Stochastic processes in dynamical systems

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Stochastic processes in dynamical systems are considered for the case of the weak interaction of a small system (for example, one particle) with a large system.[1)]

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INTRODUCTION

In Ref. 1, published in 1939, Krylov and the present author investigated the possibility of a stochastic process in a dynamical system that is under the influence of a large system. The behavior of a classical system was investigated on the basis of the Liouville equation for the probability distribution in the phase space; for a quantum-mechanical system, the investigation was based on the analogous equation for the von Neumann statistical operator. In Ref. 1 a method was developed that makes it possible to obtain Fokker-Planck equations already in the first approximation. In developing and using the method in Ref. 1, we did not give a systematic and mathematically rigorous justification of it; in Ref. 2, a detailed investigation was made of a particular model whose dynamical behavior was described by exactly integrable equations. On this rigorous mathematical basis, it was possible to analyze the approximations proposed earlier. Analogous results for quantum-mechanical systems were obtained in Ref. 3.

In my lectures given in Fall 1974, I presented a slightly modified version of the method developed in Ref. 1 and discussed its connection with the theory of two-time Green's functions.

Preparing the present work for publication on the basis of these lectures, I have taken into account in the final variant of the method the results of some important investigations in the theory of the interaction of one particle with a large system which have been made in the last decade.

In this connection, it seemed to me appropriate to introduce some important modifications at a number of places in the original text.

SECTION 1

We consider a small system S, which could be, for example, simply an individual particle, which interacts weakly with a large system Σ. Initially, we study this case in the framework of classical mechanics.

Following the usual procedure adopted in classical statistical mechanics, we introduce the probability distribution function in the phase space of the complete system $S+\Sigma$:

$$\mathcal{D}_t = \mathcal{D}_t(S, \Sigma) = \mathcal{D}_t(\Omega_S, \Omega_\Sigma), \quad (1)$$

where Ω_S and Ω_Σ are the phase points corresponding to the phase spaces of the systems S and Σ, respectively.

We now consider the situation when the system Σ at the initial time $t=0$ is in a state of statistical equilibrium and at this time the interaction between S and Σ is switched on. Thus, we assume that

$$\mathcal{D}_0(S, \Sigma) = f_0(S)\,\mathcal{D}(\Sigma), \quad (2)$$

where

$$\mathcal{D}(\Sigma) = \mathcal{D}_{eq}(\Sigma) = Z^{-1}\exp[-H_\Sigma(\Omega_\Sigma)/\theta],$$
$$Z = \int d\Omega_\Sigma \exp[-H_\Sigma(\Omega_\Sigma)/\theta]$$

is the equilibrium distribution in the phase space of Σ. Here, $H_\Sigma = H_\Sigma(\Omega_\Sigma)$ is the energy of Σ.

The evolution of the probability distribution is, of course, determined by the Liouville equation, which we write in the form

$$\partial\mathcal{D}_t/\partial t = \mathcal{L}\mathcal{D}_t; \quad (3)$$

the normalization condition for $\mathcal{D}_t$ is given by the equation

$$\int \mathcal{D}_t\, d\Omega_S\, d\Omega_\Sigma = 1.$$

Applied to the functions $(\Omega_S, \Omega_\Sigma)$, the Liouville operator $\mathcal{L}$ can be determined by the Poisson brackets

$$\mathcal{L}\mathcal{D}_t = [H, \mathcal{D}_t], \quad (4)$$

where H is the total Hamiltonian of the system $S+\Sigma$.

Note that we shall consider only the cases when $\mathcal{L}$ does not depend explicitly on the time t.

Usually, the total Hamiltonian H is represented in the form of the sum

$$H = H_S^0 + H_\Sigma + H_{int}$$

of the intrinsic Hamiltonians of the systems S and Σ augmented by a term which describes the interaction between these two systems. Accordingly, we choose the Liouville operator in the form

$$\mathcal{L} = \mathcal{L}_{S+\Sigma} = \mathcal{L}_S^0 + \mathcal{L}_\Sigma + \mathcal{L}_{int}. \quad (5)$$

Below, the interaction term $\mathcal{L}_{int}$ of $\mathcal{L}$ will be regarded as a weak perturbation, i.e., we shall assume that it contains a small parameter.

We now give some concrete examples of S, Σ, and $\mathcal{L}$.

1) This paper is an edited translation of the author's English preprint: "On the stochastic processes in the dynamical systems," E17-10514, JINR, Dubna (1977).

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We consider the case when S is one particle, and Σ is a system consisting of N identical particles, so that

$$\Omega_S=(\mathbf{r}_0, \mathbf{v}_0);\ \Omega_\Sigma=(\mathbf{r}_1, \mathbf{v}_1, \ldots, \mathbf{r}_N, \mathbf{v}_N), \tag{6}$$

where $\mathbf{r}$ and $\mathbf{v}$ are the positions and velocities of the corresponding particles.

As usual, all these particles are assumed to be enclosed within a very large cube of macroscopic volume V; the usual cyclic boundary conditions are imposed.

We take the following expressions for Л_S^0, Л_{int}:

$$\text{Л}_S^0=-\mathbf{v}_0\partial/\partial\mathbf{r}_0; \tag{7}$$

$$\text{Л}_{\text{int}}=\text{Л}_{\text{int}}^{(\Phi)}=\sum_{(1\leqslant j\leqslant N)}\frac{\partial\Phi(\mathbf{r}_0-\mathbf{r}_j)}{\partial\mathbf{r}_0}\left(\frac{1}{m}\frac{\partial}{\partial\mathbf{v}_0}-\frac{1}{M}\frac{\partial}{\partial\mathbf{v}_j}\right), \tag{8}$$

where $\Phi(\mathbf{r})$ is some radially symmetric potential function proportional to the small parameter, m is the mass of the particle S, and M is the mass of one of the particles of the system Σ.

We consider also the important special case when the interaction between the particle S and a particle of Σ can be defined as the interaction between corresponding impenetrable spheres.

Formally, the interaction between impenetrable spheres can be characterized by a special choice of $\Phi(r)$:

$$\left.\begin{array}{ll}\Phi(r)\to+\infty, & \text{if}\quad r<a;\\ \Phi(r)=0, & \text{if}\quad r\geqslant a,\end{array}\right\} \tag{9}$$

where a is the sum of the radii of particle S and a particle of Σ, or, which is the same thing, a is the distance between the centers of these particles at the instant of collision.

For such a potential function, the expression (8) is obviously singular and inconvenient to use. It has, however, been found that the dynamics of interacting hard spheres can be correctly described by means of an integrated Liouville operator of the form

$$\text{Л}_{\text{int}}^{\text{coll}}=\sum_{(1\leqslant j\leqslant N)}\overline{T}(0, j), \tag{10}$$

where

$$\overline{T}(0, 1)=a^2\int_{(\mathbf{v}_{0,1}\sigma)>0}(\mathbf{v}_{0,1}\sigma)\,\{\sigma(\mathbf{r}_0-\mathbf{r}_1-a\sigma)\,B_{\mathbf{v}_0,\mathbf{v}_1}(\sigma)-\delta(\mathbf{r}_0-\mathbf{r}_1+a\sigma)\}\,d\sigma; \tag{11}$$

$\mathbf{v}_{0,1}=\mathbf{v}_0-\mathbf{v}_1$; σ is a unit vector; $B_{\mathbf{v}_0,\mathbf{v}_1}(\sigma)$ is an operator which is applied to the function $F(\mathbf{v}_0, \mathbf{v}_1)$ and carries its arguments $\mathbf{v}_0$ and $\mathbf{v}_1$ into

$$\left.\begin{array}{l}\mathbf{v}_0\to\mathbf{v}_0^*=\mathbf{v}_0-\dfrac{2M}{M+m}\,\sigma(\mathbf{v}_{0,1}\sigma);\\ \mathbf{v}_1\to\mathbf{v}_1^*=\mathbf{v}_1+\dfrac{2M}{M+m}\,\sigma(\mathbf{v}_{0,1}\sigma).\end{array}\right\} \tag{12}$$

The expression (8) can be replaced by the integrated operator (10) because the interaction of impenetrable solid bodies is instantaneous in classical mechanics. It is worth mentioning in this connection that in the analogous situation in the quantum-mechanical description the replacement of the Poisson brackets $[H_{\text{int}}, \mathfrak{D}]$ by an operator interpreted in terms of collisions and applied to $\mathfrak{D}$ must be regarded as an approximation that holds only when the effective collision time (which is here appreciably greater than zero) is negligible compared with the characteristic time of the process. In contrast, we do not make any approximations for systems described by classical mechanics when we use $\text{Л}_{\text{int}}^{\text{coll}}$ instead of (8), though we must of course eliminate unphysical overlapping configurations by requiring that $\mathfrak{D}$ vanish for them.

One can also consider the case when the hard-sphere interaction is augmented by a further regular two-body interaction $(0, j)$ described by a smooth function $\Phi(r)$ proportional to a small parameter that is defined for $r\geqslant a$ and extended formally for $r<a$ by the requirement

$$\Phi'(r)=0 \quad \text{for} \quad r<a.$$

In this case,

$$\text{Л}_{\text{int}}=\text{Л}_{\text{int}}^{\text{coll}}+\text{Л}_{\text{int}}^{(\Phi)}. \tag{13}$$

Note that to regard $\text{Л}_{\text{int}}^{\text{coll}}$ as a small perturbation we must assume that the corresponding mean free path $\sim(Na^2/v)^{-1}$ is much greater than a:

$$Na^3/v\ll 1. \tag{14}$$

We emphasize that the condition (14) does not presuppose that the interaction between the particles of the system Σ is small.

We consider a model in which S is a neutron interacting only with the nuclei of the particles of the system Σ (we shall also simulate these nuclei by impenetrable spheres), and Σ is a liquid consisting of impenetrable spheres between which van der Waals forces act, their diameters a_Σ being many orders of magnitude greater than the diameters of their nuclei. In this model, $a_\Sigma \gg a$. Of course, many real aspects of the diffusion of a neutron in a liquid must be treated quantum mechanically. However, in some cases diffusion can also be treated in the quasiclassical approximation. It is then merely necessary to replace the operator $\overline{T}(0, j)$ in (11) by the corresponding collision operator found by solving the quantum-mechanical two-body problem. It has a very simple form if only S-wave scattering is taken into account.

Since all the particles of the system Σ are identical, the Liouville operator Л_Σ must be symmetric in the phase variables of these particles. The term Л_{int} (13) describing the interaction is also symmetric in this sense, and therefore the total Liouville operator Л is symmetric with respect to the phase variables of the system Σ. Noting that the initial distribution $\mathfrak{D}_0$ given by Eq. (2) has this symmetry, we conclude that $\mathfrak{D}_t$ is a symmetric function of $\mathbf{r}_1, \mathbf{v}_1, \ldots, \mathbf{r}_N, \mathbf{v}_N$.

We now turn to the general equation (3), which determines the evolution of the probability distribution $\mathfrak{D}_t$ in phase space. It is convenient to introduce the notation

$$(\overline{\mathcal{U}})_S=\int\mathcal{U}\,d\Omega_S;\ (\overline{\mathcal{U}})_\Sigma=\int\mathcal{U}\,d\Omega_\Sigma;\ (\overline{\mathcal{U}})_{S+\Sigma}=\int\mathcal{U}\,d\Omega_S\,d\Omega_\Sigma. \tag{15}$$

We now consider a dynamical variable $A(S)$ that refers only to the system S: $A(S)=A(\Omega_S)$. Its mean value at time t is given by the expression .

$$\langle A(S)\rangle_t = \overline{(A(S)\mathcal{D}_t(S,\Sigma))}_{S+\Sigma},$$

which can be transformed to

$$\langle A(S)\rangle_t = \overline{(A(S)f_t(S))}_S = \int A(\Omega_S) f_t(\Omega_S)\, d\Omega_S, \tag{16}$$

where

$$f_t(S) = \overline{(\mathcal{D}_t(S,\Sigma))}_\Sigma. \tag{17}$$

Thus, the probability density in the phase space s at the time t is given by the reduced distribution $f_t(S)$. It is clear that to calculate the mean value of the dynamical variable $A(S)$ it is only necessary to know the reduced probability distribution $f_t(S)$, and not the complete distribution $\mathfrak{D}_t(S,\Sigma)$.

We now turn to a method for obtaining an approximate equation for $f_t(S)$ in closed form. We proceed from the Liouville equation (13) written in the form

$$\partial\mathcal{D}_t/\partial t = (\Pi_S^0 + \Pi_\Sigma + \Pi_{\rm int})\mathcal{D}_t \tag{18}$$

with the initial condition (2).

We introduce

$$\mathcal{D}_t - f_t\mathcal{D}(\Sigma) = \Delta_t \tag{19}$$

and note that with allowance for (17)

$$\overline{(\Delta_t)}_\Sigma = 0. \tag{20}$$

Integrating (18) with respect to Ω_Σ and noting that $\overline{(\Pi_\Sigma\mathfrak{D}_t)}_\Sigma = 0$ identically, we obtain

$$\partial f_t/\partial t = \{\Pi_S^0 + \overline{(\Pi_{\rm int}\mathcal{D}(\Sigma))}_\Sigma\} f_t + \overline{(\Pi_{\rm int}\Delta_t)}_\Sigma. \tag{21}$$

Equations (18), (19), and (21) lead to

$$\begin{aligned}\frac{\partial\Delta_t}{\partial t} = \frac{\partial\mathcal{D}_t}{\partial t} - \frac{\partial f_t}{\partial t}\mathcal{D}(\Sigma) &= (\Pi_S^0 + \Pi_\Sigma + \Pi_{\rm int}) f_t\mathcal{D}(\Sigma)\\ &+ (\Pi_S^0 + \Pi_\Sigma + \Pi_{\rm int})\Delta_t - \\ &- \{(\Pi_S^0 + \overline{(\Pi_{\rm int}\mathcal{D}(\Sigma))}_\Sigma f_t + \overline{(\Pi_{\rm int}\Delta_t)}_\Sigma\}\mathcal{D}(\Sigma).\end{aligned}$$

By definition, $\mathfrak{D}(\Sigma)$ is an equilibrium distribution for Π_Σ: $\Pi_\Sigma\mathfrak{D}(\Sigma)=0$, so that $\Pi_\Sigma f_t(S)\mathfrak{D}(\Sigma) = f_t(S)\Pi_\Sigma\mathfrak{D}(\Sigma)=0$.

We introduce the notation

$$\left.\begin{aligned}\Pi_S &= \Pi_S^0 + \overline{(\Pi_{\rm int}\mathcal{D}(\Sigma))}_\Sigma;\\ \Gamma &= \Pi_{\rm int} - \overline{(\Pi_{\rm int}\mathcal{D}(\Sigma))}_\Sigma\end{aligned}\right\} \tag{22}$$

and note that

$$\begin{aligned}\Pi_S + \Gamma = \Pi_S^0 + \Pi_{\rm int};\ \overline{(\Gamma\Delta_t)}_\Sigma &= \overline{(\Pi_{\rm int}\Delta_t)}_\Sigma\\ - \overline{(\Pi_{\rm int}\mathcal{D}(\Sigma))}_\Sigma\,\overline{(\Delta_t)}_\Sigma &= \overline{(\Pi_{\rm int}\Delta_t)}_\Sigma.\end{aligned}$$

Taking this into account, we arrive at an equation for Δ_t:

$$\partial\Delta_t/\partial t = (\Pi_S + \Pi_\Sigma)\Delta_t + \Gamma\Delta_t - \overline{(\Gamma\Delta_t)}_\Sigma\mathcal{D}(\Sigma) + \Gamma f_t\mathcal{D}(\Sigma), \tag{23}$$

and we rewrite Eq. (21) in the form

$$\partial f_t/\partial t = \Pi_S f_t + \overline{(\Pi_{\rm int}\Delta_t)}_\Sigma. \tag{24}$$

The initial conditions (2) now take the form

$$\Delta_t = 0 \quad \text{for} \quad t=0. \tag{25}$$

The first thing that comes to mind if one considers Eq. (23) with the initial condition (25) is that Δ_t is, roughly speaking, proportional to the contribution of the interaction Γ.

Thus, in the framework of this semi-intuitive and simple supposition the term $\Gamma\Delta_t - \overline{(\Gamma\Delta_t)}_\Sigma\mathfrak{D}(\Sigma)$ in (23) can be regarded as a second-order term.

Retaining in the exact equation (23) only the principal term in the interaction, we obtain the approximate equation

$$\partial\Delta_t/\partial t = (\Pi_S + \Pi_\Sigma)\Delta_t + \Gamma f_t(S)\mathcal{D}(\Sigma) \tag{26}$$

with the same initial condition (25), whose formal solution is given by

$$\Delta_t = \int_0^t \exp[(\Pi_S + \Pi_\Sigma)(t-\tau)]\,\Gamma f_\tau(S)\mathcal{D}(\Sigma)\,d\tau.$$

Substituting this expression in (24), we obtain

$$\frac{\partial f_t}{\partial t} = \Pi_S f_t + \int_0^t \overline{(\Pi_{\rm int}\exp[(\Pi_S + \Pi_\Sigma)(t-\tau)]\,\Gamma\mathcal{D}(\Sigma))}_\Sigma f_\tau\,d\tau \tag{27}$$

or

$$\begin{aligned}\partial f_t/\partial t = \Pi_S f_t + \int_0^t \overline{(\Pi_{\rm int}\exp[(\Pi_S + \Pi_\Sigma)(t-\tau)]\{[\Pi_{\rm int}}\\ \times \overline{-(\Pi_{\rm int}\mathcal{D}(\Sigma))_\Sigma]\,\mathcal{D}(\Sigma)\})}_\Sigma f_\tau\,d\tau.\end{aligned} \tag{27}$$

Thus, we have obtained an approximate non-Markov kinetic equation for the reduced distribution function $f_t(S)$ in a closed form, in the sense that there is here no dependence on the total distribution for the complete system $S+\Sigma$.

This equation has been established in the framework of classical mechanics. To obtain an analog of it for the case when the dynamical behavior of the system $S+\Sigma$ is treated quantum mechanically, we must have recourse to some obvious modifications.

First, we use the representation of the von Neumann statistical operator in the matrix form

$$\mathcal{D}_t = \mathcal{D}_t(X_S, X'_S; X_\Sigma, X'_\Sigma), \tag{28}$$

where X_S and X_Σ are complete sets of values of commuting variables that characterize the states of the dynamical systems S and Σ, respectively; X'_S and X'_Σ are sets of values of the same variables.

The Liouville operators Π, Π_S^0, Π_Σ, $\Pi_{\rm int}$ must be regarded as operators which act on expressions of the type (28) defined as classical functions of the variables X_S, X'_S, X_Σ, X'_Σ. These Π operators can be defined by the quantum-mechanical Poisson brackets: $[H,\mathfrak{D}] = \Pi\mathfrak{D}$. Further, the corresponding mean values (15) must be replaced by the operations

$$\begin{aligned}\overline{(\mathcal{U})}_S &= \mathrm{Sp}_{(S)}\mathcal{U} = \int \mathcal{U}(X_S, X_S; X_\Sigma, X'_\Sigma)\,dX_S;\\ \overline{(\mathcal{U})}_\Sigma &= \mathrm{Sp}_{(\Sigma)}\mathcal{U} = \int \mathcal{U}(X_S, X'_S; X_\Sigma, X_\Sigma)\,dX_\Sigma;\\ \overline{(\mathcal{U})}_{S+\Sigma} &= \mathrm{Sp}_{(S+\Sigma)}\mathcal{U} = \int \mathcal{U}(X_S, X_S; X_\Sigma, X_\Sigma)\,dX_S\,dX_\Sigma.\end{aligned}$$

In particular, $f_t(S) = f_t(X_S, X'_S) = \mathrm{Sp}_{(\Sigma)}\mathfrak{D}_t$.

As variables X_S and X_Σ, one frequently chooses the coordinates $\mathbf{r}$ and the spins of all the particles of the system, or, alternatively, their momenta and spins. The integration with respect to X_S or X_Σ is understood as an integration with respect to all the coordinates X, which vary continuously in some region, and a summation over all the discrete components.

We can then repeat verbatim the arguments given above, starting from the quantum-mechanical Liouville equation, and we thus obtain an approximate equation for the reduced statistical operator $f_t(S)$, which has the same form as (27).

It is obvious that the method sketched here is a slightly modified version of the method proposed in Ref. 1 and developed further by Shelest.[5]

SECTION 2

We now turn to the discussion of the kinetic equation (27) for some concrete examples of dynamical systems S and Σ treated in the framework of classical mechanics.

We first return to the example mentioned in Sec. 1 when $(\Omega_S, \Omega_\Sigma)$, $\mathcal{L}_S^0$, $\mathcal{L}_{\rm int}$ are given by Eqs. (6)–(8). We now concentrate our attention on the case when the statistical equilibrium of Σ is described by the Gibbs distribution $\mathfrak{D}(\Sigma)$ corresponding to a spatially homogeneous state. Thus, we do not consider a situation in which Σ is a crystal in a state of statistical equilibrium. We shall assume further that the function which describes the interaction potential and is proportional to a small parameter is regular. We shall use the Fourier representation

$$\Phi(\mathbf{r}) = \frac{1}{V} \sum_{(k)} \exp(i\mathbf{k}\mathbf{r})\, \nu(\mathbf{k}), \tag{29}$$

where

$$\nu(\mathbf{k}) = \int \exp(-i\mathbf{k}\mathbf{r})\, \Phi(\mathbf{r})\, d\mathbf{r}. \tag{30}$$

As usual, the summation in (29) is over the quasidiscrete spectrum of wave numbers $\mathbf{k}$ corresponding to the volume V:

$$\mathbf{k} = (2\pi n_1/L,\ 2\pi n_2/L,\ 2\pi n_3/L),$$

in which n_1, n_2, n_3 are integers and $L^3 = V$. Since $\Phi(\mathbf{r})$ is radially symmetric, the Fourier transform $\nu(\mathbf{k})$ is a real function invariant under reflection:

$$\nu(\mathbf{k}) = \nu^*(\mathbf{k}) = \nu(-\mathbf{k}). \tag{31}$$

We rewrite our kinetic equation in the form

$$\frac{\partial f_t}{\partial t} = \mathcal{L}_S f_t + \int_0^t K(t-\tau) f_\tau\, d\tau; \tag{32}$$

$$K(T) = \overline{(\mathcal{L}_{\rm int} \exp[(\mathcal{L}_S + \mathcal{L}_\Sigma) T]\, [\mathcal{L}_{\rm int} - (\mathcal{L}_{\rm int}\mathscr{D}(\Sigma))_\Sigma]\, \mathscr{D}(\Sigma))_\Sigma}. \tag{33}$$

To investigate this equation, we establish some properties of expressions of the type $\overline{(\mathcal{L}_{\rm int} F(S, \Sigma))_\Sigma}$. Using the definition (8), we obtain

$$\overline{(\mathcal{L}_{\rm int} F(S, \Sigma))_\Sigma} = \sum_{(j)} \overline{\left(\frac{\partial \Phi(\mathbf{r}_0 - \mathbf{r}_j)}{\partial \mathbf{r}_0} \frac{1}{m} \frac{\partial}{\partial \mathbf{v}_0} F(S, \Sigma) \right)_\Sigma} - \sum_{(j)} \frac{1}{M} \overline{\left(\frac{\partial \Phi(\mathbf{r}_0 - \mathbf{r}_j)}{\partial \mathbf{r}_0} \frac{\partial}{\mathbf{v}_j} F(S, \Sigma) \right)_\Sigma}.$$

However, the second term on the right-hand side of this relation is identically zero since it contains the expression $\partial/\partial \mathbf{v}_i F(S, \Sigma)$ integrated over the complete space of the velocities $\mathbf{v}_j$.

Thus,

$$\overline{(\mathcal{L}_{\rm int} F(S, \Sigma))_\Sigma} = \frac{1}{m} \frac{\partial}{\partial \mathbf{v}_0} \sum_{(j)} \overline{\left(\frac{\partial \Phi(\mathbf{r}_0 - \mathbf{r}_j)}{\partial \mathbf{r}_0} F(S, \Sigma) \right)_\Sigma}. \tag{34}$$

We apply the obtained identity in the case when $F(S, \Sigma) = \mathfrak{D}(\Sigma)$. Substitution of the Fourier representation (29) in (34) gives

$$\overline{(\mathcal{L}_{\rm int}\mathscr{D}(\Sigma))_\Sigma} = \frac{1}{m} \frac{\partial}{\partial \mathbf{v}_0} \cdot \frac{1}{V} \sum_{(k)} i\mathbf{k} \exp(i\mathbf{k}\mathbf{r}_0)\, \nu(k) \overline{\left(\sum_{(j)} \exp(-i\mathbf{k}\mathbf{r}_j)\, \mathscr{D}(\Sigma) \right)_\Sigma}. \tag{35}$$

We now note that because the statistical equilibrium of Σ described by the Gibbs distribution $\mathfrak{D}(\Sigma)$ is spatially homogeneous, the expression $(\exp(-i\mathbf{k}\mathbf{r}_j)\mathfrak{D}(\Sigma))_\Sigma$ must be invariant under arbitrary spatial translations: $\mathbf{r}_j \to \mathbf{r}_j + \mathbf{r}$. Therefore,

$$\overline{(\exp(-i\mathbf{k}\mathbf{r}_j)\, \mathscr{D}(\Sigma))_\Sigma} = \exp(-i\mathbf{k}\mathbf{r})\, \overline{(\exp(-i\mathbf{k}\mathbf{r}_j)\, \mathscr{D}(\Sigma))_\Sigma}.$$

Since $\mathbf{r}$ is an arbitrary vector, $\overline{(\exp(-i\mathbf{k}\mathbf{r}_j)\mathfrak{D}(\Sigma))_\Sigma} = 0$ if $\mathbf{k} \neq 0$, and, taking into account (35), we have

$$\overline{(\mathcal{L}_{\rm int}\mathscr{D}(\Sigma))_\Sigma} = 0. \tag{36}$$

Therefore, (22) is rewritten as

$$\mathcal{L}_S = \mathcal{L}_S^0. \tag{37}$$

Further, we apply the identity (34) to the expression (33). Taking into account (36) and (37), we obtain

$$K(T) = \frac{1}{m} \frac{\partial}{\partial \mathbf{v}_0} Q(T); \tag{38}$$

$$Q(T) = \sum_{(j, j_1)} \overline{\left(\frac{\partial \Phi(\mathbf{r}_0 - \mathbf{r}_j)}{\partial \mathbf{r}_0} \exp[(\mathcal{L}_S^0 + \mathcal{L}_\Sigma) T]\, \frac{\partial \Phi(\mathbf{r}_0 - \mathbf{r}_{j_1})}{\partial \mathbf{r}_{j_1}} \left(\frac{1}{m} \frac{\partial}{\partial \mathbf{v}_0} + \frac{\mathbf{v}_j}{\theta} \right) \mathscr{D}(\Sigma) \right)_\Sigma}. \tag{39}$$

Here, we have also used the fundamental property

$$-\frac{1}{M} \frac{\partial}{\partial \mathbf{v}_j} \mathscr{D}(\Sigma) = \frac{\mathbf{v}_j}{\theta} \mathscr{D}(\Sigma). \tag{40}$$

Substitution of (29) in (30) gives

$$Q(T) = \frac{1}{V^2} \sum_{(k, k_1)} \sum_{(j, j_1)} \mathbf{k}\nu(k)\, \nu(k_1)\, \mathscr{E}(\mathbf{k}, \mathbf{k}_1), \tag{41}$$

where

$$\mathscr{E}(\mathbf{k}, \mathbf{k}_1) = \overline{\left(\exp(i\mathbf{k}\mathbf{r}_0) \exp(-i\mathbf{k}\mathbf{r}_j) \exp[(\mathcal{L}_S^0 + \mathcal{L}_\Sigma) T] \exp(i\mathbf{k}_1\mathbf{r}_0) \exp(-i\mathbf{k}_1\mathbf{r}_{j_1})\, \mathbf{k}_1 \left(\frac{1}{m} \frac{\partial}{\partial \mathbf{v}_0} + \frac{\mathbf{v}_j}{\theta} \right) \mathscr{D}(\Sigma) \right)_\Sigma}.$$

However, $\mathfrak{D}(\Sigma)$ is invariant under the translations $\mathbf{r}_j \to \mathbf{r}_j + \mathbf{r}$, $j = 1, 2, \ldots, N$, where $\mathbf{r}$ is an arbitrary vector of space.

Therefore,

$$\mathscr{E}(\mathbf{k}, \mathbf{k}_1) = \exp[-i(\mathbf{k} + \mathbf{k}_1)\mathbf{r}]\, \mathscr{E}(\mathbf{k}, \mathbf{k}_1),$$

from which it follows that $\mathscr{E}(\mathbf{k}, \mathbf{k}_1) = 0$ if $\mathbf{k} + \mathbf{k}_1 \neq 0$. Thus, we see that in the sum (41) we must retain only the terms with $\mathbf{k}_1 = -\mathbf{k}$.

