Dynamics of a relativistic string

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The classical and quantum dynamics of a relativistic string that is either free or has charges and masses at its end is reviewed. The general solution of the Cauchy problem for the equations of motion of the string, and also a number of particular solutions, are discussed. Besides the standard methods of quantization, an approach is presented that makes it possible to construct a relativistically invariant quantum theory of a string without restrictions on the space-time dimension and without tachyon states. The nonrelativistic limit of a string with masses at the ends is investigated. Stringlike solutions in field models are considered and the relation between a relativistic string and the problem of quark confinement in hadrons is discussed. The connection between the nonlinear Born-Infeld models and a string are examined.

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INTRODUCTION

Modern ideas about the structure of hadrons and the mechanism of their interaction have led to the study of a one-dimensional extended object—a relativistic string.

Relativistic strings were introduced for the first time as the dynamical basis of dual resonance models.[1-4] In the dual approach, it is assumed that the hadron spectrum is equidistant and consists of an infinite number of resonances with zero width. This spectrum is generated by a countably infinite set of oscillator creation and annihilation operators $a^{+}_{n\mu}, a_{n\mu}, n=1,2,3,\ldots,$ each of which is a Lorentz vector (see Appendix 1). Such a set of operators can be obtained by quantizing a one-dimensional extended relativistic object of finite length (string). From the point of view of dual models, a direct generalization to the relativistic case of an ordinary linear string is unsuitable, since its quadratic Lagrangian does not yield restrictions on the physical state vectors that could be identified with the Virasoro conditions in dual models. A nonlinear Lagrangian,[5,6] was therefore proposed for a relativistic string.

A relativistic string is also of interest as a model of quark confinement in hadrons, representing a simplification of the corresponding quantum field theory, i.e., quantum chromodynamics, which is currently regarded as the theory which should describe the interaction of quarks in hadrons.[7-10] Indeed, in quantum chromodynamics one can explain the main features of quark behavior, namely, that at short distances they hardly interact with one another but nevertheless cannot exist in a free state outside hadrons.

In chromodynamics, the interaction between quarks is transmitted by Yang-Mills vector meson fields. It turns out that it is energetically advantageous when the fields do not fill all space but are concentrated along lines joining the quarks.[10,11] The energy of two quarks bound by such a tube of gluon field is proportional to the distance between them. Therefore, the forces of attraction between quarks do not decrease with the distance, but remain constant. Thus, no external disturbance can break the bond and lead to the creation of free quarks.

An example of such field configurations already encountered in physics is provided by Abrikosov's magnetic vortices in the Landau-Ginzburg theory of superconductivity.[12,13] When the external magnetic field exceeds a certain critical value, it begins to penetrate into type II superconductors in the form of thin filaments of magnetic lines of force. It should be said that stringlike solutions have not been directly obtained in chromodynamics, but such solutions have been found in a number of simpler field models.[14-19]

A relativistic string with point masses at the ends simulates gluon field configurations localized along the lines connecting quarks. A relativistic string is much simpler than the extremely complicated quantum field model of chromodynamics, but the string model reproduces the main predictions obtained in the field approach.[20,21] Therefore, a relativistic string can be used as a comparatively simple model of a composite hadron that agrees with chromodynamics. In particular, a relativistic string joining two massive particles leads to a potential between them that increases linearly with the distance.

A relativistic string is also of interest as an example of the simplest extended relativistic object.[22]

The present review does not pretend to an exhaustive exposition of the theory of relativistic strings in all aspects. A number of reviews have already been published on this theme.[23-26] Therefore, our main attention is concentrated on questions that have not hitherto been adequately illuminated.

It is a pleasant duty to thank D. I. Blokhintsev and N. A. Chernikov for fruitful discussions on the subject.

1. LAGRANGIAN OF A RELATIVISTIC STRING, EQUATIONS OF MOTION, AND THEIR SOLUTION

Variational Principle. The action of a relativistic string is constructed by analogy with the action of a point particle, which is proportional to the length of the world line of the particle in Minkowski space:

$$S_m = -mc \int_{\tau_1}^{\tau_2} \sqrt{\dot{x}^2(\tau)}\, d\tau. \tag{1}$$

This principle is genralized to a one-dimensional extended object (string) by the assumption that the action of the string is proportional to the area of the work surface which it sweeps out during its motion. If $x_\mu(\sigma, \tau)$ is the parametric specification of this surface in Minkowski space, the action of the string has the form

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$$S=\int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \mathcal{L}_0 = -\gamma \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{(\dot{x}x')^2-\dot{x}^2x'^2}, \quad (2)$$

where

$$\dot{x}_\mu=\partial x_\mu(\sigma,\tau)/\partial\tau,\ x'_\mu=\partial x_\mu(\sigma,\tau)/\partial\sigma.$$

The parameter σ labels the points of the string, and τ plays the part of an evolution parameter, which finds reflection in the conditions $\dot{x}^2\geqslant 0$, $x'^2<0$.[1)] The requirement $\dot{x}^2\geqslant 0$ means that the velocity of the points of the string cannot exceed the velocity of light. Indeed,

$$\dot{x}^2=(\partial t/\partial\tau)^2-(\partial\mathbf{x}/\partial\tau)^2=(\partial t/\partial\tau)^2[1-(\partial\mathbf{x}/\partial t)^2]\geqslant 0,$$

from which it follows that $(\partial\mathbf{x}(\sigma,\tau)/\partial t)^2\leqslant 1$. The constant γ has the dimensions of mass divided by time; the functions $\sigma_i(\tau), i=1,2$, describe the motion of the ends of the string in terms of the coordinates σ and τ.

One can show (see Appendix 2) that the string action (2) goes over into the action (1) of a point particle if the string length tends to zero.

The principle of least action $\delta S=0$ for the string, considered from the geometrical point of view, is tantamount to the solution of the Plateau problem,[27,28] i.e., the problem of finding an extremal surface in space-time $(t,\mathbf{x})$ with fixed initial $x_\mu(\sigma,\tau_1)$ and final $x_\mu(\sigma,\tau_2)$ position of the string. In Minkowski space, the Plateau problem reduces to an equation of hyperbolic type, and not elliptic type, as in Euclidean space.

Variation of the action (2) with the requirement that $\delta x_\mu(\sigma,\tau_1)=\delta x_\mu(\sigma,\tau_2)=0$, leads to the equations of motion

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}_0}{\partial\dot{x}_\mu}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}_0}{\partial x'_\mu}\right)=0 \quad (3)$$

and the boundary conditions

$$\partial\mathcal{L}_0/\partial x'_\mu-(\partial\mathcal{L}_0/\partial\dot{x}_\mu)\dot{\sigma}=0,\ \sigma=\sigma_i(\tau),\ i=1,2. \quad (4)$$

If S is varied with respect to the functions $\sigma_i(\tau)$, $i=1,2$, we obtain

$$\mathcal{L}_0(\sigma_1(\tau),\tau)=\mathcal{L}_0(\sigma_2(\tau),\tau)=0. \quad (5)$$

The string action (2) is invariant under arbitrary replacement of the parameters σ and τ by new parameters $\tilde{\sigma}$ and $\tilde{\tau}$:

$$\tilde{\sigma}=f_1(\sigma,\tau);\ \tilde{\tau}=f_2(\sigma,\tau);\ \partial(\tilde{\sigma},\tilde{\tau})/\partial(\sigma,\tau)\neq 0. \quad (6)$$

These transformations contain the two functions f_i, so that in accordance with the second part of Noether's theorem[30] the left-hand sides of the equations of motion (3) must satisfy two identities, which state that their projections onto the vectors $\dot{x}_\mu$ and x'_μ vanish (see Appendix 3). Therefore, the functions $x_\mu(\sigma,\tau)$ are not completely determined by Eqs. (3). On the required solutions, one can impose two subsidiary conditions, which are conveniently taken to be

$$\dot{x}^2+x'^2=0;\ \dot{x}x'=0, \quad (7)$$

or equivalently

$$(\dot{x}\pm x')^2=0. \quad (7')$$

[1)]We use a metric in which $x^2=x_0^2-\mathbf{x}^2$.

These requirements can be regarded as conditions on the choice of the parameters σ and τ on the world surface of the string (conditions of orthogonal gauge).

With allowance for (7), the equations of motion go over into the d'Alembert equations for the vector $x_\mu(\sigma,\tau)$:

$$\ddot{x}_\mu(\sigma,\tau)-x''_\mu(\sigma,\tau)=0, \quad (8)$$

and the boundary conditions (4) become

$$x'_\mu+\dot{x}_\mu\dot{\sigma}(\tau)=0;\ \sigma=\sigma_i(\tau);\ i=1,2. \quad (9)$$

The requirement (5) is now a simple consequence of the boundary conditions (9) and the subsidiary conditions (7).

The variational principle does not give any equations for the functions $\sigma_i(\tau)$, $i=1,2$, which describe the motion of the ends of the string in the σ,τ plane. Therefore, without loss of generality, we can take $\dot{\sigma}_i(\tau)=0$ and $\sigma_1(\tau)=0$, $\sigma_2(\tau)=l$. The boundary conditions now take the form

$$x'_\mu(0,\tau)=x'_\mu(l,\tau)=0. \quad (10)$$

It would seem that the functions $\sigma_i(\tau)$ could be taken to be constants from the very start. However, in a number of cases, as will be shown below for the example of a string in an electromagnetic field, boundary conditions (4) with functions $\sigma_i(\tau)$ that are not fixed are helpful.

Covariant Formalism. A solution of the equations of motion (8) satisfying the boundary conditions (10) can be obtained in the form of the Fourier series

$$x_\mu(\sigma,\tau)=\frac{i}{\sqrt{\pi\gamma}}\sum_{n\neq 0}\exp\left(-i\frac{n\pi}{l}\tau\right)\frac{\alpha_{n\mu}}{n}\times\cos\left(\frac{n\pi}{l}\sigma\right)+Q_\mu+P_\mu\frac{\tau}{l\gamma},\quad \alpha_{-n\mu}=\alpha^+_{n\mu}. \quad (11)$$

Substitution of (11) in the subsidiary conditions (7') gives

$$L_n=-\frac{1}{2}\sum_{m=-\infty}^{+\infty}\alpha_{n-m}\alpha_m=0,\ n=0,1,2,\ldots, \quad (12)$$

where $\alpha_{0\mu}=P_\mu/\sqrt{\pi\gamma}$. These conditions actually reduce to restrictions on the initial data $x_\mu(\sigma,0)$ and $\dot{x}_\mu(\sigma,0)$, which determine the Fourier amplitudes α_n and the constants P_μ and Q_μ in the expansion (11):

$$\alpha_{n\mu}=2\sqrt{\frac{\gamma}{\pi}}\int_0^l d\sigma\cos\left(\frac{n\pi}{l}\sigma\right)\left\{\dot{x}_\mu(\sigma,0)-i\frac{n\pi}{l}x_\mu(\sigma,0)\right\},$$

$$\alpha_{-n\mu}=\alpha^+_{n\mu},\ n>0;$$

$$Q_\mu=\frac{1}{l}\int_0^l x_\mu(\sigma,0)\,d\sigma;\ P_\mu=\gamma\int_0^l \dot{x}_\mu(\sigma,0)\,d\sigma.$$

This method of solution has become known as the covariant method.

Noncovariant Method. A different approach to this problem is to express two components of the vector $x_\mu(\sigma,\tau)$ in terms of the others by means of the subsidiary condition (7) and the so-called gauge conditions. The point is that the subsidiary conditions (7) do not completely fix the parameters σ and τ. Equations (7), (8), and (9) admit the transformations $\tilde{\sigma}\pm\tilde{\tau}=f_\pm(\sigma\pm\tau)$

with arbitrary functions $f_\pm$. By a definite choice of $f_\pm$, we can achieve

$$n\dot{x}=nP/\gamma l,\quad nx'=0, \tag{13}$$

where n_μ is an arbitrary constant vector and P_μ is the total string momentum $P_\mu=\int_0^l d\sigma\partial\mathscr{L}_0/\partial\dot{x}^\mu$. With allowance for the subsidiary conditions (7), $P_\mu=\gamma\int_0^l\dot{x}_\mu(\sigma,\tau)d\sigma$. Equations (13) can be integrated and reduced to the single equation

$$nx=nP\tau/\gamma l+nQ. \tag{14}$$

The gauge conditions (14) finally fix the parameters σ and τ.

There is here an analogy with the arbitrariness in the choice of the potential $A_\mu(x)$ in electrodynamics, which can be replaced by $A_\mu(x)+\partial_\mu\lambda(x)$ with an arbitrary function $\lambda(x)$. If we impose on $A_\mu(x)$ the Lorentz condition

$$\partial_\mu A^\mu(x)=0, \tag{15}$$

then the only remaining admissible transformations must satisfy $\partial_\mu\partial\mu\lambda(x)=0$. Finally, the choice of $A_\mu(x)$ can be fixed by requiring not only (15) but also $A_0=0$ (the Coulomb gauge).

Using (14) and the subsidiary conditions (7′), we can find the partial derivatives with respect to σ and τ of the two components of the vector x_μ as functions of the remaining components $\mathbf{x}_\perp$. For this, it is convenient to take n_μ to be an isotropic vector $n^2=0$ with two nonzero components: the time component $n^0=1$ and the space component $n^1=1$. In the light-cone variables $x^\pm=(x^0\pm x^1)/\sqrt{2}$, $x^\mu=(x^\pm,\mathbf{x}_\perp)$ we obtain

$$\left.\begin{aligned}\dot{x}^+&=l\gamma(\dot{\mathbf{x}}_\perp^2+\mathbf{x}_\perp'^2)/2P^-;\quad x'^+=l\gamma\dot{\mathbf{x}}_\perp\mathbf{x}_\perp'/P^-;\\ \dot{x}^-&=P^-/l\gamma;\quad x'^-=0.\end{aligned}\right\} \tag{16}$$

At this stage, we have obviously lost the manifest relativistic invariance of the theory. In terms of the Fourier amplitudes, Eqs. (16) become

$$\alpha_n^{(+)}=(\pi\gamma/P^-)L_{n\perp},\quad n=0,\ \pm1,\ \pm2,\ \ldots;\quad \alpha_k^{(-)}=0;\quad k\neq0, \tag{17}$$

where

$$L_{n\perp}=\frac{1}{2}\sum_{m=-\infty}^{+\infty}\alpha_{n-m}\alpha_{m\perp};\quad \alpha_{0\perp}=\frac{\mathbf{P}_\perp}{\sqrt{\pi\gamma}};\quad \alpha_0^{(\pm)}=\frac{P^\pm}{\sqrt{\pi\gamma}}. \tag{18}$$

In particular, for the mass of the complete string we obtain from Eqs. (17) for $n=0$ the expression

$$M^2=P^2=-\mathbf{P}_\perp^2+2P^+P^-=\pi\gamma\sum_{m=0}\alpha_{-m\perp}\alpha_{m\perp}. \tag{19}$$

It follows immediately that M^2 is positive definite.

If in Eq. (14) we fix the gauge by taking a different vector n_μ, square roots appear when the subsidiary conditions (7) are solved for the independent components of x_μ, and this greatly complicates the transition to the quantum theory.

The initial data for the transverse components of x_μ can be specified arbitrarily, but they uniquely determine the initial conditions for the dependent components $x^\pm$ in accordance with Eqs. (16).

