter—the ratio of the characteristic inhomogeneity lengths of the perturbations and the background, and solutions can therefore be constructed in the form of expansions in powers of this small parameter.[94,125]

The high-frequency perturbation χ^B of some field component χ is represented in the approximation of geometrical optics in the form

$$\chi^B = \hat{\chi}\exp(i\omega S), \quad (169)$$

where the dimensionless constant parameter $\omega \gg 1$ is the wave frequency multiplied by some characteristic inhomogeneity length of the background, $\omega S(x^i)$ is the rapidly varying phase of the wave, and $\hat{\chi}$ is its smoothly varying amplitude.

We shall consider the propagation of the waves in the linear approximation in the perturbations and ignore their back reaction on the background field[31)] and assume that in the considered region there are no sources of the perturbations.

Since the perturbations are taken to be small, we shall assume that their amplitudes $\hat{\chi}$ (see Eq. (169)) are $O(1/\omega)$.[32)]

In the first approximation in ω, from the perturbed Maxwell equations (A.38)–(A.41) we obtain

29) This phenomenon of mutual transformation of electromagnetic and gravitational waves, which occurs when a wave packet enters a region of strong homogeneous magnetic field, was first pointed out by Gertsenshtein.[133] The mutual transformation of waves in the Kerr–Newman field in the approximation of geometrical optics was first considered in Ref. 125 (see also Ref. 125 for the history of this question). For the approximation of geometrical optics, see the following subsection of the present section.

30) We mention here that in the field of a nonrotating charged black hole (the Reissner–Nordström field) Eqs. (168) for high-frequency waves without any restrictions of the type (165) and (166), as well as the much more complicated systems of equations for waves of arbitrary frequency,[122,123] admit separation of the variables and can in principle be used to analyze the behavior of perturbations. However, this problem for the Reissner–Nordström field can be solved by a method analogous to the one used in Refs. 128 and 129 for the Schwarzschild field; this method is more cumbersome but leads to a simpler and more complete solution.[130–132]

31) "Warping" of the background can be taken into account by defining the effective contribution to the energy–momentum tensor from the high-frequency perturbations (averaged over a distance much greater than the wavelength) and then solving simultaneously the problem of the behavior of the waves and of finding the background field from field equations whose right-hand sides include the effective energy–momentum tensor of the waves.[94]

32) In fact, the value of the negative power of ω is not important; all that counts is the relative order of the different quantities.

$$\begin{cases} S_l\varnothing_1 - S_{\bar m}\varnothing_0 = 0; \\ S_m\varnothing_1 - S_n\varnothing_0 = 0; \end{cases} \begin{cases} S_l\hat\varnothing_2 - S_{\bar m}\hat\varnothing_1 = 0; \\ S_m\varnothing_2 - S_n\varnothing_1 = 0, \end{cases}$$

where $S_l = l^iS_{,i}, S_n = n^iS_{,i}, S_m = m^iS_{,i}$. These equations have a nontrivial solution for $\hat\phi_0, \hat\phi_1, \hat\phi_2$ only if $S_lS_n - S_mS_{\bar m} = 0$, i.e., in accordance with (3) the wave fronts $S(x^i) = \text{const}$ are null surfaces. The same conclusion can be drawn from the Bianchi identities (A.1), (A.4), (A.7), and (A.8) for the gravitational waves.

We choose a null tetrad in such a way that the vector l^i is tangent to the null geodesics (rays) on the surfaces $S(x^i) = \text{const}$ and the complete tetrad is parallelly transported along the rays (see Sec. 1). Then for this tetrad

$$\varkappa = \varepsilon = \pi = 0; \ \rho = \bar\rho \qquad (170)$$

and, in addition, $DS = 0, \delta S = 0, \bar\delta S = 0$, since $l^i, m^i, \overline{m}^i$ are tangent to the fronts. When the operators $D, \delta, \bar\delta$ are applied to a function of the form (169), they do not raise its order in ω. And since $\Delta S = n^il_i = 1$, in the principal approximation in ω the action of the operator Δ on (169) reduces to multiplication by $i\omega$.

