Quantum hydrodynamic description of collective nuclear states

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The nature of low-lying collective nuclear states is analyzed in the framework of the Fermi-liquid approach. It is shown that in a drop of Fermi liquid one has not only the ordinary zero-sound branch but also a new collective mode resulting from the spontaneous breaking of the translational invariance. These are quantum capillary waves that have much in common with ordinary surface excitations of a classical drop. Numerical calculations show that the first collective levels of nuclei belong to this branch.

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INTRODUCTION

The theory of collective nuclear excitations has so far passed through two stages. The original notions about collective motions in nuclei were based on the liquid-drop model,[1,2] which successfully explained the general overall features: constancy of the nuclear density, the dependence of the total binding energy on the charge Z, the mass A, etc. According to this model, a nucleus can be regarded as a drop of an incompressible classical liquid. Its collective states—capillary waves—were studied thoroughly long ago. Their main characteristics, the frequencies ω_L and the excitation probabilities $B(EL)$, are determined once a single parameter—the coefficient ε of the surface tension—has been specified, and depend smoothly on A and Z. The experiments confirm the existence in nuclei of low-lying collective states, but a detailed comparison of the theoretical predictions with the experimental data reveals numerous discrepancies.[3] For example, classical hydrodynamics is incapable of explaining the experimentally observed irregularities in the behavior of the frequencies and excitation probabilities of the first 2^+ and 3^- states of even–even nuclei; its predictions for the sum rules for electromagnetic transitions also contradict the experiments. Finally, it gives incorrect moments of inertia of deformed nuclei.

The contradictions arise because one of the conditions of applicability of classical hydrodynamics is not satisfied in a nucleus: the mean free path λ_0 of the particles is not small compared with the wavelength λ of a collective excitation. The experiments show that at low excitation energies λ_0 is greater than the nuclear diameter.

This result can be explained by the Pauli principle and is reflected in the theory of a Fermi liquid.[4-6] One of the main results of this theory is that in the limit $T \to 0$ ordinary sound ceases to propagate in a Fermi liquid, but instead there is a new collective branch: zero sound. These are volume excitations and in no way similar to capillary waves. Notions about collective motion taken from Fermi-liquid theory were first applied to nuclear physics by Belyaev.[7,8] His papers were the first microscopic investigations into the theory of collective motion in nuclei. Belyaev constructed an analytically solvable model,[8] in which, by introducing effective "pairing + QQ interaction" forces, he succeeded in explaining the observed irregularities in the spectra of 2^+ states of even–even nuclei. These papers found a wide and favorable response, and the interpretation of the first collective levels of nuclei as volume excitations of zero-sound nature as proposed in Ref. 8 was widely accepted. However, this interpretation is entirely based on a definite hypothesis about the form of the effective forces.

A quantitative theory of nuclear phenomena constructed on Fermi-liquid principles was created by Migdal[9] in his theory of finite Fermi systems. The main conclusion of this theory is that in weak external fields a nucleus behaves like a gas of interacting quasiparticles in a box, the self-consistent potential playing the role of the box. In this theory, the effective interaction between the quasiparticles is not calculated but parametrized by means of a small number of universal constants, which are determined by comparing theory and experiment. Numerical calculations in the framework of this theory made it possible to describe many nuclear properties (magnetic and quadrupole moments, isotope shifts, probabilities of single-particle transitions, dipole photoabsorption, nuclear β decay, μ capture, etc) and thereby demonstrated that the properties of a Fermi-liquid drop at low excitation energies can be understood on the basis of a gas picture.

A difficulty arose only when it was attempted to determine the change in the radius R of the system when particles are added (R can be defined as the distance from the symmetry center to the point at which the density ρ has decreased by a factor 2). It was found that the gas picture here leads to a paradoxical result: The radius R hardly increases when particles are added successively, whereas in a drop it increases in accordance with $R = r_0 A^{1/3}$. Thus, the microscopic quantum theory was incapable of explaining something that is readily deduced in the classical theory; for the density of a drop is determined by the condition that the pressure satisfy $p = -(\partial E/\partial V)_A \sim -(\partial E/\partial R)_A = 0$, but for a gas of quasiparticles confined in a box there is no such condition, the size of the box being an external parameter.

From the microsopic point of view, $(\partial E/\partial R)_A$ can be interpreted[10] as the derivative of the collective Hamiltonian $E(R)$ with respect to the collective parameter R.

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Then the vanishing of $(\partial E/\partial R)_A$ is an obvious consequence of the structure of this Hamiltonian, which is a quadratic form: $E(R)=C(R-R_0)^2/2+B\dot{R}^2/2$. It is well known that in classical hydrodynamics the radius R is a collective parameter and that the Hamiltonian $E(R)$ describes the low-lying collective branch, the capillary waves. But the equation $(\partial E/\partial R)_A=0$ holds for any macroscopic system irrespective of the laws by which it is described, classical or quantum. It is therefore natural to assume that the vanishing of the derivative $(\partial E/\partial R)_A$ indicates the existence of a low-lying surface branch in a Fermi-liquid drop as well.

At the first glance, the appearance of low-frequency surface excitations in a Fermi-liquid drop appears incomprehensible; for if we form a narrow wave packet in coordinate space, we lose kinetic energy on account of the uncertainty principle. However, on the surface of the drop, where the density is low, the particles are attracted to one another since they interact almost as in vacuum. Therefore, the loss δT in the kinetic energy can be compensated by the gain δU in the potential energy.

We know from our experience that such compensation does indeed occur in the case of displacement of the system as a whole: The frequency of dipole excitations of the center of mass satisfies $\omega_1=\delta T_1+\delta U_1=0$ for any stable system. In other words, dipole deformation of the surface does not change the total energy of the system. In the framework of the Fermi-liquid approach, one can, regarding the displacement as an external field, obtain expressions for δT_1 and δU_1 that include the derivatives of the density ρ and the self-consistent field U, and also the density-dependent effective interaction between the quasiparticles. Then the relation $\omega_1=\delta T_1+\delta U_1=0$ can be regarded as a condition of consistency between these quantities that is equivalent in the macroscopic problem to the relation $p=-(\partial E/\partial V)_A=0$. It turns out that the "dipole" condition of consistency ($\omega_1=0$) is much more convenient than the "monopole" condition $[(\partial E/\partial R)=0]$.

From the general point of view, the consistency condition which follows from the vanishing of the dipole frequency is a consequence of spontaneous breaking of a symmetry, the translational invariance in this case. If an entire branch of low-lying collective states of surface type is to arise, the rigidity coefficient C_L of these vibrations with multipolarity $L\ll R/r_0$ must increase more slowly with increasing number A of particles than the mass coefficient $B_L\sim A$. Then in the limit $A\to\infty$, the frequency of the excitations tends to zero, $\omega_L\sim\sqrt{C_L/B_L}\to 0$, i.e., they must have a gapless spectrum. It can be shown on the basis of general theorems[11] that this is the case for all systems in which there is no long-range interaction such as the gravitational or Coulomb force. We therefore conclude that in a Fermi-liquid drop there must be a low-frequency branch of collective excitations in addition to the zero-sound branch.

The numerical calculations made in Refs. 12 and 13 confirm this. They showed that the first collective levels of nuclei—the very same low-lying collective states that were previously regarded as zero-sound excitations—belong to this branch. In many respects they are similar to ordinary capillary waves, but the interaction with the zero-sound branch and with individual noncollective particle–hole excitations distorts some of their properties (for example, volume corrections appear in the form factors, and the collective motion itself becomes partly a vortex motion[13]). These excitations, which can be interpreted naturally as quantum capillary waves, were called capons in Ref. 14. Their existence resolves the paradox relating to the behavior of the radius in Migdal's theory, and gives a new approach to the description of sum rules for electromagnetic transitions.[15]

The surface nature of the low-lying collective states in nuclei is also confirmed by Hartree–Fock calculations with density-dependent effective forces[1)] made recently by Bertsch *et al.*[17,18] Finally, evaluation[19] of some recent precision experiments[20] on the inelastic scattering of fast particles on nuclei also indicates that the collective motion in the first excitations of even–even nuclei has a surface nature.

The field of investigation of collective excitations of a drop in which both classical and quantum effects are clearly manifested can be appropriately called quantum hydrodynamics. The present review is devoted to the problems of the quantum hydrodynamic description of the properties of low-lying collective nuclear states. Based on a microscopic treatment, this description synthesizes the notions of classical hydrodynamics and Fermi-liquid ideas. Consistency conditions are considered in the first sections. We then analyze in detail the properties of the capons and their relation to the excitations of a classical drop. Numerical calculations of the properties of nuclei are made at the end of the review. The basic principles of the treatment are taken from the theory of finite Fermi systems. However, all the necessary explanations are made as we proceed. It is assumed only that the reader is familiar with the Green's function technique. We devote our main attention to the first low-lying collective nuclear states; the properties of highly excited states—various giant resonances—have been treated in detail on a number of occasions[21-24] both in the framework of Migdal's theory and by other methods. So as not to overburden the review, we decided not to discuss the properties of capons in nonmagic nuclei, in which pairing begins to play an important role, nor anharmonic effects; they are treated on the basis of the quantum hydrodynamic approach in Ref. 25.

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[1)] This method was also used by Feibelman[16] to describe surface excitations of a plane layer of Fermi liquid.

1. CONSISTENCY CONDITIONS IN SYSTEMS WITH SPONTANEOUSLY BROKEN SYMMETRY

The majority of systems studied in physics have spontaneously broken symmetry. Under certain transformations of the ψ operators, the total energy of such systems remains unchanged, whereas many others characteristics—the density $\rho(\mathbf{r}) = (0|\psi^+(\mathbf{r})\psi(\mathbf{r})|0)$, the self-consistent field $U(\mathbf{r})$, and so forth—change. If, for example, we take a drop of liquid, and in one of its hemispheres cut out a narrow crescent of liquid and slowly displace it into the other hemisphere in such a way that no other changes apart from the displacement of the center of mass take place [which corresponds to the transformation $\psi(\mathbf{r}) \to \exp(i\mathbf{p}\delta\mathbf{R})\psi(\mathbf{r}) = \psi(\mathbf{r}+\delta\mathbf{R})$], then, obviously, the internal energy of the drop remains unchanged but the density is altered: $\delta\rho = -(\partial\rho/\partial\mathbf{r})\delta\mathbf{R}$, where $\delta\mathbf{R}$ is the shift of the center of mass. From the formal point of view, this means the Hamiltonian H of the complete system is invariant under the shift operation ($[H,\mathbf{p}]=0$, where $\mathbf{p}$ is the momentum operator), but the density is not invariant ($[\rho,\mathbf{p}] = i\partial\rho/\partial\mathbf{r} \neq 0$). The situation in deformed nuclei and in other nonspherical systems is completely analogous: The Hamiltonian H commutes with the angular momentum operator L, i.e., it is invariant under rotation, but the commutator of L with either the density ρ or the mass operator Σ, or, as one frequently says, with the self-consistent field, is nonzero. The list of such objects can be extended: crystals, ferromagnetics, superconductors, etc.; these are all systems with spontaneously broken symmetry.

The investigation of their general properties began comparatively recently,[11] but it is now proceeding apace in many branches of physics, including nuclear physics, so that an extensive literature on the subject has already developed.[26-30] The distinctive feature of these systems is the existence of a macroscopic parameter (drop radius, intrinsic quadrupole moment, etc.) which characterizes the components of the density or self-consistent field that are noninvariant under the given symmetry transformation. As we have already said, this transformation carries the system into a different state without changing its energy, and therefore the ground state of systems with spontaneously broken symmetry is degenerate. Because of this, the standard methods of quantum field theory do not apply to them directly; one must first in some manner lift the degeneracy by, say, applying to the system a weak external field that does not commute with the operator of the symmetry transformation.[31] Because of the presence of this noninvariant macroscopic parameter, this field can be made so small that it hardly changes the properties of the system but is nevertheless sufficient to "freeze" the degree of freedom corresponding to the transformation (translational motion of the center of mass, rotation of the system as a whole, etc.). It is this feature of systems with spontaneous symmetry breaking that leads to the appearance of consistency conditions.