2. CAUCHY PROBLEM

The Cauchy problem for a relativistic string can also be solved without the use of Fourier series, and an expression analogous to the d'Alembert formula in the theory of an ordinary string is obtained.[31]

Let us consider first the actual formulation of the Cauchy problem for a relativistic string. It is here convenient to use the geometrical interpretation.[29] We first consider an infinite string $-\infty<\sigma<+\infty$. Suppose that an arbitrary spacelike curve describing the initial position of the string is given in the parametric form $x_\mu(\sigma,0)=\rho_\mu(\sigma)$, $\rho_\mu'^2(\sigma)<0$. Further, suppose that at each point of this curve the vector $v_\mu(\sigma)$, which is not parallel to $\rho_\mu'(\sigma)$, is specified, and that the plane through the vectors $\rho_\mu'(\sigma)$ and $v_\mu(\sigma)$ intersects the light cone in two straight lines; this can be expressed by the hyperbolicity condition

$$(\rho'v)^2-\rho'^2v^2>0. \tag{20}$$

As will be shown in what follows, the component of $V_\mu(\sigma)$ perpendicular to $\rho_\mu'(\sigma)$ determines the string velocity $\dot{x}_\mu(\sigma,0)$ at the initial time.

The Cauchy problem then consists of finding a solution to the string equations of motion (8) that satisfies the subsidiary conditions (7) and describes a world surface that passes through the curve $\rho_\mu(\sigma)$ at $\tau=0$ and touches at every point of this curve the plane through the vectors $\rho_\mu'(\sigma)$ and $v_\mu(\sigma)$. We note from this formulation that not the complete vector $v_\mu(\sigma)$, but only its component perpendicular to $\rho_\mu'(\sigma)$ is relevant for the initial data:

$$v(\sigma)=v_{\parallel}(\sigma)+v_\perp(\sigma),\quad v_\perp(\sigma)\rho'(\sigma)=0. \tag{21}$$

Indeed, only $v_\perp(\sigma)$ is used in the specification of the tangent plane, and it is only this component that occurs in the hyperbolicity condition (20), which in conjunction with (21) takes the form

$$(\rho'v_{\parallel})^2-\rho'^2v_{\parallel}^2-\rho'^2v_\perp^2=-\rho'^2v_\perp^2>0\quad\text{or}\quad v_\perp^2>0.$$

Thus, different vectors $v_\mu(\sigma)$ with equal perpendicular components $v_\perp(\sigma)$ lead to the same string motions. Therefore, the v_μ component parallel to ρ_μ' has no physical meaning.

A solution of the Cauchy problem for an infinite string was obtained in Ref. 31:

$$x_\mu(\sigma,\tau)=\frac{1}{2}[\rho_\mu(\sigma+\tau)+\rho_\mu(\sigma-\tau)]+\frac{1}{2}\int_{\sigma-\tau}^{\sigma+\tau}\frac{(\rho'v)\rho_\mu'-\rho'^2v_\mu}{\sqrt{(\rho'v)^2-\rho'^2v^2}}d\lambda. \tag{22}$$

As expected, (22) contains only the perpendicular component of v_μ. If we use the expansion (21), then the integral in (22) can be written as

$$\frac{1}{2}\int_{\sigma-\tau}^{\sigma+\tau}v_\perp(\lambda)\sqrt{-\frac{\rho'^2(\lambda)}{v_\perp^2(\lambda)}}\,d\lambda.$$

It can be shown that, except for a constant factor γ, the integrand in (22) contains the density of the string's canonical momentum $\pi_\mu(\sigma,\tau=0)$ at the initial time, the canonical momentum being given by

$$\pi_\mu(\sigma,\tau)=-\partial\mathscr{L}_0/\partial\dot{x}^\mu=\gamma[(\dot{x}x')x_\mu'-x'^2\dot{x}_\mu]/\sqrt{(\dot{x}x')^2-\dot{x}^2x'^2}. \tag{23}$$

Substituting $x_\mu'(\sigma,0)$ and $\dot{x}_\mu(\sigma,0)$ from (22) in (23),

$$x_\mu'(\sigma,0)=\rho_\mu'(\sigma),\quad \dot{x}_\mu(\sigma,0)=[(\rho'v)\rho_\mu'-\rho'^2v_\mu]/\sqrt{(\rho'v)^2-\rho'^2v^2}, \tag{24}$$

we obtain

$$\pi_\mu(\sigma,0)=\gamma[(\rho'v)\rho_\mu'-\rho'^2v_\mu]/\sqrt{(\rho'v)^2-\rho'^2v^2}. \tag{25}$$

Note that the velocities of the points of the string at the initial time $\dot{x}_\mu(\sigma, 0)$ found in accordance with (22) are equal to $v_\mu(\sigma)$ only if the initial data $\rho'_\mu(\sigma)$ and $v_\mu(\sigma)$ satisfy the subsidiary conditions (7):

$$\rho'^2(\sigma) + v^2(\sigma) = 0, \quad \rho'(\sigma)\, v(\sigma) = 0. \tag{26}$$

If these conditions are satisfied, the integrand in (22) is simply the string velocity at the initial time:

$$x_\mu(\sigma, \tau) = \frac{1}{2}\Big[\rho_\mu(\sigma+\tau) + \rho_\mu(\sigma-\tau) + \int_{\sigma-\tau}^{\sigma+\tau} v_\mu(\lambda)\, d\lambda\Big] \quad \dot{x}_\mu(\sigma, 0) = v_\mu(\sigma). \tag{27}$$

In the general case, we have in accordance with (24)

$$\dot{x}_\mu(\sigma, 0) = v_{\perp\mu}(\sigma)\left(-\rho'^2(\sigma)/v_\perp^2(\sigma)\right)^{1/2},$$

despite the fact that $\dot{x}_\mu(\sigma, 0)$ and $v_\mu(\sigma)$ occur in the same way in (23) and (25). There is no contradiction here because (23) cannot be inverted, i.e., one cannot express the velocities $\dot{x}_\mu(\sigma, \tau)$ in terms of the momenta $\pi_\mu(\sigma, \tau)$. This is due to the singularity of the string Lagrangian[32-34]:

$$\det(\partial\pi_\mu/\partial\dot{x}_\nu) \equiv \det(-\partial^2\mathcal{L}_0/\partial\dot{x}^\mu\,\partial\dot{x}_\nu) = 0.$$

The solution of the Cauchy problem for a finite string can also be obtained from (22). To satisfy the boundary conditions (10), it is sufficient to continue the initial data $\rho_\mu(\lambda)$ and $\pi_\mu(\lambda)$ [Eq. (25)] outside the interval $0 < \lambda < l$ in an even manner with respect to the points $0, l$.

In physical problems, it is usual to specify the position of the system and its velocity at the initial time as initial data. If the motion of the relativistic string is considered in the same manner, then $\rho_\mu(\sigma) = x_\mu(\sigma, 0)$ and $v_\mu(\sigma) = \dot{x}_\mu(\sigma, 0)$ must be chosen so that the conditions (26) are satisfied, and the solution is then constructed in accordance with the d'Alembert formula (27).

3. EXAMPLES OF CLASSICAL MOTIONS OF A RELATIVISTIC STRING

The solution of the Cauchy problem for a relativistic string makes it possible to study string motions from given initial positions.[31,35] The nonlinear behavior of such objects leads to a number of interesting features of their motion. Above all, the string length may change during the motion and the string may even contract to a point. The velocity of the free ends of the string is always equal to the velocity of light, which is a consequence of the boundary conditions (10) and the orthogonal gauge conditions (7), in accordance with which $\dot{x}^2(\sigma_i, \tau) = 0$, $\sigma_1 = 0$, $\sigma_2 = l$. Setting $t = \tau$, we obtain $d\mathbf{x}(\sigma_i, t)/(dt)^2 = 1$. If the string is initially at rest, $x_\mu(\sigma, 0) = 0$, then during the motion its free ends remain on the initial string configuration.

Let us consider some of the simplest examples of string motions.

Suppose that at the initial time the string is in the shape of a circle of radius R and is at rest in the x, y plane[31]:

$$x_0(\lambda) = R\cos(\lambda/R); \quad y_0(\lambda) = R\sin(\lambda/R);$$
$$z_0(\lambda) = 0; \quad t_0(\lambda) = 0; \quad \mathbf{v}(\lambda) = 0; \quad v_t(\lambda) = 1.$$

From (27), we obtain

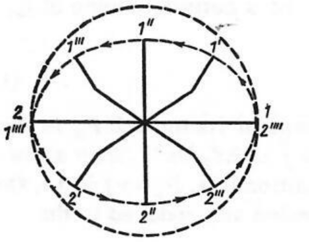

FIG. 1.

$$t = \tau; \quad x(\sigma, t) = R\cos(t/R)\cos(\sigma/R);$$
$$y(\sigma, t) = R\cos(t/R)\sin(\sigma/R); \quad z = 0.$$

Thus, the ring remains in the x, y plane and pulsates with period πR. In this case, it is immaterial whether one regards the string as infinite and wound into a ring or as finite but closed.

A string that initially is a rectilinear segment at rest begins to oscillate, contracting to a point and then returning to its original length. Its ends periodically change places.

We now consider the rotation of a string in a plane. We take the initial data in the form

$$x^\mu(\sigma, 0) = (0, (1/\sqrt{\gamma}\,\omega)\sin(\omega, \sigma), 0, 0);$$
$$\dot{x}^\mu(\sigma, 0) = (1/\sqrt{\gamma}, 0, (1/\sqrt{\gamma})\sin(\omega, \sigma), 0), \quad -L < \sigma < L.$$

The shape of the string at subsequent times is shown in Fig. 1. The string rotates, remaining in the x, y plane, but not all of its points have the same angular velocity. A point of inflection therefore appears in the string profile, and the length of the string changes with the time. But if $2\omega L = \pi$, the string rotates with angular velocity ω as a rigid rod. The square of the string mass is given by

$$M^2 = P_0^2 = 4\gamma L^2.$$

The angular momentum of the string is

$$J = \gamma\int_{-L}^{+L} d\sigma\,(x\dot{y} - y\dot{x}) = \frac{1}{\omega^2}\left[\omega L - \sin(2\omega L)\right].$$

If $2\omega L = \pi$, the string has maximal angular momentum, and

$$J = (1/2\pi\gamma)\, M^2 = \alpha' M^2.$$

Thus, we have a linearly rising Regge trajectory with slope $\alpha' = (2\pi\gamma)^{-1}$.

Figure 2 shows the profiles of a string which at the

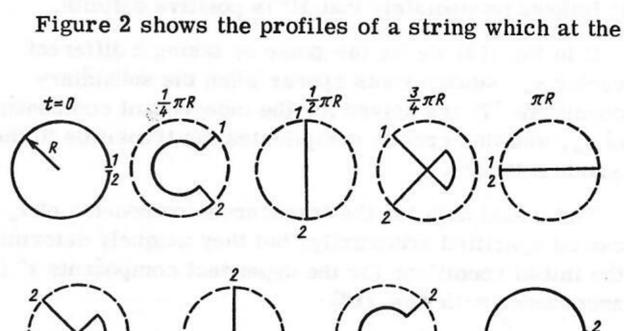

FIG. 2.

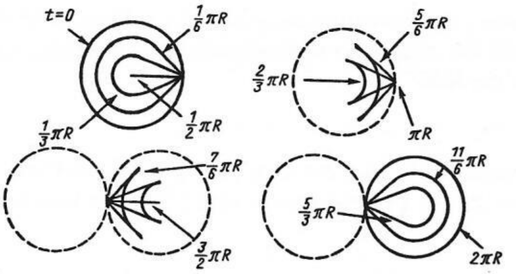

FIG. 3.

initial time has the form of a circle cut at one point. The ends of the string remain free during the motion, and they move along the original profile of the string. The motion of the same string but with fixed ends is shown in Fig. 3. In these examples, the dimensional constant γ is taken equal to 1.

4. DIFFERENT METHODS OF QUANTIZATION OF A RELATIVISTIC STRING

To go over to the quantum theory, it is first necessary to construct a Hamiltonian formalism for the classical string dynamics. As we have already pointed out, the string Lagrangian (2) is singular,[32-34] since

$$\det(\partial^2 \mathcal{L}_0/\partial \dot{x}_\mu \partial \dot{x}^\nu) = 0.$$

This is manifested in the constraints on the canonical variables $x_\mu(\sigma, \tau)$ and $\pi_\nu(\sigma,\tau) = -\partial \mathcal{L}_0/\partial \dot{x}^\nu(\sigma,\tau)$:

$$\varphi_1 = \gamma^{-2}\pi^2 + x'^2 = 0, \quad \varphi_2 = \gamma^{-1}\pi_\mu x'^\mu = 0. \tag{28}$$

Further, $\mathcal{L}_0$ is a homogeneous function of the first degree in the velocities $\dot{x}_\nu$, and therefore in accordance with Euler's theorem

$$\dot{x}_\nu \partial \mathcal{L}_0/\partial \dot{x}_\nu = -\dot{x}_\nu \pi^\nu = \mathcal{L}_0. \tag{29}$$

Therefore, the Hamiltonian vanishes identically:

$$\mathcal{H}_0 = -\dot{x}_\nu \pi^\nu - \mathcal{L}_0 \equiv 0. \tag{30}$$

There are various other approaches to the problem of constructing a quantum theory of systems with constraints.[32-34]

Noncovariant Method. The simplest way of taking into account constraints in the theory is to eliminate the dependent variables by means of the constraints. For this, as in the Lagrangian method, Eqs. (28) must be augmented by two conditions that finally fix the gauge. To achieve a direct connection to the Lagrangian approach, these conditions can be

$$n\pi = nP/l, \quad nx' = 0,$$

where n is a constant isotropic vector with the two nonvanishing components $n^0 = n^1 = 1$. Then, using (16), we obtain

$$\pi^+ = l[(\gamma^{-1}\pi_\perp)^2 + \mathbf{x}_\perp'^2]/2P^-; \quad x^{+\prime} = l\pi_\perp \mathbf{x}_\perp'/P^-; \quad \pi^- = P^-/l; \quad x^{-\prime} = 0.$$

The Hamiltonian for the dynamics of the independent canonical variables $\mathbf{x}_\perp$ and $\pi_\perp$ can be taken in the form

$$H = \int_0^l d\sigma \mathcal{H} = \frac{P^-}{l}\int_0^l d\sigma \pi^+(\sigma,\tau) = \frac{\gamma}{2}\int_0^l d\sigma\{(\gamma^{-1}\pi_\perp')^2 + \mathbf{x}_\perp'^2\}. \tag{31}$$

This is because the variational principle in the Hamiltonian formalism

$$\delta S = \delta \int_{\tau_1}^{\tau_2} d\tau \int_0^l d\sigma (\pi_\perp \dot{\mathbf{x}}_\perp - \mathcal{H}) = 0$$

gives the canonical equations of motion

$$\dot{\mathbf{x}}_\perp = \frac{\partial \mathcal{H}}{\partial \pi_\perp} = \gamma^{-1}\pi_\perp, \quad \dot{\pi}_\perp = \frac{\partial}{\partial\sigma}\left(\frac{\partial \mathcal{H}}{\partial \mathbf{x}_\perp'}\right) = \gamma \mathbf{x}_\perp'' \tag{32}$$

and boundary conditions

$$\partial \mathcal{H}/\partial \mathbf{x}_\perp' = \gamma \mathbf{x}_\perp' = 0, \quad \sigma = 0, \ l,$$

which are equivalent to the corresponding equations in the Lagrangian method.

Using this, as solutions for $\mathbf{x}_\perp(\sigma,\tau)$ and $\pi_\perp(\sigma,\tau)$ we can take the expansions that follow from (11) and (32):

$$\left.\begin{aligned} \mathbf{x}_\perp(\sigma,\tau) &= \frac{i}{\sqrt{\pi\gamma}} \sum_{n\neq 0} \frac{\exp(-i\omega_n\tau)}{n} \alpha_{n\perp} \cos(\omega_n\sigma) + \mathbf{Q}_\perp + \frac{\mathbf{P}_\perp}{l\gamma}\tau; \\ \pi_\perp(\sigma,\tau) &= \frac{\sqrt{\pi\gamma}}{l} \sum_{n\neq 0} \exp(-i\omega_n\tau)\alpha_{n\perp}\cos(\omega_n\sigma) + \frac{\mathbf{P}_\perp}{l}, \end{aligned}\right\} \tag{33}$$

where $\omega_n = n\pi/l$ and $\alpha_{-n} = \alpha_n^+$.

