Sturm-Liouville expansion method in nuclear physics

J. Bang

Niels Bohr Institute, Copenhagen

F. A. Gareev and A. P. Ivanova

Joint Institute for Nuclear Research, Dubna

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The expansion of the form factors of one- and two-nucleon transfer reactions with respect to the Sturm-Liouville functions corresponding to the Woods-Saxon potential is discussed. Completeness and the convergence of the expansion as $r\to\infty$ are elucidated. The theory of direct nucleon-transfer reactions is briefly presented, special attention being paid to the role of the form factor. A number of applications of the method, to single-particle transfers to deformed nuclei and two-particle transfers to spherical nuclei, are discussed. The advantages of the Sturm-Liouville method are revealed by a comparison with the exact expression for the asymptotic behavior of the form factors and with other methods of calculation.

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INTRODUCTION

Recently, the method of expansion of wave functions of the Schrödinger equation with respect to Sturm-Liouville basis functions has been widely used in problems of atomic[1-8] and nuclear[9-18] physics. Therefore, there is need for a detailed discussion of the properties of the expansions that are particularly useful in practical calculations.

In mathematical physics, the expression "Sturm-Liouville problem" is used for a large class of eigenvalue problems:

$$-\Delta f_i+\sum_s \lambda_{i,s}V_s f_i=0, \tag{1}$$

where f_i satisfy definite boundary conditions, and this can be a multidimensional problem or even one considered in an infinite region. The coefficient V_s of the eigenvalue $\lambda_{i,s}$ depends on the choice of the coordinate system, and the Sturm-Liouville problem consists of determining $\lambda_{i,s}$ if V_s depends on the ordinary "physical" coordinates. In the one-dimensional case, one can always reduce the eigenvalue problem to a form such that $V_s=1$. However, this entails the introduction of new additional coordinates, which complicate the problem.

In this review, we consider the application of the method of expansion of Schrödinger wave functions with respect to Sturm-Liouville basis functions in some problems of nuclear physics. The usefulness of Sturm-Liouville functions in quantum mechanics is intimately related to some physical facts. All known interaction potentials between particles have a finite range. This means that the Schrödinger equation that describes the motion of particles interacting through such forces will have a continuous and, possibly, a discrete spectrum (bound states).

Further, we very frequently encounter a situation in which the functions that are the eigenfunctions of a complicated problem can be represented in the form of expansions with respect to basis eigenfunctions of a simpler problem. If these basis functions are solutions of a Schrödinger equation, then it is necessary in the general case to include continuum functions since only then can one guarantee completeness of the system of functions. In practice, it is convenient to work with basis functions that have only a discrete spectrum, for example, with oscillator functions or with the functions corresponding to a rectangular well with infinite walls. But because these potentials are unphysical, they lead to difficulties related to poor convergence of the resulting series in certain regions of space.

In general, it can be said that if a physical problem is solved by means of the method of expansion with respect to a complete system of functions, then the convergence will be best if the basis functions, or at least some of them, are close to the physical functions, i.e., if the basis functions are solutions of equations of the same type. In this sense, the Sturm-Liouville method has definite advantages since it enables one to find Sturm-Liouville functions with a discrete set of quantum numbers for which the corresponding expansions with respect to them satisfy conditions of rapid convergence (for example, uniform logarithmic convergence as determined in Sec. 2).

The method of expansion using Sturm-Liouville functions was first introduced into quamtum mechanics by Epstein[1] to describe the Stark effect, immediately after Schrödinger's work. His aim was to obtain a simple expansion, the basis functions being solutions of the Schrödinger equation for the hydrogen atom with fixed energy and variable charges:

$$\left(-\Delta+\frac{2m}{\hbar^2}\left[\frac{z_i e^2}{r}-\varepsilon\right]\right)f_i=0. \tag{2}$$

The situation is simplified because the eigenfunctions have an analytic form. This example is interesting since the potential is singular at the origin and has a long range.

Basically, we shall be concerned with the Woods-Saxon potential, which is a good approximation to the nucleon–nucleus interaction potential, i.e., we shall consider equations of the type

$$\left.\begin{aligned}&\left(-\frac{\hbar^2}{2m}\Delta+V_0+\lambda_i V-\varepsilon\right)f_i=0;\\ &V=-(1+\exp[(r-R)/a])^{-1},\end{aligned}\right\} \tag{3}$$

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where V_0 may include spin–orbit, Coulomb, and other interactions. The main properties of the potential, which make it convenient for problems in nuclear physics, is that it has a finite range, is almost constant in the interior region, and varies smoothly in a definite surface region of order a. It does not have singularities and is nonzero in the whole of space, which is very helpful for our further investigations.

The aim of this review is to discuss the asymptotic behavior of the wave functions that describe bound states and the amplitudes of transfer reactions; to elucidate the cases in which it is particularly important to take into account the correct asymptotic behavior; and to give a method for obtaining correct wave functions in the limit $r \to \infty$ for different cases of reactions.

In the following sections, we introduce form factors (overlap integrals of the nuclear wave functions) and discuss their connection to the wave functions obtained in calculations of nuclear structure (Sec. 1). We then consider in detail the convergence properties of the Sturm-Liouville expansion (Sec. 2). In particular, we discuss the expansion of a form factor with respect to eigenfunctions of Eq. (3), which are particularly interesting for us.

The most important questions relate to the convergence of the expansion at large distances. These questions are discussed in Sec. 3, in which we briefly give the results of different methods of description of nuclear reactions and draw particular attention to the importance of form factors. Note that it is impossible to distinguish completely between the effect of introducing more accurate form factors and the effects of correct treatment of the reaction mechanisms. This is demonstrated in Sec. 4, in which we consider the form factors of transfers to deformed nuclei, studying particularly the case of $\Delta N = \pm 2$ mixing.

In Sec. 5 we shall consider transfer reactions to spherical nuclei and, in particular, discuss the calculation of the form factors of two-nucleon transfer reactions, which have recently attracted much attention; of particular interest are transfers in reactions induced by heavy ions.

1. FORM FACTORS OF TRANSFER REACTIONS

The questions discussed here are related to the calculation of stationary wave functions of nuclear states. These functions must be calculated with sufficient accuracy in the limit $r \to \infty$ to be useful in calculations of direct transfer reaction cross sections.

Nuclear transfer reactions take place in the surface region of the nucleus, and, depending on the particular reaction, one can specify fairly accurately a range of distances that is important for the given process. For example, in sub-barrier proton-stripping reactions, this range of distances extends from one to four or five radii of the nucleus. The densities of nucleons in the nucleus are small at these distances, and it is therefore not surprising that in spectroscopic calculations the contribution from such small quantities can be ignored, but in the calculation of reaction amplitudes they become important.

Modern understanding of the mechanism of nuclear transfer reactions is far from perfect. We are concerned with a problem that involves at least three bodies. The boundary conditions in such problems are complicated in the coupled-channel method as well as in iterative schemes such as the Born approximation. Since the corresponding Faddeev equations and, *a fortiori*, equations with more particles cannot be solved on a computer, they are usually solved under certain approximations. One of the standard approximations is to replace the interaction, making the Faddeev equations solvable. However, the approximate interaction is nonlocal, and, as shown in Ref. 19, the solutions of the Fadeev equation in this case are less accurate than the equations obtained in the well known iterative schemes. It is clear that in any iterative scheme one must consider the convergence, and this problem is not yet solved, though investigations have recently been published (see Low's review, Ref. 20, and the bibliography given there) in which the convergence of iterations for various cases has been considered.

A common feature of all these schemes is that the amplitude of the reaction in which $1, 2, 3, \ldots$ particles are transferred from the target nucleus $|A\rangle$ to the product nucleus $|B\rangle$ contains integrals of the type

$$\int d\mathbf{r}_1 d\mathbf{r}_2 \ldots \mathcal{J}(\mathbf{r}_1, \mathbf{r}_2, \ldots) \langle A | B \rangle \equiv \int d\mathbf{r}_1 d\mathbf{r}_2 \ldots \mathcal{J} C F(\mathbf{r}_1, \mathbf{r}_2, \ldots), \tag{4}$$

where $\mathcal{J}$ includes wave functions of the incoming and outgoing nuclei and the interactions corresponding to these transfers, $\mathcal{J}$ making the main contribution in the region of space determined by the inequalities

$$r_1 \geqslant R, \quad r_2 \geqslant R, \ldots. \tag{5}$$

Here, R is the radius of the target nucleus or the product nucleus (of the state $|A\rangle$ or $|B\rangle$). The reason why the conditions (5) arise becomes clear if one studies in more detail the approximate expressions for the amplitudes (Figs. 1 and 2).[21] We emphasize that to calculate the integrals (4) we must know the functions F (or $\langle A | B \rangle$) at large distances $r_1, r2, \ldots$. The factorization $\langle A | B \rangle = CF$, where C is the spectroscopic amplitude, is introduced to demonstrate the dependence of the inte-

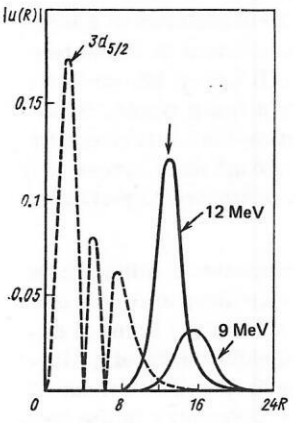

FIG. 1. Spatial distribution of the differential cross section of the $^{238}U(d,p)^{239}U$ transfer reaction to the ground state at $\theta = 135°$ for two energies of the incident deuteron.

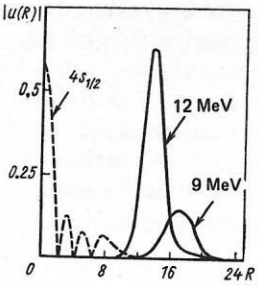

FIG. 2. The same as in Fig. 1 for $^{238}U(d,t)^{237}U$.

grals (4) on all the coordinates of the nucleons in the target nucleus $|A\rangle$. The coefficients C are well determined only in the case when a corresponding normalization of F is required. To calculate the transfer amplitudes, it is necessary to antisymmetrize all states appropriately and sum the transfer amplitudes over all identical particles.

Note (see, for example, Ref. 22) that systematic calculation of the form factors $\langle A|B\rangle$ is not the same as calculation of the functions $|B\rangle$ ($|A\rangle$). This is really so since some properties of the functions F can be studied on the basis of the fact that in the region (5), in which the transfer takes place, the interaction between the transferred nucleons and the other nucleons is weak. However, the coefficients C are obtained only in structure calculations.

