Statistical approach to analytic extrapolations in strong interactions

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After an introduction to the problems of analytic extrapolation, the basic ideas of the statistical approach to analytic extrapolations and to the representation of data by analytic functions are reviewed. Some applications of the approach in strong interactions are discussed.

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INTRODUCTION

Analyticity of scattering amplitudes and form factors has already been in use in strong-interaction physics for more than 20 years. In the last decade, great attention has been devoted to the theoretical justification of these methods. One can now say that in the case of a single complex variable the situation is clear; Cutkosky, Deo, and Ciulli have found conformal mappings which are optimal for the expansion of amplitudes in convergent series of polynomials; in addition, they have studied and solved the problem of optimizing weighted dispersion relations.

But these already classic problems do not correspond to the statistical nature of experimental data. In 1968, Cutkosky proposed a new method for representing statistical experimental data by analytic functions. This method was later developed by Cutkosky himself, Pietarinen, and others. These statistical methods are frequently used in phase-shift and amplitude analyses to determine the basic parameters of the strong interactions at low energies (coupling constants, scattering lengths, etc.).

Good and comparatively exhaustive reviews of the classical problems of analytic extrapolations (polynomial expansions and weighted dispersion relations) have recently been published.[1-7]

The aim of the present paper is to review the basic ideas of the statistical approach to the representation of experimental data by analytic functions and to briefly recall some of the applications of this approach. The main attention will be concentrated on the principles; the original literature must be consulted for the technical details. In addition, simple examples illustrating the main ideas will frequently be given. They will be given preference to a striving for mathematical rigor.

First, we briefly recall the problems of analytic extrapolations, optimal conformal mappings, expansions in power series, and weighted dispersion relations. Many of these results also find important use in the statistical approach, which will be discussed later. The most important applications of this approach are phase-shift and amplitude analyses of hadron–hadron collisions. The analyticity of the scattering amplitudes is then tested and its singularities are determined. The practical applications include the modern methods for testing microscopic causality, the determination of the parameters of πN and $\pi\pi$ resonances, which are regarded as poles on the unphysical plane, the determination of scattering lengths, etc.; the results of investigations by the Bratislava group are also given.

The review can in no way be regarded as exhaustive, and I therefore apologize to authors whose work does not here receive due attention.

1. PROBLEMS OF ANALYTIC EXTRAPOLATIONS. POLYNOMIAL EXPANSION

Stability Problems. In this paper, we shall be concerned with extrapolations in simply connected domains. Such domains can always be mapped conformally onto the unit disk. In practice, the analyticity domain is usually the entire complex plane with one or two cuts along the real axis, and the conformal mapping onto the unit disk is simple. We denote the original analyticity domain and its boundary by $\mathfrak{D}$ and $\mathfrak{B}$, and the unit disk and unit circle by D and B.

The most important problem of analytic extrapolations concerns the stability of the result against small variations of the original data. From this point of view, extrapolation problems can be classified[4,7] according as we extrapolate a function to the boundary of the analyticity domain or to points within the domain.

We illustrate the stability problem and methods of solving it by the following simple examples.

Example 1. Suppose that the function $f(z)$ is analytic in the unit disk D. Consider extrapolation from data on the interval (a,b) to the point c (Fig. 1). Let $f_{\exp}(z)$ be the experimental data, $f_1(z)$ a function analytic in the unit disk, and suppose that

$$|f_1(z) - f_{\exp}(z)| \leqslant \varepsilon, \quad z \in (a, b),$$

where ε is comparable with the experimental errors. The function $f_1(z)$ gives a good description of the data in

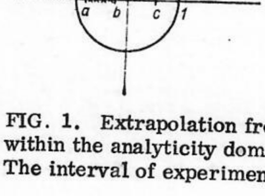

FIG. 1. Extrapolation from the interval (a, b) to the point c within the analyticity domain and to the point d on the boundary. The interval of experimental data is shown by the crosses.

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the interval (a,b), and we can therefore take $f_1(c)$ as a solution of our problem. What error should we expect?

It is readily seen that

$$f_2(z) = f_1(z) + \varepsilon \exp[K(z-b)]$$

also gives a good description (for any $K>0$) of the data in the interval (a,b). At the point c the difference $|f_2(c) - f_1(c)|$ takes an arbitrarily large value (it is sufficient to take K large). The difference between $f_2(z)$ and $f_1(z)$ is greatest at the point $z=1$. Therefore, requiring that the extrapolation from (a,b) to the point c be stable, we must restrict the class of considered analytic functions. A simple solution of the problem is, for example, the condition $|f(z)|<M$ (where M is some given value), where the inequality must hold in the whole of D.

We can add the following comment to this example. The constant M in the condition $|f(z)|<M$ is as a rule determined arbitrarily; we rely more on *a priori* intuition than on concrete information; since $|f(z)|$ attains its maximum on the boundary, it is sufficient to impose a restriction on $|f(z)|$ only on the boundary; stability of the extrapolation can be achieved by imposing bounds on the derivative of $f(z)$, for example, $|f'(z)|<M'$, etc. (this is readily seen).

Example 2. We consider the situation depicted in Fig. 1 and attempt to extrapolate to the point d on the boundary. Suppose again that $f_1(z)$ has analytic properties and is in agreement with the data within the experimental errors. Then

$$f_2(z) = f_1(z) + \varepsilon/(1+\varepsilon+iz) \quad (1)$$

gives a good description of the data, is bounded by the same constant on the boundary, but $f_2 - f_1 = 1$ at the point d. Except in the neighborhood of d, the functions f_2 and f_1 differ very little from one another. The difference is due to the pole at the point $z=(1+\varepsilon)i$, which lies outside the analyticity domain and near the boundary. The pole is manifested as fluctuations of $f_2(z)$ in the neighborhood of d.

Note. In the case of extrapolation to the boundary, it is impossible without additional conditions to reproduce fluctuations that arise from singularities with small residues outside the analyticity domain near the boundary. Such fluctuations are localized in a small part of the boundary. A stable extrapolation can be achieved in this case if we are interested in not the value of the function at a definite point on the boundary, but the mean value along a definite arc on the boundary.

Differentiating both sides of Eq. (1), we obtain

$$f_2'(z) = f_1'(z) - i\varepsilon/(1+\varepsilon+iz)^2.$$

At $z=i$, we have $|f_2' - f_1'| = 1/\varepsilon$. It can be seen from this that bounds on the derivative of the function (or higher derivatives) can stabilize extrapolation to the boundary.

Example 3. Suppose that $F(z) = \varphi(z) + g/(z-z_0)$, where $\varphi(z)$ is analytic in the unit disk D, the point z_0 lies in D, and Γ is part of the unit circle (Fig. 2). In accordance with the generalized Weierstrass theorem[8] there exists a sequence of polynomials $\{P_N(z)\}$ that

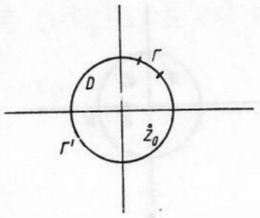

FIG. 2. Approximation of functions with a pole by an analytic function.

converges uniformly on Γ to $F(z)$. However, this sequence cannot be bounded on the complete boundary and the series $\max|P_N(z)|$ for $z \in \Gamma'$ must diverge.

Note. If we approximate a nonanalytic function on part of the boundary Γ very accurately by an analytic function, we do not achieve an accurate result on the remaining part of the boundary: the approximant "blows up" on Γ'.

Nevanlinna's Principle. The intuitive ideas about the instability of extrapolations found their expression in Nevanlinna's principle. Suppose we attempt to extrapolate from the arc Γ to the complete analyticity domain D (Fig. 3). Let $F(z) = f_1(z) - f_2(z)$ be the difference of two functions that give a good description of the data on Γ. Then $|F(z)| < \varepsilon$ for all $z \in \Gamma$. Suppose also that on the basis of *a priori* information (or intuition) $|F(z)| < M$ for $z \in \Gamma'$. Let us find out what is $|F(z)| = |f_1(z) - f_2(z)|$ in D. For this, we must, in the first place, construct a harmonic measure $\omega(z)$, which is a real harmonic function in D and on the boundary satisfies $\omega(z)=0$, $z\in\Gamma$ and $\omega(z)=1$, $z\in\Gamma'$. The function $\omega(z)$ is defined uniquely by these conditions. If $F(z)$ has no zeros in D, then $\varphi(z) = \ln|F(z)| - (1-\omega(z))\ln\varepsilon - \omega(z)\ln M$ and $\varphi(z)$ is a harmonic function in D. It is also readily seen that $\varphi(z) \leqslant 0$ on Γ and Γ'. Since a harmonic function attains its maximum on the boundary and $\varphi(z) \leqslant 0$ in D,

$$|F(z)| = |f_1(z) - f_2(z)| \leqslant M^{\omega(z)}\varepsilon^{1-\omega(z)}. \quad (2)$$

We obtain the same result if $F(z)$ has zeros in D.

The extrapolation is stable for all points within the analyticity domain but deteriorates strongly as the arc Γ' is approached (the factor $1-\omega$ tends to zero).

Extrapolation by a Polynomial Expansion. Optimal Conformal Mappings. In the first practical applications of the extrapolation procedure, polynomial expansions of amplitudes were used. However, it soon became clear that the convergence of expansions in powers can be accelerated by a suitable conformal mapping.[9-11]

The optimization idea is based on the following considerations (Fig. 4). Let $F(t)$ be a function which is analytic in the complex t plane with a cut (b,∞). Sup-

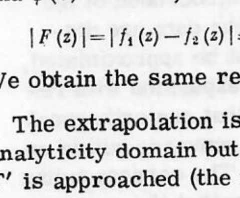

FIG. 3. Extrapolation from the arc Γ to the boundary.

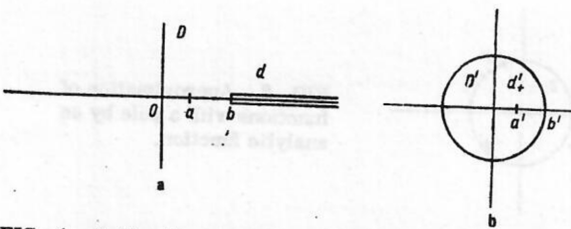

FIG. 4. Optimal conformal mapping for problem in which the data are given in only a small neighborhood of the origin.

pose also (we are considering an idealization) we have information on $F(t)$ in a small neighborhood of the point $t=0$. Then the function can be calculated at the point $t=a$ by means of the Taylor expansion

$$F(a) \approx F_N(a) \equiv \sum_{n=0}^{N} c_n a^n. \quad (3)$$

It follows from Cauchy's theorem that $|c_n| < Mb^{-n}$, where M is an upper bound of $|F(z)|$ on the boundary and b is the distance to the nearest singularity. For the remainder in (3), we obtain the estimate $|F(a) - F_N(a)| \leq K(a/b)^{N+1}$.

The accuracy of the extrapolation from the neighborhood of the origin to the point a is thus determined by the ratio $(a/b)^{N+1}$, where b is the distance from the origin to the nearest singularity. The Taylor series (3) will converge only for points within a circle of radius b. Ciulli and Fischer showed that a mapping of the cut plane onto the unit disk minimizes the partial derivative of a/b and in this sense is optimal. Moreover, after the conformal mapping the Taylor series will converge for all points within the analyticity domain (cf. d and d' in Figs. 4a and 4b).

An idealization in this case consists of the assumption that the data lie in a small neighborhood of the origin. In more realistic cases, the data are distributed along an interval and must be approximated, not by a Taylor series, but by an expansion with respect to a system of polynomials that are orthogonal on this interval. Such expansions converge within an ellipse, and not a circle (Fig. 5). The corresponding mapping is optimal[12,13] in the sense that the expansion with respect to the orthogonal polynomials converges at every point within the analyticity domain and the rate of convergence at every point is asymptotically maximal.

The optimal mappings of Cutkosky and Deo[12] and Ciulli[13] completed the first stage in the application of the theory of analytic extrapolations in strong interactions.

A problem of expansion in polynomials is to estimate

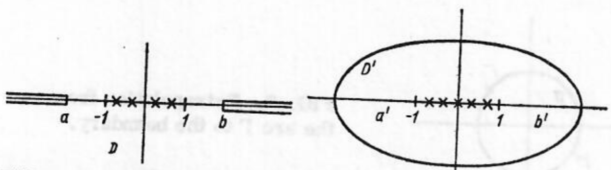

FIG. 5. Optimal mapping of plane with two cuts onto an ellipse.

the degree of the polynomials that must be used in a concrete situation. If the degree is too low, the remainder of the series may be too large; if the degree is too high, the polynomial may reproduce the noise due to the errors of the data. It turns out that this problem can be resolved consistently only by applying a statistical approach to the representation of data by analytic functions. Optimized methods of conformal mappings are frequently used with success for practical calculations as well.[1,2,4-6]

2. METHOD OF WEIGHT FUNCTIONS (WEIGHTED DISPERSION RELATIONS)

Formulation of the Problem. The method of weighted dispersion relations is essentially a generalization of various modifications used earlier.[14,15] However, it was only at the end of the sixties that it proved possible to formulate the problem in a sufficiently general form[16-20] and later solve it completely.[21,1,2] This problem was formulated independently by groups of theoreticians at Bucharest and Bratislava. In the present paper, we describe the method of the Bratislava group, taking as a basis Ref. 16 and the later Refs. 18–20.