The Fourier amplitudes of the dependent variables $\alpha_n^{(\pm)}$ are obviously determined by Eqs. (17) and (18).

The Hamiltonian (31) with the expansions (33) takes the form

$$H = \frac{\pi}{l}L_0 = \frac{\pi}{2l}\sum_{n=-\infty}^{+\infty} \alpha_{-n\perp}\alpha_{n\perp} = \frac{\mathbf{P}_\perp^2}{2l\gamma} + \frac{\pi}{2l}\sum_{n\neq 0}\alpha_{-n\perp}\alpha_n. \tag{34}$$

On the transition to the quantum theory, the amplitudes $\alpha_{n\perp}$ become operators with the commutation relations

$$\left.\begin{aligned} &[\alpha_n^i, \alpha_m^j] = n\delta_{ij}\delta_{n+m,0}; \quad [Q^i, P^j] = \delta_{ij}; \\ &n, m = \pm 1, \pm 2, \ldots; \quad i, j = 2, 3, \ldots; D-1, \end{aligned}\right\} \tag{35}$$

where D is the dimension of the space-time in which the string moves. The ordinary creation and annihilation operators $\mathbf{a}_{n\perp}^+$ and $\mathbf{a}_{n\perp}$, $n = 1, 2, \ldots$, with the commutators

$$[a_n^i, a_m^{+j}] = \delta_{n,m}\delta_{ij},$$
$$[\mathbf{a}_n^+, \mathbf{a}_m^+] = [\mathbf{a}_n, \mathbf{a}_m] = 0$$

are related to the α_n by

$$\alpha_{n\perp} = \sqrt{n}\,\mathbf{a}_{n\perp}; \quad \alpha_{-n\perp} \equiv \alpha_n^+ = \sqrt{n}\,\mathbf{a}_n^+, \ n = 1, 2, 3 \ldots.$$

Equations (35) are equivalent to the "equal-time" commutators

$$[x^i(\sigma,\tau), \pi^j(\sigma',\tau)] = i\delta_{ij}\delta(\sigma-\sigma'),$$
$$[x^i(\sigma,\tau), x^j(\sigma',\tau)] = [\pi^i(\sigma,\tau), \pi^j(\sigma',\tau)] = 0.$$

To avoid infinities in the quantum theory, it is necessary to go over in (17) and (18) to the normal product of the operators $\alpha_{n\perp}$ and $\alpha_{m\perp}$. To the classical expression, it is necessary to add a constant, which is usually denoted by $-\alpha(0)$:

$$\alpha_n^{(+)} = \frac{\sqrt{\pi\gamma}}{P^-}[L_{n\perp} - \delta_{n,0}\alpha(0)],$$

where $L_{n\perp} = \frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m\perp}\alpha_{m\perp}$. The same constant must be introduced into the expression for the square of the string mass (19):

$$M^2 = \pi\gamma \sum_{m\neq 0} :\alpha_{-m\perp}\alpha_{m\perp}: - 2\pi\gamma\alpha(0) \tag{36}$$

and into the Hamiltonian (34):

$$H = \frac{\pi}{2l}\sum_{n=-\infty}^{+\infty} :\alpha_{-n\perp}\alpha_{n\perp}: - \frac{\pi}{l}\alpha(0).$$

It is easy to obtain the quantum equations of motion. For example, for $\alpha_n(\tau)$

$$d\alpha_{n\perp}(\tau)/d\tau = \mathrm{i}[H, \alpha_{n\perp}(\tau)] = -\mathrm{i}(n\pi/l)\alpha_{n\perp}(\tau) = -\mathrm{i}\omega_n\alpha_{n\perp}(\tau),$$

i.e., $\alpha_{n\perp}(\tau) = \alpha_{n\perp}\exp(-i\omega_n\tau)$, which corresponds to the expansions (33).

In the considered case, the norm of the state vectors is obviously positive, since these vectors are obtained by applying only the operators $\mathbf{a}^{+}_{n\perp}$ to the vacuum. These vectors coincide exactly with the so-called transverse states in Veneziano's dual resonance model.[1-4]

The main problem in the noncovariant approach is to prove the relativistic invariance of the theory in the quantum case. For this, it is necessary to show that the generators P_μ and $M_{\mu\nu}$ of the Poincaré group constructed by means of the dynamical variables of the string satisfy the well-known commutation relations. The total string momentum P_μ is the generator of translations, and the string angular momentum tensor

$$M_{\mu\nu} = \int_0^l (x_\mu\pi_\nu - x_\nu\pi_\mu)\, d\sigma = Q_\mu P_\nu - Q_\nu P_\mu - \frac{\mathrm{i}}{2}\sum_{n\neq 0}\frac{1}{n} \times (\alpha_{-n\mu}\alpha_{n\nu} - \alpha_{-n\nu}\alpha_{n\mu}) \tag{37}$$

is the generator of Lorentz rotations. It can be shown that all the commutation relations have the correct values except for the commutator

$$[M^{+i}, M^{+j}] = \frac{2\pi\gamma}{P^{-2}}\sum_{m=1}^{\infty}\Big[m\Big(1-\frac{1}{24}(D-2)\Big) + \frac{1}{m}\Big(\frac{1}{24}(D-2)-\alpha(0)\Big)\Big](\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m),$$

where $i, j = 2, 3, \ldots, D-1$; D is the dimension of space-time. The algebra of the Poincaré group requires $[M^{+i}, M^{+j}] = 0$. Therefore, the unique possibility for reconciling this theory with relativistic invariance is to require $\alpha(0) = 1$ and $D = 26$. It follows, in particular, that, in accordance with (36), the string ground state has imaginary mass (is a tachyon state).

Covariant Formalism. One can preserve the manifest relativistic invariance of the theory by treating all the components of the vectors $x_\mu(\sigma,\tau)$ and $\pi_\nu(\sigma,\tau)$ on an equal footing and imposing the subsidiary conditions (28) on the physical state vectors. For systems with constraints, the Hamiltonian formalism and the transition to the quantum theory without elimination of the dependent variables were developed mainly by Dirac.[33] Following this method, we take the Hamiltonian of the relativistic string to be the linear combination of constraints (28):

$$H = \int_0^l d\sigma\,[f_1(\sigma,\tau)\varphi_1(\sigma,\tau) + f_2(\sigma,\tau)\varphi_2(\sigma,\tau)],$$

where f_1 and f_2 are arbitrary functions.[2)] If we set $f_1 = 0$ and $f_2 = -\gamma/2$, then

$$H = -\frac{\gamma}{2}\int_0^l d\sigma\,[(\gamma^{-1}\pi)^2 + x'^2].$$

[2)]According to Dirac's classification, the constraints (28) are primary constraints, since they follow directly from the Lagrangian; simultaneously, they are constraints of the first class, since their Poisson brackets can be expressed in terms of φ_i.

In this case, the Hamiltonian equations (32) go over directly into the equations of motion in the covariant Lagrangian method:

$$\dot{x}_\mu = \gamma^{-1}\pi_\mu;\quad \ddot{x}_\mu - x''_\mu = 0;\quad x'_\mu = 0,\quad \sigma = 0, l.$$

The expansion (11) is a solution for $x_\mu(\sigma,\tau)$, and for $\pi_\mu(\sigma,\tau)$ we have, in accordance with (11), the Fourier series

$$\pi_\mu(\sigma,\tau) = \gamma\dot{x}_\mu = \frac{\sqrt{\pi\gamma}}{l}\sum_{n\neq 0}\exp(-\mathrm{i}\omega_n\tau)\,\alpha_{n\mu}\cos(\omega_n\sigma) + \frac{P_\mu}{l},$$

where $\omega_n = n\pi/l$.

In quantum theory, the commutators

$$[\alpha_{m\mu}, \alpha_{n\nu}] = -m g_{\mu\nu}\delta_{n+m,0},\quad [Q_\mu, P_\nu] = -\mathrm{i}g_{\mu\nu}, \tag{38}$$

where $g_{00} = -g_{ii} = 1$, are postulated.

The time components α^{+}_{n0}, $n > 0$, if applied to the vacuum, lead to state vectors with negative norm. Physical states must satisfy the conditions

$$\varphi_i|\Phi\rangle = 0,\quad i = 1, 2,$$

which are equivalent to

$$[L_n - \delta_{n,0}\alpha(0)]|\Phi\rangle = 0,\quad n = 0, 1, 2, \ldots, \tag{39}$$

where $L_n = -\frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m}\alpha_m:$; $L^{+}_n = L_{-n}$. These conditions are identical with the Virasoro conditions in Veneziano's dual resonance model (see Appendix 1).

From (39) for $n = 0$ the string mass is determined:

$$M^2 = P^2 = -\pi\gamma\sum_{m\neq 0}\alpha_{-m}\alpha_m - 2\pi\gamma\alpha(0). \tag{40}$$

The operators L_n satisfy the algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}n(n^2-1)\delta_{n+m,0}. \tag{41}$$

An important point is the appearance of a c-number term in the commutator (41). In the classical theory, in which the Poisson brackets play the part of the commutators, this term is not present:

$$\{L_n, L_m\}_{\mathrm{PB}} = \mathrm{i}(n-m)L_{n+m}. \tag{42}$$

The algebra (42) is isomorphic to the Lie algebra of the conformal group on a plane.[23]

The Schwinger term appears in the commutator (41) because of the transition to the normal product of the operators α_n in L_n. The easiest way of obtaining this term is to use Wick's theorem[36] to calculate the vacuum expectation value of the commutator (41), remembering that the operator pairing $\alpha^\rho_k\alpha^\sigma_j$ is equal to $-g^{\rho\sigma}\theta(k)k\delta_{k+j,0}$.

A physical space of state vectors with positive norm can be constructed[25] only in space-time with dimension $D = 26$ and if $\alpha(0) = 1$. In accordance with Eq. (40), the string ground state has imaginary mass (is a tachyon state).

It should be noted that the lightlike gauge (14) is not apparently used explicitly here, although the so-called transverse states needed to prove the theorem that there are no ghosts is based on precisely this gauge.

The mathematical formalism of the quantum theory of relativistic strings considered here is identical with the operator formalism in Veneziano's dual resonance model (see Appendix 1). This makes it possible to regard a relativistic string as the dynamical basis of the dual resonance approach.

The noncovariant method in the theory of strings is analogous to quantization of the electromagnetic field in the Coulomb gauge; the covariant approach is analogous to Fermi's method in quantum electrodynamics. Note, however, the important difference that the subsidiary conditions on the vector potential in Maxwell's theory are linear, whereas in the theory of strings the conditions (28) are quadratic in x_μ and π_ν.

The unusual results in the quantum theory of relativistic strings such as the restriction on the dimension of space-time and the presence of tachyon states are perfectly acceptable from the point of view of dual models. Indeed, they are even necessary if a relativistic string is to be regarded as a dynamical basis of dual models. The Fock space constructed in the operator formalism for dual resonance amplitudes was actually transferred unmodified into the quantum theory of relativistic strings.[37]

But, of course, such a situation cannot be regarded as satisfactory from the point of view of physics. It is hard to understand why a string, an object which has been so well studied in classical theory and in nonrelativistic quantum mechanics, cannot be treated consistently at the quantum level in real four-dimensional space-time. Attempts have been made to find other quantum solutions in the problem of a relativistic string that avoid these difficulties. Interesting results have been obtained by Rohrlich.[38-40]

Rohrlich's Quantum Theory of Relativistic Strings. There are two key elements in Rohrlich's approach:

1) the choice of the gauge condition that fixed the parameter τ:

2) the use of this condition to eliminate states with negative norm, with the physical state vectors being constructed in the center-of-mass system of the string. This frame of reference is uniquely distinguished for a relativistic composite object such as a string. In all the earlier attempts at string quantization, this fact was entirely ignored.

The "light-cone" gauge (14) with $n^2=0$ was explicitly or implicitly used in all previous methods of string quantization. It has a serious shortcoming[35,41] in that it does not enable one to describe motions of the string for which

$$n(\dot{x}\pm x')=0. \tag{43}$$

Rohrlich[40] therefore proposed that one should take, not the arbitrary vector n_μ in the conditions (13), but the total string momentum P_μ:

$$P\dot{x}=P^2/\gamma l,\quad Px'=0, \tag{44}$$

which is equivalent to the requirement

$$Px=P^2\tau/\gamma l+PQ. \tag{45}$$

One can assume that $P^2\neq 0$, and then

$$P(\dot{x}\pm x')\neq 0;$$

thus, the new gauge does not lead to any restrictions on the string motion.

Substituting the expansion (11) in (45), we obtain

$$\alpha_{n\mu}P^\mu=0,\quad n\neq 0. \tag{46}$$

This means that in the center-of-mass system of the string, in which $\mathbf{P}=0$, the time Fourier components α_{n0} are zero—and it is these components that lead to states with negative norm on quantization. Therefore, in the quantum theory it is convenient to construct the physical space of state vectors in this frame of reference.

Equations (46) can now be regarded as conditions on the state vectors $|\mathscr{F}\rangle_{CM}$:

$$G_n|\mathscr{F}\rangle_{CM}=\alpha_{n\mu}P^\mu|\mathscr{F}\rangle_{CM}=\alpha_{n0}P^0|\mathscr{F}\rangle_{CM}=0, \tag{47}$$

and it is sufficient to require fulfillment of this equation for $n>0$. In other words, only the negative-frequency components of the condition (46) are imposed on the state vectors, just as is done with the Lorentz condition in quantum electrodynamics.

Assuming that $P^0|\mathscr{F}\rangle_{CM}\neq 0$, we obtain from (47)

$$\alpha_{n0}|\mathscr{F}\rangle_{CM}=0,\quad n>0,$$

i.e., in the quantum case too the time components of the operators $\alpha_{n\mu}$, $n>0$, are actually zero.[3)] Thus, the state vectors in the center-of-mass system of the string are constructed by applying only the spatial components of the operators to the vacuum:

$$a^+_{m\mu}=\alpha^+_{m\mu}/\sqrt{m},\quad m>0;$$

$$|\mathscr{F}\rangle_{CM}=\prod_{n=1}^{\infty}\frac{(a^+_{nx})^{\lambda_{nx}}}{\sqrt{\lambda_{nx}!}}\frac{(a^+_{ny})^{\lambda_{ny}}}{\sqrt{\lambda_{ny}!}}\frac{(a^+_{nz})^{\lambda_{nz}}}{\sqrt{\lambda_{nz}!}}|0\rangle\equiv|\lambda_1,\lambda_2,\ldots\rangle,$$

where the vectors $\lambda_n=(\lambda_{nx},\lambda_{ny},\lambda_{nz})$ have non-negative integral components.

In an arbitrary frame, the state vectors are obtained from $|\mathscr{F}\rangle_{CM}$ by applying the unitary operator

$$U=\exp\{ia_\mu P^\mu+(i/2)\,\omega_{\mu\nu}M^{\mu\nu}\}, \tag{48}$$

where a_μ is the displacement vector, $\omega_{\mu\nu}$ are the parameters of the Lorentz rotation, P_μ is the total string momentum, and $M_{\mu\nu}$ is the angular momentum tensor (37) of the string. Since $[U,G_n]=0$, it is obvious that the state vectors retain their positive definiteness.

The physical states, which we denote by $|\Phi\rangle$, must also satisfy the subsidiary conditions (12):

$$L_n|\Phi\rangle=\Lambda_n|\Phi\rangle-\frac{1}{\sqrt{\pi\gamma}}G_n|\Phi\rangle=\Lambda_n|\Phi\rangle=0,\ n>0; \tag{49}$$

$$L_0|\Phi\rangle=\Lambda_0|\Phi\rangle-(P^2/2\pi\gamma)|\Phi\rangle=(-m_0^2/2\pi\gamma)|\Phi\rangle, \tag{50}$$

where $\Lambda_n=-\frac{1}{2}\sum_{m\neq 0,n}:\alpha_{n-m}\alpha_m:$.