In some investigations, attempts are made to calculate the functions F and the coefficients C independently of one another. Such attempts are related to the successes of the shell and other models based on shell states as basis states without allowance for the continuum contribution. In this case, the wave function is a combination of solutions of the single-particle Schrödinger equation with potential $V(r)+V_{s.o}+V_{Coul}$:

$$(H_0-E_i)f_i=\left(-\frac{\hbar^2}{2m}\Delta+V(r)+V_{s.o.}+V_{Coul}-E_i\right)f_i=0. \quad (6)$$

Then the wave function $|B\rangle$ ($|A\rangle$) can be written in the form

$$|B\rangle=\sum_{ij}c_{ij}^{B}\ldots f_if_j\ldots, \quad (7)$$

where the coefficients c_{ij}^{B} are determined by diagonalizing the Hamiltonian:

$$H=\sum_i H_0(r)+V_{res}, \quad (8)$$

and the summation is over all the valence nucleons. It follows that the overlap integrals $\langle A|B\rangle$ must have the form

$$\langle A|B\rangle=CF=\sum_{ij}c_{ij}^{AB}\ldots f_if_j. \quad (9)$$

We have already said that at large distances r for short-range potentials the function F is determined by the binding energy; for example, in the single-particle case it is necessary to obtain[22]

$$\lim_{r\to\infty}F(r)\sim\exp\left(-\sqrt{\frac{2m}{\hbar^2}|E_{BA}|}\,r\right), \quad (10)$$

where $E_{BA}=E_B-E_A$ is the single-particle binding energy. For one-particle transfer, the expression (9) simplifies:

$$\langle A|B\rangle=\sum_i c_i^{AB}f_i. \quad (11)$$

In (9), the summation over i is truncated, i.e., for fixed values of l and j only one term of the sum over n is taken, and the contribution from the continuum is ignored under the assumption that the corresponding mixing coefficients in the expansion (9) are small.[23-25] The asymptotic behavior of the expression (11) in the limit $r\to\infty$ differs from (10):

$$\langle A|B\rangle\approx\sum a_i\exp\left(-\sqrt{\frac{2m}{\hbar^2}|E_i|}\,r\right), \quad (12)$$

where E_i is the binding energy of the single-particle state in the potential of the shell model. It can be seen from the expression (12) that for large values of r the contribution from the large components a_i of the expansion may be smaller than the contribution from the more weakly bound orbitals, since the smallness of the expansion coefficients will be compensated by the weaker damping of the corresponding component. In the majority of nuclear spectroscopic calculations, the asymptotic behavior of the wave functions is so unimportant that correct results can be obtained by using harmonic-oscillator functions instead of functions of the Woods-Saxon potential. In calculations of direct nuclear reactions, the asymptotic behavior of the wave functions becomes very important. However, in this case it is also necessary to know the wave functions in the inner region in order to find the correct normalization factor. Therefore, one is faced with the problem of a fairly accurate calculation of the wave functions in both the inner and outer regions of the nucleus. We can put this differently as follows. The asymptotic behavior of the radial part of the form factor is determined by the nucleon separation energy, which is related to the Q value of the reaction. At the same time, the functional behavior of $\langle A|B\rangle$ inside the nucleus changes in accordance with the intensity and range of the residual interactions; therefore, the normalization of the asymptotic part of $\langle A|B\rangle$ depends on its interior part. In our opinion, this is the fundamental difference between the spectroscopic criteria of accuracy and the criteria to be applied to wave functions used in direct-transfer reactions.

Different methods have been proposed for calculating the overlap integral $\langle A|B\rangle$ with a view to improving the asymptotic behavior of the function F, coefficients C from structure calculations being used as well. Such approximate methods cannot be regarded as consistent, and they frequently lead to incorrect results. However, one of them must be mentioned since it has been widely used in calculations of direct reactions. In this method, which was proposed in Ref. 26, the depth of the potential is varied until the calculated energy E_i is equal to the experimental value of the single-particle binding energy E_{BA}, i.e., one solves the equation

$$\left(-\frac{\hbar^2}{2m}\Delta+\lambda_iV+V_{s.o.}+V_{Coul}-E_{BA}\right)f_i=0, \quad (13)$$

the residual interactions being ignored.

This is known as the well-depth prescription (WDP). However, Eq. (13) is an equation for the eigenvalue and eigenfunction of a Sturm-Liouville problem. The properties of the solutions of Eq. (13) have been described in detail in Refs. 10, 11, 16, 17, 22, 27, and 28; the important thing for us now is that the functions f_i satisfying Eq. (13) form a complete discrete set in the Hilbert space of square-integrable functions. It is worth

noting that the WDP approximation actually consists of retaining a single principal component in the expansion of the total wave function with respect to the basis set of Sturm-Liouville functions, and its applicability has been investigated fairly fully in Refs. 27 and 28. One can therefore say that the solutions of Eq. (13) are better suited for the calculations of the integrals $\langle A|B\rangle$ than the solutions of Eq. (6). The use of Sturm-Liouville functions in nuclear physics problems has different aspects, and we shall consider them briefly below.

A difference between Schrödinger equations and equations for Sturm-Liouville functions is that in the latter the eigenvalues λ_i are multiplied by the function $V(r)$, which depends on the coordinates. By a transformation of the coordinate system, the equation can be reduced to a form in which the λ_i appear with constant coefficients.[28] Under such a change of variables, the interval $0 \leqslant r \leqslant \infty$ is transformed into a finite interval, and the resulting equation is in general fairly complicated compared with the original equation. The Coulomb case is an exception; for it, the transformed equation is that of a harmonic oscillator.

The main advantage of expansion with respect to Sturm-Liouville functions is that in this expansion all the basis functions have the same required asymptotic behavior (which can be seen from Eq. (13), which determines the WDP approximation). In this sense, the method of expansion with respect to Sturm-Liouville functions recalls the coupled-channel method,[29-32] but, in contrast to that method, we here have a system of algebraic equations, which facilitates the calculations. If the coupled-channel method or other methods are used, including an iterative procedure, some serious difficulties appear in the calculation of bound states.

The Schrödinger equation for bound states is an eigenvalue problem in which the eigenvalue is either the energy or (if the energy is fixed) certain parameters of the Hamiltonian, as is shown below in numerous examples. This means that the above procedure must include at every step of the iterations an analogous procedure for changing the eigenvalues.

However, it is only in the one-dimensional case that there exists a lower limit for the distance between two eigenvalues (for the energy or depth of the potential). Therefore, it is in principle difficult to specify *a priori* an algorithm of the iteration procedure that will automatically converge to the correct result except for the one-dimensional case and the case when one seeks only the lowest eigenvalue (energy, depth of the potential, etc). In practice, this is complicated in both deformed and spherical nuclei.

In contrast to these iterative methods, methods based on diagonalization in a complete basis will automatically give all solutions in the given interval of eigenvalues. And then, on the basis of some physical criterion, one can choose the correct solution. In such a situation, the fact that the diagonalization methods give several unphysical solutions is in fact an advantage over the other methods.

We point out once more that the WDP approximation differs from the Sturm-Liouville expansion method not only in the different method of determining the coefficient C but also in that the expansion is not restricted to components corresponding to the quantum numbers of bound shell functions. Theoretically, the number of these components is infinite, but in practice the good convergence of the method enables one to operate in a restricted configuration space.

2. SOME MATHEMATICAL PROPERTIES OF THE STURM-LIOUVILLE EXPANSION METHOD

The general theory of expansion with respect to Sturm-Liouville functions was developed long ago,[33, 34] and it is in principle very similar to the theory of expansion in a Fourier series, of which it is a generalization. It was first successfully used in determining the general motion of vibrational systems; in particular, in the one-dimensional case the vibration of a string was treated by means of an expansion with respect to normal modes.

Examples of generalization of this theory to nuclear-physics problems are given in Refs. 17, 27, and 28, and therefore we shall give here some of the useful properties we need of Sturm-Liouville expansions essentially without the proofs, which can be found in the quoted literature.

We proceed from Eq. (3), setting $V_0 \equiv V_{s.o} + V_{Coul}$; $\hbar^2/2m = 1$; $V(r) < 0$, i.e., $V(r)$ is an attractive potential—the Woods–Saxon potential; ε is a fixed negative number. We require that the functions f_i satisfy the boundary conditions

$$f_i(r) \to 0 \quad \text{as} \quad r \to 0 \text{ and } r \to \infty. \tag{14}$$

We write down Eq. (3) for two eigenvalues:

$$(-\Delta + \lambda_i V + V_0 - \varepsilon) f_i = 0; \tag{15}$$

$$(-\Delta + \lambda_j V + V_0 - \varepsilon) f_j = 0; \tag{16}$$

then, multiplying (15) by f_j^* and (16) by f_i^* and integrating with respect to $\mathbf{r}$ with allowance for the Hermiticity of $-\Delta + V_0$, we obtain

$$(\lambda_i - \lambda_j) \int f_i^* V f_j \, d\tau = 0 \tag{17}$$

or

$$\int f_i^* V f_j \, d\tau = K_i \delta_{ij} \tag{18}$$

(for convenience, we set the normalization constant K_i equal to -1). Therefore, the functions

$$g_i = |V|^{1/2} f_i \tag{19}$$

form an orthonormalized basis. In nuclear physics, the generally adopted potential is the Woods–Saxon potential (3), for which $|V| = -V \neq 0$ in the whole of space.

It is natural to ask whether there is a connection between the Sturm-Liouville functions and the functions of the Schrödinger equation. If $\lambda_{0lj} = 1$ (in appropriate units) and ε_{0lj} is an eigenvalue of the Schrödinger equation, then $f_{0lj}(r)$ is a function of the ground ($n = 0$) state of the Schrödinger equation and will be the first eigenfunction of the Sturm–Liouville problem. For fixed l

and j, the potential well becomes deeper with increasing λ_{nlj}, and if we can choose λ_{nlj} such that the energy of the n-th level is equal to ϵ_{0lj}, then the resulting wave function will be the eigenfunction $f_{nlj}(r)$ of the Sturm–Liouville problem. Therefore, the Sturm–Liouville equations are similar to the Schrödinger equation, but the energy ε_{0lj} is a fixed parameter of the problem and the same for all λ_{nlj} (lj fixed, and n takes the values $0, 1, 2, \ldots, n-1, n, n+1$), and the wave functions $f_{nlj}(r)$ for $n \neq 0$ are not physical. The eigenfunctions $f_{0lj}(r)$ of the Sturm–Liouville problem in the considered special case when ε_{nlj} is taken the same for given values of lj (in general, the Sturm–Liouville functions can be chosen differently), coincide for the states $1s, 1p, 1d, \ldots$ with the wave functions of the Schrödinger equation. Then the series

$$\chi_{nlj}(r) = \sum_{n'}^{\infty} d^{n}_{n'lj} f_{n'lj}(r)$$

($\chi_{nlj}(r)$ is the wave function of the Schrödinger equation) for the states $1s, 1p, 1d$ contains only the single term $d^{0}_{0lj} = 1$ of the expansion, and all the remaining $d^{0}_{n'lj} = 0$. Even in this simple example one can see the main advantages of an expansion with respect to Sturm–Liouville functions over all the other methods: Already in the first approximation this method enables one to encompass all the main characteristic features of the investigated problem.