We note further that $\mathcal{L}_S^0$ commutes with $\mathcal{L}_\Sigma$, $\mathbf{r}_j$ and $\mathcal{L}_\Sigma$ commutes with $\mathbf{r}_0$. Therefore, the expression (41) can be rewritten as

$$Q(T)=\frac{1}{V^2}\sum_{(k)}\mathbf{k}\nu^2(k)\exp(i\mathbf{k}\mathbf{r}_0)\exp(\text{Л}_S^0T)\exp(-i\mathbf{k}\mathbf{r}_0)$$

$$\times\overline{\left(\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j)\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\,\mathbf{k}\left(\frac{1}{m}\frac{\partial}{\partial\mathbf{v}_0}+\frac{1}{\theta}\mathbf{v}_j\right)\mathscr{D}(\Sigma)\right)_\Sigma}. \tag{42}$$

Considering motions in the isolated system Σ corresponding to the Liouville operator Л_Σ, we obtain

$$\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)(\mathbf{k}\mathbf{v}_j)=\sum_{(j)}\exp[i\mathbf{k}\mathbf{r}_j(-T)](\mathbf{k}\mathbf{v}_j(-T))$$

$$=-\sum_{(j)}\exp[i\mathbf{k}\mathbf{r}_j(-T)]\frac{d}{dT}(\mathbf{k}\mathbf{r}_j(-T))$$

$$=i\frac{d}{dT}\sum_{(j)}\exp[i\mathbf{k}\mathbf{r}_j(-T)]=i\frac{d}{dT}\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j),$$

which, with allowance for (42), leads to the expression

$$Q(T)=\frac{1}{V^2}\sum_{(k)}\mathbf{k}\nu^2(k)\exp(i\mathbf{k}\mathbf{r}_0)\exp(\text{Л}_S^0T)\exp(-i\mathbf{k}\mathbf{r}_0)$$

$$\times\left\{U_k(T)\frac{1}{m}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}\right)+\frac{i}{\theta}\frac{\partial U_k(T)}{\partial T}\right\}. \tag{43}$$

where

$$U_k(T)=\overline{\left(\sum_{(j)}\exp(-i\mathbf{k}\mathbf{v}_j)\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\mathscr{D}(\Sigma)\right)_\Sigma}$$

$$=N\overline{\left(\exp(-i\mathbf{k}\mathbf{r}_1)\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\mathscr{D}(\Sigma)\right)_\Sigma}=NR_k(T);$$

$$R_k(T)=\overline{\left(\exp(-i\mathbf{k}\mathbf{r}_1)\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\mathscr{D}(\Sigma)\right)_\Sigma}. \tag{44}$$

Introducing the mean density of particles

$$n=N/V \tag{45}$$

and rewriting the expression (43) by means of (44), we obtain

$$Q(T)=n\frac{1}{V}\sum_{(k)}\mathbf{k}\nu^2(k)\exp(i\mathbf{k}\mathbf{r}_0)\exp(\text{Л}_S^0T)\exp(-i\mathbf{k}\mathbf{r}_0)$$

$$\times\left\{\frac{1}{m}R_k(T)\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}\right)+\frac{i}{\theta}\frac{\partial R_k(T)}{\partial T}\right\}. \tag{46}$$

In this notation, our kinetic equation (33), (38) is reduced to the form

$$\frac{\partial f_t(\mathbf{r}_0,\mathbf{v}_0)}{\partial t}=-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}f_t(\mathbf{r}_0,\mathbf{v}_0)$$

$$+\frac{1}{m}\frac{\partial}{\partial\mathbf{v}_0}\int_0^t Q(t-\tau)f_\tau(\mathbf{r}_0,\mathbf{v}_0)\,d\tau. \tag{47}$$

We go over to the Fourier representation

$$f_t(\mathbf{r}_0,\mathbf{v}_0)=\frac{1}{V}\sum_{(l)}\exp(-i\mathbf{l}\mathbf{r}_0)f_l(t,\mathbf{v}_0) \tag{48}$$

and note that

$$\exp(i\mathbf{k}\mathbf{r}_0)\exp(\text{Л}_S^0T)\exp[-i(\mathbf{k}+\mathbf{l})\mathbf{r}_0]=\exp(-i\mathbf{l}\mathbf{r}_0)\exp(\mathbf{k}+\mathbf{l})\mathbf{v}_0T.$$

In such a case, it is readily seen that Eq. (47) reduces to separate equations for each component $f_l(t,\mathbf{v}_0)$:

$$\frac{\partial f_l(t,\mathbf{v}_0)}{\partial t}=i(\mathbf{l}\mathbf{v}_0)f_l(t,\mathbf{v}_0)+\frac{1}{m}\frac{\partial}{\partial\mathbf{v}_0}\int_0^t Q_l(t-\tau)f_l(\tau,\mathbf{v}_0)\,d\tau, \tag{49}$$

where

$$Q_l(T)=n\frac{1}{V}\sum_{(k)}\mathbf{k}\nu^2(k)\exp[i(\mathbf{k}+\mathbf{l})\mathbf{v}_0T]\left\{R_k(T)\frac{1}{m}\,\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}+\frac{i}{\theta}\frac{\partial R_k(T)}{\partial T}\right\}. \tag{50}$$

Carrying out the usual limiting process in statistical mechanics and going over in (50) from the summation over k to an integration, we obtain

$$Q_l(T)=\frac{n}{(2\pi)^3}\int\mathbf{k}\nu^2(k)\exp[i(\mathbf{k}+\mathbf{l})\mathbf{v}_0T]$$

$$\times\left\{R_k(T)\frac{1}{m}\,\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}+\frac{i}{\theta}\frac{\partial R_k(T)}{\partial T}\right\}d\mathbf{k}. \tag{51}$$

It is convenient to investigate Eq. (49) by using the Laplace transform

$$\int_0^\infty\exp(-zt)f_l(t,\mathbf{v}_0)\,dt=f_{l,z}(\mathbf{v}_0)$$

$$(z=\varepsilon-i\omega,\ \operatorname{Re}z=\varepsilon>0). \tag{52}$$

Applying Laplace transforms to both sides of Eq. (49), we find

$$zf_{l,z}(\mathbf{v}_0)=i(\mathbf{l}\mathbf{v}_0)f_{l,z}(\mathbf{v}_0)$$

$$+\frac{1}{m}\frac{\partial}{\partial\mathbf{v}}\int_0^\infty Q_l(T)\exp(-zT)\,dTf_{l,z}(\mathbf{v}_0)+f_l(0,\mathbf{v}_0); \tag{53}$$

$$\int_0^\infty Q_l(T)\exp(-zT)\,dT=\frac{n}{(2\pi)^3}\int\mathbf{k}\nu^2(k)$$

$$\times\left\{\int_0^\infty R_k(T)\exp\{[i(\mathbf{k}+\mathbf{l})\mathbf{v}_0-z]T\}\,dT\right\}\frac{1}{m}\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}\right)d\mathbf{k}$$

$$+\frac{n}{(2\pi)^3}\int\mathbf{k}\nu^2(k)\left\{\frac{i}{\theta}\int_0^\infty\exp\{[i(\mathbf{k}+\mathbf{l})\mathbf{v}_0-z]T\}\frac{\partial R_k(T)}{\partial T}\,dT\right\}d\mathbf{k}. \tag{53a}$$

However, on the one hand,

$$\frac{i}{\theta}\int_0^\infty\exp\{[i(\mathbf{k}+\mathbf{l})\mathbf{v}_0-z]T\}\frac{\partial R_k(T)}{\partial T}\,dT=-\frac{i}{\theta}R_k(0)$$

$$+\frac{1}{\theta}[(\mathbf{k}+\mathbf{l})\mathbf{v}_0+iz]\int_0^\infty R_k(T)\exp\{[i(\mathbf{k}+\mathbf{l})\mathbf{v}_0-z]T\}\,dT,$$

and on the other, it follows from (44) that

$$R_k(0)=\frac{1}{N}\overline{\left(\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\mathscr{D}(\Sigma)\right)_\Sigma},$$

which enforces fulfillment of the relation $R_k(0)=R_{-k}(0)$. Since the function $\nu(k)$ has, in accordance with (31), a similar symmetry property, we readily see that $\int\mathbf{k}\nu^2(k)R_k(0)d\mathbf{k}=0$. Thus, Eq. (53) can be written in the form

$$zf_{l,z}(\mathbf{v}_0)=i(\mathbf{l}\mathbf{v}_0)f_{l,z}(\mathbf{v}_0)$$

$$+\frac{n}{m(2\pi)^3}\int\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}}\right)\nu^2(k)\left\{\int_0^\infty R_k(T)\exp\{i[(\mathbf{k}+\mathbf{l})\mathbf{v}_0-z]T\}\,dT\right\}$$

$$\times\left(\frac{1}{m}\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}+\frac{(\mathbf{k}+\mathbf{l})\mathbf{v}_0-iz}{\theta}\right)d\mathbf{k}f_{l,z}(\mathbf{v}_0)+f_l(0,\mathbf{v}_0). \tag{54}$$

Note that the integral term on the right-hand side of (54) containing $\nu^2(k)$ is formally proportional to the square of the small parameter. If we consider the case of small z and l, then we can ignore the corresponding terms in the integral and obtain the very simple approximate equation

$$zf_{l,z}(\mathbf{v}_0)=i(\mathbf{l}\mathbf{v}_0)f_{l,z}(\mathbf{v}_0)$$

$$+\frac{n}{m(2\pi)^3}\int\nu^2(k)\left(\mathbf{k}\frac{\partial}{\partial\mathbf{v}_0}\right)\int_0^\infty R_k(T)\exp(i\mathbf{k}\mathbf{v}_0T)\,dT\mathbf{k}$$

$$\times\left(\frac{1}{m}\frac{\partial}{\partial\mathbf{v}_0}+\frac{\mathbf{v}_0}{\theta}\right)d\mathbf{k}f_{l,z}(\mathbf{v}_0)+f_l(0,\mathbf{v}_0). \tag{55}$$

It must be emphasized that Eq. (54) does not contain terms of higher powers in the interaction, for which one cannot preclude singular behavior in the neighborhood of $z=0$, $l=0$.[2)] For this reason, Eq. (55) may not give the correct asymptotic behavior of $f_{l,z}(\mathbf{v}_0)$ as $l\to 0$, $z\to 0$.

On the other hand, it is interesting that Eq. (55) can be obtained formally from the equation for the reduced probability distribution

$$\frac{\partial f_t(\mathbf{r}_0,\mathbf{v}_0)}{\partial t}=-\mathbf{v}_0\frac{\partial}{\partial \mathbf{r}_0}f_t(\mathbf{r}_0,\mathbf{v}_0)+\frac{n}{m(2\pi)^3}\int \nu^2(k)\left(\mathbf{k}\frac{\partial}{\partial \mathbf{v}_0}\right)\times\int_0^\infty R_k(T)\exp(i\mathbf{k}\mathbf{v}_0T)\,dT\,\mathbf{k}\left(\frac{1}{m}\frac{\partial}{\partial \mathbf{v}_0}+\frac{\mathbf{v}_0}{\theta}\right)d\mathbf{k}f_t(\mathbf{r}_0,\mathbf{v}_0), \tag{56}$$

by using the Fourier expansion (48) and Laplace transform with respect to the variable t. Therefore, the two equations (55) and (56) are completely equivalent: one of them corresponds to the (z, l) representation, and the other to the $(t, \mathbf{r}_0)$ representation. It is clear that (56) is a typical Fokker-Planck equation for a Markov stochastic process. Obviously, (56) also permits the existence of a spatially homogeneous solution $f_t(\mathbf{v}_0)$, which must satisfy the equation

$$\frac{\partial f_t(\mathbf{v}_0)}{\partial t}=\frac{n}{m(2\pi)^3}\int \nu^2(k)\left(\mathbf{k}\frac{\partial}{\partial \mathbf{v}_0}\right)\int_0^\infty R_k(T)\exp(i\mathbf{k}\mathbf{v}_0T)\,dT\times\mathbf{k}\left(\frac{1}{m}\frac{\partial}{\partial \mathbf{v}_0}+\frac{\mathbf{v}_0}{\theta}\right)d\mathbf{k}f_t(\mathbf{v}_0), \tag{57}$$

which enables us to conclude that in the given simple situation $f_t(\mathbf{v}_0)$ approaches the Maxwellian velocity distribution as the time increases.

We have already noted that for $l=0$ the correction terms to the solutions of (54) or (55) can become singular as $z\to 0$. Similarly, in the t representation Eq. (57) may not give the correct asymptotic behavior of the difference $f_t(\mathbf{v}_0)-f_{\max}(\mathbf{v}_0)$ for sufficiently large t. This question will be discussed in detail in Sec. 4.

We now establish some useful properties of the function $R_k(T)$. We consider the equilibrium mean value for the system Σ:

$$\langle\rho(t,\mathbf{r})\rho(0,\mathbf{r}')\rangle_\Sigma=\overline{(\rho(t,\mathbf{r})\rho(0,\mathbf{r}')\mathscr{D}(\Sigma))_\Sigma}, \tag{58}$$

where $\rho(t,\mathbf{r})$ is the microscopic density of the Σ particles: $\rho(t,\mathbf{r})=\sum_{(1\leq j\leq N)}\delta(\mathbf{r}-\mathbf{r}_j(t))$. Since the equilibrium mean value is invariant under time translations, the expression (58) is equivalent to

$$\langle\rho(0,\mathbf{r})\rho(-t,\mathbf{r})\rangle_\Sigma=\langle\rho(0,\mathbf{r})\exp(\Pi_\Sigma t)\rho(0,\mathbf{r}')\rangle_\Sigma. \tag{59}$$

In such a case, using the Fourier representation, we obtain

$$\langle\rho(t,\mathbf{r})\rho(0,\mathbf{r}')\rangle_\Sigma=\frac{1}{V^2}\sum_{(k)}\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')]\times\overline{\left(\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j)\exp(\Pi_\Sigma t)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\mathscr{D}(\Sigma)\right)_\Sigma}$$

$$=n^2+\frac{1}{V^2}\sum_{(k\neq 0)}\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')]\times\overline{\left(\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j)\exp(\Pi_\Sigma t)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\mathscr{D}(\Sigma)\right)_\Sigma},$$

which with allowance for (44) gives

$$\langle\rho(t,\mathbf{r})\rho(0,\mathbf{r}')\rangle_\Sigma=n^2+n\frac{1}{V}\sum_{k\neq 0}R_k(t)\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')] \tag{60}$$

or, in the thermodynamic limit $V\to\infty$, $n=\text{const}$,

$$\langle\rho(t,\mathbf{r})\rho(0,\mathbf{r}')\rangle_\Sigma=n^2+\frac{n}{(2\pi)^3}\int R_k(t)\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')]\,d\mathbf{k}. \tag{61}$$

Since the microscopic particle density is a real function, the left-hand side of (61) must also be real, and therefore

$$R_k^*(t)=R_{-k}(t). \tag{62}$$

We now consider the integral term in Eq. (56) and rewrite it as

$$\frac{n}{(2\pi)^3 m}\int \nu^2(k)\left(\mathbf{k}\frac{\partial}{\partial \mathbf{v}_0}\right)\frac{1}{2}\left\{\int_0^\infty R_k(T)\exp(i\mathbf{k}\mathbf{v}_0T)\,nT+\int_0^\infty R_{-k}(T)\exp(-i\mathbf{k}\mathbf{v}_0T)\,dT\right\}\mathbf{k}\left(\frac{1}{m}\frac{\partial}{\partial \mathbf{v}_0}+\frac{\mathbf{v}_0}{\theta}\right)d\mathbf{k}f_t(\mathbf{r}_0,\mathbf{v}_0).$$

However, the relation (62) leads to the equation

$$\frac{1}{2}\left\{\int_0^\infty R_k(T)\exp(i\mathbf{k}\mathbf{v}_0T)\,dT+\int_0^\infty R_{-k}(T)\exp(-i\mathbf{k}\mathbf{v}_0T)\,dT\right\}=\operatorname{Re}\int_0^\infty R_k(T)\exp(i\mathbf{k}\mathbf{v}_0T)\,dT.$$

Finally, Eq. (56) for the reduced probability distribution can be rewritten as

$$\frac{\partial f_t(\mathbf{r}_0,\mathbf{v}_0)}{\partial t}=-\mathbf{v}_0\frac{\partial}{\partial \mathbf{r}_0}f_t(\mathbf{r}_0,\mathbf{v}_0)+\frac{n}{m(2\pi)^3}\int \nu^2(k)\left(\mathbf{k}\frac{\partial}{\partial \mathbf{v}_0}\right)\times F(\mathbf{k}\mathbf{v}_0)\,\mathbf{k}\left(\frac{1}{m}\frac{\partial}{\partial \mathbf{v}_0}+\frac{\mathbf{v}_0}{\theta}\right)d\mathbf{k}f_t(\mathbf{r}_0,\mathbf{v}_0), \tag{63}$$

where

$$F(\omega)=\operatorname{Re}\int_0^\infty R_k(t)\exp(i\omega t)\,dt. \tag{64}$$

We see that we must determine the function (64) in order to have a completely defined equation. In Sec. 3, we shall sketch a method of explicit calculation of this function in some frequently encountered situations. Here we merely point out that the equivalence of (58) and (59) enforces fulfillment of the equation $R_k(-t)=R_{-k}(t)$, which leads to the relation

$$F(\omega)=\frac{1}{2}\int_{-\infty}^{\infty}R_k(t)\exp(i\omega t)\,dt. \tag{65}$$

In such a case, since $F(\omega)$ is the Fourier transform of the equilibrium mean value of the correlation function of two conjugate dynamical variables:

$$R_k(t)=\frac{1}{n}\lim_{V\to\infty}\frac{1}{V}\left\langle\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j)\exp(\Pi_\Sigma t)\sum_{(j)}\exp(i\mathbf{k}\mathbf{r}_j)\right\rangle_\Sigma, \tag{66}$$

we have

$$F(\omega)\geqslant 0. \tag{67}$$

We now analyze the following example: all conditions are the same, but instead of the regular interaction (8) we consider an interaction between impenetrable spheres. In this case, we must take

$$\Pi_{\text{int}}=\Pi_{\text{int}}^{\text{coll}}. \tag{68}$$

[2)] Indeed, there are weighty indications that there is a large probability for the realization of such a possibility.

where the concrete form of $\Pi_{\mathrm{int}}^{\mathrm{coll}}$ is given by Eqs. (10) and (11).

In Sec. 1 we have already noted that the fundamental Liouville equation (18) gives an exact description of the dynamics of the system, provided that unphysical overlapping configurations are eliminated. Of course, if nonoverlapping configurations were absent at the initial time $t=0$, they could not arise at subsequent times $t \neq 0$. Therefore, we must impose the condition $\mathfrak{D}_0(S, \Sigma) = 0$ (for $t=0$) if for at least one j of the set $j=1, 2, \ldots, N$

$$|\mathbf{r}_0 - \mathbf{r}_j| < a. \qquad (69)$$

If this condition holds, it is automatically satisfied for $\mathfrak{D}_t$ when $t>0$.[3)]

The difficulty that arises is now obvious as soon as we attempt to use Eq. (27) with Π_{int} given by (68). An important part of the derivation of this equation was the use of the initial condition (2), $\mathfrak{D}_0(S, \Sigma) = f_0(S)\mathfrak{D}(\Sigma)$, and such a form of $\mathfrak{D}_0$ precludes imposition of the condition (69). Therefore, the probability of overlapping configurations is not zero. Nevertheless, we shall use the equation of overlapping of the diameters of the particle S and a particle of the system Σ proportional to the small quantity na^3, assuming that the part played by this overlapping is negligible if we calculate for this equation the contribution of the corrections in the case of low density, especially when the l in $f_l(t, v_0)$ are sufficiently small and t is sufficiently large.

Further, in Sec. 4, we propose a different form of the choice for $\mathfrak{D}_0(S, \Sigma)$ which automatically eliminates overlapping configurations and which will therefore give an *a posteriori* justification of the procedure we have used here.

In order to particularize the approximate equation under consideration, we first substitute (7), (10), and (11) in Eq. (22):

$$\Pi_S = \Pi_S^0 + \sum_{(1\leq j\leq N)} \overline{(\overline{T}(0, j)\, \mathscr{Z}(\Sigma))_\Sigma} = \Pi_S^0 + N\,\overline{(\overline{T}(0, 1)\, \mathscr{D}(\Sigma))_\Sigma}. \qquad (70)$$

We note that the equilibrium distribution $\mathfrak{D}(\Sigma)$ for the classical dynamical system Σ has the form

$$\mathscr{D}(\Sigma) = W(\mathbf{r}_1, \ldots, \mathbf{r}_N) \prod_{(1\leq j\leq N)} \Phi_\Sigma(v_j), \qquad (71)$$

where

$$\Phi_\Sigma(v) = \left(\frac{M}{2\pi\theta}\right)^{3/2} \exp(-Mv^2/2\theta); \quad \int \Phi_\Sigma(v)\, d\mathbf{v} = 1 \qquad (72)$$

is the normalized Maxwellian velocity distribution.

[3)] We emphasize that the expression (11) for $\overline{T}(0,1)$ can be used only to study the evolution of D_t for $t>0$. But if we wish to investigate this evolution for the opposite time direction ($t<0$), we must use a different form for $\overline{T}(0,1)$. The direction of the time in these operators is particularized by the convention of whether $\mathbf{v}$ and $\mathbf{v}^*$ are the velocities before and after collision or whether they are taken in the reverse order. A more detailed discussion of this question can be found in Ref. 4.

The normalization condition $\int \mathfrak{D}(\Sigma) d\Omega_\Sigma = 1$ leads to

$$\int W(\mathbf{r}_1, \ldots, \mathbf{r}_N)\, d\mathbf{r}_1 \ldots d\mathbf{r}_N = 1. \qquad (73)$$

We consider the equilibrium mean value of the microscopic particle density in Σ at the point $\mathbf{r}$:

$$n = \langle \rho(\mathbf{r}) \rangle_\Sigma = \sum_{1\leq j\leq N} \int \delta(\mathbf{r} - \mathbf{r}_j)\, \mathscr{D}(\Sigma)\, d\Omega_\Sigma$$
$$= N \int \delta(\mathbf{r} - \mathbf{r}_1)\, \mathscr{D}(\Sigma)\, d\Omega_\Sigma = N \int \delta(\mathbf{r} - \mathbf{r}_1)\, W\, d\mathbf{r}_1 \ldots d\mathbf{r}_N.$$

Taking into account the requirement of spatial homogeneity, we see that this mean density does not depend on $\mathbf{r}$ and, therefore, $N\int \delta(\mathbf{r} - \mathbf{r}_1) W d\mathbf{r}_1 \ldots d\mathbf{r}_N = n$. By virtue of this relation, Eqs. (11), (71), and (72) enable us to conclude that

$$N\,\overline{(\overline{T}(0, 1)\, \mathscr{Z}(\Sigma))_\Sigma} = na^2 \int (\mathbf{v}_{0,1}\sigma)\, \theta(\mathbf{v}_{0,1}\sigma) \{B_{v_0, v_1}(\sigma) - 1\} \times \Phi_\Sigma(v_1)\, d\sigma\, d\mathbf{v}_1, \qquad (74)$$

where

$$\theta(\tau) = \begin{cases} 1 & \text{for} \quad \tau > 0; \\ 0 & \text{for} \quad \tau \leqslant 0. \end{cases}$$

We have therefore arrived at the Lorentz-Boltzmann collision operator acting only on the function v_0:

$$N\,\overline{(\overline{T}(0, 1)\, \mathscr{Z}(\Sigma))_\Sigma} f(S)$$
$$= na^2 \int (\mathbf{v}_{0,1}\sigma)\, \theta(\mathbf{v}_{0,1}\sigma) \{B_{v_0, v_1}(\sigma) - 1\}\, \Phi_\Sigma(v_1)\, f(\mathbf{r}_0, \mathbf{v}_0)\, n\sigma\, d\mathbf{v}_1.$$

It is convenient to introduce the notation

$$f(S) = \chi(S)\, \Phi_0(v_0), \qquad (75)$$

where $\Phi_0(v_0)$ is the normalized Maxwellian distribution for S:

$$\Phi_0(v_0) = [m/(2\pi\theta)]^{3/2} \exp[-mv^2/(2\theta)].$$

Then, remembering that

$$B_{v_0, v_1}(\sigma)\, \Phi_0(v_0)\, \Phi_\Sigma(v_1)$$
$$= [m/(2\pi\theta)]^{3/2}\, [M/(2\pi\theta)]^{3/2} \exp\{-mv^{*2}/2\theta - Mv_1^{*2}/2\theta\}$$
$$= \Phi_0(v_0)\, \Phi_\Sigma(v_1), \qquad (76)$$

we obtain from (70)

$$\Pi_S f(S) = \Pi_S \chi(S)\, \Phi_0(v_0)$$
$$= \Phi_0(v_0) \left\{ -\mathbf{v}_0 \frac{\partial \chi(\mathbf{r}_0, \mathbf{v}_0)}{\partial \mathbf{r}_0} + na^2 L_S \chi \right\}, \qquad (77)$$

where

$$L_S \chi = \int (\mathbf{v}_{0,1}\sigma)\, \theta(\mathbf{v}_{0,1}\sigma)\, \Phi_\Sigma(v_1) \{B_{v_0, v_1}(\sigma) - 1\}\, \chi(\mathbf{r}_0, \mathbf{v}_0)\, d\sigma\, d\mathbf{v}_1.$$

We return once more to Eq. (27), which we represent in the form

$$\frac{\partial f_t(\mathbf{r}_0, \mathbf{v}_0)}{\partial t} = \Phi_0(v_0) \left\{ -\mathbf{v}_0 \frac{\partial \chi_t(\mathbf{r}_0, \mathbf{v}_0)}{\partial \mathbf{r}_0} + na^2 L_S \chi \right\}$$
$$+ \int_0^t K(t - \tau)\, \chi_\tau(\mathbf{r}_0, \mathbf{v}_0)\, \Phi_0(v_0)\, d\tau, \qquad (78)$$

where

$$K(t)$$
$$= \overline{\Big(\sum_{(j)} \overline{T}(0, j) \exp[(\Pi_S + \Pi_\Sigma)\, t] \sum_{(j)} (\overline{T}(0, j) - \overline{(\overline{T}(0, j)\, \mathscr{Z}(\Sigma))_\Sigma} \mathscr{D}(\Sigma)\Big)_\Sigma}$$
$$= \overline{N\,(\overline{T}(0, 1) \exp(\Pi_S t) \exp(\Pi_\Sigma t) \sum_{(j)} (\overline{T}(0, j) - \overline{(\overline{T}(0, j)\, \mathscr{D}(\Sigma))_\Sigma})\, \mathscr{D}(\Sigma))_\Sigma}.$$
$$(79)$$

Note that here Λ_S commutes with Λ_Σ and in the general case Λ_S commutes with the variables Ω_Σ, whereas Λ_Σ commutes with the variables Ω_S.