We now consider the system of Newman–Penrose equations (A.1)–(A.11), (A.12)–(A.29) and the Maxwell equations (A.38)–(A.41), linearizing both systems with respect to the perturbations. Because all amplitudes of the perturbations are $O(1/\omega)$, and none of the operators except Δ raise the order of a perturbed function in ω, the amplitudes of all quantities in these equations to which Δ is applied must have a higher order, namely $O(1/\omega^2)$. Thus, from (A.1)–(A.11),

$$\hat\Psi_0, \hat\Psi_1, \hat\Psi_2, \hat\Psi_3, \hat\Phi_{00}, \hat\Phi_{10}, \hat\Phi_{20}, \hat\Phi_{01}, \hat\Phi_{11}, \hat\Phi_{21} \sim O(1/\omega^2). \qquad (171)$$

From Eqs. (A.12)–(A.29), we obtain similarly

$$\hat\varkappa, \hat\varepsilon, \hat\pi, \hat\lambda, \hat\mu, \hat\alpha, \hat\beta, \hat\sigma, \hat\rho, \sim O(1/\omega^2), \qquad (172)$$

and from the Maxwell equations (A.38)–(A.41), using (172),

$$\varnothing_0, \varnothing_1, \varnothing_2 \sim O(1/\omega^2). \qquad (173)$$

In addition, considering the perturbations of the commutators (1.6) applied to some scalar function, we find

$$\hat D, \hat\delta, \hat{\bar\delta}, \sim O(1/\omega^2). \qquad (174)$$

Therefore, only the amplitudes $\hat\Psi_4, \hat\tau, \hat\gamma, \hat\nu, \hat\Delta$ can be $O(1/\omega)$.

Before we consider the propagation of waves on an arbitrary electrovac background, we consider the much simpler case when there is no electromagnetic field of the background.

Geometrical Optics in an Arbitrary Vacuum Gravitational Field. From Eqs. (A.7) and (A.40), using (170)–(174) on the background of a vacuum gravitational field, i.e., under the condition $\Phi_{inn} = 0$ for the background, we can obtain in the approximation $\sim 1/\omega$ the two independent equations

$$(D-\rho)\hat\Psi_4 = 0; \ (D-\rho)\hat\varnothing_2 = 0, \qquad (175)$$

which describe, respectively, gravitational and electromagnetic waves and are the ordinary equations of "transfer of the amplitude" of waves along rays in the approximation of geometrical optics written in a somewhat different form.[94]

In accordance with (170), $\rho = \bar\rho = -\nabla_i l^i/2$, so that from Eqs. (175) we obtain

$$l^i\nabla_i\hat\chi + (1/2)(\nabla_i l^i)\hat\chi = 0 \quad \text{or} \quad \nabla_i(l^i|\hat\chi|^2) = 0, \ l^i\nabla_i(\arg\hat\chi) = 0, \qquad (176)$$

where $\hat\chi$ is either $\hat\Psi_4$ or $\hat\phi_2$. The last of the equations means that as the waves propagate their polarization state remains the same relative to the tetrad parallelly transported along the rays (the real and imaginary parts of $\hat\chi$ describe the different polarization states).

Since l^i is a tangent vector parallelly transported along the rays, the operator $D = d/dr$ is the total derivative along the rays with respect to their affine parameter r, and if we introduce the "luminosity parameter" $\boldsymbol{r}$,[4], which is defined by the equation $d\boldsymbol{r}/dr = -\rho\boldsymbol{r}$, then from (176) we obtain

$$|\hat\chi|^2 = A/\boldsymbol{r}^2, \text{ where } dA/dr = 0. \qquad (177)$$

The quantity $|\hat\chi|^2$ characterizes the energy flux in the wave that arrives on unit area of the section of some thin ray tube, and $\boldsymbol{r}^2$ in the null geodesic coordinate system (see Sec. 1) is equal to $\sqrt{g}$ and therefore determines the change in the area of the section of this tube as one moves along the rays. Thus, (177) is simply the analog of the well-known inverse square law for wave intensities expressing the energy conservation law of the wave.