For simplicity, we assume that the function $f(z)$ is analytic in the unit disk D and that experimental data on it are distributed along the arc Γ (see Fig. 3). Our task is to estimate the value of the function $f(z)$ at a point $z=a$ inside D. We first write down Cauchy's theorem:

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma+\Gamma'} \frac{f(z)}{z-a} dz. \quad (4)$$

In this form, it cannot be effectively exploited, since the value of $f(z)$ on Γ' is unknown. Therefore, we require a method to suppress the contribution from Γ' to the integral (4). Let $h(z)$ be a function analytic in the unit disk and such that

$$0 = \frac{1}{2\pi i} \int_{\Gamma+\Gamma'} f(z)\, h(z)\, dz. \quad (5)$$

From Eqs. (4) and (5),

$$f(a) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \left[\frac{1}{z-a} - h(z)\right] dz + \frac{1}{2\pi i} \int_{\Gamma'} f(z) \left[\frac{1}{z-a} - h(z)\right] dz. \quad (6)$$

If $h(z)$ is chosen such that $|(z-a)^{-1} - h(z)|$ is small on Γ', then the influence of the arc Γ' on the value of $f(a)$ can be suppressed.

This method was considered in Ref. 16 and used to solve a more complicated problem. There, one can also find a calculation of the function $h(z)$ that minimizes a certain upper bound of the analog of the right-hand side of Eq. (6) under the assumption that the error of $f(z)$ on Γ is approximately equal to ε and that $f(z)$ on Γ' is bounded by the constant M.

When written in the form of the relation (6), the problem of choosing the optimal function $h(z)$ reduces to that of approximating $1/(z-a)$ on part of the boundary. It was shown[17-21] that the problem can be reformulated in a more convenient form.

Introducing the function

$$g(z)=1-(z-a)\,h(z), \tag{7}$$

we can rewrite Eq. (6) as

$$f(a)=\frac{1}{2\pi i}\int\limits_B \frac{f(z)\,g(z)}{z-a}\,dz,\quad B=\Gamma+\Gamma'. \tag{8}$$

The function $g(z)$ is analytic in D, $g(a)=1$, and $g(z)$ is small on Γ' (and large on Γ), as follows from (7) and the properties of $h(z)$.

We call $g(z)$ in (8) a *weight function.* An optimal weight function is one that minimizes the error of $f(a)$ for given information on $f(z)$. If Refs. 16 and 19, such an optimal weight function was determined as the minimum of a certain upper bound of the error of $f(a)$. The estimates of the error did not contradict Nevanlinna's principle. However, it was later found that such an approach cannot be regarded as convenient.

Optimal Weight Functions. The problem of finding an optimal weight function was solved very elegantly by Ciulli and Fischer[21] and is formulated as follows.

Suppose that the function $f(z)$ is given on the arc Γ (see Fig. 3) by experimental values $f_{\exp}(z)$ with error $\delta(z)$. Therefore

$$|f(z)-f_{\exp}(z)|<\delta(z). \tag{9}$$

On Γ', only an upper bound is given:

$$|f(z)|<M(z). \tag{10}$$

Equations (9) and (10) can be written together as

$$|f(z)-F_e(z)|<\varepsilon(z),\quad z\in\Gamma+\Gamma', \tag{11}$$

where

$$\varepsilon(z)=\begin{cases}\delta(z), & z\in\Gamma;\\ M(z), & z\in\Gamma';\end{cases}\qquad F_e(z)=\begin{cases}f_{\exp}(z), & z\in\Gamma;\\ 0, & z\in\Gamma'.\end{cases}$$

If $\varepsilon(z)\equiv 1$, the error on the boundary is everywhere the same and intuitively it appears obvious that the most exact representation for $f(a)$ is the expression

$$f(a)=\frac{1}{2\pi i}\int\limits_B \frac{F_e(z)}{z-a}\,dz.$$

The idea of Ciulli and Fischer is to transform the general case (11) into the case with unit error. To this end, one introduces a function $g(z)$ which is analytic in D, has no zeros in D, and satisfies $|g(z)|=\varepsilon(z)$ for $z\in B$. Such a function is uniquely specified by the expression derived below.

If $g(z)$ is known, we construct the function

$$\varphi(z)=f(z)/g(z),\quad \varphi_e(z)=F_e(z)/g(z). \tag{12}$$

Instead of (11), we obtain

$$|\varphi(z)-\varphi_e(z)|<1,\quad z\in B.$$

The function $\varphi_e(z)$ represents the extremal values of $\varphi(z)$, and the error of such a determination is unity. Therefore, the best estimate of $\varphi(a)$ is

$$\varphi(a)=\frac{1}{2\pi i}\int\limits_B \frac{\varphi_e(z)}{z-a}\,dz.$$

If we substitute here $\varphi_e(z)$ from (12), we obtain

$$f(a)=\frac{1}{2\pi i}\,g(a)\int\limits_B \frac{F_e(z)}{g(z)}\,\frac{dz}{z-a}. \tag{13}$$

The function $g(z)$ is given explicitly[22] by

$$g(z)=\exp\left[\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\frac{\exp(i\vartheta)+z}{\exp(i\vartheta)-z}\ln\varepsilon[\exp(i\vartheta)]\,d\vartheta\right]. \tag{14}$$

It was found later that this optimal weight function (we shall call it the *Ciulli-Fischer weight function*) is also very helpful for the statistical approach to extrapolations.

Difficulties in Extrapolation to the Boundary. Extrapolation on the Average. The classic case of extrapolation to the boundary is the determination of the spectral functions of the nucleon form factor from data on the scattering of electrons on nucleons. An analogous problem is however also encountered in other problems. In the past, many practical extrapolations of the nucleon form factor were made[23–25]; the problem was also studied theoretically.[23,26] It was found (this follows from Nevanlinna's principle and the examples given in Sec. 1 as well as otherwise) that extrapolation to the boundary is in principle unstable and, unless restrictions are imposed on the smoothness of the derivatives, only the average values of the results can be regarded as valid. The method of weight functions is especially suitable for extrapolations on the average. Following Ref. 18, we describe the practically employed version of the method of extrapolation on the average. The analyticity domain of the form factor is the cut t plane with cut (t_0,∞) (Fig. 6a). Suppose that the data lie in the interval (t_1,t_2), where $t_2\leqslant 0$. For our purposes, it is expedient to cut the plane along the interval (t_1,t_2) and to map the resulting t plane with two cuts onto an annulus (Fig. 6b) in the complex z plane.

If $f(z)$ is the form factor and $g(z)$ is analytic in the annulus, then on the basis of Cauchy's theorem

$$\int\limits_{\Gamma_1} f(z)\,g(z)\,dz+\int\limits_{\Gamma_2} f(z)\,g(z)\,dz=0. \tag{15}$$

The first integral can be calculated from the experimental values; in the second, the function $g(z)$ can be chosen in such a way as to obtain from it approximately the mean value of the form factor in the neighborhood of a definite point z_0 on Γ_2. It is easy to construct such a function $g(z)$. If the radius of the circle Γ_2 is equal to unity, a weight function suitable for extrapolation to the average value on an arc in the neighborhood of $z_0=\exp(i\vartheta_0)$ is

$$g(z)=(K/iz)\exp\{-A[(z+1/z)/2-\cos\vartheta_0]^2\}. \tag{16}$$

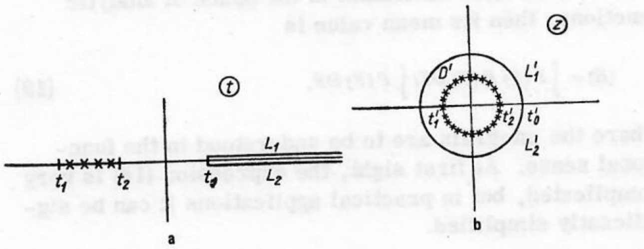

FIG. 6. Mapping of the t plane with cut $(t_0,\ \infty)$ and with additional cut (t_1,t_2) onto an annulus in the complex z plane.

By a suitable choice of K, we select the desired combination of the real and imaginary parts of the form factor; using the constant A, we adjust the length of the interval over which we average. Detailed estimates of the errors of the extrapolation are given in Ref. 18. By an analogous technical formalism, Prešnajder and the present author[19] considered extrapolation from part of the boundary (of a simply connected domain) to the average value on a chosen arc on the remaining part of the boundary. In Ref. 20, weight functions were used to study certain types of modified dispersion relations.

3. STATISTICAL APPROACH TO THE REPRESENTATION OF DATA BY ANALYTIC FUNCTIONS

Basic Ideas. Cutkosky's Work. The basic ideas of the statistical approach to the representation of data by analytic functions were formulated by Cutkosky.[27]

It is obvious from the examples given in Sec. 1, and also from Nevanlinna's principle, that an extrapolation is unstable unless one introduces *a priori* restrictions on the function $f(z)$ in regions where there are no experimental data. In the simplest case, these restrictions can be formulated in the form of an absolute bound of the type $|f(z)| \leq M(z)$, etc. At first sight, this appears completely natural, but a deeper analysis reveals certain difficulties. For example, it is hard to explain why the least upper bound of the function $|f(z)|$ must be precisely $M(z)$ and why one cannot choose $M(z) = (1+10^{-3})$. Moreover, and this is even more important, if we formulate the *a priori* restrictions in the form of rigorous mathematical upper bounds, we thereby create a framework in which it is naturally impossible to include the statistical nature of the experimental values and their errors. Cutkosky[27] therefore proposed the introduction in the space of functions analytic in the considered domain of an *a priori* probability determined by the behavior of the function on the boundary.

Proceeding on the basis of intuition (or information given *a priori* for the given problem) we introduce a "reference function" $F_0(z)$ and non-negative kernel $K(z, z') = K^*(z', z)$; with every function $F(z)$ we associate the (un-normalized) probability

$$P(F) \sim \exp\left[-\int_B \Delta F^*(z) K(z, z') \Delta F(z') |dz||dz'|\right], \quad (17)$$

where $\Delta F(z) = F(z) - F_0(z)$, and we take the integral along the boundary B of the analyticity domain.

If $R(F)$ is some functional in the space of analytic functions, then its mean value is

$$\langle R\rangle = \int P(F) R(F) DF / \int P(F) DF, \quad (18)$$

where the integrals are to be understood in the functional sense. At first sight, the expression (18) is very complicated, but in practical applications it can be significantly simplified.

The nature of the restrictions introduced into the space of analytic probability functions $P(F)$ is determined by the kernel $K(z, z')$. Below, we give some examples that illustrate these restrictions.

In the simplest case

$$K(z, z') = \delta(z - z')/\varepsilon^2(z), \ \varepsilon(z) > 0, \quad (19)$$

the probability is

$$P(F) \sim \exp\left[-\int_B \frac{|F(z) - F_0(z)|^2}{\varepsilon^2(z)} |dz|\right], \quad (20)$$

and, as we shall show later, this expression admits a simple physical interpretation in which F_0 plays a role that corresponds formally to the data and $\varepsilon(z)$ is the analog of the errors.

An important advantage of this approach is that the actual data can also be readily understood as factors that introduce a probability into the space of analytic functions. Both sources of information—*a priori* and *a posteriori*—are formally described by the same statistical formalism.

Experimental Data and Probability in the Space of Analytic Functions. Hypotheses as "Data with Errors". Following Ref. 28, we show how experimental data introduce probability into the space of analytic functions. Then, we discuss the approximation of the experimental values and their errors at discrete points by continuous functions and, finally, we show that Cutkosky's *a priori* probabilities can be obtained if the *a priori* restrictions on the amplitudes are construed as hypothetical experimental values with errors.

Suppose we have measured the amplitude $F(z)$ at the single point z_1 and as a result obtained the number Y_i with error ε_i (the same for the real and imaginary parts). If $F_T(z_i)$ is the actual value of the amplitude at z_i, then in repeated measurements the results Y_i are random variables with a Gaussian distribution

$$P(Y_i/F_T(z_i)) \sim \exp[-|Y_i - F_T(z_i)|^2/2\varepsilon_i^2].$$

If a single measurement is made, and the actual value is not known, it is natural to interpret the result Y_i with error ε_i as a factor which introduces into the space of analytic functions the probability

$$P(F) \sim \exp[-|F(z_i) - Y_i|^2/2\varepsilon_i^2]. \quad (21)$$

If there are m independent measurements at the points z_i with results Y_i and errors $\varepsilon_i (i = 1, 2, \ldots, m)$, then for the probability in the function space we obtain

$$P(F) \sim \exp\left\{-\sum_{i=1}^{m} \frac{1}{2}[|F(z_i) - Y_i|^2/\varepsilon_i^2]\right\}. \quad (22)$$

It is easy to verify the internal consistency of this introduction of $P(F)$. The probability induced by two measurements with results (Y_1, ε_1) and (Y_2, ε_2) obtained at the same point is equal to the probability that we should find from one measurement with the result (Y, ε) if the relations $\varepsilon^{-2} = \varepsilon_1^{-2} + \varepsilon_2^{-2}$, $Y\varepsilon^{-2} = Y_1\varepsilon_1^{-2} + Y_2\varepsilon_2^{-2}$ held. But the two measurements (Y_1, ε_1) and (Y_2, ε_2) are equivalent to the single measurement (Y, ε).