The constraints G_n, $n>0$, and L_m, $m\geqslant 0$, taken together are first-class constraints in Dirac's classification,[33] since they form a closed algebra.

$$[G_n,G_m]=0,\ n>0,\ m>0;$$
$$[L_n,L_m]=(n-m)L_{n+m},\ n,m\geqslant 0;$$
$$[G_k,L_n]=kG_{n+k},\ k>0,\ n\geqslant 0.$$

[3)]Note that the condition (47) was used, for example, to eliminate states with negative norm in the quark model of hadrons with the potential of a relativistic oscillator.[42]

Therefore, the conditions (47), which eliminate states with negative norm, and the conditions (49) and (50), which define the physical state vectors, are consistent. Note that there is no c-number term in the commutator $[L_n, L_m]$, since $n \geqslant 0$ and $m \geqslant 0$.

The conditions (49) and (50), which select from the Fock space of vectors with positive norm $\mathscr{F}^+$ the subspace Φ of physical vectors $|\Phi\rangle$, do not violate the relativistic invariance of the theory, since $[L_n, U]=0$, $n \geqslant 0$, where U is the operator (48) that realizes the transformations of the inhomogeneous Lorentz group.

We mention finally that it remains an open question whether a dual model can be constructed on the basis of Rohrlich's quantum theory of a relativistic string.

5. STRING IN AN EXTERNAL ELECTROMAGNETIC FIELD

Interaction Lagrangian. The main principle in the choice of the interaction Lagrangian is to preserve the reparametrization invariance possessed by the action of the free string.[43] In the case of the electromagnetic field, this condition is satisfied by the Lagrangian

$$\mathscr{L}_{\rm int} = g x'_\mu \dot{x}_\nu F^{\mu\nu}(x). \tag{51}$$

The corresponding action is

$$S_{\rm int} = g \int_{\tau_1}^{\tau_2} d\tau \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma\, x'_\mu \dot{x}_\nu F^{\mu\nu}(x) = -g \int_{\tau_1}^{\tau_2} d\tau \frac{dx_\nu}{d\tau} A^\nu(x)\Big|_{\sigma=\sigma_1(\tau)} + g \int_{\tau_1}^{\tau_2} d\tau \frac{dx_\nu}{d\tau} A^\nu(x)\Big|_{\sigma=\sigma_2(\tau)}, \tag{52}$$

where $\partial x_\nu/\partial\tau = \dot{x}_\nu + x'_\nu \dot{\sigma}(\tau)$. Thus, $S_{\rm int}$ describes the interaction with the electromagnetic field of two point charges at the ends of the string. These charges are equal in magnitude and opposite in sign: $g_1 = -g_2 = g$.

It follows from Eq. (52) that the electromagnetic field changes neither the equations of motion (8) nor the subsidiary conditions (7), which remain the same as in the free case. The boundary conditions are obtained from Eqs. (4) by replacing $\mathscr{L}_0$ by $\mathscr{L}_0 + \mathscr{L}_{\rm int}$ and are

$$\gamma x'_\nu + g F_{\nu\mu} \dot{x}^\mu + (\dot{x}_\nu + g F_{\nu\mu} x'^\mu)\dot{\sigma} = 0, \quad \sigma = \sigma_i(\tau), \quad i = 1, 2. \tag{53}$$

In this case too, the variational principle does not determine the explicit form of the functions $\sigma_i(\tau)$, $i = 1, 2$, and therefore we can again set $\sigma_1(\tau) = 0$, $\sigma_2(\tau) = l$. This appreciably simplifies the boundary conditions (53):

$$\gamma x'_\mu + g F_{\mu\nu} \dot{x}^\nu = 0, \quad \sigma = 0, l. \tag{54}$$

One can find a solution of the equations of motion (8) satisfying the boundary conditions (54) in two cases: when $F_{\mu\nu} = \text{const}$[44-46] (a constant homogeneous electromagnetic field) and and in the field of a plane monochromatic electromagnetic wave.[47]

These examples show that interaction with the electromagnetic field does not change the equidistant nature of the string mass spectrum. We should mention some features in the formulation of this problem. In the action (52), the external electromagnetic field is assumed given, and the radiation of the charges of the string, their interaction with one another through the electromagnetic field, and the masses of the charges are ignored. This has the consequence, in particular, that the charged ends of the string can move with the velocity of light.

Constant Homogeneous Electromagnetic Field. Covariant Formalism. If $F_{\mu\nu} = \text{const}$, then the solution can be represented by the Fourier series[45]

$$x^\mu(\sigma, \tau) = \frac{i}{\sqrt{\pi\gamma}} \sum_{n \neq 0} \frac{1}{2n} \Big\{ \alpha_n^\mu \exp[-i\omega_n(\sigma+\tau)] + [(1-f)^{-1}]^\mu_\rho (1+f)^{\rho\beta} \alpha_{n\beta} \exp[i\omega_n(\sigma-\tau)] + [(1-f^2)^{-1}]^\mu_\rho Q^\rho + \frac{1}{l\gamma} P^\mu \tau - f^{\mu\rho} \frac{P_\rho}{l\gamma}\left(\sigma - \frac{l}{2}\right) \Big\}, \tag{55}$$

where $\omega_n = n\pi/l$; $f_{\mu\nu} = gF_{\mu\nu}/\gamma$; $\alpha_{-n\mu} = \alpha^*_{n\mu}$. This expression describes the motion of a relativistic string in fields for which

$$\det \|(1-f)^\mu_\nu\| = \det \|(1+f)^\mu_\nu\| = 1 + (g/\gamma)^2 (H^2 - E^2) - (g/\gamma)^4 (\mathbf{EH})^2 \neq 0.$$

In particular, (55) becomes meaningless[44] when $(g/\gamma)^2 E^2 = 1$ and $\mathbf{H} = 0$. The explicit form of the inverse matrices $(1-f)^{-1}$ and $(1-f^2)^{-1}$ in (53) will not be required in what follows.[4)]

Substitution of the solution (55) in the subsidiary conditions (7′) leads to the same restrictions on the Fourier amplitudes $\alpha_{n\mu}$ as in the free case:

$$L_n = -\frac{1}{2} \sum_{m=-\infty}^{+\infty} \alpha_{n-m\mu} \alpha_m^\mu = 0, \; n = 0, 1, 2, \ldots; \; L_{-n} = L_n^*. \tag{56}$$

The only difference is that now the tensor $f_{\mu\nu}$ occurs in the zeroth mode $\alpha_{0\mu}$:

$$\alpha_{0\mu} = \frac{1}{\sqrt{\pi\gamma}} (1-f)_{\mu\rho} P^\rho.$$

It was not surprising that the subsidiary conditions (56) were the same as the Virasoro conditions, since $S_{\rm int}$ (52) has the same reparametrization invariance as the free string action (2).

The density of the canonical momentum is

$$\pi_\mu = -\partial\mathscr{L}/\partial\dot{x}^\mu = \gamma(\dot{x}_\mu + f_{\mu\nu} x'^\nu) = \frac{\sqrt{\pi\gamma}}{l} \sum_{n=-\infty}^{+\infty} (1+f)_{\mu\rho} \alpha_n^\rho \exp(-i\omega_n\tau)\cos(\omega_n\sigma). \tag{57}$$

The total canonical momentum of the string

$$\Pi_\mu = \int_0^l d\sigma\, \pi_\mu(\sigma, \tau) \tag{58}$$

is conserved. Indeed,

$$\frac{d\Pi_\mu}{d\tau} = -\int_0^l d\sigma \frac{\partial}{\partial\tau}\left(\frac{\partial\mathscr{L}}{\partial\dot{x}^\mu}\right) = \int_0^l d\sigma \frac{\partial}{\partial\sigma}\left(\frac{\partial\mathscr{L}}{\partial x'^\mu}\right) = \frac{\partial\mathscr{L}}{\partial x'^\mu}\Big|_{\sigma=0}^{\sigma=l} = 0.$$

In deriving this equation, we used the equations of motion (3) and the boundary conditions (4) with $\mathscr{L}_0$ replaced by $\mathscr{L}$.

Substituting the expansion (57) in (58), we obtain

$$\Pi_\mu = (1-f^2)_{\mu\rho} P^\rho. \tag{59}$$

For a free particle, we could assume that the mechanical and the canonical momenta are the same (by analogy with the theory of a free point particle), but for a string with charges in an external field this is not so.

[4)] We adopt here the definition $(A^{-1})^\mu_\rho A^\rho_\nu = \delta^\mu_\nu$, where δ^μ_ν is the Kronecker delta symbol for the inverse matrix A^{-1}. The first index of the Lorentz tensor $A_{\mu\nu}$ is always raised: $A^\mu_\nu = g^{\mu\rho} A_{\rho\nu}$. The matrices $(1+f)^{-1}$ and $(1-f^2)^{-1}$ were found in Ref. 48.

The question arises of how the mass of the string should be defined in this case. One could, for example, take $M^2 = \Pi^2$. But, as was shown in Ref. 49, this leads to tachyon states already in the classical theory. This difficulty can be avoided by assuming that

$$M^2 = P^2,$$

where P_μ is the constant of the term linear in τ in the expansion (55). Using the condition (56) for $n = 0$, we obtain

$$M^2 = P^2 = P_\mu (f^2)^{\mu\nu} P_\nu - \gamma\pi \sum_{n\neq 0} \alpha_{-n}\alpha_n. \tag{60}$$

It will be shown in what follows that this quantity really is positive definite.

The above method is a covariant formalism for a relativistic string in an external field. One can also obtain a noncovariant solution of this problem as in the theory of the free string.

Noncovariant Formalism.[44] It is first of all necessary to find gauge conditions analogous to (13), and they must be compatible with the requirement $\dot{\sigma}_i(\tau) = 0$, $i = 1, 2$. In the choice of the gauge, it is convenient to use the boundary conditions (53), in which the functions $\sigma_i(\tau)$ are not yet fixed. We project Eqs. (53) onto the constant vector n:

$$nx' + nf\dot{x} + (n\dot{x} + nfx')\dot{\sigma} = 0, \quad \sigma = \sigma_i(\sigma), \quad i = 1, 2. \tag{61}$$

We now choose the gauge by specifying the conditions

$$nx' + nf\dot{x} = 0; \quad n\dot{x} + nfx' = n\Pi/\gamma l, \tag{62}$$

where Π_μ is the total canonical momentum of the string (58), (59). It follows from (61) and (62) that $\dot{\sigma}_i = 0$, i.e., the gauge (62) uniquely fixes the functions $\sigma_i(\tau)$.

One can show[44] that the conditions (62) in fact amount to a transition to new parameters $\tilde{\sigma} \pm \tilde{\tau} = f_\pm(\sigma \pm \tau)$ with functions $f\pm$ defined by the actual motion of the string.

Equations (62) and (7) make it possible to express two components of the vector $x_\mu(\sigma, \tau)$ in terms of the other two. For this, it is necessary to consider a definite configuration of the external electromagnetic field and to choose the constant vector n_μ.

An arbitrary electromagnetic field can be reduced by an appropriate Lorentz transformation to one of the following four cases[50]:

1) $\mathbf{E} \neq 0$, $\mathbf{H} = 0 (E^2 - H^2 > 0$, $\mathbf{E} \cdot \mathbf{H} = 0)$;

2) $\mathbf{E} = 0$, $\mathbf{H} \neq 0 (E^2 - H^2 < 0$, $\mathbf{E} \cdot \mathbf{H} = 0)$;

3) $\mathbf{E}$ and $\mathbf{H}$ are nonzero and parallel (the invariant $E^2 - H^2$ takes arbitrary values, and $\mathbf{E} \cdot \mathbf{H} = 0$);

4) the electric and the magnetic field in all frames of reference are equal in magnitude and perpendicular to each other ($E^2 - H^2 = 0$ and $\mathbf{E} \cdot \mathbf{H} = 0$).

If only the electric or only the magnetic field is nonzero, then the matrix $(F^2)^\mu_\nu$ is a multiple of the unit matrix and from the boundary conditions (54), using (7), we find that the velocity of the ends of the string is in this case equal to the velocity of light. For other configurations of the electromagnetic field, the boundary conditions do not uniquely determine the velocity of the ends of the string.

In all the four cases given above, one can obtain a solution in the form of a Fourier series and determine the operator of the square of the string mass.[44] It can be shown that an electric field increases the distance between the equidistant levels of this operator. It can be seen from the solutions that there exist limiting values of the electric field at which the solutions change their nature. These values are $E = \pm E_{cr}$, where $E_{cr} = (2\pi\hbar c\alpha' g)^{-1}$. If we assume that the charges at the ends of the string are in absolute magnitude equal to the electron charge, and $\alpha' \approx 0.9\ \mathrm{GeV}^{-2}$ (as is done when a free relativistic string is confronted with dual models), then $E_{cr} \approx 10^{22}$ V/cm. For comparison, we note that this is 10^{12} times stronger than the electric field acting on the electron in the hydrogen atom. For external field values $|\mathbf{E}| \ll E_{cr}$, $|\mathbf{H}| \ll E_{cr}$, all solutions for the string in the external field go over into the free solutions.

As an example, let us consider the case when $\mathbf{E}$ and $\mathbf{H}$ are nonzero and parallel. The problem is solved similarly in the remaining cases.

We direct the fields $\mathbf{E}$ and $\mathbf{H}$ along the x axis:

$$f_{01} = gE/\gamma = e, \quad f_{32} = gH/\gamma = h.$$

The constant vector n in the gauge conditions (62) can be conveniently taken to be

$$n^\mu = (1, 1, 0, 0). \tag{63}$$

Substituting (63) in (62), we obtain

$$x'^- = e\dot{x}^-; \quad \dot{x}^- = ex'^- + \sqrt{2}\,\Pi^-/\gamma l. \tag{64}$$

The boundary conditions (54) in component form are

$$x'^\pm \pm e\dot{x}^\pm = 0; \quad y' + h\dot{z} = 0; \quad z' - h\dot{y} = 0 \tag{65}$$

for $\sigma = 0$ and $\sigma = l$.

We shall assume that y and z are independent variables. The equations of motion (8) and boundary conditions (65) are satisfied by the following expansions of y and z in Fourier series:

$$\begin{gathered} y(\sigma, \tau) = [Q_y + P_y \tau/\gamma l - hP_z(\sigma - l/2)/\gamma l](1 + h^2)^{-1/2} \\ + \frac{i}{\sqrt{\pi\gamma(1 + h^2)}} \sum_{m\neq 0} \frac{\exp(-i\omega_m\tau)}{m} [\alpha_{my}\cos(\omega_m\sigma) + ih\beta_{mz}\sin(\omega_m\sigma)]; \\ z(\sigma, \tau) = [Q_z + P_z\tau/\gamma l + hP_y(\sigma - l/2)/\gamma l](1 + h^2)^{-1/2} \\ + \frac{i}{\sqrt{\pi\gamma(1 + h^2)}} \sum_{m\neq 0} \frac{\exp(-i\omega_m\tau)}{m} [\alpha_{mz}\cos(\omega_m\sigma) - ih\alpha_{my}\sin(\omega_m\sigma)], \end{gathered} \tag{66}$$

where $\omega_m = m\pi/l$.