It follows from (177), in particular, that if the rays approach a caustic and, therefore, $\boldsymbol{r} = {}^4\sqrt{g} \to 0$ (nearby rays begin to intersect), the wave amplitudes increase without limit. However, near the caustic the approximation of geometrical optics already breaks down [since the basic assumption (169) is violated], and other approximate methods must be used here.[134] A more detailed investigation shows[125] that the wave amplitudes remain bounded near caustics.

Propagation of Waves in an Arbitrary Electrovac Gravitational Field. Mutual Transformation of Electromagnetic and Gravitational Waves. On the background of an arbitrary electrovac gravitational field, Eqs. (A.7) and (A.40), linearized with respect to the perturbations in the approximation $\sim 1/\omega$, contain in addition to the amplitudes $\hat\Psi_4$ and ϕ_2 the quantities $\hat\tau, \hat\nu, \hat\gamma$ and $\hat\Delta$, which, as we have said above, can be $O(1/\omega)$ and cannot be made to vanish simultaneously by small rotations of the basis tetrad in the perturbed space. However, one can find combinations of these variables with respect to which the linearized equations (A.7) and (A.40) in the approximation $\sim 1/\omega$ form a closed system. Indeed, using the Maxwell equations for the perturbations, we obtain from (A.7)

$$(D-\rho)\hat\Psi + (\varkappa/4\pi)\bar\varnothing_0\hat\Phi = 0, \qquad (178)$$

where $\psi = \hat\psi_4^B$, and $\hat\Phi$ is the amplitude of the quantity $\Phi = [(\Delta + 2\gamma)\phi_2 - 2\nu\phi_1]^B$. A second equation for Ψ and Φ follows from (A.40), to which the operator Δ is applied, after which the perturbations are superimposed. By virtue of the commutation relations (6), and also (A.21) and the remaining Maxwell equations, we obtain in the $\sim 1/\omega$ approximation

$$(D-\rho)\hat{\Phi}-\varnothing_0\hat{\Psi}=0. \quad (179)$$

It is interesting to note that $\hat{\Phi}$ and $\hat{\Psi}$ in the closed system (178)–(179) are invariant in the considered approximation under small rotations of the null tetrad (46) belonging to the groups A and B.

Equations (178) and (179) describe the interaction of gravitational and electromagnetic waves in the presence of a background electromagnetic field. In the absence of a background electromagnetic field, this interaction vanishes in the considered approximation, which is expressed by the independence of the equations in (175). Equations (178) and (179) are equivalent to the equations obtained in Ref. 125 in a study of high-frequency perturbations of the metric for a special choice of the coordinates in the perturbed space.

In the general case, the interaction of these waves has the consequence that the arguments of the complex functions $\hat{\Psi}$ and $\hat{\Phi}$ do not remain constant along the rays, and there is therefore a rotation of the polarization planes of the wave relative to a parallelly transported tetrad. In addition, for the electromagnetic and gravitational waves separately an energy conservation law of the form (176) no longer holds; there is now such a law only for the total energy transmitted by waves of both types.[125] To see this, we note that from (178) and (179)

$$\nabla_i\,[l^i\,(|\hat{\Psi}|^2+(\varkappa/4\pi)\,|\hat{\Phi}|^2)]=0.$$

The distribution of this energy between the electromagnetic (the function Φ) and gravitational (the function Ψ) components of the wave changes periodically along the ray, i.e., the two components are continuously being transformed into one another.

There is an interesting special case[125] when the background field or the ray trajectories are such that the argument of the complex function ϕ_0, which describes the electromagnetic field of the background in (178) and (179), is constant along the rays.[33)] In this case, we can introduce the new variables

$$\hat{Y}_\pm=\hat{\Psi}\mp i\,[\sqrt{\varkappa/4\pi}\exp(-i\arg\varnothing_0)]\hat{\Phi}, \quad (180)$$

for which (178) and (179) decompose into two independent equations[34)]:

$$(D-\rho\pm i\sqrt{\varkappa/4\pi}\,|\varnothing_0|)\,\hat{Y}_\pm=0. \quad (181)$$