Here, we shall obtain a consistency condition, that will be important for the following exposition, due to the spontaneous breaking of translational invariance. We consider a system in a static external field v_0^{st}, which restricts the center of mass motion. We apply to it in addition a weak alternating "shift" field $v_0(t) \sim \mathbf{p}\exp(i\omega t)$ with frequency ω close to the frequency ω_{cm} of the characteristic oscillations executed by the center of mass of the system in the field v_0^{st}. Then the complete effect of applying this alternating field reduces to the resonant excitation of the zero-point oscillations of the center of mass, so that the change in the mass operator is

$$\delta\tilde{\Sigma}(\mathbf{x},\mathbf{y},\varepsilon,\omega) = \tilde{\Sigma}(\mathbf{x}+\delta\mathbf{R}(\omega),\mathbf{y}+\delta\mathbf{R}(\omega),\varepsilon) - \tilde{\Sigma}(\mathbf{x},\mathbf{y},\varepsilon) \approx \left(\frac{\partial}{\partial\mathbf{x}}+\frac{\partial}{\partial\mathbf{y}}\right)\tilde{\Sigma}(\mathbf{x},\mathbf{y},\varepsilon)\delta\mathbf{R}(\omega), \tag{1}$$

and the change of the density matrix is

$$\delta\tilde{\rho}(\mathbf{x},\mathbf{y},\varepsilon,\omega) = \tilde{\rho}(\mathbf{x}+\delta\mathbf{R}(\omega),\mathbf{y}+\delta\mathbf{R}(\omega),\varepsilon) - \tilde{\rho}(\mathbf{x},\mathbf{y},\varepsilon) \approx \left(\frac{\partial}{\partial\mathbf{x}}+\frac{\partial}{\partial\mathbf{y}}\right)\tilde{\rho}(\mathbf{x},\mathbf{y},\varepsilon)\delta\mathbf{R}(\omega), \tag{2}$$

where

$$\delta\mathbf{R}(\omega) \sim \alpha/(\omega-\omega_{cm}) \tag{3}$$

(the tilde above a letter means that the corresponding quantity is taken in the field v_0^{st}). On the other hand, $\delta\Sigma$ can be found by means of the well known equation[9,32] for the vertex part T[2]:

$$\mathrm{T} = V_0 + \delta\Sigma \equiv V_0 + \mathfrak{U}\delta G, \tag{4}$$

where V_0 is the external field [in our case $V_0 = v_0^{st} + v_0(t)$]; $\mathfrak{U}$, the block of the two-particle interaction, is irreducible in the particle-hole channel; $\delta G = GTG$ is the change of the Green's function G in the field V_0. Therefore, the equation for T can also be written in the form

$$\mathrm{T} = V_0 + \mathfrak{U}GTG. \tag{4'}$$

The frequencies of the collective excitations of the system—the poles of the vertex part T—are eigenfrequencies of the homogeneous equation

$$g = \mathfrak{U}GgG, \tag{5}$$

where g, the residue of T at the pole, is the production amplitude for the excitation. According to (1), (3), and (4), T in the present case has a pole at the point $\omega = \omega_{cm}$ with residue $\tilde{g}(\mathbf{x},\mathbf{y},\varepsilon,\omega_{cm}) \sim (\partial/\partial\mathbf{x}+\partial/\partial\mathbf{y}) \times \tilde{\Sigma}(\mathbf{x},\mathbf{y},\varepsilon)$. Substituting this solution in (5), we obtain

$$\left(\frac{\partial}{\partial\mathbf{x}_1}+\frac{\partial}{\partial\mathbf{x}_2}\right)\tilde{\Sigma}(\mathbf{x}_1,\mathbf{x}_2,\omega_{cm}) = \int \tilde{\mathfrak{U}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,|\varepsilon,\varepsilon') \times \tilde{G}\left(\mathbf{x}_3,\mathbf{x}_5,\varepsilon'-\frac{\omega_{cm}}{2}\right)\left[\left(\frac{\partial}{\partial\mathbf{x}_5}+\frac{\partial}{\partial\mathbf{x}_6}\right) \times \tilde{\Sigma}(\mathbf{x}_5,\mathbf{x}_6,\varepsilon')\right]\tilde{G}\left(\mathbf{x}_6,\mathbf{x}_4,\varepsilon'+\frac{\omega_{cm}}{2}\right) \times d\mathbf{x}_3\,d\mathbf{x}_4\,d\mathbf{x}_5\,d\mathbf{x}_6\,\frac{d\varepsilon'}{2\pi i}. \tag{6}$$

For stable systems of finite radius, the field v_0^{st} can be made so small that ω_{cm} will be much lower than the characteristic frequencies ω_{sp} of the single-particle

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[2] To shorten the equations, we shall frequently use the same symbolic notation as in Migdal's book, Ref. 9.

transitions in the system.[3)] Then in (6) the dependence of all these quantities on ω_{cm} and v_0^{st} can be ignored, and we obtain the consistency condition

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\Sigma(x_1, x_2, \varepsilon)=\int \mathcal{U}(x_1, x_2, x_3, x_4, \varepsilon, \varepsilon') \times G(x_3, x_5, \varepsilon')\left[\left(\frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_6}\right) \times \Sigma(x_5, x_6, \varepsilon')\right] G(x_6, x_4, \varepsilon')\, dx_3\, dx_4\, dx_5\, dx_6 \frac{d\varepsilon'}{2\pi i}. \quad (7)$$

This condition means that the frequency of the dipole oscillations of the center of mass of a stable system is zero, i.e., its dipole rigidity vanishes, $C_1=0$. The relation (7) characterizes the main difference between the properties of a Fermi-liquid drop, as a system that confines itself, and a Fermi gas enclosed in a box of the same size. Although their behavior is described by the same equations, C_1 is zero for the liquid but not for the gas. Note that the dipole rigidity also vanishes in classical hydrodynamics. It is for this reason, as we have already pointed out in the introduction, that many classical features must also be manifested in the properties of a quantum drop, especially in nuclei at low excitation energies.

Taking into account the connection $G^{-1}(\mathbf{x}, \mathbf{y}, \varepsilon) = (\varepsilon - p^2/2m)\delta(\mathbf{x}-\mathbf{y}) - \Sigma(\mathbf{x}, \mathbf{y}, \varepsilon)$, we can rewrite the condition (7) in the more compact form

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\Sigma(x_1, x_2, \varepsilon) = \int \mathcal{U}(x_1, x_2, x_3, x_4, \varepsilon, \varepsilon')\left(\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}\right) \times G(x_3, x_4, \varepsilon')\, dx_3\, dx_4 \frac{d\varepsilon'}{2\pi i}, \quad (7')$$

or, symbolically, most succinctly as

$$[\Sigma, \mathbf{p}] = \mathcal{U}[G, \mathbf{p}]. \quad (7'')$$

For systems with broken rotational symmetry, the consistency condition is similar, it being necessary only to replace the operator $(\partial/\partial \mathbf{x} + \partial/\partial \mathbf{y})$ in (7) and (7′) by $(\mathbf{x}\times\partial/\partial \mathbf{x} + \mathbf{y}\times\partial/\partial \mathbf{y})$, i.e., to replace the momentum operator $\mathbf{p}$ in (7) by the angular momentum operator $\mathbf{L}$. Generalizing, we can write the consistency conditions in the form[27]

$$[\Sigma, Q_i] = \mathcal{U}[G, Q_i], \quad (8)$$

where Q_i is the operator corresponding to the i-th transformation with respect to which the spontaneous symmetry breaking has occurred; in the general case there may be several such broken symmetries for the system, and then several conditions (8) must be satisfied simultaneously.

The relations we have obtained can be generalized to the case of pairing[27] and to multicomponent systems (ionic crystals, nuclei, etc); then G and Σ become matrices. For example, for a nucleus (7) is replaced by the system of equations

$$\left.\begin{aligned}\frac{\partial \Sigma^n}{\partial R} &= \mathcal{U}^{nn} G^n \frac{\partial \Sigma^n}{\partial R'} G^n + \mathcal{U}^{pn} G^p \frac{\partial \Sigma^p}{\partial R'} G^p;\\ \frac{\partial \Sigma^p}{\partial R} &= \mathcal{U}^{pn} G^n \frac{\partial \Sigma^n}{\partial R'} G^n + \mathcal{U}^{pp} G^p \frac{\partial \Sigma^p}{\partial R'} G^p.\end{aligned}\right\} \quad (9)$$

Here, we have introduced the coordinate $\mathbf{R}=(\mathbf{x}+\mathbf{y})/2$; the index n refers to the neutrons and p to the protons.

Let us consider briefly some of the consequences of the above relations. Note first that the consistency conditions are nonlinear since the amplitude $\mathcal{U}$ and the mass operator Σ are functionals of the Green's function G. It is obvious that Eqs. (9) do not always have a nontrivial solution—the system acquires a surface only under certain conditions, in particular, only for definite values of the vacuum amplitude $\mathcal{U}_{ex}$. In order to get a feeling for these conditions, let us consider a simple model: Suppose that the forces have zero range and do not depend on the velocities or the energy [$\mathcal{U}^{ik} = \mathcal{F}^{ik}(r_1)\delta(r_1-r_2)\delta(r_1-r_2)\delta(r_1-r_3)\delta(r_3-r_4)$]. Then for $N=Z$, we obtain from (9)

$$\partial U^+/\partial r = \mathcal{F}^+(r)\,\partial \rho^+/\partial r. \quad (10)$$

where $\mathcal{F}^+ = \mathcal{F}^{nn} + \mathcal{F}^{np}$, $U^+ = (\Sigma^n + \Sigma^p)/2$ and $\rho^+ = (\rho^n + \rho^p)/2$. From this the first condition follows immediately: In vacuum there must be attraction, i.e., the amplitude $\mathcal{F}^+_{ex}$ must be negative (ther derivatives $\partial U^+/\partial r$ and $\partial \rho^+/\partial r$ have opposite signs on the surface). Further, we write down an approximate Thomas-Fermi relationship between the density ρ^+ and the self-consistent field U^+: $(\rho^+(r) = [2m(\mu - U^+(r))]^{3/2}/3\pi^2 \equiv p_F^3(r)/3\pi^2$. Differentiating it with respect to r and substituting the result in (10), we obtain

$$1 = -\mathcal{F}^+(r)\, p_F(r)\, m/\pi^2. \quad (11)$$

It can be seen from this expression that as we move into the drop (the nucleus) the attraction must decrease since the density is increasing, and we therefore conclude that if the system has a bound state then for it $p_F > \pi^2/|\mathcal{F}^+_{ex}|m$, i.e., its density will be the higher, the smaller is the vacuum amplitude![4)] This, at the first glance, paradoxical result is confirmed by the example of ^{3}He, in which the attractive forces are so weak that they cannot ensure a bound state of two atoms. However, the relative density of liquid ^{3}He is very high: The ratio of the distance between the particles to the radius of the repulsive core is ≈0.9, whereas in a nucleus this parameter is ≈0.3.

[3)] We can estimate ω_{cm} and v_0^{st} as follows. We place a spherical nucleus in a rectangular one-dimensional well of depth v_0 and width $2R$, where $R = r_0 A^{1/3}$ is the radius of the nucleus. In this well, the nucleus as a whole will execute oscillations along the x axis with frequency $\omega_{cm} = \sqrt{C/B}$ and amplitude $x_0 = (CB)^{-4}$, where $B = mA$ is the mass coefficient, and the rigidity is related to the density ρ by $C \approx \rho \pi R v_0 \approx v_0 A^{1/3} r_0^{-2}$ [when the nucleus is displaced through Δx, particles in a volume of spherical segment $\approx \pi R(\Delta x)^2$ come into the region of action of the field v_0]. The field v_0 must be sufficiently weak to avoid exciting the intrinsic degrees of freedom of the nucleus; but if we are to be able to assume that the center of mass is fixed, the amplitude x_0 of the oscillations in the field v_0 must not exceed the shortest characteristic spatial scale over which the density, mass operator, etc., change appreciably.
This characteristic scale is of the order of the mean distance r_0 between the particles. Taking $x_0 \gtrsim r_0$, we obtain $v_0 \gtrsim (mr_0^2)^{-1} A^{-4/3} \approx \epsilon_F A^{-4/3}$ and $\omega_{cm} \gtrsim \epsilon_F A^{-1}$.
At the same time, the frequencies of the single-particle transitions are $\omega_{sp} \approx \epsilon_F A^{-1/3}$, where ϵ_F is the Fermi energy.

[4)] In fact, of course, the equilibrium density of the system is determined not only by the value of $\mathcal{F}^+_{ex}$ but also the behavior of $\mathcal{F}^+$ in the complete transition layer. It may be that $\mathcal{F}^+$ changes sign too rapidly, becoming significantly positive in this layer already. Then the condition (11) is not satisfied and a bound state does not arise.