The independent components $x^\pm$ are determined from Eqs. (64) and (7):

$$\left.\begin{aligned} \dot{x}^+ &= \frac{\gamma l}{2\Pi^-}(\dot{\mathbf{x}}_\perp^2 + \mathbf{x}_\perp'^2 - 2e\dot{\mathbf{x}}_\perp \mathbf{x}_\perp'); \quad \dot{x}^- = \frac{\Pi^-}{\gamma l}\frac{1}{1 - e^2}; \\ x'^+ &= -\frac{\gamma l}{2\Pi^-}[e(\dot{\mathbf{x}}_\perp^2 + \mathbf{x}_\perp'^2) - 2\dot{\mathbf{x}}_\perp \mathbf{x}_\perp']; \quad x'^- = \frac{\Pi^-}{\gamma l}\frac{e}{1 - e^2}, \end{aligned}\right\} \tag{67}$$

where $\mathbf{x}_\perp$ is a two-dimensional vector with components y and z. The dependent components can also be represented by Fourier series satisfying the boundary conditions (65):

$$\begin{gathered} x^\pm(\sigma, \tau) = \frac{i}{2\sqrt{\pi\gamma}} \sum_{n\neq 0} \frac{\exp(-i\omega_n\sigma)}{n} [(1 \pm e)\alpha_n^{(\pm)}\exp(-i\omega_n\tau) \\ - (1 \mp e)\alpha_{-n}^{(\pm)}\exp(i\omega_n\tau)] + \frac{p^\pm}{l\gamma}(\tau \mp e\sigma). \end{gathered} \tag{68}$$

Substituting (66) and (68) in (67), we obtain

$$\alpha_n^{(+)}=(\sqrt{\pi\gamma}/\Pi^-)L_{n\perp},\quad n=0,\pm1,\pm2,\ldots;\ \alpha_k^{(-)}=0;\ k\neq0;$$
$$\Pi^-=P^-(1-e^2), \tag{69}$$

where

$$L_{n\perp}=\frac{1}{2}\sum_{m=-\infty}^{+\infty}\alpha_{n-m\perp}\alpha_{m\perp};\ \alpha_0^\mu=\frac{P^\mu}{\sqrt{\pi\gamma}};\quad \mu=\pm,\ y,\ z.$$

The density of the canonical momentum (57) has the components

$$\pi_y=\gamma(\dot{y}+hz');\ \pi_z=\gamma(\dot{z}-hy');\ \pi^\pm=\gamma(\dot{x}^\pm\pm ex^{\pm\prime}).$$

The expansions (66) and (68) enable us to represent π_μ in the form

$$\pi_\perp=\frac{\sqrt{1+h^2}}{l}\Big[\mathbf{P}_\perp+\sqrt{\pi\gamma}\sum_{m\neq0}\exp(-i\omega_m\tau)\,\alpha_\perp\cos(\omega_m\sigma)\Big];$$
$$\pi^+=\frac{1-e^2}{l}\Big[P^++\sqrt{\pi\gamma}\sum_{m\neq0}\exp(-i\omega_m\tau)\,\alpha_m^{(+)}\cos(\omega_m\sigma)\Big];$$
$$\pi^-=\frac{1-e^2}{l}P^-.$$

The dynamics of the independent canonical variables $\mathbf{x}_\perp$ and $\pi_\perp$ is determined by the Hamiltonian function

$$H=\frac{\gamma}{2}\int_0^l d\sigma\{(\gamma^{-1}\pi_y-hz')^2+(\gamma^{-1}\pi_z+hy')^2+\mathbf{x}_\perp'^2\}$$
$$=\frac{\pi}{l}(1-e^2)L_{0\perp}.$$

For the square of the string mass $M^2=P^2$ we obtain from (69)

$$M^2=(1-e^2)^{-1}\,[\pi\gamma\sum_{m\neq0}\alpha_{-m\perp}\alpha_{m\perp}+e^2\mathbf{P}_\perp^2]. \tag{70}$$

Thus, M^2 is positive definite, since we assume that $e^2<1$. In the limit $e\to0$, Eq. (70) goes over into the expression for the square of the free string mass (19).

Quantum Theory. The quantization of a relativistic string in a constant homogeneous electromagnetic field is basically similar to the quantization of the free string. The total Lagrangian with allowance for the interaction (51) is singular, and therefore the canonical variables $x_\mu(\sigma,\tau)$ and $\pi_\mu(\sigma,\tau)$ satisfy the conditions

$$\left.\begin{aligned}\varphi_1&=(\gamma^{-1}\pi_\mu-f_{\mu\nu}x'^\nu)^2+x_\mu'^2=0;\\ \varphi_2&=(\gamma^{-1}\pi_\mu-f_{\mu\nu}x'^\nu)\,x'^\mu=0.\end{aligned}\right\} \tag{71}$$

The Hamiltonian constructed in accordance with the usual rules (30) vanishes identically. Therefore, the Hamiltonian function is taken to be a linear combination of the constraints (71):

$$H=-\frac{\gamma}{2}\int_0^l d\sigma\,[(\gamma^{-1}\pi_\mu-f_{\mu\nu}x'^\nu)^2+x_\mu'^2]. \tag{72}$$

The first Hamilton equation

$$\dot{x}_\mu=-\partial\mathscr{H}/\partial\pi^\mu=\gamma^{-1}\pi_\mu-f_{\mu\nu}x'^\nu \tag{73}$$

establishes a connection between the canonical momentum π_μ and the coordinate x_μ. The second Hamilton equation has the form

$$\dot{\pi}_\mu=\gamma x_\mu''-(\pi_\sigma'-\gamma f_{\sigma\nu}x''^\nu)f_\mu^\sigma. \tag{74}$$

It follows from (72) and (73) that

$$\ddot{x}_\mu-x_\mu''=0.$$

The boundary conditions

$$\partial\mathscr{H}/\partial x_\mu'=0,\ \sigma=0,\ l$$

with allowance for (73) are written as

$$x_\mu'+f_{\mu\nu}\dot{x}^\nu=0,\quad \sigma=0,\ l.$$

Therefore, the choice of the Hamiltonian function in the form (72) reduces the Hamiltonian formalism to the Lagrangian dynamics in the orthogonal gauge. Therefore, as solutions for $x_\mu(\sigma,\tau)$ and $\pi_\mu(\sigma,\tau)$ we can take the expansions (55) and (57). Doing this, we obtain the following expression for the Hamiltonian function (72):

$$H=\frac{\pi}{l}L_0=-\frac{\pi}{2l}\sum_{m=-\infty}^{+\infty}\alpha_{-m}\alpha_m.$$

The constraints (71) reduce, after the solutions (55) and (57) have been substituted in them, to the conditions (56) on the Fourier amplitudes α_n.

On the transition to the quantum theory, as in the free case, the commutation relations (38) are postulated, and the commutators between Q_μ, P_ν, and $\alpha_{n\rho}$ are assumed to be zero. This is equivalent to the usual "equal-time" commutators

$$[x_\mu(\sigma,\tau),\ \pi_\nu(\sigma',\tau)]=-ig_{\mu\nu}\delta(\sigma-\sigma');$$
$$[x_\mu(\sigma,\tau),\ x_\nu(\sigma',\tau)]=[\pi_\mu(\sigma,\tau),\ \pi_\nu(\sigma',\tau)]=0.$$

The problem now reduces to the elimination of states with negative norm. We shall follow Rohrlich's method and impose on the solution (55) conditions analogous to (44). For this, in the equations of the lightlike gauge for a string in an external field (62) we replace the constant vector n_μ by the vector K_μ, which is related to the total mechanical momentum P_μ of the string: $K_\mu=(1+f)^{-1}_{\mu\rho}P^\rho$. We now specify the gauge by

$$K_\mu x'^\mu+K_\mu f^{\mu\nu}\dot{x}_\nu=0;\ K_\mu\dot{x}^\mu+K_\mu f^{\mu\nu}x_\nu'=K_\mu\Pi^\mu/l\gamma. \tag{75}$$

Substituting (55) and (59) in (75), we obtain the requirement

$$\alpha_{n\mu}P^\mu=0,\quad n\neq0, \tag{76}$$

which means that in the center-of-mass system of the string ($\mathbf{P}=0$) the time Fourier components α_{n0} are zero: $\alpha_{n0}=0$ for $n\neq0$. Therefore, the space of state vectors with positive norm in the center-of-mass system of the string is constructed in the same way as for the free string. Further, these vectors must also satisfy the conditions (56) of the orthogonal gauge, which with allowance for (76) take the form

$$L_n\,|\,\Phi\rangle=\Big(\Lambda_n+\frac{1}{\sqrt{\pi\gamma}}\alpha_{n\mu}f^{\mu\nu}P_\nu\Big)\,|\,\Phi\rangle=0,\quad n>0; \tag{77}$$

$$L_0\,|\Phi\rangle=\Big(\Lambda_0-\frac{1}{2\pi\gamma}[P^2-P^\mu(f^2)_{\mu\nu}P^\nu]\Big)\,|\,\Phi\rangle=\alpha(0)\,|\,\Phi\rangle, \tag{78}$$

where $\Lambda_n=(-\frac{1}{2})\sum_{m\neq0,n}:\alpha_{n-m}\alpha_m:$. The operators in Eqs. (76)–(78) form a closed algebra, i.e., their commutators in the weak sense[33] vanish. Thus, this system of constraints is consistent.

Using the condition (78) written in the system in which $\mathbf{P}=0$, we can represent the expression for the square of the string mass in an external electromagnetic field in the form

$$M^2=P_0^2=\frac{2\pi\gamma}{1-(gE/\gamma)^2}\sum_{n=1}^{\infty}n{:}a_n^+a_n{:}+m_0^2, \tag{79}$$

where

$$m_0^2 = -2\pi\gamma\alpha(0)/[1-(gE/\gamma)^2].$$

The sign of m_0^2 is determined by the following considerations. The operator P_0^2 in the string center-of-mass system obviously has a positive spectrum, and the eigenvalues N of the operator $\sum_{n=1}^{\infty} n{:}a_n^+ a_n{:}$ can be equal to any positive integer $N=0,1,2,\ldots$. If $(gE/\gamma)^2<1$, then, applying the left- and right-hand sides of Eq. (79), respectively, to $|\Phi\rangle_{CM}$, we obtain the condition $m_0^2>0$ or $\alpha(0)<1$, i.e., the string has no tachyon states. But if $(gE/\gamma)^2>1$, then this leads to the requirement

$$m_0^2 > 2\pi\gamma N/\left|1-\left(\frac{g}{\gamma}E\right)^2\right|,$$

which cannot be satisfied by any finite value of m_0^2 since N can be arbitrarily large. We therefore arrive at the conclusion obtained earlier in the noncovariant formalism, namely, that the relativistic string in an external electromagnetic field can be studied in the orthogonal gauge only if $(gE/\gamma)^2<1$.

It follows from (79) that the operator M^2 has an equidistant spectrum, but that the distance between its levels is $[1-(gE/\gamma)^2]^{-1}$ times as large as in the free case.

Thus, Rohrlich's method makes it possible to construct a quantum theory of a relativistic string interacting with a constant electromagnetic field that is relativistically invariant and has no tachyon states. We shall not consider here quantization of the model by the noncovariant method,[44] since it is completely analogous to the theory for a free string.

Plane Monochromatic Wave. An exact solution of the equations of motion was obtained in Ref. 47 for a relativistic string in the field of a plane monochromatic electromagnetic wave. We shall not give the solution explicitly but merely mention its characteristic features. The ends of the string, as in the free case, move with the velocity of light. The solution has a resonance nature. The condition for the occurrence of resonance is

$$2kP = n,$$

where k is the wave vector of the electromagnetic field, P is the total momentum of the string, and n is any integer. The mass spectrum of the string is the same as in the free case.

6. RELATIVISTIC STRING WITH MASSES AT THE ENDS

Besides a string with charges, various models of a massive relativistic string have been considered.[51-57] The requirement of reparametrization invariance rules out a mass distribution along the string, just as it does a charge distribution. There remains the only possibility of placing the masses at the ends of the string. But if the reparametrization invariance is abandoned, then one cannot consistently introduce into the theory constraints on the dynamical variables (gauge conditions) by means of which states with negative definite norm can be eliminated.[53]

The action of a string with masses at the ends has the form

$$S = -\gamma\int_{\tau_1}^{\tau_2} d\tau \int_0^l d\sigma \sqrt{(\dot{x}x')^2-\dot{x}^2x'^2} - m\sum_{i=1}^{2}\int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^2(\sigma_i,\tau)}, \qquad (80)$$

where $\sigma_1=0$ and $\sigma_2=l$.

The equations of motion (8) and the subsidiary conditions (7) remain the same as in the free case. The boundary conditions however are now essentially nonlinear:

$$\left.\begin{aligned} m\frac{d}{d\tau}\left(\frac{\dot{x}_\nu}{\sqrt{\dot{x}^2}}\right) &= \gamma x'_\nu, \quad \sigma=0;\\ m\frac{d}{d\tau}\left(\frac{\dot{x}_\nu}{\sqrt{\dot{x}^2}}\right) &= -\gamma x'_\nu, \quad \sigma=l. \end{aligned}\right\} \qquad (81)$$

A particular solution was obtained for a string with masses at the ends rotating with angular velocity ω as a rigid rod.[52] By analogy with the free string case, we expect this motion to give the leading Regge trajectory, i.e., for given mass, the string has maximal angular momentum.

Setting $\tau=t$, we can represent the solution as

$$\mathbf{x}(\sigma,t)=\rho(\sigma)(\cos\omega t, \sin\omega t, 0),$$

in which the explicit form of the function $\rho(\sigma)$ is unimportant because of the invariance of the theory under σ and τ reparametrization. The only condition on the function $\rho(\sigma)$ is

$$\omega\rho(0) = -\omega\rho(l) = \omega R,$$

where $\omega R = \sqrt{1+(M\omega/2\gamma)^2} - m\omega/2\gamma$.

The expressions for the energy E and the angular momentum J have a fairly complicated form[5)]

$$E = (2\gamma/\omega)\left[\arcsin(\omega R) + m\omega/\gamma\sqrt{1-R^2\omega^2}\right];$$
$$J = (\gamma/\omega^2)\left[\arcsin(\omega R) + \omega R\sqrt{1-\omega^2R^2}\right] + 2m\omega R^2/\sqrt{1-R^2\omega^2}.$$

The connection between J and E^2 is now nonlinear.

It is not possible to obtain a general solution of the equations of motion (8) satisfying the nonlinear boundary conditions (81). But if we restrict ourselves to string motions for which the parameter τ is the proper time of the massive points at the ends of the string,[54] i.e.,

$$\dot{x}^2(0,\tau) = \dot{x}^2(l,\tau) = m^{-2}, \qquad (82)$$

then the boundary conditions (81) become linear:

$$\ddot{x}_\nu(0,\tau) = qx'_\nu(0,\tau); \quad \ddot{x}_\nu(l,\tau) = -qx'_\nu(l,\tau), \quad q=\gamma/m^2. \qquad (83)$$

By means of the change of variables $\sigma\pm\tau\to f_\pm(\sigma\pm\tau)$, under which the equations of motion (8) are invariant, the subsidiary conditions (7), and the boundary conditions (81), we can show that Eqs. (82) are satisfied if

$$\dot{x}^2(0,\tau+l) = \dot{x}^2(l,\tau); \quad \dot{x}^2(\sigma_i,\tau) = \dot{x}^2(\sigma_i,\tau+2l), \quad i=1,2.$$

Note that fixing the gauge by the single condition (82) would not lead to a restriction on the string motions if an infinite string were loaded with a point mass or if a mass were placed at the end of a semi-infinite string.[57]

Setting $x(\sigma,\tau)=\exp(-i\omega\tau)u(\sigma)$, we arrive at the following boundary-value problem for the function $u(\sigma)$:

$$u''(\sigma)+\omega^2u(\sigma)=0;$$
$$\omega^2u(0) = -qu'(0); \quad \omega^2u(l) = qu'(l).$$

The solution has the form

[5)] *Translator's note*. The Russian notation for the trigonometric, inverse trigonometric, hyperbolic trigonometric functions, etc., is retained here and throughout the article in the displayed equations.

$$u_n(\sigma) = N_n[\cos(\omega_n\sigma) - (\omega_n/q)\sin(\omega_n\sigma)], \quad (84)$$

where N_n are normalization factors and ω_n are the roots of the transcendental equation

$$\mathrm{tg}(\omega_n l) = 2\omega_n q/(\omega_n^2 - q^2),$$

which is equivalent to the two equations

$$\mathrm{tg}(\omega_n l/2) = -\omega_n/q \quad (n \text{ even});$$
$$\mathrm{ctg}(\omega_n l/2) = \omega_n/q \quad (n \text{ odd}).$$

The same equations, except for a different sign, were obtained in Ref. 53 in a study of a relativistic string with masses in the framework of a different approach.