If the data are distributed densely in a certain region, it is expedient to make the substitution

$$\sum_{i=1}^{m} \varepsilon_i^{-2} \,|\, F(z_i) - Y_i \,|^2 \to \int_L \varepsilon^{-2}(z) \,|\, F(z) - Y(z) \,|^2 \,|\, dz \,|, \tag{23}$$

where $Y(z)$ is a smooth interpolation of the data on the arc L, and $\varepsilon(z)$ must in accordance with (23) be chosen such that

$$\rho(z_i)\,\varepsilon_i^{-2} \to \varepsilon^{-2}(z_i), \tag{24}$$

where $\varepsilon(z)$ is a continuous error function; ε_i is the error at the point z_i; $\rho(z_i)$ is the density of the experimental values in the neighborhood of the point z_i.

The relation (24) is also important for a realistic determination of the error in the "corridor" approach, in which we understand the error as a rigorous mathematical constraint of the type $|F(z) - Y(z)| < \varepsilon(z)$. Although the error in this case does not have a statistical nature, it is introduced quantitatively instead of the intuitive estimate. It is also easy to show that $\varepsilon(z)$ has the correct transformation properties under conformal mappings.

By means of the error introduced in this way, the probability (22) can be rewritten in the form

$$P(F) \sim \exp\left\{ -\frac{1}{2}\left[\int_L \varepsilon^{-2}(z) \,|\, F(z) - Y(z) \,|^2 \,|\, dz \,| \right] \right\}. \tag{25}$$

If we have experimental values on the complete boundary, then we use (25) directly. If part of the boundary is not covered by measurements, then on this boundary we must keep the function "under control". Cutkosky's proposal[20] leads exactly to a probability in the form (25) if the reference function $F_0(z)$ is identified with the hypothetical data and $\varepsilon(z)$ with the hypothetical errors ε_i. Such an interpretation of $F_0(z)$ and $\varepsilon(z)$ shows that in the statistical approach the *a priori* probabilities can be understood as hypothetical (not realized) measurements with errors.

After the experimental values have been augmented in this manner by the *a priori* probabilities, we arrive at the final expression for the probability $P(F)$ in the space of analytic functions:

$$P(F) \sim \exp\left\{ -\frac{1}{2}\left[\sum_{i=1}^{m} \varepsilon_i^{-2} \,|\, F(z_i) - Y_i \,|^2 + \frac{1}{L} \int_B |\, F(z) - Y(z) \,|^2 \varepsilon^{-2}(z) \,|\, dz \,| \right] \right\}, \tag{26}$$

where the first term is a consequence of the actual measurements at the points z_i within the analyticity domain and the second derives from the actual measurements on part of the boundary and from the *a priori* hypotheses on the remainder of the boundary.

For convenience in the following exposition, we "renormalize" in (26) the error by the substitution $\varepsilon^{-2}(z) \to \varepsilon^{-2}(z)/L$, where L is the length of the boundary. In some cases, the first term in (26) can be replaced by an integral, while in others there are data only on the boundary; then the first term on the right-hand side of (26) disappears. It is now completely natural to introduce the considered function space in such a way that it contains all functions analytic in D and such that

$$\|F\|^2 \equiv \frac{1}{L} \int_B |F(z)|^2 \varepsilon^{-2}(z) \,|\, dz \,| < \infty. \tag{27}$$

If the first term on the right-hand side of (26) is absent (which we assume in what follows), then in such a space one introduces the scalar product

$$(F,\, G) = \frac{1}{L} \int_B \varepsilon^{-2}(z)\, F^*(z)\, G(z) \,|\, dz \,|. \tag{28}$$

Integration in the function space can be significantly simplified by introducing a basis which is orthonormalized with respect to the scalar product (28).

The Most Probable Function. Extrapolation from Part of the Boundary to a Point Inside the Analyticity Domain. Estimate of the Error. Mapping conformally the analyticity domain onto the unit disk D, we formulate our problem as follows.[28] We consider the space fo functions that are analytic in D and such that $\|F\|^2 < \infty$, where $\|F\|^2$ is given by (27). In this space of functions there is the probability

$$\left.\begin{aligned} &P(F) \sim \exp\left[-\chi^2(F)/2\right]; \\ &\chi^2(F) = \frac{1}{2\pi} \int_B |\, F(z) - Y(z) \,|^2 \varepsilon^{-2}(z) \,|\, dz \,|. \end{aligned}\right\} \tag{29}$$

We now find the "most probable function"[28] and we determine its value at the point $z = 0$ and the corresponding error.

The problem can be considerably simplified by introducing a suitable weight function. Suppose $g(z)$ is analytic in the unit disk D, has no zeros in D, and on the boundary satisfies the condition $|g(z)| = \varepsilon(z)$. This function is identical with the Ciulli-Fischer function (14), which is known from the weighted dispersion relations. We now define

$$f(z) = F(z)\, g^{-1}(z); \quad y(z) = Y(z)\, g^{-1}(z). \tag{30}$$

The function $f(z)$ is analytic in D, and it can be written in the form of the Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in D$; for $y(z)$, we use the Laurent expansion $y(z) = \sum_{n=-\infty}^{\infty} C_n z^n$. If these expansions are substituted by means of (30) in (29), we obtain

$$P(f) \sim \exp\left\{ -\frac{1}{2} \sum_{n=0}^{\infty} |\, a_n - C_n \,|^2 \right\}, \tag{31}$$

where we have omitted the multiplicative factor $\exp \sum_{-\infty}^{-1} |C_n|^2$. It can be seen from (31) that the coefficients a_n are independent random variables with a Gaussian distribution having the mean value and variance

$$\langle a_n \rangle = C_n; \quad \langle (\Delta a_n)^2 \rangle = 1. \tag{32}$$

Therefore, the "most probable function", i.e., the function that in the space of analytic functions minimizes $\chi^2(F)$, is

$$\bar{F}(z) = \bar{f}(z)\, g(z), \tag{33}$$

where

$$\bar{f}(z) = \sum_{n=0}^{\infty} c_n z^n = \frac{1}{2\pi i} \int_B \frac{y(z')}{z' - z}\, dz'. \tag{34}$$

If $\bar{F}(z)$ is known, we can readily find the mean values and the variances of the linear functionals.

For extrapolation of the amplitude to the origin, the functional $R(F)=F(0)$ is suitable. It can be seen from Eqs. (33) and (34) that

$$\langle F(0)\rangle = C_0 g(0);\ \langle(\Delta F(0))^2\rangle = g^2(0). \tag{35}$$

The treatment of the functional $R(F)=F(z_0)$ for z_0 within the analyticity domain is somewhat more complicated.[28] The final result

$$\left.\begin{aligned}\overline{F}(z_0) &= \overline{f}(z_0)\,g(z_0) = g(z_0)\sum_{n=0}^{\infty} c_n z_0^n;\\ \langle(\Delta F(z_0))^2\rangle &= |g(z_0)|^2/(1-|z_0|^2)\end{aligned}\right\} \tag{36}$$

does not contradict Nevanlinna's principle.[28]

An important simplification of the complete analysis was the assumption that the errors of the real and imaginary parts are equal. A more general formulation of the problem is given below.

Other Possibilities of A Priori Probabilities in the Space of Analytic Functions. Although *a priori* probabilities in the space of analytic functions are very natural and arise because of the need to stabilize the problem, they are to a certain extent arbitrary. As a result, they are introduced in different ways. A very perspicuous representation of the significance of these *a priori* probabilities is given by a mechanical analogy contained in unpublished lectures of Pietarinen (I was acquainted with them through discussions with J. L. Petersen, to whom I am grateful).

Suppose it is necessary to lay a rod through points whose position is known with definite errors, so that the rod need not pass directly through the points. Therefore, we depict the situation as follows. To every point we fix a spring (the rigidity of the spring is inversely proportional to the error in the determination of the position of the given point). At the same time, the rod itself has a certain rigidity (Fig. 7).

If the rod is very rigid, it will reproduce the positions of the individual points only very approximately and inaccurately. If the rigidity is too low, it will reproduce all the fluctuations (inaccuracies) in the determination of the positions of the points. Intuition suggests that the rigidity of the rod must be comparable with that of the springs. This example illustrates well the circumstance that the actual problem of *a priori* probabilities is to stabilize the problem, to avoid reproducing the fluctuations, and to make it possible to obtain a faithful representation of the experimental data. Thus, the qualitatively formulated framework is sufficiently large to permit the introduction of *a priori* probability in several ways.

We now discuss in more detail some simple examples of the introduction of *a priori* probabilities. We write

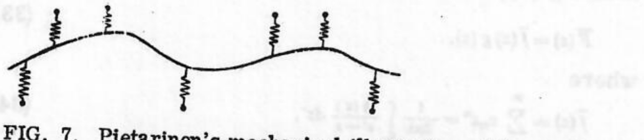

FIG. 7. Pietarinen's mechanical illustration of the problem of *a priori* and experimental probabilities.

the *a priori* probabilities in a form analogous to

$$P(F) \sim \exp[-\chi_c^2(F)/2]; \tag{37}$$

$$\chi_c^2(F) \sim \int_B F^*(z)\,K(z, z')\,F(z')\,|dz|\,|dz'|, \tag{38}$$

and after this we express the probability by means of coefficients of the Taylor expansion of $F(z)$. The symbol χ_c^2 denotes the so-called *Cutkosky chi squared.* The analyticity domain is the unit disk D and the boundary is the unit circle B.

Example 1. Suppose that

$$\chi_c^2(F) = \frac{1}{2\pi}\int_B \frac{|F(z)|^2}{M^2}\,|dz| \tag{39}$$

and M is a constant.

If we set

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \tag{40}$$

we obtain

$$\chi_c^2(F) = \frac{1}{M^2}\sum_{n=0}^{\infty} |a_n|^2. \tag{41a}$$

After substitution in (37) we see that a_n is an independent variable with Gaussian distribution [if $F(z)$ is a real analytic function, then a_n are real numbers].

Example 2. If

$$\chi_c^2(F) = \frac{1}{2\pi}\int_B \frac{|F(z)-F_0(z)|^2}{M^2}\,|dz|,$$

then after expansion of $F(z)$ in a Taylor series and $F_0(z)$ in a Laurent series we obtain

$$\chi_c^2(F) = \frac{1}{M^2}\sum_{n=0}^{\infty}|a_n - c_n|^2 + \frac{1}{M^2}\sum_{n=-\infty}^{-1}|c_n|^2, \tag{41b}$$

where a_n are introduced in (40) and c_n are the coefficients in the Laurent expansion of $F_0(z)$.

Example 3. In both the preceding examples the kernel was $K(z, z') \sim \delta(z - z')$. We now choose a case in which the kernel is "smeared". Suppose $z \equiv \exp(i\varphi)$; $z' \equiv \exp(i\psi)$. We set

$$\left.\begin{aligned}&K(z, z') \equiv K(\exp(i\varphi),\ \exp(i\psi)) = k(\varphi-\psi);\\ &k(\varphi-\psi) = \frac{1}{2\pi}\,\frac{1-b^2}{1-2b\cos(\varphi-\psi)+b^2},\quad 0<b<1.\end{aligned}\right\} \tag{42}$$

The function $k(\varphi-\psi)$ is shown in Fig. 8. It is chosen in such a way that $k(\alpha)\to\delta(\alpha)$ for $b = 1-\varepsilon$, $\varepsilon\to 0$. If we use the expansion

$$(1-b^2)/(1-2b\cos\alpha+b^2) = 1 + 2\sum_{n=1}^{\infty} b^n \cos n\alpha,$$

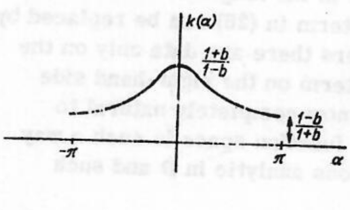

FIG. 8. Form of the function $k(\alpha)$ in Eq. (42).

then from (38) with the kernel (42) we obtain

$$\chi_c^2(F)=\sum_{n=0}^{\infty} b^n (a_n)^2. \tag{43}$$

Since $b<1$, such a χ_c^2 represents weaker restrictions on the coefficients a_n with larger n than the simple χ_c^2 from Eqs. (41).

Example 4. We now consider what *a priori* probabilities we obtain if we introduce derivatives of the function $F(z)$ in χ_c^2. Suppose

$$\chi_c^2(F)=\frac{1}{2\pi}\frac{1}{M^2}\int_B \left|\frac{dF}{dz}\right|^2 |dz|. \tag{44}$$

After substitution of the expansion (40),

$$\chi_c^2(F)=\frac{1}{M^2}\sum_{n=1}^{\infty}(na_n)^2. \tag{45}$$

The examples given show that the most stringent restrictions on the coefficients a_n with large n are obtained if we impose by means of χ_c^2 restrictions on the derivatives of the function $F(z)$ on the boundary [see Eq. (44)]. Restrictions on $|F(z)|$ lead to restrictions on the coefficients a_n with large n weaker than (41), and "smeared" $K(z,z')$ lead to even weaker restrictions.