The eigenvalues λ_i form an infinite sequence of positive (negative) discrete numbers satisfying the condition

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_{i-1} < \lambda_i < \lambda_{i+1} < \ldots \tag{20}$$

Note that the fact that the eigenvalues have a definite sign for $\varepsilon < 0$ can be readily obtained from Eq. (3) by rewriting it in the form

$$\lambda_i = -\langle f_i | -\Delta - \varepsilon | f_i \rangle \Big/ \int f_i^* V f_i \, d\tau.$$

The matrix element in the numerator for $\varepsilon < 0$ is always positive, and the sign of λ_i is determined by the sign of $V(r)$. For $V(r) < 0$ (attractive potential) $\lambda_i > 0$, and for $V(r) > 0$ (repulsive potential) $\lambda_i < 0$. Here, we have assumed $V_0 = 0$ but the conclusions are unchanged if $V_0 \neq 0$.[27-28]

Equation (3) can be rewritten in the integral form

$$\operatorname{sign}(V)\lambda_i |V|^{1/2} (\varepsilon - H_0)^{-1} |V|^{1/2} g_i = g_i, \tag{21}$$

where $H_0 = -\Delta + V_0$ and the operator $(\epsilon - H_0)^{-1}$ is an integral Green's operator. For further use, Eq. (21) can be conveniently written in the form

$$\lambda_i \int K(\mathbf{r}, \mathbf{r}') g_i(\mathbf{r}') \, d\mathbf{r}' = g_i(\mathbf{r}). \tag{22}$$

If H_0 is a self-adjoint operator, then K is symmetric. This property of the kernel K is fundamental for the proof of the expansion theory. Mercer's theorem[35] asserts that if all but a finite number of the eigenvalues of K have the same sign, then the convergence of the expansion is uniform and absolute. This means that if we have

$$g(\mathbf{r}) = \sum_{n=0}^{\infty} c_n g_n(\mathbf{r}); \tag{23}$$

$$g_N(\mathbf{r}) = \sum_{n=0}^{N} c_n g_n(\mathbf{r}), \tag{24}$$

we can find a number $N_0(\xi)$ for arbitrarily small ξ such that

$$|g_N - g| \leqslant \xi \quad \text{for} \quad N > N_0(\xi). \tag{25}$$

Of course, $g_i(r)$ (or f_i) form a complete set only in the region where $V(r) \neq 0$.

We emphasize that, in contrast to the Schrödinger equation, the Sturm–Liouville problem does not have a continuum. Therefore, the advantage of the expansion with respect to the functions g_i is that they constitute a discrete series of functions and one is therefore relieved of the difficulties with allowing for the influence of the continuum on the bound states.

In some cases a function that one can expand in a series with respect to a complete Sturm–Liouville basis satisfies a Schrödinger equation with the same energy as the Sturm–Liouville equation but the Hamiltonian contains a residual interaction V_1 which in the whole of space satisfies

$$|V_1| < c|V|. \tag{26}$$

We solve the equation

$$(-\Delta + V_0 + \lambda V + V_1 - E)\Psi = 0, \tag{27}$$

and the expansion

$$\Psi_N = \sum_{n=0}^{N} c_n f_n \quad (E = \varepsilon) \tag{28}$$

has the interesting property

$$|\Psi_N / \Psi^{-1}| \leqslant \xi \tag{29}$$

for $N \geqslant N_0(\xi)$ for all r separated by a finite distance from the zeros of the function ψ. This property is stronger than uniform convergence, and makes it possible to obtain approximate wave functions with equal relative accuracy in the whole of space even for a finite number N. Therefore, the transfer amplitude, even in the cases when it gets its main contribution from the regions of arbitrarily large distances between the transferred nucleon and the core, can be correctly calculated. It is obviously an advantage to use basis functions with the same binding energy as the expanded function.

The properties (25) and, in particular (29) are true, strictly speaking, only for motion of a particle in a static potential when the condition (26) is satisfied, for example, in the adiabatic limit of a deformed nucleus for different types of deviation from spherical symmetry of the potential. The main results of the method of expansion with respect to Sturm–Liouville functions can be generalized to the multidimensional case, but in such problems the boundary conditions become more complicated and the generalized Sturm–Liouville functions lose their simplicity. It is therefore sensible to attempt to keep the Sturm–Liouville functions as simple as possible.

The use of Sturm–Liouville expansions in systematic structure calculations involves substituting a truncated expansion into the Schrödinger equation with total Hamiltonian

$$(H - E)\sum c_{ij\ldots} f_i f_j \ldots = 0. \tag{30}$$

Then, using the equations for f_i, we obtain a system of linear algebraic equations for the mixing coefficients c_{ij}. The condition for the existence of solutions of this system leads to the requirement that the determinant

of the system vanish, which is equivalent in such a restricted basis to the problem of determining an approximate eigenvalue of the Schrödinger equation (upper bound). Here, however, we have several possibilities. In problems of the type of the Stark effect in the hydrogen atom all the terms of the Hamiltonian are known exactly, and the energy can be regarded as an eigenvalue of the problem which must be calculated. In usual problems of nuclear physics, a model Hamiltonian, for example, of the type (8), represents the physical situation rather crudely, and, as a rule, contains a certain number of parameters that cannot be established from basic principles but in the best case only from confrontation with experimental data. One such set of well known experimental data is provided by the binding energies, and then the parameters of the potential are adjusted to reproduce these energies. Another set of experimental data, for example, the transfer reaction cross sections, can be used to test the adequacy of the chosen parametrization. Confirmation that the parameters are optimal (in the framework of the considered model) can be obtained by using a truncated Sturm–Liouville basis if the basis energies are equal to or near the experimental energies, since in this case there is good convergence of the method in the asymptotic region.

We now use the orthogonality relation for the Sturm–Liouville functions and briefly discuss two ways in which Eq. (30) can be solved. If we regard E as an eigenvalue, we then multiply (30) by $\Pi_i V_i(r_i) f_n^*(r_i)$ and integrate over all the coordinates, obtaining by means of equations of the type (3) for f_n matrices whose principal diagonal terms contain the quantities $(E-\sum_i \epsilon_i)$. Then, changing certain parameters of the Hamiltonian (for example, the parameters of the residual interaction), we can obtain

$$E=\sum \varepsilon_i \tag{31}$$

equal to the experimental value.

However, the total matrix is in general non-Hermitian, which can lead to certain difficulties in the calculations since the resulting eigenfunctions may be biorthogonal.[36] One can multiply Eq. (30) by $\Pi_i f_n^*(r_i)$, and then integrate. The resulting matrix is Hermitian. If the interaction parameters that we wish to determine occur linearly in the Hamiltonian (depth of the potential, coupling constants, or deformation parameters), then they can be selected by diagonalizing the matrix of the considered problem. Two possibilities for diagonalizing the matrix of the Hamiltonian are more conveniently investigated in concrete examples, as will be done in Secs. 4 and 5. It is clear that without a truncation of the basis both methods give the exact result, but for a truncated basis, as is shown in Ref. 28, the second method of diagonalization gives a more accurate result than the first. However, for the potentials usually used and a sufficiently large configuration space, this difference is not very important. The numerical results (see Sec. 5) agree with this conclusion.

3. DIRECT NUCLEON-TRANSFER REACTIONS

We briefly describe here the different methods used to calculate transfer-reaction cross sections; our aim will be to study the influence of the properties of the wave functions of bound states of the transferred nucleons on these quantities.

We denote the ingoing reaction channel by α and the outgoing channels in which we are interested by β. The fragments in channel α may consist of $a+A$; then if we are interested in the transfer of nucleons $n_1, n_2, \ldots$ we can write

$$a=b+n_1+n_2+\ldots; \quad B=A+n_1+n_2+\ldots. \tag{32}$$

We can consider other two-fragment channels, for example, the γ channel:

$$c=b+n_1; \quad C=A+n_2. \tag{33}$$

Three-fragment channels must be described[37-38] as unbound states of two-fragment channels. We write down the Hamiltonian in each channel:

$$H=H_a+H_A+T_{aA}+V_{aA}\equiv H_\alpha+T_\alpha+V_\alpha=H_\beta+T_\beta+V_\beta, \tag{34}$$

where $T_\alpha=T_{aA}$ are the relative kinetic energies of the fragments; $V_\alpha(\mathbf{r}_\alpha)\equiv V_{aA}(\mathbf{r}_{aA})$ is the interaction between the fragments, in principle, the sum of the interactions of all particles in a and all particles in A; $H_\alpha(\varepsilon_\alpha)\equiv H_a+H_A$ are the internal Hamiltonians of the fragments.

We write the total wave function in the form

$$\Phi=\sum_{\varepsilon=\alpha,\beta,\gamma\ldots} \Psi_\varepsilon(\xi_\varepsilon)\, U_\varepsilon(\mathbf{r}_\varepsilon), \tag{35}$$

where

$$H_\varepsilon \Psi_\varepsilon(\xi_\varepsilon)=E_\varepsilon \Psi_\varepsilon(\xi_\varepsilon); \tag{36}$$

$U_\varepsilon(\mathbf{r}_\varepsilon)$ is a function of the relative motion in channel ε. For example, if we take into account only one outgoing channel β, then the total wave function of the system is

$$\Phi=\Psi_\alpha(\xi_\alpha)\, U_\alpha(\mathbf{r}_\alpha)+\Psi_\beta(\xi_\beta)\, U_\beta(\mathbf{r}_\beta), \tag{37}$$

where $\Psi_\alpha(\xi_\alpha)=\Psi_a(\xi_a)\Psi_A(\xi_A)$, etc.

As an illustration, let us write down the Hamiltonian H_B:

$$H_B=H_A+\sum_n (V_{nA}+T_n)+\sum_{nn'} V_{nn'}, \tag{38}$$

where V_{nA} describes the interaction between the transferred nucleons and the core; $V_{nn'}$ describes the interaction between the valence nucleons, i.e., H_B is the Hamiltonian of the shell model with residual interactions (in Sec. 5, we shall write it out in more detail for a concrete case).

If $\sum V_{nA}$ if replaced by a potential, then the depth of this potential must depend on the energy[39,40] if it is to give the correct binding energies. In other words, these potentials include both exchange effects and rearrangement effects. The solutions of the Schrödinger equation must be antisymmetric with respect to all the coordinates of the particles, at least for the considered fragment. We discuss the methods to find the amplitude of the reaction cross section.

Multiplying the left-hand side of the Schrödinger equation

$$(H-E)\,\Phi=0 \tag{39}$$

by Ψ_α and integrating with respect to the internal coordinates ξ_α, we obtain

$$[T_\alpha+\langle\Psi_\alpha|V_\alpha|\Psi_\alpha\rangle-(E-E_\alpha)]U_\alpha(r_\alpha)$$
$$=-\sum_{\alpha'}\langle\Psi_\alpha|V_\alpha|\Psi_{\alpha'}\rangle U_{\alpha'}(r_{\alpha'})-\sum_{\beta}\langle\Psi_\alpha|H-E|\Psi_\beta\rangle U_\beta(r_\beta), \quad (40)$$

where the channels with the same distributions of particles in the fragments are denoted by α (α' are the inelastic channels); for them,

$$\langle\Psi_\alpha|\Psi_{\alpha'}\rangle=\delta_{\alpha\alpha'}. \quad (41)$$

It is clear that there is no orthogonality between the inelastic channels α' and the transfer channels β [the last term on the right-hand side of Eq. (40)]; this question is discussed in Refs. 20 and 41–43.

The system of coupled integrodifferential equations can be solved with boundary conditions for which there are outgoing waves in all the channels α' and β, and in channel α an ingoing wave is added. It is fairly difficult to solve the system, especially if the channels contain more than two fragments forming a continuous spectrum. This is due to the need to integrate an infinite system of equations. At the same time, if one also includes in the treatment fragment-disintegration channels (this must be done in order to obtain a complete system of equations), then the resulting system will be overdetermined.