The behavior of the amplitude $\mathcal{F}^+$ in the transition region is completely determined by the many-particle correlations, and to study it one requires not only the relation (7) but also more complicated consistency condition between the two- and three-particle characteristics of the system. These conditions can be obtained from the relation between the change $\delta\mathcal{U}$ of the two-particle block in an external field and the change δG of the Green's function

$$\delta\mathcal{U} = \mathcal{K}^{(3)}\delta G, \tag{12}$$

where $\mathcal{K}^{(3)}$ is the block of the three-particle interaction and irreducible in the same sense as $\mathcal{U}$. In our case, the external field is a "shift" field $v_0(t) \sim \mathbf{p}\exp(i\omega t)$, whose role reduces to replacing all the coordinates $\mathbf{x}_i$ in $\mathcal{U}$ and G by $\mathbf{x}_i + \delta\mathbf{R}(\omega)$. Repeating the arguments that led from the relation (6) to the condition (7), we obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial \mathbf{x}_1} + \ldots + \frac{\partial}{\partial \mathbf{x}_4}\right)\mathcal{U}(\mathbf{x}_1, \ldots, \mathbf{x}_4; \varepsilon_1, \varepsilon_2)\\ &= \int \frac{d\varepsilon_3}{2\pi i} d\mathbf{x}_5\, d\mathbf{x}_6\, \mathcal{K}^{(3)}(\mathbf{x}_1, \ldots, \mathbf{x}_6; \varepsilon_1, \varepsilon_2, \varepsilon_3)\\ &\quad\times\left(\frac{\partial}{\partial \mathbf{x}_5} + \frac{\partial}{\partial \mathbf{x}_6}\right) G(\mathbf{x}_5, \mathbf{x}_6; \varepsilon_3).\end{aligned} \tag{13}$$

The block $\mathcal{K}^{(3)}$ is related by the consistency condition to the block $\mathcal{K}^{(4)}$ of the four-particle interaction:

$$\begin{aligned}&\left(\frac{\partial}{\partial \mathbf{x}_1} + \ldots + \frac{\partial}{\partial \mathbf{x}_6}\right)\mathcal{K}^{(3)}(\mathbf{x}_1, \ldots \mathbf{x}_6; \varepsilon_1, \varepsilon_2, \varepsilon_3)\\ &= \int \frac{d\varepsilon_4}{2\pi i} d\mathbf{x}_7\, d\mathbf{x}_8\, \mathcal{K}^{(4)}(\mathbf{x}_1, \ldots, \mathbf{x}_8; \varepsilon_1, \ldots, \varepsilon_4)\\ &\quad\times\left(\frac{\partial}{\partial \mathbf{x}_7} + \frac{\partial}{\partial \mathbf{x}_8}\right) G(\mathbf{x}_7, \mathbf{x}_8; \varepsilon_4).\end{aligned} \tag{14}$$

Proceeding further in accordance with this scheme, we obtain a hierarchy of relations between ever more complicated blocks. If at some stage we make definite assumptions about the properties of the block of a many-particle interaction, then, going backward through the hierarchy, we can also construct the two-particle amplitude. For example, assuming in the framework of the same simple model that the block $\mathcal{K}^{(3)}$ is local and does not depend on the energy [$\mathcal{K}^{(3)} = b\delta(\mathbf{x}_1 - \mathbf{x}_2)\ldots\delta(\mathbf{x}_1 - \mathbf{x}_6)$], we obtain explicitly the density dependence of the two-particle local amplitude[5]:

$$\mathcal{F}(r) = a + b\rho(r), \tag{15}$$

where a and b are certain constants. Of course, the block $\mathcal{K}^{(3)}$ is not strictly δ-functional, and therefore the amplitude $\mathcal{F}$ in general depends on not only the density $\rho(\mathbf{r})$ but also on the total density matrix $\rho(\mathbf{x}, \mathbf{y})$, so that $\mathcal{F}$ may acquire terms containing gradients of ρ (the possibility that terms of the type $(\partial\rho/\partial r)^2$ could appear in $\mathcal{F}$ was noted in Ref. 34).

In principle, knowledge of $\mathcal{F}$ as a function of ρ enables us to construct the equation of state of nuclear matter at $T = 0$, i.e., find the pressure p as a function of the density ρ. We show this for the example of the same simple model. From the relations $\delta\Sigma/\delta\rho = \mathcal{F}(\rho)$, using (15), we obtain $\Sigma = a\rho + b/2\rho^2$ and find the chemical potential $\mu = p_F^2/2m + \Sigma = \alpha\rho^{2/3} + a\rho + b/2\rho^2$, where $\alpha = (3\pi^2)^{2/3}/2m$, and m is the nucleon mass. The energy of a large system is $E = Nf(N/V)$ (N is the number of particles and V is the volume[35]), and $\mu = (\partial E/\partial N)_V$, so that for f we obtain a differential equation whose solution is $f = 3\alpha\rho^{2/3}/5 + a\rho/2 + b\rho^2/6$. Going over in the usual manner[9] to dimensionless constants,

$$f_{ex}^+ = p_0 ma/\pi^2, \quad f_{in}^+ - f_{ex}^+ = p_0 m\rho_0 b/\pi^2,$$

where ρ_0 and p_0 are the density and Fermi momentum of normal nuclear matter, and introducing the dimensionless variable $x = \rho/\rho_0$, we arrive at the equation of state

$$p = -\left(\frac{\partial E}{\partial V}\right)_N = \left(\frac{\pi^4\rho_0^5}{3m^3}\right)^{1/3} x^2\left[\frac{3}{5}x^{-1/3} + \frac{1}{2}f_{ex}^+ + \frac{1}{3}(f_{in}^+ - f_{ex}^+)x\right]. \tag{16}$$

This equation must give $p = 0$ at $x = 1$; it is only under this condition that it corresponds to the equation of state of real nuclear matter. From this we obtain the following relation between the constants of the interaction:

$$2f_{in}^+ + f_{ex}^+ = -18/5. \tag{17}$$

Since the amplitude $\mathcal{F}$ must in addition reproduce the other characteristics of nuclear matter, for example, the correct value of the chemical potential $\mu = \mu_0$ at $\rho = \rho_0$, from the formula for μ we obtain one further connection between the constants[6]:

$$f_{in}^+ + f_{ex}^+ = 6m\mu_0/p_0^2 - 3 \approx -4.4. \tag{18}$$

(Here, we have used the values $\mu_0 = -16$ MeV and $r_0 = 1.18$ F from Ref. 36.) Writing down the deviation ΔE of the energy of nuclear matter from the equilibrium value (per particle) in the form $\Delta E/N = k\cdot(x-1)^2/2$, for the compressibility

$$k = \rho_0^2(\partial^2 f/\partial\rho^2)_{\rho=\rho_0} = \rho_0(\partial\mu/\partial\rho)_{\rho=\rho_0} = ms^2$$

(s is the velocity of sound) in this model with allowance for (17) and (18) we find

$$k = p_0^2(1 + f_{in}^+)/3m = -2\mu_0 + 2p_0^2/15m \approx 41\,\text{MeV}. \tag{19}$$

It is interesting to note that the values of the constants f_{in}^+ and f_{ex}^+ obtained in this crude model agree reasonably with the same values found by comparing the nuclear characteristics calculated in the framework of the theory of finite Fermi systems[9,38] and the experimental data. However, it must be borne in mind that the constant f_{in}^+ is determined not at all accurately from (17) and (18) since it is the difference of two large numbers. As we shall see in Sec. 8, where we make realistic calculations, $f_{in}^+ \approx 0.4$, and therefore a more probable value is

$$k = 2\varepsilon_F(1 + f_{in}^+)/3 \approx 33\,\text{MeV}. \tag{19'}$$

2. CONSISTENCY CONDITIONS AND COLLECTIVE MODES

As we have already noted, the consistency conditions in a system with spontaneously broken symmetry re-

[5] That the density dependence of the two-particle interaction must be taken into account in calculations of nuclear characteristics was first pointed out in Refs. 9 and 33.

[6] Somewhat different restrictions on the constants were also obtained in Ref. 37 by a numerical analysis of the consistency conditions for specific nuclei.

flect the fact that its rigidity C vanishes with respect to some external field that sets into coherent motion an appreciable fraction of the particles, i.e., gives rise to a perturbation with large mass coefficient B. It is intuitively clear that if the configuration of this external field is changed very slightly, then with respect to it the rigidity will not differ strongly from zero, and in the system one can therefore have collective excitations with low frequency $\omega \approx \sqrt{C/B}$. To make these arguments more precise, we consider Eq. (5) for the amplitude g. We seek a solution of this equation for small ω in the form

$$g_k(\omega) = g_0 f_k + \varphi_k(\omega), \tag{20}$$

where f_k is a smooth function [in the case of spherical symmetry, one can take $f_k = Y_{km}(\mathbf{n})$; for planar geometry, $f_k = \exp(ikx)$], and φ_k is some correction that, as we shall see, has a quantum origin. For $\omega = 0$, the solution is in accordance with (8) known: $f_k = 1$, $\varphi_k = 0$, and $g_0 = [\Sigma, \hat{Q}]$.

We now use the circumstance that in a Fermi-liquid drop the block $\mathfrak{U}$ is local and depends weakly on the energy variables (the region of appreciable variation of $\mathfrak{U}$ with respect to the momenta is $\delta q \sim p_F$ and with respect to the energy $\delta\omega \sim \varepsilon_F$; we shall discuss somewhat later the role played by long-range interactions, in particular, the Coulomb forces $\sim q^{-2}$) and $\mathfrak{U}$ does not contain momentum operators (in nuclei, the velocity forces are small), and we find that the solution of Eq. (5) depends quadratically on the frequency ω and at small ω we have

$$\begin{aligned} g_0 f_k + \varphi_k(0) = {} & \mathfrak{U}(GG)_{\omega=0}(g_0 f_k + \varphi_k(0)) \\ & + \omega_k^2 \left\{ \mathfrak{U}\left(\frac{\partial GG}{\partial \omega^2}\right)_{\omega=0}(g_0 f_k + \varphi_k(0)) \right. \\ & \left. + [\mathfrak{U}(GG)_{\omega=0} - 1]\left(\frac{\partial \varphi_k}{\partial \omega^2}\right)_{\omega=0} \right\}. \end{aligned} \tag{21}$$

Let us assume that the operator $\mathfrak{U}(GG)_{\omega=0}$ is strictly δ-functional. It then follows from the consistency condition $g_0 = \mathfrak{U}(GG)_{\omega=0} \cdot g_0$ that $\mathfrak{U}(GG)_{\omega=0}$ is simply the identity operator, and therefore the left-hand side of (21) cancels exactly with the first term on the right-hand side, and as a result we obtain the solution $\omega_k = 0$ for all f_k and φ_k, i.e., the spectrum of collective excitations is in this case infinitely degenerate. We can understand this: If the system is constructed in such a way that the field applied at the point $\mathbf{r}$ changes the density only at this point [because $\mathfrak{U}(GG)$ is δ-functional], the particles will not "feel" the difference between perturbations with different k; it is only when "information" can be exchanged over a distance in the system that the particles are capable of "distinguishing" between, say, a dipole and a quadrupole perturbation. In reality, the response of the system is not a pure δ function, and the spectrum is not degenerate. First, the irreducible block $\mathfrak{U}$ of the two-particle interaction has nonzero range, being spread out over a distance of order r_0, and already because of this $\mathfrak{U}(GG)_{\omega=0} g_0 f_k - g_0 f_k \sim k^2 A^{-2/3}$. Second, the particle-hole propagator $\int (GG)_{\omega=0} d\varepsilon$ is also not a δ function, having two essentially different components,[39] of which one is local, with the same range r_0 as $\mathfrak{U}$, and the other is a nonuniversal quantum long-range interaction $\mathfrak{A}$ of small amplitude $\sim R^{-3}$, which generates the quantum correction φ_k. For a spherical nucleus, $g_0 \sim \partial\Sigma/\partial r$ is a function that is appreciably nonzero in a surface layer of width r_0, and this correction has the estimate $\varphi_k \sim \int \mathfrak{U}\mathfrak{A} g_0 d^3 r \sim \varepsilon_F/R$ [$\mathfrak{U} \sim \varepsilon_F r_0^3$; $\mathfrak{A} \sim \varepsilon_F^{-1}(r_0 R)^{-3}$; $g_0 \sim \varepsilon_F/r_0$ and also $r_0 R^2$ from the integration].