To simplify the expression (79), we use the Fourier representation

$$\delta(\mathbf{r}-\mathbf{r}_j)=\sum_k \exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}_j)]/V$$

and obtain

$$\overline{T}(0,\ j)=\sum_k \exp[i\mathbf{k}(\mathbf{r}_0-\mathbf{r}_j)]\,\overline{T}_k(v_0,\ v_j)/V, \tag{80}$$

where

$$\overline{T}_k(v_0,v_j)=a^2\int(\mathbf{v}_{0,\,j}\sigma)\,\theta(\mathbf{v}_{0,\,j}\sigma)\{\exp[-ia(\mathbf{k}\sigma)]\,B_{v_0,\,v_j}(\sigma)-\exp[ia(\mathbf{k}\sigma)]\}\,d\sigma;$$
$$\theta(x)=\begin{cases}1, & x>0;\\ 0, & x\leqslant 0.\end{cases} \tag{81}$$

On the other hand, using the identity (76), we find

$$\overline{T}_k(v_0,v_j)\,\chi(\mathbf{r}_0,\mathbf{v}_0)\,\Phi_0(v_0)\,\mathscr{D}(\Sigma)=\{\overline{T}_k(v_0,v_j)\,\chi(\mathbf{r}_0,\mathbf{v}_0)\}\,\Phi_0(v_0)\,\mathscr{D}(\Sigma).$$

The upshot is

$$K(t)\,\chi\Phi_0=n\int\overline{T}(0,1)\exp(\Lambda_S t)\,Q(0,1)\,d\mathbf{r}_1\,d\mathbf{v}_1, \tag{82}$$

where

$$Q(0,1)=\sum_{(k\neq 0)}\int\exp(\Lambda_\Sigma t)\sum_{(j)}\exp[i\mathbf{k}(\mathbf{r}_0-\mathbf{r}_j)]\times\{\overline{T}_k(v_0,v_j)\,\chi\}\,\Phi_0(v_0)\,\mathscr{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N$$
$$+\int\exp(\Lambda_\Sigma t)\sum_{(j)}\Big\{\overline{T}_0(v_0,v_j)\,\chi-\int\{\overline{T}_0(v_0,v_j)\,\chi\}\,\Phi_\Sigma(v_j)\,d\mathbf{v}_j\Big\}\,\Phi_0(v_0)\times\mathscr{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N. \tag{83}$$

The first term on the right-hand side of (83) can be represented as

$$\left.\begin{aligned}&Q_1(\mathbf{r}_0,\mathbf{v}_0;\ \mathbf{r}_1,\mathbf{v}_1)=\sum_{(k\neq 0)}Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{r}_1,\mathbf{v}_1)\exp(i\mathbf{k}\mathbf{r}_0);\\&Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{r}_1\mathbf{v}_1)=\int\exp(\Lambda_\Sigma t)\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j)\\&\times\{\overline{T}_k(v_0,v_j)\chi\}\,\Phi_0(v_0)\,W(\mathbf{r}_1,\mathbf{r}_2,\dots,\mathbf{r}_N)\\&\times\prod_{(1\leqslant j\leqslant N)}\Phi_\Sigma(v_j)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N.\end{aligned}\right\} \tag{84}$$

Let $\mathbf{r}$ be an arbitrary vector. Changing the variables of integration through

$$\mathbf{r}_2\to\mathbf{r}_2+\mathbf{r},\ \dots,\ \mathbf{r}_N\to\mathbf{r}_N+\mathbf{r},$$

we obtain

$$Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{r}_1+\mathbf{r},\mathbf{v}_1)=\exp(-i\mathbf{k}\mathbf{r})\int\exp(\Lambda_\Sigma t)\sum_{(1\leqslant j\leqslant N)}\exp(-i\mathbf{k}\mathbf{r}_j)$$
$$\times\{\overline{T}_k(v_0,v_j)\chi\}\Phi_0(v_0)\,W(\mathbf{r}_1+\mathbf{r},\ \mathbf{r}_2+\mathbf{r},\ \dots,\ \mathbf{r}_N+\mathbf{r})\times\prod_{(1\leqslant j\leqslant N)}\Phi_\Sigma(v_j)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N.$$

However, because of the spatial homogeneity, the function $W(\mathbf{r}_1+\mathbf{r},\dots,\mathbf{r}_N+\mathbf{r})$ is equivalent to $W(\mathbf{r}_1,\dots,\mathbf{r}_N)$. Thus,

$$\exp(i\mathbf{k}\mathbf{r})\,Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0,\mathbf{r}_1+\mathbf{r},\mathbf{v}_1)=Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{r}_1,\mathbf{v}_1).$$

For $\mathbf{r}=-\mathbf{r}_1$, this gives

$$Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{r}_1,\mathbf{v}_1)=\exp(-i\mathbf{k}\mathbf{r}_1)\,Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_1), \tag{85}$$

where

$$Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_1)=Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ 0,\ \mathbf{v}_1).$$

Considering the second term in (83), we find by the same arguments that it does not depend on $\mathbf{r}_1$:

$$\int\exp(\Lambda_\Sigma t)\sum_{(1\leqslant j\leqslant N)}\widetilde{\chi}(\mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_j)\,\Phi_0(v_0)\,\mathscr{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N=Q_2(\mathbf{r}_0,\mathbf{v}_0;\ v_1), \tag{86}$$

where we have introduced the abbreviated notation

$$\widetilde{\chi}(\mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_j)=\overline{T}_0(v_0,v_j)\,\chi(\mathbf{r}_0,\mathbf{v}_0)-\int\{\overline{T}_0(v_0,v_j)\,\chi(\mathbf{r}_0,\mathbf{v}_0)\}\,\Phi_\Sigma(v_j)\,d\mathbf{v}_j. \tag{87}$$

Note that the function $\widetilde{\chi}$ satisfies the equation

$$\int\widetilde{\chi}(\mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v})\,\Phi_\Sigma(v)\,d\mathbf{v}=0. \tag{88}$$

Summing now our results (82), (85), and (86), we obtain

$$K(t)\,\chi\Phi_0=n\sum_{(k\neq 0)}\int\overline{T}(0,1)\exp(-i\mathbf{k}\mathbf{r}_1)\exp(\Lambda_S t)\exp(i\mathbf{k}\mathbf{r}_0)\times Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_1)\,d\mathbf{r}_1\,d\mathbf{v}_1+n\int\overline{T}(0,1)\exp(\Lambda_S t)\,Q_2(\mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_1)\,d\mathbf{r}_1\,d\mathbf{v}_1.$$

On the other hand,

$$\int\overline{T}(0,1)\exp(-i\mathbf{k}\mathbf{r}_1)\,d\mathbf{r}_1=T_{-k}(v_0,v_1)\exp(-i\mathbf{k}\mathbf{r}_0)$$

and therefore

$$K(t)\,\chi\Phi_0=n\sum_{(k\neq 0)}\int T_{-k}(v_0,v_1)\exp(-i\mathbf{k}\mathbf{r}_0)\exp(\Lambda_S t)\exp(i\mathbf{k}\mathbf{r}_0)\times Q_1(k;\ \mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_1)\,d\mathbf{v}_1+n\int T_0(v_0,v_1)\exp(\Lambda_S t)\,Q_2(\mathbf{r}_0,\mathbf{v}_0;\ \mathbf{v}_1)\,d\mathbf{v}_1. \tag{89}$$

We can now transform the expressions for Q_1 and Q_2 to a more convenient form.

We consider the integral

$$\int\exp(\Lambda_\Sigma t)\sum_{(1\leqslant j\leqslant N)}-\exp(i\mathbf{k}\mathbf{r}_j)\,\delta(\mathbf{v}_j-\mathbf{v})\,\mathscr{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N.$$

Using the arguments given above, we find that it depends on $\mathbf{r}_1$ through $\exp(-i\mathbf{k}\mathbf{r}_1)$, and therefore the function $U_k(t,\mathbf{v}_1,\mathbf{v})$ can be determined as follows:

$$\int\exp(\Lambda_\Sigma t)\sum_{(1\leqslant j\leqslant N)}\exp(-i\mathbf{k}\mathbf{r}_j)\,\delta(\mathbf{v}_j-\mathbf{v})\,\mathscr{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N=\exp(-i\mathbf{k}\mathbf{r}_1)\,\frac{1}{V}\,\Phi_\Sigma(v_1)\,U_k(t;\ \mathbf{v}_1,\ \mathbf{v}). \tag{90}$$

Naturally, U_k depends on V. The relation (90) leads to

$$\int\exp(\Lambda_\Sigma t)\sum_{(1\leqslant j\leqslant N)}\exp(-i\mathbf{k}\mathbf{r}_j)\,\Phi(\mathbf{v}_j)\,\mathscr{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2\dots d\mathbf{r}_N\,d\mathbf{v}_N=\exp(-i\mathbf{k}\mathbf{r}_1)\,\frac{1}{V}\,\Phi_\Sigma(v_1)\int U_k(t;\ \mathbf{v}_1,\ \mathbf{v}_1')\,\Phi(\mathbf{v}_1')\,d\mathbf{v}_1'. \tag{91}$$

It is convenient to regard the expression $U_k(t,\mathbf{v}_1,\mathbf{v}_1')$ as a matrix representation of the operator $U_k(t;1)$ acting only on the function $\mathbf{v}_1$ in accordance with

$$U_k(t;1)\,f(\mathbf{v}_1)=\int U_k(t;\ \mathbf{v}_1,\ \mathbf{v}_1')\,f(\mathbf{v}_1')\,d\mathbf{v}_1'. \tag{92}$$

Thus, we can write

$$\int \exp(\text{Л}_\Sigma t) \sum_{(1\le j\le N)} \exp(-i\mathbf{k}\mathbf{r}_j)\,\Phi(\mathbf{v}_j)\,\mathcal{D}(\Sigma)\,d\mathbf{r}_2\,d\mathbf{v}_2 \ldots d\mathbf{r}_N\,d\mathbf{v}_N = \exp(-i\mathbf{k}\mathbf{r}_1)\frac{1}{V}\Phi_\Sigma(v_1)\,U_k(t;\,1)\,\Phi(v_1). \quad (93)$$

Taking into account (85) and (86), we have

$$Q_1(k;\,\mathbf{r}_0,\,\mathbf{v}_0;\,\mathbf{v}_1) = \Phi_0(v_0)\,\Phi_\Sigma(v_1)\frac{1}{V}U_k(t;\,1)\,\overline{T}(v_0,\,v_1)\,\chi(\mathbf{r}_0,\,\mathbf{v}_0);$$

$$Q_2(\mathbf{r}_0,\,\mathbf{v}_0;\,\mathbf{v}_1) = \Phi_0(v_0)\,\Phi_\Sigma(v_1)\frac{1}{V}U_0(t;\,1)\,\widetilde{\chi}(\mathbf{r}_0,\,\mathbf{v}_0;\,\mathbf{v}_1).$$

These expressions must be substituted in (89). We transform first the expression of the type $\exp(\text{Л}_S t)\Phi_0(v_0)\,h(\mathbf{r}_0,\mathbf{v}_0)$ in (89). Using (77), we obtain

$$\exp(\text{Л}_S t)\,\Phi_0(v_0)\,h(\mathbf{r}_0,\,\mathbf{v}_0) = \Phi_0(v_0)\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right)t\right]h(\mathbf{r}_0,\,\mathbf{v}_0).$$

We note also that $\Phi_\Sigma(v_1)$ commutes with $\exp(\text{Л}_S t)$ and

$$\overline{T}_{-k}(v_0,\,v_1)\,\Phi_0(v_1)\,\Phi_\Sigma(v_1) = \Phi_0(v_1)\,\Phi_\Sigma(v_1)\,\overline{T}_{-k}(v_0,\,v_1).$$

With allowance for this, we finally obtain from (89)

$$K(t)\,\chi(S)\,\Phi_0(v_0) = \Phi_0(v_0)\,n\frac{1}{V}\sum_{(k\ne 0)}\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,T_{-k}(v_0,\,v_1) \times \exp(-i\mathbf{k}\mathbf{r}_0)\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right)t\right] \times \exp(i\mathbf{k}\mathbf{r}_0)\,U_k(t,\,1)\,\overline{T}_k(v_0,\,v_1)\,\chi(\mathbf{r}_0,\,\mathbf{v}_0) + \Phi_0(v_0)\,n\frac{1}{V}\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_0(v_0,\,v_1) \times \exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right)t\right]U_0(t;\,1)\,\widetilde{\chi}(\mathbf{r}_0,\,\mathbf{v}_0;\,\mathbf{v}_1)$$

and we thus reduce Eq. (78) to the form

$$\begin{aligned}
\frac{\partial\chi_t(\mathbf{r}_0,\,\mathbf{v}_0)}{\partial t} &= \left\{-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right\}\chi(\mathbf{r}_0,\,\mathbf{v}_0) \\
&+ n\frac{1}{V}\sum_{(k\ne 0)}\int_0^t d\tau\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_{-k}(v_0,\,v_1)\exp(-i\mathbf{k}\mathbf{r}_0) \\
&\times\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right)(t-\tau)\right] \\
&\times\exp(i\mathbf{k}\mathbf{r}_0)\,U_k(t-\tau;\,1)\,\overline{T}_k(v_0,\,v_1)\,\chi_\tau(\mathbf{r}_0,\,\mathbf{v}_0) \\
&+ n\frac{1}{V}\int_0^t d\tau\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_0(v_0,\,v_1) \\
&\times\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right)(t-\tau)\right]U_0(t-\tau;\,1) \\
&\times\widetilde{\chi}_\tau(\mathbf{r}_0,\,\mathbf{v}_0,\,\mathbf{v}_1); \\
f_t(\mathbf{r}_0,\,\mathbf{v}_0) &= \Phi_0(v_0)\,\chi_t(\mathbf{r}_0,\,\mathbf{v}_0).
\end{aligned} \quad (94)$$

We note that

$$\exp(-i\mathbf{k}\mathbf{r}_0)\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2L_S\right)(t-\tau)\right]\exp[i(\mathbf{k}+\mathbf{l})\,\mathbf{r}_0] = \exp(i\mathbf{l}\mathbf{r}_0)\exp[(-i\mathbf{v}_0(\mathbf{k}+\mathbf{l})+na^2L_S)(t-\tau)].$$

In this case, it is easy to see, applying a Fourier transform

$$\chi_t(\mathbf{r}_0,\,\mathbf{v}_0) = \frac{1}{V}\sum_{(l)}\exp(i\mathbf{l}\mathbf{r}_0)\,\chi_l(t,\,\mathbf{v}_0), \quad (95)$$

that from (97) for each component χ_l we obtain the equation

$$\begin{aligned}
\frac{\partial\chi_l(t,\,\mathbf{v}_0)}{\partial t} &= \{-i(\mathbf{l}\mathbf{v}_0)+na^2L_S\}\,\chi_l(t,\,\mathbf{v}_0) \\
&+ n\frac{1}{V}\sum_{(k\ne 0)}\int_0^t d\tau\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_{-k}(v_0,\,v_1) \\
&\times\exp[(-i\mathbf{v}_0(\mathbf{k}+\mathbf{l})+na^2L_S)(t-\tau)] \\
&\times U_k(t-\tau;\,1)\,\overline{T}_k(v_0,\,v_1)\,\chi_l(\tau,\,\mathbf{v}_0) \\
&+ n\frac{1}{V}\int_0^t d\tau\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_0(v_0,\,v_1) \\
&\times\exp[(-i\mathbf{v}_0\mathbf{l}+na^2L_S)(t-\tau)]\,U_0(t-\tau;\,1)\,\widetilde{\chi}_l(\tau,\,\mathbf{v}_0,\,\mathbf{v}_1).
\end{aligned} \quad (96)$$

In particular, for $l=0$

$$\begin{aligned}
\frac{\partial\chi(t,\,\mathbf{v}_0)}{\partial t} &= na^2L_S\chi(t,\,\mathbf{v}_0) \\
&+ n\frac{1}{V}\sum_{(k\ne 0)}\int_0^t d\tau\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_{-k}(v_0,\,v_1) \\
&\times\exp[(-i\mathbf{v}_0\mathbf{k}+na^2L_S)(t-\tau)]\,U_k(t-\tau;\,1)\,\overline{T}_k(v_0,\,v_1) \\
&\times\chi(\tau,\,\mathbf{v}_0) + n\frac{1}{V}\int_0^t d\tau\int d\mathbf{v}_1\,\Phi_\Sigma(v_1)\,\overline{T}_0(v_0,\,v_1) \\
&\times\exp[na^2L_S(t-\tau)]\,U_0(t-\tau;\,1)\,\widetilde{\chi}(\tau,\,\mathbf{v}_0,\,\mathbf{v}_1).
\end{aligned} \quad (97)$$

In these equations, the kernels of the integral expressions $\int_0^t\ldots d\tau$ are functions of $t-\tau$; we can therefore use the Laplace transform method.

For later actual use of these equations, we must establish the explicit expressions for the operator $U_k(t;\,1)$, which is determined solely by the dynamics of the isolated system Σ. This problem will be considered in Sec. 3. Here we note only that, using the operator U_k, we can calculate the functions $R_k(t)$ in the expressions given above. Indeed, from (44) we obtain

$$R_k(T) = V\exp(i\mathbf{k}\mathbf{r}_1)\int\exp(\text{Л}_\Sigma T)\sum_{(j)}\exp(-i\mathbf{k}\mathbf{r}_j) \times\mathcal{D}(\Sigma)\,d\mathbf{v}_1\,d\mathbf{r}_2\,d\mathbf{v}_2\ldots d\mathbf{r}_N\,d\mathbf{v}_N \quad (98)$$

and using the definition (90) we obtain

$$R_k(T) = \int\Phi_\Sigma(v_1)\,U_k(T;\,\mathbf{v}_1,\,\mathbf{v}_1')\,d\mathbf{v}_1\,d\mathbf{v}_1'. \quad (99)$$

SECTION 3

In this section, we concentrate our attention on studying equilibrium correlation mean values. Let Σ be a dynamical system whose behavior is described by classical mechanics and whose canonical distribution we denote, as before by $\mathfrak{D}(\Sigma)$.

We consider a dynamical variable as a function of the point of phase space, $U=U(\Omega_\Sigma)$, and denote it at time t by $U(t)=U(\Omega_\Sigma(t))$, where $\Omega_\Sigma(t)$ is the solution of the dynamical equations whose value at the initial time $t=0$ is equal to Ω_Σ, i.e., $\Omega_\Sigma(0)=\Omega_\Sigma$. Note that for a general nonequilibrium distribution $\mathfrak{D}_t(\Sigma)$ satisfying the Liouville equation $\partial\mathfrak{D}_t/\partial t=\text{Л}_\Sigma\mathfrak{D}_t$, $\mathfrak{D}_t=\mathfrak{D}_0$ for $t=0$, we have the well-known relation

$$\langle\mathcal{U}\rangle_t = \int\mathcal{U}(t)\,\mathcal{D}_0(\Sigma)\,d\Omega_\Sigma = \int\mathcal{U}(\Omega_\Sigma)\,\mathcal{D}_t(\Sigma)\,d\Omega_\Sigma. \quad (100)$$

We now investigate the equilibrium correlation mean values of two dynamical variables:

$$\langle\mathcal{U}(t)\,\mathcal{B}(\tau)\rangle = \overline{(\mathcal{U}(t)\,\mathcal{B}(\tau)\,\mathcal{D}(\Sigma))}_\Sigma = \int\mathcal{U}(t)\,\mathcal{B}(\tau)\,\mathcal{D}(\Sigma)\,d\Omega_\Sigma. \quad (101)$$

The invariance of such equilibrium mean values under time translations gives

$$\langle\mathcal{U}(t)\,\mathcal{B}(\tau)\rangle = \langle\mathcal{U}(t-\tau)\,\mathcal{B}\rangle.$$

Thus, the Fourier integral of this quantity can be written in the form

$$\langle\mathcal{U}(t)\,\mathcal{B}(\tau)\rangle = \int_{-\infty}^{+\infty} J_{\mathcal{U},\mathcal{B}}(\omega)\exp[-i\omega(t-\tau)]\,d\omega. \quad (102)$$

As in the quantum-mechanical case, we have the well-known inequality

$$J_{\mathcal{U}^*, \mathcal{U}}(\omega) \geqslant 0. \tag{103}$$

In the quantum-mechanical treatment of problems in statistical mechanics, a very important part is played by the method of two-time Green's functions defined by the relations

$$\left.\begin{aligned} G_{\text{ret}}(t-\tau) &= \theta(t-\tau)\langle[\mathcal{U}_t, \mathcal{B}_\tau]\rangle; \\ G_{\text{adv}}(t-\tau) &= -\theta(\tau-t)\langle[\mathcal{U}_t, \mathcal{B}_\tau]\rangle, \end{aligned}\right\} \tag{104}$$

where $[\ldots, \ldots]$ denotes the quantum-mechanical Poisson brackets. N. N. Bogolyubov, Jr., and Sadovnikov in Ref. 6 extended this method to classical mechanics. Their definition of the two-time Green's functions is also given by the expressions (104), except that the Poisson brackets (104) must be understood in the classical sense. They introduced the function

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_l = \frac{1}{2\pi\theta}\int_{-\infty}^{\infty} J_{\mathcal{U}, \mathcal{B}}(\omega')\frac{\omega'}{-\omega'+\nu}\,d\omega', \tag{105}$$

which is regular in the whole of the complex plane of the variable ν except for the real axis. The function (105) determines the frequency representation

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_\omega^{r,a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{r,a}(t)\exp(i\omega t)\,dt$$

of the retarded and advanced Green's functions by means of the relations

$$\left.\begin{aligned} \langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_\omega^r &= \langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega+i0+}; \\ \langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_\omega^a &= \langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega-i0+}, \end{aligned}\right\} \tag{106}$$

which leads to the result

$$J_{\mathcal{U}, \mathcal{B}}(\omega) = i\frac{\theta}{\omega}\{\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega+i0+} - \langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega-i0+}\}. \tag{107}$$

It should be noted that one must first make the usual passage to the limit $V \to \infty$ of statistical mechanics and then perform the limiting procedure of taking the variable ν to its values on the real axis.

We make some comments concerning the possibility of effective determination of the Green's functions. One of the methods developed in Ref. 6 can be briefly summarized as follows.

To the Hamiltonian H_Σ, we add an infinitesimally small term whose explicit time dependence is given by

$$\delta H_t = \exp(\varepsilon t - i\omega t)\,\mathcal{B}(\Omega_\Sigma)\,\delta\xi + \exp(\varepsilon t + i\omega t)\,\mathcal{B}^*(\Omega_\Sigma)\,\delta\xi^*, \quad \varepsilon > 0 \tag{108}$$

so that $H_t = H_\Sigma + \delta H$. Note that because of the choice of ε, $\delta H_t \to 0$ when $t \to -\infty$.

We proceed from the corresponding Liouville equation

$$\partial\mathcal{D}_t/\partial t = \Pi_\Sigma\mathcal{D}_t + [\delta H_t, \mathcal{D}_t]$$

with initial conditions as $t \to -\infty$: $\mathfrak{D}_{-\infty} = \mathfrak{D}(\Sigma)$. In other words, as $t \to -\infty$ we have statistical equilibrium and the infinitesimally small perturbation (108) is switched on adiabatically. Of course, $\mathfrak{D}_t = \mathfrak{D}(\Sigma) + \delta\mathfrak{D}_t$.

In this case, if we consider the time average of the dynamical variable $\mathfrak{U} = \mathfrak{U}(\Omega_\Sigma)$, we find that

$$\langle \mathcal{U}\rangle_t = \langle \mathcal{U}\rangle_{\text{eq}} + \delta\langle \mathcal{U}\rangle_t, \tag{109}$$

where

$$\delta\langle \mathcal{U}\rangle_t = \exp[-(\omega+i\varepsilon)it]\,2\pi\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega+i\varepsilon}\delta\xi + \exp[-i(-\omega+i\varepsilon)t]\,2\pi\langle\langle \mathcal{U}, \mathcal{B}^*\rangle\rangle_{-\omega+i\varepsilon}\delta\xi^*.$$

Thus, to obtain the expression for the Green's function in the upper half-plane of ν it is sufficient to calculate the variation $\delta\langle\mathfrak{U}\rangle_t$ of the time average of the given variable induced by an infinitesimally small perturbation of the form (108) in the Hamiltonian.

Note further that, as a consequence of (108), $J_{\mathfrak{B},\mathfrak{U}}(\omega) = J_{\mathfrak{U},\mathfrak{B}}(-\omega)$, whence

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega-i\varepsilon} = \langle\langle \mathcal{B}, \mathcal{U}\rangle\rangle_{-\omega+i\varepsilon}. \tag{110}$$

Therefore, the frequency representation of the Green's function in the lower half-plane can be obtained in exactly the same way by interchanging the roles of $\mathfrak{U}$ and $\mathfrak{B}$. The method proposed above is very fruitful, especially when one is dealing with the so-called hydrodynamic approximation. However, we shall here have recourse to a different procedure associated with the Laplace transform method, which is now widely used in statistical-mechanical problems of classical systems. By means of the expression for the distribution in a state of statistical equilibrium, $\mathfrak{D}(\Sigma) = Z^{-1}\exp[-H_\Sigma(\Omega_\Sigma)/\theta]$, we readily establish

$$[\mathcal{U}(t); \mathcal{D}(\Sigma)] = -\frac{1}{\theta}[\mathcal{U}(t); H_\Sigma]\,\mathcal{D}(\Sigma) = -\frac{1}{\theta}\frac{d\mathcal{U}(t)}{dt}\mathcal{D}(\Sigma).$$

In such a case, the identity $[\mathfrak{U}(t); \mathfrak{B}]\mathfrak{D}(\Sigma) + [\mathfrak{U}(t); \mathfrak{D}(\Sigma)]\mathfrak{B} \equiv [\mathfrak{U}(t); \mathfrak{B}\mathfrak{D}(\Sigma)]$ satisfied by the Poisson brackets and the relation $([\mathfrak{U}(t); \mathfrak{B}\mathfrak{D}(\Sigma)])_\Sigma = 0$ lead to

$$\langle[\mathcal{U}(t); \mathcal{B}]\rangle = \overline{([\mathcal{U}(t); \mathcal{B}]\,\mathcal{D}(\Sigma))_\Sigma} = -\frac{1}{\theta}\frac{d}{dt}\langle \mathcal{U}(t)\mathcal{B}\rangle.$$

On the other hand, (104) and (105) give

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega+i\varepsilon} = \frac{1}{2\pi}\int_0^\infty \exp[(-\varepsilon+i\omega)t]\,\langle[\mathcal{U}(t), \mathcal{B}]\rangle\,dt,$$

from which it follows that

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega+i\varepsilon} = \frac{1}{2\pi\theta}\int_0^\infty \exp(-zt)\frac{d}{dt}\langle \mathcal{U}(t)\mathcal{B}\rangle\,dt, \tag{111}$$

where

$$z = \varepsilon - i\omega \tag{112}$$

or

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega+i\varepsilon} = \frac{1}{2\pi\theta}\left\{z\int_0^\infty \exp(-zt)\langle \mathcal{U}(t)\mathcal{B}\rangle\,dt - \langle \mathcal{U}\mathcal{B}\rangle\right\}. \tag{113}$$

Taking into account (110), we also obtain

$$\langle\langle \mathcal{U}, \mathcal{B}\rangle\rangle_{\omega-i\varepsilon} = \frac{1}{2\pi\theta}\left\{\int_0^\infty z^*\exp(-z^*t)\langle \mathcal{U}(t)\mathcal{B}\rangle\,dt - \langle \mathcal{U}\mathcal{B}\rangle\right\}. \tag{114}$$

We see that the Green's functions in the upper and lower half-planes can be directly determined by means of a Laplace transform of equilibrium correlation mean values of the type

$$\langle \mathcal{U}_t\mathcal{B}\rangle, \quad t \geqslant 0. \tag{115}$$

To reduce the problem of finding such correlation mean values to the problem of calculating equal-time mean values, we proceed from the standard Liouville equation

$$\partial \mathcal{D}_t/\partial t = \text{Л}_\Sigma \mathcal{D}_t, \quad t \geqslant 0, \tag{116}$$

with the initial condition

$$\mathcal{D}_0 = \mathcal{D}(\Sigma) + \mathcal{B}(\Omega_\Sigma)\delta\xi, \quad t = 0, \tag{117}$$

which means that the initial expression for $\mathfrak{D}_t$ (for $t = 0$) differs only infinitesimally from the equilibrium distribution. In this case, $\mathfrak{D}_t = \mathfrak{D}(\Sigma) + \delta\mathfrak{D}_t$ and, using (100), we obtain

$$\begin{aligned}\delta\langle \mathcal{U}\rangle_t &= \int \mathcal{U}(t)\,\mathcal{B}\mathcal{D}(\Sigma)\,d\Omega_\Sigma\delta\xi = \langle \mathcal{U}(t)\,\mathcal{B}\rangle_{eq}\,\delta\xi \\ &= \int \mathcal{U}\delta\mathcal{D}_t\,d\Omega_\Sigma = \int \mathcal{U}\exp(\text{Л}_\Sigma t)\,\mathcal{B}\mathcal{D}(\Sigma)\,d\Omega_\Sigma\,d\xi.\end{aligned} \tag{118}$$

Note that in the framework of this approximation we are dealing with a Liouville operator Л_Σ which does not depend on the time. The variation is introduced, not in Л_Σ, but in the initial value $\mathfrak{D}$.

To investigate a more concrete situation, we shall consider, as in the previous section, a dynamical system Σ consisting of N identical particles of mass M. Further, we shall assume that the Liouville operator has the form

$$\text{Л}_\Sigma = \sum_{(1 \leqslant j \leqslant N)} \text{Л}_j^{(0)} + \sum_{(1 \leqslant j_1 < j_2 \leqslant N)} \text{Л}_{j_1, j_2}, \tag{119}$$

where

$$\text{Л}_j^0 = -\mathbf{v}_j \partial/\partial \mathbf{r}_j, \tag{120}$$

and

$$\text{Л}_{j_1, j_2} = \text{Л}_{j_1, j_2}^{(\Phi_\Sigma)}, \tag{121}$$

or

$$\text{Л}_{j_1, j_2} = \text{Л}_{j_1, j_2}^{(\text{coll})}, \tag{122}$$

or

$$\text{Л}_{j_1, j_2} = \text{Л}_{j_1, j_2}^{(\Phi)} + \text{Л}_{j_1, j_2}^{(\text{coll})} \tag{123}$$

(the notation is the same as in the previous section).

We now concentrate our attention on the method of reduced distribution frunctions in the form already developed by myself and set forth in the monograph of Ref. 8. These reduced distribution functions are introduced as follows:

$$\left.\begin{aligned}&F_1(t;1) = F_1(t;\mathbf{r}_1,\mathbf{v}_1) = V\int \mathcal{D}_t\,d\mathbf{r}_2\,d\mathbf{v}_2\ldots d\mathbf{r}_N\,d\mathbf{v}_N;\\ &F_2(t;1,2) = F_2(t;\mathbf{r}_1,\mathbf{v}_1;\mathbf{r}_2,\mathbf{v}_2)\\ &= V^2(1-1/N)\int \mathcal{D}_t\,d\mathbf{r}_3\,d\mathbf{v}_3\ldots d\mathbf{r}_N\,d\mathbf{v}_N;\\ &F_s(t;1,2,\ldots,s) = F_s(t;\mathbf{r}_1,\mathbf{v}_1;\ldots,\mathbf{r}_s,\mathbf{v}_s)\\ &= V^s(1-1/N)\ldots[1+(1-s)/N]\int \mathcal{D}_t\,d\mathbf{r}_{s+1}d\mathbf{v}_{s+1}\ldots d\mathbf{r}_N\,d\mathbf{v}_N.\end{aligned}\right\} \tag{124}$$

Because of the symmetry of $\mathfrak{D}_t$, the F_s are symmetric functions of the phases $(1),\ldots,(s)$. Since $\mathfrak{D}_t = \exp(\text{Л}_\Sigma t)\,\mathfrak{D}_0$, we can also write

$$F_1(t;\mathbf{r}_1,\mathbf{v}_1) = V\int \exp(\text{Л}_\Sigma t)\,D_0\,d\mathbf{r}_2\,d\mathbf{v}_2\ldots d\mathbf{r}_N\,d\mathbf{v}_N. \tag{125}$$

It is readily seen that the functions $F_1(t;1)$, $F_2(t;1,2)$, ... give, respectively, the probability density for finding one particle with the phase $(\mathbf{r}_1,\mathbf{v}_1)$, two particles with the phase $(\mathbf{r}_1,\mathbf{v}_1;\mathbf{r}_2,\mathbf{v}_2)$, etc.

We consider the additive dynamical variable

$$\mathcal{U} = \sum_{(1 \leqslant j \leqslant N)} A(\mathbf{r}_j,\mathbf{v}_j). \tag{126}$$

Proceeding from the definition (124) and using the symmetry properties, we find

$$\langle \mathcal{U}\rangle_t = n\int A(\mathbf{r}_1,\mathbf{v}_1)\,F_1(t;\mathbf{r}_1,\mathbf{v}_1)\,d\mathbf{r}_1\,d\mathbf{v}_1, \tag{127}$$

or, in a more compact form

$$\langle \mathcal{U}\rangle_t = n\int A(1)\,F_1(t;1)\,d(1).$$

Similarly, the mean value of a dynamical variable of binary type can be expressed by means of $F_2(t;1,2)$, etc.