The eigenfunctions (84) satisfy the orthogonality condition[58]

$$\int_0^l d\sigma u_n(\sigma)\xi(\sigma)u_m(\sigma) = \delta_{n,m}$$

and the completeness condition

$$\sum_{n=0}^{\infty} u_n(\sigma)u_n(\sigma')\xi(\sigma') = \delta(\sigma, \sigma'), \quad (85)$$

where $\xi(\sigma) = 1 + (1/q)[\delta(\sigma) + \delta(\sigma - l)]$, and the function $\delta(\sigma, \sigma')$ is determined by the requirement that

$$\int_0^l d\sigma'\delta(\sigma, \sigma')f(\sigma') = f(\sigma)$$

for any sufficiently smooth function $f(\sigma)$ defined on $0 \leq \sigma \leq l$.

The normalization factors are

$$N_n^{-2} = (l/2)(1 + \omega_n^2/q^2) + 1/q, \quad n > 0; \quad N_0^{-2} = l + 2/q.$$

For $x_\mu(\sigma, \tau)$, we now obtain the expansion

$$x_\mu(\sigma, \tau) = Q_\mu + \frac{q}{2+ql}\frac{P_\mu\tau}{\gamma} + \frac{i}{\sqrt{2\gamma}}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}\frac{\exp(-i\omega_n\tau)}{\omega_n}\alpha_n u_n(\sigma). \quad (86)$$

Thus, $x_\mu(\sigma, \tau)$ is an almost periodic function of σ and τ.[59]

When the masses at the ends of the string tend to zero, then $q \to \infty$,

$$\omega_n \to n\pi/l, \quad u_n(\sigma) \to \sqrt{2/l}\cos(n\pi\sigma/l)$$

and the expansion (86) goes over into the free string solution (11). Substituting (86) in the subsidiary conditions (7), we obtain

$$\sum_{n, m=-\infty}^{+\infty} \exp[-i(\omega_n + \omega_m)(\tau \pm \sigma)]\alpha_n\alpha_m N_n N_m$$
$$\times\left(1 \mp i\frac{\omega_n}{q}\right)\left(1 \mp i\frac{\omega_m}{q}\right), \quad (87)$$

where again $\alpha_{0\mu} = \sqrt{2/\gamma}P_\mu$. Since the frequencies ω_n, in contrast to the case of the free string, are not multiples of $\overset{\circ}{\omega} = \pi/l$, restrictions on the amplitudes α_n follow from (87):

$$\left.\begin{aligned}
&\alpha_n\alpha_m = 0 \quad \text{for } n \neq -m;\ n, m = \pm 1, \pm 2, \ldots; \\
&\alpha_n P = 0, \quad n \neq 0; \\
&P^2 = -\frac{\gamma}{2}\sum_{n\neq 0}\left(\frac{N_n}{N_0}\right)^2\left(1 + \frac{\omega_n^2}{q^2}\right)\alpha_{-n}\alpha_n.
\end{aligned}\right\} \quad (88)$$

The restrictions (88) are stronger than the Virasoro conditions (12) in the theory of a massless string. This result does not contradict the fact that the original action (80) has the same reparametrization invariance as the Nambu action (2) for a free string. The point is that here we consider, not all motions of a string with masses at the ends, but only the motions that satisfy the conditions (82). In the case of a massless string, $\overset{\circ}{\omega}_n = n\overset{\circ}{\omega}$, and the Virasoro conditions follow from (87):

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{+\infty}\alpha_{n-m}\alpha_m = 0, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Besides the restrictions (88), at the ends of the string the relations (82) must also hold, and these lead to

$$\frac{1}{2\gamma}\sum_{n\neq 0}N_n^2\frac{\omega_n^2}{q^2}\alpha_{-n}\alpha_n = -\frac{1}{m^2}. \quad (89)$$

The conserved total momentum of the string with masses at the ends is

$$\Pi_\mu = \int_0^l \pi_\mu(\sigma, \tau)d\sigma = \gamma\int_0^l \xi(\sigma)\dot{x}_\mu(\sigma, \tau)d\sigma = P_\mu.$$

The angular momentum tensor $M_{\mu\nu}$ is

$$M_{\mu\nu} = (Q_\mu P_\nu - P_\mu Q_\nu) - \frac{i}{2}\sum_{n\neq 0}\frac{1}{\omega_n}(\alpha_{-n\mu}\alpha_{n\nu} - \alpha_{-n\nu}\alpha_{n\mu}). \quad (90)$$

The square of the string mass is obviously

$$M^2 = P^2 = -\frac{\gamma}{2}\sum_{n\neq 0}\left(\frac{N_n}{N_0}\right)^2\alpha_{-n}\alpha_n + 4m^2\left(1 + \frac{ql}{2}\right)^2. \quad (91)$$

Even if the string does not execute any vibrational motions ($\alpha_n = 0$, $n \neq 0$), the square of its mass is nevertheless different from $4m^2$ and is equal to

$$M_0^2 = 4m^2(1 + lq/2)^2. \quad (92)$$

On the transition to the quantum theory, one postulates the commutation relations

$$[\alpha_{m\mu}, \alpha_{n\nu}] = \omega_m\delta_{m+n,0}g_{\mu\nu}; \quad [Q_\mu, P_\nu] = ig_{\mu\nu}.$$

With the completeness condition (85), these lead to the usual "equal-time" commutators for $x_\mu(\sigma, \tau)$ and $\pi_\mu(\sigma, \tau)$:

$$[x_\mu(\sigma, \tau), \pi_\nu(\sigma', \tau)] = [x_\mu(\sigma, \tau), \gamma\xi(\sigma')\dot{x}_\nu(\sigma'\tau)] = ig_{\mu\nu}\delta(\sigma, \sigma').$$

The remaining commutators vanish.

Using the solution (86), we can also find the commutators for different τ and τ', for example,

$$[x_\mu(\sigma, \tau), x_\nu(\sigma', \tau')] = i\frac{g_{\mu\nu}q}{\gamma(2+ql)}(\tau' - \tau)$$
$$+ \frac{g_{\mu\nu}}{2\gamma}\sum_{m\neq 0}\frac{\exp(-i\omega_m(\tau - \tau'))}{\omega_m}u_m(\sigma)u_m(\sigma').$$

As the Hamiltonian of the system, giving the correct equations of motion for $\alpha_n(\tau)$, we can take

$$H = \frac{P^2}{2}\frac{q}{\gamma(2+ql)} + \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\alpha_n. \quad (93)$$

The relations (88) and (89) for the amplitudes α_n are imposed in the quantum theory as conditions on the physical state vectors:

$$L_{nm}|\Phi\rangle \equiv \alpha_n\alpha_m|\Phi\rangle = 0, \quad n > 0, \ m \neq -n, \ 0; \quad (94)$$
$$G_n|\Phi\rangle \equiv \alpha_n P|\Phi\rangle = 0, \quad n > 0; \quad (95)$$

$$\left.\begin{aligned}
&P^2|\Phi\rangle = \left\{-\frac{\gamma}{2}\sum_{n\neq 0}\left(\frac{N_n}{N_0}\right)^2\left(1 + \frac{\omega_n^2}{q^2}\right) : \alpha_{-n}\alpha_n : + \beta_1\right\}|\Phi\rangle; \\
&\left\{\frac{1}{2\gamma}\sum_{n\neq 0}N_n^2\frac{\omega_n^2}{q^2} : \alpha_{-n}\alpha_n : + \beta_n\right\}|\Phi\rangle = -m^{-2}|\Phi\rangle,
\end{aligned}\right| \quad (96)$$

where the constants β_1 and β_2 arose on the transition to the operator normal product $:\alpha_{-n}\alpha_n:$. It is sufficient to require fulfillment of the conditions (94) and (95) for only the values of n and m specified in (94) and (95). In this case, the operators in (94)–(96) and the Hamiltonian (93) form a closed algebra. Therefore, the conditions (94)–(96) on the state vectors are compatible, and the evolution of the system in time does not result in these conditions ceasing to hold.

The condition (94) can be used, as it was by Rohrlich[40] in the case of a free string, to construct a physical space of state vectors with positive definite norm in the string center-of-mass system, in which $\mathbf{P}=0$. Since G_n and L_{nm} commute with the operator U [see (48)], which realizes transformations in the inhomogeneous Lorentz group, the physical space of vectors constructed in this way is relativistically invariant.

In particular, for the operator M^2 in the string center-of-mass system we have

$$M^2=\gamma\sum_{n=1}^{\infty}\left(\frac{N_n}{N_0}\right)^2\omega_n a_n^+ a_n+m_0^2,$$

$$a_n^+=\alpha_{-n}/\sqrt{\omega_n};\quad n>0,$$

where $m_0^2=\beta_1+\beta_2+M^2$; and m_0^2 is determined in (92).

The expressions obtained for the square of the string mass and the angular momentum tensor, (91) and (90), are exactly the same as in the theory of a massless string. However, the frequencies ω_n are no longer multiples of $\overset{\circ}{\omega}$, and therefore the mass spectrum is richer in this case. The major part of its characteristic degeneracy for a free string is lifted. The restrictions (94) on the state vectors differ appreciably from the Virasoro conditions (39). They can probably be satisfied only by a finite set of amplitudes α_n, and the number of nonzero α_n must not exceed the dimension of the space-time in which the string moves.

A similar problem was considered in Ref. 55, in which, however, Andreo and Rohrlich did not use the reparametrization-invariant Lagrangian (80) but rather a linearized action for a string with masses at its ends:

$$S=-\frac{\gamma}{2}\int_{\tau_1}^{\tau_2}d\tau\int_0^l d\sigma\,(\dot{x}^2-x'^2)-\frac{m^2}{2}\sum_{i=1}^{2}\int_{\tau_1}^{\tau_2}d\tau\ddot{x}^2(\sigma_i,\tau),\qquad(97)$$

$$\sigma_1=0;\quad\sigma_2=l.$$

This action leads immediately to the linear boundary conditions (83). However, the following difficulties arise here. First, it is not clear what relationship the obtained solutions have to a string with masses at its ends described by the ordinary action function (80); second, from the action (97) one cannot directly deduce gauge conditions by means of which states with negative-definite norm can be eliminated. The use[55] for this purpose of the Virasoro conditions from the theory of a massless string is rather artificial.

Nonrelativistic Limit in the Theory of a String with Massive Ends. This limit was investigated in detail by Chernikov and Shavokhina in Ref. 22. They showed that two massive point particles joined by a string move in the nonrelativistic limit in the same way as if an attractive potential that increases linearly with the distance acted between them:

$$V(\mathbf{x}_1,\mathbf{x}_2)=\gamma c\,|\mathbf{x}_2(t)-\mathbf{x}_1(t)|,\qquad(98)$$

where γ is the constant in the action (2) for the relativistic string.

The potential (98) generated by the string is independent of the properties of the particles at the ends of the string (for example, their masses). The linear potential deduced from quantum chromodynamics when fermion loops are ignored has the same universality.

A linear potential (at least for large distances between the quarks) is widely used in phenomenological composite quark models of hadrons.[60-64] In the framework of this approach, one can, for example,[62-64] describe the mass spectrum of the ψ/J mesons by choosing appropriately the parameters of the theory, in particular, the constant γ in (98) that determines the strength of the potential.

It is interesting to compare the values of the constant γ that are obtained in the phenomenological quark models and in the dual resonance approach. For this, one should consider quark models in which the strength of the potential is taken to be universal for the interaction of all quarks, viz., the light p and n quarks, the strange λ quarks, and the charmed quarks. Such a model was proposed in Ref. 61 by Kang and Schnitzer, who obtained for γ the value

$$\gamma=0.3(\mathrm{GeV})^2.\qquad(99)$$

On the other hand, the constant γ can be related to the universal slope α' of the Regge trajectories if the relativistic string is regarded as the dynamical basis of the dual models (see Appendix 1). In this approach, $\gamma=(2\pi\alpha')^{-1}$. Taking the experimental value $\alpha'=0.9$ GeV^{-2} for the slope of the Regge trajectories, we obtain

$$\gamma=0.2(\mathrm{GeV})^2.\qquad(100)$$

This agreement between the values of γ found in the composite quark model (99) and in the dual resonance approach (100) can hardly be fortuitous. It should probably be regarded as an indication that these two approaches have the same dynamical basis.

7. STRINGLIKE SOLUTIONS OF CLASSICAL FIELD EQUATIONS

The relativistic string defined by the action (2) is not to be taken as a real physical object but rather as a mathematical abstraction. There are in fact situations in quantum field systems for which a relativistic string can serve as an approximate model. We are referring here to the so-called vortex solutions of classical equations in certain field theory models. In this case, the string can be regarded as a field concentrated along a line.

Stringlike Solutions in the Theory of Superconductivity. Stringlike solutions are well known in the theory of superconductivity[12,13] and superfluidity.[65,66] The localization of magnetic field in one dimension can be pictured as follows. Take a long cavity within a superconductor and place two magnetic charges of opposite sign in it. Since the magnetic field does not penetrate into the superconductor, it will be completely localized within the cavity.

A more realistic example, which does not use hypothetical magnetic charges, is the penetration of magnetic field into type II superconductors (Abrikosov's vortices). It is known that when the external magnetic field has a certain value, it penetrates into a type II superconductor, but only in the form of thin filaments of magnetic lines of force. The central part of the filament is in the nonsuperconducting state, while the bulk of the conductor is in the superconducting state.

These vortex, or filament, solutions were found by Abrikosov in the Landau-Ginzburg superconductivity equations.[12,13] This phenomenological theory is based on the following expression for the free energy f of the superconductor:

$$f=f_0+\alpha(T)\,|\Psi|^2+\frac{\beta(T)}{2}\,|\Psi|^4+\frac{1}{2m^*}\left|\left(\frac{\hbar}{i}\nabla-\frac{e^*}{c}\mathbf{A}\right)\Psi\right|^2+\frac{\mathbf{B}^2}{8\pi}, \quad (101)$$

where f_0 is the free energy of the ground state, $\Psi(\mathbf{r},t)$ is the wave function of a Cooper pair of electrons, $e^*=2e$ is the charge of this pair, m^* is the effective mass of the pair, $\mathbf{B}$ is the external magnetic field, and T is the temperature.

Varying $\int d\mathbf{r}\,dt f(\mathbf{r},t)$ with respect to Ψ^*, we obtain the Landau-Ginzburg equation

$$\alpha(T)\Psi+\beta(T)|\Psi|^2\Psi+\frac{1}{2m^*}\left(-i\hbar\nabla-\frac{e^*}{c}\mathbf{A}\right)\Psi=0,$$

which must be augmented by the connection between the magnetic field and the current in Maxwell's theory:

$$\frac{c}{4\pi}\operatorname{curl}\mathbf{B}=\mathbf{J}=\frac{e^*\hbar}{2m^*i}(\Psi^*\nabla\Psi-\Psi\nabla\Psi^*)-\frac{e^{*2}}{m^*c}|\Psi|^2\mathbf{A}.$$

Investigation of these equations for a type II superconductor [the parameters of such a superconductor satisfy the condition $\varkappa^2=(1/2\pi)(m^*c/e^*\hbar)\beta>1/2$] that fills the whole of space and is in an external magnetic field parallel to the z axis shows that in this case there exists an approximate solution in which the magnetic field penetrates the superconductor in the form of thin filaments.[12] Such a solution in a cylindrical coordinate system (z,r,θ) has the form

$$B(z,r,\theta)=B(\rho)=\text{const}\,K_0(\rho),$$

where $\rho=r/\lambda$, λ is the characteristic length, and $K_0(\rho)$ is the Kelvin function, which satisfies $K_0(\rho)\sim\exp(-\rho)$ as $\rho\to\infty$. Thus, the magnetic field intensity decreases exponentially from the center of the filament.