Again, as in (176), introducing the luminosity parameter r along the rays, we write Eqs. (181) in the form $d(r\hat{Y}_\pm)/dr=\mp\sqrt{\varkappa/4\pi}\,|\phi_0|(r\hat{Y}_\pm)$, from which we obtain

$$\hat{Y}_\pm=\left(\frac{B_\pm}{r}\right)\exp\left[\mp i\sqrt{\frac{\varkappa}{4\pi}}\int_{r_0}^{r}|\varnothing_0|\,dr\right], \quad (182)$$

where $B_\pm$ do not depend on the parameter r along the rays. After this, from (180) we can readily find the gravitational and electromagnetic components:

$$\hat{\Psi}=(\hat{Y}_+ + \hat{Y}_-)/2;\ \exp(-i\arg\varnothing_0)\hat{\Phi}=(i/2)(\hat{Y}_+-\hat{Y}_-)\sqrt{4\pi/\varkappa}. \quad (183)$$

If a source at the point of the ray with $r=r_0$ emits, for example, only electromagnetic waves, then in (182) $B_+=-B_-=iB$ and from (183) we obtain

$$\hat{\Psi}=\left(\frac{-B}{r}\right)\sin\left(\sqrt{\frac{\varkappa}{4\pi}}\int_{r_0}^{r}|\varnothing_0|\,dr\right);$$

$$\hat{\Phi}=\sqrt{\frac{4\pi}{\varkappa}}\exp(i\arg\varnothing_0)\frac{B}{r}\cos\left(\sqrt{\frac{\varkappa}{4\pi}}\int_{r_0}^{r}|\varnothing_0|\,dr\right).$$

Similar expressions can also be obtained for waves from a source of purely gravitational perturbations. These expressions clearly demonstrate how the transformation of the waves takes place, namely, the amplitudes of the electromagnetic and gravitational components of the high-frequency wave are modulated, when the wave reaches the region of the strong electromagnetic field, by smooth sinusoids with phases of the form

$$\sqrt{\frac{\varkappa}{4\pi}}\int|\varnothing_0|\,dr,$$

shifted by a quarter period.

An appreciable intensity of this process can be achieved in the immediate neighborhood of a charged black hole. In particular, if some localized packet of, for example, gravitational waves approaches a charged black hole, makes a few orbits around it (moving near the photon circular orbit), and then goes off again to infinity, it may happen that as a result of the transformations of the waves this packet becomes "visible," i.e., is transformed into a packet of purely electromagnetic waves (without distortion of the original form), having undergone during the process of motion complete (possibly multiple) transformations of its components.[121]

CONCLUSIONS

We have considered by no means the full gamut of problems in general relativity in which the Newman–Penrose formalism or individual elements of it in conjunction with other methods of different nature are widely used. The numerous helpful applications of this formalism indicate that it is one of the most powerful methods of modern gravitational theory.

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33) Precisely this property is possessed by all rays in the field of a nonrotating charged black hole – the Reissner–Nordström field – and also rays that move near the equatorial plane in the field of a rotating charged black hole (the Kerr–Newman field).

34) Such a decomposition already occured in Eqs. (167) for high-frequency waves in the Kerr–Newman field. With regard to Eqs. (167) and (181), we note here that the approximation (181) and (182) of geometrical optics is suitable only outside the neighborhoods of the caustic surfaces for the rays (or, in the case of separation of the variables, outside the neighborhoods of the turning points for the radial and angular potential barriers). Equations (167), in contrast to (181), do apply for the description of perturbations in the whole of space, including the neighborhoods of caustics and regions "under the potential barrier," which makes it possible to use (167) to describe the coupling of the sources to the wave field, i.e., to determine the radiation emitted by the sources.[121,124]

35) *Translator's note.* A misprint in the Russian text, of k instead of κ, has been corrected in the main body of the text but is not corrected in the equations of the Appendices in order to simplify the composition.