Pietarinen's Remark Concerning A Priori Probabilities. Pietarinen[29] noted that if we use

$$P(F)=\exp\left[-\frac{1}{2}\frac{1}{2\pi}\int_B |F(z)|^2|dz|\right]=\exp\left[-\frac{1}{2}\sum_{n=0}^{\infty}a_n^2\right], \tag{46}$$

we find that the probability integrated over the complete space of functions satisfying the condition

$$\int |F(z)|^2|dz|<\infty \tag{47}$$

is zero. As an alternative, he suggested defining the space of considered analytic functions by means of the condition

$$\int_B |F'(z)|^2|dz|=\sum_{n=0}^{\infty}(na_n)^2<\infty$$

and using as probability in this space the expression

$$P(F)=\exp(-\chi^2/2);\ \chi^2=\sum_{n=0}^{\infty}(a_n/\Delta_n)^2, \tag{48}$$

where

$$\Delta_k=\lambda^{-1/2}(k+1)^{-3/2-\varepsilon};\ \varepsilon>0.$$

It should be said that λ is the scale parameter which regulates the relative weight of the *a priori* and *a posteriori* probabilities.

No practical difficulties arise even when we take $P(F)$ based on (46) and (47). The point is that in calculating the values of the functionals and their fluctuations we use limiting procedures of the type

$$\langle R\rangle=\lim_{N\to\infty}\frac{\int R(F_N)P(F_N)\prod_{n=1}^{N}da_n}{\int P(F_N)\prod_{n=1}^{N}da_n},$$

where R is a functional and F_N is the sum of the first N terms in the Taylor expansion of $F(z)$. Other reasons for using Gaussian *a priori* probabilities of the type (38) are given in Ref. 30.

Different Errors of the Real and Imaginary Parts of the Amplitude. All the foregoing arguments were based on the assumption that the errors of the real and imaginary parts in the determination of the scattering amplitude are equal. Such an approach is close to reality for the partial-wave amplitudes but does not hold for the forward scattering amplitude, in which the imaginary part is determined by means of the optical theorem and σ_T, whereas the real part is obtained by other, as a rule less accurate methods. It is therefore desirable to have methods that determine the most probable functions and the most important functionals:

$$P(F)\sim\exp(-\chi_c^2/2);$$

$$\chi_c^2=\sum_{i=1}^{N}\left[\frac{(\mathrm{Re}\,F(z_i)-\mathrm{Re}\,y_i)^2}{\varepsilon_{Ri}^2}+\frac{(\mathrm{Im}\,F(z_i)-\mathrm{Im}\,y_i)^2}{\varepsilon_{Ii}^2}\right] + \frac{1}{2\pi}\int_B\left[\frac{(\mathrm{Re}\,F(z)-\mathrm{Re}\,y(z))^2}{\varepsilon_R^2(z)}+\frac{(\mathrm{Im}\,F(z)-\mathrm{Im}\,y(z))^2}{\varepsilon_I^2(z)}\right]|dz|, \tag{49}$$

where y_i are the results of measurements at the points z_i and ε_{Ri} and ε_{Ii} are the errors in the determination of Re y_i and Im y_i.

The second term in (49) corresponds to data on part of the boundary and hypotheses on the part of the boundary on which we have no data. The physical formulation of the problem is the same as for the case with equal errors, but from the technical point of view the situation is now more complicated, since the asymmetry of ReF and ImF in (49) prevents our using a weight function and thereby simplifying the problem. However, in this case too solutions are known. The first solution, corresponding to the case when there are no data within the analyticity domain [the first term on the right-hand side of (49) is absent] is due to Ross.[31] His solution does not have sufficient generality but applies for realistic problems. A more general solution was given by Shih and Sheppard.[32,33] The case with data within the analyticity domain was described by Shih and Nenciu,[32,34] and in a general formulation corresponding to (49) by Prešnajder.[35]

4. AMPLITUDE AND PHASE-SHIFT ANALYSES IN STRONG INTERACTIONS

The most important applications of modern statistical methods of representing data by analytic functions are to amplitude and phase-shift analyses. Actually, one should not speak of "applications," since these methods arose[27] and were developed[29,36-40] precisely in the solution of amplitude and phase-shift analyses. The new methods were first used as accelerated convergence expansions (ACE), which were studied and used by Cutkosky *et al.*[36-38]

An important second application is found in amplitude analyses of meson-nucleon scattering at fixed t or u (Refs. 29 and 39-42) and, finally, in the amplitude analysis of pion-pion scattering.[43,44]

Cutkosky's Accelerated Convergence Expansions.

Phase-shift analyses of πN, KN, or NN scattering can be divided into energy-dependent and energy-independent analyses. In the first case, the partial-wave amplitudes are parametrized as functions of the energy and the free parameters are determined by comparison with all the data simultaneously. In the second case, phase-shift analyses are made at different energies and the solutions at different energies are then "linked" under the assumption that the energy dependences are "smooth". Such "smooth linking" is very often carried out when one uses the analytic properties of the partial-wave amplitudes.

Cutkosky's accelerated convergence expansions are a method for simplifying and accelerating an energy-independent phase-shift analysis at a definite fixed energy. In such an analysis, the amplitude at fixed energy is expanded in Legendre polynomials and the free coefficients are fitted to the data on the effective differential cross sections. One minimizes the expression

$$\sum_{i=1}^{N} [(d\sigma/dz)_i - |F(z_i)|^2] \frac{1}{\varepsilon_i^2} \qquad (50)$$

where the subscript i labels the data at $z_i = \cos\vartheta_i$;

$$F(z) = \sum_{l=0}^{L} A_l P_l(z), \qquad (51)$$

and A_l are the partial-wave amplitudes. However, the expansion (50) is not the most effective representation of the amplitude if one takes into account the analytic properties of $F(z)$.

The point is that $F(z)$ is analytic in the complex z plane with cuts $(-\infty, -z_1)$ and (z_2, ∞), where $z_1, z_2 > 1$. But the expansion on the right-hand side of (51) converges only within the ellipse shown in Fig. 9a; moreover, asymptotically it does not converge most rapidly. Therefore, Cutkosky proposed that, first, the z plane with two cuts should be mapped onto an ellipse in the v plane in such a way that the interval $(-1, 1)$ is again mapped onto $(-1, 1)$. Such a mapping ensures the fastest asymptotic convergence. After

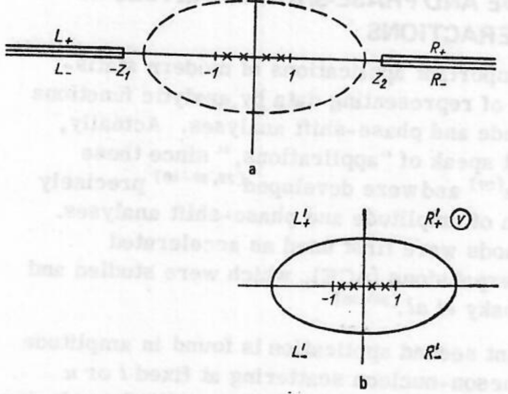

FIG. 9. Mapping of the complex z plane with the cuts $(-\infty, -z_1)$ and (z_2, ∞) onto the interior of the ellipse in the v plane. The dashed ellipse is the domain of convergence of the expansion (51).

this, the amplitude $F(z)$ is written, not in the form (51), but as

$$F(z) = F[v(z)] = \sum_{n=0}^{M} B_n T_n(v), \qquad (52)$$

where $\{T_n(v)\}$ is an orthonormal system of polynomials in the interval $-1 \leqslant v \leqslant 1$. In practical calculations it is sufficient to take 20–50% fewer terms than in the expansion (51) if a good fit is to be obtained.

We obtain the partial waves corresponding to the amplitude (52) by using the inverse relation

$$\sum_{l=0}^{\infty} A_l P_l(z) = \sum_{n=0}^{M} B_n T_n(v(z))$$

and therefore

$$A_l = \sum_{n=0}^{M} B_n \frac{2}{2l+1} \int_{-1}^{1} T_n(v(z)) P_l(z)\, dz. \qquad (53)$$

From this, some interesting properties of the ACE expansions are obvious. At a given energy, all partial waves are represented; there is a correlation between the partial waves: if a sensitive energy dependence is manifested in a determined $B_n = B_n(E)$, then it propagates through the relation (53) to all the partial waves. Although this fact at first appears to contradict our prejudices, it does not contradict physics, in which the dynamics predicts such correlations through analyticity.

The actual problems are only technical in nature. The unitarity condition is true only for the partial waves A_l and not for B_n. Therefore, in a phase-shift analysis one must impose additional conditions to ensure that the A_l obtained by means of (53) do not contradict the restrictions that follow from unitarity. In the practical application of accelerated convergence expansions, the sum of the expression (50) is minimized by a function similar to (41a). This additional term prevents the function $F(z)$ reproducing fluctuations of the data. After this, the partial waves $A_l(E)$ at different energies E_i are linked to make a single solution, usually[36–38] on the basis of the analyticity of $A_l(E)$.

Pietarinen Amplitudes of πN Scattering at Fixed t or u. Adopting Cutkosky's approach[27] to the representation of data by analytic functions, Pietarinen proposed[29] and successfully used[39–42] a very effective and mathematically elegant method for reproducing amplitudes (at fixed t or u) on the basis of experimental data. We shall briefly describe the principle of this method.[1)]

Let $F(s)$ be the scattering amplitude at fixed t, analytic in the complex s plane with the cuts $(-\infty, -s'_0)$ and (s_0, ∞). By the transformation $z = z(s)$, we map this plane onto the unit disk in the z plane.

Information about the amplitude consists of an *a priori* probability and the experimental values. These values make in χ^2 a contribution in the form of the term

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1)Pietarinen's method is very clearly presented in the reviews of Hamilton and Petersen.[45,44] The argumentation in Ref. 44 is succinct and clear. A more detailed description can be found in Pietarinen's original paper Ref. 29.

$$\chi^2_{exp} = \sum_{i=1}^{M} (\mathrm{Re}\, F(z_i) - R_i)^2 \varepsilon_{Ri}^{-2} + \sum_{i=M+1}^{N} (\mathrm{Im}\, F(z_i) - I_i)^2 \varepsilon_{Ii}^{-2}, \quad (54)$$

where z_i, $i=1,2,\ldots,M$, are the points at which $\mathrm{Re}F(z_i)$ was measured with the results R_i and errors ε_{Ri}; z_i, $i=M+1,\ldots,N$, are the points at which $\mathrm{Im}F(z_i)$ was measured with results I_i and errors ε_{Ii}.

The *a priori* "penalty function" [see the text after Eq. (53)] makes in χ^2 the contribution

$$\chi^2_{a\ priori} \equiv \Phi = \lambda \sum_{n=0}^{\infty} (n+1)^3 a_n^2, \quad (55)$$

where a_n are the coefficients in the Taylor expansion of the function $F(z)$. The parameter λ determines the relative contribution of the two terms to χ^2.

Therefore, the coefficients a_n are determined by minimizing the expression

$$\chi^2 = \chi^2_{exp} + \Phi, \quad (56)$$

where we replace $F(z)$ in (54) by the "truncated" Taylor series

$$F(z) = \sum_{n=0}^{K} a_n z^n$$

and we also terminate the series (55) at $n=K$. Thus, χ^2 in (56) is a quadratic function of the coefficients $a_0, \ldots, a_k$. We then obtain the coefficients $a_0, \ldots, a_k$ that minimize (56) by matrix inversion. On the basis of some attempts, the parameter λ can be chosen in such a way that χ^2_{exp} and Φ at the minimum make approximately equal contributions to χ^2.

Using this method, Pietarinen[41] analyzed πN amplitudes at fixed t. The amplitudes obtained at individual energies were used as additional restrictions on the phase shifts, which were determined in an energy-independent analysis (therefore, at the energy E Pietarinen's input was not only $d\sigma/d\Omega$ and the polarizations but also the values of the amplitude at several t_i). Such use of amplitudes with fixed t ensures continuity of the partial-wave amplitudes as functions of the energy.

Pion-Pion Amplitudes at Fixed t or u. Pion-Pion amplitudes at fixed t were found by Froggatt and Petersen[43] using Pietarinen's method. We shall briefly review their method, following the review article by Petersen.[44]

In determining the amplitudes at fixed t, it is important to have a good and smooth description of the amplitude at large s. Therefore, the amplitude F^{+-} of $\pi^+\pi^-$ scattering is written as

$$F^{+-}(\nu, t) = F_R^{+-}(\nu, t) + \mathcal{F}^{+-}(\nu, t), \quad (57)$$

where $\nu = (s-u)/4$ and $F_R^{+-}(\nu, t)$ is the sum of the contributions of the Regge poles from the Pomeranchuk trajectories ρ and f. The form of $F_R^{+-}(\nu, t)$ is chosen to be suitable at small ν. Then the complex ν plane with two cuts along the real axis is mapped conformally onto the unit disk in the z plane and the function $\mathcal{F}^{+-}(\nu, t)$ in (57) is then parametrized in the form

$$\mathcal{F}^{+-}(\nu, t) = A^{+-}(\nu, t)\, R(\nu, t)\, \Psi_{+-}(\nu, t). \quad (58)$$

The first term on the right-hand side, $A^{+-}(\nu, t)$, is a simple, explicitly given expression that ensures the expected asymptotic behavior at large ν. We have already subtracted the leading Regge poles, so that the asymptotic behavior of $A^{+-}(\nu, t)$ corresponds to lower trajectories. The following factor $R(\nu, t)$ may contain the poles outside the unit disk corresponding to resonances in direct channel, and only $\Psi_{+-}(\nu, t)$ is expanded in a Taylor series:

$$\psi_+(z, t) = \sum_{n=0}^{N} a_n z^n. \quad (59)$$

In the same way, one must construct the amplitude $F^{++}(\nu, t)$, which is related by crossing symmetry to $F^{+-}(\nu, t)$.