The Liouville equation leads to the hierarchy of equations

$$\left.\begin{aligned}&\partial F_1(t;1)/\partial t = \text{Л}_1^{(0)}F_1(t;1)\\ &+ n\int \text{Л}_{1,2}F_2(t;1,2)\,d(2);\\ &\partial F_2(t;1,2)/\partial t = (\text{Л}_1^{(0)} + \text{Л}_2^{(0)} + \text{Л}_{1,2})\,F_2(t;1,2)\\ &+ n\int (\text{Л}_{1,3} + \text{Л}_{2,3})\,F_3(t;1,2,3)\,d(3);\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\partial F_s(t;1,2,\ldots,s)/\partial t\\ &= \Big(\sum_{(1\leqslant j\leqslant s)} \text{Л}_j^{(0)} + \sum_{(1\leqslant j_1<j_2\leqslant N)} \text{Л}_{j_1,j_2}\Big)\,F_s(t;1,2,\ldots,s)\\ &+ n\int \sum_{(1\leqslant j\leqslant s)} \text{Л}_{j,s+1}F_{s+1}(t;1,2,\ldots,s,s+1)\,d(s+1).\end{aligned}\right\} \tag{128}$$

When we deal with the reduced distributions F_s, it is usually assumed that for $V\to\infty$, $N/V = n = \text{const}$ they have definite limits that also satisfy Eqs. (128).

In the case of an equilibrium distribution, this assumption was rigorously justified for a large class of physically admissible short-range potential functions $\Omega_\Sigma(r)$ if the particle density is sufficiently low.[7] Under these conditions, the analyticity of F_s as a function of n was also proved.[7]

Note that investigation into the behavior of the equilibrium F_s is greatly simplified by virtue of the fact that they factorize:

$$F_{eq}(1,\ldots,s) = f(\mathbf{r}_1,\ldots,\mathbf{r}_s)\prod_{(1\leqslant j\leqslant s)} \Phi_\Sigma(v_j). \tag{129}$$

So far as we know, the behavior of nonequilibrium F_s has not been studied at a mathematically rigorous level.

We now consider Eqs. (128), going to the limit $V\to\infty$. From the formal point of view, we arrive at a system of linear equations for the reduced distribution functions F_s. It must, however, be borne in mind that not all solutions of these equations are physically admissible.

We take, for example, $F_s(t;1,\ldots,s)$ and combine the indices $1,\ldots,s$ into l groups $[j_1],\ldots,[j_l]$ containing, respectively, $s_1,\ldots,s_l$ numbers: $F_s(t;1,\ldots,s) = F_{s_1+\ldots+s_l}(t;[j_1],\ldots,[j_l])$, $s = s_1+\cdots+s_l$. We assume that the distances between the particles belonging to different groups tend to infinity. In this case, it is natural from a physical point of view to expect the correlation between the sets $[j_1],\ldots,[j_l]$ of the particles of the system Σ to vanish:

$$F_{s_1+\ldots+s_l}(t;[j_1],\ldots,[j_l]) - F_{s_1}(t;[j_1])\ldots F_{s_l}(t;[j_l]) \to 0, \quad (130)$$

when

$$|\mathbf{r}_{j_p} - \mathbf{r}_{j_{p'}}| \to \infty;\ p,\ p' = 1,\ \ldots,\ l;\ j_p \in [j_p];\ j_{p'} \in [j_{p'}].$$

These relations, which express the fundamental principle of correlation weakening,[8] can be regarded as a certain kind of boundary condition[4)] imposed on F_s.

Of course, these boundary conditions are nonlinear. To make them linear,[8] we introduce Green's functions $G_s(t; 1,\ldots,s)$ $(s=2, 3,\ldots)$, setting

$$\left.\begin{aligned}
&F_2(t;1,2) = F_1(t;1)F_1(t;2) + G_2(t;1,2);\\
&F_3(t;1,2,3) = F_1(t;1)F_1(t;2)F_1(t;3)\\
&+F_1(t;1)G_2(t;2,3) + F_1(t;2)G_2(t;1,3)\\
&+F_1(t;3)G_2(t;1,2) + G_3(t;1,2,3).
\end{aligned}\right\} \quad (131)$$

In such a case, (130) leads to the linear relations

$$\left.\begin{aligned}
&G_2(t;1,2)\to 0, \ \text{if}\ \ |\mathbf{r}_1-\mathbf{r}_2|\to\infty;\\
&G_3(t;1,2,3)\to 0,\\
&\text{if}\ \max\{|\mathbf{r}_1-\mathbf{r}_2|,\ |\mathbf{r}_1-\mathbf{r}_3|,\ |\mathbf{r}_2-\mathbf{r}_3|\}\to\infty.
\end{aligned}\right\} \quad (132)$$

Using the definitions (131), we arrive at a hierarchy of nonlinear equations for F_1, G_2, $G_3\ldots$:

$$\left.\begin{aligned}
&\partial F_1(t;1)/\partial t = \text{Л}_1^{(0)}F_1(t;1)\\
&+n\int \text{Л}_{1,2}\{F_1(t;1)F_1(t;2)+G_2(t;1,2)\}\,d(2);\\
&\partial G_2(t;1,2)/\partial t = (\text{Л}_1^{(0)}+\text{Л}_2^{(0)}+\text{Л}_{1,2})\,G_2(t;1,2)\\
&+\text{Л}_{1,2}F_1(t;1)F_1(t;2) + n\int \text{Л}_{1,3}\{F_1(t;3)G_2(t;1,2)\\
&+F_1(t;1)G_2(t;2,3)+G_3(t;1,2,3)\}\,d(3)\\
&+n\int \text{Л}_{2,3}\{F_1(t;3)G_2(t;1,2)+F_1(t;2)G_2(t;1,3)\\
&+G_3(t;1,2,3)\}\,d(3)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right| \quad (133)$$

We now turn to the problem of calculating equilibrium mean values. We have to deal with two dynamical variables of additive type: $U=\sum_{(1\leq j\leq N)}A(j)$ and $\mathscr{B}=\sum_{(1\leq j\leq N)}B(j)$, for which

$$\int B(1)\,F_1^{(\mathrm{eq})}(1)\,d(1)=0, \quad (134)$$

or, equivalently,

$$\langle\mathscr{B}\rangle_{\mathrm{eq}}=0. \quad (135)$$

We consider a solution of the Liouville equation that differs only infinitesimally from the equilibrium Gibbs distribution:

$$\mathscr{D}_t=\mathscr{D}(\Sigma)+\delta\mathscr{D}_t, \quad (136)$$

proceeding from the initial distribution

$$\left.\begin{aligned}
&\mathscr{D}_0=\mathscr{D}(\Sigma)+\delta\mathscr{D}_0;\\
&\delta\mathscr{D}_0=\sum_{(1\leq j\leq N)}B(j)\,\delta\xi,
\end{aligned}\right\} \quad (137)$$

and we introduce the corresponding reduced distributions

$$F_1^{(\mathrm{eq})}(1)+\delta F_1(t;1);\ \ldots\ F_s^{\mathrm{eq}}(1,\ldots,s)+\delta F_s(t;1,\ldots,s);\ \ldots$$

Then in accordance with (118),

$$\langle \mathcal{U}(t)\,\mathscr{B}\rangle\,\delta\xi = n\int A(1)\,\delta F_1(t;1)\,d(1), \quad (138)$$

and Eq. (125) gives

$$\delta F_1(t;1)=V\int \exp(\text{Л}_\Sigma t)\,\delta(\mathscr{D}_0)\,d(2)\ldots d(N). \quad (139)$$

Variation of the relations (131) enables us to introduce $\delta G_2(t;1,2);\ldots\delta G_s(t;1,2,\ldots,s)$. Note that variation of the nonlinear equations (133) leads to linear equations for $\delta F_1(t;1)$; $\delta G_2(t;1,2),\ldots\delta G_s(t;1,2,\ldots,s)$; $\ldots$, in which the coefficients depend on the equilibrium functions.

We now turn to the derivation of initial expressions for these variations. Thus, from (136) we obtain

$$\begin{aligned}
&(1/\delta\xi)\,\delta F_1(0;1)=B(1)F_1(1)+n(1-1/N)\int B(3)F_2(1,3)\,d(3);\\
&(1/\delta\xi)\,\delta F_2(0;1,2)=\{B(1)+B(2)\}\,F_2(1,2)\\
&\qquad+n(1-2/N)\int B(3)F_3(1,2,3)\,d(3),
\end{aligned}$$

where for brevity we have omitted the index eq. of $F_s(1,\ldots,s)$. With allowance for (134),

$$\begin{aligned}
\int B(3)F_2(1,3)\,d(3)&=\int B(3)\{F_2(1,3)-F_1(1)F_1(3)\}\,d(3)\\
&=\int B(3)\,G_2(1,3)\,d(3)
\end{aligned}$$

and therefore

$$\delta F_1(0;1)=\left\{B(1)F_1(1)+n(1-1/N)\int B(3)\,G_2(1,3)\,d(3)\right\}\delta\xi.$$

We also have

$$\begin{aligned}
&\delta G_2(0;1,2)=\delta F_2(0;1,2)-F_1(1)\,\delta F_1(0;2)\\
&-F_1(2)\,\delta F_1(0;1)=\{B(1)+B(2)\}\,G_2(1,2)\\
&+n(1-1/N)\int B(3)\{F_3(1,2,3)-F_1(1)F_1(2)F_1(3)\\
&-F_1(1)G_2(2,3)-F_1(2)G_2(1,3)-F_1(3)G_2(1,2)\}\,d(3)\\
&-(n/N)\int B(3)\{F_3(1,2,3)-F_2(1,2)F_1(3)\}\,d(3).
\end{aligned}$$

Thus, ignoring terms of order $1/N$, we obtain

$$\begin{aligned}
&\delta G_2(0;1,2)=\Big\{(B(1)+B(2))\,G_2(1,2)\\
&\quad+n\int B(3)\,G_3(1,2,3)\,d(3)\Big\}\,\delta\xi\\
&\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

As we noted earlier, we consider here only the case when the state of statistical equilibrium in the system Σ is spatially homogeneous. Therefore

$$\begin{aligned}
&F_1(1)=\Phi_\Sigma(v_1);\\
&G_2(1,2)=g_2(\mathbf{r}_1-\mathbf{r}_2)\,\Phi_\Sigma(v_1)\,\Phi_\Sigma(v_2);\\
&G_3(1,2,3)=g_3(\mathbf{r}_1-\mathbf{r}_3,\ \mathbf{r}_2-\mathbf{r}_3)\,\Phi_\Sigma(v_1)\,\Phi_\Sigma(v_2)\,\Phi_\Sigma(v_3)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

In such a case, we see that the condition (134) can be rewritten in the form

$$\int B(\mathbf{r},\ \mathbf{v})\,\Phi_\Sigma(\mathbf{v})\,d\mathbf{r}\,d\mathbf{v}=0. \quad (140)$$

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4) In a purely mathematical discussion of the properties of the system (128) there arise various difficult questions; for example, in what sense must one understand the relations (130); what other conditions imposed on F_s must be taken into account; which initial conditions at $t=0$ must be imposed on F_s?

Also,

$$\left.\begin{aligned}\delta F_1(0;\ 1) &= \Phi_\Sigma(v_1)\left\{B(\mathbf{r}_1,\ \mathbf{v}_1)+n\int g_2(\mathbf{r}_1-\mathbf{r}_2)B(\mathbf{r}_2,\ \mathbf{v})\right.\\ &\quad\left.\times\Phi_\Sigma(v_2)\,d\mathbf{r}_2\,d\mathbf{v}_2\right\}\delta\xi;\\ \delta G_2(0;\ 1,\ 2) &= \Phi_\Sigma(v_1)\Phi_\Sigma(v_2)\left\{(B(\mathbf{r}_1\ \mathbf{v})+B(\mathbf{r}_2,\ \mathbf{v}_2))\right.\\ &\quad\times g_2(\mathbf{r}_1-\mathbf{r}_2)+n\int g_3(\mathbf{r}_1-\mathbf{r}_3,\ \mathbf{r}_2-\mathbf{r}_3)\\ &\quad\left.\times B(\mathbf{r}_3,\ \mathbf{v}_3)\Phi_\Sigma(v_3)\,d\mathbf{r}_3\,d\mathbf{v}_3\right\}\delta\xi\end{aligned}\right\} \tag{141}$$

We now consider the special case when

$$B(\mathbf{r},\ \mathbf{v})=B_k(\mathbf{r},\ \mathbf{v})=\exp(-i\mathbf{k}\mathbf{r})\,\phi(\mathbf{v}), \tag{142}$$

and we note that for $k\neq 0$ the condition (140) is satisfied automatically and for $k=0$ this condition requires fulfillment of

$$\int\phi(\mathbf{v})\Phi_\Sigma(\mathbf{v})\,d\mathbf{v}=0;\quad \phi(\mathbf{v})=B_0. \tag{143}$$

Then

$$\delta F_1(0;\ 1)=\exp(-i\mathbf{k}\mathbf{r}_1)\Phi_\Sigma(v_1)\left\{\phi(\mathbf{v}_1)+n\int g(\mathbf{r})\exp(i\mathbf{k}\mathbf{r})\,d\mathbf{r}\int\phi(\mathbf{v})\Phi_\Sigma(v)\,d\mathbf{v}\right\}d\xi \tag{144}$$

and

$$\delta G_s(0;\ \mathbf{r}_1+\mathbf{r},\ \mathbf{v}_1;\ \ldots\ \mathbf{r}_s+\mathbf{r}_s)=\exp(-i\mathbf{k}\mathbf{r})\,\delta G_s(0;\ \mathbf{r}_1,\ \mathbf{v}_1;\ \ldots\ \mathbf{r}_s,\ \mathbf{v}_s).$$

Since the linear equations obtained from (133) for

$$\delta F_1(t;\ 1);\ \ldots\ \delta G_s(t;\ 1,\ \ldots,\ s);\ \ldots,$$

are invariant under spatial translations, we have

$$\begin{aligned}&\delta F(t;\ 1)=\exp(-i\mathbf{k}\mathbf{r}_1)\Phi_k(t,\ \mathbf{v}_1)\,\delta\xi;\\ &\delta G_s(t;\ \mathbf{r}_1+\mathbf{r},\ \mathbf{v}_1;\ \ldots\ \mathbf{r}_s+\mathbf{r},\ \mathbf{v}_s)\\ &=\exp(-i\mathbf{k}\mathbf{r})\,\delta G_s(t;\ \mathbf{r}_1,\ \mathbf{v}_1;\ \ldots\ \mathbf{r}_s,\ \mathbf{v}_s).\end{aligned} \tag{145}$$

Here, $\Phi_k(t,\mathbf{v}_1)$, like δG_s, are linear functionals of $\phi(\mathbf{v})$. Using the relations (91), (137), and (139), we obtain

$$\Phi_k(t,\ \mathbf{v})=\Phi_\Sigma(v_1)\int U_k(t,\ \mathbf{v}_1,\ \mathbf{v}_1')\,\phi(\mathbf{v}_1')\,d\mathbf{v}_1', \tag{146}$$

where for $k=0$ the condition (143) must be satisfied. It also follows from (99) that

$$R_k(t)=\int\Phi_k(t,\ \mathbf{v}_1)\,dv_1\ \text{ for }\ \phi(\mathbf{v})=1,\ k\neq 0. \tag{147}$$

We now turn to the case when

$$Л_{1,\,2}=Л_{1,\,2}^{(\Phi_\Sigma)};\quad Л_{\rm int}=Л_{\rm int}^{(\Phi)}. \tag{148}$$

We recall that to reduce the previously formulated approximate equations (56) and (57) or the kinetic equations to a completely definite form, it is necessary to calculate $R_k(t)\ (k\neq 0)$ explicitly. In the case

$$Л_{1,\,2}=Л_{1,\,2}^{(\rm coll)};\quad Л_{\rm int}=Л_{\rm int}^{(\rm coll)} \tag{149}$$

the corresponding approximate equations (96) and (97) become completely definite if we can obtain explicit expressions for U_k.

Thus, we see that in both cases (148) and (149) we must calculate explicitly $\Phi_k(t,v_1)$. To achieve this aim, we restrict ourselves to the simplest approximation in the system of nonlinear equations (133) and consider only the first of them, ignoring the correlation function $G_2(t;1,2)$. In such an approximation, we are dealing with only a single nonlinear equation

$$\partial F_1(t;\ 1)/\partial t=Л_1^{(0)}F_1(t;\ 1)+n\int Л_{1,\,2}F_1(t;\ 1)\,F(t;\ 2)\,d(2). \tag{150}$$

It is obvious that for (148) this equation goes over into the well-known Vlasov equation

$$\begin{aligned}&\partial F_1(t;\ \mathbf{r}_1,\ \mathbf{v})/\partial t=-\mathbf{v}_1\,\partial F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1)/\partial\mathbf{r}_1\\ &+\frac{n}{M}\left\{\frac{\partial}{\partial\mathbf{r}_1}\int\Phi_\Sigma(\mathbf{r}_1-\mathbf{r}_2)\,\bar\rho(t;\ \mathbf{r}_2)\right\}\frac{\partial F_1(t;\mathbf{r}_1,\mathbf{v}_1)}{\partial\mathbf{v}_1},\end{aligned} \tag{151}$$

where

$$\bar\rho(t;\ \mathbf{r})=\int F_1(t;\ \mathbf{r},\ \mathbf{v})\,d\mathbf{v}_1.$$

A single-component Vlasov equation of this type is used, for example, to describe the simplest model of an electron plasma, namely, a classical electron gas consisting of negatively charged point particles in a compensating homogeneous positively charged background. In this model

$$\Phi_\Sigma(r)=e^2/r. \tag{152}$$

Note that for the state of statistical equilibrium $\bar\rho_{\rm eq}=1$. To take into account the external field due to the positive background, it is necessary to subtract the constant charge density from the charge density of the electrons. This leads to the replacement of the expression (151) for the particle density by $\bar\rho(t;\mathbf{r})=\int F_1(t;\mathbf{r},\mathbf{v}_1)d\mathbf{v}_1-1$.

In the state of statistical equilibrium, the total charge density is zero, and, therefore, the equation for the variation is

$$\begin{aligned}&\partial\delta F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1)/\partial t=-\mathbf{v}_1\,\partial\delta F_1(t;\ \mathbf{r}_1,\mathbf{v}_1)/\partial\mathbf{r}_1\\ &+\frac{n}{M}\frac{\partial}{\partial\mathbf{r}_1}\int\Phi_\Sigma(r_1-\mathbf{r}_2)\,\delta\bar\rho(t;\mathbf{r}_2)\,d\mathbf{r}_2\,\frac{\partial\Phi_\Sigma(v_1)}{\partial\mathbf{v}_1}.\end{aligned} \tag{153}$$

Since we are considering here the case $\phi(\mathbf{v})=1$, and since in the adopted approximation we must omit in (144) the term containing the correlation function $g(r)$, we obtain

$$\delta F_1(0;\ \mathbf{r}_1,\mathbf{v}_1)=\exp(-i\mathbf{k}\mathbf{r}_1)\Phi_\Sigma(v_1)\,\delta\xi.$$

Equations (145) and (147) enable us to reduce (153) to the form

$$\left.\begin{aligned}&\frac{\partial\Phi_k(t;\mathbf{v})}{\partial t}=i(\mathbf{k}\mathbf{v})\left\{\Phi_k(t;\ \mathbf{v})+\frac{4\pi e^2 n}{\theta k^2}R_k(t)\Phi_\Sigma(\mathbf{v})\right\};\\ &\Phi_k(0;\ \mathbf{v})=\Phi_\Sigma(v).\end{aligned}\right\} \tag{154}$$

To solve this equation, we use the Laplace transform

$$\left.\begin{aligned}&\int_0^\infty\Phi_k(t;\ \mathbf{v})\exp(-zt)\,dt=\bar\Phi_k(z;\ \mathbf{v}),\\ &\operatorname{Re}z>0;\\ &\int_0^\infty R_k(t)\exp(-zt)\,dt=\int_0^\infty\bar\Phi_k(z,\ \mathbf{v})\,d\mathbf{v}=\bar R_k(z),\end{aligned}\right\} \tag{155}$$

which reduces (154) to

$$(z-i(\mathbf{k}\mathbf{v}))\,\bar\Phi_k(z,\ \mathbf{v})=i\mathbf{k}\mathbf{v}\frac{4\pi e^2 n}{\theta k^2}\bar R_k(z)\,\Phi_\Sigma(v)+\Phi_\Sigma(v)$$

and for $\bar\Phi_k(z,v)$ gives the expression

$$\tilde{\Phi}_k(z, v) = \frac{\Phi_\Sigma(v)}{z - i(kv)} + \frac{ikv}{z - i(kv)} \frac{4\pi e^2 n}{\theta k^2} R_k(z) \Phi_\Sigma(v).$$

Taking into account (155), we obtain

$$\int^\infty R_k(t) \exp(-zt)\, dt = \int \frac{\Phi_\Sigma(v)}{z - i(kv)} dv \left\{1 - \frac{4\pi e^2 n}{\theta k^2} \int \frac{ikv}{z - i(kv)} \Phi_\Sigma(v)\, dv\right\}^{-1}$$

or

$$\int_0^\infty R_k(t) \exp(-zt)\, dt = \int \frac{\Phi_\Sigma(v)}{z - i(kv)} \times dv \left\{1 + \frac{4\pi e^2 n}{\theta k^2} - \frac{4\pi e^2 n}{\theta k^2} z \int \frac{\Phi_\Sigma(v)}{z - i(kv)} dv\right\}^{-1} \quad (\mathrm{Re}\, z > 0). \tag{156}$$

It is the left-hand side of (156) that occurs in Eqs. (54) and (63).

We can now obtain a more definite expression for the integral

$$\int \Phi_\Sigma(v)\, dv/[z - i(kv)], \tag{157}$$

by noting that here $\Omega_\Sigma(v)$ is the normalized Maxwellian velocity distribution. To this end, it is convenient to choose the direction of the vector **k** as the direction of the z axis in the integration space for (157).

Then we obtain

$$\int \Phi_\Sigma(v)\, dv/[z - i(kv)] = \left(\frac{M}{2\pi\theta}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-Mu^2/2\theta)\, du\, (z - iku)^{-1}.$$

Here

$$(z - iku)^{-1} = \int_0^\infty \exp[-\tau(z - iku)]\, d\tau, \quad \mathrm{Re}\, z > 0.$$

Integration with respect to u leads to the expression

$$\left(\frac{M}{2\pi\theta}\right)^{1/2} \int_{-\infty}^{\infty} \exp[i\tau k u - Mu^2/2\theta]\, du = \exp(-\tau^2 k^2 u_{\mathrm{eq}}^2), \quad u_{\mathrm{eq}} = \sqrt{\theta/2M},$$

from which it follows that

$$\int \frac{\Phi_\Sigma(v)}{z - ikv} dv = \int_0^\infty \exp(-\tau z - u_{\mathrm{eq}}^2 k^2 \tau^2)\, d\tau = \frac{1}{ku_{\mathrm{eq}}} \int_0^\infty \exp\left(-\tau \frac{z}{ku_{\mathrm{eq}}}\right) \exp(-\tau^2)\, d\tau$$

and, in particular,

$$\lim_{\varepsilon \to 0} \int \frac{\Phi_\Sigma(v)}{\varepsilon - i\omega - ikv} dv = \frac{1}{ku_{\mathrm{eq}}} \int_0^\infty \exp(-\tau^2) \times \left\{\cos\frac{\omega\tau}{ku_{\mathrm{eq}}} + i \sin\frac{\omega\tau}{ku_{\mathrm{eq}}}\right\} d\tau = \frac{1}{ku_{\mathrm{eq}}} \left\{\frac{\sqrt{\pi}}{2} \exp\left[-\frac{\omega^2}{4k^2 u_{\mathrm{eq}}^2}\right] + i \int_0^\infty \exp(-\tau^2) \sin\frac{\omega\tau}{ku_{\mathrm{eq}}}\, d\tau\right\}.$$

Thus, Eqs. (64) and (156) give

$$F(\mathbf{k}\mathbf{v}_0) = \mathrm{Re}\, \frac{[1/(ku_{\mathrm{eq}})] \left\{(\sqrt{\pi}/2) \exp[-(\sigma\cdot\mathbf{v}_0)^2/4u_{\mathrm{eq}}^2] + i \int_0^\infty \exp(-\tau^2) \sin[\omega\tau/(ku_{\mathrm{eq}})]\, d\tau\right\}}{1 + \frac{4\pi e^2 n}{\theta k^2} \left\{1 - \left[\frac{(\sigma\cdot\mathbf{v}_0)}{u_{\mathrm{eq}}}\right] \int_0^\infty \exp(-\tau^2) \sin\left[\tau \frac{(\sigma\cdot\mathbf{v}_0)}{u_{\mathrm{eq}}}\right] d\tau\right\} \to + \left[\frac{4\pi e^2 n}{(\theta k^2)}\right] i \left[\frac{\sigma\cdot\mathbf{v}_0}{2}\right] \sqrt{\pi} \exp\left[-\frac{(\sigma\cdot\mathbf{v}_0)^2}{(4u_{\mathrm{eq}}^2)}\right]}. \tag{158}$$

We now consider Eq. (63) for the case when a point particle S with charge Ze interacts with particles of the system Σ solely through the Coulomb law. Then

$$\nu(k) = 4\pi Z e^2/k^2. \tag{159}$$

Substituting (158) and (157) in (63), we obtain a kinetic equation of Markov type.

In a simpler approximation, an analogous kinetic equation was found by Temko.[9] Its generalization to the quantum case was considered by Klimontovich and Temko.[10] It is obvious that the main field of application of this equation is to the description of the motion of a charged particle in a classical electron plasma.

Note, however, that all our equations have been derived from the general approximate equation (27), which itself was obtained under the assumption that the interaction between the systems S and Σ is weak.

If we assume that e^2 can indeed be regarded as a small parameter, then in the denominator of (158) we must omit all the terms except the unity since they are all proportional to e^2 and $\nu^2(k)$, quantities that already contain this parameter. In this case, we obtain the very simple expression

$$F(\mathbf{k}\cdot\mathbf{v}_0) = \frac{1}{ku_{\mathrm{eq}}} \frac{\sqrt{\pi}}{2} \exp\left[-\frac{(\sigma\cdot\mathbf{v}_0)^2}{4u_{\mathrm{eq}}^2}\right],$$

which is proportional to $1/k$.

In Eq. (63), $d\mathbf{k} = k^2 dk\, d\sigma$, so that the integration with respect to k takes the form

$$\int_0^\infty \frac{1}{k^4} k^2 \frac{1}{k} k^2\, dk = \int_0^\infty \frac{dk}{k}.$$

We see that it diverges logarithmically for both small and large k. In the language of quantum field theory, we have here both infrared and ultraviolet divergences. It is easy to trace the physical origin of this divergence to the Coulomb interaction which we are considering in this case.

We note first that the potential energy of the interaction between the particle S and the system Σ is small compared with their mean kinetic energy when $1/r \ll \theta/(|Z|e^2)$. In such a case, a correctly calculated contribution of the **k** space to the integral is obtained only in the region where

$$k \ll k_{\max} = \theta/(|Z|e)^2. \tag{160}$$

Second, it is necessary to take into account the charge screening in the plasma at large distances of the order of the Debye radius. It is the neglect of this effect which is responsible for the divergence at small k.

If we take the complete expression, including the terms omitted in the denominator, we see that for small k the function $F(\mathbf{k}\cdot\mathbf{v}_0)$ is of order k, which annihilates the "infrared" divergence. However, for $k \to \infty$

$$F(\mathbf{k}\cdot\mathbf{v}_0) \sim \frac{1}{ku_{\mathrm{eq}}} \exp\left[-\frac{(\sigma\cdot\mathbf{v}_0)^2}{4u_{\mathrm{eq}}^2}\right]$$

and for large k the logarithmic divergence remains.

Therefore, to make the integral on the right-hand side of Eq. (63) converge, we can use a cutoff procedure, integrating with respect to k in the interval $(0, k_{\max})$ instead of $(0, +\infty)$.

In order to develop a self-consistent approximation procedure, we do not have to adopt the cutoff procedure *ad hoc*. We must improve our approximation by separating out, for example, from the short-range part of the Coulomb interaction a Liouville operator of special type describing collisions. We shall not consider this question here but turn to investigation of the case (149).

Then Eq. (150) goes over into the Boltzmann-Enskog equation for the interaction of hard spheres:

$$\partial F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1)/\partial t = -\mathbf{v}_1\, \partial F_1(t;\ \mathbf{r}_1,\ \mathbf{v})/\partial \mathbf{r}_1 + na_0^2 \int \mathbf{v}_{1,2}\cdot\sigma\theta(\mathbf{v}_{1,2}\cdot\sigma)\{\delta(\mathbf{r}_1-\mathbf{r}_2-a_0\sigma)\, b_{v_1v_2}(\sigma) - \delta(\mathbf{r}_1-\mathbf{r}_2+a_0\sigma)\} \times F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1)\, F_1(t;\ \mathbf{r}_2,\ \mathbf{v}_2)\, d\sigma\, d\mathbf{r}_2\, d\mathbf{v}_2. \qquad (161)$$

Here, $b_{\nu_1,\nu_2}(\sigma)$ is an operator which is applied to the function $f(\mathbf{v}_1, \mathbf{v}_2)$ and replaces its arguments as follows:

$$\left.\begin{aligned} \mathbf{v}_1 &\to \mathbf{v}_1^* = \mathbf{v}_1 - \sigma(\mathbf{v}_{1,2}\cdot\sigma); \\ \mathbf{v}_2 &\to \mathbf{v}_2^* = \mathbf{v}_2 + \sigma(\mathbf{v}_{1,2}\cdot\sigma), \end{aligned}\right\} \qquad (162)$$

where σ is a unit vector; $a_0 = a_\Sigma$ is the diameter of the hard spheres that characterizes the interaction of the Σ particles.