To find the collective frequency ω_k, we apply the operator $(g_0 f_k + \varphi_k(0))(GG)_{\omega=0}$ to both sides of Eq. (21) and integrate with respect to all coordinates. Then the square of the frequency can be represented in the form $\omega_k^2 = C_k/B_k$. We shall not write out here the explicit expressions for C_k and B_k (see Sec. 5 for details), but merely point out that at small k there is, because of the consistency conditions, a strong cancellation of the local terms in C_k, as a result of which their contribution is proportional to k^2, while the main contribution to C_k in large systems is made by volume terms of the type $(\varphi_k \mathfrak{A} \varphi_k)$ and $(\partial\Sigma/\partial r\, \mathfrak{A}\, \partial r)$ containing the long-range components φ_k and $\mathfrak{A}$—they are all $\sim \varepsilon_F R/r_0^3$, so that for large $\mathfrak{A}$ we have $C_k \sim \varepsilon_F A^{1/3}/r_0^2$. At the same time, in the mass coefficient B_k there is no cancellation, and to estimate it we can set $g_k = g_0 \approx \partial\Sigma/\partial r$, obtaining

$$B_k \approx B_1 = -\left(\frac{\partial\Sigma}{\partial r}\frac{\partial (GG)}{\partial \omega^2}\frac{\partial\Sigma}{\partial r}\right)_{\omega=0} \sim mA. \tag{22}$$

The upshot is that in the limit $A \to \infty$ we have the estimate $\omega_k \sim \varepsilon_F A^{1/3}$, from which it can be seen that the energy of the collective state tends to zero with increasing number of particles, i.e., the spectrum of these excitations is gapless. For macroscopic systems without long-range interaction in which spontaneous symmetry breaking has occurred, this conclusion is essentially a simple consequence of Goldstone's well known theorem.[11]

If there are unscreened long-range interactions of gravitational type ($\mathfrak{U} \sim 1/r$) in the system, the estimate of the mass coefficient is not changed, but the rigidity is changed appreciably—at large A, the main contribution to it is now made by the long-range interaction, and the difference $\mathfrak{U}(GG)_{\omega=0} g_0 f_k - g_0 f_k$ no longer decreases with increasing A but increases as $A^{1/3}$. Therefore, for a spherical system, for example, the contribution of the long-range interaction to C_L is[40]

$$C_L^{long} = \alpha\rho^2 V[3/(2L+1) - 1] \tag{23}$$

(for a nucleus, $\alpha = e^2$ and $\rho = Z/V$ is the density of protons; for the gravitational interaction $\alpha = -\gamma m^2$, where γ is the gravitational constant and $\rho = A/V$ is the density of particles with mass m). For $\rho = \text{const}$, this contribution increases linearly with the volume V of the system, and as a result the ratio C_L/B_L no longer tends to zero. However, in the considered situation C_L cannot be positive for all L, and at some angular momentum L_c it becomes negative and the system ceases to be stable[7]; in the case of a long-range attraction, the system collapses ($C_0 < 0$), whereas in the case of repulsion it is deformed and undergoes fission ($C_2 < 0$).

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[7] The situation in electrically neutral media is different – in this case a gap appears in the spectrum of collective excitations of a stable system (for example, the plasmon branch in a solid[41]).

3. RENORMALIZATION OF THE BASIC EQUATIONS

In order to describe nuclear phenomena quantitatively, it is necessary to solve equations of the type (5), which contain the Green's function G, the interaction amplitude $\mathfrak{U}$, etc. They can be found in the framework of the graphical approach by summation of appropriate diagrams. In systems with strong interaction, this is a very complicated and not yet completely solved problem Therefore, in many microscopic theories there is an element of phenomenology: certain sets of diagrams are not calculated but parametrized on the basis of experimental data. And here a fundamental role is played by the experimental fact that in many phenomena Fermi systems, whether they are nuclei, electrons in a metal, or liquid ^{3}He, behave like a gas of Fermi particles. This circumstance stimulated Hartree-Fock calculations, in which the functional $E(\rho)$ is sought on the class of Fermi-gas wave functions—Slater determinants that describe noninteracting quasiparticles. Recently, in this method the vacuum interaction has been widely replaced by density-dependent effective forces[42,43] with parameters determined by fitting to the Weizsäcker formula for the nuclear binding energies. It should be emphasized that the procedure in which effective forces are used instead of the vacuum forces but the wave function of the system is taken as before in the form of a Slater determinant is frequently self-contradictory. This contradiction can be seen for the example of a system with a pairing Hamiltonian by comparing the Hartree-Fock formula for the energy density

$$e_{\mathrm{HF}} = t + \rho v(\rho)\rho/2, \tag{24}$$

where t is the density of the kinetic energy, and $v(\rho)$ is the effective interaction potential, with the exact Galitskii–Migdal expression[44]

$$e_{\mathrm{GM}} = t + \frac{1}{2}\int \Sigma G \frac{d\varepsilon}{2\pi i}, \tag{25}$$

where Σ is the mass operator, its role in the Hartree-Fock method being played by the following self-consistent potential, which does not depend on the energy variable ε:

$$\Sigma_{\mathrm{HF}} = \rho v + \frac{1}{2}\rho\frac{\partial v}{\partial \rho}\rho. \tag{26}$$

If we substitute Σ_{HF} in the exact expression (25) and remember that $\int G\, d\varepsilon/2\pi i = \rho$, we obtain

$$e = t + \frac{1}{2}\rho v\rho + \frac{1}{2}\left(\rho^2\frac{\partial v}{\partial \rho}\rho + \rho\frac{\partial v}{\partial \rho}\rho^2\right). \tag{27}$$

We see that a term which is absent in (24) has appeared in the energy density! This result shows that one cannot reduce everything to a renormalization of the interaction without complicating the wave function—the two effects are related and in a rigorous approach they must be taken into account simultaneously.

The theory of finite Fermi systems,[9] which is based on the ideas of the theory of a Fermi liquid,[4-6] is free of such contradictions. In it, the Green's function of a particle is represented as a sum of two terms: $G = aG^q + G^R$. The pole quasiparticle term G^q, which enters with weight $0 < a < 1$, corresponds to the ordinary Slater determinant, and the regular part G^R describes the contribution of more complicated configurations. One of the main assumptions of Migdal's theory is that at low excitation energies (for $\omega \ll \varepsilon_F$) the fluctuations in the properties of the nuclei are related to the pole parts of G^q, while the regular parts G^R make a smooth contribution. This provides the basis of the renormalization idea: In all equations, the product G^qG^q of the pole parts is separated out from the product GG, and the remainder is removed to universal blocks—the local charge e_q and the effective interaction amplitude $\mathcal{F}$. This procedure is presented for finite systems by Migdal in his book, Ref. 9. Here, we give only the results for the case when in $\mathcal{F}$ one can ignore the terms that depend on the energies, velocities, and spins of the particles[8] and we shall restrict ourselves to the dependence on only the momentum transfer q (this last is important since the transition density of these excitations has a surface peak of width $\sim r_0$ and the momentum transfer in the particle–hole channel is $\sim p_F$). The consistency condition in this case takes the form

$$\frac{\partial U^i}{\partial \mathbf{r}} = \int \mathcal{F}^{ik}(\mathbf{r}, \mathbf{r}_1; \rho)\, A^k(\mathbf{r}_1, \mathbf{r}_2; \omega = 0)\frac{\partial U^k}{\partial \mathbf{r}_2}\, d\mathbf{r}_1\, d\mathbf{r}_2. \tag{28}$$

The equation for the effective field[9] can be written in the form

$$V^i(\mathbf{r}; \omega) = e_q^i V_0^i(\mathbf{r}) + \int \mathcal{F}^{ik}(\mathbf{r}, \mathbf{r}_1)\, A^k(\mathbf{r}_1, \mathbf{r}_2; \omega)\, V^k(\mathbf{r}_2; \omega)\, d\mathbf{r}_1\, d\mathbf{r}_2. \tag{29}$$

Here, e_q^i is the local charge and V_0 is the external field. For the transition amplitude g we have the equation

$$g^i(\mathbf{r}; \omega) = \int \mathcal{F}^{ik}(\mathbf{r}, \mathbf{r}_1)\, A^k(\mathbf{r}_1, \mathbf{r}_2; \omega)\, g^k(\mathbf{r}_2; \omega)\, d\mathbf{r}_1\, d\mathbf{r}_2 \tag{30}$$

with the normalization condition[9]

$$\left(g\frac{dA}{d\omega}g\right) = -1. \tag{31}$$

Equations (28)–(31) contain the propagator

$$A(\mathbf{r}_1, \mathbf{r}_2; \omega) = \int G^q(\mathbf{r}_1, \mathbf{r}_2; \varepsilon - \omega/2)\, G^q(\mathbf{r}_2, \mathbf{r}_1; \varepsilon + \omega/2)\frac{d\varepsilon}{2\pi i} = \sum_{\lambda\lambda'} \varphi_\lambda^*(\mathbf{r}_1)\varphi_{\lambda'}(\mathbf{r}_1)\frac{n_\lambda - n_{\lambda'}}{\varepsilon_\lambda - \varepsilon_{\lambda'} - \omega}\varphi_{\lambda'}^*(\mathbf{r}_2)\varphi_\lambda(\mathbf{r}_2), \tag{32}$$

where G^q is the quasiparticle Green's function, and it satisfies the equation

$$(\varepsilon - p^2/2m^* - U(\mathbf{r}_1))\, G^q(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) = \delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{33}$$

Here, m^* is the effective mass of the quasiparticle, U is the self-consistent field, φ_λ, ε_λ, and n_λ are, respectively, the wave functions of the quasiparticles, their energies, and the population numbers of the quasiparticle levels; $\lambda = (n, l, j, m) \equiv (\nu, m)$ are the single-particle quantum numbers.

4. SOLUTION OF THE EQUATIONS AND THE MAIN COMPUTATIONAL FORMULAS

To study quantum capillary waves (capons) it is convenient to separate out the main, surface component

[8] The more general case is considered in the Appendix.

in the amplitude g for creation of the collective state. For this, we transform Eq. (30), having first separated in it the angular variables. Since the properties of capons in nuclei are determined mainly by the scalar part of the interaction amplitude $\mathcal{F}$, retaining only this part, for excitation with multipolarity L in the case of spherical symmetry we obtain $g_L(\mathbf{r};\omega)=g_L(r,\omega)Y_{LM}(\mathbf{n})$ and from (30) we obtain

$$g_L^i(r;\omega)=\int \mathcal{F}_L^{ik}(r,r_1)A_L^k(r_1,r_2;\omega)g_L^k(r_2;\omega)r_1^2dr_1r_2^2dr_2, \tag{34}$$

where

$$\mathcal{F}_L^{ik}(r,r')=\int \mathcal{F}^{ik}(\mathbf{r},\mathbf{r}')P_L(\mathbf{nn}')\frac{d\mathbf{n}\,d\mathbf{n}'}{4\pi}; \tag{35}$$

$$A_L^k(r,r';\omega)=\int A^k(\mathbf{r},\mathbf{r}';\omega)P_L(\mathbf{nn}')\frac{d\mathbf{n}\,d\mathbf{n}'}{4\pi}. \tag{36}$$

The consistency condition (28) is then written in the form

$$\frac{\partial U^i}{\partial r}=\int \mathcal{F}_1^{ik}(r,r_1)A_1^k(r_1,r_2;\omega=0)\frac{\partial U^k}{\partial r_2}r_1^2dr_1r_2^2dr_2, \tag{37}$$

and the equation conjugate to it has the form

$$\frac{\partial\rho^i}{\partial r}=\int A_1^i(r,r_1;\omega=0)\mathcal{F}_1^{ik}(r_1,r_2)\frac{\partial\rho^k}{\partial r_2}r_1^2dr_1r_2^2dr_2. \tag{38}$$

Here, we have used the relation

$$\frac{\partial\rho^i}{\partial r}=\int A_1^i(r,r_1;\omega=0)\frac{\partial U^i}{\partial r_1}r_1^2dr_1. \tag{39}$$

In order to separate explicitly in the amplitude g_L the main component proportional to $\partial U/\partial r$, we expand the kernal $\mathcal{F}_LA_L$ of the integral equation (34) with respect to the eigenfunctions of Eqs. (37) and (38):

$$\int \mathcal{F}_L^{ik}(r_1,r_3)A_L^k(r_3,r_2;\omega)r_3^2dr_3 = \varkappa_L(\omega)\frac{\partial U^i}{\partial r_1}\frac{\partial\rho^k}{\partial r_2}+\mathcal{R}_L^{ik}(r_1,r_2;\omega), \tag{40}$$

where $\mathcal{R}_L$, the orthogonal complement of the first term on the right-hand side, is essentially the volume part of the kernel $\mathcal{F}_LA_L$ since it is determined by the condition

$$\left(\frac{\partial\rho}{\partial r}\mathcal{R}_L\frac{\partial U}{\partial r}\right)=0 \tag{41}$$