As input data, Froggatt and Petersen[43] used amplitudes from phase-shift analyses, but only where such analyses were sufficiently reliable (below the threshold of the $K\bar{K}$ reaction). They used the solution obtained in the analysis of Ref. 44 and supplemented these data with the amplitude calculated from the scattering lengths at low energies.

This input information served as the basis for the expression χ^2_{exp}[see (54)]. The second part of χ^2 arises from the a priori probabilities and was calculated by means of (55), the series being truncated at a definite N. Minimizing χ^2, they determined the coefficients a_n.

Speaking in the language of analytic extrapolations, Froggatt and Petersen essentially extrapolated the amplitude $F(\nu, t)$ (for fixed t) from the region below the $K\bar{K}$ threshold to higher energies. At these higher energies, where the phase-shift analyses are not entirely unique, the calculated values of $F(\nu, t)$ were used as additional information in the phase-shift analyses. It was shown that by such a method one can virtually eliminate the ambiguity and arrive at phase-shift analyses that are unique from the threshold at $W = 2m_\pi$ to $W = 1.8$ GeV.[43,44]

5. A TEST OF ANALYTICITY AND DETERMINATION OF THE SINGULARITIES OF SCATTERING AMPLITUDES

Introductory Remarks. Locality (microscopic causality) of interactions is one of the principles of modern elementary-particle physics. From the mathematical point of view, locality is expressed through analyticity of the scattering amplitudes. Therefore, great attention has been devoted to tests of analyticity since the middle of the fifties. In this connection, it is natural to ask how analyticity can be most effectively tested and how one can estimate the errors of the individual methods of verification. Optimizing the test of analyticity is a very delicate matter and the extent of success can be gauged only by comparing the hypothesis of analyticity with the alternative hypothesis that the amplitude has singularities of a definite kind. The errors can be estimated correctly only when the employed method takes into account the statistical nature of experimental data.

To the author at least, the test of analyticity is one of the central problems in the description of data by analytic functions. Many other problems can be reduced to this one. Let us give some examples.

Extrapolation from part of the boundary to the remainder of the boundary: to supplement the values of the amplitude on a "known" part of the boundary by values on the "unknown" part in such a way that the result is the boundary value of an analytic function (in order that the result satisfy the analyticity test).

Determination of coupling constants as residues at poles. If $f(z)$ is an amplitude with a pole at the point z_0, then the residue can be determined from the condition that

$$F(z) = f(z) - g(z - z_0)^{-1}, \tag{60}$$

satisfy the analyticity test.

Determination of the positions of resonances. Resonances can be determined consistently and unambiguously as poles on the unphysical sheets of the partial-wave amplitudes. The determination of the resonance parameters (the position in the complex plane and the residue) can be understood as testing the analyticity of expressions obtained in such a way that the resonance poles are subtracted from the amplitude.

Determination of the zeros of scattering amplitudes. The zeros of the amplitude $f(z)$ are poles of the function $F(z) = 1/f(z)$.

In this section, we shall discuss tests of analyticity and in the following section the use of these tests to solve the above problems.

Tests of Analyticity.[2] Suppose that the analyticity domain of the function $F(s)$ has already been mapped onto the unit disk D. By measuring the amplitude, we obtain [inside D and/or on the boundary B] values Y_i at points z_i with errors ε_i. In addition, we assume that the data are sufficiently dense for us to be able to introduce on the boundary a continuous error $\varepsilon(z)$ and function $Y(z)$. If part of the boundary is not covered by measurements, we must supplement the measurements by hypotheses about $Y(z)$ and $\varepsilon(z)$ on this part of the boundary (see the discussion in Sec. 3).

The expression $\exp(-\chi^2/2)$, where

$$\chi^2 = \sum_{i=1}^{N} \frac{|F(z_i) - Y_i|^2}{\varepsilon_i^2} + \frac{1}{L}\int_B \frac{|F(z_i) - Y(z)|^2}{\varepsilon^2(z)}|dz|, \tag{61}$$

can be interpreted in two ways: either as a probability in the space of analytic functions induced by the results of measurements [denoted by $P(F/Y_1, Y_2, \ldots, Y_N, Y(z))$], or as the probability $P(Y_1, Y_2, \ldots, Y_N, Y(z)/F)$ of a measurement of the value $Y_1, \ldots, Y_N, Y(z)$ for a given function $F(z)$. In the present paper, we shall adhere to the *second* interpretation.

For simplicity, we suppose that the data (if necessary, supplemented with *a priori* hypotheses) are known only on the boundary. Then

$$P(Y(z)/F) \sim \exp\left\{-\frac{1}{2}\frac{1}{2\pi}\int_B \frac{|F(z) - Y(z)|^2}{\varepsilon^2(z)}|dz|\right\}. \tag{62}$$

[2]This section is based on Ref. 46.

We now wish to know whether $Y(z)$, measured with errors $\varepsilon(z)$, is consistent with the analyticity of $F(z)$. It is first expedient to rewrite the expression (62) by means of the weight function (14), which is analytic in D, has no zeros in D, and on the boundary satisfies the condition $|g(z)| = \varepsilon(z)$. Introducing

$$F(z) = g(z) f(z), \quad Y(z) = g(z) y(z), \tag{63}$$

we immediately obtain from (62)

$$P(y(z)/f) \sim \exp\left\{-\frac{1}{2}\frac{1}{2\pi}\int_B |y(z) - f(z)|^2 |dz|\right\}. \tag{64}$$

The function $f(z)$ must be analytic in D, and $y(z)$ must be sufficiently smooth on B. Therefore, $f(z)$ can be expanded in a Taylor series, and $y(z)$ in a Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n; \tag{65a}$$

$$y(z) = \sum_{n=1}^{\infty} Q_n z^{-n} + \sum_{n=0}^{\infty} q_n z^n, \tag{65b}$$

where

$$Q_n = \frac{1}{2\pi i}\int_B \frac{Y(z)}{g(z)} z^{n-1}\, dz = \frac{1}{2\pi i}\int_B y(z) z^{n-1}\, dz. \tag{66}$$

Substituting (65) in (64), we obtain[3]

$$P(y(z)/f) \sim \exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty} Q_n^2 - \frac{1}{2}\sum_{n=0}^{\infty}(q_n - a_n)^2\right\}. \tag{67}$$

It can be seen from this that if $f(z)$ is analytic the coefficients Q_n are random, normally distributed variables with zero mean value and unit variance. This assertion follows solely from the analyticity of $f(z)$ and in no way depends on the properties of $f(z)$, since Q_n are the coefficients of the singular part of the Laurent series. In this sense, the coefficients Q_n characterize only the nonanalytic part of the measured results of $Y(z)$.

The Q_n are random variables with a well known and simple distribution and they can therefore be used to test analyticity. For example,

$$K_N \equiv \sum_{n=1}^{N} Q_n^2 \tag{68}$$

has [for analytic $f(z)$] a χ^2 distribution with N degrees of freedom. Then

$$\langle K_N \rangle = N, \ \langle (\Delta K_n)^2 \rangle = 2N. \tag{69}$$

Comparison of the value of K_N calculated from the data with the expectations in (69) is a well defined test of analyticity.

Another method can also be used. One can choose a set of non-negative coefficients $\{c_n\}$ and study the statistical distribution

$$T_N = \sum_{n=1}^{N} c_n Q_n^2 \tag{70}$$

or introduce a vector $(d_1, \ldots, d_N)$ and consider the random variable

$$S_N = |\sum_{n=1}^{N} d_n Q_n|^2. \tag{71}$$

[3]We assume that $f(z)$ is a real analytic function, and therefore the coefficients a_n, q_n, and Q_n are real.

It can be shown that T_N or S_N appear only when we wish to find the optimal value for testing the hypothesis that $f(z)$ is analytic by comparing it with the alternative hypothesis that $f(z)$ has singularities of a definite kind. We shall return to this question later.

Determination of the Singularities of Scattering Amplitudes.[4)] We first sketch the general method and we then consider in more detail the determination of the poles of an amplitude.

If the function $f(z)$ consists of a part that is analytic in the unit disk D and a part containing singularities which depend on the parameters $\beta_1, \ldots, \beta_N$, then

$$f(z)=\sum_{n=0}^{\infty} a_n z^n+\sum_{n=1}^{\infty} \eta_n(\beta_1, \ldots, \beta_N) z^{-n}. \qquad (72)$$

Repeating the previous method, beginning with Eq. (64) and ending with (67), we obtain a distribution for the coefficients:

$$P(Q_1, \ldots, Q_n, \ldots/f) \sim \exp\Big[-\frac{1}{2}\sum_{n=1}^{\infty} |Q_n-\eta_n(\beta_1, \ldots, \beta_N)|^2\Big]. \qquad (73)$$

The parameters $\beta_1, \ldots, \beta_N$ can be determined by minimizing the expression

$$\chi_M^2(\beta_1, \ldots, \beta_N)=\sum_{n=1}^{M} |Q_n-\eta_n(\beta_1, \ldots, \beta_N)|^2 \qquad (74)$$

as a function of the parameters $\beta_1, \ldots, \beta_N$.

For practical purposes, the most important case is that in which the series for $f(z)$ has besides the analytic part only a pair of complex-conjugate poles. Then

$$f(z)=\sum_{n=0}^{\infty} a_n z^n+\frac{\alpha}{z-\lambda}+\frac{\alpha^*}{z-\lambda^*}=\sum_{n=0}^{\infty} a_n z^n+\sum_{k=1}^{\infty}[2\,\mathrm{Re}(\alpha\lambda^{k-1})]\, z^{-k}. \qquad (75)$$

The parameters α and λ can be determined by minimizing the expression

$$\chi_N^2(\alpha, \lambda)=\sum_{n=1}^{N} |Q_n-2\,\mathrm{Re}(\alpha\lambda^{n-1})|^2, \qquad (76)$$

the expression $\chi_N^2(\alpha, \lambda)$ depending on four real parameters: $\alpha_1, \alpha_2, \lambda_1, \lambda_2 (\alpha=\alpha_1+i\alpha_2, \lambda=\lambda_1+i\lambda_2)$.

Before the numerical minimization of $\chi_N^2(\alpha, \lambda)$, it is expedient to estimate approximately the parameters α and λ and establish whether the set of coefficients $\{Q_N\}$ is consistent with the hypothesis that there is only a single pair of poles. In this case, it is helpful to forget the errors for a while and attempt to set directly

$$Q_i=2\,\mathrm{Re}(\alpha\lambda^{i-1}). \qquad (77)$$

If we take four such equations with $i=n, n+1, n+2, n+3$, then after transformations we obtain

$$\left.\begin{aligned}\mathrm{Re}\,\lambda&=\frac{1}{2}(Q_{n+1}Q_{n+2}-Q_nQ_{n+3})/(Q_{n+1}^2-Q_nQ_{n+2});\\ |\lambda|^2&=(Q_{n+2}^2-Q_{n+1}Q_{n+3})/(Q_{n+1}^2-Q_nQ_{n+2}).\end{aligned}\right\} \qquad (78)$$

However, these equations are more readily obtained by transforming the equation $\int_B f(z)(z-\lambda)(z-\lambda^*)z^k dz = 0$, which holds for $f(z)$ given by (75). Practical problems of determining resonance poles will be considered later.

Optimization of the Test of Analyticity; Determination of Singularities. The test of analyticity can be optimized only if the hypothesis of analyticity of the amplitude is compared with the alternative hypothesis that a definite singularity is present. As an illustration, we give the simplest alternative: $f(z)$ has a pole at the point $z=\mu$. The pole at the point μ can be written as

$$\left.\begin{aligned}\frac{1}{z-\mu}&\equiv\frac{1}{\sqrt{1-|\mu|^2}}\,\varphi_\mu(z);\\ \varphi_\mu(z)&=\sqrt{1-|\mu|^2}\sum_{k=1}^{\infty}\mu^{k-1}z^{-k}.\end{aligned}\right\} \qquad (79)$$

The function $\varphi_\mu(z)$ is normalized: $\frac{1}{2}\pi\int_B\varphi_\mu^*(z)\varphi_\mu(z)|dz|=1$, and it determines a "direction" in the space of the singular parts of the Laurent expansion of the data $y(z)$. If we wish to test the hypothesis that $f(z)$ is analytic by comparing it with the alternative hypothesis that there is a pole at the point μ (μ is fixed), the most advantageous tactic is to construct the random variable

$$(y, \varphi_\mu)=\frac{1}{2\pi}\int_B y^*(z)\,\varphi_\mu(z)\,|dz|=\sqrt{1-|\mu|^2}\sum_{k=1}^{\infty} Q_k\mu^{k-1}. \qquad (80)$$

If $y(z)$ is the boundary value of a function analytic in D, then (y, φ_μ) will have the distribution $N(0,1)$. If $f(z)$ has a pole at the point μ with residue α, then (y, φ_μ) will have the distribution $N(\alpha, 1)$. We obtain the optimal difference between these hypotheses when we use the data to calculate

$$S_\mu=|(y, \varphi_\mu)|^2 \qquad (81)$$

and compare the result with what we expect on the basis of the two hypotheses.