It was pointed out above that when $Л_{1,2} = Л_{1,2}^{\Phi\Sigma}$ and $\Phi_\Sigma(r)$ corresponds to short-range repulsion it is possible to obtain for $F_1(t; 1)$ a kinetic equation that contains an operator which takes into account collisions, using for this purpose the second equation of the system (133) and ignoring the term proportional to the particle density. Here, we shall treat only the simplest variant of the Boltzmann-Enskog equation (161), which describes the dynamics of hard spheres. The corresponding generalization of the discussion does not lead to any essential difficulties.

Varying Eq. (161) in the neighborhood of the equilibrium solution, we obtain for $F_1(t; 1)$ the equation

$$\frac{\partial \delta F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1)}{\partial t} = -\mathbf{v}_1 \frac{\partial}{\partial \mathbf{r}_1}\delta F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1) + na_0^2 \int (v_{1,2}\cdot\sigma)\,\theta(\mathbf{v}_{1,2}\cdot\sigma) \times\{\delta(\mathbf{r}_1-\mathbf{r}_2-a_0\sigma)\, b_{v_1,v_2}(\sigma) - \delta(\mathbf{r}_1-\mathbf{r}_2+a_0\sigma)\} \times[\Phi_\Sigma(v_1)\,\delta F_1(t;\ \mathbf{r}_2,\ \mathbf{v}_2) + \Phi_\Sigma(v_2)\,\delta F_1(t;\ \mathbf{r}_1,\ \mathbf{v}_1)]\, d\sigma\, d\mathbf{r}_2\, d\mathbf{v}_2. \qquad (163)$$

As we noted earlier, the initial condition is given by (144).

In the framework of the low-density approximation, we must retain only the first term and, therefore, $\delta F_1(0; \mathbf{r}_1, \mathbf{v}_1) = \exp(-i\mathbf{k}\mathbf{r}_1)\,\phi(\mathbf{v}_1)\Phi_\Sigma(v_1)$.

From (145), we have $\delta F_1(t; \mathbf{r}_1, \mathbf{v}_1) = \exp(-i\mathbf{k}\mathbf{r}_1)\Phi_k(t; \mathbf{v}_1)\delta\xi$. Thus, setting here

$$\Phi_k(t;\ \mathbf{v}_1) = \Phi_\Sigma(v_1)\, X_k(t;\ \mathbf{v}_1), \qquad (164)$$

we reduce Eq. (163) to the form

$$\frac{\partial X_k(t;\mathbf{v}_1)}{\partial t} = i\mathbf{k}\mathbf{v}_1 X_k(t;\ \mathbf{v}_1) + na^2 L_k(\mathbf{v}_1)\, X_k(t;\ \mathbf{v}_1); \qquad (165)$$

$$X_k(0;\ \mathbf{v}_1) = \phi(\mathbf{v}_1), \qquad (166)$$

where $L_k(\mathbf{v}_1)$ is an operator applied to the function $f(\mathbf{v}_1)$ in accordance with the rule

$$L_k(\mathbf{v}_1)\, f(\mathbf{v}_1) = \int (\mathbf{v}_{1,2}\cdot\sigma)\,\theta(\mathbf{v}_{1,2}\cdot\sigma) \times\{\exp[ia_0(\mathbf{k}\cdot\sigma)]\, f(\mathbf{v}_2^*) - \exp[-ia_0(\mathbf{k}\cdot\sigma)]\, f(\mathbf{v}_2) + f(\mathbf{v}_1^*) - f(\mathbf{v}_1)\} \times \phi_0(v_0)\,\Phi_\Sigma(v_1)\, d\sigma\, d\mathbf{v}_1.$$

To find the solution of (165), we introduce the Laplace transforms

$$\left.\begin{aligned} &\int_0^\infty \exp(-zt)\, X_k(t;\ \mathbf{v})\, dt = \widetilde{X}_k(z;\ v); \\ &\int_0^\infty \exp(-zt)\, \Phi_k(t;\ \mathbf{v})\, dt = \Phi_\Sigma(v)\, \widetilde{X}_k(z;\ v), \end{aligned}\right\} \qquad (167)$$

by means of which Eq. (165) with the initial condition (166) takes the form $(z - i\mathbf{k}\mathbf{v}_1)\widetilde{X}_k(z; v_1) = na_0^2 L_k(v_1)\widetilde{X}_k(z; v_1) + \phi(v_1)$. Thus

$$\widetilde{X}_k(z;\ v_1) = \{z - i(\mathbf{k}\cdot\mathbf{v}_1) - na_0^2 L_k(v_1)\}^{-1}\,\phi(v_1). \qquad (168)$$

Using (146), (167), and (168), we obtain

$$\int_0^\infty \exp(-tz)\, U_k(t, \mathbf{v}_1)\, dt = \{z - i\mathbf{k}\cdot\mathbf{v}_1 - na_0^2 L_k(v_1)\}^{-1}. \qquad (169)$$

Here it must be borne in mind that this operator relation was obtained using the initial condition (144), so that (169) is valid in all cases when $k \neq 0$, while for $k = 0$ it remains valid only if applied to a function $f(v_1)$ satisfying the condition (143).

We recall further that each of the equations (96) and (97) contains only one term with $U_0(t - \tau; 1)$. This operator must be applied to some expression $\bar{\chi}$, which, as a function of v_1, satisfies the condition (143) if (88) is taken into account. We note also that the mentioned terms are proportional to $1/V$. To be specific, we now investigate Eq. (97). If we use a Laplace transform and go to the limit $V \to \infty$, we arrive at the equation

$$(z - na^2 L_S(\mathbf{v}_0))\,\overline{\chi}(z;\ \mathbf{v}_0) = \chi(\mathbf{v}_0) + \frac{n}{(2\pi)^3}\int d\mathbf{k}\int d\mathbf{v}_1 \Phi_\Sigma(v_1)\, \overline{T}_{-k}(v_0,\ v_1)\, W_k(z;\ 1) \times T_k(v_0,\ v_1)\,\overline{\chi}(z;\ \mathbf{v}_0), \quad \chi(\mathbf{v}_0) = \chi(0,\ \mathbf{v}_0), \qquad (170)$$

where

$$L_S(\mathbf{v}_0)\, f(\mathbf{v}_0) = \int (\mathbf{v}_{0,1}\cdot\sigma)\,\theta(\mathbf{v}_{0,1}\cdot\sigma)\,\Phi_\Sigma(v_1) \times\{B_{v_0,v_1}(\sigma) - 1\}\, d\sigma\, d\mathbf{v}_1 f(\mathbf{v}_0); \qquad (171)$$

$$\overline{\chi}(z;\ \mathbf{v}_0) = \int_0^\infty \exp(-zt)\,\chi(t;\ v_0)\, dt, \quad \operatorname{Re} z > 0;$$

$$W_k(z;\ 1) = \int_0^\infty \exp\{-[z + i\mathbf{v}_0\mathbf{k} - na^2 L_S(v_0)]\, t\}\, U_k(t,\ \mathbf{v}_1)\, dt.$$

Since the operators $i\mathbf{v}_0\cdot\mathbf{k} - na^2 L_S(\mathbf{v}_0)$, $i\mathbf{k}\cdot\mathbf{v}_1 + na_0^2 L_k(\mathbf{v}_1)$ act on functions of different arguments, they commute, so that (169) gives

$$W_k(z,\ 1) = \{z + i\mathbf{v}_0\cdot\mathbf{k} - na^2 L_S(v_0) - i\mathbf{v}_1\cdot\mathbf{k} - na^2 L_k(\mathbf{v}_1)\}^{-1}. \qquad (172)$$

In such a case, we can reduce Eq. (170) to the form

$$\{z - na^2 L_S(\mathbf{v}_0) - R(z;\ v_0)\}\,\overline{\chi}(z;\ \mathbf{v}_0) = \lambda(\mathbf{v}_0), \qquad (173)$$

where

$$R(z;\ \mathbf{v}_0) = \frac{n}{(2\pi)^3}\int dk \int d\mathbf{v}_1 \Phi_\Sigma(v_1)\, \overline{T}_{-k}(v_0,\ v_1) \times\{z + i(\mathbf{v}_0 - \mathbf{v}_1)\,\mathbf{k} - na^2 L_S(\mathbf{v}_0) - na_0^2 L_k(\mathbf{v}_1)\}^{-1}\, \overline{T}_k(v_0,\ v_1). \qquad (174)$$

We now consider a function $F(\mathbf{v}_0)$. Repeating the arguments (see Sec. 1) that led us to Eqs. (15)–(17), we find

$$\int F(\mathbf{v}_0)\,\Phi_0(\mathbf{v}_0)\,\chi(t;\ \mathbf{v}_0)\,d\mathbf{v}_0 = \int F(\mathbf{v}_0)\,f(t;\ \mathbf{v}_0)\,d\mathbf{v}_0$$
$$= \frac{1}{V}\int F(\mathbf{v}_0)\,f(t;\ \mathbf{v}_0)\,d\mathbf{r}_0\,d\mathbf{v}_0 = \frac{1}{V}\int F\{\mathbf{v}_0(t)\}\,\mathscr{D}_0(S,\ \Sigma)\,d\Omega_S\,d\Omega_\Sigma$$
$$= \int F\{\mathbf{v}_0(t)\}\,\chi(\mathbf{v}_0)\,\mathscr{D}_{ae}(S,\ \Sigma)\,d\Omega_S\,d\Omega_\Sigma,$$

where

$$\mathscr{D}_{ae}(S,\ \Sigma) = \Phi_0(v_0)\,\mathscr{D}_{eq}(\Sigma)/V;$$
$$\int \mathscr{D}_{ae}(S,\ \Sigma)\,d\Omega_S\,d\Omega_\Sigma = 1.$$

Thus, we see that the expression

$$\langle F(\mathbf{v}_0(t))\,\chi(\mathbf{v}_0)\rangle_{ae} = \int \Phi_0(u_0)\,F(\mathbf{v}_0)\,\chi(t;\ \mathbf{v}_0)\,d\mathbf{v}_0$$

is a two-time correlation mean value taken with respect to the approximately equilibrium probability distribution $D_{ae}(S, \Sigma)$, which differs from the exact equilibrium distribution $D_{eq}(S, \Sigma)$ for the complete system $S+\Sigma$ by neglect of the correlations between the particles of S and Σ.

But it must be emphasized that we are considering here the case when the probability of collision between the S and Σ particles is small, $na^3 \ll 1$, and in such a situation we can, when calculating the principal term, ignore the corresponding correlation effects. Thus, in this approximation we can set

$$\langle F(\mathbf{v}_0(t))\,\chi(\mathbf{v}_0)\rangle_{eq} = \int F(\mathbf{v}_0)\,\Phi_0(v_0)\,\chi(t;\ \mathbf{v}_0)\,d\mathbf{v}_0. \tag{175}$$

We take, for example, $F(\mathbf{v}_0) = \chi(\mathbf{v}_0) = v_{0,x}$; then in the adopted approximation

$$\int_0^\infty \exp(-zt)\,\langle v_{0,x}(t)\,v_{0,x}\rangle\,dt = \int \phi_0(v_0)\,v_{0,x}\chi(z;\ \mathbf{v}_0)\,d\mathbf{v}_0, \tag{176}$$

where $\chi(z;\mathbf{v}_0)$ is determined by Eq. (173), in which $\chi(v_0) = v_{0,x}$.

The validity of the approximations (175) and (176) is discussed in Sec. 4, in which the initial condition for $\mathfrak{D}_t(S, \Sigma)$ is taken in the form

$$\mathscr{D}_0(S,\ \Sigma) = \chi(S)\,\mathscr{D}_{eq}(S,\ \Sigma), \tag{177}$$

and not in the form (2).

Note that Eq. (173) is completely analogous to the equations established by Dorfman and Cohen[11] for a low-density gas, and it can therefore be considered by means of the procedure developed by them. They assumed $M=m$ and $a_0=a$, so that the particle S can be regarded as a probe particle in the large system Σ. However, this circumstance is in no way important for the validity of the assertions and they can be repeated almost verbatim for Eq. (171). For this reason, we shall not discuss this group of questions.

It bears repeating that Eq. (171) follows from (96) and (97), for whose study no assumptions were made about the weakness of the interaction between the particles of the system Σ. Of course, to derive (96) and (97) in a completely definite form, we must know the expression for the operator $U_k(t;1)$. However, such an expression can be found for not only the case when the Boltzmann-Enskog equation for hard spheres is used. It is perfectly possible to use other and more complicated kinetic equations.

We can also use the so-called hydrodynamic approximation, which does not depend on the assumption that the interaction in Σ is relatively weak, in order to find the explicit expression for the operator $U_k(t;1)$ in the region

$$k \ll l_\Sigma^{-1},\quad t \gg t_\Sigma, \tag{178}$$

where l_Σ and t_Σ are the mean free path and mean free time for particles in the system Σ. It is easy to show that it is precisely this region which is important for determining the behavior of correlation mean values of the type (175) at large times.

SECTION 4

We continue our investigations into the interaction of the particle S with the large system Σ under the same conditions as in Secs. 1 and 2, except that now we choose not the requirement (2), but an initial expression for $\mathfrak{D}_t(S, \Sigma)$ in the form

$$\mathscr{D}_0(S_0\Sigma) = h(S)\,\mathscr{D}_{eq}(S,\ \Sigma),$$

where $\mathfrak{D}_{eq}(S, \Sigma)$ is the distribution function corresponding to complete statistical equilibrium of the total system.

In the considered situation

$$\mathscr{D}_{eq}(S,\ \Sigma) = W(\mathbf{r}_0,\ \mathbf{r}_1,\ \ldots,\ \mathbf{r}_N)\,\Phi_0(v_0)\prod_{(1\leq j\leq N)}\Phi_\Sigma(v_j) \tag{179}$$

with the normalization condition

$$\overline{(\mathscr{D}_{eq}(S,\ \Sigma))}_{S+\Sigma} = 1.$$

Thus,

$$\int_V \cdots \int_V W(\mathbf{r}_0,\ \mathbf{r}_1,\ \ldots,\ \mathbf{r}_N)\,d\mathbf{r}_0\,d\mathbf{r}_1\ldots d\mathbf{r}_N = 1.$$

Since W is translationally invariant, this last equation gives

$$\int_V \cdots \int W(\mathbf{r}_0,\ \mathbf{r}_1,\ \ldots,\ \mathbf{r}_N)\,d\mathbf{r}_1\ldots d\mathbf{r}_N = 1/V. \tag{180}$$

Thus,

$$\overline{(\mathscr{D}_0(S,\ \Sigma))}_\Sigma = h(S)\,\overline{(\mathscr{D}_{eq}(S,\ \Sigma))}_\Sigma = h(S)\frac{1}{V}\Phi_0(v_0). \tag{181}$$

Note that in the case considered earlier, when the initial value is given by (2),

$$\overline{(\mathscr{D}_0(S,\Sigma))}_\Sigma = f(S) = \chi(S)\,\Phi_0(v_0). \tag{182}$$

Therefore, if we wish to retain this originally adopted normalization, we must set $h(S) = V\chi(S)$ in (181). In such a case, the initial value

$$\mathscr{D}_0(S,\ \Sigma) = V\chi(S)\,\mathscr{D}_{eq}(S,\ \Sigma) \tag{183}$$

will satisfy the same relation (182) as in the case (2).

We determine the time evolution of $\mathfrak{D}_t(S, \Sigma)$ by means of the Liouville equation (18):

$$\partial \mathcal{D}_t/\partial t = (Л^{(0)}_S + Л_\Sigma + Л_{int})\, \mathcal{D}_t,$$

using the initial condition in the form (183). We now introduce the function $\chi_t(S)$:

$$\overline{(\mathcal{D}_t)}_\Sigma = \chi_t(S)\,\Phi_0(v_0) = f_t(S), \qquad (184)$$

and we note that it can be used to calculate equilibrium correlation mean values of the type $\langle F(\Omega_S(t))\chi(\Omega_S)\rangle_{eq}$. Indeed, it is easy to see that

$$V\langle F(\Omega_S(t))\,\chi(\Omega_S)\rangle_{eq} = \overline{(F(\Omega_S(t))\,V\chi(S)\,\mathcal{D}_{eq}(S,\Sigma))}_{S+\Sigma} = \overline{(F(\Omega_S(t))\,\mathcal{D}_0(S,\Sigma))}_{S+\Sigma}$$
$$= \overline{(F(\Omega_S)\,\mathcal{D}_t(S,\Sigma))}_{S+\Sigma} = \overline{(F(S)\,(\mathcal{D}_t(S,\Sigma))_\Sigma)}_S$$

and therefore

$$V\langle F(\Omega_S(t))\,\chi(\Omega_S)\rangle_{eq} = \overline{(F(S)\,f_t(S))}_S$$
$$= \int F(\mathbf{r}_0,\mathbf{v}_0)\,\chi(t;\mathbf{r}_0,\mathbf{v}_0)\,\Phi_0(v_0)\,d\mathbf{r}_0\,d\mathbf{v}_0. \qquad (185)$$

Noting that

$$\langle f(S)\rangle = \overline{(f(S)\,\mathcal{D}_{eq}(S,\Sigma))}_{S+\Sigma} = \frac{1}{V}\overline{(f(S)\,\Phi_0(v_0))}_S,$$

we can also write

$$\frac{\langle F(\Omega_S(t))\,\chi(\Omega_S)\rangle_{eq}}{\{\langle |F(\Omega_S)|^2\rangle_{eq}\langle|\chi(\Omega_S)|^2\rangle_{eq}\}^{1/2}}$$
$$= \frac{\int F(\mathbf{r}_0,\mathbf{v}_0)\,\chi(t;\mathbf{r}_0,\mathbf{v}_0)\,\Phi_0(v_0)\,d\mathbf{r}_0\,d\mathbf{v}_0}{\{\int|F(\mathbf{r}_0,\mathbf{v}_0)|^2\,\Phi_0(v_0)\,d\mathbf{r}_0\,d\mathbf{v}_0\int|\chi(\mathbf{r}_0,\mathbf{v}_0)|^2\,\Phi_0(v_0)\,d\mathbf{r}_0\,d\mathbf{v}_0\}^{1/2}}; \qquad (186)$$

the form of this expression is obviously independent of the normalization $\chi(S)$.

We now exploit the method developed in Sec. 1 to obtain an approximate equation for $\chi_t(S)$. We denote

$$\mathcal{D}_t - V\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma) = \Delta_t. \qquad (187)$$

With allowance for (181), (183), and (184),

$$\overline{(\Delta_t)}_\Sigma = 0, \quad \Delta_0 = 0. \qquad (188)$$

Integrating (18) with respect to Ω_Σ and using the identity

$$\overline{(Л_\Sigma F(S,\Sigma))}_\Sigma \equiv 0, \qquad (189)$$

we obtain

$$\Phi_0(v_0)\,\partial\chi_t(S)/\partial t = Л^0_S\chi_t(S)\,\Phi_0(v_0)$$
$$+ V\overline{(Л_{int}\chi_t(S)\,D_{eq}(S,\Sigma)}_\Sigma + \overline{(Л_{int}\Delta_t)}_\Sigma,$$

which reduces to the form

$$\frac{\partial\chi_t(S)}{\partial t} = Л^0_S\chi_t(S) + V\frac{1}{\Phi_0(v_0)}$$
$$\times\overline{(Л_{int}\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma))}_\Sigma + \frac{1}{\Phi_0(v_0)}\overline{(Л_{int}\Delta_t)}_\Sigma, \qquad (190)$$

since

$$Л^0_S\chi_t(S)\,\Phi_0(v_0) = \Phi_0(v_0)\,Л^0_S\chi_t(S).$$

We now introduce the operator $Л^{(1)}_S$, which acts only on functions $f(S)$ of the phase Ω_S:

$$Л^{(1)}_S f(S) = V\,\overline{{}_{int}(Л f(S)\,\Phi_0^{-1}(v_0)\,\mathcal{D}_{eq}(S,\Sigma))}_\Sigma. \qquad (191)$$

Then (190) is reduced to the form

$$\frac{\partial\chi_t(S)}{dt} = Л^0_S\chi_t(S) + \frac{1}{\Phi_0(v_0)}Л^{(1)}_S\chi_t(S)\,\Phi_0(v_0) + \frac{1}{\Phi_0(v_0)}\overline{(Л_{int}\Delta t)}_\Sigma. \qquad (192)$$

From (18), (187), and (188), we obtain

$$\frac{\partial\Delta_t}{\partial t} = (Л^0_S + Л_\Sigma + Л_{int})\,\Delta_t + V(Л^0_S + Л_\Sigma$$
$$+ Л_{int})\,\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma) - V\Big\{Л^0_S\chi_t(S)$$
$$+ \frac{1}{\Phi_0(v_0)}Л^{(1)}_S\chi_t(S)\,\Phi_0(v_0)$$
$$+ \frac{1}{\Phi_0(v_0)}\overline{(Л_{int}\Delta_t)}_\Sigma\Big\}\,\mathcal{D}_{eq}(S,\Sigma), \quad \Delta_0 = 0. \qquad (193)$$

It is easy to see that

$$(Л^0_S + Л_\Sigma)\,\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma) = \{Л^0_S\chi_t(S)\}\,\mathcal{D}_{eq}(S,\Sigma)$$
$$+ \chi_t(S)\,(Л^0_S + Л_\Sigma)\,\mathcal{D}_{eq}(S,\Sigma).$$

However,

$$(Л^0_S + Л_\Sigma + Л_{int})\,\mathcal{D}_{eq}(S,\Sigma) = 0$$

and therefore

$$(Л^0_S + Л_\Sigma)\,\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma)$$
$$= \{Л^0_S\chi_t(S)\}\,\mathcal{D}_{eq}(S,\Sigma) - \chi_t(S)\,Л_{int}\mathcal{D}_{eq}(S,\Sigma).$$

It now follows from (193) that

$$\frac{\partial\Delta_S}{\partial t} = (Л^0_S + Л_\Sigma + Л_{int})\,\Delta_t - \frac{V}{\Phi_0(v_0)}\{\overline{(Л_{int}\Delta_t)}_\Sigma\}\,\mathcal{D}_{eq}(S,\Sigma)$$
$$+ V\{Л_{int}\chi_t(S)\,\mathcal{D}_{eq}(S,\Sigma) - \chi_t(S)\,Л_{int}\mathcal{D}_{eq}(S,\Sigma)\}$$
$$- \frac{V}{\Phi_0(v_0)}\{Л^{(1)}_S\chi_t(S)\,\Phi_0(v_0)\}\,\mathcal{D}_{eq}(S,\Sigma),$$

or

$$\frac{\partial\Delta_t}{\partial t} = (Л_S + Л_\Sigma + \Gamma)\,\Delta_t - \frac{V}{\Phi_0(v_0)}\{\overline{(\Gamma\Delta_t)}_\Sigma\}\,\mathcal{D}_{eq}(S,\Sigma)$$
$$+ V(Л_{int}\chi_t(S) - \chi_t(S)\,Л_{int})\,\mathcal{D}_{eq}(S,\Sigma)$$
$$- V\mathcal{D}_{eq}(S,\Sigma)\Big\{\frac{1}{\Phi_0(v_0)}Л^{(1)}_S\chi_t(S)\,\Phi_0(v_0)\Big\}, \quad \Delta_0 = 0, \qquad (194)$$

where

$$\Gamma = Л_{int} - Л^{(1)}_S; \qquad (195)$$

$$Л_S = Л^{(0)}_S + Л^{(1)}_S. \qquad (196)$$

We consider the case when

$$Л_{int} = \sum_{(1\leqslant j\leqslant N)} Л(0,j). \qquad (197)$$

Here, $Л(0,j)$ is the Liouville operator corresponding to the interaction between S and particle j of Σ. For example,

$$Л^{(coll)}_{int} = \sum_{(1\leqslant j\leqslant N)} \overline{T}(0,j).$$

We consider the expression

$$V\overline{(Л(0,j)\,\mathcal{D}_{eq}(S,\Sigma)\,f(S))}_\Sigma. \qquad (198)$$

Note that (179) gives

$$V\overline{(Л(0,j)\,\mathcal{D}_{eq}(S,\Sigma)\,f(S))}_\Sigma$$
$$= V\int Л(0,j)\,F_{S,\Sigma}(0,j)\,f(S)\,\Phi_0(v_0)\,\Phi_\Sigma(v_j)\,d\mathbf{r}_j\,d\mathbf{v}_j, \qquad (199)$$

where

$$F_{S,\Sigma}(0,j) = \int_V \ldots \int_V \delta(\mathbf{r}_0 - \mathbf{r}')\,\delta(\mathbf{r}_j - \mathbf{r}_j')$$
$$\times W(\mathbf{r}_0',\mathbf{r}_1',\ldots,\mathbf{r}_N')\,d\mathbf{r}_0'\,d\mathbf{r}_1'\ldots b\,d\mathbf{r}_N'.$$

Taking into account the symmetry of the functions

$$W(\mathbf{r}_0',\mathbf{r}_1',\ldots,\mathbf{r}_N')$$

with respect to the variables $\mathbf{r}_1',\ldots,\mathbf{r}_N'$, we see that

$$F_{S,\Sigma}(0, j)=\int\limits_V \ldots \int\limits_V \delta(\mathbf{r}_0-\mathbf{r}_0')\,\delta(\mathbf{r}_j-\mathbf{r}_1')$$
$$\times W(\mathbf{r}_0', \mathbf{r}_1', \ldots, \mathbf{r}_N')\,d\mathbf{r}_0'\,d\mathbf{r}_1' \ldots d\mathbf{r}_N'$$
$$=\int\limits_V \ldots \int\limits_V W(\mathbf{r}_0, \mathbf{r}_j, \mathbf{r}_2', \ldots, \mathbf{r}_N')\,d\mathbf{r}_2' \ldots d\mathbf{r}_N'. \qquad (200)$$

We introduce the reduced spatial correlation function with the usual normalization condition

$$W(\mathbf{r}_0, \mathbf{r}_1)=V^2\int\limits_V \ldots \int\limits_V W(\mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N)\,d\mathbf{r}_2 \ldots d\mathbf{r}_N. \qquad (201)$$

It follows from the translational invariance and isotropy of space that this function is radially symmetric: $w(\mathbf{r}_0, \mathbf{r}_1)=w(|\mathbf{r}_0-\mathbf{r}_1|)$.

The limiting expression (for $V\to\infty$) of the function $w(r)$ has the property of correlation weakening: $w(r)\to 1$, $r\to\infty$. If there is no interaction at all between S and Σ, this function must be equal to 1.

In the considered case of weak interaction, $w(r)$ is close to unity almost everywhere except in the range of variation in which large repulsive forces act.

Returning to (200) and (201), we obtain with allowance for (199)

$$V\,\overline{(\mathfrak{I}(0, j)\,\mathscr{D}_{eq}(S, \Sigma)\,f(S))_\Sigma}$$
$$=\frac{1}{V}\int \widetilde{\mathfrak{I}}(0, j)\,f(S)\,\Phi_0(v_0)\,\Phi_\Sigma(v_j)\,d\mathbf{r}_j\,d\mathbf{v}_j$$
$$=V\left(\widetilde{\mathfrak{I}}(0, j)\,f(S)\,\frac{\Phi_0(v)}{V}\,\mathscr{D}_{eq}(\Sigma)\right)_\Sigma, \qquad (202)$$

where

$$\widetilde{\mathfrak{I}}(0, j)=\mathfrak{I}(0, j)\,w(|\mathbf{r}_0-\mathbf{r}_j|) \qquad (203)$$

and therefore

$$V\,\overline{(\mathfrak{I}_{int}f(S)\,\mathscr{D}_{eq}(S, \Sigma))_\Sigma}$$
$$=V\left(\widetilde{\mathfrak{I}}_{int}f(S)\,\frac{\Phi_0(v_0)}{V}\,\mathscr{D}_{eq}(\Sigma)\right)_\Sigma. \qquad (204)$$

Here

$$\mathfrak{I}_{int}=\sum_{(1\leqslant j\leqslant N)}\widetilde{\mathfrak{I}}(0, j). \qquad (205)$$

Thus, we can formulate a rule: If $\mathfrak{D}_{eq}(S, \Sigma)$ is replaced by its approximation in which the correlation between S and Σ is completely ignored,

$$\mathscr{D}_{eq}(S, \Sigma)\to\frac{\Phi_0(v_0)}{V}\,\mathscr{D}_{eq}(\Sigma), \qquad (206)$$

then the renormalization of the interaction, i.e., the substitution

$$\mathfrak{I}_{int}\to\widetilde{\mathfrak{I}}_{int}, \qquad (207)$$

makes it possible to take into account the effect of the correlation ignored in the operation (206).

This rule is at least valid when it is applied to the construction of the operator $\mathfrak{I}_S^{(1)}$. We can see from (202) that all these expressions for $j=1, \ldots, N$ are identical and, therefore, taking into account the definition of $\mathfrak{I}_s^{(1)}$, we obtain

$$\mathfrak{I}_S^{(1)}f(S)=\frac{n}{\Phi_0(v_0)}\int \widetilde{\mathfrak{I}}(0, 1)\,\Phi_\Sigma(v_1)\,f(S)\,d\mathbf{r}_1\,d\mathbf{v}_1. \qquad (208)$$

We now turn to the calculation of the correction term on the right-hand side of (190):

$$\frac{1}{\Phi_0(v_0)}\,\overline{(\mathfrak{I}_{int}\,\Delta_t)_\Sigma}. \qquad (209)$$

To this end, we return to (193) and (194). To obtain from (194) an approximate expression Δ_t that could be used in (209), we ignore in (194) the second-order terms, thus regarding Δ_t as a quantity of first order.