APPENDICES

A. First group of Newman-Penrose equations:

$$(D-4\rho-2\varepsilon)\Psi_1-(\bar{\delta}-4\alpha+\pi)\Psi_0+3k\Psi_2+(\delta-2\beta-2\bar{\alpha}+\bar{\pi})\Phi_{00}$$
$$-(D-2\bar{\rho}-2\varepsilon)\Phi_{01}-2k\Phi_{11}+2\sigma\Phi_{10}-\bar{k}\Phi_{02}=0; \quad (A.1)$$

$$(D-3\rho)\Psi_2-(\bar{\delta}+2\pi-2\alpha)\Psi_1+2k\Psi_3+\lambda\Psi_0+(\delta-2\bar{\alpha}+\bar{\pi})\Phi_{10}$$
$$-(D-2\bar{\rho})\Phi_{11}-k\Phi_{21}-\bar{k}\Phi_{12}-\mu\Phi_{00}+\pi\Phi_{01}+\sigma\Phi_{20}-D\Lambda=0; \quad (A.2)$$

$$(D-2\rho+2\varepsilon)\Psi_3-(\bar{\delta}+3\pi)\Psi_2+2\lambda\Psi_1+k\Psi_4+(\delta-2\bar{\alpha}+2\beta+\bar{\pi})\Phi_{20}$$
$$-(D-2\bar{\rho}+2\varepsilon)\Phi_{21}-2\mu\Phi_{10}+2\pi\Phi_{11}-\bar{k}\Phi_{22}-2\bar{\delta}\Lambda=0; \quad (A.3)$$

$$(\delta-4\tau-2\beta)\Psi_1-(\Delta-4\gamma+\mu)\Psi_0+3\sigma\Psi_2+(\delta-2\beta+2\bar{\pi})\Phi_{01}$$
$$-(D-2\varepsilon+2\bar{\varepsilon}-\bar{\rho})\Phi_{02}-2k\Phi_{12}+2\sigma\Phi_{11}-\bar{\lambda}\Phi_{00}=0; \quad (A.4)$$

$$(\delta-3\tau)\Psi_2-(\Delta+2\mu-2\gamma)\Psi_1+2\sigma\Psi_3+\nu\Psi_0+(\delta+2\bar{\pi})\Phi_{11}$$
$$-(D+2\bar{\varepsilon}-\bar{\rho})\Phi_{12}-k\Phi_{22}-\mu\Phi_{01}+\pi\Psi_{02}+\sigma\Phi_{21}-\bar{\lambda}\Phi_{10}-\delta\Lambda=0; \quad (A.5)$$

$$(\delta+2\beta-2\tau)\Psi_3-(\Delta+3\mu)\Psi_2+2\nu\Psi_1+\sigma\Psi_4+(\delta+2\beta+2\bar{\pi})\Phi_{21}$$
$$-(D+2\varepsilon+2\bar{\varepsilon}-\bar{\rho})\Phi_{22}-2\mu\Phi_{11}+2\pi\Phi_{12}-\bar{\lambda}\Phi_{20}-2\Delta\Lambda=0; \quad (A.6)$$

$$(D+4\varepsilon-\rho)\Psi_4-(\bar{\delta}+4\pi+2\alpha)\Psi_3+3\lambda\Psi_2+(\Delta+2\gamma-2\bar{\gamma}+\bar{\mu})\Phi_{20}$$
$$-(\bar{\delta}+2\alpha-2\bar{\tau})\Phi_{21}-2\nu\Phi_{10}+2\lambda\Phi_{11}-\bar{\sigma}\Phi_{22}=0; \quad (A.7)$$

$$(\delta+4\beta-\tau)\Psi_4-(\Delta+2\gamma+4\mu)\Psi_3+3\nu\Psi_2+(\Delta+2\gamma+2\bar{\mu})\Phi_{21}$$
$$-(\bar{\delta}+2\alpha+2\bar{\beta}-\bar{\tau})\Phi_{22}-2\nu\Phi_{11}+2\lambda\Phi_{12}-\bar{\nu}\Phi_{20}=0; \quad (A.8)$$