(here and in what follows, the round brackets denote integration with respect to all coordinates and summation over all indices). From (37), (40), and (41) we obtain the relations

$$\varkappa_L(\omega)=\left(\frac{\partial\rho}{\partial r}\mathcal{F}_LA_L(\omega)\frac{\partial U}{\partial r}\right)\left(\frac{\partial\rho}{\partial r}\frac{\partial U}{\partial r}\right)^{-2}; \tag{42}$$

$$\varkappa_1(0)=\left(\frac{\partial\rho}{\partial r}\frac{\partial U}{\partial r}\right)^{-1}; \tag{43}$$

$$\int\frac{\partial\rho^i}{\partial r_1}\mathcal{R}_1^{ik}(r_1,r;\omega=0)r_1^2dr_1=\int\mathcal{R}_1^{ki}(r,r_1;\omega=0)\frac{\partial U^i}{\partial r_1}r_1^2dr\equiv 0. \tag{44}$$

Substituting (40) in (34), we obtain

$$g_L^i(r;\omega)=\varkappa_L(\omega)\left(\frac{\partial\rho}{\partial r}g_L\right)\Omega_L^i(r;\omega), \tag{45}$$

where we have introduced the function Ω_L, which satisfies the equation

$$\Omega_L^i(r;\omega)=\frac{\partial U^i}{\partial r}+\int\mathcal{R}_L^{ik}(r,r_1;\omega)\Omega_L^k(r_1;\omega)r_1^2dr_1, \tag{46}$$

and, as can be seen from the expressions we have written down,

$$\Omega_1^i(r;\omega=0)=\partial U^i/\partial r. \tag{47}$$

Multiplying (45) by $\partial\rho^i/\partial\gamma$, integrating with respect to the coordinates, and summing over the neutrons and protons, we obtain an equation from which the frequencies ω_L of the collective excitations can be determined:

$$\Theta_L(\omega)\equiv 1-\varkappa_L(\omega)\left(\frac{\partial\rho}{\partial r}\Omega_L(\omega)\right)=0 \tag{48}$$

Following the treatment of Sec. 2, we write

$$\Omega_L^i(r;\omega)=\partial U^i/\partial r+\phi_L^i(r;\omega). \tag{49}$$

Then using (44) and (46), we obtain an equation for ϕ_L^i:

$$\phi_L^i(r;\omega)=\int(\mathcal{R}_L^{ik}(r,r_1;\omega)-\mathcal{R}_L^{ik}(r,r_1;\omega=0))\frac{\partial U^k}{\partial r_1}r_1^2dr_1 +\int\mathcal{R}_L^{ik}(r,r_1;\omega)\phi_L^k(r_1;\omega)r_1^2dr_1. \tag{50}$$

Because of the presence in this equation of the term containing $\mathcal{R}_1^{ik}(\omega=0)$, the vanishing of the correction function $\phi_L^i(r;\omega)$ for $\omega=0$ and $L=1$ is ensured, i.e., $g_1^i(r;0)\sim\partial U^i/\partial r$ is automatically obtained. As can be seen from (48) and (43), the eigenfrequency is $\omega_1=0$, as must be for dipole excitation.

Making similar transformations in Eq. (29) for the effective field, we represent its solution in the form

$$V_L^i(r;\omega)=v_L^i(r;\omega)+\varkappa_L(\omega)\left(\frac{\partial\rho}{\partial r}v_L\right)\Theta_L^{-1}(\omega)\Omega_L^i(r;\omega). \tag{51}$$

where v_L is a function satisfying the equation

$$v_L^i(r;\omega)=e_q^iV_{0L}^i(r)+\int\mathcal{R}_L^{ik}(r,r_1;\omega)v_L^k(r_1;\omega)r_1^2dr_1. \tag{52}$$

For completeness, we write down here the expressions for the transition density and the reduced probability $B(EL)$. The square of the matrix element of the transition $0^+\to L^\pi$, summed over the projections of the angular momentum L in the final state, is given by the usual expression[9]

$$|M_{0L}|^2=-\frac{1}{\pi}\operatorname{Im}(e_qV_{0L}A_LV_L)=(e_qV_{0L}A_Lg_L)^2,$$

and therefore from (51), (45), and (31) we obtain

$$|M_{0L}|^2=\left[\varkappa_L\left(\frac{\partial\rho}{\partial r}v_L\right)(e_qV_{0L}A_L\Omega_L)\left(\frac{d\Theta_L}{d\omega}\right)^{-1}\right]_{\omega=\omega_L} =\left[-(e_qV_{0L}A_L\Omega_L)^2\left(\Omega_L\frac{dA_L}{d\omega}\Omega_L\right)^{-1}\right]_{\omega=\omega_L}. \tag{53}$$

Note that at the point $\omega=\omega_L$

$$\left(\Omega_L\frac{dA_L}{d\omega}\Omega_L\right)=-\left(\Omega_LA_L\frac{\partial U}{\partial r}\right)\frac{d\Theta_L}{d\omega}. \tag{53'}$$

Introducing in accordance with (53) the transition density

$$\rho_{tr}^i(r;\omega_L)=\left(-\Omega_L\frac{dA_L}{d\omega}\Omega_L\right)_{\omega=\omega_L}^{-1/2} \times\int e_q^iA_L^i(r,r_1;\omega_L)\Omega_L^i(r_1;\omega_L)r_1^2dr_1, \tag{54}$$

we obtain for $V_{0L}^i=e^ir^LY_{LM}$ an expression for $B(EL)$:

$$B(EL:0^+\to L^\pi)=(2L+1)\left[\int e^i\rho_{tr}^i(r;\omega_L)r^{L+2}dr\right]^2. \tag{55}$$

5. PROPERTIES OF CAPONS IN NUCLEI

The spontaneous breaking of translational invariance, which has the consequence that the collective branch of

low-lying surface excitations arises in the Fermi-liquid drop, holds in a wide range of temperatures—from zero to the boiling point—and therefore such a branch exists in both the quantum and the classical region. To the latter, there correspond ordinary capillary waves. As the temperature is reduced and we go over to the quantum region, their properties are changed by the interaction with the zero-sound branch which then arises, but the main feature—the surface nature of the excitations—remains. The interaction with the zero sound has a stronger influence on the small quantities—the rigidities C_L and the frequencies ω_L, and it is therefore not surprising that, when applied to a nucleus, the classical model,[1,2] gives the wrong position of the first low-lying collective states, although the corresponding transition densities differ in their form little, as is shown by the recently performed experiments,[19] from the classical $\rho_{tr}^{cl} \sim (\partial\rho/\partial r) Y_{LM}$.

To find the properties of low-lying collective states in nuclei, one must use a computer to solve Eq. (34) for the amplitude g simultaneously with the consistency condition (37), i.e., one must implement the scheme developed in the previous section. However, many properties of these collective states can be analyzed without recourse to complicated calculations (which are considered in Sec. 8). Because the capons are formed basically on the surface, where the nucleons attract one another, their frequencies ω_L, in contrast to zero-sound excitations, are always lower than the corresponding single-particle differences ω_{sp}. Therefore, the capons form the lowest branch of collective excitations in nuclei and their main properties can be studied qualitatively using the adiabatic approximation $\omega_L/\omega_{sp} \ll 1$. Then, expanding (48) near $\omega = 0$ in the series

$$\Theta_L(\omega_L) = \Theta_L(0) + \omega_L^2 (d\Theta_L/d\omega^2)_{\omega=0},$$

we obtain the standard expression for the frequency in the adiabatic approximation:

$$\omega_L^2 = C_L/B_L, \tag{56}$$

where

$$C_L = -\left[1 - \varkappa_L(0)\left(\frac{\partial\rho}{\partial r}\Omega_L\right)_{\omega=0}\right]\Big/\varkappa_1(0); \tag{57}$$

$$B_L(0) = \left(\frac{d\Theta}{d\omega^2}\right)_{\omega=0}\Big/\varkappa_1(0). \tag{58}$$

The normalization is chosen here in such a way that the mass coefficient B_L has the dimensions of mass. On the other hand, if Θ_L is expanded in a series near $\omega = \omega_L$, and we write $\Theta_L(0) = \Theta_L(\omega_L) - (d\Theta_L/d\omega^2)_{\omega=\omega_L}\omega_L^2$, then, using the condition $\Theta_L(\omega_L) = 0$, we can also obtain for the determination of the mass coefficient

$$B_L(\omega_L) = \left(\frac{d\Theta_L}{d\omega^2}\right)_{\omega=\omega_L}\Big/\varkappa_1(0). \tag{58'}$$

Comparing the values of $B_L(0)$ and $B_L(\omega_L)$ calculated in accordance with (58) and (58'), we can gauge the accuracy of the adiabatic approximation. This comparison will be made below, and for the moment we consider the C_L. Using (43) and (49), we can reduce Eq. (57) to the form

$$C_L = \frac{\varkappa_L(0) - \varkappa_1(0)}{\varkappa_1^2(0)} + \frac{\varkappa_L(0)}{\varkappa_1(0)}\left(\frac{\partial\rho}{\partial r}\phi_L\right)_{\omega=0}, \tag{59}$$

which with allowance for the consistency conditions (37)–(39) and the definition (42) of $\varkappa_L$ gives

$$\begin{aligned} C_L = &\left(\frac{\partial U}{\partial r}(A_L - A_1)\frac{\partial U}{\partial r}\right)_{\omega=0} + \left(\frac{\partial\rho}{\partial r}(\mathcal{F}_L - \mathcal{F}_1)\frac{\partial\rho}{\partial r}\right) \\ &+ \left(\frac{\partial\rho}{\partial r}(\mathcal{F}_L - \mathcal{F}_1)(A_L - A_1)\frac{\partial U}{\partial r}\right)_{\omega=0} + \frac{\varkappa_L(0)}{\varkappa_1(0)}\left(\frac{\partial\rho}{\partial r}\phi_L\right)_{\omega=0}. \end{aligned} \tag{60}$$

The estimates and numerical calculations show that, as a rule, the last two terms play a small role (Table I), and the main contribution to C_L is made by the first and second terms. To calculate them, it is convenient to separate in the differences $\mathcal{F}_L - \mathcal{F}_1$ and $A_L - A_1$ the local and the long-range components. The interaction $\mathcal{F}$ contains two terms—the local nuclear amplitude F and the Coulomb long-range interaction $\mathcal{F}^c$ for the protons. The particle-hole propagator A has two components—the local universal part A^u with radius $\sim r_0$ and the quantum long-range interaction $\mathcal{A}$ with radius $\sim R$.[39] The contribution to C_L of the local parts can be found by using the circumstance that for the integration of $A^u(\mathbf{r}_1, \mathbf{r}_2, 0)$ and $F(\mathbf{r}_1, \mathbf{r}_2)$ with $\partial\rho/\partial r$ and $\partial U/\partial r$ the main contribution in the first two terms of (60) is made by the region $|\mathbf{r}_1 - \mathbf{r}_2| \sim r_0$ and $|\mathbf{r}_1| \sim |\mathbf{r}_2| \sim R$, in which $1 - x_0 \sim (r_0/R)^2$, i.e., $1 - x \sim A^{-2/3}$ (here, $x = \cos \mathbf{n}_1 \cdot \mathbf{n}_2$). Therefore, when $L \ll p_F R$, to calculated F_L and A_L^u in accordance with (35) and (36) we can use an expansion of the Legendre polynomial near $x = 1$, which has the form $P_L(x) \approx 1 - L(L+1)(1-x)/2$, and then

$$\left.\begin{aligned} F_L(r_1, r_2) &\approx F_0(r_1, r_2) - \pi L(L+1)\int F(r_1, r_2, x)(1-x)\,dx; \\ A_L^u(r_1, r_2; \omega = 0) &\approx A_0^u(r_1, r_2; \omega = 0) - \\ &- \pi L(L+1)\int A^u(r_1, r_2, x; \omega = 0)(1-x)\,dx. \end{aligned}\right\} \tag{61}$$

Substituting these expressions in the first two terms in (60), we find that the contribution of F and A^u to C_L is

$$C_L^{loc} = \sigma(L(L+1) - 2), \tag{62}$$

where

$$\begin{aligned} \sigma = &-\pi\int \frac{\partial U}{\partial r_1} A^u(r_1, r_2, x; \omega = 0)\frac{\partial U}{\partial r_2}(1-x)\,dx r_1^2 dr_1 r_2^2 dr_2 \\ &-\pi\int \frac{\partial\rho}{\partial r_1} F(r_1, r_2, x)\frac{\partial\rho}{\partial r_2}(1-x)\,dx r_1^2 dr_1 r_2^2 dr_2. \end{aligned} \tag{63}$$

Remembering that the Coulomb harmonic

$$\mathcal{F}_L^c(r_1, r_2) = \frac{4\pi e^2}{2L+1}\frac{r_<^L}{r_>^{L+1}}, \tag{64}$$

where $r_<$ and $r_>$ are, respectively, the smaller and the greater of the coordinates r_1 and r_2, is a smooth function, and $\partial\rho/\partial r$ is a rapidly changing function that is nonzero only for $r \sim R$, we obtain from the second term in (60) a formula for the Coulomb part of the rigidity

TABLE I. Contribution to the ridigity of the last two terms in Eq. (60) for some magic nuclei.