It is therefore natural to attempt to solve the system of equations (40) under certain simplifying assumptions. If the elastic scattering determined by the diagonal term $\langle\Psi_\alpha|V_\alpha|\Psi_\alpha\rangle$ is large, and the second-order effects determined by the nondiagonal terms $\langle\Psi_\alpha|V_\alpha|\Psi_{\alpha'}\rangle$ are small, then their influence can be ignored. Then to calculate the elastic-scattering amplitude we solve the equation

$$(T_\alpha+\langle\Psi_\alpha|V_\alpha|\Psi_\alpha\rangle-(E-E_\alpha))U_\alpha^0(r_\alpha)=0 \quad (42)$$

with appropriate boundary conditions (ingoing plus outgoing waves), and when taking into account inelastic scattering (channel α') and transfer (channel β) it is necessary to solve the inhomogeneous equation

$$(T_{\alpha'}+\langle\Psi_{\alpha'}|V_\alpha|\Psi_{\alpha'}\rangle-(E-E_{\alpha'}))U_{\alpha'}(r_{\alpha'})=-\langle\Psi_{\alpha'}|V_\alpha|\Psi_\alpha\rangle U_\alpha^0, \quad (43)$$

and then in the asymptotic region there are only outgoing waves. Such an approximation is also justified by the Coulomb repulsion between the ions a and A (b and B). In this way we arrive at the distorted-wave Born approximation (DWBA).[44]

The individual reaction channels are neglected because the corresponding cross sections are small. Therefore, such channels can be included in the DWBA by perturbation theory if they are small. If they are large, they can be taken into account in the way this is done in the coupled-channel method (see, for example, the case of strong coupling of rotational states in deformed nuclei[12, 22, 30, 45]). A third possibility is to consider high orders of distorted waves in the Born approximation in the framework of the DWBA. Such a description is possible[46, 47] if one is studying two-nucleon transfer reactions. We mention these papers particularly because of the importance of the form factors obtained in them, which lead to interesting results.

However, we shall dwell in detail only on the first method since it is used in the majority of investigations[16, 17, 22]; other methods are a generalization of it, and the role of the form factors in them is essentially the same as in the DWBA.

If all the nondiagonal terms in (40) are small, we solve Eq. (42) since elastic scattering is the main reaction channel:

$$\Phi\sim\Phi^0=\Psi_\alpha\chi_\alpha^+(k_\alpha, r_\alpha)\delta_{\varepsilon\alpha}, \quad (44)$$

where $\chi_\alpha^+=U_\alpha^0$. Then, substituting U_α^0 in the right-hand side of Eq. (43), we find the small components in channel β, and, therefore, the transfer amplitude:

$$f_{\beta\alpha}(\hat{k}_\beta)=-\frac{\mu_\beta}{2\pi\hbar^2}\langle\chi_\beta^-\Psi_\beta|H-E|\chi_\alpha^+\Psi_\alpha\rangle; \quad (45)$$

here, χ_β^- is obtained by complex conjugation of χ_β^+; μ_β is the reduced mass in channel b,B. Neglect of the nondiagonal terms in (40) does not mean that the channel coupling is ignored in the solution of the equation. It is taken into account in an averaged manner—by the introduction of a complex absorption potential.

Using Eq. (43) and the energy conservation condition, we obtain the amplitude in the *post* and *prior* representations:

$$f_{\beta\alpha}=-\frac{\mu_\beta}{2\pi\hbar^2}\langle\chi_\beta^-\Psi_\beta|V_\alpha-\langle\alpha|V_\alpha|\alpha\rangle|\chi_\alpha^+\Psi_\alpha\rangle$$
$$=-\frac{\mu_\beta}{2\pi\hbar^2}\langle\chi_\beta^-\Psi_\beta|V_\beta-\langle\beta|V_\beta|\beta\rangle|\chi_\alpha^+\Psi_\alpha\rangle. \quad (46)$$

If, for example, we consider deuteron stripping, we can show that the optical potential $\langle\beta|V_\beta|\beta\rangle$ describes the interaction between the outgoing proton and the core, and the difference $V_\beta-\langle\beta|V_\beta|\beta\rangle$ is due to the interaction between the proton and the transferred neutron. This is a short-range interaction since the transfer amplitude is roughly proportional to the three-dimensional overlap integral of the product of the wave function of the relative motion of the deuteron the target nucleus and the proton product nucleus and the wave function of the neutron $\langle\Psi_B|\Psi_A\rangle$, all of them being taken at the same points.

For heavy ions, such a zero-range approximation for the interaction is invalid,[48, 49] and it is therefore necessary to calculate the multidimensional integrals (46) when both representations have the same form. On the one hand, for fixed relative distances between the heavy ions, the overlap integrals $\langle A|B\rangle$, and also $V_\alpha-\langle\alpha|V_\alpha|\alpha\rangle(V_\beta-\langle\beta|V_\beta|\beta\rangle)$ are decreasing functions for increasing corresponding radius vectors, while on the other hand the wave functions of the relative motion are small at short relative distances. And this means that the distances between the cores and the transferred nucleons that make the largest contribution to the transfer amplitude must be fairly large.

Deuteron stripping clearly reveals the cause of the small contribution from the short distances between the neutron and the core. We have already emphasized that the overlap integrals $\langle A|B\rangle$ are in fact the wave functions of bound states, i.e., they decrease exponentially at large distances and oscillate at small distances. The wave functions of the relative motion of the fragments at short distances will also oscillate strongly (if they do not completely disappear because of the imaginary

part of the potential) even at low energy of the relative motion. For a short-range interaction at short distances the integrand in (46) is a product of three oscillating functions, and therefore the integral gets its value at large distances.

The complexity of the calculations and the unsatisfactory nature of the approximations described above stimulated the development of other methods to describe nuclear reactions between complex ions. Here, we briefly describe only one of many such attempts.[50]

In the case when the DWBA is justified, i.e., when the Born series with respect to the coupling constant converges, a method equivalent to it[44] is to describe the colliding systems by means of wave packets and use nonstationary perturbation theory. In fact, however, such a description brings little advantage. For reactions induced by heavy ions, the motion of the wave packets can be replaced by motion along the classical trajectories, but even then Ψ can be regarded as the wave function. This approximation is justified by the smallness of the wavelength compared with the range of the forces responsible for the scattering. If only one impact parameter ρ contributes to the transfer cross section for given angle θ, this cross section has the simple form

$$d\sigma/d\Omega = p\,(\rho/\sin\theta)\,|\,d\rho/d\theta\,|, \tag{47}$$

where p is the probability of the process, equal in the lowest order to

$$p = \left|\exp\int_{-\infty}^{\infty}\frac{i}{\hbar}\,(V_\alpha(t) - \langle\alpha\,|\,V_\alpha\,|\,\alpha\rangle)\,dt\right|^2. \tag{48}$$

The classical cross section of elastic scattering

$$d\sigma_{el}\ /d\Omega = (\rho/\sin\theta)\,(d\rho/d\theta) \tag{49}$$

is obtained from the requirement that the corresponding phase be stationary:

$$\varphi = \frac{i}{\hbar}\int\left(\frac{1}{2}m\dot{q}^2 - \langle\alpha\,|\,V\,|\,\alpha\rangle\right)dt. \tag{50}$$

It is clear that to confront these results with DWBA calculations it is necessary to introduce an imaginary part of the potential; then in the cross section a damping factor appears:

$$a = \left|\exp\int\frac{W}{\hbar}\,dt\right|^2, \tag{51}$$

and its role reduces to suppressing the contribution to the transfer cross section of small impact parameters. This once more emphasizes the importance of large distances in transfer reactions.

In this section, we have discussed a number of simple methods for calculating transfer amplitudes in which the role of the wave functions of the transferred particles is fairly perspicuous. The situation is the same as in the more complicated approaches,[49] but in them one must calculate multidimensional integrals (six-dimensional in single-nucleon and nine-dimensional in two-nucleon transfer reactions between complex ions).

It is, however, worth emphasizing that in many-nucleon transfer reactions questions related to the asymptotic behavior of the wave functions of the transferred nucleons become more important and more complicated than for single-nucleon transfers. For the latter, the asymptotic behavior of the wave functions can be established by simple considerations (see Secs. 4 and 5). For two-nucleon transfers, progress has been achieved only very recently (see Sec. 5), and for a larger number of transferred nucleons the problem of finding the correct asymptotic behavior of the wave functions of the transferred nucleons has not yet been solved, except for the special case of cluster transfer.

4. DEFORMED NUCLEI

Study of stripping and pickup reactions on strongly deformed nuclei has provided the best interpretation of the structure of certain populated states, especially single-particle states. This is because in nuclei that are far from magic one can separate an average deformed field and the single-particle states corresponding to it from the collective motions of the nucleus as a whole. In many nuclei, the collective and single-particle motions are separated and certain states on the Fermi surface are single-particle states to a good accuracy. We describe briefly the method of calculation of single-particle wave functions with required asymptotic behavior in the deformed average field and the application of the resulting wave functions to the calculation of single-nucleon transfer cross sections.

The general equations for the form factors obtained from the Schrödinger equations for nuclei $|A\rangle$ and $|B\rangle$:

$$\left.\begin{aligned} H_A\,|\,A\rangle &= E_A\,|\,A\rangle;\\ (H_A + \textstyle\sum_i [V_i + T_i] + V_{\text{res}})\,|\,B\rangle &= E_B\,|\,B\rangle, \end{aligned}\right\} \tag{52}$$

have a different form

$$(E_B - E_A)\,\langle A\,|\,B\rangle = \textstyle\sum [V_i + T_i]\,\langle A\,|\,B\rangle + \langle A\,|\,V_{\text{res}}\,|\,B\rangle \tag{53}$$

depending on the particular problem considered. Note that by V_{res} we can understand all the interactions, irrespective of whether they commute with H_A. If V_{res} does not commute with H_A, then we obtain an infinite system of coupled equations of the type (53), including different states $|A_i\rangle$ ($|B_i\rangle$). Since these states can be populated in pickup or stripping reactions, we investigate the form factors, which describe the corresponding experimental data.

If odd and even nuclei B and A, respectively, are strongly deformed and axisymmetric, then they have large moments of inertia J, which leads to appreciable simplifications in calculations when one is studying states which are not very highly excited. The first simplification is due to the smallness of the Coriolis mixing for the case when there exist regular rotational bands in both system A and system B. These states are characterized by the quantum number K, the projection of the total angular momentum I onto the symmetry axis of the nucleus. In the internal coordinate system, the Coriolis interaction has the form

$$H_{\text{Cor}} = (-\hbar^2/2J)\,(j_+I_- + j_-I_+), \tag{54}$$

where j_+ and j_- are single-particle operators; I_+ and I_- are the operators of the total angular momentum. The smallness of the Coriolis interaction follows from the following ratio, which is obtained in perturbation theory:

$$\left|\frac{\langle K\pm1|H_{\text{Cor}}|K\rangle}{E(K\pm1)-E(K)}\right|\ll 1, \tag{55}$$

in which the denominator is usually 20–100 times larger than the numerator[51,52] [$E(K)$ and $E(K\pm1)$ are the single-particle energies]. At the end of this section, we shall consider in more detail the influence of the Coriolis interaction on the transfer cross sections; here we just point out that the Coriolis interaction does not have a radial dependence. The second simplification arises from neglect of the rotational energies, and then the equations for the form factors reduce in practice to equations of single-particle motion in the deformed average field. In this case (we ignore all the residual interactions) the wave functions corresponding to different energies are orthogonal and it is very simple to take into account the Pauli principle.