Vortex Solutions in the Higgs Model. A relativistic generalization of the field model (101), which is the basis of the Ginzburg-Landau theory of superconductivity, is the Higgs Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-|(\partial_\mu+ieA_\mu)\Phi|^2+a|\Phi|^2-b(|\Phi|^2)^2, \quad (102)$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and a and b are positive constants. It is therefore natural that stringlike solutions localized in one dimension have also been found in the Euler equations corresponding to the Higgs Lagrangian.[14-18] We denote a static vortex solution by A^0_μ and Φ^0. It corresponds to a cylindrically symmetric vortex line along the z axis:

$$A^0_\mu(x_1,x_2)=\varepsilon_{03\mu\nu}(x^\nu/r)A_n(r);\quad \Phi^0(x_1,x_2)=\exp(in\theta)R_n(r),$$

where $A_n(r)$ and $R_n(r)$ are real functions. As a result of a complete rotation around the z axis, the phase Φ^0 changes by $2\pi n$, where n is an integer. If it is assumed that $|\Phi|\to$ const as $r\to\infty$, the vortex solution in the model (102) has the form

$$R_n(r)\approx\lambda+c_S\exp(-m_Sr);$$
$$A_\mu(r)\approx n/er+(c_V/\sqrt{er})\exp(-m_Vr),$$

where $m_S=\sqrt{2a}$, $m_V=\sqrt{2}e\lambda$, $\lambda=\sqrt{2a/2b}$, and c_S and c_V are constants. The magnetic field is parallel to the vortex line and equal to

$$H_n=\frac{1}{r}\frac{d}{dr}(rA_r)\sim(c_V/\sqrt{er})\exp(-m_Vr).$$

The transverse dimensions of the vortex are determined by the mass parameters m_S^{-1} and m_V^{-1}.

If there is a stringlike solution in a field model, one can show that the Lagrangian corresponding to this solution is the Nambu Lagrangian (2) for a relativistic string; for in this case the field functions are nonzero only along the string, and therefore

$$S_{\text{string}}\sim\int dt\int ds\sqrt{1-\mathbf{v}_\perp^2}, \quad (103)$$

where t is the time; s is a parameter which coincides with the string length, $(\partial\mathbf{x}(t,s)/\partial s)^2=1$; the factor $\sqrt{1-\mathbf{v}_\perp^2}$ is introduced in order to take into account the Lorentz contraction due to the motion of the string with the transverse velocity

$$\mathbf{v}_\perp=\frac{\partial\mathbf{x}}{\partial t}-\frac{\partial\mathbf{x}}{\partial s}\left(\frac{\partial\mathbf{x}}{\partial t}\frac{\partial\mathbf{x}}{\partial s}\right). \quad (104)$$

Substituting (104) in (103), we obtain

$$S_{\text{string}}\sim\int dt\int ds\sqrt{\left[1-\left(\frac{\partial\mathbf{x}}{\partial t}\right)^2\right]\left(\frac{\partial\mathbf{x}}{\partial s}\right)^2+\left(\frac{\partial\mathbf{x}}{\partial t}\frac{\partial\mathbf{x}}{\partial s}\right)^2}.$$

If we go over in the formula from t and s to arbitrary parameters τ and σ on the world surface of the string, we obviously obtain the Nambu action (2).

In the Higgs model (102), stringlike solutions could be infinite in space or closed, and in this latter case the radius of curvature of the string must be appreciably larger than its transverse dimensions. To obtain in this approach a finite string or vortex, it is necessary to introduce sources of the magnetic field—magnetic charges, or monopoles—on which the magnetic lines of force of the string could end.

Magnetic charges were first introduced into electrodynamics by Dirac.[67] From the point of view of the theory of a relativistic string, Dirac monopoles are also of interest because in electrodynamics with magnetic charges one can consider one-dimensional extended objects—Dirac filaments or strings.

Strings in Dirac's Theory of Magnetic Charges. In Dirac's theory, strings are a purely ancillary mathematical concept; they do not carry energy, and all observable quantities are independent of their motion. However, the situation is significantly changed if the electromagnetic field in Dirac's theory is replaced by a vector field with mass. In this field model, strings are physical objects, since they carry energy. The Lagrangian describing their motion is almost identical with the Nambu Lagrangian (2). Let us consider this model briefly.

In Dirac's electrodynamics with magnetic charges, Maxwell's equations are generalized as follows:

$$\partial F_{\mu\nu}/\partial x_\nu = -j^e_\mu; \tag{105}$$

$$\partial \tilde{F}_{\mu\nu}/\partial x_\nu = -j^m_\mu, \tag{106}$$

where $\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}/2$ is the dual tensor of $F_{\mu\nu}$; j^e_μ and j^m_μ are the currents generated by the electric and magnetic charges, respectively:

$$j^e_\mu(z) = \sum_e e \int \frac{dx_\mu}{ds} \delta^{(4)}[z - x(s)]\, ds;$$

$$j^m_\mu(z) = \sum_g g \int \frac{dx_\mu}{ds} \delta^{(4)}[z - x(s)]\, ds.$$

The difference from the ordinary theory is that on the right-hand side of Eq. (106) we do not have zero but $-j^m_\mu$. If there are magnetic charges, it is impossible to use the ordinary definition for the electromagnetic field tensor $F_{\mu\nu}$ in terms of the vector potential A_μ:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{107}$$

since it follows from (107) that

$$\mathbf{H} = \operatorname{curl} \mathbf{A} \text{ and } \operatorname{div} \mathbf{H} = 0,$$

whereas one must have

$$\operatorname{div} \mathbf{H} = \rho^m,$$

where ρ^m is the density of the magnetic charges.

Dirac assumed that Eq. (107) at each instant of time is violated at one point on a closed surface surrounding the magnetic charge. Since such a surface can be chosen arbitrarily, the relationship (107) is actually not satisfied along a filament or a string, which joins magnetic charges of opposite signs or goes away to infinity at one end. Each magnetic pole must be at the end of such a string.

In Minkowski space, a Dirac string covers a two-dimensional surface $x_\mu(\sigma, \tau)$, on which Eq. (3) is not satisfied. Dirac added to the right-hand side of this equation a tensor field $G_{\mu\nu}(z)$ localized on the world surface of the strings:

$$F_{\mu\nu}(z) = \partial_\mu A_\nu(z) - \partial_\nu A_\mu(z) + \tilde{G}_{\mu\nu}(z). \tag{108}$$

Substitution of (108) in (106) leads to an equation for $G_{\mu\nu}(z)$:

$$\partial_\nu G_{\mu\nu}(z) = j^m_\mu(z) = \sum_g g \int \frac{dx_\mu}{ds} \delta^{(4)}(z - x(s))\, ds.$$

The solution of this equation has the form

$$G_{\mu\nu}(z) = \int\int d^2v\, \delta^{(4)}(z - x(\sigma, \tau))\, \sigma_{\mu\nu}(\sigma, \tau), \tag{109}$$

where $\sigma_{\mu\nu} = \partial(x_\mu, x_\nu)/\partial(\tau, \sigma)$ and $d^2v = d\tau d\sigma$. The motion of the electric charges is described by the same equation as in Maxwellian electrodynamics:

$$m(d^2 z_\nu/ds^2) = e(dx^\mu/ds) F_{\nu\mu}(z). \tag{110}$$

A similar equation is postulated for the magnetic charges:

$$m d^2 z_\nu/ds^2 = g(dx^\mu/ds) \tilde{F}_{\nu\mu}(z). \tag{111}$$

The action in Dirac's theory is given by

$$S = -\sum_{e,g} m \int ds - \frac{1}{4} \int d^4z F_{\mu\nu} F^{\mu\nu} - e \int A^\mu(z) \frac{dz_\mu(s)}{ds} ds. \tag{112}$$

When this action is varied, $F_{\mu\nu}$ must be regarded as a function of the vector potential A_μ and the variables of the string in accordance with Eqs. (108) and (109). Variation of S leads to Eqs. (105), (110), and (111), whereas Eq. (106) is a consequence of (108) and (109), in complete analogy with Maxwell's theory. It is important that no equations arise for the string variables $x_\mu(\sigma, \tau)$, which reflects the unphysical nature of these variables.

Connection between Dirac's Strings and Dual Strings. In Ref. 19, Nambu considered the action (112) for two magnetic charges of opposite sign $\pm g$, but in contrast to Dirac's theory he assumed that the mass of the vector field $A_\mu(x)$ is nonzero and equal to m_V.[6)] One can, for example, assume that this mass arises because of the Higgs mechanism through an interaction of the field A_μ with the scalar field Φ in accordance with the Lagrangian (102). Equation (105) is now replaced by the Klein-Gordon equation with right-hand side

$$(\Box - m_V^2) A^\nu = -\partial_\mu \tilde{G}^{\mu\nu} \tag{113}$$

and the Lorentz condition $\partial_\mu A^\mu = 0$ is imposed on the field A_μ. It follows from (113) that

$$A^\nu(z) = -\int \Delta(z - y)\, \partial_\mu \tilde{G}^{\mu\nu}(y)\, d^4y, \tag{114}$$

where $\Delta(z)$ is the Green's function of Eq. (113). Using (114) to eliminate the field A_μ from (112), we obtain an expression for the effective action that depends only on the string variables $x_\mu(\sigma, \tau)$:

$$\int d^2v \mathcal{L}_{\text{eff}} = \frac{1}{4} g^2 m_V^2 \int\int d^2v\, d^2v' \sigma_{\mu\nu} \Delta(x - x') \sigma'^{\mu\nu}$$
$$+ \frac{1}{2} \sum_{i,j=1}^{2} \int\int g^{(i)} g^{(j)} \dot{x}^{(i)\mu} \Delta(x^{(i)} - x^{(j)'}) \dot{x}^{(j)'}_\mu\, d\tau\, d\tau'$$
$$- \sum_{i=1}^{2} \int m^{(i)} \sqrt{\dot{x}^{(i)}_\mu \dot{x}^{(i)\mu}}\, d\tau. \tag{115}$$

The first term in this equation is the Yukawa interaction of the two world surfaces $x_\mu(\sigma, \tau)$ and $x'_\mu(\sigma', \tau')$ with the interaction transmitted by the vector field $A_\mu(x)$. The second term is due to the interaction of the magnetic currents with one another and to their self-interaction. Finally, the third term is the kinetic term of the magnetic monopoles with masses $m^{(i)}$, $i = 1, 2$.

The string variables $x_\mu(\sigma, \tau)$ disappear from the effective action (115) if the vector field is massless, $m_V = 0$, and in this case the string becomes an unphysical entity, as we have already said.

Noting the δ-functional nature of the Green's function $\Delta(z)$, we see that the first term in Eq. (115) is proportional to the world surface area of the string $x_\mu(\sigma, \tau)$ joining the monopoles. A more detailed calculation[19] shows that this term really does reduce to the string Lagrangian

$$\mathcal{L}_{\text{string}} = -\gamma \sqrt{|\det|\sigma_{\mu\nu}||}, \tag{116}$$

where the constant γ is equal to

$$\gamma = 1/2\pi\alpha' = (g^2/8\pi)\, m_V^2 \ln(1/r_\perp m_V^2 + 1),$$

where $r_\perp$ characterizes the transverse dimensions of the

6)The name "magnetic charge" is in this case merely derived from the analogy between this model and the theory of the Dirac monopole.

string. As we showed above when we considered the Higgs Lagrangian (102), the transverse dimensions of the string are characterized by m_S^{-1}, where m_S is the mass of the scalar Higgs field, and we can therefore set $r_\perp \sim m_S^{-1}$ (see also Refs. 14-18).

If the distance between the magnetic charges is large compared with m_V^{-1}, then the string Lagrangian (116) will be dominant in Eq. (115). In this case, the first and third terms in (115) give the action (80) of a relativistic string with masses at its ends. At short distances, the Yukawa interaction between the monopoles, which is described by the second term in (115), becomes predominant. In the static case, the Yukawa attractive potential $(-g^2/4\pi)\exp(-m_V r)/r$ acts between the monopoles. This interaction will obviously shift the low lying levels in the string mass spectrum.

An indication of the possible existence of stringlike solutions in more realistic models of quantum field theory, and in particular in chromodynamics, is provided by the localized solutions in Yang-Mills models carrying magnetic charge.[68,69] An intimate connection with a string model was traced in two-dimensional models with the $U(N)$ color gauge group as $N \to \infty$ (Ref. 70) in scalar electrodynamics and in the Schwinger model.[71]

An important question, which still remains completely uninvestigated, is that of the stability of the stringlike solutions in both the classical case[72,73] and after quantization.[74]

8. NONLINEAR BORN-INFELD MODELS AND RELATIVISTIC STRINGS

A relativistic string is intimately related to the nonlinear Born-Infeld models in two-dimensional space-time.[31] The simplest model of such type for a scalar massless field $\varphi(x,t)$ is given by the Lagrangian

$$\mathscr{L} = -\varkappa^2 \sqrt{1+\varkappa^{-2}(\varphi_x^2 - \varphi_t^2)}, \tag{117}$$

where $\varphi_x = \partial\varphi(x,t)/\partial x$, $\varphi_t = \partial\varphi(x,t)/\partial t$, and $\varkappa$ is a constant with the dimensions of an inverse length. The Lagrangian (117) was considered, in particular, by Heisenberg[75] in mesodynamics in an investigation of meson showers.

Variation of this Lagrangian leads to a nonlinear equation for the field $\varphi(x,t)$:

$$(\varkappa^2 - \varphi_t^2)\varphi_{xx} + 2\varphi_x\varphi_t\varphi_{xt} - (\varkappa^2 + \varphi_x^2)\varphi_{tt} = 0. \tag{118}$$

As $\varkappa \to \infty$, the Lagrangian (117) goes over into the free field Lagrangian $\varphi(x,t)$, and Eq. (118) reduces to the d'Alembert equation.

The problem can be formulated differently, in a parametric form. The function $\varphi(x,t)$ describes a surface in the three-dimensional space-time with coordinates $t, x, y = \varkappa^{-1}\varphi(x,t)$. This surface can be specified parametrically if we introduce a Lorentz vector x_μ, $\mu = 0, 1, 2$, which depends on the two parameters σ and τ and has the components

$$x^\mu(\sigma,\tau) = (t(\sigma,\tau),\ x(\sigma,\tau),\ y(\sigma,\tau) = \varkappa^{-1}\varphi(x(\sigma,\tau),\ t(\sigma,\tau))).$$

Remembering that

$$\varphi_x = \varkappa^2 \frac{\begin{vmatrix} y' & t' \\ \dot{y} & \dot{t} \end{vmatrix}}{\begin{vmatrix} x' & t' \\ \dot{x} & \dot{t} \end{vmatrix}}; \quad \varphi_t = \varkappa^2 \frac{\begin{vmatrix} x' & y' \\ \dot{x} & \dot{y} \end{vmatrix}}{\begin{vmatrix} x' & t' \\ \dot{x} & \dot{t} \end{vmatrix}},$$

and that on the transition to an integration with respect to the variables σ and τ the product $dxdt$ must be replaced by $(x'\dot{t} - \dot{x}t')d\sigma\, d\tau$, we obtain the following expression for the action function corresponding to the Lagrangian (117):

$$S = -\varkappa^2 \int dt \int dx \sqrt{1+\varkappa^{-2}(\varphi_x^2 - \varphi_t^2)}$$
$$= -\varkappa^2 \int d\tau \int d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}. \tag{119}$$

This is then the action (2) of an infinite relativistic string in the three-dimensional space-time $t, x, y = \varkappa^{-1} \times \varphi(x,t)$.