The hypothesis put up to compare with analyticity of $f(z)$ is too narrow. Assuming that the pole can lie within a circle of diameter Λ, to test analyticity we can take

$$S_\Lambda=\frac{1}{\pi\Lambda^2}\int_{|\mu|<\Lambda} d^2\mu\, S_\mu=\frac{1}{\Lambda^2}\sum_{n=1}^{\infty}|Q_n|^2\Big[\frac{\Lambda^{2n}}{n}-\frac{\Lambda^{2n+2}}{n+1}\Big]. \qquad (82)$$

A similar method can also be used in more complicated cases. Such sophisticated tests of analyticity have not hitherto been used in practice, so that we shall not consider them further.

For the determination of the resonance $\Delta(1236)$ and pion-pion resonances, minimization of the expression (76) was used and also a simpler alternative, which is as follows.

If $f(z)$ has a pair of complex-conjugate poles at the points λ and λ^* then

$$f(z)\,\frac{z-\lambda}{1-\lambda^* z}\,\frac{z-\lambda^*}{1-\lambda z} \qquad (83)$$

does not have poles. Moreover, $|(z-\lambda)/(1-\lambda^* z)|=1$ for $|z|=1$ and the multiplication factors in (83) do not change the error $\varepsilon(z)$. Therefore, to determine the position of the poles we can use, instead of $y(z)$, the expression (83), in which $f(z)$ is replaced by $y(z)$,

4)This subsection is based on Refs. 46 and 47.

to calculate the coefficients Q_n. Then, using the coefficients $\{Q_n\}$ we calculate $\chi^2=\sum|Q_n(\lambda)|^2$ and minimize this expression as a function of λ.

Test of Analyticity when the Errors of the Real and the Imaginary Part of the Scattering Amplitude are not Equal. The principle of the method is the same as for the case of equal errors. But technical difficulties appear, since one cannot achieve the simplification by the introduction of a weight function. For a special but practically adequate form of $\varepsilon_R(z)$ and $\varepsilon_I(z)$, the problem was solved in Ref. 48. We shall not go into technical details but merely sketch the essence of the problem.

Let $\mathfrak{L}$ be the Hilbert space of functions defined for $|z|=1$ and real in the sense $f(z^*)=f^*(z)$. We define the scalar product by the condition

$$(f, g)=\frac{1}{2\pi}\int_B\left[\frac{\operatorname{Re} f \operatorname{Re} g}{\varepsilon_R^2(z)}+\frac{\operatorname{Im} f \operatorname{Im} g}{\varepsilon_I^2(z)}\right]|dz|.$$

We adopt the notation H for the subspace of $\mathfrak{L}$ consisting of boundary values of functions that are analytic in D. In addition, let $\tilde{S}$ be the complement of H.

We require a basis in $\tilde{S}$. We denote it by $\{S_n(z)\}$. If $f(z)$ is analytic in D, then $Q_n=(S_n, y)$, and the experimentally determined data $y(z)$ will have the distribution $N(0,1)$. Then in the test of analyticity Q_n will play the same role as Q_n in the case when the errors of the real and imaginary parts are equal. If we wish to find the parameters of the pole, we must first calculate $A_n(\alpha,\lambda)=(S_n(z), \alpha/z-\lambda+\alpha^*/z-\lambda^*)$ and then minimize the expression $\chi^2(\alpha,\lambda)=\sum_{n=1}^N|Q_n-A_n(\alpha,\lambda)|^2$ as a function of the parameters α and λ.

6. TEST OF ANALYTICITY OF πN AMPLITUDES FOR FORWARD SCATTERING. COUPLING CONSTANT AND PARAMETERS AT LOW ENERGIES

Test of Analyticity and Determination of the Coupling Constant. To test analyticity, we use the methods of Refs. 47 and 49 set forth above. Let $F(\omega)$ be the amplitude for forward scattering with charge exchange ($\pi^-p\to\pi^0n$) divided by ω:

$$F(\omega)=\frac{1}{\omega}[A_-(\omega)-A_+(\omega)+\omega B_-(\omega)-\omega B_+(\omega)], \qquad (84)$$

where ω is the laboratory energy of the incident pion in units of $m_\pi c^2$; the subscripts $\pm$ denote, respectively, $\pi^+p\to\pi^+p$ and $\pi^-p\to\pi^-p$; $A(\omega)$ and $B(\omega)$ are the invariant amplitudes in the usual notation. The function $F(\omega)$ is analytic in the complex ω plane with cuts $(-\infty,-1)$ and $(1,\infty)$, it is crossing symmetric, $F(\omega)=F(-\omega)$, and at the points $\omega=\pm\omega_0=\pm\frac{1}{2}M$ it has a pole corresponding to a nucleon in the direct channel. Using the crossing symmetry of $F(\omega)$, we can conformally map the right-hand half of the ω plane onto the unit disk (Fig. 10).

The mapping is given explicitly by

$$z=z(\omega)=(\sqrt{1-\omega_0^2}-\sqrt{1-\omega^2})/(\sqrt{1-\omega_0^2}+\sqrt{1-\omega^2}), \qquad (85)$$

and the pole at the point ω_0 is transformed onto the origin in this case. Except for this pole, $F(z)=F(\omega(z))$ is analytic in the unit disk D.

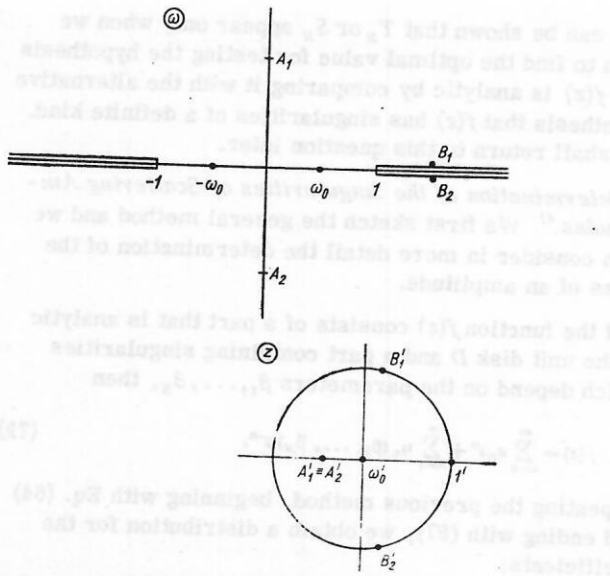

FIG. 10. Mapping of the right-hand half of the ω plane onto the unit disk in the z plane. The image of the point A under the mapping is denoted by A'. By crossing symmetry, $F(A_1)=F(A_2)$, and therefore there is no cut in the interval $(-1, 0)$ in the z plane.

Experimental data $y(z)$ about the amplitude on the boundary B are obtained from the following sources:

a) phase-shift analyses[50] at total c.m.s. energies in the range $1096<W<2189$ MeV;

b) for $W>2200$ MeV, the imaginary part of the amplitude is found from information about the total effective cross section, and the real part from data on Coulomb interference at small angles;

c) the value of $F(z)$ in the neighborhood of the threshold is obtained from the recommended values[51] for the scattering length $a^{(-)}$. The errors at individual points are calculated from the errors given in the phase-shift analysis,[50] and in constructing the averaged error $\varepsilon(z)$ we have followed (24). For given $\varepsilon(z)$, we construct a weight function $g(z)$ that is analytic in D, has no zeros in D, and on the boundary satisfies $|g(z)|=\varepsilon(z)$. In accordance with (67), the coefficients

$$Q_n=\frac{1}{2\pi}\int_B\frac{y(z)}{g(z)}z^n|dz|, \quad n=2, 3, \ldots, \qquad (86)$$

must be random variables with zero mean value and unit variance. The mean value of the coefficient Q_1 is nonvanishing since the conformal mapping $z=z(\omega)$ transformed the nucleon pole onto the origin in the z plane. Therefore, the amplitude $F(z)/g(z)$ takes the form

$$\frac{F(z)}{g(z)}=\frac{1}{z}\frac{f^2}{2g(0)}\frac{1}{1-(m/2M)^2}+\sum_{n=0}^{\infty}a_nz^n;$$

where m is the pion mass, M is the nucleon mass, and f^2 is the πN coupling constant.

The calculated coefficients $Q_1,\ldots,Q_{10}$ have the following values[47]:

n	1	2	3	4	5	6	7	8	9	10
Q_n	41.9	1.6	−2.7	1.7	0.1	1.8	0.6	3.5	−3.4	−2.3

(87)

Considering these coefficients, we note two interesting features:

1) the coefficient Q_1 is clearly greater than the remaining Q_n. If we did not know that the amplitude $F(\omega)$ had a pole, the result (87) would clearly indicate the existence of a singularity near the origin of z;

2) the coefficients $Q_2, \ldots, Q_{10}$ are somewhat greater than our expectation based on their having the distribution (0, 1). It can be seen from this that the errors in the phase-shift analyses were clearly underestimated.

From the value of Q_1 we can obtain the coupling constant f^2. Assuming that the mean error is $Q_1 = 2$, we obtain $f^2 = 0.0835 \pm 0.0040$. In an independent analysis using the same method, Lichard and Prešnajder[52] obtained the value $f^2 = 0.0803 \pm 0.0010$. The two determinations of f^2 agree to within the indicated errors.

Estimating the errors of the phase-shift analyses realistically, we see that the coefficients $Q_2, \ldots, Q_{10}$ indicate that the scattering amplitude is analytic. More rigorous assertions require a complete analysis of the experimental data and a detailed consideration of the possible inaccuracies of the numerical methods (interpolation and numerical integration).

Note that, using the method of Ref. 48 to test the analyticity for unequal errors of the real and imaginary parts of the amplitude, we obtained $f^2 = 0.0804 \pm 0.0014$ and $\sum_2^{20} Q_n^2 = 13.6$, which indicates that the amplitude does not contradict analyticity.

Determination of scattering lengths by testing analyticity of the πN amplitudes. The πN scattering lengths cannot be directly measured, and therefore, as a rule, they are determined from sum rules, dispersion relations, etc.

Following Ref. 49, we show how the scattering lengths can be determined by means of a statistical test of analyticity. Lichard[49] worked with the amplitudes $F^+(\omega)$, $B^+(\omega)/\omega$, $F^-(\omega)/\omega$, $B^-(\omega)$. They are all crossing symmetric. The amplitudes were chosen in such a way that some of them are sensitive to the values of the s-wave scattering lengths and others to the p-wave scattering lengths. By means of the conformal mapping (85), the right-hand half of the ω plane is mapped onto the unit disk. The direct experimental data lie on the part of the boundary marked with crosses in Fig. 11. This region corresponds to pion kinetic energy between 88 MeV and 2 GeV. In the region $0 < \varphi < 60°$, which corresponds to $T_\pi < 88$ MeV, the amplitudes of the s and p partial waves were expressed by means of the parametrization

$$q^{2l+1} \cot\delta_l = 1/a_l + q^2 r_l / 2,$$

where the scattering lengths a_l and the effective ranges r_l were regarded as free parameters, and q is the meson momentum in the center of mass system. For $\varphi > 172°$ ($T_\pi > 2$ GeV), the available information on the πN interaction at high energies was used.

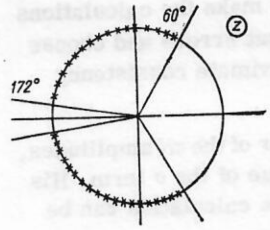

FIG. 11. Conformal mapping of the ω plane onto the z plane with identification of individual regions on the boundary. The arc $60 < \varphi < 172°$ corresponds to 88 MeV $< T_\pi < 2$ GeV, and the data on it are taken from phase-shift analyses. For $\varphi < 60°$, the amplitude is parametrized by means of the scattering lengths.

If we know a smooth error function $\varepsilon(z)$, then the subsequent operations are in principle simple. From the known $y(z)$ for $\varphi > 60°$ and the parametrization $y(z, a_l, r_l)$ we calculate the coefficients

$$Q_n(a_l, r_l) = \frac{1}{2\pi} \int_B \frac{y(z)}{g(z)} z^n |dz| \tag{88}$$

and minimize[5)]

$$\chi^2(a_l, r_l) = \sum_{n=2}^{N} Q_n^2(a_l, r_l) \tag{89}$$

as a function of the parameters a_l and r_l. By such a minimization, we determine the scattering lengths a_l and estimate their errors.