In such an approximation, we first omit in (194) the terms containing $\Gamma_{\Delta t}$. In what follows, for $\mathfrak{D}_{eq}(S, \Sigma)$ we use the zeroth approximation, namely, the expression (206). In order to compensate in some measure the result of these procedures, we can attempt to use the rule just formulated and make the substitution

$$\mathfrak{I}_{int}\to\widetilde{\mathfrak{I}}_{int} \qquad (210)$$

in Eqs. (194) and (209). We then obtain the approximate equations

$$\frac{\partial\Delta_t^{(a)}}{\partial t}=(\mathfrak{I}_S+\mathfrak{I}_\Sigma)\,\Delta_t^{(a)}+(\widetilde{\mathfrak{I}}_{int}\chi_t(S)$$
$$-\chi_t(S)\,\widetilde{\mathfrak{I}}_{int})\,\Phi_0(v_0)\,\mathscr{D}_{eq}(\Sigma)$$
$$-\mathscr{D}_{eq}(\Sigma)\,\{\mathfrak{I}_S^{(1)}\chi_t(S)\,\Phi_0(v_0)\},\quad \Delta_t^{(a)}=0 \text{ for } t=0, \qquad (211)$$

and from (192), since $\chi_t(S)\Phi_0(v_0)=f_t(S)$, we find

$$\frac{\partial f_t(S)}{\partial t}=\mathfrak{I}_S f_t(S)+\overline{(\mathfrak{I}_{int}\Delta_t^{(a)})_\Sigma}. \qquad (212)$$

It should be emphasized that the procedure carried out to take into account the correlation between S and Σ particles may not be formally self-consistent.

Indeed, we have retained here only some correction terms, whereas others, which formally have the same order, were ignored. Nevertheless, the procedure can be justified by means of the same intuitive physical arguments as were employed by Enskog in his theory of dense gases whose molecules are assumed to be impenetrable spheres. Thus, the correlation function becomes negligibly small in the region in which large repulsive forces are effective. Its introduction through the substitution (210) ensures that the probability of finding $|\mathbf{r}_0-\mathbf{r}_j|$ within this region is small.

Returning to (211), we readily obtain

$$\Delta_t^{(a)}=\int\limits_0^t \exp\left[(\mathfrak{I}_S+\mathfrak{I}_\Sigma)(t-\tau)\right]((\widetilde{\mathfrak{I}}_{int}\chi_\tau(S)$$
$$-\chi_\tau(S)\,\widetilde{\mathfrak{I}}_{int})\,\Phi_0(v_0)\,\mathscr{D}_{eq}(\Sigma)-\mathscr{D}_{eq}(\Sigma)\,\{\mathfrak{I}_S^{(1)}\chi_\tau(S)\,\Phi_0(v_0)\})\,d\tau. \qquad (213)$$

On the other hand

$$\{\mathfrak{I}_S^{(1)}\chi_\tau(S)\,\Phi_0(v_0)\}=\overline{(\widetilde{\mathfrak{I}}_{int}\chi_\tau(S)\,\Phi_0(v_0)\,\mathscr{D}_{eq}(\Sigma))_\Sigma};$$
$$\overline{(\widetilde{\mathfrak{I}}_{int}\Phi_0(v)\,\mathscr{D}_{eq}(\Sigma))_\Sigma}=V\,\overline{(\mathfrak{I}_{int}\mathscr{D}_{eq}(S, \Sigma))_\Sigma}$$
$$=-V\,\overline{([\mathfrak{I}_S^0+\mathfrak{I}_\Sigma]\,\mathscr{D}_{eq}(S, \Sigma))_\Sigma}=-\mathfrak{I}_S^0\Phi_0(v_0)$$
$$-V\,\overline{(\mathfrak{I}_\Sigma\mathscr{D}_{eq}(S,\Sigma))_\Sigma}=0.$$

Thus, (213) can be written in the form

$$\Delta_t^{(a)}=\int\limits_0^t \exp\left[(\mathfrak{I}_S+\mathfrak{I}_\Sigma)(t-\tau)\right]$$
$$\times\{(\widetilde{\mathfrak{I}}_{int}\chi_\tau(S)-\chi_\tau\widetilde{\mathfrak{I}}_{int})\,\Phi_0(v_0)\,\mathscr{D}_{eq}(\Sigma)$$
$$-\mathscr{D}_{eq}(\Sigma)\,\overline{((\widetilde{\mathfrak{I}}_{int}\chi_\tau(S)-\chi_\tau(S)\,\widetilde{\mathfrak{I}}_{int})\,\Phi_0(v_0)\,\mathscr{D}_{eq}(\Sigma))_\Sigma}\}\,d\tau. \qquad (214)$$

Since this function is symmetric with respect to the particles 1, 2, . . . , N of Σ, we obtain from (197) and (212)

$$\frac{\partial f_t(S)}{\partial t} = (\text{Л}_S^0 + \text{Л}_S^{(1)}) f_t(S) + N\,\overline{(\tilde{\text{Л}}(0,\,1)\,\Delta_t^{(a)})}_\Sigma. \tag{215}$$

Substitution of (214) in (215) leads to an approximate equation for $\chi_t(S)$ in closed form.

We now turn to the detailed derivation and investigation of this equation for the interaction of hard spheres given by Eq. (10). We note first that $\tilde{\text{Л}}(0, j) = w(a)\overline{T}(0, j)$ in accordance with (11), from which it follows that

$$\text{Л}_S^{(1)} = w(a)\, na^2 \mathfrak{C}_S, \tag{216}$$

where the operator $\mathfrak{C}_S$ is defined below in (223). We note further that

$$\tilde{\text{Л}}_{\rm int} = w(a)\,\text{Л}_{\rm int} = w(a) \sum_{(1\leq j\leq N)} T(0,\, j), \tag{217}$$

and we consider the expression

$$\overline{T}(0,\, j)\,\chi(S)\,\Phi_0(v_0)\,\mathscr{D}_{\rm eq}(\Sigma) - \chi(S)\,\overline{T}(0,\, j)\,\Phi_0(v_0)\,\mathscr{D}_{\rm eq}(\Sigma)$$
$$= a^2 \int \theta(\mathbf{v}_{0,\,j}\cdot\sigma)\,\mathbf{v}_{0,\,j}\cdot\sigma\,\{\delta(\mathbf{r}_0 - \mathbf{r}_j - a\sigma)\,B_{v_0,\,v_j}(\sigma)\,\chi(S)\,\Phi_0(v_0)$$
$$\times \mathscr{D}_{\rm eq}(\Sigma) - \delta(\mathbf{r}_0 - \mathbf{r}_j - a\sigma)\,\chi(S)\,B_{v_0,\,v_j}(\sigma)\,\Phi_0(v_0)\,\mathscr{D}_{\rm eq}(\Sigma)\}\,d\sigma.$$

Since

$$B_{v_0,\,v_j}(\sigma)\,\Phi_0(v_0)\,\Phi_\Sigma(v_j) \equiv \Phi_0(v_0)\,\Phi_\Sigma(v_j),$$

we have as a consequence

$$B_{v_0,\,v_j}\Phi_0(v_0)\,\mathscr{D}_{\rm eq}(\Sigma) = \Phi_0(v_0)\,\mathscr{D}_{\rm eq}(\Sigma).$$

Thus

$$\overline{T}(0,\, j)\,\chi(S)\,\Phi_0(v)\,\mathscr{D}_{\rm eq}(\Sigma) - \chi(S)\,\overline{T}(0,\, j)\,\Phi(v_0)$$
$$\times \mathscr{D}_{\rm eq}(\Sigma) = T(0,\, j)\,\chi(S)\,\Phi_0(v)\,\mathscr{D}_{\rm eq}(\Sigma) = T(0,\, j)\,f(S)\,\mathscr{D}_{\rm eq}(\Sigma), \tag{218}$$

where the operator $T(0, j)$ is given by

$$T(0,\, 1) = a^2 \int \theta(\mathbf{v}_{0,\,1}\cdot\sigma)\,\mathbf{v}_{0,\,1}\cdot\sigma\,\delta(\mathbf{r}_0 - \mathbf{r}_1 - a\sigma)\,\{B_{v_0,\,v_1}(\sigma) - 1\}\,d\sigma. \tag{219}$$

Taking into account (214), (217), and (218), we can rewrite (215) in the form

$$\frac{\partial f_t(S)}{\partial t} = \left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,\mathfrak{C}_S\right) f_t(S)$$
$$+ w^2(a) \int_0^t K(t-\tau)\,f_\tau(S)\,d\tau, \tag{220}$$

where $K(t)$ is an operator which acts on $f(s)$ and is defined by the relation

$$K(t)$$
$$= \overline{N(\overline{T}(0,1)\exp[(\text{Л}_S + \text{Л}_\Sigma)t] \sum_{(1\leq j\leq N)} [T(0,\, j) - \overline{(T(0,\, j)\,\mathscr{D}_{\rm eq}(\Sigma))}_\Sigma]\,\mathscr{D}_{\rm eq}(\Sigma))}_\Sigma;$$
$$\text{Л}_S = -\mathbf{v}_0\,\partial/\partial\mathbf{r}_0 + \text{Л}_S^{(1)}; \tag{221}$$

$$\left.\begin{aligned} \text{Л}_S^{(1)} &= nw(a) \int \overline{T}(0,\, 1)\,\Phi_\Sigma(v_1)\,d\mathbf{r}_1\,d\mathbf{v}_1 = na^2 w(a)\,\mathfrak{C}_S, \\ \mathfrak{C}_S &= \int \theta(\mathbf{v}_{0,\,1}\cdot\sigma)\,(\mathbf{v}_{0,\,1}\cdot\sigma)\,\{B_{v_0,\,v_j}(\sigma) - 1\}\,\Phi_\Sigma(v_1)\,dv_1. \end{aligned}\right\} \tag{222}$$

It is desirable to note the connection between the operators $\mathfrak{C}_S$ (222) and L_S, which acts on the function $\chi(S)$ in accordance with (77) and in operator form can be written as

$$L_S = \int \theta(\mathbf{v}_{0,\,1}\cdot\sigma)\,(\mathbf{v}_{0,\,1}\cdot\sigma)\,\Phi_\Sigma(v_1)\,\{B_{v_0,\,v_1}(\sigma) - 1\}\,dv_1. \tag{223}$$

Since[5] $\mathfrak{C}_S\Phi_0(v_0)h(S) = \Phi_0(v_0)L_S h(S)$,

$$\left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,\mathfrak{C}_S\right) \Phi_0(v_0)\,h(S)$$
$$= \Phi_0(v_0) \left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,L_S\right) h(S),$$

which leads to the identity

$$\exp\left[t\left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,\mathfrak{C}_S\right)\right] \Phi_0(v_0)\,h(S)$$
$$\equiv \Phi_0(v_0) \exp\left[t\left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,L_S\right)\right] h(S). \tag{224}$$

Returning to (220)–(221), we see that this equation is virtually the same as the one determined above by Eqs. (78) and (79) with the only difference that, discounting the Enskog factor $w(a)$, the operator $T(0, 1)$ has appeared instead of the operator $\overline{T}(0, 1)$, which occurs in (79), on the right-hand side of (221). Therefore, we can use the same procedure as we did in Secs. 2 and 3.

In this case, we obtain

$$\frac{\partial \chi_t(\mathbf{r}_0,\,\mathbf{v}_0)}{\partial t} = \left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,L_S\right) \chi_t(\mathbf{r}_0,\,\mathbf{v}_0)$$
$$+ w^2(a) \int_0^t Q(t-\tau)\,\chi_\tau(\mathbf{r}_0,\,\mathbf{v}_0)\,d\tau, \tag{225}$$

where

$$Q(t) = \frac{n}{(2\pi)^3} \int d\mathbf{k} \int d\mathbf{v}_1 \Phi_\Sigma(\mathbf{v}_1)\,\overline{T}_{-k}(v_0,\, v_1)$$
$$\times \exp(-i\mathbf{k}\mathbf{r}_0) \exp\left[\left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,L_S\right) t\right]$$
$$\times \exp(i\mathbf{k}\mathbf{r}_0)\,U_k(t,\, 1)\,T_k(v_0,\, v_1); \tag{226}$$

$$\left.\begin{aligned} T_k(v_0,\, v_1) &= a^2 \int (\mathbf{v}_{0,\,1}\cdot\sigma)\,\theta(\mathbf{v}_{0,\,1}\cdot\sigma) \\ &\quad\times \exp(-ia\mathbf{k}\cdot\sigma)\,(B_{v_0,\,v_1}(\sigma) - 1)\,d\sigma; \\ T_{-k}(v_0,\, v_1) &= a^2 \int (\mathbf{v}_{0,\,1}\cdot\sigma)\,\theta(\mathbf{v}_{0,\,1}\cdot\sigma)\,(\exp(ia\mathbf{k}\sigma)\,B_{v_0,\,v_1}(\sigma) \\ &\quad - \exp(-ia\mathbf{k}\sigma))\,d\sigma. \end{aligned}\right| \tag{227}$$

The operator $U_k(t; 1)$ can be determined in the same way as was shown in Sec. 3, namely, by means of an infinitesimally small variation of the reduced distribution functions for the system Σ of the type (142). In such a case

$$\delta F_1(t;\, 1) = \exp(-i\mathbf{k}\mathbf{r}_1)\,\Phi_k(t,\, v_1)\,\delta\xi$$

and

$$\Phi_k(t,\, \mathbf{v}_1) = \Phi_\Sigma(v_1)\,U_k(t;\, 1)\,\phi(v_1)$$
$$= \Phi_\Sigma(v_1) \int U_k(t;\, \mathbf{v}_1,\, \mathbf{v}_1')\,\phi(\mathbf{v}_1')\,d\mathbf{v}_1'.$$

It is interesting that if we introduce the different operator $U_k'(t; 1)$ by setting

$$U_k'(t;\, \mathbf{v}_1,\, \mathbf{v}_1')\,\Phi_\Sigma(v_1') = \Phi_\Sigma(v_1)\,U_k(t;\, \mathbf{v}_1,\, \mathbf{v}_1), \tag{228}$$

then (220) could be written in the form

$$\frac{\partial f_t(\mathbf{r}_0,\,\mathbf{v}_0)}{\partial t} = \left(-\mathbf{v}_0 \frac{\partial}{\partial \mathbf{r}_0} + na^2 w(a)\,\mathfrak{C}_S\right)$$
$$\times f_t(\mathbf{r}_0,\,\mathbf{v}_0)\,w^2(a) \int_0^t Q'(t-\tau)\,f_\tau(\mathbf{r}_0,\,\mathbf{v}_0)\,d\tau, \tag{229}$$

where

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[5] This equation follows from the relation $B_{v_0,\,v_1}(\sigma)\,\Phi_0(v_0)\,\Phi_\Sigma(v_1)\,h(S) = \Phi_0(v_0)\,\Phi_\Sigma(v_1)\,B_{v_0,\,v_1}(\sigma)\,h(S)$.

$$Q'(t)=\frac{n}{(2\pi)^3}\int dk\int d\mathbf{v}_1\overline{T}_{-k}(v_0,v_1)\exp(-i\mathbf{k}\mathbf{r}_0)$$
$$\times\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2w(a)\mathfrak{C}_S\right)t\right]\exp(i\mathbf{k}\mathbf{r}_0)U'_k(t;1)$$
$$\times\Phi_\Sigma(v_1)T_k(v_0,v_1). \tag{230}$$

The two representations (225) and (229) are equivalent by virtue of (224).

It is also easy to see that the operators $\exp(-i\mathbf{k}\mathbf{r}_0)$ $\times\exp[(-\mathbf{v}_0\,\partial/\partial\mathbf{r}_0+na^2w(a)L_S)t]\exp(i\mathbf{k}\mathbf{r}_0)$ and $U_k(t;1)$ commute, since they act on functions of different variables, namely on $h(S)$ and $F(\mathbf{v}_1)$.

We now consider the different identity

$$\exp(-i\mathbf{k}\mathbf{r}_0)\exp\left[\left(-\mathbf{v}_0\frac{\partial}{\partial\mathbf{r}_0}+na^2w(a)L_S\right)t\right]\exp[i(\mathbf{k}+\mathbf{l})\mathbf{r}_0]$$
$$=\exp(i\mathbf{l}\mathbf{r}_0)\exp[(-i\mathbf{v}_0(\mathbf{k}+\mathbf{l})+na^2w(a)L_S)t],$$

from which it follows that (225) has a solution of the form

$$\chi_t(\mathbf{r}_0,\mathbf{v}_0)=\exp(i\mathbf{l}\mathbf{r}_0)\chi_l(t,\mathbf{v}_0), \tag{231}$$

where χ_l satisfies the equation

$$\frac{\partial\chi_l(t,v_0)}{\partial t}=(-i\mathbf{l}\mathbf{v}_0+na^2w(a)L_S)\chi_l(t,\mathbf{v}_0)$$
$$+w^2(a)\int_0^t Q_l(t-\tau)\chi_l(\tau,\mathbf{v}_0)\,d\tau, \tag{232}$$

with

$$Q_l(t)=\frac{n}{(2\pi)^3}\int d\mathbf{k}\int d\mathbf{v}_1\Phi_\Sigma(v_1)\overline{T}_{-k}(v_0,v_1)$$
$$\times U_k(t;1)\exp[(-i\mathbf{v}_0(\mathbf{k}+\mathbf{l})+na^2w(a)L_S)t]T_k(v_0,v_1). \tag{233}$$

In particular, for $l=0$ we have the equation

$$\frac{\partial\chi(t,\mathbf{v}_0)}{dt}=na^2w(a)L_S\chi(t,\mathbf{v}_0)+w^2(a)\int_0^t Q_0(t-\tau)\chi(\tau,\mathbf{v}_0)\,d\tau. \tag{234}$$

where

$$Q_0(t)=\frac{n}{(2\pi)^3}\int d\mathbf{k}\int d\mathbf{v}_1\Phi_\Sigma(v_1)\overline{T}_{-k}(v_0,v_1)$$
$$\times U_k(t;1)\exp[(-i\mathbf{v}_0\mathbf{k}+na^2w(a)L_S)t]T_k(v_0,v_1). \tag{235}$$

For an arbitrary initial expression $\chi_0(\mathbf{r}_0,\mathbf{v}_0)$ we can use the Fourier representation and, using (232), consider each Fourier component separately.

We now obtain the hydrodynamic approximation for $U_k(t;1)$. We proceed from the local-equilibrium distribution

$$F_1^{(\mathrm{hyd})}(t,\mathbf{r},\mathbf{v})=\frac{\rho}{n}\left(\frac{M}{2\pi\theta}\right)^{3/2}\exp\left[-\frac{M(\mathbf{v}-\mathbf{u})^2}{2\theta}\right];$$
$$\theta=k_BT, \tag{236}$$

where

$$\rho=\rho(t,\mathbf{r}),\quad T=T(t,\mathbf{r}),\quad \mathbf{u}=\mathbf{u}(t,\mathbf{r})$$

are the local particle density, the temperature, and the velocity vector. These must be very slowly varying functions over distances of the order of the mean free path l_Σ and over time intervals of the order of the mean free time t_Σ, which guarantees that the correction term on the right-hand side of (236) is small.

All that is here required is that we consider a situation in which the local equilibrium differs only infinitesimally from the completely equilibrium state:

$$\left.\begin{array}{l}\rho(t,\mathbf{r})=n+\delta\rho(t,\mathbf{r});\quad T(t,\mathbf{r})=T+\delta T(t,\mathbf{r});\\ \mathbf{u}(t,\mathbf{r})=\delta\mathbf{u}(t,\mathbf{r});\quad n,\ T=\mathrm{const};\end{array}\right\} \tag{237}$$

where $\delta\rho$, δT, and $\delta\mathbf{u}$ are infinitesimally small. In such a case, the principal term $\delta F_1^{(\mathrm{hyd})}$ obtained by substituting (237) in (236) can be written in the form

$$\delta F_1^{(\mathrm{hyd})}(t,\mathbf{r},\mathbf{v})=\Phi_\Sigma(v)\left\{\frac{\delta\rho(t,\mathbf{r})}{n}+\frac{Mv^2-3\theta}{2\theta}\frac{\delta T(t,\mathbf{r})}{T}+\frac{M(\mathbf{v}\delta\mathbf{u}(t,\mathbf{r}))}{\theta}\right\}, \tag{238}$$

where $\delta\rho$, δT, and $\delta\mathbf{u}$ satisfy the well-known linearized Navier–Stokes equations. In general, the correction terms to the right-hand side of (238) are proportional to the gradients $l_\Sigma\partial/\partial\mathbf{r}$, and $t_\Sigma\partial/\partial t$ of the variations $\delta\rho$, δT, and $\delta\mathbf{u}$. Because the equations are linear, these variations can be regarded as complex quantities with the real and imaginary parts separately satisfying the equations.

We set

$$\delta\rho(t,\mathbf{r})=\exp(-i\mathbf{k}\mathbf{r})n\sigma_k(t)\delta\xi;\quad \delta T(t,\mathbf{r})=\exp(-i\mathbf{k}\mathbf{r})\tau_k(t)\delta\xi;$$
$$\delta\mathbf{u}(t,\mathbf{r})=\exp(-i\mathbf{k}\mathbf{r})\psi_k(t)\delta\xi.$$

In this case,

$$\delta F_1^{(\mathrm{hyd})}(t,\mathbf{r},\mathbf{v})=\Phi_\Sigma(v)\exp(-i\mathbf{k}\mathbf{r})$$
$$\times\left\{\sigma_k(t)+\frac{Mv^2-3\theta}{2\theta}\frac{\tau_k(t)}{T}+\frac{M(\mathbf{v}\psi_k(t))}{\theta}\right\}\delta\xi, \tag{239}$$

where by virtue of the linearized Navier–Stokes equations

$$\left.\begin{array}{l}\dfrac{1}{k}\dfrac{\partial\sigma_k}{\partial t}=i(\mathbf{e}\cdot\psi_k);\\ \dfrac{1}{k}\dfrac{\partial\psi_k}{\partial t}=i\dfrac{c_0^2}{\gamma}\mathbf{e}\sigma_k-\nu k\psi_k-k(D_l-\nu)\mathbf{e}(\mathbf{e}\psi_k)+\dfrac{c_0^2\alpha}{\gamma}i\mathbf{e}\tau_k;\\ \dfrac{1}{k}\dfrac{\partial\tau_k}{\partial t}=i\dfrac{\gamma-1}{\alpha}\mathbf{e}\cdot\psi_k-\gamma D_Tk\tau_k;\quad \mathbf{e}=\mathbf{k}/k.\end{array}\right| \tag{240}$$

Here, c_0 is the velocity of sound in the long-wavelength limit; $\gamma=C_p/C_v$ is the ratio of the specific heats at constant pressure and volume per particle, C_p and C_v; $\alpha=\partial p/\partial T(n\,\partial p/\partial n)^{-1}$ is the coefficient of thermal expansion; $p=p(n,T)$ is the equilibrium pressure; ν is the kinematic viscosity; D_T is the coefficient of thermal diffusion; $D_l=(4/3)\nu+\zeta(nM)^{-1}$; ζ is the bulk viscosity.

It is well known that (240) has solutions corresponding to five modes: two shear waves, a thermal mode, and two acoustic modes. The time dependence of these modes is given by the exponentially decreasing functions

$$\begin{array}{l}\exp(-\nu k^2t)\ \text{(shear or viscosity waves)};\\ \exp(-D_Tk^2t)\ \text{(thermal mode)};\\ \exp[-(\pm ic_0k+\Gamma_Sk^2/2)t]\ \text{(acoustic waves)},\end{array} \tag{241}$$

where $\Gamma_S=D_l+(\gamma+1)D_T$. Therefore, any solution of (240), and also the expression in the brackets on the right-hand side of (231), regarded as functions of t, are linear combinations of the expressions (241).

We note also that ν, D_T, and Γ_S are of order $l_\Sigma^2t_\Sigma^{-1}$. It can be seen from this that these functions vary very slowly with t/t_Σ when k is sufficiently small;

$$kl_\Sigma\ll 1,\quad kc_0t_\Sigma\ll 1. \tag{242}$$

We return again to variations of the reduced distribution functions with respect to the equilibrium distributions for the special choice of $B(r,v)$ in accordance with (142). We consider first (145) and (146), and make the following assertions: For sufficiently small k satisfying (242), the function $\Phi_k(t,\mathbf{v})$ rapidly approaches the expression

$$\Phi_\Sigma(v)\left\{\sigma_k(t)+\frac{Mv^2-3\theta}{2\theta}\frac{\tau_k(t)}{T}+\frac{M}{\theta}(\mathbf{v}\cdot\psi_k(t))+\text{correction term}\right\}, \tag{243}$$

so that after a definite relaxation time $t_{rel} \gg t_\Sigma$ the function $\Phi_k(t,\mathbf{v})$ virtually coincides with (243) and the hydrodynamic regime is established. Here, the correction term contains the factor k and its time dependence is given by a linear combination of functions of the type (241).

By virtue of (146), this assertion leads to the conclusion that asymptotically

$$\int U_k(t,\mathbf{v},\mathbf{v}')\phi(\mathbf{v}')\,d\mathbf{v}'=\sigma_k(t)+\frac{Mv^2-3\theta}{2\theta}\frac{\tau_k(t)}{T}+\frac{M}{\theta}(\mathbf{v}\cdot\psi_k(t))+\text{correction term} \tag{244}$$

for

$$t>t_{rel}\gg t_\Sigma;\quad k\ll 1/l_\Sigma,\quad 1/c_0t_\Sigma.$$

It is worth emphasizing that in the situation when one can use a kinetic equation, such as the Boltzmann–Enskog equation or the Enskog equation for dense gases, the assertion made above can be formally justified; for if we have at our disposal such a kinetic equation and find that Φ_k is proportional to δF_1, then we merely need to analyze the corresponding linearized equation obtained by means of Φ_k. From this linearized kinetic equation there follows the validity of not only the above assertion. It also becomes possible to derive the linearized Navier–Stokes equation and to calculate its coefficients explicitly. Such a program was carried out in the classical study of Chapman and Enskog.

However, we also emphasize that in the case when the kinetic-equation method is not valid, as is the case for a liquid, our assertion concerning the behavior of $\Phi_k(t,v)$ is merely the usually adopted assumption, and the coefficients in the Navier–Stokes equations must be determined experimentally.

Before we turn to the calculation of the principal asymptotic term in (244), we first make a simple remark concerning integrals of the type

$$\int_0^{k_{\max}}\exp(-\xi k^2t)(1+\alpha_1k+\alpha_2k^2+\ldots)k^2k,\quad \xi>0, \tag{245}$$

which arise in the expression $Q_0(t)$. Making the change of variables $k=q/\sqrt{\xi t}$, we transform (245) to

$$\frac{1}{(\xi t)^{3/2}}\int_0^{\sqrt{\xi t}\,k_{\max}}\exp(-q^2)\left(1+\alpha_1\frac{q}{\sqrt{\xi t}}+\alpha_2\frac{q^2}{\xi t}+\ldots\right)q^2\,dq.$$

Thus, for large t, we have the asymptotic behavior

$$\frac{1}{(\xi t)^{3/2}}\int_0^\infty\exp(-q^2)q^2\,dq=\frac{\sqrt{\pi}}{4(\xi t)^{3/2}}. \tag{246}$$

It is obvious that the correction terms $\alpha_1k+\alpha_2k^2+\ldots$ in (245) do not contribute to this result. The same situation arises when we consider the more complicated integrals that are encountered in an investigation of $Q_I(t)$. For this reason, it is necessary to calculate only the principal terms of the coefficients that appear in (244) with the functions (241) and ignore the terms proportional to $O(k)$.

We now find an explicit expression for the right-hand side of (244). We note first that it was here assumed that $\sigma_k(t)$, $\tau_k(t)$, and $\psi_k(t)$ satisfy Eqs. (240), but we did not particularize the choice of the initial values $\sigma_k(0)$, $\tau_k(0)$, $\psi_k(0)$. By virtue of (244), we know only that the initial values are linear functionals of $\phi(\mathbf{v})$. To solve this problem and determine the linear functionals, we use the arguments invoked by Ernst, Hauge, and van Leeuwen.[12] We consider variations of the particle density, momentum density, and energy density.

We have

$$\delta\rho(t,\mathbf{r})=n\int\delta F_1(t,\mathbf{r},\mathbf{v})\,d\mathbf{v}=\exp(-i\mathbf{kr})\,n\int\Phi_k(t,\mathbf{v})\,d\mathbf{v};$$

$$\delta\mathbf{j}(t,\mathbf{r})=nM\int\mathbf{v}\delta F_1(t,\mathbf{r},\mathbf{v})\,d\mathbf{v}=\exp(-i\mathbf{kr})\,nM\int\mathbf{v}\Phi_k(t,\mathbf{v})\,d\mathbf{v};$$

$$\delta E(t,\mathbf{r})=n\frac{M}{2}\int v^2\delta F_1(t,\mathbf{r},\mathbf{v})\,d\mathbf{v}+\frac{n^2}{2}\int\Phi(\mathbf{r}-\mathbf{r}')\,\delta f_2(t,\mathbf{r},\mathbf{r}')\,d\mathbf{r}',$$

where

$$\delta f_2(t,\mathbf{r},\mathbf{r}')=\delta\int F_2(t,\mathbf{r},\mathbf{v},\mathbf{r}',\mathbf{v}')\,d\mathbf{v},\,d\mathbf{v}'. \tag{247}$$

We recall that we here consider the case (141).