L^π	Contribution to the rigidity, %		
	^{88}Sr	^{132}Sn	^{208}Pb
2^+	4.7	0.2	1.3
3^-	−5.9	−0.5	−2.5

that coincides with (23). Combining (62) and (23), we arrive at an expression that corresponds exactly to the one obtained in the classical hydrodynamics of a drop:

$$C_L^{\mathrm{cl}} = \sigma(L-1)(L+2) - (3Z^2e^2/2\pi R^3)(L-1)/(2L+1). \quad (65)$$

To find the total rigidity, we must add to it the contribution of the quantum long-range interaction $\mathfrak{A}$. Note that with increasing L the harmonics of this interaction $\mathfrak{A}_L$ decrease rapidly, and therefore Eq. (65) essentially determines the complete rigidity of the system in the classical region $1 \ll L \ll p_F R$. This makes it possible to interpret the parameter σ introduced in accordance with (63) as the coefficient of surface tension. A numerical calculation of σ in accordance with (63) was made in Ref. 46 (for more details, see Sec. 8). The local part A^{u} of the propagator A is always negative, and the surface part of the isoscalar nuclear amplitude F^+, which contributes to σ, is also negative (note that in (63) summation over the neutrons and protons is assumed). Therefore, the parameter σ is a positive quantity. For $L>1$, as can be seen from (65), the "local" part of the rigidity is positive, and the Coulomb part is negative. For $L=0$, the situation is reversed—the local contribution is negative and the Coulomb part is positive. It is well known that if C_0 becomes less than zero, collapse commences. Therefore, in the absence of the Coulomb interaction only the quantum long-range interaction $\mathfrak{A}$ in the particle-hole propagator A can save the nucleus from collapse. The importance of this long-range interaction can be judged on the basis of the data[9] in Table II, which gives the values of C_L for a number of magic nuclei. The contribution of the long-range component $\mathfrak{A}$ to the rigidity is characterized by the difference between the values calculated in accordance with (60) and the hydrodynamic values found from (65) (at the same time, it is assumed that $4\pi r_0^2\sigma = 20.8$ MeV and $r_0 = 1.18$ F (Ref. 36); we recall that the factor R^2 is not present in C_L and B_L for the chosen normalization). It can be seen from Table II that C_0 and C_2 appreciably exceed the corresponding values of C^{cl}. At the same time, C_3 does not differ so strongly from C_3^{cl}. This behavior of the rigidities is explained by shell effects. Indeed, the main contribution to C_L of the quantum long-range interaction $\mathfrak{A}$ derives from the first term in (60) and is equal to $\delta C_L = (\partial U/\partial r(\mathfrak{A}_L - \mathfrak{A}_1)\partial U/\partial r)$; as can be seen from (32), its value depends on the characteristic energy differences $\omega_{sp} = \epsilon_\lambda - \epsilon_{\lambda'}$ for the single-particle levels λ near the Fermi surface on different sides of it. If the frequency $(\omega_{sp})_L$ occurring in $\mathfrak{A}_L$ is less than $(\omega_{sp})_1$ in $\mathfrak{A}_1$, then δC_L is negative, and it is positive in the opposite case. (For $\omega = 0$ and $r_1 \approx r_2$, the contribution of each term to the sum (32) is negative, and in δC_L only the region $r_1 \approx r_2 \approx R$ is important.) In magic nuclei for odd L we have $(\omega_{sp})_L \sim \omega_0$, where ω_0 is the distance between the shells, while for even L, $(\omega_{sp})_L \sim 2\omega_0$. Therefore, for $L = 0, 2, 4, \ldots$ the part of the rigidity due to the quantum long-range interaction in the propagator A is positive, and as a result C_0 and C_2 are much larger than the hydrodynamic values. For this reason, even in a nucleus with $Z = 114$ the rigidity C_2 is positive, although C_2^{cl} is here nearly zero (Table III); this circumstance is an additional argument in favor of the existence of superheavy nuclei. For odd L, the importance of the long-range interaction in the difference $A_L - A_1$ is less, but, as can be seen from Table I, its contribution to C_L is also positive, and this is one of the reasons why magic nuclei have a stable spherical shape.

TABLE II. Rigidities of magic nuclei with respect to monopole (C_0), quadrupole (C_2), and octupole (C_3) perturbations (the hydrodynamic values are given in brackets).

Nucleus	^{40}Ca	^{88}Sr	^{132}Sn	^{208}Pb
C_0 (C_0^{cl}), MeV/F^2	29 (3.0)	42.5 (5.7)	50.5 (6.7)	61.5 (12.3)
C_2 (C_2^{cl}), MeV/F^2	18 (3.9)	8.5 (3.4)	19.7 (3.2)	28.0 (2.0)
C_3 (C_3^{cl}), MeV/F^2	6.9 (0.7)	7.4 (9.9)	14.5 (9.6)	9.9 (8.0)

[9] All the calculations in this section were made for a self-consistent potential of Woods-Saxon type with parameters taken from Ref. 45. The interaction $\mathfrak{F}$ was taken with Gaussian radial dependence and with linear interpolation with respect to the density (for details, see Sec. 8).

The situation is different for nuclei that are far from magic. In them, an important role is played by intrashell transitions between levels of the same parity for which the energy differences ω_{sp} are less than the frequency ω_0 characteristic of the single-particle transitions with odd L. And whereas in magic nuclei the long-range interaction plays its most important role in the propagator A_1, as the shells are filled the long-range interaction makes an even greater contribution to A_2 and eventually $\mathfrak{A}_2$ becomes so large that C_2 vanishes and the nucleus loses its spherical shape. The first model of this phenomenon was constructed in Ref. 8. For proper calculations of the rigidities in nonmagic nuclei, it is necessary to take into account systematically the effect of pairing.[8,25] Here, we shall not consider this problem; the corresponding expressions can be found in Ref. 25.

We now consider the mass coefficients. We calculate first B_1. As we have already said more than once, if $L=1$ then $\omega_1 = 0$ and $\Omega_1 = \partial U/\partial r$. In this case, the two determinations (58) and (58′) coincide, and then with allowance for (53′) we obtain

$$B_1 = -\left(\frac{\partial U}{\partial r}\frac{dA_1}{d\omega^2}\frac{\partial U}{\partial r}\right)_{\omega=0}. \quad (66)$$

It is convenient to make the further calculations in the λ representation. Using (32) and (36), we find

$$B_1 = -\frac{1}{4\pi}\sum_{\lambda\lambda'}\left(\frac{\partial U}{\partial r}\right)_{\lambda\lambda'}\frac{n_\lambda - n_{\lambda'}}{(\varepsilon_\lambda - \varepsilon_{\lambda'})^3}\left(\frac{\partial U}{\partial r}\right)_{\lambda'\lambda}. \quad (67)$$

If we now remember that $(\partial U/\partial r)_{\lambda\lambda'} = m(\epsilon_\lambda - \epsilon_{\lambda'})^2(\mathbf{r})_{\lambda\lambda'}$,

TABLE III. Comparison of rigidity C_L of superheavy nuclei with the hydrodynamic values C^{cl} (MeV/F^2).

L^π	C_L (^{298}114)	C_L^{cl} (^{298}114)	C_L (^{342}114)	C_L^{cl} (^{342}114)
0$^+$	70.0	17.0	73.1	14.7
2$^+$	24.7	1.1	16.3	1.6
3$^-$	13.0	6.7	24.1	7.3

TABLE IV. Mass coefficients for the first 2^+ and 3^- collective excitations in magic nuclei (hydrodynamic values given in parentheses).

Nucleus	Mass coefficients, $MeV^{-1} \cdot F^{-2}$			
	^{40}Ca	^{88}Sr	^{132}Sn	^{208}Pb
B_2 (0)	0.03	3.5	0.55	0.44
B_2 (ω_2) (B_2^{cl})	−(0.11)	8.53 (0.25)	4.23 (0.58)	3.48 (0.60)
B_3 (0)	0.25	1.05	0.55	1.33
B_3 (ω_3) (B_3^{cl})	0.55 (0.08)	2.34 (0.17)	1.46 (0.25)	2.10 (0.40)

we obtain immediately

$$B_1 = 3m(N+Z)/4\pi = 3mA/4\pi, \tag{68}$$

which coincides except for the normalization factor R^2 with the expression for B_1^{cl} obtained in the drop metal. This result is not changed when velocity forces are taken into account as well.[14]

For $L \neq 1$, the values of B_L calculated in accordance with (58) and (58′) are in general different, and the difference is more appreciable, the worse is the accuracy of the adiabatic approximation. It should be noted that in real nuclei this approximation does not, as a rule, work well since the adiabaticity parameter is not small: $\omega_L/\omega_{sp} \gtrsim 0.5$. As can be seen from Table IV, this leads to a large difference between $B_L(0)$ and $B_L(\omega_L)$, and we always have $B_L(\omega_L) > B_L(0)$; in addition, both these mass coefficients are very different from the hydrodynamic value:

$$B_L^{cl} = B_1/L = 3mA/4\pi L, \tag{69}$$

which is obtained in the drop model for irrotational flow of an ideal incompressible fluid. This last circumstance also explains why the experimentally observed probabilities $B(EL)$ depart strongly from the predictions of the hydrodynamic model. To see why, we rewrite Eq. (55) for $B(EL)$ in the form

$$B(EL) = \frac{2L+1}{2\omega_L B_L(\omega_L)} \frac{(e_q r^L \partial_L A_L \Omega_L)^2_{\omega=\omega_L}}{\varkappa_1(0)\left(\frac{\partial U}{\partial r} A_L \Omega_L\right)_{\omega=\omega_L}} = \frac{2L+1}{2\omega_L B_L(\omega_L)} M_L^2. \tag{70}$$

Here we have used the determination (58′) of the mass coefficient $B_L(\omega_L)$, the relation (53′), and the expression (53) for the square of the matrix element of the transition $0^+ \to L^\pi$. We calculate the value of M_L^2 in (70) under the assumption that the unnormalized transition density $\int A_L^i(r, r_1; \omega_L)\Omega_L^i(r_1; \omega_L) r_1^2 dr_1$ differs little from the classical $\partial\rho^i/\partial r$ (in the case $L=1$, as can be seen from (39) and (47), they coincide exactly; the extent to which this assumption is justified for other L can be deduced from the data in Table V). Then, recalling (43), we obtain $(\partial U/\partial r A_L \Omega_L)\varkappa_1(0) \approx 1$ and for $V_0 = e r^L Y_{LM}(\mathbf{n})$ we have $M_L \approx e\int \partial\rho p/\partial r\, r^{L+2} dr \equiv M_L^{cl}$, so that

TABLE V. Unnormalized matrix elements M_L of the transition $0^+ \to L^\pi$ with excitation of the first 2^+ and 3^- states in magic nuclei (hydrodynamic values given in parentheses).

Nucleus	^{40}Ca	^{88}Sr	^{132}Sn	^{208}Pb
M_2 (M_2^{cl}), $e \cdot F$	—	56.7 (47.6)	68.7 (71.7)	134 (137)
M_3 (M_3^{cl}), $e \cdot F^2$	109 (77.8)	327 (250)	464 (431)	1072 (987)

$$B(EL) = \frac{2L+1}{2\omega_L B_L(\omega_L)} (M_L^{cl})^2 \approx \frac{2L+1}{2\omega_L B_L(\omega_L)} \left(\frac{3Ze}{4\pi} R^{L-1}\right)^2. \tag{71}$$

This expression has the same form as the one obtained in the liquid-drop model.[2] It can be seen from Table V that the ratio $(M_L/M_L^{cl})^2$ is not strongly different from unity, which cannot be said of the ratio $B_L(\omega_L)/B_L^{cl}$, which, as follows from Table IV, may be of order 10 or even more.