$$(D-2\rho-2\bar{\rho})\Phi_{11}-(\delta-2\bar{\alpha}-2\tau+\bar{\pi})\Phi_{10}-(\bar{\delta}-2\bar{\tau}-2\alpha+\pi)\Phi_{01}$$
$$+(\Delta-2\gamma-2\bar{\gamma}+\mu+\bar{\mu})\Phi_{00}+\bar{k}\Phi_{12}+k\Phi_{21}-\bar{\sigma}\Phi_{02}-\sigma\Phi_{20}+3D\Lambda=0; \quad (A.9)$$

$$(D-2\rho+2\bar{\varepsilon}-\bar{\rho})\Phi_{12}-(\delta+2\bar{\pi}-2\tau)\Phi_{11}-(\bar{\delta}+2\bar{\beta}-2\alpha-\bar{\tau}+\pi)\Phi_{02}$$
$$+(\Delta+2\bar{\mu}-2\gamma+\mu)\Phi_{01}+k\Phi_{22}-\bar{\nu}\Phi_{00}+\bar{\lambda}\Phi_{10}-\sigma\Phi_{21}+3\delta\Lambda=0; \quad (A.10)$$

$$(D+2\bar{\varepsilon}+2\varepsilon-\rho-\bar{\rho})\Phi_{22}-(\bar{\delta}+2\bar{\beta}+2\pi-\bar{\tau})\Phi_{12}-(\delta+2\bar{\pi}$$
$$+2\beta-\tau)\Phi_{21}+(\Delta+2\bar{\mu}+2\mu)\Phi_{11}-\bar{\nu}\Phi_{10}-\nu\Phi_{01}+\bar{\lambda}\Phi_{20}+\lambda\Phi_{02}+3\Delta\Lambda=0. \quad (A.11)$$