As we have already said, the parameters of the average field of the nuclei must in the general case depend on the energy. However, the calculations then become fairly cumbersome: The wave functions will be orthogonal with complicated weights, the effective potentials will be nonlocal, and so forth. Therefore, in many problems of nuclear physics (see Refs. 39 and 40) the energy dependence of the parameters of the potential is ignored. Moreover, it is very frequently assumed that the depth of the average-field potential for deformed nuclei is approximately the same as for spherical nuclei.[53-55]

For the single-particle states of deformed nuclei, it is very important that for equilibrium or large values of the deformation parameters certain states with different $N=2n+l$ become almost degenerate (see the review Ref. 53, and also Refs. 11 and 56–58). It is known that states with the same parity and the same projection of the angular momentum for an oscillator potential (without ls and l^2 terms) are completely degenerate. The level scheme and the wave functions are sensitive to the extent to which the realistic potential employed differs from that of a pure oscillator. Here, it is important to take into account in the deformation potential a higher order than the quadrupole orders, and the deviation of the radial dependence of the potential from r^2. Since the mixing of states with different N has been ignored until recently, we give briefly the results of Ref. 59.

We write down the Schrödinger equation

$$[-\Delta+cW(\mathbf{r})]\,\Psi_{\Omega,\pi}(\mathbf{r})=E\Psi_{\Omega,\pi}(\mathbf{r}), \tag{56}$$

where

$$\left.\begin{aligned} &CW(\mathbf{r})=C[V(\mathbf{r})+\tilde{V}_{\text{s.o.}}(\mathbf{r})]+V_{\text{s.o.}}(r)+V_{\text{Coul}}(\mathbf{r});\\ &\tilde{V}_{\text{s.o.}}(\mathbf{r})=V_{\text{s.o.}}(\mathbf{r})-V_{\text{s.o.}}(r);\\ &V(\mathbf{r})=-V_0\{1+\exp[(r-R(\theta))/a]\}^{-1};\\ &R(\theta)=R_0(1+\beta_0+\beta_{20}Y_{20}+\beta_{40}Y_{40}). \end{aligned}\right\} \tag{57}$$

All the parameters are taken from Ref. 53. We represent the single-particle wave function $\Psi_{\Omega,\pi}$ of a bound state of a deformed nucleus as a superposition over Sturm–Liouville basis functions:

$$\Psi_{\Omega,\pi}=\sum_{nlj}^{M}a^{\Omega}_{nlj}\frac{f_{nlj}(r)}{r}|ljM\rangle. \tag{58}$$

For the further treatment, as we pointed out at the end of Sec. 2, there are two possibilities. One can assume that E in (56) is an eigenvalue of the problem and that the parameter C is fixed (we can set $C=1$). Then, substituting the expansion (58) in (56), multiplying $V(r)$ from the left by f_i^*, and using the orthogonality of the functions f_i with a weight, we integrate with respect to r,θ,φ, obtaining the system of equations

$$\sum_i A_{ij}a_j=\sum_j\langle i|V(H-E)|j\rangle a_j=\sum_j(\langle i|VH|j\rangle+E\delta_{ij})a_j. \tag{59}$$

If we multiply from the left by f_i^*, then after integration we find

$$\sum_j A'_{ij}a_j=\sum_j\langle i|H-E|j\rangle a_j=\sum_j(\langle i|-\Delta-E|j\rangle+C\langle i|W|j\rangle)a_j. \tag{60}$$

However, as generalized eigenvalues we have the parameter C, and the energy $E=\varepsilon$ is taken from the experimental data.

It is readily seen that the matrix A_{ij} is non-Hermitian, while A'_{ij} is Hermitian. One can therefore say[27] that diagonalization of the matrix A'_{ij} is equivalent to using a variational principle. Further, since for (60) we take all the basis functions f_i for $E=\varepsilon$ equal to the experimental value, convergence of the expansion in the asymptotic region too [see the relation (29) and the discussion after it] is guaranteed even for small values of M. This was to be expected since in the coupled-channel method all the components of the wave function must have the same energy,[29-32] this being equal to an eigenvalue of the problem (under the condition that all interactions decrease sufficiently rapidly at large distances). In this sense, the system (60) is analogous to the system of coupled-channel equations. It is therefore preferable to work with the matrix A'_{ij}.

In Table I, we give the mixing coefficients a^{Ω}_{nlj} for some neutron states of the nucleus ^{155}Sm, obtained by solving the system (60) for fixed values of E equal to the energy for separation of a neutron from the nucleus ^{155}Sm. For this nucleus, the levels with $N=5$ (for odd states) are situated at the Fermi surface, and in the Nilsson model only these states participate in the diagonalization procedure. In our method too the subshell with $N=5$ makes the main contribution, i.e., the largest mixing coefficients correspond to states in this subshell. However, as can be seen from Table I, it is necessary to take into account basis states from the subshells with $N=3$, 7, and sometimes with $N=9$.

In Fig. 3, we represent the results of calculations of the ^{154}Sm$(d,p)^{155}$Sm reaction cross sections at $E_d=12$ MeV made with the wave functions of the bound neutron states given in Table I. The transfer cross sections were obtained in the coupled-channel method, the optical parameters of the ingoing and outgoing channels being determined from elastic and inelastic scattering of deuterons and protons at different energies and on the corresponding targets (see Ref. 60 and the references in it). In this sense, the coupled-channel calculations are self-consistent since there is no free parameter, and therefore comparison of the theoretically obtained transfer cross sections with the experimental ones indicates the correctness (or incorrectness) of our assumptions about the structure of the nucleus. The coupled-channel calculations differ from those of the

TABLE I. Coefficients of expansion with respect to Sturm-Liouville functions for bound neutron states of the nucleus ^{155}Sm.

Band	Basis states	Expansion coefficients			
		$N=1$	$N=3$	$N=5$	$N=7$
3/2⁻[521]	$p_{3/2}$	−0.043	−0.115	0.344	0.035
	$f_{5/2}$	—	0.006	−0.266	−0.046
	$f_{7/2}$	—	0.007	0.799	0.045
	$h_{9/2}$	—	—	0.165	−0.058
	$h_{11/2}$	—	—	−0.313	0.123
	$i_{13/2}$	—	—	—	0.041
	$i_{15/2}$	—	—	—	−0.046
1/2⁻[521]	$p_{1/2}$	0.025	0.065	−0.500	−0.036
	$p_{3/2}$	0.002	0	0.308	0.007
	$f_{5/2}$	—	−0.061	−0.109	−0.059
	$f_{7/2}$	—	−0.042	−0.514	0.031
	$h_{9/2}$	—	—	0.494	0.009
	$h_{11/2}$	—	—	0.255	−0.095
	$i_{13/2}$	—	—	—	0.103
	$i_{15/2}$	—	—	—	0.035
Band	Basis states	Expansion coefficients			
		$N=3$	$N=5$	$N=7$	$N=9$
5/2⁻[523]	$f_{5/2}$	0.055	−0.279	−0.024	−0.007
	$f_{7/2}$	0.043	0.770	−0.033	−0.007
	$h_{9/2}$	—	−0.361	−0.060	0.016
	$h_{11/2}$	—	−0.407	0.120	0.029
	$i_{13/2}$	—	—	−0.055	−0.021
	$i_{15/2}$	—	—	−0.047	0.001
	$l_{17/2}$	—	—	—	−0.007
5/2⁻[512]	$f_{5/2}$	−0.070	0.161	0.027	0.014
	$f_{7/2}$	0.083	0.467	0.028	0.001
	$h_{9/2}$	—	0.815	0.003	0.002
	$h_{11/2}$	—	0.051	0.058	0.019
	$i_{13/2}$	—	—	0.151	0.042
	$i_{15/2}$	—	—	0.005	0.001
	$l_{17/2}$	—	—	—	0.020
		$N=2$	$N=4$	$N=6$	$N=8$
5/2⁺[642]	$d_{5/2}$	−0.014	−0.063	0.072	0.018
	$g_{7/2}$	—	0.030	−0.038	−0.016
	$g_{9/2}$	—	−0.198	0.379	0.062
	$i_{11/2}$	—	—	−0.076	−0.035
	$i_{13/2}$	—	—	0.820	0.074
	$k_{15/2}$	—	—	—	−0.034
	$k_{17/2}$	—	—	—	0.135

DWBA, in which spectroscopic information is extracted when the theoretical cross sections are compared with the experimental data. The results given clearly reveal a good description of the cross section of transfers to low-lying states, but the agreement for the cross sections of transfers to highly excited states is much less good. It is clear that in this case it is necessary to take into account a larger number of channels and also the complexity of the structure of the occupied states.

In Refs. 56–59, an investigation was made into the structure of the quasicrossing levels 1/2⁺[660] and 1/2⁺[400] and also 3/2⁺[651] and 3/2⁺[402] of nuclei at the start of the rare earth region of elements; these lie very close to each other in energy and have a strong mixing of the components ($\Delta N=\pm 2$ mixing). These states were observed in (d,p) and (d,t) reactions.[61] In

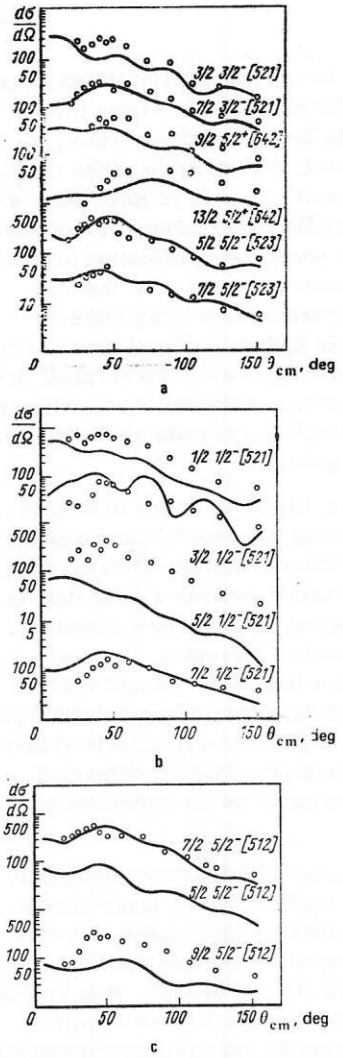

FIG. 3. Comparison of differential cross sections calculated by the coupled-channel method (continuous curves) and the experimental data (open circles) of the reaction ^{154}Sm$(d,p)^{155}$Sm for energy $E=12$ MeV (for details, see Ref. 60).