Thus, the variations of each reduced distribution function have the form

$$\delta F_S(t,\mathbf{r}_1,\mathbf{v}_1,\ldots,\mathbf{r}_S,\mathbf{v}_S)=\exp(-i\mathbf{kr}_1)\,\Phi_k^{(S)}(t,r_1,\mathbf{v}_1,\ldots,\mathbf{r}_S,\mathbf{v}_S)\,\delta, \tag{248}$$

where $\Phi_k^{(2)}$ is an invariant under spatial translations. Therefore, we can write

$$\left.\begin{aligned}&\delta f_2(t,\mathbf{r},\mathbf{r}')=\exp(-i\mathbf{kr}_1)\,\overline{\Phi}_k^{(2)}(t,\mathbf{r}-\mathbf{r}')\,\delta\xi;\\&\overline{\Phi}_k^{(2)}(t,\mathbf{r}_1-\mathbf{r}_2)=\int\Phi_k^{(2)}(t,\mathbf{r}_1,\mathbf{v}_1,\mathbf{r}_2,\mathbf{v}_2)\,d\mathbf{v}_1\,d\mathbf{v}_2.\end{aligned}\right\} \tag{249}$$

Hence

$$\left.\begin{aligned}&\delta\rho(t,\mathbf{r})=\exp(-i\mathbf{kr})\,n\int\Phi_k(t,\mathbf{v})\,d\mathbf{v}\,\delta\xi;\\&\delta\mathbf{j}(t,\mathbf{r})=\exp(-i\mathbf{kr})\,nM\int\mathbf{v}\Phi_k(t,\mathbf{v})\,d\mathbf{v}\,\delta\xi;\\&\delta E(t,\mathbf{r})=\exp(-i\mathbf{kr})\left\{\frac{nM}{2}\int v^2\Phi_k(t,\mathbf{v})\,d\mathbf{v}+\frac{n^2}{2}\int\Phi(\mathbf{r}-\mathbf{r}')\,\overline{\Phi}_k^{(2)}(t,\mathbf{r}-\mathbf{r}')\,d\mathbf{r}'\right\}\delta\xi.\end{aligned}\right| \tag{250}$$

Note that in the limit $k\to 0$ we arrive at the spatially homogeneous case and the variations (250) of the particle, momentum, and energy densities must be exact integrals of the motion.

In the considered case of sufficiently small k, we can analyze the time derivatives $\partial/\partial t$ determined by (250). Using the hierarchy of equations for δF_S and taking into account (248), we conclude that these derivatives are proportional to k. Thus, (250) may be called quasi-integrals, i.e., they are virtually conserved in time intervals that are the longer the smaller are the k values under consideration.

We fix a definite time $t_0\geq t_{rel}$ when the transition to the hydrodynamic regime has been achieved. In this case, we can find a k_0 such that to terms of order $O(k)$:

$$\left.\begin{aligned}&\int \Phi_k(t_0, \mathbf{v})\, d\mathbf{v} = \int \Phi_k(0, \mathbf{v})\, d\mathbf{v};\\ &\int \mathbf{v}\Phi_k(t_0, \mathbf{v})\, d\mathbf{v} = \int \mathbf{v}\Phi_k(0, \mathbf{v})\, d\mathbf{v};\\ &\frac{nM}{2}\int v^2\Phi_k(t_0, \mathbf{v})\, d\mathbf{v} + \frac{n^2}{2}\int \Phi(\mathbf{r}-\mathbf{r}')\,\Phi_k^2(t_0, \mathbf{r}-\mathbf{r}')\, d\mathbf{r}'\\ &= \frac{nM}{2}\int v^2\Phi_k(0, \mathbf{v})\, d\mathbf{v} + \frac{n^2}{2}\int \Phi(\mathbf{r}-\mathbf{r}')\,\overline{\Phi}_k^{(2)}(0, \mathbf{r}-\mathbf{r}')\, d\mathbf{r}'\end{aligned}\right\} \tag{251}$$

for $k \leqslant k_0$.

On the other hand, since the hydrodynamic regime is achieved at t_0, we have

$$\left.\begin{aligned}&\delta\rho(t_0, \mathbf{r}) = \exp(-i\mathbf{kr})\, n\sigma_k(t_0)\,\delta\xi;\\ &\delta\mathbf{j}(t_0, \mathbf{r}) = \exp(-i\mathbf{kr})\, nM\psi_k(t_0)\,\delta\xi;\\ &\delta E(t_0, \mathbf{r}) = \frac{\partial\varepsilon(n, T)}{\partial n}\,\delta\rho(t_0, \mathbf{r}) + \frac{\partial\varepsilon(n, T)}{\partial T}\,\delta T(t_0, \mathbf{r})\\ &= \exp(-i\mathbf{kr})\left\{n\frac{\partial\varepsilon(n, T)}{\partial n}\,\delta_k(t_0) + \frac{\partial\varepsilon(n, T)}{\partial T}\,\tau_k(t_0)\right\}\delta\xi,\end{aligned}\right\} \tag{252}$$

where $\varepsilon(n, T)$ is the equilibrium energy density.

We note further that since $\sigma_k(t), \tau_k(t), \psi_k(t)$ is a linear combination of the functions (241), we can write asymptotically

$$\sigma_k(t_0) = \sigma_k(0);\quad \tau_k(t_0) = \tau_k(0);\quad \psi_k(t_0) = \psi_k(0) \tag{253}$$

for

$$k \ll \frac{1}{ct_0},\quad \frac{1}{\sqrt{D_T t_0}},\quad \frac{1}{\sqrt{\Gamma_S t_0}},\quad \frac{1}{\sqrt{\nu t_0}}.$$

Thus, by virtue of (251) and the asymptotic equations given by the expressions (250) and (252), which are valid from the time t_0 for sufficiently small k, we obtain to terms $O(k)$:

$$\left.\begin{aligned}&\sigma_k(0) = \int \Phi_k(0, \mathbf{v})\, d\mathbf{v};\quad \psi_k(0) = \int \mathbf{v}\Phi_k(0, \mathbf{v})\, d\mathbf{v};\\ &n\frac{\partial\varepsilon(n, T)}{\partial n}\,\sigma_k(0) + \frac{\partial\varepsilon(n, T)}{\partial T}\,\tau_k(0)\\ &= \frac{nM}{2}\int v^2\Phi_k(0, \mathbf{v})\, d\mathbf{v} + \frac{n^2}{2}\int \Phi(\mathbf{r}-\mathbf{r}')\,\overline{\Phi}_k^{(2)}(0, \mathbf{r}-\mathbf{r}')\, d\mathbf{r}'\end{aligned}\right\} \tag{254}$$

for $k \leqslant k_1$, where

$$k_1 \ll k_0;\ k_1 \ll \frac{1}{ct_0},\quad \frac{1}{\sqrt{D_{T}t_0}},\quad \frac{1}{\sqrt{\Gamma_S t_0}},\quad \frac{1}{\sqrt{\nu t_0}}.$$

In (254)

$$\partial\varepsilon(n, T)/\partial T = nC_V, \tag{255}$$

where C_V is the specific heat per particle at constant density.

We make some remarks concerning $\varepsilon(n, T)$. We have

$$\varepsilon(n, T) = \frac{3\theta}{2}\, n + \frac{n^2}{2}\int \Phi(r)\, f_2^{(\text{eq})}(\mathbf{r})\, d\mathbf{r}, \tag{256}$$

where

$$f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2) = \int F_2^{(\text{eq})}(1, 2)\, d\mathbf{v}_1\, d\mathbf{v}_2$$

is the spatial binary reduced distribution function in the state of statistical equilibrium. Of course, $f_2^{(\text{eq})}$ depends on n and T. It is convenient to introduce the chemical potential $\mu = \mu(n, T)$, $n = n(\mu, T)$. Using then the properties of fluctuations in a state of equilibrium, we find

$$\left.\begin{aligned}&\frac{\theta}{n}\left(\frac{\partial n}{\partial\mu}\right)_T = 1 + n\int g_2(\mathbf{r})\, d\mathbf{r};\quad g_2(\mathbf{r}) = f_2^{(\text{eq})}(\mathbf{r}) - 1;\\ &\frac{\theta}{n}\left(\frac{\partial}{\partial\mu}\, n^2 f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2)\right)_T = 2nf_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2)\\ &+ n^2\int [f_3^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2, \mathbf{r}_1-\mathbf{r}_3) - f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2)]\, d\mathbf{r}_3\end{aligned}\right\} \tag{257}$$

and therefore

$$n\frac{\partial\varepsilon(n, T)}{\partial n} = \frac{3\theta}{2}\, n + \frac{n^2}{2}\int \Phi(\mathbf{r}_1-\mathbf{r}_2)\Big[2f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2) + n\int \{f_3^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2, \mathbf{r}_1-\mathbf{r}_3) - f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2)\}\, d\mathbf{r}_3\Big]\, d\mathbf{r}_2\left(\theta\frac{\partial n}{\partial\mu}\right)_T^{-1}.$$

We can now represent the third equation (254) in the form

$$\begin{aligned}C_V\tau_k(0) = \int \frac{MV^2 - 3\theta}{2}\,\Phi_k(0, \mathbf{v})\, d\mathbf{v} + \frac{n}{2}\int \Phi(\mathbf{r}_2-\mathbf{r}_2)\\ \times\Big\{\overline{\Phi}_k^{(2)}(0, \mathbf{r}_1-\mathbf{r}_2) - \Big[2f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2) + n\int \big(f_3^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2, \mathbf{r}_1-\mathbf{r}_3)\\ - f_2^{(\text{eq})}(\mathbf{r}_1-\mathbf{r}_2)\big)\Big]\, d\mathbf{r}_3\sigma_k(0)\, n\left(\theta\frac{\partial n}{\partial\mu}\right)_T^{-1}\Big\}\, d\mathbf{r}_2.\end{aligned} \tag{258}$$

To obtain expressions for $\Phi_k(0, \mathbf{v})$; and $\overline{\Phi}_k^{(2)}(0, \mathbf{r}_1-\mathbf{r}_2)$, we use our previous results (see Sec. 3). Thus, from (144) and (145),

$$\Phi_k(0, \mathbf{v}) = \Phi_\Sigma(v)\left\{\phi(\mathbf{v}) + n\int g_2(\mathbf{r})\exp(i\mathbf{kr})\, d\mathbf{r} \times \int \phi(\mathbf{v}')\,\Phi_\Sigma(\mathbf{v}')\, d\mathbf{v}'\right\}. \tag{259}$$

Thus,

$$\int \frac{Mv^2 - 3\theta}{2}\,\Phi_k(0, \mathbf{v})\, d\mathbf{v} = \int \frac{Mv^2 - 3\theta}{2}\,\Phi_\Sigma(v)\,\phi(\mathbf{v})\, d\mathbf{v} \tag{260}$$

and (254) reduces to

$$\left.\begin{aligned}&\sigma_k(0) = \left(1 + n\int g_2(\mathbf{r})\exp(i\mathbf{kr})\, d\mathbf{r}\right)\int \phi(\mathbf{v})\,\Phi_\Sigma(v)\, d\mathbf{v};\\ &\psi_k(0) = \int \mathbf{v}\Phi_\Sigma(v)\,\phi(\mathbf{v})\, d\mathbf{v}.\end{aligned}\right\} \tag{261}$$

Note that the equilibrium correlation function $g_2(\mathbf{r})$ effectively vanishes when $\mathbf{r}$ becomes much greater than the correlation length.

If the equilibrium system Σ is not near a critical point, which we here assume, then the correlation length is of the order of the range a_Σ of the interparticle interaction. For a liquid, l_Σ is of order a_Σ; for gases $a_\Sigma \ll l_\Sigma$.

In any case, since $k \ll l_\Sigma^{-1}$, we see that up to terms of order $O(k^2)$ the following asymptotic equality holds:

$$\int g_2(\mathbf{r})\exp(i\mathbf{kr})\, d\mathbf{r} = \int g_2(\mathbf{r})\, d\mathbf{r}.$$

Thus, (261) leads in the framework of the adopted approximation to

$$\sigma_k(0) = \frac{\theta}{n}\left(\frac{\partial n}{\partial\mu}\right)_T\int \phi(\mathbf{v})\,\Phi_\Sigma(v)\, d\mathbf{v}. \tag{262}$$

To obtain an expression for $\overline{\Phi}_k^{(2)}(0, \mathbf{r}_1-\mathbf{r}_2)$, we shall proceed from (141). These formulas give

$$\begin{aligned}\delta F_2(0; 1, 2) = \Phi_\Sigma(v_1)\,\Phi_\Sigma(v_2)\{(\exp(-i\mathbf{kr}_1)\,\phi(\mathbf{v}_1) + \exp(i\mathbf{kr}_2)\,\phi(\mathbf{v}_2))\, f_2(\mathbf{r}_1-\mathbf{r}_2)\\ + n\int [f_3(\mathbf{r}_1-\mathbf{r}_2, \mathbf{r}_1-\mathbf{r}_3) - f_2(\mathbf{r}_1-\mathbf{r}_2)]\\ \times\exp(-i\mathbf{kr}_3)\, d\mathbf{r}_3\int \phi(\mathbf{v})\,\Phi_\Sigma(\mathbf{v})\, d\mathbf{v}\}\,\delta\xi.\end{aligned}$$

It follows from (249) that

$$\begin{aligned}\overline{\Phi}_k^{(2)}(0, \mathbf{r}_1-\mathbf{r}_2) = \{(1 + \exp[i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)])\, f_2(\mathbf{r}_1-\mathbf{r}_2)\\ + n\int [f_3(\mathbf{r}_1-\mathbf{r}_2, \mathbf{r}_1-\mathbf{r}_3) - f_2(\mathbf{r}_1-\mathbf{r}_2)]\\ \times\exp[i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_3)]\, d\mathbf{r}_3\}\int \phi(\mathbf{v})\,\Phi_\Sigma(\mathbf{v})\, d\mathbf{v}.\end{aligned} \tag{263}$$

We require this expression here only to calculate the integral

$$\frac{n}{2}\int \Phi(\mathbf{r}_1-\mathbf{r}_2)\,\overline{\Phi}_k^{(2)}(0,\,\mathbf{r}_1-\mathbf{r}_2)\,d\mathbf{r}_2.$$

Since the relative distances $|\mathbf{r}_1-\mathbf{r}_2|$ are of the order of the range a_Σ of the interparticle interaction, in (263) the factor $1+\exp i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)$ can be replaced by 2. Further, when $|\mathbf{r}_1-\mathbf{r}_3|\gg a_\Sigma$ and therefore $|\mathbf{r}_2-\mathbf{r}_3|\gg a_\Sigma$, the combination $f_3^{(eq)}(\mathbf{r}_1-\mathbf{r}_2,\mathbf{r}_1-\mathbf{r}_3)-f_2^{(eq)}(\mathbf{r}_1-\mathbf{r}_2)$, which characterizes the correlation between particles at the point $\mathbf{r}_3$ and particles in the neighborhood of $\mathbf{r}_1$ and $\mathbf{r}_2$, is effectively zero.

Thus, in our approximation

$$\frac{n}{2}\int \Phi(\mathbf{r}_1-\mathbf{r}_2)\,\overline{\Phi}_k^{(2)}(0,\,\mathbf{r}_1-\mathbf{r}_2)\,d\mathbf{r}_2$$
$$=\frac{n}{2}\int \Phi(\mathbf{r}_1-\mathbf{r}_2)\Big\{2f_2^{(eq)}(\mathbf{r}_1-\mathbf{r}_2)+n\int[f_3^{(eq)}(\mathbf{r}_1-\mathbf{r}_2,\,\mathbf{r}_1-\mathbf{r}_3)-f_2^{(eq)}(\mathbf{r}_1-\mathbf{r}_2)]\,d\mathbf{r}_3\Big\}\,d\mathbf{r}_2\int\phi(\mathbf{v})\,\Phi_\Sigma(\mathbf{v})\,d\mathbf{v}. \qquad (264)$$

However, by virtue of (262),

$$n\sigma_k(0)\left(\theta\frac{\partial n}{\partial\mu}\right)_T^{-1}=\int\phi(\mathbf{v})\,\Phi_\Sigma(\mathbf{v})\,d\mathbf{v},$$

and therefore the second term on the right-hand side of (258) is zero. Note also that $(\partial n/\partial\mu)_T=n(\partial p/\partial n)_T^{-1}$.

Collecting our results (258), (261), (262), and (264), we can finally write down expressions that are adequate to the values of the initial quantities calculated to terms $O(k)$:

$$\left.\begin{aligned}
&\sigma_k(0)=\theta\left(\frac{\partial p}{\partial n}\right)_T^{-1}\int\Phi_\Sigma(v')\,\phi(\mathbf{v}')\,d\mathbf{v}';\\
&\tau_k(0)=C_V^{-1}\int\Phi_\Sigma(v')\,\frac{Mv'^2-3\theta}{2}\,\phi(\mathbf{v}')\,d\mathbf{v}';\\
&\psi_k(0)=\int\Phi_\Sigma(v')\,\mathbf{v}'\phi(\mathbf{v}')\,d\mathbf{v}'.
\end{aligned}\right\} \qquad (265)$$

We now proceed to find solutions of Eqs. (240). These equations contain the unit vector $\mathbf{e}=\mathbf{k}/k$. We introduce two further unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in such a way that the three vectors $\mathbf{e},\mathbf{e}_1,\mathbf{e}_2$ are mutually orthogonal. Then

$$\psi_k=\mathbf{e}_1(\mathbf{e}_1\psi_k)+\mathbf{e}_2(\mathbf{e}_2\psi_k)+\mathbf{e}(\mathbf{e}\psi_k) \qquad (266)$$

and it follows from (240) that

$$\frac{d}{dt}(\mathbf{e}_j\psi_k(t))=-\nu k^2(\mathbf{e}_j\psi_k(t)),\quad j=1,\,2.$$

Therefore

$$(\mathbf{e}_j\psi_k(t))=\exp(-\nu k^2t)(\mathbf{e}_j\psi_k(0))$$
$$=\exp(-\nu k^2t)\int\Phi_\Sigma(v')(\mathbf{e}_j\mathbf{v}')\,\phi(\mathbf{v}')\,d\mathbf{v}',\quad j=1,\,2. \qquad (267)$$

It remains to find the three functions

$$\sigma_k(t),\quad s_k(t)=(\mathbf{e}\psi_k(t)),\quad \tau_k(t). \qquad (268)$$

The system of equations (240) can be rewritten as

$$\left.\begin{aligned}
&\frac{1}{k}\frac{\partial\sigma_k}{\partial t}=is_k;\\
&\frac{1}{k}\frac{\partial s_k}{\partial t}=i\frac{c_0^2}{\gamma}\sigma_k-D_lks_k+i\frac{c_0^2\alpha}{\gamma}\tau_k;\\
&\frac{1}{k}\frac{\partial\tau_k}{\partial t}=i\frac{\gamma-1}{\alpha}s_k-\gamma D_Tk\tau_k.
\end{aligned}\right\} \qquad (269)$$

To solve these equations, we introduce three independent combinations A_H and $A_\pm$ formed from the functions (268) in such a way that (269) takes the form

$$\partial A(t)/\partial t=-\Omega A(t)$$

and

$$A(t)=\exp(-\Omega t)A(0).$$

We calculate Ω in such a way as to take into account the terms proportional to k^2, since they are the ones responsible for the damping of the functions (268). On the other hand, in calculating the coefficients of the linear forms A_H and $A_\pm$ we must ignore terms of order $O(k)$, since the initial values (268) were themselves calculated to only this order. The upshot is

$$\left.\begin{aligned}
&A_H(t)=\gamma^{-1}((\gamma-1)\sigma_k(t)-\alpha\tau_k(t));\quad \Omega_H=D_Tk^2;\\
&A_\pm(t)=\gamma^{-1}(\sigma_k(t)+\alpha\tau_k(t))/2\mp c_0^{-1}s_k(t)/2;\\
&\Omega_\pm=\pm ic_0k+\Gamma_Sk^2/2.
\end{aligned}\right\} \qquad (270)$$

Inverting (270), we have

$$\sigma_k(t)=A_H(t)+A_+(t)+A_-(t);$$
$$\tau_k(t)=-\alpha^{-1}A_H(t)+(\gamma-1)\alpha^{-1}(A_+(t)+A_-(t));$$
$$s_k(t)=c_0(A_-(t)-A_+(t)).$$

Thus

$$\left.\begin{aligned}
&\sigma_k(t)=\exp(-\Omega_Ht)A_H(0)+\exp(-\Omega_+t)A_+(0)+\exp(-\Omega_-t)A_-(0);\\
&\tau_k(t)=-\alpha^{-1}\exp(-\Omega_Ht)A_H(0)+(\gamma-1)\alpha^{-1}\exp(-\Omega_+t)\times A_+(0)+(\gamma-1)\alpha^{-1}\exp(-\Omega_-t)A_-(0);\\
&(\mathbf{e}\psi_k(t))=s_k(t)=c_0A_-(0)\exp(-\Omega_-t)-c_0A_+(0)\exp(-\Omega_+t).
\end{aligned}\right\} \qquad (271)$$

With allowance for (265) and (270),

$$\left.\begin{aligned}
&A_H(0)=\int\left\{\left(1-\gamma^{-1}\right)\theta\left(\frac{\partial p}{\partial n}\right)_T^{-1}-\gamma^{-1}\alpha C_V^{-1}\frac{Mv'^2-3\theta}{2}\right\}\times\Phi_\Sigma(v')\,\phi(\mathbf{v}')\,dv';\\
&A_\pm(0)=\int\left\{\frac{1}{2}\gamma^{-1}\theta\left(\frac{\partial p}{\partial n}\right)_T^{-1}+\frac{1}{2}(\gamma C_V)^{-1}\alpha\frac{Mv'^2-3\theta}{2}\mp\frac{1}{2}c_0^{-1}(\mathbf{e}\mathbf{v}')\right\}\Phi_\Sigma(v')\,\phi(\mathbf{v}')\,dv'.
\end{aligned}\right\} \qquad (272)$$

Substituting (266), (267), and (271) in (244), we obtain

$$\int U_k(t,\,\mathbf{v},\,\mathbf{v}')\,\phi(\mathbf{v}')\,d\mathbf{v}'$$
$$=\exp(-\nu k^2t)\frac{M}{\theta}(\mathbf{v}_1\mathbf{e}_1)\int(\mathbf{v}_1'\mathbf{e}_1)\,\Phi_\Sigma(v_1')\,\phi(\mathbf{v}_1')\,d\mathbf{v}_1'$$
$$+\exp(-\nu k^2t)\frac{M}{\theta}(\mathbf{v}_1\mathbf{e}_2)\int(\mathbf{v}_1'\mathbf{e}^2)\,\Phi_\Sigma(v_1')\,\phi(\mathbf{v}_1')\,d\mathbf{v}_1'$$
$$+\exp(-\Omega_Ht)\left\{1-\frac{Mv^2-3\theta}{2\theta}(\alpha T)^{-1}\right\}A_H(0)$$
$$+\exp(-\Omega_+t)\left\{1+\frac{Mv^2-3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1)-\frac{M}{\theta}c_0(\mathbf{v}\cdot\mathbf{e})\right\}A_+(0)$$
$$+\exp(-\Omega_-t)\left\{1+\frac{Mv^2-3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1)+\frac{M}{\theta}c_0(\mathbf{v}\cdot\mathbf{e})\right\}A_-(0). \qquad (273)$$

To unify the notation, we define

$$\left.\begin{aligned}
&\theta_1^{(L)}(\mathbf{e},\,\mathbf{v})=\theta_1^{(R)}(\mathbf{e},\,\mathbf{v})=\sqrt{\frac{M}{\theta}}(\mathbf{e}_1\mathbf{v});\\
&\theta_2^{(L)}(\mathbf{e},\,\mathbf{v})=\theta_2^{(R)}(\mathbf{e},\,\mathbf{v})=\sqrt{\frac{M}{\theta}}(\mathbf{e}_2\mathbf{v});\quad \omega_1(k)=\omega_2(k)=\nu k^2;\\
&\theta_3^{(L)}(\mathbf{e},\,\mathbf{v})=\left(\frac{Mv^2-3\theta}{2\theta}-\alpha T\right)\left(\frac{k_B}{C_p}\right)^{1/2};\\
&\theta_3^{(R)}(\mathbf{e},\,\mathbf{v})=\left(\frac{Mv^2-3\theta}{2\theta}-(\gamma-1)\frac{nC_v}{(\partial p/\partial T)_n}\right)\left(\frac{k_B}{C_p}\right)^{1/2};\\
&\omega_3(k)=\Omega_H=D_Tk^2.\\
&\theta_{\binom{4}{5}}^{(L)}(\mathbf{e},\,\mathbf{v})=\left(1+\frac{Mv^2-3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1)\mp\frac{M}{\theta}c_0(\mathbf{v}\mathbf{e})\right)(1/2)^{1/2};\\
&\theta_{\binom{4}{5}}^{(R)}(\mathbf{e},\,\mathbf{v})=\left(\theta\gamma^{-1}\left(\frac{\partial p}{\partial n}\right)^{-1}+(\gamma C_V)^{-1}\alpha\frac{Mv^2-3\theta}{2}\pm\frac{1}{C_0}(\mathbf{v}\mathbf{e})\right)\left(\frac{1}{2}\right)^{1/2};\\
&\omega_4(k)=\Omega_+=ic_0k+\Gamma_Sk^2/2;\\
&\omega_5(k)=\Omega_-=-ic_0k+\Gamma_Sk^2/2,
\end{aligned}\right\} \qquad (274)$$

where k_B is Boltzmann's constant.

Then (272) and (274) enable us to rewrite (273) as

$$\int U_k(t, \mathbf{v}, \mathbf{v}') \phi(\mathbf{v}') d\mathbf{v}'$$
$$= \sum_{(1\leq j\leq 5)} \theta_j^{(L)}(\mathbf{e}, \mathbf{v}') \exp(-\omega_j(k) t) \int \theta_j^{(R)}(\mathbf{e}, \mathbf{v}')$$
$$\times \Phi_\Sigma(v') \phi(\mathbf{v}') dv', \quad t > t_{\rm rel}; \quad k < k_1. \tag{275}$$

We emphasize that in the cases when a Boltzmann or Enskog type kinetic equation holds[6] the same result as (275) can be obtained. Strictly speaking, for this one requires, not the complete form of any kinetic equation, but only its linearized version.

These linearized equations lead to the relation (275) if for $\omega_j(k)$ the contributions of terms proportional to k^2 are calculated, whereas in the calculation of the coefficients $\theta_j^{(L)}$ and $\theta_j^{(R)}$ the terms of order k are ignored. In such an approximation, the actual values of both the equilibrium means and the transport coefficients (ν, D_T, Γ_S) are obtained in accordance with the same approximations in which the kinetic equation is established. We now use (275) to reduce Eqs. (232) and (234) to a more convenient form.

We consider first the expression $Q_l(t)\chi(\mathbf{v}_0)$ and note that it contains the operator $\exp[(-i\mathbf{v}_0\lambda + na^2w(a)L_S)t]$, $\boldsymbol{\lambda} = \mathbf{k}+\mathbf{l}$, applied to functions of $\mathbf{v}_0$. We introduce a scalar product for such functions:

$$(g, h) = \int \Phi_0(v_0) g(\mathbf{v}_0) h(\mathbf{v}_0) d\mathbf{v}_0; \tag{276}$$

the corresponding Hilbert scalar product is given by

$$(g, h)_H = (g^*, h). \tag{277}$$

By definition, the operator

$$na^2w(a) L_S \tag{278}$$

is symmetric and Hermitian:

$$(g, L_S h) = (L_S g, h); \quad (g, L_S h)_H = (L_S g, h)_H.$$

It is also well known that its spectrum consists of a negative part and a nondegenerate zero eigenvalue corresponding to the normalized eigenfunction $\varphi(v) = 1$: $L_S \cdot 1 = 0$. The gap between the negative part and the zero for (278) is of order t_0^{-1}, where

$$t_0 = (m/\pi\theta)^{1/2}/[4na^2w(a)] \tag{279}$$

is the mean free time of the particle S in the Enskog approximation.

Of course, the eigenfunctions $\psi(\mathbf{v})$ of the operator (278) corresponding to its negative eigenvalues are orthogonal to 1:

$$\int \Phi_0(v) \psi(\mathbf{v}) d\mathbf{v} = 0 \tag{280}$$

The operator $E_\lambda = -i\mathbf{v}\lambda + na^2w(a)L_S$, is obviously non-Hermitian, though it preserves the symmetry properties: $(g, E_\lambda h) = (E_\lambda g, h)$.