B. Second group of Newman-Penrose equations:

$$\Phi_{00}=(D-\rho-\varepsilon-\bar{\varepsilon})\rho-(\bar{\delta}-3\alpha-\bar{\beta}+\pi)k-\sigma\bar{\sigma}+\tau\bar{k}; \quad (A.12)$$
$$\Psi_0=(D-\rho-\bar{\rho}-3\varepsilon+\bar{\varepsilon})\sigma-(\delta-\tau+\bar{\pi}-\bar{\alpha}-3\beta)k; \quad (A.13)$$
$$\Psi_1+\Phi_{01}=(D-\rho-\varepsilon+\bar{\varepsilon})\tau-(\Delta-3\gamma-\bar{\gamma})k-\rho\bar{\pi}-\sigma\bar{\tau}-\pi\sigma; \quad (A.14)$$
$$\Phi_{10}=(D-\rho-\bar{\varepsilon}+2\varepsilon)\alpha-(\bar{\delta}-\bar{\beta}+\pi)\varepsilon-\beta\bar{\sigma}+k\lambda+\bar{k}\gamma-\pi\rho; \quad (A.15)$$
$$\Psi_1=(D-\bar{\rho}+\bar{\varepsilon})\beta-(\delta-\bar{\alpha}+\bar{\pi})\varepsilon-(\alpha+\pi)\sigma+(\mu+\gamma)k; \quad (A.16)$$
$$-\Lambda+\Psi_2+\Phi_{11}=(D+\varepsilon+\bar{\varepsilon})\gamma-(\Delta-\gamma-\bar{\gamma})\varepsilon-(\tau+\bar{\pi})\alpha-(\pi+\bar{\tau})\beta-\pi\tau+\nu k;$$
$$(A.17)$$
$$\Phi_{20}=(D-\rho+3\varepsilon-\bar{\varepsilon})\lambda-(\bar{\delta}+\pi+\alpha-\bar{\beta})\pi-\mu\bar{\sigma}+\nu\bar{k}; \quad (A.18)$$
$$2\Lambda+\Psi_2=(D-\bar{\rho}+\varepsilon+\bar{\varepsilon})\mu-(\delta+\bar{\pi}-\bar{\alpha}+\beta)\pi-\sigma\lambda+\nu k; \quad (A.19)$$
$$\Psi_3+\Phi_{21}=(D+3\varepsilon+\bar{\varepsilon})\nu-(\Delta+\mu+\gamma-\bar{\gamma})\pi-\mu\bar{\tau}-(\bar{\pi}+\tau)\lambda; \quad (A.20)$$
$$-\Psi_4=(\Delta+\mu+\bar{\mu}+3\gamma-\bar{\gamma})\lambda-(\bar{\delta}+3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu; \quad (A.21)$$
$$-\Psi_1+\Phi_{01}=(\delta-\bar{\alpha}-\beta-\tau)\rho-(\bar{\delta}-3\alpha+\bar{\beta})\sigma+\tau\bar{\rho}-(\mu-\bar{\mu})k; \quad (A.22)$$
$$\Lambda-\Psi_2+\Phi_{11}=(\delta-\bar{\alpha}+2\beta)\alpha-(\bar{\delta}+\bar{\beta})\beta-\mu\rho+\sigma\lambda-(\rho-\bar{\rho})\gamma-\varepsilon(\mu-\bar{\mu}); \quad (A.23)$$
$$-\Psi_3+\Phi_{21}=(\delta-\bar{\alpha}+3\beta)\lambda-(\bar{\delta}+\pi+\alpha+\bar{\beta})\mu-(\rho-\bar{\rho})\nu+\pi\bar{\mu}; \quad (A.24)$$
$$\Phi_{22}=(\delta-\tau+3\beta+\bar{\alpha})\nu-(\Delta+\mu+\gamma+\bar{\gamma})\mu-\lambda\bar{\lambda}+\pi\bar{\nu}; \quad (A.25)$$
$$\Phi_{12}=(\delta-\tau+\bar{\alpha}+\beta)\gamma-(\Delta-\gamma+\bar{\gamma}+\mu)\beta-\mu\tau+\sigma\nu+\varepsilon\bar{\nu}-\alpha\bar{\lambda}; \quad (A.26)$$
$$\Phi_{02}=(\delta-\tau-\beta+\bar{\alpha})\tau-(\Delta+\mu-3\gamma+\bar{\gamma})\sigma-\bar{\lambda}\rho+k\bar{\nu}; \quad (A.27)$$
$$-2\Lambda-\Psi_2=(\Delta+\bar{\mu}-\gamma-\bar{\gamma})\rho-(\bar{\delta}+\bar{\beta}-\alpha-\bar{\tau})\tau+\sigma\lambda-\nu k; \quad (A.28)$$
$$-\Psi_3=(\Delta-\bar{\gamma}+\bar{\mu})\alpha-(\bar{\delta}+\bar{\beta}-\bar{\tau})\gamma-(\rho+\varepsilon)\nu+(\tau+\beta)\lambda. \quad (A.29)$$

C. Third group of Newman-Penrose equations (coordinate equations):

$$\delta V-D\omega=(\bar{\alpha}+\beta-\bar{\pi})V+kU-\sigma\bar{\omega}-(\bar{\rho}+\varepsilon-\bar{\varepsilon})\omega; \quad (A.30)$$
$$\delta Y^{\alpha}-D\xi^{\alpha}=(\bar{\alpha}+\beta-\bar{\pi})Y^{\alpha}+kX^{\alpha}-\sigma\bar{\xi}^{\alpha}-(\bar{\rho}+\varepsilon-\bar{\varepsilon})\xi^{\alpha}; \quad (A.31)$$
$$\Delta Y^{\alpha}-DX^{\alpha}=(\gamma+\bar{\gamma})Y^{\alpha}+(\varepsilon+\bar{\varepsilon})X^{\alpha}-(\tau+\bar{\pi})\bar{\xi}^{\alpha}-(\bar{\tau}+\pi)\xi^{\alpha}; \quad (A.32)$$
$$\Delta V-DU=(\gamma+\bar{\gamma})V+(\varepsilon+\bar{\varepsilon})U-(\tau+\bar{\pi})\bar{\omega}-(\bar{\tau}+\pi)\omega; \quad (A.33)$$
$$\delta U-\Delta\omega=-\bar{\nu}V+(\tau-\bar{\alpha}-\beta)U+\bar{\lambda}\bar{\omega}+(\mu-\gamma+\bar{\gamma})\omega; \quad (A.34)$$
$$\delta X^{\alpha}-\Delta\xi^{\alpha}=-\bar{\nu}Y^{\alpha}+(\tau-\bar{\alpha}-\beta)X^{\alpha}+\bar{\lambda}\bar{\xi}^{\alpha}+(\mu-\gamma+\bar{\gamma})\xi^{\alpha}; \quad (A.35)$$
$$\bar{\delta}\omega-\delta\bar{\omega}=(\bar{\mu}-\mu)V+(\bar{\rho}-\rho)U-(\bar{\alpha}-\beta)\bar{\omega}-(\bar{\beta}-\alpha)\omega; \quad (A.36)$$
$$\bar{\delta}\xi^{\alpha}-\delta\bar{\xi}^{\alpha}=(\bar{\mu}-\mu)Y^{\alpha}+(\bar{\rho}-\rho)X^{\alpha}-(\bar{\alpha}-\beta)\bar{\xi}^{\alpha}-(\bar{\beta}-\alpha)\xi^{\alpha}. \quad (A.37)$$