Ref. 62, which is devoted to this question, differential transfer cross sections were calculated in the DWBA. However, as was emphasized in Ref. 12, the differential cross sections calculated in the DWBA are insensitive to the mixing coefficients of the single-particle wave functions since the contribution of only one basis state is taken into account in the transition amplitude and all the remaining states are ignored (if the target nucleus is even). It is clear from this that definite information about the $\Delta N=\pm 2$ mixing coefficients can be obtained only by studying the transfer-reaction cross sections by means of the coupled-channel method. Moreover, the mixing of the rotational bands due to the Coriolis interaction must be investigated separately since for $\Delta N=\pm 2$ mixed states near the quasicrossing of two levels the Coriolis interaction may become particularly important.[52]

As is shown in Ref. 56, the intensity of $\Delta N=\pm 2$ mixing depends strongly on the accuracy in the solution of the Schrödinger equations. In order to increase the accuracy in the calculation of the eigenvalues and eigen-

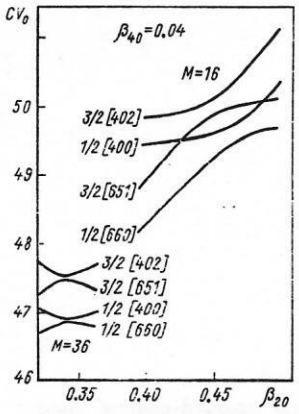

FIG. 4. Eigenvalues CV_0 as function of the deformation parameter β_{20} and the rank M of the matrix of the diagonalization (60) for $\beta_{40}=0.04$.

functions $\Psi_{\Omega,\pi}$ for $\Delta N=\pm 2$ mixed states, it is sufficient to increase the rank M of the matrix of the system (60). Figure 4 shows the eigenvalues CV_0 of Ref. 59 as a function of the deformation parameter β_{20} for $\beta_{40}=0.04$. The value of β_{40} is taken from Ref. 56, where β_{40} was determined by the requirement that the calculated relative distances of the quasicrossing levels and the experimental levels be equal. The rank of the matrix M is equal to 16 ($N=0, 2, 4, 6$) and 36 ($N=0, 2, 4, 6, 8, 10$), respectively. Figure 5 shows the behavior of the $3s_{1/2}$ and $1i_{13/2}$ components of the single-particle wave function as a function of the quadrupole deformation parameter for $\beta_{40}=0.04$, and the corresponding differential cross sections are in Figs. 6 and 7. These figures clearly illustrate the strong dependence of the mixing coefficients on β_{20} in the region of quasicrossing of the levels, and the corresponding cross sections in the same region vary by an order of magnitude (for example, when β_{20} varies by 0.007 the cross section of transfer to the state 1/2 1/2⁺[660] varies by two orders of magnitude). From evaluation of experimental data on elastic and inelastic scattering, it is obviously impossible to determine deformation parameters with such accuracy. For the considered case at deformation $\beta_{20}=0.3296$ both the relative and absolute values of the cross sections agree well with the experimental data, and therefore one can obtain more precise deformation parameters when studying transfer reactions to such quasicrossing levels. This is not a particularly trivial

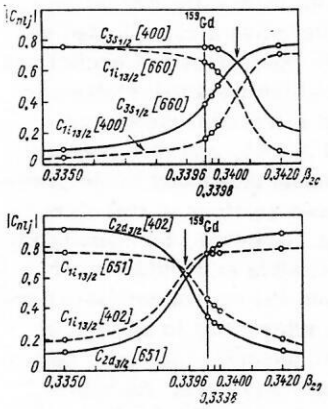

FIG. 5. Behavior of the $3s_{1/2}$, $2d_{3/2}$, and $1i_{13/2}$ components of the single particle wave function as functions of the quadrupole deformation parameter β_{20} for $\beta_{40}=0.04$.

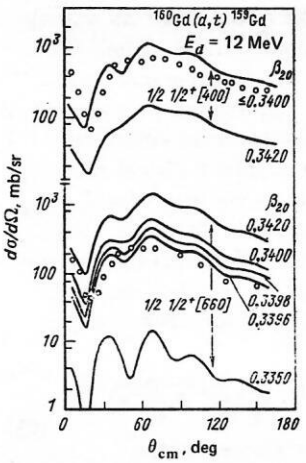

FIG. 6. Calculated differential reaction cross sections of transfers to the states 1/2 1/2⁺ [400] and 1/2 1/2⁺ [660] for different values of the quadrupole deformation parameter β_{20} for $\beta_{40}=0.04$. The experimental data are the open circles.

result since structure (spectroscopic) calculations[56] do not enable one to obtain definite information about the order in which these $\Delta N=\pm 2$ mixed levels are arranged or consequently, to determine more accurately the deformation parameters or other parameters of the problem.

The interaction of various types of motion that, as we are confident, must be taken into account in real calculations destroys the simple single-particle picture of excitations in deformed nuclei. The Coriolis interaction is one of the most important and best understood residual interactions leading to a mixing of the single-particle wave functions. In the DWBA, the differential cross section has a fairly simple form[61] (even–even target):

$$d\sigma/d\Omega = 2\Big|\sum_N \alpha^{(N)} C_{jl}^{(N)} (\Phi_l^{(N)})^{1/2}\Big|^2. \qquad (61)$$

It can be seen from (61) that the cross section is a sum of simple cross sections obtained in the framework of the DWBA which are multiplied by appropriate coefficients $(\alpha^{(N)}C_{jl}^{(N)})^2$. Very frequently, these coefficients are calculated in Kerman's model,[51] in which only two rotational bands are mixed.

We write the wave function of the final nucleus B in the most general form[52]:

$$\Phi^*_{J_B M_B}(r,\ \theta_i) = \sqrt{\frac{2J_B+1}{16\pi^2}} \sum_{K_B \alpha} C^{J_B}_{K_B \alpha} \times \Big\{ D^{I^*_n}_{M_B K_B}(\theta_i)\, \Psi^{(\alpha)}_{K_B=\Omega,\pi}(r) + (-1)^{J_B+K_B+1} \times D^{J_B}_{M_B K_B}(\theta_i)\, \Psi^{(\alpha)}_{K_B=-\Omega,\pi}(r) \Big\}, \qquad (62)$$

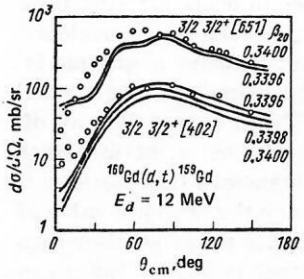

FIG. 7. The same as in Fig. 6 for transfers to the states 3/2 3/2⁺ [402] 3/2 3/2⁺ [651].

where $C^{J_B}_{K_B\alpha}$ are the coefficients of the Coriolis mixing calculated in the semi-microscopic model of Ref. 52. In (62), the summation is over the nonconserved projections K_B of the total angular momentum J_B and over the index α, which distinguishes different rotational states with the same K_B. If we consider single-nucleon transfer reactions to an even–even nucleus with $J_A = K_A = 0$, then $J_B = J$ and $K_B = \Omega$, where J is the transferred angular momentum; Ω is its projection onto the symmetry axis of the nucleus. Then in the coupled-channel approximation, the differential transfer cross section can be written in the form[63]

$$\sigma^{MCK}_{LJ\Omega}(\theta) = \frac{1}{2}\frac{M_d M_p}{(2\pi\hbar^2)^2}\frac{D_n p k_p}{2\pi k_d}(2L+1)^{-1}\sum_m \Big|\sum_{(\Omega\alpha)} C^J_{\Omega\alpha}\sum_{nlj} a^{(\Omega_i\alpha)}_{nlj} \times \sum_\sigma (LS\Omega - \sigma\sigma \,|\, J\Omega)(ls\Omega - \sigma\sigma \,|\, j\Omega) B_{L\Omega}(nlj\sigma m_p)\Big|^2. \quad (63)$$

The amplitudes $B_{L\Omega}$ were determined in Ref. 12, and the coefficients $a^{(\Omega_i\alpha)}_{nlj}$ are solutions of the system of equations (60).

Comparing the cross sections in the coupled-channel approximation (63) and the DWBA (61), we see that the cross section in the latter is the coherent sum of all possible transfers to rotational states coupled through the Coriolis interaction. Calculation of the differential transfer cross sections with allowance for the Coriolis mixing in accordance with the method of Ref. 52 shows that although the admixtures for each state are small the cross section is changed appreciably. For example, the ratios of the cross sections for transfer to the states $1/2^+[400]$ and $1/3^+[600]$ with and without allowance for Coriolis mixing differ by 30%. It is well known that in rotational bands the Coriolis mixing increases with increasing spin of the state, and it is therefore of interest to investigate the cross sections of transfer to excited rotational states with large K. In this case it is necessary to take into account the mixing of states with $\Delta K = \pm 1, \pm 2, \ldots$.

Thus, in many cases, the cross section of single-nucleon transfer can be obtained with the same accuracy as the experimental data. Obviously, this can be achieved only when the populated states are to a good accuracy single-particle or quasiparticle states, but then[55] the cross section must be multiplied by the factor u^2 or v^2.

Recently, papers have been published[15, 18] in which the cross sections of two-nucleon transfer reactions to rare-earth nuclei are investigated. The form factors of the two-nucleon transfer reactions are expanded with respect to Sturm–Liouville functions in the WDP approximation with $\epsilon = E/2$, where E is the energy of separation of two nucleons, pairing is used in the approximation $G = \text{const}$, and allowance is made for virtual excitations of 0^+, 2^+, and 4^+ states of the target nucleus and the product nucleus, but the transfer amplitude is calculated in the first Born approximation. As a result, for the reaction $^{182}W(^{12}C, ^{14}C)^{180}W$ at energy 70 MeV of the relative motion a good description is obtained of the differential cross sections of transfers that populate the 0^+ and 2^+ states (Fig. 8), although the absolute value of the theoretical cross section is 1.5 times smaller than the experimental. This difference is small, and it can

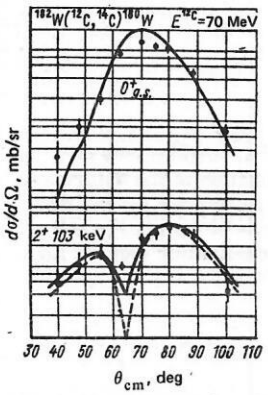

FIG. 8. Experimental and theoretical (coupled-channel method) differential cross sections of $^{182}W(^{12}C, ^{14}C)^{180}W$ transfers to 0^+ and 2^+ states of the ground-state rotational band of ^{180}W. The calculations include all inelastic processes and take into account simultaneously transfer of two neutrons to the 0^+, 2^+, and 4^+ states of the ground-state bands of the nuclei ^{182}W and ^{180}W. The theoretical curves have been multiplied by 1.5. The continuous curves correspond to form factors calculated for $E/2$, where E is the energy of separation of two neutrons in ^{182}W. The dashed curve gives energies equal to the separation energies of one neutron in ^{182}W (see Ref. 18).

be reduced by taking into account the second order of the Born approximation.[41]

The method described above can also be used to investigate transfer reactions to nuclei having a vibrational level spectrum. However, in these cases the adiabatic approximation is inapplicable, so that for each channel of multistep stripping it is necessary to have a basis set of Sturm–Liouville functions with different energies determined from the energy conservation law. Details of the calculations can be found in Ref. 14.