One of the main reasons for this large difference between $B_L(\omega_L)$ and B_L^{cl} [Eqs. (58′) and (69)] is the interference between the surface waves and the zero-sound excitations and also with individual particle-hole excitations, this interference leading to a strong change in the nature of the collective motion: Whereas in classical hydrodynamics this motion corresponds to irrotational flow of an incompressible fluid, in a nucleus the collective motion becomes solenoidal and penetrates into the interior. This is confirmed by a direct calculation of the transition current:

$$\mathbf{j}_L = \frac{1}{2mi}\left[\left(\frac{\partial}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{r}'}\right)\rho_L(\mathbf{r}, \mathbf{r}'; \omega_L)\right]_{\mathbf{r}'\to\mathbf{r}} \equiv \rho_0 \mathbf{v}_L, \tag{72}$$

where

$$\rho_L(\mathbf{r}, \mathbf{r}'; \omega_L) = \int \frac{d\varepsilon}{2\pi i} d\mathbf{r}_1 G^q\left(\mathbf{r}, \mathbf{r}_1; \varepsilon - \frac{\omega_L}{2}\right) \times g_L(\mathbf{r}_1; \omega_L) G^q\left(\mathbf{r}_1, \mathbf{r}'; \varepsilon + \frac{\omega_L}{2}\right) \tag{73}$$

is the matrix of the transition density; $\mathbf{v}_L$ is the collective velocity; ρ_0 is the density of the quasiparticles, and $g_L = \alpha_L \Omega_L Y_{LM}(\mathbf{n})$ is the transition amplitude with normalization coefficient

$$\alpha_L = \left[-\left(\Omega_L \frac{dA}{d\omega}\Omega_L\right)_{\omega=\omega_L}\right]^{-1/2} = \left[2\omega_L B_L(\omega_L)\left(\frac{dU}{dr} A_L \Omega_L\right)\varkappa_1(0)\right]^{-1/2} \tag{74}$$

[see (53)–(55) and (58′)]. In classical hydrodynamics, $\alpha_L = (2\omega_L B_l^{cl})^{-1/2}$ and $\Omega_L = dU/dr$.

The collective velocity can be conveniently written as

$$\mathbf{v}_L(\mathbf{r}; \omega_L) = i\omega_L \alpha_L (F_L^{(1)}\mathbf{n} + r F_L^{(2)}\nabla) Y_{LM}(\mathbf{n}). \tag{75}$$

Then

$$\operatorname{curl} \mathbf{v}_L = i\omega_L \alpha_L R_L(r) Y_{LM}(\mathbf{n}), \tag{76}$$

where

$$R_L(r) = \frac{1}{r}\left(\frac{d(rF_L^{(2)})}{dr} - F_L^{(1)}\right). \tag{77}$$

After separation of the angular variables, we obtain for the form factors $F_L^{(1)}$ and $F_L^{(2)}$

$$F_L^{(1)}(r) = \frac{1}{2m\rho_0\omega_L} \sum_{l_1 j_1 l_2 j_2} \frac{\langle j_1 l_1 \| Y_{LM} \| j_2 l_2\rangle^2}{2L+1} \times \int \frac{d\varepsilon}{2\pi i} r_1^2 dr_1 \Omega_L(r_1; \omega_L)\left[G^q_{j_2 l_2}\left(r_1, r; \varepsilon + \frac{\omega_L}{2}\right)\frac{\partial}{\partial r} \times G^q_{j_1 l_1}\left(r, r_1; \varepsilon - \frac{\omega_L}{2}\right) - G^q_{j_1 l_1}\left(r, r_1; \varepsilon - \frac{\omega_L}{2}\right) \times \frac{\partial}{\partial r} G^q_{j_2 l_2}\left(r_1, r; \varepsilon + \frac{\omega_L}{2}\right)\right]; \tag{78}$$

$$F_L^{(2)}(r) = \frac{1}{2m\rho_0\omega_L} \sum_{l_1 j_1 l_2 j_2} \frac{\langle j_1 l_1 \| Y_{LM} \| j_2 l_2\rangle^2}{2L+1} \times \frac{l_1(l_1+1) - l_2(l_2+1)}{L(L+1)} \frac{1}{r} \int \frac{d\varepsilon}{2\pi i} r_1^2 dr_1 \Omega_L(r; \omega_L) \times G^q_{j_1 l_1}\left(r, r_1; \varepsilon - \frac{\omega_L}{2}\right) G^q_{j_2 l_2}\left(r_1, r; \varepsilon + \frac{\omega_L}{2}\right). \tag{79}$$

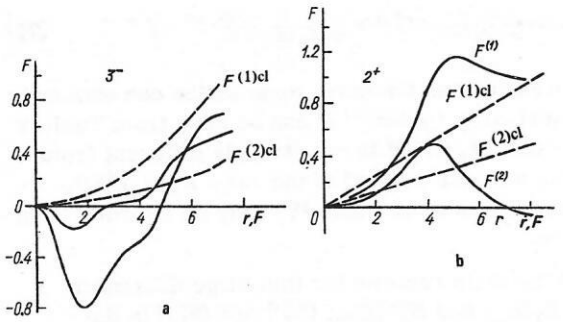

FIG. 1. Form factors of the radial distribution of the charged transition current for 3^- (*a*) and 2^+ (*b*) collective states in ^{208}Pb.

Here, G^q_{jl} is the Green's function of the radial Schrödinger equation (see Sec. 8 for its determination). In classical hydrodynamics, $\mathbf{v}^{cl} = \nabla\varphi$, where $\varphi = i\alpha_L\omega_L R/L(r/R)^L Y_{LM}(\mathbf{n})$ is the potential of the velocities, and therefore $F_L^{(1)cl} = (r/R)^{L-1}$; $F_L^{(2)cl} = 1/L(r/R)^{L-1}$ and curl $\mathbf{v}^{cl} = 0$. The calculated form factors $F_L^{(1)}$ and $F_L^{(2)}$ for the 3^- and 2^+ states in ^{208}Pb are given in Fig. 1, and the distribution of the curl of the velocity, $R_L(r)$, is given in Fig. 2. We see that the distribution of the currents in the quantum case differs appreciably from the classical case, as a result of which curl v is appreciably nonzero. This then leads to an increase of the mass coefficient $B_L(\omega_L)$ compared with B_L^{cl}.

The difference between $B_L(\omega_L)$ and B_L^{cl} makes it possible to explain why a problem arises in the liquid-drop model in the description of the sum rules for electromagnetic transitions in nuclei. In the absence of velocity forces (and they do not play an important role at low excitation energies of the nucleus) the sum $S_L = \sum (E_s - E_0)\,|(Q_L)_{0s}|^2$ can be calculated in terms of the double commutator: $S_L = \frac{1}{2}\langle 0\,|[Q_L, [H, Q_L]]\,|0\rangle$ and for $Q_L = er^L Y_{L0}$ one can obtain

$$S_L = Ze^2 L R^{2L-2}/8\pi m \tag{80}$$

(see, for example, Ref. 24). On the other hand, in classical hydrodynamics, as can be seen from (71) and (69), $B^{cl}(EL) = (2L+1)(M_L^{cl})^2/(2\omega_L B_L^{cl})$ and

$$\sum_M \omega_L\,|(Q_L)_{0^+\to L^\pi M}|^2 = \omega_L B^{cl}(EL)/(2L+1) = Ze^2 L R^{2L-2}/8\pi m, \tag{81}$$

so that the sum rule (80) in the liquid-drop model is

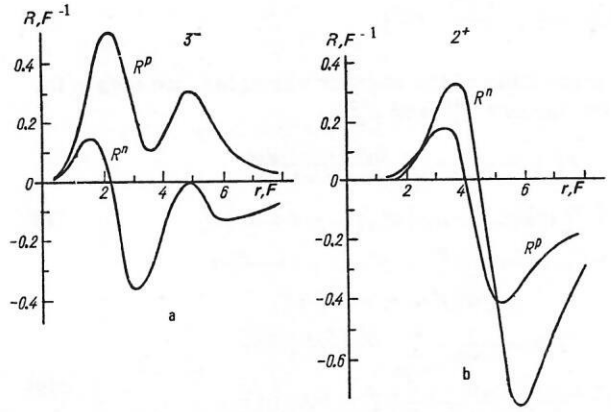

FIG. 2. Radial distribution of curl v of the collective motion for the first 3^- (*a*) and 2^+ (*b*) states in ^{208}Pb (see Ref. 76).

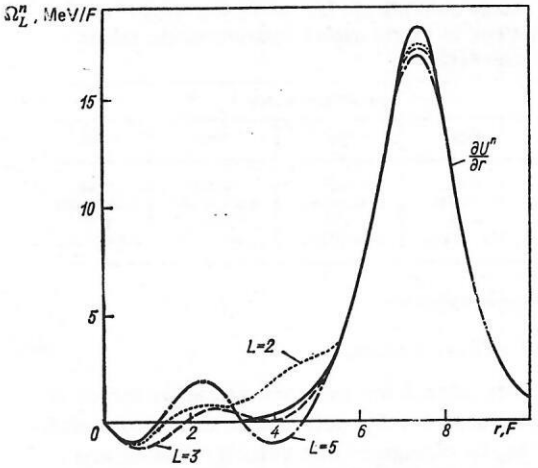

FIG. 3. Comparison of the functions Ω_L^n characterizing the radial dependence of the neutral component of the transition amplitude for the first 3^-, 2^+, and 5^- collective levels in ^{208}Pb with the dipole (classical) amplitude $\Omega_1^n = dU^n/dr$.

completely exhausted by the transition to the single-phonon state. However, the experiments show that the first low-lying collective states in nuclei by no means exhaust S_L, and this occurs precisely because the real value of B_L is much larger than B_L^{cl}.

Since there is not only the capillary branch (capons) in a Fermi-liquid drop but also the zero-sound collective branch, we must also take into account the contribution from it to the sum rule. Estimates[15] indicate that the two branches make comparable contributions to S_L, and this eliminates the sum rule paradox.

The coordinate dependence of the amplitude g_L for capon production is determined by the function Ω_L, which is very close to the classical amplitude $\partial U/\partial r$. This is demonstrated in Fig. 3, which compares the functions $\Omega_1 = \partial U/\partial r$ and Ω_L for the low-lying collective states in ^{208}Pb. Calculations for other magic nuclei lead to a similar conclusion. The same can also be said of the transition densities: They have a classical surface peak $\sim\partial\rho/\partial r$ (see Sec. 8). It is therefore not surprising that if the frequency ω_L and the mass coefficient B_L (characterizing the degree of collectivization) are regarded as empirical parameters, i.e., if the normalization α_L (74) is chosen from the experiments, one can in the framework of the classical collective model[2] describe qualitatively the interaction between the particles and phonons that is, in particular, responsible for the energy splitting of levels in multiplets.[47]

The theory developed here makes it possible to take into account automatically the quantum volume corrections, which are absent in the classical picture. The theory enables one to construct self-consistently an interaction between the particles and the phonons; as will be shown in Sec. 9, one then finds important differences from the approach based on the collective model.[2]

6. CHANGE IN THE RADIUS AND SHAPE OF NUCLEI IN EXTERNAL FIELDS

The existence of low-lying surface excitations of the Fermi-liquid drop—the quantum capillary waves—also affects the behavior of the drop in static external fields. When such fields are applied or when particles are added, there is a virtual production of capons, which results in a transfer of density from the inner regions to the surface of the drop, which in the classical language can be described as a change in the radius and shape of the drop under the influence of the external forces. The quantum hydrodynamic approach makes it possible to calculate these effects quantitatively and derive the corresponding classical laws microscopically.

Paradox Related to the Increase in the Drop Radius when Particles are Added. It is well known at least for the example of liquid ^{3}He that the radius of a Fermi-liquid drop increases in accordance with the usual law $R = r_0 A^{1/3}$. If the simple formula $\rho = \rho_0/[1 - \exp((r-R)/d)]$ is taken for the distribution of the density ρ, then when one particle is added $\delta R = R/3A$ and the change in the density is

$$\delta\rho(r) = -\frac{\partial\rho}{\partial r}\delta R = \frac{\exp((r-R)/d)}{[1+\exp((r+R)/d)]^2}\frac{\rho_0}{d}\frac{2A}{3A}, \tag{82}$$

i.e., within the drop $\delta\rho$ is an exponentially small quantity, and on the surface, where $|r-R| \sim d$, $\delta\rho(r)_{\lambda_0} \sim \rho_0 A^{-2/3}$. On the other hand, the density of the particle added in the state λ_0 is a volume function: $\delta_0\rho = |\varphi_{\lambda_0}|^2 \sim 1/R^3$.