D. Maxwell equations in the Newman-Penrose formalism:

$$(D-2\rho)\varnothing_1-(\bar{\delta}-2\alpha+\pi)\varnothing_0+k\varnothing_2=(-2\pi/c)J_{(l)}; \quad (A.38)$$
$$(\delta-2\tau)\varnothing_1-(\Delta-2\gamma+\mu)\varnothing_0+\sigma\varnothing_2=-(2\pi/c)J_{(m)}; \quad (A.39)$$
$$(D+2\varepsilon-\rho)\varnothing_2-(\bar{\delta}+2\pi)\varnothing_1+\lambda\varnothing_0=-(2\pi/c)J_{(\bar{m})}; \quad (A.40)$$
$$(\delta+2\beta-\tau)\varnothing_2-(\Delta+2\mu)\varnothing_1+\nu\varnothing_0=-(2\pi/c)J_{(n)}. \quad (A.41)$$

For the projections of the four-current, the following notation is adopted:

$$J_{(l)}=J_il^i; \quad J_{(m)}=J_im^i; \quad J_{(\bar{m})}=J_i\bar{m}^i; \quad J_{(n)}=J_in^i. \quad (A.42)$$

E. Transformations of the tetrad components of the Weyl and Maxwell tensors under Lorentz rotations of the null tetrad:

$$\Psi_0'=G^2\exp(2iH)\Psi_0; \quad (A.43)$$
$$\Psi_1'=G\exp(iH)\Psi_1; \quad (A.44)$$
$$\Psi_2'=\Psi_2; \quad (A.45)$$
$$\Psi_3'=G^{-1}\exp(-iH)\Psi_3, \quad (A.46)$$
$$\Psi_4'=G^{-2}\exp(-2iH)\Psi_4; \quad (A.47)$$
$$\Psi_0'=\Psi_0; \quad (A.48)$$
$$\Psi_1'=\Psi_1+\bar{A}\Psi_0; \quad (A.49)$$
$$\Psi_2'=\Psi_2+2\bar{A}\Psi_1+\bar{A}^2\Psi_0; \quad (A.50)$$
$$\Psi_3'=\Psi_3+3\bar{A}\Psi_2+3\bar{A}^2\Psi_1+\bar{A}^3\Psi_0; \quad (A.51)$$
$$\Psi_4'=\Psi_4+4\bar{A}\Psi_3+6\bar{A}^2\Psi_2+4\bar{A}^3\Psi_1+\bar{A}^4\Psi_0. \quad (A.52)$$
$$\varnothing_0'=G\exp(iH)\varnothing_0; \quad (A.53)$$
$$\varnothing_1'=\varnothing_1; \quad (A.54)$$
$$\varnothing_2'=G^{-1}\exp(-iH)\varnothing_2; \quad (A.55)$$
$$\varnothing_0'=\varnothing_0; \quad (A.56)$$
$$\varnothing_1'=\varnothing_1+\bar{A}\varnothing_0; \quad (A.57)$$
$$\varnothing_2'=\varnothing_2+2\bar{A}\varnothing_1+\bar{A}^2\varnothing_0. \quad (A.58)$$

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