If in the Born-Infeld electrodynamics

$$\mathscr{L} = -\varkappa^2 \sqrt{1+\varkappa^{-2}F - \varkappa^{-4}G^2},$$

where

$$F = (1/2) F_{\mu\nu}F^{\mu\nu};\ G = (1/4)\,\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma};\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

we consider only plane wave, then the problem reduces to investigating the Lagrangian of a relativistic string. Suppose that the wave propagates along the x axis. The potential $A_\mu(x)$ in this case depends only on t and x. If we eliminate the electric field directed along the x axis, $F_{01} = 0$, then the Born-Infeld Lagrangian contains only two components of the potential $A_\mu(x)[A_y(x,t)$ and $A_z(x,t)]$:

$$\mathscr{L} = -\varkappa^2 \sqrt{\Big(1+\varkappa^2 \sum_{i=y,z} A_{i,x}^2\Big)\Big(1-\varkappa^2 \sum_{i=y,z} A_{i,t}^2\Big) + \Big(\varkappa^2 \sum_{i=y,z} A_{i,t}A_{i,x}\Big)^2}.$$

The transition from the variables (x,t) to (σ,τ) again leads to the action (119) for a relativistic string in the four-dimensional space $(t, x, \varkappa^{-1}A_y, \varkappa^{-1}A_z$. The most general form of the Lagrangian of n fields of the Born-Infled type in two-dimensional space (x,t), which reduces to the Lagrangian of a string in a space of $n+2$ dimensions when the parameters σ and τ are introduced, is given by the expression

$$\mathscr{L} = -\varkappa^2 \sqrt{\Big(1+\varkappa^{-2}\sum_{i=1}^{n} \varphi_{i,x}^2\Big)\Big(1-\varkappa^{-2}\sum_{i=1}^{n} \varphi_{i,t}^2\Big) + \Big(\varkappa^{-2}\sum_{i=1}^{n} \varphi_{i,t}\varphi_{i,x}\Big)^2}. \tag{120}$$

The action (120), which describes simultaneously a system of nonlinear Born-Infeld scalar fields and an infinite relativistic string, was investigated in detail in Refs. 76 and 77. The Cauchy problem was solved for the equations of motion and the scattering of two plane waves was investigated in the classical case. A method was also proposed for quantization in which nonlinear subsidiary conditions are imposed on the state vectors. In the same papers, an algebra of constraints was obtained that includes as a special case the Virasoro algebra (41) in the theory of a finite string.

CONCLUSIONS

The various aspects of the dynamics of a relativistic string considered here show that this object has numerous properties of undoubted interest from the point of view of the physics of elementary particles. Above all, a string is an example of a comparatively simple but fairly realistic model of quark confinement in hadrons. The string model has been developed in this direction by the introduction of internal quantum numbers of quarks placed at the ends of the string,[20,21] and also by attempts to construct a string model of baryons.[78-80]

Developing the theory of a string, one can attempt to extend the geometrical approach that provides its basis. On the world surface of the string one can consider not only the area of the surface but also other invariants; for example, to the Nambu action (2) one can add a term proportional to the integral Gaussian curvature of the string surface.[54] It is interesting that in this case only the boundary conditions change, and the equation of motion of the string is as before the d'Alembert equation. However, the nonlinear nature of the boundary conditions makes it impossible to obtain an exact solution of this problem.

One can consider relativistic objects of more dimensions such as membranes[81] or three-dimensional objects. The coordinates of such objects in Minkowski space are given by a Lorentz vector $x_\mu(\xi^0, \xi^1, \ldots, \xi^n)$, which for $n=1$ describes a string, for $n=2$ a membrane, etc. By analogy with the theory of a relativistic string, the action can be taken to be proportional to the area of the hypersurface that this object sweeps out in Minkowski space:

$$S = -\varkappa \int \ldots \int d\xi^0 \ldots d\xi^n \left(|\det g_{ij}(\xi^0, \xi^1, \ldots, \xi^n)|\right)^{1/2}, \qquad (121)$$

where $g_{ij} = (\partial x_\mu/\partial \xi^i)(\partial x^\mu/\partial \xi^j)$ is the metric tensor on the hypersphere and $\varkappa$ is a constant with the dimensions l^{-n-1}. On the variables $x_\mu(\xi^0, \xi^1, \ldots \xi^n)$ one can impose $n+1$ conditions in accordance with the number of the parameters ξ^i. However, to linearize the equations of motion that follow from the action (121) we need $(n^2+3n)/2$ conditions. Therefore, it is not possible to linearize the equations of motion for relativistic objects with more dimensions than a string.

APPENDIX 1

Dual models

To construct dual amplitudes, a special mathematical formalism was developed[1-4]; it is an operator formalism similar to the Feynman diagram technique in quantum field theory. In this approach, one can most readily trace the connection between dual models and relativistic strings.

In the operator formalism, one introduces an infinite set of creation and annihilation operators, $a^+_{n\nu}$ and $a_{m\nu}$, which satisfy the commutation relations

$$[a_{m\mu}, a^+_{n\nu}] = -g_{\mu\nu}\delta_{m,n}, \qquad (A.1)$$

where $g_{00} = -g^{11} = \ldots = -g^{D-1,D-1}$, and D is the dimension of the pseudo-Euclidean space on which the operators $a_{m\mu}$ act. With the Veneziano amplitude B_N that has N external lines there is associated the tree diagram in Fig. 4. The amplitude B_N is constructed in accordance with the following correspondence rules. The external

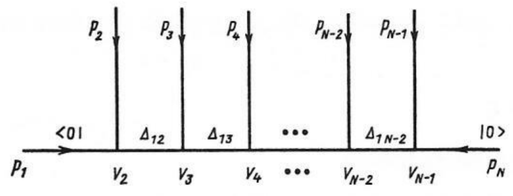

FIG. 4.

ends with momenta p_1 and p_N are associated with the vacuum states $\langle 0|$ and $|0\rangle$, which are, as usual, defined by the requirement

$$a_{m\mu}|0\rangle = \langle 0|a^+_{m\mu} = 0.$$

To each vertex there corresponds a vertex operator $V(p_i)$, $i = 2, 3, \ldots, N-1$:

$$V(p_i) = \exp\left(i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{p_i a_n^+}{\sqrt{n}}\right)\exp\left(i\sqrt{2\alpha'}\sum_{n=1}^{\infty}\frac{p_i a_n}{\sqrt{n}}\right),$$

where α' is the slope of the Regge trajectories. With the internal lines in the diagram there is associated the propagator

$$\Delta_{ij} = [s_{ij} + \alpha' M^2 + \alpha(0)]^{-1},$$

where $s_{ij} = (p_i + p_{i+1} + \ldots + p_j)^2$, $\alpha(0)$ is the intercept of the Regge trajectory, $\alpha(s_{ij}) = \alpha(0) + \alpha' s_{ij}$, and $\alpha' M^2 = \sum_{n=1}^{\infty} n a_n^+ a_n$, is the mass operator.

In accordance with these rules, we obtain for the amplitude B_N the expression

$$B_N = \langle 0|V(p_2)\Delta_{12}V(p_3)\Delta_{13}\ldots V(p_{N-1})|0\rangle.$$

Calculating the vacuum expectation value in this expression, we can transform B_N to the ordinary integral representation for the N-point Veneziano amplitude[1-4]:

$$B_N = \int_0^1 \ldots \int_0^1 \prod_{i=1}^{N-3} dx_i x_i^{-\alpha(s_{1i})-1} \prod_{1\le i<j\le N} (1-x_{ij})^{-p_i p_j}, \qquad (A.2)$$

where $x_{ij} = x_{i-1}x_i \ldots x_{j-2}$.

The state vectors in this formalism are constructed, as usual, by applying the creation operators $a^+_{m\mu}$ to the vacuum. However, because of the metric tensor $g_{\mu\nu}$ in the commutator (A.1), vectors with negative-definite norm are obtained. The physical state vectors $|\Phi\rangle$ with positive-definite norm are determined by the Virasoro conditions

$$L_n|\Phi\rangle = 0, \quad n = 1, 2, 3, \ldots;$$
$$[L_0 - \alpha(0)]|\Phi\rangle = 0,$$

where

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{+\infty} :\alpha_{n-m}\alpha_m:; \quad \alpha_{0\mu} = \sqrt{2\alpha'}\, p_\mu;$$

$$\alpha_{-k} = \alpha_k^+ = \sqrt{k}\, a_k^+, \quad k = 1, 2, \ldots .$$

To eliminate states with negative-definite norm, we must require $\alpha(0) = 1$ and that the dimension D of the space on which the operators $a_{n\mu}$ act be 26.

Thus, the operator formalism in the dual approach is identical with the quantum theory of a relativistic string that uses the lightlike gauge. The constant γ in the Nambu gauge (2) is related to the universal slope α' of the Regge trajectories: $\gamma = (2\pi\alpha')^{-1}$. Moreover, it has been shown that the dual amplitude (A.2) can be obtained as a quantum-mechanical probability amplitude in a theory which considers the joining and separation of strings.[82]

APPENDIX 2

Point limit in string theory

We show that the relativistic string action (2) goes over into the action (1) of a point particle if the length of the string tends to zero. First, we represent the dimensional constant γ in Eq. (2) in the form

$$\gamma = m_0 c / l_0,$$

where m_0 and l_0 are constants with the dimensions of mass and length, respectively, and c is the velocity of light. Then in the string action (2) we separate the integration with respect to σ in such a way that it gives the string length:

$$S = -m_0 c \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^2(\sigma^*, \tau)} \frac{1}{l} \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma \sqrt{\frac{(\dot{x}x')^2}{\dot{x}^2} - x'^2}$$

$$= -m_0 c \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^2(\sigma^*, \tau)} \frac{L(\tau)}{l_0}, \qquad \text{(A.3)}$$

where $L(\tau) = \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} d\sigma [\dot{x}x')^2/\dot{x}^2 - x'^2]^{1/2}$ is the ordinary three-dimensional length of the string written in covariant form,[6,50] and $\sigma^*(\tau)$ is a point in the interval $[\sigma_1(\tau), \sigma_2(\tau)]$, $\tau_1 \leqslant \tau \leqslant \tau_2$. If we now assume that $L(\tau) \to 0$ and accordingly $l_0 \to 0$, with $(L(\tau)/l_0) \to a < \infty$, then Eq. (A.3) goes over into the action of a relativistic point particle with mass am_0.

APPENDIX 3

Noether identities in string theory

The invariance of the relativistic string action (2) under transformation of the parameters σ and τ leads to identities that must be satisfied by the left-hand sides of the equations of motion (the second part of Noether's theorem[30]).

We shall regard $x_\mu(\sigma, \tau)$ as fields defined in the two-dimensional space $\tau = \xi_1$, $\sigma = \xi_2$, and choose the Lagrangian of the system $L(x, x_{,i})$, where $x_{,i} = \partial x / \partial \xi_i$, in such a way that the action $S = \int d^2\xi \mathcal{L}(x, x_{,i})$ is invariant under the coordinate transformations $\xi_i \to \tilde{\xi}_i = f_i(\xi), i = 1, 2$. These transformations are determined by two arbitrary parameters, and therefore in accordance with Noether's theorem (second part) the left-hand sides of the Euler equations must satisfy two identities.

For a finite string $0 \leqslant \xi_2 \leqslant l$, and therefore the functions f_i satisfy in this case the conditions $f_i(\xi_1, 0) = 0$ and $f_2(\xi_1, l) = l$, from which it follows that the infinitesimally small variations $\varepsilon_i(\xi)$ of the coordinates:

$$\tilde{\xi}_i = \xi_i + \varepsilon_i(\xi)$$

must vanish for $\sigma = 0$ and $\sigma = l$. As will be seen from what follows, these requirements do not lead to any restrictions in the results obtained.

The variation δS of the action, which is zero, has the form

$$\delta S = \int d^2\xi [\delta \mathcal{L} + \partial_i (\mathcal{L} \varepsilon_i)], \qquad \text{(A.4)}$$

where $\delta \mathcal{L}$ is the variation of the form of the Lagrangian:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} \delta x_\mu + \frac{\partial \mathcal{L}}{\partial x_{\mu, i}} \delta x_{\mu, i}.$$

Denoting the left-hand sides of the Euler equations by

$$L^\mu = \frac{\partial \mathcal{L}}{\partial x_\mu} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial x_{\mu, i}} \right)$$

and remembering that $\delta x_\mu = -x_{\mu,j}\varepsilon_j$, we transform Eq. (A.4) to

$$\delta S = \int d^2\xi \left\{ \left[\left(\mathcal{L} \delta_{ij} - \frac{\partial \mathcal{L}}{\partial x_{\mu, i}} x_{\mu, j} \right) \varepsilon_j \right]_{,i} - L^\mu x_{\mu, j} \varepsilon_j \right\} = 0. \qquad \text{(A.5)}$$

Initially, we consider variations ε_j of the independent variables ξ_j that vanish on the boundary of the domain of integration. It is then obvious that the term in the square brackets in Eq. (A.5) does not contribute to δS, and, as a consequence of this, we obtain the two identities

$$L_\mu \dot{x}^\mu = L_\mu x'^\mu \equiv 0. \qquad \text{(A.6)}$$

Thus, the projections of the left-hand sides of the Euler equations onto $\dot{x}_\mu$ and x'_μ vanish identically.

Equation (A.5) contains, in the curly brackets, the energy-momentum tensor with respect to shifts in the space ξ_i:

$$t_{ij} = \frac{\partial \mathcal{L}}{\partial x_{\mu, i}} x_{\mu, j} - \delta_{ij} \mathcal{L}.$$

If ε_j are constants and x_μ satisfy the equations of motion $L_\mu(x) = 0$, then from (A.5) we obtain the conservation laws

$$\partial_i t_{ij} = 0, \quad j = 1, 2. \qquad \text{(A.7)}$$

These equations are in fact improper conservation laws,[30,83] since S is invariant under arbitrary coordinate transformations, and therefore Eqs. (A.7) are satisfied identically by virtue of (A.6), and not only on solutions of the equations of motion. Moreover, we shall show below that in fact $t_{ij} \equiv 0$.

The point is that the conservation laws (A.7) and the Noether identities (A.6) do not exhaust all the consequences of the invariance of the action S under arbitrary transformations of the coordinates ξ_i.[84,85] To see this, we first take the parameters of the functions ε_i in (A.5) to be constants, and then linear functions in ξ_i; then, using (A.6), we obtain the new identities

$$t_{ij,i} \equiv 0, \quad j = 1, 2; \quad t_{ij} \equiv 0, \quad i, j = 1, 2.$$

This procedure is equivalent to equating to zero the coefficients of $\varepsilon_i(\xi)$ in Eq. (A.5) for arbitrary $\varepsilon_{i,j}(\xi)$ as well.

We write the identities $t_{ij} \equiv 0$ in the explicit form

$$\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{x}^\mu = 0; \quad \mathcal{L} - \frac{\partial \mathcal{L}}{\partial x'^\mu} x'^\mu = 0;$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} x'^\mu = 0; \quad \frac{\partial \mathcal{L}}{\partial x'^\mu} \dot{x}^\mu = 0.$$

The first equation means that the Hamiltonian constructed in accordance with the canonical rules vanishes in the given case [cf. Eqs. (29) and (30)]. The third identity is a connection between the canonical variables x_μ and π_μ: $\pi_\mu = -\partial \mathcal{L} / \partial \dot{x}^\mu$, $x'_\mu \pi^\mu = 0$.

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