The calculations showed that χ^2 is hardly sensitive to the parameters r_l, and in the subsequent analysis they were therefore not used.

Using as input data a combination of Refs. 53 and 54, Lichard obtained for the s-wave scattering lengths

$$a_1 = 0.208 \pm 0.020; \quad a_3 = -0.091 \pm 0.017 \tag{90}$$

and for the p waves

$$\begin{rcases} a_{11} = -0.109 \pm 0.035; & a_{31} = -0.063 \pm 0.022 \\ a_{13} = -0.045 \pm 0.035; & a_{33} = 0.186 \pm 0.022. \end{rcases} \tag{91}$$

These results agree with many other determinations of the scattering lengths, and also with the values recommended in the reviews Ref. 51. It should be emphasized once more that the errors in (90) and (91) are purely statistical in nature and do not contain other possible sources of errors.[55]

We should also add that in calculations of the type made in Ref. 49 the consistency of the input and output errors is a very delicate matter. The point is that the method requires an *a priori* specification of $\varepsilon(z)$. In the region in which we do not know the data (i.e., $0 < \varphi < 60°$ in Fig. 11), the error $\varepsilon(z)$ is chosen on the basis of preliminary estimates of the errors of the low energy parameters. Then, minimizing (89), we calculate the scattering lengths and their errors. From this, we can obtain the errors of the amplitude in the low energy region. These "new" or "output" errors must be approximately equal to the old, or input er-

5) The sum on the right-hand side does not contain Q_1, which is determined mainly by the nucleon pole. Thus, the scattering lengths do not depend on f^2.

rors. It is therefore expedient to make the calculations with different estimates of the input errors and choose the estimate which leads to approximate consistency of the old and the new errors.[49]

Studying the low energy behavior of the πN amplitudes, Lichard[56] also calculated the value of the σ term. His result was $\sigma = 64 \pm 9$. Details of the calculation can be found in Ref. 56.

7. DETERMINATION OF THE POSITIONS AND WIDTHS OF RESONANCES AS POLES ON THE SECOND SHEET

Introductory Comments. In the determination of the basic parameters (masses and widths) of resonances, the most frequently used procedure is to reproduce the data on the effective cross section of a partial wave by means of a sum of a resonance term and a background. Such a method is frequently ambiguous, since the method used to parametrize the background may influence the results of the determination of M and Γ.

We obtain a unique determination of M and Γ when we construe the resonance as a pole on the unphysical sheet of the partial-wave amplitude. The position of the pole corresponding to the $\Delta(1236)$ resonance was determined in Refs. 57 and 58, in which it was shown that the pole lies on the second sheet at

$$W \approx M - i\Gamma/2 = (1211 - i50)\ \mathrm{MeV}. \qquad (92)$$

It was found that the position of the pole is remarkably insensitive to reasonable changes of the parametrization.[57,58] Nevertheless, such methods depend on the variables of the parametrizations and the problem of separating the resonances and background cannot be avoided.

We give here, first, a method for determining resonance poles,[46] then a method and results for determining the position of the $\Delta(1236)$ resonance.[59] Finally, we mention the method and results of Boháčik and Kuehnelt obtained when they found the positions of the resonances in $\pi\pi$ scattering.

The method does not depend on the chosen model in the sense that it determines the position of the pole only from the singular part of the Laurent expansion independently of the analytic background.

Principle of the Method. Conformal Mapping. In the usual notation, the partial-wave amplitude on the first sheet is

$$f^{\mathrm{I}}(s) = [\eta \exp(2i\delta) - 1]/(2iq). \qquad (93)$$

We arrive on the second sheet by passing through the elastic cut:

$$f^{\mathrm{II}}(s) = f^{\mathrm{I}}(s)/[1 + 2iqf^{\mathrm{I}}(s)]. \qquad (94)$$

On the second sheet, the amplitude has (except for a pair of complex-conjugate poles) the same analytic properties as on the first.

The analyticity domain of $f^{\mathrm{I}}(s)$ and $f^{\mathrm{II}}(s)$ is shown[61] in Fig. 12 (poles corresponding to resonances are not shown). We map the domain D in Fig. 12 conformally

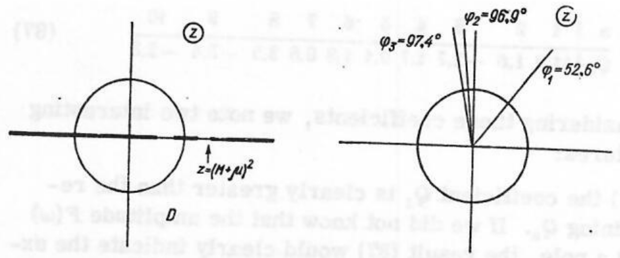

FIG. 12. Mapping of the analyticity domain of the πN partial-wave scattering amplitude onto the unit disk. The arc (φ_1, φ_2) is the image of the region $1096 < W < 2203$ MeV, in which there are phase-shift analyses[53]; the arc (φ_2, φ_3) is the image of 2203 MeV $< W < \infty$; the arc (φ_3, π) corresponds to the left-hand cuts. On the arc $(0, \varphi_1)$ the amplitude is determined from the scattering lengths.

onto the unit disk in the z plane. The method is conformally invariant and therefore the mapping is unique. In our calculations,[59] we chose the mapping in such a way that the expected pole on the second sheet came approximately on the imaginary axis in the z plane.

As result of the conformal mapping, we obtained the amplitude $f^{\mathrm{II}}(z)$, and using (94), data on the arc $0 < \varphi < 97°$. On the arc $96.9 < \varphi < 180°$ we do not dispose of experimental data. In the course of the calculations, a simple form of the amplitude was here chosen and a larger error $\varepsilon(z)$ imposed on it. Then, we constructed the weight function $g(z)$ and using the data calculated the coefficients

$$Q_n = \frac{1}{2\pi} \int \frac{y(z)}{g(z)} z^n |dz|. \qquad (95)$$

We then calculated the position of the poles, first "roughly" by means of (78) and then, if everything was in order, "more accurately" by minimizing the expression (76).

Before determining the position of the $\Delta(1236)$ pole, we tested the method and its accuracy on a simple example of an amplitude with a pole on the second sheet. In this illustrative example, we chose

$$F^{\mathrm{I}}(s) = \frac{s(k) -}{2iq}, \quad k = \sqrt{s - (M+m)^2}/2$$

and parametrized $s(k)$ by means of the Jost functions $s(k) = f(k)/f(-k)$, with $f(k) = (k - a - ib)(k + a - ib)$. Choosing the real constants a and b, we can adjust the position of the pole on the second sheet of the complex s plane. The pole was chosen in such a way that it lay approximately where the $\Delta(1236)$ pole was expected. Using this $s(k)$, we calculated the phases $s(k) = \exp[2i\delta(k)]$. On the latter, we imposed realistic random errors and used them as input data for determining the pole. It was found that the statistical deviations are small compared with the systematic errors due to the calculation and the not completely correct allowance for the left-hand cut. Generalizing the experience of the illustrative example, we concluded that the position of the pole can hardly be determined with accuracy greater than ±10 MeV. Therefore, in subsequently determining the $\Delta(1236)$ we assign this error.

Determination of the Pole of the Δ (1236) Resonance.

The position of the Δ(1236) resonance pole on the second sheet was found by the above method in Ref. 59. The expression (95) was used to determine the coefficients Q_n, and the position of the pole was found by minimizing the expression (76).

The value of $\chi^2(\alpha,\lambda)$ for $\alpha=0$, i.e., when there is no pole, is approximately equal to 10^6; if a pole is introduced, then $\chi^2(\alpha,\lambda)$ falls to 10–100, which is an acceptable value. This rapid decrease of χ^2 is a cogent argument that on the second sheet $f^{II}(s)$ has a singularity, and indeed a pole.

Calculating the parameters of the pole in accordance with the simplified method based on Eqs. (78), we immediately see [see Ref. 59] that the lowest coefficients Q_n are not entirely reliable since the influence of the left-hand cut is strongly manifested in them. The coefficients Q_n with $n>30$ are also unreliable because the rapid oscillations of the factor z^n in (95) can lead to inaccuracies in the numerical calculation. It is therefore most expedient to use only Q_n with $10<n<30$ in $\chi^2(\alpha,\lambda)$ [see (76)]. In the second paper of Ref. (59), we used $13 \leqslant n \leqslant 26$ and obtained the result

$$M=1214.5\pm10\,\text{MeV}\quad \Gamma=97.2\pm10\,\text{MeV}. \tag{96}$$

where the error is not of statistical nature but derives from the experience with the illustrative example.

The result agrees with the data of Ball *et al.*[57,58] and coincides remarkably accurately with the result of Spearman,[62] who used a different method to determine the pole, but again one that did not depend on a chosen model.

Subsequently, the position of the Δ(1236) resonance pole was recalculated.[63] More accurate computational methods were employed and the left-hand cut was taken into account more accurately. First, verifying the analyticity of $f^I(s)$, we obtained approximate information about the partial-wave amplitude on the left-hand cut on the first sheet, and then, using (94), transferred this information to the second sheet. The results show that in this way one can halve the error (by which we mean the systematic errors). Moreover, the coefficients with low n are determined reliably. In this case, the parameters of the Δ(1236) pole position are

$$M=1226\pm5\,\text{MeV},\quad \Gamma=110\pm5\,\text{MeV}. \tag{97}$$

In the future, we can anticipate a significant improvement in the accuracy of phase-shift analyses. A new determination of the Δ(1236) position or other resonance poles should be based on new phase shifts.

"Perturbation Method" for Determining Corrections to Singularities. We here consider an interesting and simple problem that may be important for finding electromagnetic mass differences of resonances in a definite multiplet.

Suppose $f_1(z)$ is analytic in the unit disk D except for a single pole:

$$f_1(z)=\alpha/(z-\lambda)+\sum_{n=0}^{\infty}a_nz^n, \tag{98}$$

and suppose $f_2(z)$ deviates only slightly from $f_1(z)$:

$$f_2(z)=f_1(z)+\delta f(z)=(\alpha+\delta\alpha)/(z-\lambda-\delta\lambda)+\sum_{n=0}^{\infty}(a_n+\delta_n)z^n. \tag{99}$$

Our task is, using data on f_1 and f_2, to determine the parameters (α,λ) and the corrections ($\delta\alpha$ and $\delta\lambda$) to them.

At first sight it would seem simplest to determine the positions of the poles in f_1 and f_2, and then subtract one pole from the other. But in this way we should obtain the small quantities $\delta\alpha$ and $\delta\lambda$ as the differences of two large quantities, and the result would be sensitive to statistical and, above all, systematic errors.

An alternative method[64] is to determine the positions of the poles from the amplitude f_1, and $\delta\alpha$ and $\delta\lambda$ from f_1-f_2. The singular part of δf is equal to

$$(\alpha+\delta\alpha)/(z-\lambda-\delta\lambda)-\alpha/(z-\lambda)$$
$$=\sum_{n=1}^{\infty}z^{-n}[\lambda^{n-1}\delta\alpha+\alpha(n-1)\lambda^{n-2}\delta\lambda], \tag{100}$$

where we have retained only the first-order part in $\delta\alpha$ and $\delta\lambda$. Indeed, it is the coefficients of z^{-n} on the right-hand side of (100) that are the Q_n for the difference $\delta f=f_2-f_1$.

Equations (99) and (100) can be readily generalized to the more realistic case of two complex-conjugate poles.

Basic Parameters of Pion–Pion Resonances. Recent experiments[65,66] with high statistics have made it possible to obtain more accurate information about the pion–pion interaction. The phase-shift analysis of $\pi\pi$ scattering up to $W=2$ GeV can now be regarded as unambiguous.[44] In these circumstances, one can attempt to determine the positions of the $\pi\pi$ resonances by model-independent methods as poles on the second sheets of the partial-wave amplitudes.

In Refs. 60, Boháčik and Kuehnelt used the method presented here earlier for determining the positions of resonances, making some modifications.

The analyticity domain of the $\pi\pi$ scattering partial wave (Fig. 13a) was mapped conformally onto the unit disk D (Fig. 13b). The data (direct or reconstructed from the scattering lengths) cover the arcs $0<\varphi<90°$ and $270<\varphi<360°$ on the boundary B. The partial-wave amplitude $f^{II}(s)$ on the second sheet is related to $f^I(s)$ by

$$f^{II}(s)=f^I(s)/[1+2iqf^I(s)];\quad q=\sqrt{s-4}/2. \tag{101}$$

The errors in the region covered by the data were calculated by the usual method, and the region in which data are not known was suppressed by a weight function.