Thus, one- and two-nucleon transfer reactions to deformed nuclei that populate low-lying states can be investigated by means of wave functions obtained by an expansion with respect to Sturm–Liouville basis functions in the couple-channel approximation. As we have already seen, one cannot use this method directly for high excitations. Obviously, the structure of nuclear states loses its simplicity with increasing excitation energy, and the wave function becomes a complicated superposition of single-particle excitations and excitations of the core. The structure of highly excited states can be described by means of the semi-microscopic approach developed in Refs. 64 and 65, and then one can directly apply one of the methods described in the present paper to calculate the cross sections of transfers to each of the considered states. However, for such fairly high excitations it is more sensible to consider averaged characteristics of the nucleus over a definite energy range, and it is therefore interesting to attempt to apply a statistical approach to describe the cross sections of transfers to highly excited states. As before, for weakly bound single-particle states of deformed nuclei, Sturm–Liouville functions provide better bases since the asymptotic behavior of the wave functions in this case is very important (see Figs. 1 and 2). In Ref.

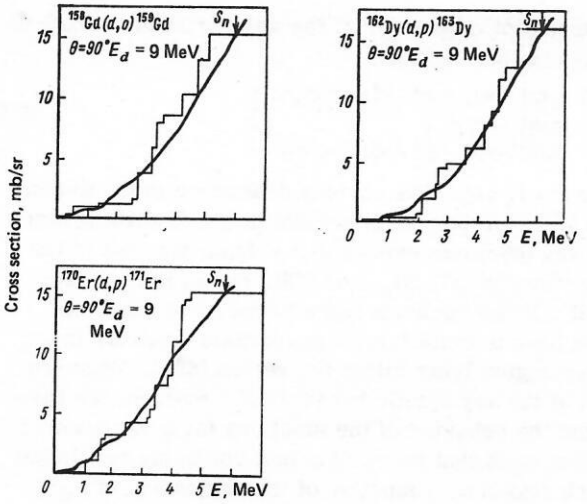

FIG. 9. Experimental cross sections (histogram) and theoretical cross sections as function of the excitation energy (S_n is the neutron separation energy).

66, the cross sections obtained with such form factors in the coupled-channel approximation were averaged over the Lorentz distribution. Figure 9 shows the integrated cross sections and one can see that there is reasonable agreement with experiment, despite the rather approximate nature of the calculations.

The usefulness of statistical averaging and the success of comparison of the averaged characteristics obtained in the semi-microscopic model of Refs. 64 and 65 with experiment suggest that for such averaging the fine structure of the states and the accuracy of the approximations adopted are not too important. If this is so, then one can introduce a complex energy-dependent average field in the same way as is done in the optical model or in a calculation of the total energy of a nucleus and in quasielastic nucleon-knockout reactions.[39,40] Then the imaginary part of the complex potential will be responsible for the averaging, and factors of the type of the Lorentz distribution appear in the expressions for the cross section,[39,40] these being equal to unity for purely single-particle stripping and less than unity and smeared with respect to the energy when a complex state is populated as a result of the transfer.

The results presented in this section show clearly that the Sturm–Liouville method of calculating the single-particle states in the deformed average field can, in conjunction with the coupled-channel method, be used to describe transfers to states with low excitation energy and to highly excited states, the transfer cross section being obtained in many cases with as good accuracy as the experimental data.

5. SPHERICAL NUCLEI

The present state of the theory of single-nucleon transfers to spherical nuclei is fairly well presented in the review Ref. 22. The use of Sturm–Liouville functions to calculate the form factors for such reactions made it possible to establish the accuracy of other methods and indicate the limits of their applicability.

In this section, we discuss some problems that arise in an investigation of two-nucleon transfer reactions on spherical nuclei. The situation here is much more complicated than in the single-nucleon case. The absolute values of the theoretical cross sections of two-nucleon transfer reactions between complex ions are smaller than the experimental values by 3–10^3 times. This is due to a lack of knowledge of the very mechanism of the two-nucleon transfer reactions and to the complexity in the description of the asymptotic behavior of the two-nucleon form factors. It was the hope for a long time that the interactions between the reaction channels are weak, so that one need take only the first perturbation order. But it was found (see, for example, the review Ref. 45 and the references there) that transitions to low-lying collective states are strongly affected by inelastic excitations. Therefore, the single-step DWBA was modified in such a way as to include inelastic excitations in the treatment. This is none other than the coupled-channel method, which nevertheless still regards the process of nucleon transfer as a single-step process. (This approximation was completely satisfactory for single-nucleon transfers; see Ref. 22 and the results of Sec. 4.) It is natural that this treatment of inelastic scattering and the transfer process cannot be satisfactory. Subsequently, the assumption of a single-step process was removed in the study of transfer reactions,[20,41-43] and it was found that the second order of the Born approximation makes a contribution to the cross section comparable with the first order. Details and numerical examples can be found in the papers quoted above.

We consider now the simplest case of two identical particles interacting with one another and an inert core. The main assumption is that the residual interactions between the particles are weak, so that bound states of the two particles do not exist. We assume that the inert core is infinitely heavy, and thus our system consists of two identical particles outside filled shells (for example, ^{42}Ca, ^{210}Pb). Despite its simplicity, this system preserves the main characteristic features of many-particle systems, and methods of solution suitable for solving the simple case can be generalized.

Thus, let us solve the Schrödinger equation

$$(H_0+\lambda^{(1)}V(r_1)+\lambda^{(2)}V(r_2)+\gamma V_{12}(r_{12})-E)\,\Psi=0, \qquad (64)$$

where V_{12} are the two-particle residual interactions; $V(r_1)$ and $V(r_2)$ can be chosen in the form (3). For simplicity, we ignore the Coulomb and spin–orbit forces, and omit everywhere the spin indices, although in concrete calculations they are included explicitly.[17,24,25,68] A number of approximate methods have been proposed for solving Eq. (64), but only two are candidates for being the most accurate: expansion with respect to harmonic-oscillator functions[41,69] and expansion with respect to Sturm–Liouville basis functions. In Sec. 2, we have given some arguments concerning the convergence of the various methods, and in this case too the expansions with respect to Sturm–Liouville functions must converge rapidly. The similarity of the principal components of the Sturm–Liouville expansion and the shell-function expansion facilitates comparison of these two

methods and ensures that the description of certain properties of the nucleus determined by the internal parts of the wave functions is approximately the same. But the identity of the principal components does not mean that other properties of the nucleus determined by small mixing coefficients correspond.

It can be assumed that the depths of the potentials $\lambda^{(1)}$ and $\lambda^{(2)}$ are approximately known, and then the energies E can be fitted to the experimental values by varying the parameters of the residual interaction. This procedure is used in the shell model, and we proceeded in the same manner, but we also investigated the transfer amplitudes, requiring a good description of the cross sections. We also used the well known fact that the residual interactions between valence nucleons do not coincide with the residual interactions between free particles, and this difference cannot be established from basic principles.

Another difficulty is the computational complexity if we wish to use more realistic inter-nucleon forces. Very frequently, this complexity is used to justify the use of model forces such as pairing forces or δ-functional forces. However, if we work with model forces we must take a certain care, especially if we change the dimension of the configuration space in which the Hamiltonian is diagonalized. In general, model forces contain certain parameters, which are fixed in a definite space of basis functions in such a way that a number of quantities (for example, the energies) have values agreeing with the experimental ones. But if the basis is extended, these parameters must be changed in such a way as to preserve the existing agreement. It is well known that for pairing with $G = \text{const}$ the process diverges in the sense that if the pairing gap is fixed then the constant of the pairing forces will tend to zero if the number of basis states is increased indefinitely. The situation is the same for the δ-functional forces.[70] For certain physical properties of the nucleus, a divergence of this type is unimportant, as for example in spectroscopic calculations with allowance for pairing. However, if the mixing coefficients for high orbitals are not well determined, then they can no longer be used to calculate transfer amplitudes, since this may lead to an unphysical result. In such a case, it is necessary to introduce more realistic forces or operate in a truncated basis, as is usually done in concrete calculations.

The asymptotic behavior of the wave functions of bound states of three-particle systems have been fairly fully studied in Ref. 71, and we here briefly compare our results with the results of Ref. 71. Suppose the single-particle binding energy is

$$-E_1=\varkappa_1^2,$$

and we denote the two-particle binding energy by

$$E_2=-E=\varkappa^2.$$

We split the six-dimensional configuration space of our system into three regions:

$$\left.\begin{array}{l}\text{a)}\ r_2^2(E_2-E_1)<r_1^2E_1;\\ \text{b)}\ r_1^2E_1<r_2^2(E_2-E_1);\quad r_2^2E_1<r_1^2(E_2-E_1);\\ \text{c)}\ r_1^2(E_2-E_1)<r_2^2E_1.\end{array}\right\}\tag{65}$$

The asymptotic behavior of the wave function in each of the regions has the form

$$\left.\begin{array}{l}\text{a)}\ \exp[-\varkappa_1 r_2-(\varkappa^2-\varkappa_1^2)^{1/2}r_1]/r_1r_2;\\ \text{b)}\ \exp[-\varkappa r]/r^{5/2};\\ \text{c)}\ \exp[-\varkappa_1 r_1-(\varkappa^2-\varkappa_1^2)^{1/2}r_2]/r_1r_2,\end{array}\right\}\tag{66}$$

where $r^2=r_1^2+r_2^2$. The obvious difference in the dimensions between these expressions is not disturbing since (66a) and (66c) correspond to a volume element in the space $r_1^2dr_1r_2^2dr_2d\Omega_1d\Omega_2$, and (66b) to one in the space $r^5drd\Omega$. If two nucleons had a bound state, then we should have to consider the asymptotic behavior in one further region lying within the region (66b). When we speak of the asymptotic behavior of functions, we have in mind the behavior of the functions for a variation of r_1 and r_2 such that $r\to\infty$, this function being multiplied in each region by a function of the angles and of r_1/r_2.

Equations (66) are valid on five-dimensional surfaces for constant r; for fewer dimensions for fixed r other surfaces are possible. In particular, in the transition zones between regions a and b and between b and c the coordinate asymptotic behavior can be described by means of error functions. Thus, the asymptotic behavior of the wave functions of the bound states is too complicated for one to construct a corresponding Sturm–Liouville basis. One can construct a simple basis for each of the three regions, but it is not possible to obtain a simple basis for all three regions simultaneously.

Actually, however, one can use the fact that the binding energy due to the residual forces is positive and much smaller than the total binding energy:

$$\varkappa^2=2(\varkappa_1^2+\varkappa'^2),\quad 0<\varkappa'^2\leqslant\varkappa^2.\tag{67}$$

Therefore, one can introduce a set of functions with the asymptotic properties

$$\left.\begin{array}{l}\psi\sim\exp[-\varkappa_h(r_1+r_2)]/r_1r_2;\\ \varkappa_h^2=\varkappa^2/2.\end{array}\right\}\tag{68}$$

Then the difference between the asymptotic behavior (68) and the correct (66) will be determined by the following relations in the corresponding regions:

$$\left.\begin{array}{l}\text{a)}\ a_a\sim\exp\{r_2(\varkappa_1-\varkappa_h)+r_2[(\varkappa^2-\varkappa_1^2)^{1/2}-\varkappa_h]\}\\ \qquad\approx\exp[\varkappa'^2(r_1-r_2)/2\varkappa_1];\\ \text{b)}\ a_b\sim\exp|-[(r_1+r_2)+\sqrt{2(r_1^2+r_2^2)}](\varkappa_1^2+\varkappa'^2)^{1/2}|;\\ \text{c)}\ a_c\sim\exp\{r_1(\varkappa_1-\varkappa_h)+r_2[(\varkappa^2-\varkappa_1^2)^{1/2}-\varkappa_h]\}\approx\\ \qquad\approx\exp[\varkappa'^2(r_2-r_1)/2\varkappa_1].\end{array}\right\}\tag{69}$$

The relations (69) are valid if

$$1-2\varkappa'^2/\varkappa_1^2<r_2/r_1<1+2\varkappa'^2/\varkappa_1^2.\tag{70}$$

However, this condition has been derived approximately.