We consider the eignfunction

$$E_\lambda \varphi_\lambda(\mathbf{v}) = -\omega_0(\lambda) \psi_\lambda(\mathbf{v}),$$

for which $\omega_0(\lambda) \to 0$ when $\lambda \to 0$. Using ordinary perturbation theory, we readily find

$$\left.\begin{aligned} &\psi_\lambda(\mathbf{v}) = 1 + \frac{1}{na^2w(a)} L_S^{-1}(\lambda, \mathbf{v}) + O(\lambda^2);\\ &\omega_0(\lambda) = D_0\lambda^2 + O(\lambda^2);\\ &D_0 = -\int \Phi_0(\mathbf{v}) v_x L_S^{-1} v_x d\mathbf{v} (na^2w(a))^{-1}. \end{aligned}\right\} \tag{281}$$

We note here that the functions v_x, v_y, v_z belong to the class (280), on which the inverse operator L_S^{-1} is well defined. In the first Enskog approximation

$$D_0 = \frac{3}{8na^2w(a)} \left(\frac{m}{\pi\theta}\right)^{-1/2}. \tag{282}$$

Ignoring for $t \gg t_0$ the rapidly decaying terms in the exponentials due to the negative part of the spectrum (278), we write

$$\exp(E_\lambda t) \chi(\mathbf{v}) = \exp(-\omega_0(\lambda) t) \psi_\lambda(\mathbf{v})$$
$$\times \int \Phi_0(v) \psi_\lambda(\mathbf{v}) \chi(\mathbf{v}) d\mathbf{v}.$$

We must, however, bear in mind that the gap between the zero and the negative part of the spectrum (278) is of order t_0^{-1}. Therefore, for the validity of this asymptotic relation we require that $D_0\lambda^2 \ll t_0^{-1}$ or

$$\lambda \ll l_0^{-1} = (3/2)^{-1/2} 4na^2w(a) \tag{283}$$

Adhering in such a case to the adopted scheme, we ignore the terms of order $O(\lambda)$ in ψ_λ and terms of higher order than $O(\lambda^2)$ in $\omega_0(\lambda)$, and we set

$$\psi_\lambda(\mathbf{v}) = 1; \quad \omega_0(\lambda) = D_0\lambda^2. \tag{284}$$

Proceeding in this way, we obtain

$$\exp[(-i\mathbf{v}_0\lambda + na^2w(a) L_S) t] \chi(\mathbf{v}_0)$$
$$= \exp[-\omega_0(\lambda) t] \int \Phi_0(v) \chi(\mathbf{v}) d\mathbf{v}, \tag{285}$$

when $t \gg t_0$.

Before we use this result in (233), it is helpful to note that (233) contains the operators

$$\overline{T}_k, \quad T_k, \tag{286}$$

whose k dependence is determined by the factors $\exp[\pm iak(\mathbf{e}\cdot\sigma)]$. However, $ka \ll al_0^{-1} \ll 1$, so that for self-consistency of the employed approximations we must replace (286) by $T_0 = \overline{T}_0$. On the other hand, the integration with respect to $\mathbf{k}$ in (233) obviously requires a cutoff:

$$k < k_{\max}, \tag{287}$$

where $k_{\max} < k_1$ and $k_{\max} \ll l_0^{-1}$, since we are investigating here only the part of $Q_l(t)$ that decreases less weakly than any exponential $\exp(-t/t_f)$ with fixed t_f, and since the entire scheme of our approximation depends strictly on this last condition [see, for example, (275) and (283)].

We now substitute our results in (233). First, it follows from (285) that

$$\exp[(-i\mathbf{v}_0(\mathbf{k}+\mathbf{l}) + na^2w(a) L_S) t]$$
$$\times T_k(v_0, v_1) \chi(\mathbf{v}_0) = \exp[-t\omega_0(\mathbf{k}+\mathbf{l})]$$
$$\times \int d\mathbf{v}' \Phi_0(v_0') T_0(v_0', v_1) \chi(\mathbf{v}_0).$$

Here, the right-hand side is a function of $\mathbf{v}_1$. Therefore, using (275), we obtain

[6]This is the case for a gas of hard spheres of moderate density.

$$U(t;\ 1)\exp[(-i\mathbf{v}_0(\mathbf{k}+\mathbf{l})+na^2w(a)L_S)t]\,T_k(v_0,\ v_1)\,\chi(\mathbf{v}_0)$$
$$=\sum_{(1\leqslant j\leqslant 5)}\exp\{-[\omega_j(k)+\omega_0(\mathbf{k}+\mathbf{l})]\,t\}\,\theta_j^{(L)}(\mathbf{e},\ \mathbf{v}_1)$$
$$\times\int d\mathbf{v}_0'\,d\mathbf{v}_1'\,\Phi_0(v_0')\,\Phi_\Sigma(v_1')\,\theta_j^{(R)}(\mathbf{e},\ \mathbf{v}_1')\,T_0(v_0',\ v_1')\,\chi(\mathbf{v}_0').$$

It now follows from (233) that

$$Q_l(t)\,\chi(\mathbf{v}_0)=\frac{n}{(2\pi)^3}\int_{|\mathbf{k}|<k_{\max}}d\mathbf{k}\sum_{(1\leqslant j\leqslant 5)}\exp[-(\omega_j(k)+\omega_0(\mathbf{k}+\mathbf{l}))\,t]$$
$$\times\left\{\int d\mathbf{v}_1\Phi_\Sigma(v_1)\,T_0(v_0,\ v_1)\,\theta_j^{(L)}(\mathbf{e},\ \mathbf{v}_1)\right\}$$
$$\times\left\{\int d\mathbf{v}_0'\,d\mathbf{v}_1'\,\Phi_0(v_0')\,\Phi_\Sigma(v_1')\,\theta_j^{(R)}(\mathbf{e},\ \mathbf{v}_1')\,T_0(v_0',\ v_1')\,\chi(\mathbf{v}_0')\right\}.$$

Noting that the functions

$$g(\mathbf{v}_0,\ \mathbf{v}_1)=\begin{cases} mv_0^2+Mv_1^2; \\ mv_0+Mv_1; \\ \text{const} \end{cases}$$

are invariants of a collision, we see that

$$\int d\mathbf{v}_1\Phi_\Sigma(v_1)\,T_0(v_0,\ v_1)\,\theta_j^{(L)}(\mathbf{e},\ \mathbf{v}_1)$$
$$=-\int d\mathbf{v}_1\Phi_\Sigma(v_1)\,T_0(v_0,\ v_1)\,\psi_j^{(R)}(\mathbf{e},\ \mathbf{v}_0)=-a^2L_S\psi_j^{(L)}(\mathbf{e},\ \mathbf{v}_0);$$
$$\int d\mathbf{v}_0'\,d\mathbf{v}_1'\,\Phi_0(v_0')\,\Phi_\Sigma(v_1')\,\theta_j^{(R)}(\mathbf{e},\ \mathbf{v}_1')$$
$$\times\,T_0(v_0',\ v_1)\,\chi(\mathbf{v}_0')=-\int d\mathbf{v}_0'\,d\mathbf{v}_1'\,\Phi_0(\mathbf{v}_0')$$
$$\times\,\Phi_\Sigma(v_1')\,\psi_j^{(R)}(\mathbf{e},\ \mathbf{v}_0')\,T_0(v_0',\ v_1')\,\chi(\mathbf{v}_0')$$
$$=-a^2\int d\mathbf{v}_0'\Phi_0(v_0')\,\psi_j^{(R)}(\mathbf{e},\ \mathbf{v}_0')\,L_S\chi(\mathbf{v}_0'),$$

where[7)]

$$\left.\begin{aligned}
&\psi_j^{(R)}(\mathbf{e},\ \mathbf{v})=\psi_j^{(L)}(\mathbf{e},\ \mathbf{v})=\frac{m}{(M\theta)^{1/2}}(\mathbf{e}_j\mathbf{v}),\quad j=1,\ 2;\\
&\psi_3^{(R)}(\mathbf{e},\ \mathbf{v})=\psi_3^{(L)}(\mathbf{e},\ \mathbf{v})=[(mv^2-3\theta)/2\theta]\,(k_B/C_p)^{1/2};\\
&\psi_{\binom{4}{5}}^{(L)}(\mathbf{e},\ \mathbf{v})=(1/2)^{1/2}\left(\frac{mv^2-3\theta}{2\theta}(\alpha T)^{-1}(\gamma-1)\mp\frac{m}{\theta}C_0\mathbf{v}\cdot\mathbf{e}\right);\\
&\psi_{\binom{4}{5}}^{(R)}(\mathbf{e},\ \mathbf{v})=\left(\frac{1}{2}\right)^{1/2}\left(\frac{mv^2-3\theta}{2}\frac{\alpha}{C_p}\mp\frac{m}{MC_0}\mathbf{v}\cdot\mathbf{e}\right).
\end{aligned}\right\}\qquad(288)$$

We then arrive at the completely definite expression

$$Q_l(t)\,\chi(v_0)=\frac{na^4}{(2\pi)^3}\int_{|k|k_{\max}}\sum_{(1\leqslant j\leqslant 5)}\exp\{-[\omega_j(k)+\omega_0(\mathbf{k}+\mathbf{l})]\,t\}$$
$$\times L_S\psi_j^{(L)}(\mathbf{e},\ \mathbf{v}_0)\int d\mathbf{v}_0'\Phi_0(v_0')\,\psi_j^{(R)}(\mathbf{e},\ \mathbf{v}_0')\,L_S\chi(\mathbf{v}_0'),\qquad(289)$$

when $t\gg t_0,\ t_0>t_{\rm rel}$, which can be substituted in Eqs. (232) and (234).

We consider the case when $l=0$. Then (234) leads to the equation

$$\frac{\partial\chi(t,\ \mathbf{v}_0)}{\partial t}=na^2w(a)\,L_S\chi(t,\ \mathbf{v}_0)+w^2(a)\int_0^t Q_0(t-\tau)\,\chi(\tau,\ \mathbf{v})\,d\mathbf{v};$$
$$Q_0(t-\tau)\,\chi(\mathbf{v}_0)=\frac{na^4}{(2\pi^3)}\int_0^{k_{\max}}k^2\,dk\sum_{(1\leqslant j\leqslant 5)}\exp\{-[\omega_j(k)$$
$$+\,\omega_0(k)]\,(t-\tau)\}\int d\mathbf{e}\,L_S\psi_j^{(L)}(\mathbf{e},\mathbf{v}_0)\int d\mathbf{v}_0'\Phi_0(v_0')\,\psi_j^{(R)}(\mathbf{e},\ \mathbf{v})\,L_S\chi(\mathbf{v}_0').$$
$$(290)$$

It is obvious that if $\chi(0,\mathbf{v}_0)=\text{const}$, then also $\chi(t,\mathbf{v}_0)=\chi(0,\mathbf{v}_0)=\text{const}$, since L_S const $=0$. From the physical point of view, this trivial solution corresponds to a change in the normalization of $\mathfrak{D}_{\rm eq}(S,\Sigma)$.

7) It is clear that we can add to the right-hand side of (188) any terms that do not depend on **v**, since their contribution vanishes.

Subtracting from $\chi(0,\mathbf{v}_0)$ an appropriate constant, we can achieve fulfillment of the relation

$$\int\Phi_0(v_0)\,\chi(0,\ \mathbf{v}_0)\,d\mathbf{v}_0=0.\qquad(291)$$

Note that this property is also conserved,

$$\int\Phi_0(v_0)\,\chi(t,\ \mathbf{v}_0)\,d\mathbf{v}_0=0,\qquad(292)$$

since

$$\int\Phi_0(v_0)\,L_Sg(\mathbf{v}_0)\,d\mathbf{v}_0=0.$$

For this reason, we concentrate on the functions (291), which are orthogonal to unity:

$$(1,\ \chi)=0.\qquad(293)$$

To obtain the first approximation for $\chi(t,\mathbf{v})$, we ignore in Eq. (290) the correction term containing $Q_0(t)$, and we find

$$\chi(t,\ \mathbf{v}_0)=\exp[tna^2w(a)\,L_S]\,\chi(0,\ \mathbf{v}_0).\qquad(294)$$

Since the spectrum of the operator $na^2w(a)L_S$ in the space of the functions (293) is negative and separated from zero by a gap of order t_0^{-1}, the function (294) decreases exponentially for $t\gg t_0$.

Thus, the given approximation can be represented by

$$\chi(t,\ \mathbf{v}_0)=\delta(t)\int_0^\infty\exp[tna^2w(a)\,L_S]\,dt\chi(0,\ \mathbf{v})$$
$$=-\delta(t)\,(na^2w(a))^{-1}L_S^{-1}\chi(0,\ \mathbf{v}_0).$$

Substituting it in the correction term of the right-hand side of (290), we obtain the equation

$$\frac{\partial\chi(t,\ \mathbf{v}_0)}{\partial t}=na^2w(a)\,L_S\chi(t,\ \mathbf{v}_0)$$
$$-w(a)\,(na^2)^{-1}Q_0(t)\,L_S^{-1}\chi(0,\ \mathbf{v}_0),$$

from which it follows that

$$\chi(t,\ \mathbf{v}_0)=\exp[tna^2w(a)\,L_S]\,\chi(0,\ \mathbf{v}_0)$$
$$-w(a)\,(na^2)^{-1}\int_0^t\exp[na^2w(a)\,L_S(t-\tau)]$$
$$\times\,Q_0(\tau)\,d\tau L_S^{-1}\chi(0,\ \mathbf{v}_0),$$

which leads to the following form of the correction to the rapidly decaying term:

$$\left.\begin{aligned}
&\chi_c(t,\ \mathbf{v}_0)=(na^2)^{-2}L_S^{-1}Q_0(t)\,L_S^{-1}\chi(0,\ \mathbf{v}_0);\\
&\chi(t,\ \mathbf{v}_0)=\chi_c(t,\ \mathbf{v}),\\
&\text{when }t\gg t_0.
\end{aligned}\right\}\qquad(295)$$

Equation (290) now gives

$$\chi_c(t,\ \mathbf{v}_0)=\frac{1}{(2\pi)^3n}\int_0^{k_{\max}}k^2\,dk\sum_{(1\leqslant j\leqslant 5)}\exp[-(\omega_j(k)+\omega_0(k))\,t]$$
$$\times\int d\mathbf{e}\psi_j^{(L)}(\mathbf{e},\ \mathbf{v}_0)\int d\mathbf{v}_0'\Phi_0(v_0')\,\psi_j^{(R)}(\mathbf{e},\ \mathbf{v}_0')\,\chi(0,\ \mathbf{v}_0').\qquad(296)$$

Here, with allowance for (245), the asymptotic values of the integrals

$$\int_0^{k_{\max}}\exp[-(\nu+D_0)\,k^2t]\,k^2\,dk,\qquad\int_0^{k_{\max}}\exp[-(D_T+D_0)\,k^2t]\,k^2\,dk$$

for large $t\gg t_0$ are given by

$$\frac{\sqrt{\pi}}{4\,[(\nu+D_0)\,t]^{3/2}},\qquad\frac{\sqrt{\pi}}{4\,[(D_T+D_0)\,t]^{3/2}}.$$

We note further that (288) enables us to show that

$$\int de\psi_4^{(L)}(e,\ v_0)\,\psi_4^{(R)}(e,\ v_0')=\int de\psi_5^{(L)}(e,\ v_0)\,\psi_5^{(R)}(e,\ v_0').$$

Therefore, we can combine the corresponding exponentials containing t,

$$\exp[-(\omega_4(k)+\omega_0(k))\,t]+\exp[-(\omega_5(k)+\omega_0(k))\,t]$$
$$=\exp[-(\Gamma_S/2+D_0)\,k^2t]\,[\exp(-ickt)+\exp(ickt)],$$

which leads to the integral

$$\int_{-k_{\max}}^{k_{\max}}\exp[-(\Gamma_S/2+D_0)\,k^2t]\exp(ickt)\,k^2\,dk,$$

whose asymptotic behavior for large t is

$$\frac{\sqrt{\pi}}{2\,(\xi t)^{3/2}}\exp(-c^2t/4\xi),$$

where

$$\xi=\Gamma_S/2+D_0.$$

Since the given integral decreases exponentially, we see that the acoustic modes do not contribute to the considered "hydrodynamic tail" of the asymptotic behavior, so that they can be omitted in the expression (296). There remain therefore the two viscosity modes and the one thermal mode.

Noting that

$$\int e_{j,\alpha}e_{j,\beta}\,de=\frac{4\pi}{3}\,\delta_{\alpha,\beta},\quad j=1,\ 2;\quad \alpha,\ \beta=x,\ y,\ z,$$

we can readily integrate with respect to e, and we arrive at

$$\chi_c(t,\ v)=\left(\frac{t_0}{t}\right)^{3/2}\Big\{\frac{1}{12n}\{\pi(\nu+D_0)\,t_0\}^{-3/2}\frac{m^2}{M\theta}$$
$$\times\int(\mathbf{v}\cdot\mathbf{v}')\,\chi(0,\ v')\,\Phi_0(v')\,dv'+\frac{1}{8n}\frac{k_B}{C_p}\{\pi(D_T+D_0)\,t_0\}^{-3/2}$$
$$\times\frac{mv^2-3\theta}{2\theta}\int\frac{mv'^2-3\theta}{2\theta}\chi(0,\ v')\,\Phi_0(v')\,dv'\Big\},\quad t\gg t_0. \qquad (297)$$

This asymptotic expression can be used to obtain the slowly decaying part of the equilibrium time correlation function.

Let us consider, for example, $\chi(0,v)=v_x$. With this choice, (297) leads to the expression

$$\langle v_x(t)\,v_x(0)\rangle_{\mathrm{eq}}=\int v_x\chi_c(t,\ v)\,\Phi_0(v)\,dv$$
$$=\left(\frac{t_0}{t}\right)^{3/2}\frac{m^2}{12nM\theta}\{\pi(\nu+D_0)\,t_0\}^{-3/2}\left(\int v_x^2\Phi_0(v)\,dv\right)^2$$
$$=\left(\frac{t_0}{t}\right)^{3/2}\frac{m}{12nM}\{\pi(\nu+D_0)\,t_0\}^{-3/2}\langle v_x^2\rangle_{\mathrm{eq}}. \qquad (298)$$

We consider the case when S is a particle probe for the system Σ, and the hydrodynamic part $U_k(t;1)$ is calculated using the Enskog equation for a gas of hard spheres of moderate density. Then ν in (298) must be replaced by ν_E. Since D_0 is the Enskog diffusion coefficient, we here obtain the expressions derived by Dorfman and Cohen.[13] On the other hand, if we replace D_0 by the "total" diffusion coefficient, Eq. (298) leads to the well-known result of the theory of interacting modes.

We now make some remarks concerning Eq. (232) for $l\neq 0$, in which it is necessary to substitute the expression for $Q_l(t)$ given by (289). Using the Laplace transform method, we can write it in the form

$$(z-na^2w(a)\,L_S)\,\tilde{\chi}_l(z,\ v_0)=-il v_0\chi_l(z,\ v_0)$$
$$+w^2(a)\,\tilde{Q}_l(z)\,\tilde{\chi}_l(z,\ v_0)+\chi_l(0,\ v_0), \qquad (299)$$

where

$$\tilde{\chi}_l(z,\ v_0)=\int_0^\infty\exp(-zt)\,\chi_l(t,\ v_0)\,dt; \qquad (300)$$

$$\tilde{Q}_l(z)\,g(v_0)=\frac{na^4}{(2\pi)^3}\int_{|k|<k_{\max}}dk$$
$$\times\sum_{(1\leq j\leq 5)}\frac{1}{\omega_j(k)+\omega_0(\mathbf{k}+\mathbf{l})+z}L_S\psi_j^{(L)}(e,\ v_0)$$
$$\times\int dv_0'\Phi_0(v_0)\,\psi_j^{(R)}(e,\ v_0')\,L_Sg(v_0').$$

To analyze the diffusion process, we consider the case when

$$\chi_l(0,\ v_0)=\rho_l(0) \qquad (301)$$

does not depend on v_0.

Suppose

$$\tilde{\chi}_l(z,\ v_0)=\tilde{\rho}_l(z)+\phi_l(z,\ v_0), \qquad (302)$$

where

$$\left.\begin{aligned}&\tilde{\rho}_l(z)=\int\Phi_0(z)\,\tilde{\chi}_l(z,\ v_0)\,dv_0;\\&\int\Phi_0(v_0)\,\phi_l(z,\ v_0)\,dv_0=0.\end{aligned}\right\} \qquad (303)$$

Then (299) gives

$$z\tilde{\rho}_l(z)=-il\int v_0\Phi_0(v_0)\,\phi(z,\ v_0)\,dv_0+\rho_l(0) \qquad (304)$$

and

$$(z-na^2w(a)\,L_S)\,\phi_l(z,\ v_0)=-ilv_0\tilde{\rho}_l(z)+w^2(a)\,\tilde{Q}_l(z)\,\phi_l(z,\ v_0)$$
$$-il\left(v_0\phi_l(z,\ v_0)-\int v_0'\phi_l(z,\ v_0')\,\Phi_0(v_0')\,dv_0'\right).$$

Since l by hypothesis must be fairly small, $l\ll l_0$, we need retain in $\varphi(z,v_0)$ only the terms proportional to l. In this case, the equation is written in the form

$$(z-na^2w(a)\,L_S)\,\phi_l(z,\ v_0)=-ilv_0\tilde{\rho}_l(z)+w^2(a)\,\tilde{Q}_l(z)\,\phi_l(z,\ v_0).$$

Ignoring further the correction term with $\tilde{Q}_l$, we obtain in the first approximation $\phi_l(z,v_0)=-i(z-na^2w(a)L_S)^{-1}lv_0\,\tilde{\rho}_l(z)$. Substituting this formula in the correction term, we arrive at

$$\phi_l(z,\ v_0)=-i\,(z-na^2w(a)\,L_S)^{-1}lv_0\tilde{\rho}_l(z)+$$
$$+w^2(a)\,(z-na^2w(a)\,L_S)^{-1}\tilde{Q}_l(z)\,(z-na^2w(a)\,L_S)^{-1}(-ilv_0)\,\tilde{\rho}_l(z).$$

We now recall once more that when L_S acts on functions $g(v_0)$ orthogonal to unity it has only a negative spectrum, and to investigate the behavior of the correlation functions at large times we are interested in the region $z\ll t_0^{-1}$. In this case, we can ignore z in the term $(z-na^2w(a)L_S)^{-1}$ and our approximation takes the form

$$\phi_l(z,\ v_0)=i\,(na^2w(a))^{-1}L_S^{-1}lv_0\tilde{\rho}_l(z)-i\,(na^2)^{-2}L_S^{-1}\tilde{Q}_l(z)\,L_S^{-1}lv_0\tilde{\rho}_l(z).$$

Then (304) enables us to conclude that

$$z\tilde{\rho}_l(z)=-l^2D(l,\ z)\,\tilde{\rho}_l(z)+\rho_l(0), \qquad (305)$$

where

$$D(l,\ z)=D+\Delta D(l,\ z); \qquad (306)$$

D is a renormalized diffusion coefficient:

$$D=D_0+D_1; \qquad (307)$$

$$D_0 = -(na^2 w(a))^{-1} \int \Phi_0(v) v_x L_S^{-1} v_x \, d\mathbf{v};$$

$$D_1 = -\frac{1}{(2\pi)^3 n} \int_{k<k_{\max}} d\mathbf{k} \sum_{(1 \leqslant j \leqslant 5)} \frac{\int \Phi_0(v) v_x \psi_j^{(L)}(\mathbf{e}, \mathbf{v}) \, d\mathbf{v}}{\omega_j(k) + \omega_0(k)} \times \int \Phi_0(v) \psi_j^{(R)}(\mathbf{e}, \mathbf{v}) v_x \, d\mathbf{v};$$

$$\Delta D(l, z) = \frac{1}{(2\pi^3) n} \int_{k<k_{\max}} dk \sum \frac{z + \omega_0(\mathbf{k} + \mathbf{l}) - \omega_0(\mathbf{k})}{(z + \omega_j(k) + \omega_0(\mathbf{k} + \mathbf{l}))(\omega_j(k) + \omega_0(k))} \times \int \Phi_0(v) (\hat{l}, \mathbf{v}) \psi_j^{(L)}(\mathbf{e}, \mathbf{v}) \, d\mathbf{v} \int \Phi_0(v') \psi_j^{(R)}(\mathbf{e}, \mathbf{v}') (\hat{l}, \mathbf{v}') \, d\mathbf{v}'; \quad (308)$$

$\hat{l} = \mathbf{l}/l$ is a unit vector. Calculation of the additional term D_1 due to the interaction of the hydrodynamic modes shows that it is small, quadratic in the density.

Nevertheless, it should be emphasized that D_1 contains the integral

$$\int_{k<k_{\max}} k^2 \, dk / [(\nu + D_0) k^2] = k_{\max} / (\nu + D_0),$$

which is proportional to $k_{\max}$. Since $k_{\max}$ is determined only to within a numerical factor of order unity, we see that the actual value of D_1 must also depend on the nonhydrodynamic part of our operators.

Using (288), we obtain from (308)

$$\left.\begin{aligned} &\int \Phi_0(v) (\hat{l}\mathbf{v}) \psi_j^{(L)}(\mathbf{e}, \mathbf{v}) \, d\mathbf{v} \int \Phi_0(v') (\hat{l}, \mathbf{v}) \psi_j^{(R)}(\mathbf{e}, \mathbf{v}) \, d\mathbf{v} \\ &\qquad = \frac{m}{M} (\hat{l}, \mathbf{e}_j)^2, \quad j = 1, 2; \\ &\int \Phi_0(v) \psi_{\binom{4}{5}}^{(L)}(\mathbf{e}, \mathbf{v}) (\hat{l}, \mathbf{v}) \, d\mathbf{v} \int \Phi_0(v') (\hat{l}, \mathbf{v}) \psi_{\binom{4}{5}}^{(R)}(\mathbf{e}, \mathbf{v}) \, d\mathbf{v} \\ &\qquad = \frac{1}{2} \frac{m}{M} (\hat{l}, \mathbf{e})^2. \end{aligned}\right\} \quad (309)$$

Since $\omega_1 = \omega_2 = \nu k^2$, the two terms of (308) together give

$$\frac{m}{M} \{(\hat{l}, \mathbf{e}_1)^2 + (\hat{l}, \mathbf{e}_2)^2\} = \frac{m}{M} \{1 - (\hat{l}, \mathbf{e})^2\}.$$

Taking into account the symmetry of this expression with respect to the reflection $\mathbf{e} \to -\mathbf{e}$, we write the terms corresponding to the viscosity modes in $\Delta D(l, z)$ in the form

$$\frac{m}{M} (\nu + D)^{-1} \int (1 - (\hat{l}\mathbf{e})^2) \left\{ \int_0^{k_{\max}} dk \frac{z}{z + \nu k^2 + D(k^2 + l^2 + 2kl(\hat{l}\mathbf{e}))} + kl(\mathbf{e}\hat{l}) \left[\frac{1}{z + (\nu + D) k^2 + 2Dkl(\mathbf{e}\hat{l})} - \frac{1}{z + (\nu + D) k^2 - 2Dkl(\mathbf{e}\hat{l})} \right] \right\} d\mathbf{e}$$

$$= \frac{m}{M} (\nu + D)^{-1} \int d\mathbf{e} (1 - (\hat{l}\mathbf{e})^2) \int_0^{k_{\max}} dk \left\{ \frac{z}{z + \nu k^2 + D(k^2 + l^2 + 2kl(\hat{l}\mathbf{e}))} - 4Dk^2 l^2 (\hat{l}\mathbf{e})^2 \frac{1}{\{z + \nu + D(k^2 + l^2 + 2kl(\hat{l}\mathbf{e}))\}\{z + \nu + D(k^2 + l^2 - 2kl(\hat{l}\mathbf{e}))\}^{-1}} \right\}.$$

Using the variables $k = ql$ and $\zeta = z/Dl^2$, we can conclude that for finite ζ the limit of integration with respect to q must be taken with $k_{\max}/l \to \infty$ when $l \to 0$, but, as is readily seen, this integral will diverge.

Exactly the same procedure can be formulated for the acoustic modes, but there the corresponding contribution from the factor l is less and therefore, in the proposed approximation, it can be ignored. It is obvious that the contribution of the thermal mode must be equal to zero. Note that Eq. (305) with (306), (308), and (309) is of the kind of equations considered by de Schepper[14] and can therefore be studied by the methods developed in Ref. 14.

I should like to point out that all the equations obtained in Sec. 4 and based on the initial condition in the form $\mathfrak{D}_0(S, \Sigma) = V\chi_0(S)\mathfrak{D}_{eq}(S, \Sigma)$ could also be derived from the equations established in Sec. 2, in which we used the initial condition[1] in the form

$$\mathscr{D}_0(S, \Sigma) = f_0(S) \mathscr{D}_{eq}(\Sigma); \qquad f_0(S) = \chi_0(S) \Phi_0(v_0).$$

The use of these two approaches reveals the following differences in the procedure for deriving the corresponding equations. First, if Eq. (226) is derived using (2), it must contain on its right-hand side the operator $\overline{T}_k$ instead of the T_k which is there. However, this difference disappears at the stage when we replace T_k and $\overline{T}_k$ by $T_0 = \overline{T}_0$.

A second difference—which remains—is that when (2) is used it is necessary to replace $w(a)$ by its low-density limit, i.e., unity.

Thus, all the results discussed in Sec. 4 can be obtained on the basis of our old scheme proposed and developed in Ref. 1. The main new point in the technical application of this method—and which prompted me to the new investigation—was the introduction of the collision operator in accordance with Ref. 4. It also bears emphasizing that the method developed in the present paper requires an important modification.

For whereas the operator $U(t; 1)$, which refers to the system Σ, can be calculated from any justified kinetic equation, the interaction term Π_{int} was here treated in a very crude approximation. In concrete calculations, we assumed that it is small, and we took into account correctly only the second-order terms.

Suppose we wish to consider the situation when $\Pi_{int} = \Pi_{int}^{(\Phi)}$ with $\Phi(r)$ corresponding to short-range strong repulsive forces. It is clear that such an interaction must lead to some collision operator, though formally our scheme can be used in this situation only under the condition that we replace $\Pi_{int}^{(\Phi)}$ by an *ad hoc* interaction during the collision.

It is also obvious that our scheme requires a certain improvement. This could be achieved by, for example, replacing the simplest approximation $\mathfrak{D}_t(S, \Sigma) = V\chi(S)\mathfrak{D}_{eq}(s, \Sigma)$ by the probability distribution

$$\mathscr{D}_t(S, \Sigma) = V\{\chi_t(S) + \sum_{(1 \leqslant j \leqslant N)} \eta_t(S, j)\} \mathscr{D}_{eq}(S, \Sigma),$$

where $\eta_t(S, j)$ depends on the phases of the particle S and particle j of the system Σ.

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