Boháčik and Kuehnelt first considered a series of trial problems with known poles. Of particular interest

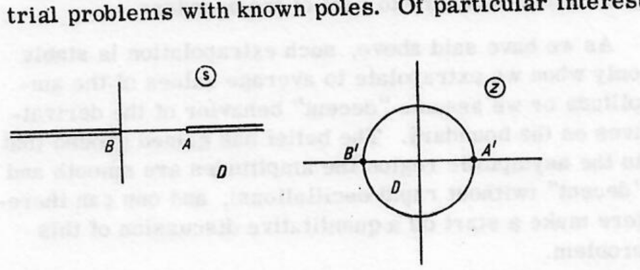

FIG. 13. Mapping of the s plane with the cuts $(-\infty,0)$ and $(4,\infty)$ onto the unit disk in the z plane.

are the results which indicated that the method is sufficiently sensitive to distinguish the case with one pole from the case with two poles on the second sheet. The method must detect two poles in the s wave with isospin zero if they really are there. They used two methods to determine the position of the pole. In the first, they minimized $\chi^2(\alpha, \lambda)$ given by the expression (76); in the second, the sum of the squares by the expression

$$Q'_n(\mu)=\frac{1}{2\pi}\int_B Y(z)\, g^{-1}(z)\, B(z,\mu)\, B^*(z,\mu)\, z^n\, |dz|, \qquad (102)$$

where $B(z, \mu) = (z-\mu)/(1-z\mu^*)$ is the Blaschke factor which cancels the pole at the point λ if $\mu=\lambda$.

For the positions of the resonances ρ, f, and S^*, Boháčik and Kuehnelt found the values

$$\left.\begin{aligned} m_\rho &= (775\pm 10)\pm i\,(70.5\pm 2)\ \text{MeV},\\ m_{S^*} &= (999\pm 1)\pm i\,(19\pm 1)\ \text{MeV},\\ m_f &= (1265\pm 2)\pm i\,(72\pm 1)\ \text{MeV}. \end{aligned}\right\} \qquad (103)$$

They did not find any indication of a resonance in the s wave with $I=0$ in the region 1200–1300 MeV.

The mass of the ρ meson has a relatively large error because the results depend to some extent on the form and value of the weight function on the left-hand cut. A significant improvement is possible only if the left-hand cut is determined at least approximately by a test of analyticity on the first sheet.

The width of the ρ meson calculated by Boháčik and Kuehnelt ($\Gamma = 141\pm 4$ MeV) is somewhat lower than the value 150 ± 2 MeV given in Ref. 66. The results of other authors relating to the parameters of the $\pi\pi$ resonances can be found in Refs. 60 and 44.

8. CAN ONE PREDICT THE HIGH ENERGY BEHAVIOR OF AN AMPLITUDE FROM DATA AT LOWER ENERGIES?

In 1973, it was announced[67] that the total effective cross section of the pp interaction increases at ISR energies. Later, this effect was also confirmed for the remaining hadron reactions at FNAL energies.

Could this effect have been predicted on the basis of the data on the real and imaginary parts of the forward scattering amplitude at lower energies? This is a natural question. If we knew the amplitude at lower energies exactly (without errors), the amplitude would then be uniquely determined in the entire analyticity domain. In reality, the situation is otherwise. We only know the amplitude with certain errors, and the problem with which we are concerned is that of extrapolation from part of the boundary to part of the boundary.

As we have said above, such extrapolation is stable only when we extrapolate to average values of the amplitude or we assume "decent" behavior of the derivatives on the boundary. The belief has gained ground that in the asymptotic region the amplitudes are smooth and "decent" (without rapid oscillations), and one can therefore make a start on a quantitative discussion of this problem.

We consider the crossed binary πN amplitude in the complex ω plane (pion laboratory energy). The analytic

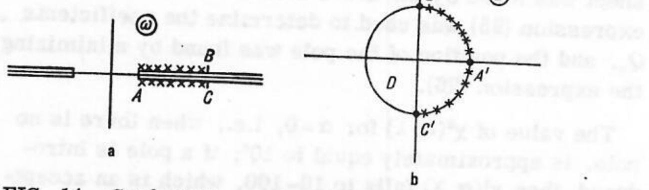

FIG. 14. Conformal mapping of the right-hand half of the ω plane onto the unit disk in the z plane. The region with data is indicated by the crosses. The left-hand semicircle in the z plane corresponds to the high energy region.

properties of the forward scattering amplitude are shown in Fig. 14a. By means of the conformal mapping

$$z=(A-\sqrt{1-\omega^2})/(A+\sqrt{1-\omega^2}) \qquad (104)$$

the right-hand half of the ω plane is carried into the unit disk in the z plane. For appropriate A, the domain covered by the data is transformed onto the right-hand semicircle in Fig. 14b.

The difficulties of extrapolating from the region covered by the data to the neighborhood of the point $z=-1$ (asymptotic region) can be seen from the following simple example. Suppose the errors of the experimental data are in order of magnitude equal to ε and suppose $F_1(z)$ gives a good description of the data on the right-hand semicircle in the z plane. Then the functions

$$\begin{aligned} F_2(z) &= F_1(z)+\varepsilon\exp(-Mz), \quad M>0;\\ F_3(z) &= F_1(z)+\varepsilon(1+\delta+z)^{-n}, \quad \delta>0, \end{aligned}$$

must also give a good description of the data for $\mathrm{Re}\, z>0$, but they will differ appreciably from F_1 in the neighborhood of $z\approx -1$ (for large M and small δ).

Pondering this or similar examples, we conclude that extrapolation to high energies is possible only when we have at our disposal fairly strong restrictions on the behavior of the scattering amplitude. It is usually assumed that at high energies the amplitude has a simple form and contains only a few parameters. Our problem can therefore be formulated as follows[68-70]

At low energies, say for $s<s_H$, we know data $F_{\exp}(s)$ on the scattering amplitude $F(s)$ (real and imaginary parts) with an error $\varepsilon(s)$. At higher energies, we assume that the amplitude can be parametrized as $F_H(s;\alpha_1,\alpha_2,\ldots,\alpha_m)$, where α_i are free parameters. We wish to obtain the answer to two questions: In the set of functions $F_H(s;\alpha_1,\ldots,\alpha_m)$ can one find one that in conjunction with $F_{\exp}(s)$ at low energies agrees with the requirement of analyticity; if so, what are the "correct" values of the parameters $\alpha_1,\ldots,\alpha_m$?

To give a quantitative answer to the posed question, we must make it more precise and complete. We introduce a function $y(z)$ defined for $|z|=1$:

$$\left.\begin{aligned} y(z) &= F_{\exp}(z); \quad |z|=1; \quad \mathrm{Re}(z)>0;\\ y(z) &= F_H(z;\alpha_1,\ldots,\alpha_m); \quad |z|=1, \quad \mathrm{Re}(z)<0; \end{aligned}\right\} \qquad (105)$$

in addition, we must construct the function

$$\begin{aligned} \varepsilon(z) &= \varepsilon_{\exp}(z); \quad |z|=1, \quad \mathrm{Re}(z)>0;\\ \varepsilon(z) &= \varepsilon_H(z); \quad |z|=1, \quad \mathrm{Re}(z)<0. \end{aligned} \qquad (106)$$

The error $\varepsilon_{exp}(z)$ is taken from the experimental data; the error $\varepsilon_H(z)$ is the function that gives us information on the error with which $F_H(z;\alpha_1,\ldots,\alpha_H)$ must describe the behavior of the amplitude at high energies. We pose this question: If we also measured the amplitude for $s>s_H$ (i.e., $|z|=1$, $\mathrm{Re}(z)<0$) and obtained the result $F_H(z)$ with error $\varepsilon(z)$, would the result, together with $F_{exp}(z)$ and $\varepsilon(z)$, be compatible with analyticity?

It is now easy to test analyticity. First, we introduce a weight function $g(z)$ [$g(z)$ is analytic in D, has no zeros in D, and for $|z|=1$ satisfies $|g(z)|=\varepsilon(z)$]. Then, we calculate the coefficients

$$Q_n=\frac{1}{2\pi}\int_B \frac{y(z)}{g(z)}z^n|dz|, \tag{107}$$

where Q_n are functions of the parameters $\alpha_1,\ldots,\alpha_m$ (105):

$$Q_n=Q_n(\alpha_1,\ldots,\alpha_m). \tag{108}$$

Finally, we minimize the expression

$$\chi^2(\alpha_1,\ldots,\alpha_n)=\sum_{n=1}^{N}|Q_n(\alpha_1,\ldots,\alpha_m)|^2 \tag{109}$$

as a function of the parameters $\alpha_1,\ldots,\alpha_m$ and evaluate the results by the usual statistical methods.

The choice of the function $\varepsilon_H(z)$ is a comparatively delicate matter. If $\varepsilon_H(z)$ is too large, then any F_H will be compatible with analyticity; but if ε_H is too small, then it will be virtually impossible to find a suitable F_H. Naively, one expects that the optimal case will be the one when F_H and F_{exp} make approximately equal contributions to the coefficients Q_n.

For the crossing-even πN amplitude in the case of constant or logarithmically increasing σ_T the asymptotic parametrization

$$F_H(s)=is[A_0+A_1(\lg s-i\pi/2)+A_2(\lg s-i\pi/2)^2] + C\exp(-i\pi/4)\sqrt{s}, \tag{110}$$

where $A_0, A_1, A_2, \ldots, C$ are free parameters, appears natural.

It must, however, be emphasized that (110) is a parametrization of the complete amplitude (real and imaginary parts) and that, using the method based on a minimization of (109), we test the consistency of the complete amplitude at high energies with the complete amplitude at low energies.

Nogová[69,70] considered the crossing-even amplitude for forward πN scattering:

$$C^+(\omega)=[C_{\pi^-p}(\omega)+C_{\pi^+p}(\omega)]/2. \tag{111}$$

The experimental data in the energy range 0-70 GeV were taken from phase-shift analyses, from measurements of the total effective cross section, and from Coulomb interference. In the region above 70 GeV, the parametrization

$$C^+(\omega)=is[a_0+a_1(\lg s-i\pi/2)^\beta]+a_2(1-i)\sqrt{s} \tag{112}$$

was used. Below the energy 70 GeV, the function $\varepsilon(z)$ was taken from the experimental data, and above it was extended continuously (after the conformal mapping, the point with $\arg(z)=165°$ corresponded to the energy $\omega=70$ GeV).

The aim of the calculation was to establish whether one could determine the parameter β by minimizing χ^2 in (109). The results showed that χ^2 is not very sensitive to the parameter β, since an increase in β can be compensated by a change of the remaining parameters. However, the question cannot be regarded as closed. It would undoubtedly be interesting to obtain the results of analogous calculations but with different functions $\varepsilon_H(z)$ or with a test of analyticity corresponding to unequal errors of the real and imaginary parts of the amplitude.

CONCLUSIONS

The phenomenology of strong interactions is based largely on analyticity of the amplitudes. The statistical approach to extrapolations, and in the general case to the description of data by analytic functions, is the most suitable way of wedding the statistical nature of data and analyticity of the amplitudes in a consistent and practically viable whole.

It was not the aim of the present review to give an idea of the present state of investigations into the problems of analytic extrapolation or to give all technical methods. The aim was more modest: to give a review of the main ideas of the statistical approach to the description of scattering amplitudes by analytic functions. This approach is best used in phase-shift and amplitude analyses, in which the problem of errors and uniqueness is central. In other problems, in which one can be satisfied by a realistic estimate of the intuitively determined error, other methods can be used.

In order to represent more fully the material on these questions, we mention some of the recent papers, in which one can find a full bibliography. Mathematical questions relating to extrapolations are discussed in the review of Ciulli *et al.*[1,2] This group has also written a system of programs for practical application of extrapolation methods.[71] A review of different practical applications of extrapolations can be found in Ref. 4. Classical and newer methods of analysis of the pion-nucleon amplitudes are discussed in detail in the lectures Ref. 45; the present state of the phenomenology of pion–pion interactions has been well described by Petersen.[44] Many aspects of pion–nucleon amplitudes are analyzed in the lectures Ref. 72.

It is evident that the phenomenology of strong interactions will for a long time continue to use analyticity of the amplitudes as one of its basic principles. In the near future, at least for an amplitude as a function of a single variable, we can expect a deepening, improvement, and extension of the numerical methods. At the present time, the principles of the methods are well known but the practical aspects have not always been carried through to completion. In the future, it is above all necessary to develop a method of extrapolation to the boundary of the analyticity domain. In addition, it is to be expected that methods will be developed which are suitable for special problems. This is the place to mention the use of an additional condition: positivity or unitarity to stabilize extrapolations.[73-74]

Another field in which progress can be expected is the representation of amplitudes by rational functions or Padé approximants.[75] But this problem is very complicated and it is not yet clear whether these methods apply in general or only in problems in which there is preliminary information about the behavior of the amplitudes outside the considered analyticity domain. A more promising problem is that of the expansion of amplitudes as functions of two complex variables.[76-78] We can also expect the development of methods of determining singularities of the scattering amplitudes.[46, 79-81] Despite the undoubted usefulness of representations of data by analytic functions and extrapolations, we must also bear in mind the limitations of this method.

An extrapolation is an attempt to replace in a certain sense a deeper theory or a direct measurement by inductive arguments. It is precisely here that we encounter the instabilities so typical of not only extrapolations but also of all inverse problems in which the cause is deduced from the effects. This is most readily seen in the example of the determination of singularities from data. Such a determination is highly unstable, whereas the determination of an amplitude from known singularities is a stable problem. It is much the same when one attempts to obtain charge distributions from measurements of the electrostatic field (an unstable problem in which the causes are determined from the effects).

For all this, the phenomenological use of analytic functions could play a part, if not crucial, in the construction of a deeper theory of strong interactions.

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