Further, we denote $r'=r_2-r_1$ and rewrite (69) in the form

$$\left.\begin{array}{l}a_a\sim a_b\sim\exp(r'\varkappa'^2/2\varkappa_1);\\ a_c\sim\exp[(\varkappa_1^2+\varkappa'^2)^{1/2}r'^2/4r_1]\approx\\ \approx\exp[(\varkappa_1^2+\varkappa'^2)^{1/2}(\varkappa'^2/\varkappa_1^2)(r'/2)]\approx\exp[r'\varkappa'^2/2\varkappa_1].\end{array}\right\}\tag{71}$$

Thus, for small $\varkappa'$ the asymptotic behavior of the wave function is too large if (68) is used by a factor $\lesssim\exp(r'\varkappa'^2/2\varkappa_1)$ and, for example, for the overlap integral (^{208}Pb, ^{210}Pb) we obtain $\varkappa'^2/\varkappa_1^2\sim 0.1$, $\varkappa_1=0.5$ F^{-1}, so

that the error is ≈25% for $r' \geq 10$ F. Obviously, the transfer amplitude will be small at distances $r' \geq 10$ F. For a single-step two-nucleon transfer the error is even smaller.

The asymptotic expression (68) determines the set of Sturm–Liouville functions satisfying the equation

$$(-\Delta + \lambda_i V + \varkappa_i^2) f_i = 0. \tag{72}$$

One is also quite free to choose two other sets of Sturm–Liouville functions satisfying the equations

$$\left.\begin{aligned} &[-\Delta_1 + \lambda_i V + (\varkappa^2 - \varkappa_1^2)] f_i(r_1) = 0; \\ &[-\Delta_2 + \lambda_j V + \varkappa_1^2] f_j(r_2) = 0. \end{aligned}\right\} \tag{73}$$

These functions will give the correct asymptotic behavior in region a but not in region c. To correct this and correctly antisymmetrize the wave functions, one can add the set of functions satisfying the same Eqs. (73) but with r_1 and r_2 interchanged. The orthogonality properties between the different sets are then lost, and in region b one also requires a different set of functions.

For physical reasons, certain regions of space are more important than others. In the single-particle limit, we investigate the region of small r_1 for arbitrary r_2 (or vice versa). For such calculations, the basis functions (73) are ideal if correctly antisymmetrized.

We use the possibility of choosing different energies for the basis functions $\{f^{(1)}f^{(1)}\}$:

$$\left.\begin{aligned} &(H_{01} + \lambda_n^{(1)} V^{(1)} - \varepsilon_1) f_n^{(1)} = 0; \\ &(H_{02} + \lambda_m^{(2)} V^{(2)} - \varepsilon_2) f_m^{(2)} = 0, \end{aligned}\right\} \tag{74}$$

and make the expansion

$$\Psi(r_1, r_2) = \sum c_{mn} f_m^{(1)} f_n^{(2)}. \tag{75}$$

Substituting (75) in (64), multiplying from the left by $f_k^{(1)*} f_l^{(2)*}$, and integrating, we obtain

$$\begin{aligned} &\sum_{mn} c_{mn} [(\varepsilon_1 + \varepsilon_2 - E) \langle kl | mn \rangle + \gamma \langle kl | V_{12} | mn \rangle \\ &- (\lambda^{(1)} - \lambda_m^{(1)}) \langle l | n \rangle_2 \delta_{km} - (\lambda^{(2)} - \lambda_n^{(2)}) \langle k | m \rangle_1 \delta_{ln}] = 0. \end{aligned} \tag{76}$$

If we also multiply Eq. (64) by $V^{(1)}V^{(2)} f_k^{(1)*} f_l^{(2)*}$, we obtain

$$\begin{aligned} &\sum_{mn} c_{mn} [(\varepsilon_1 + \varepsilon_2 - E) \delta_{kn}\delta_{lm} + \gamma \langle kl | V^{(1)}V^{(2)} V_{12} | mn \rangle \\ &- (\lambda^{(1)} - \lambda_m^{(1)}) \delta_{ln} \langle k | V^{(1)2} | m \rangle_1 - (\lambda^{(2)} - \lambda_n^{(2)}) \delta_{km} \langle l | V^{(2)2} | n \rangle_2] = 0, \end{aligned} \tag{77}$$

where $\langle l | n = \int f_l^* f_n \, d\mathbf{r}$, etc.

Again we encounter the possibility of choosing different diagonalization methods, as is described in Sec. 2. Both systems can be written in the form

$$\{EA + \lambda^{(1)} B_1 + \lambda^{(2)} B_2 + \gamma C + R\} c = 0, \tag{78}$$

where A, B_1, B_2, and C are Hermitian matrices.

However, R is Hermitian only for (76), and even if computational arguments give preference to (77) the arguments of Sec. 2 force us to choose (76). The system (76) can be diagonalized in four different ways, in each of which one of the four parameters E, $\lambda^{(1)}$, $\lambda^{(2)}$, and γ is regarded as an eigenvalue. For the truncated basis $\{f^{(1)}, f^{(2)}\}_M$ the diagonalization gives M independent functions

$$F^{(i)} = \sum^{M} c_{mn}^{(i)} f_m^{(1)} f_n^{(2)}, \tag{79}$$

so that as a result we have Sturm–Liouville functions for the two-particle problem, and these again for a complete set.

Even if all the matrices A, B_1, B_2, C, and R are Hermitian, the total matrix that must be diagonalized to obtain E as an eigenvalue is not necessarily Hermitian:

$$E + A^{(-1)} (\lambda^{(1)} B_1 + \lambda^{(2)} B_2 + \gamma C + R) c = 0. \tag{80}$$

In some important problems, it can be diagonalized simply (see, for example, Ref. 17).

In the method of expansion with respect to Sturm–Liouville functions, as in all the approximate methods of nuclear physics, we encounter the problem of allowance for the Pauli principle. It is that solutions of the exact Hamiltonian, which is symmetric with respect to all particles, must be symmetric or antisymmetric with respect to the coordinates of all the particles. This requirement is not satisfied if one is working with a truncated basis, in particular, if the filled shells are identified with the inert core of the nucleus.

One can hope that wave functions that extremalize the Hamiltonian and are orthogonal to the occupied states of the core will be better wave functions. This was followed up in Ref. 25 by the method proposed in Ref. 34. To the Hamiltonian, a term was added that is the projection operator onto occupied one- and two-particle states multiplied by a constant T:

$$\tilde{H} = H + T \sum_i |i\rangle \langle i|. \tag{81}$$

If T is very large, $\tilde{H}$ will have eigenfunctions very close to $|i\rangle$, and eigenvalues close to T:

$$\tilde{H} |i\rangle \approx T |i\rangle. \tag{82}$$

These assertions can be proved as follows. If we know a solution of the equation with the Hamiltonian H:

$$H\Psi = E\Psi, \tag{83}$$

then the solution of the equation with the effective Hamiltonian $\tilde{H}$ ($\tilde{H}\tilde{\Psi} = \tilde{E}\tilde{\Psi}$) will be

$$|\tilde{\Psi}\rangle = |\Psi\rangle + TG(E) \sum_i |i\rangle \langle i | \tilde{\Psi}\rangle, \tag{84}$$

where $G(E)$ is the operator the Green's function. From the relation (84), we readily obtain

$$\langle i | \tilde{\Psi}\rangle = \sum_j [\langle i | 1 - TG(E) | j\rangle]^{-1} \langle j | \Psi\rangle, \tag{85}$$

and therefore $\lim_{T \to \infty} \langle i | \tilde{\Psi} \rangle = 0$. This means that the resulting wave function $\tilde{\Psi}$ is orthogonal to all the one- and two-particle occupied states $|i\rangle$ of the core and gives the desired extremum of the Hamiltonian. Details and numerical examples can be found in Ref. 25. To conclude this section, we give (Fig. 10) the cross section of two-proton transfer in the reaction ^{48}Ca(^{16}O, ^{14}C)^{50}Ti at energy 58 MeV of the incident ion. The difference between the two cross sections clearly reveals the influence of an increase of the configuration space on the calculated cross sections.

As we have already said, the absolute values of the two-nucleon transfer cross sections obtained theoretically are, as a rule, smaller than the experimental values except in a few cases. However, we have not posed the problem of obtaining the absolute values of the cross sections since it would then be necessary to

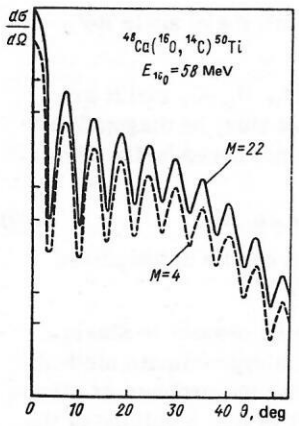

FIG. 10. Calculated cross sections of two-proton transfer for different ranks M of the matrix (76).

consider in more detail the reaction mechanism. We wished to emphasize the importance of a careful study of the asymptotic behavior of the form factors of two-nucleon transfers and demonstrate a method suitable for achieving this aim.

CONCLUSIONS

The review presented here enables one to judge the usefulness of using Sturm–Liouville functions to calculate the wave functions of bound states of nuclei and transfer-reaction amplitudes. Their advantage over other basis functions is obvious: The method using an expansion with respect to Sturm–Liouville functions enables one, even if only a few terms of the expansion are used, to obtain all the main important characteristic features of the solution to the investigated problem. Moreover, the well depth prescription, which is frequently used in calculations of the form factors of direct nuclear reactions of single-nucleon transfers, corresponds to retention in the Sturm–Liouville expansion of the one main component. In this sense, the Sturm–Liouville method is a generalization of the well depth prescription and enables one to justify its use in concrete cases.

The convergence properties (25) and (29) of the expansions of Ψ with respect to Sturm–Liouville functions mean that any continuous square-integrable function satisfying the Schrödinger equation with $\epsilon = E$ can be calculated with any preassigned accuracy in a finite region, uniform logarithmic convergence of the resulting series in the important region $r \to \infty$ being guaranteed. With regard to the problem of the eigenvalues, our investigation of the different possibilities of diagonalization proposed in the literature indicate that Hermitian methods are preferable to the others.

The difficulties of generalizing the method to many-particle systems are obvious, but the same difficulties are encountered in other methods and the connection between the Sturm–Liouville functions and the shell functions, at least in certain cases, together with the properties of completeness and uniform and absolute convergence, favor the expansion method considered here. Here, we have not discussed relativistic problems, but one can hope that the Klein–Gordon or Dirac equations could be treated in the same way as the Schrödinger equation.

An analogous method for free states is the method of expansion with respect to Weinberg functions. For positive energy, these functions form a complete set,[72] but hitherto they have not found application in nuclear physics.

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