In a Fermi gas, the change in the total (with allowance for polarization effects) density $\delta\rho(r)$ also has a volume character, i.e., both in the interior and on the surface $\delta\rho(r) \sim 1/R^3 \sim \rho_0 A^{-1}$. This means that when a particle is added to a Fermi-gas system the increase in the radius is $\delta R \sim r_0 A^{-1}$, whereas for a liquid $\delta R \sim r_0 A^{-2/3}$.

This is the nub of the radius paradox obtained when a drop of Fermi liquid is regarded as a gas of interacting quasiparticles in a potential well whose size is an external parameter of the problem.[9] However, as we already know, in such a drop there is a branch of low-lying surface excitations (not present in a gas). To calculate correctly their contribution to the increase in the radius of the system, it is necessary to use systematically the consistency conditions (37) and (38) in the equations of the theory of finite Fermi systems for the change in the density (here, we are interested only in the spherically symmetric part of $\delta\rho$):

$$\delta\rho = \delta_0\rho + A_0(\omega=0)\mathcal{F}_0\delta\rho. \tag{83}$$

This can be done almost in the same way as in Sec. 4; the only difference is that the kernel $A\mathcal{F}$ of Eq. (83) is transposed with respect to the kernel $\mathcal{F}A$ of Eq. (34). Therefore, the splitting of $A_0\mathcal{F}_0$ analogous to (40) now takes the form

$$A_0\mathcal{F}_0 = \varkappa_0\frac{\partial\rho}{\partial r}\frac{\partial U}{\partial r'} + \mathcal{R}_0^+, \tag{84}$$

where $(\partial U/\partial r \mathcal{R}_0^+ \partial\rho/\partial r') = 0$. The constant κ_0 is determined by Eq. (42) for $L = 0$.

Substituting (84) in (83) and making some simple transformations, we obtain [as in (51)]

$$\delta\rho = \delta_1\rho + \frac{\varkappa_0\left(\frac{\partial U}{\partial r}\delta_1\rho\right)}{1-\varkappa_0\left(\frac{\partial\rho}{\partial r}\Omega_0\right)}\mathcal{P}_0, \tag{85}$$

where the function $\mathcal{P}_0$, which is responsible for the increase of the density on the surface of the drop, is determined by an equation analogous to (46):

$$\mathcal{P}_0 = \frac{a\rho}{\partial r} + \mathcal{R}_0^+\mathcal{P}_0, \tag{86}$$

and

$$\delta_1\rho = \delta_0\rho + \mathcal{R}_0^+\delta_1\rho. \tag{87}$$

In deriving (85), we have used the equation $(\partial U/\partial r \mathcal{P}_0) = (\partial\rho/\partial r\,\Omega_0)$.

If we expand $\delta\rho$ with respect to the eigenfunctions χ_n of the operator $A_1(0)F_1$, one of which by virtue of the consistency condition (37) is $\chi_1 = \partial\rho/\partial r$ and the conjugate function $\chi_1^+ = \partial U/\partial r$, and we explicitly separate the term $\sim\partial\rho/\partial r$ that describes the change in the drop radius R, then we can represent $\delta\rho$ as a sum of the volume term $\delta\rho_V$ and a surface term

$$\delta\rho = \delta\rho_V - (\partial\rho/\partial r)\,\delta R, \tag{88}$$

where the constant δR is determined by the condition $\delta R = -(\partial U/\partial r\,\delta\rho)/(\partial U/\partial r\,\partial\rho/\partial r)$. Substituting here the expression (85) for $\delta\rho$, we obtain

$$\delta R = -\frac{\varkappa_1\left(\frac{\partial U}{\partial r}\delta_1\rho\right)}{1-\varkappa_0\left(\frac{\partial\rho}{\partial r}\Omega_0\right)} \equiv \left(\frac{\partial U}{\partial r}\delta_1\rho\right)\Big/ C_0, \tag{89}$$

where C_0, the monopole rigidity, is determined by Eq. (57) for $L = 0$.

In a gas, the difference $[1 - \varkappa_0(\partial\rho/\partial r\,\Omega_0)]$ is of order unity, and therefore $\delta R \sim r_0 A^{-1}$. In a liquid, by virtue of the consistency conditions, the unity and $\varkappa_0(\partial\rho/\partial r\,\Omega_0)$ in this difference almost exactly cancel (see Sec. 5)—their difference is $\sim A^{-1/3}$—and therefore $\delta R \sim r_0 A^{-2/3}$, which corresponds to the law of variation of the radius of a liquid drop. This resolves the paradox with the radius.

Analytic calculation of δR from (89) even in a large system in which the quasiclassical methods work well is rather difficult. Bearing in mind that for a large drop the main contribution to C_0 is made by the first term of (60) and ignoring the difference between $\delta_1\rho$ and $\delta_0\rho = |\varphi_{\lambda_0}|^2$, we can write the formula for δR in the simple form

$$\delta R = \varkappa_1\left(\frac{\partial U}{\partial r}\right)_{\nu_0\nu_0}\Big/\varkappa_0(k_0 - k_1), \tag{90}$$

where $\nu = (n, l, j)$,

$$k_0 - k_1 = \left[\frac{\partial U}{\partial r}(A_0 - A_1)\frac{\partial U}{\partial r}\right] = \sum_{\nu\nu'}\left(\frac{\partial U}{\partial r}\right)^2_{\nu\nu'}\frac{n_\nu - n_{\nu'}}{\varepsilon_\nu - \varepsilon_{\nu'}} - \left(\frac{\partial U}{\partial r}\frac{\partial\rho}{\partial r}\right). \tag{91}$$

Equation (90) makes it possible to calculate δR analytically in the quasiclassical approximation. This is done as follows. The matrix element $(\partial U/\partial r)_{\nu\nu'}$ can be writ-

ten on the basis of the radial Schrödinger equation in the form (we ignore the spin–orbit interaction)

$$(\partial U/\partial r)_{\nu\nu'}=(\varepsilon_{\nu'}-\varepsilon_{\nu})(\partial/\partial r)_{\nu\nu'}+(l(l+1)/mr^3)_{\nu\nu'}. \quad (92)$$

After substitution of this expression in (91) and simple transformations, we obtain

$$k_0-k_1=2\int\frac{\partial U}{\partial r}\frac{\tilde{\rho}}{r}dr+\sum_l\int\frac{l(l+1)}{mr^3}\frac{\partial\rho_l}{\partial r}dr$$
$$+\sum_{\nu\nu'}\left(\frac{l(l+1)}{mr^3}\right)_{\nu\nu'}\frac{n_\nu-n_{\nu'}}{\varepsilon_\nu-\varepsilon_{\nu'}}\left(\frac{l(l+1)}{mr^3}\right)_{\nu'\nu}, \quad (93)$$

where $\tilde{\rho}=\sum_l\rho_l$, and $\rho_l=\sum m_\nu\chi_\nu^2$ ($\chi_\nu=R_\nu/r$; the summation is over the principal quantum number for fixed l).

Replacing $(n_\nu-n_{\nu'})/(\varepsilon_\nu-\varepsilon_{\nu'})$ and $dn_\nu/d\varepsilon_\nu$ and remembering that there are no diagonal transitions in the last term of (93), we obtain

$$\sum\left(\frac{l(l+1)}{mr^3}\right)_{\nu\nu'}\frac{n_\nu-n_{\nu'}}{\varepsilon_\nu-\varepsilon_{\nu'}}\left(\frac{l(l+1)}{mr^3}\right)_{\nu'\nu}$$
$$\approx\sum_{\nu\nu'}\left(\frac{l^2}{mr^3}\right)_{\nu\nu'}\frac{dn_\nu}{d\varepsilon_\nu}\left(\frac{l^2}{mr^3}\right)_{\nu'\nu}-\sum\left(\frac{l^2}{mr^3}\right)_{\nu\nu}^2\frac{dn_\nu}{d\varepsilon_\nu}.$$

Substituting here the quasiclassical expression $\chi_\nu^2=a_\nu^2\cos^2\Phi_\nu(r)/\rho_\nu(r)$, where Φ_ν is the quasiclassical phase, and replacing $\cos^2\Phi_\nu(r)$ by $\frac{1}{2}$, we find $(l^2/mr^3)_{\nu\nu}=2(\varepsilon_\nu-U_0)/R$. Using this result, and making some simple but lengthy calculations, we obtain

$$k_0-k_1=3Ap_F^2/mR^2. \quad (94)$$

The numerator in (90) can be readily calculated by means of (92):

$$(\partial U/\partial r)_{\nu_0\nu_0}=p_F^2/mR. \quad (95)$$

Substituting (94) and (95) in (90), we find $\delta R=R/3A$, which agrees exactly with the hydrodynamic law. In a large system, this law must also be satisfied when allowance is made for the small corrections omitted in the analytic calculation. This is due to the saturation property of the nuclear forces: $\rho\to$ const as $A\to\infty$. This behavior of ρ follows from the consistency condition (37). This can be seen in the simple model with δ-functional interaction considered in Sec. 1 with the self-consistent potential $U=a\rho+b\rho^2/2$ [$a=c_0f_{ex}^+<0$, $b=(c_0/\rho_0)(f_{in}^+-f_{ex}^+)>0$; here $c_0=(dn/d\varepsilon_F)^{-1}$. If ρ increases as $A\to\infty$, then for large A the term $b\rho^2/2$ is dominant and U becomes positive, i.e., the bound state is lost. If this is not to occur, the density must tend to a constant limit.

The general expression (89) shows that the change in the radius when particles are added fluctuates as a function of the state occupied by the added particle (fluctuations of the matrix element of the force $(\partial U/\partial r)_{\nu_0\nu_0}$), and as a function of the nucleus (fluctuations of C_0). The scale of the fluctuations δR for the nucleus ^{208}Pb can be estimated from Table VI.

Hooke's Law for a Fermi-Liquid Drop. The problem just considered is related to another—calculation of the increase in the size of the system in an external field. To derive the necessary relation, we require the formula relating the change in the density ρ to the effective field V in the system. It has the form[9]

$$\delta\rho(r)=A_0V_0. \quad (96)$$

Substituting here the expression for V from (51) and carrying out operations analogous to those that led to Eq. (89), we obtain

$$\delta R=\frac{\left(\frac{\partial U}{\partial r}A_0v_0\right)\left(1-\varkappa_0\left(\frac{\partial\rho}{\partial r}\Omega_0\right)\right)+\varkappa_0\left(\frac{\partial\rho}{\partial r}v_0\right)\left(\frac{\partial U}{\partial r}A_0\Omega_0\right)}{\left(1-\varkappa_0\left(\frac{\partial\rho}{\partial r}\Omega_0\right)\right)\varkappa_1}. \quad (97)$$

Ignoring small corrections, we can transform this expression to

$$\delta R=\left(\frac{\partial\rho}{\partial r}v_0\right)\Big/C_0=-\left(\frac{\partial\mathcal{V}_0}{\partial r}\rho\right)\Big/C_0. \quad (98)$$

It has the form of Hooke's law: The change of the radius of the system is equal to the force applied to it $F=-\int\partial\mathcal{U}/\partial r\,\rho r^2dr$ ($f=-\partial\mathcal{U}/\partial r$ is the specific force) divided by the rigidity.

An analogous expression for the parameter of quadrupole deformation of the surface can be obtained by taking, not a scalar field, but a quadrupole field $V\sim Y_{2m}(\mathbf{n})$; $\delta R=\alpha_2Y_{2m}(\mathbf{n})$, where

$$\alpha_2=-\rho(\partial\mathcal{V}_2/\partial r)/C_2. \quad (99)$$

TABLE VI. Change in the radius δR_{λ_0} due to addition of a particle in the state λ_0 in the ^{208}Pb nucleus ($\delta R^{cl}\approx R/3A=1.06$ F).

Neutrons						
λ_0	$p_{3/2}$	$f_{5/2}$	$p_{1/2}$	$g_{9/2}$	$i_{11/2}$	$i_{15/2}$
δR_{λ_0}, F	0.95	0.97	0.97	1.01	0.95	1.23
Protons						
λ_0	$s_{1/2}$	$d_{3/2}$	$h_{11/2}$	$h_{9/2}$	$f_{7/2}$	$i_{13/2}$
δR_{λ_0}, F	1.15	1.07	1.26	1.01	1.26	1.44

7. TRANSITION TO THE CLASSICAL LIMIT AND COMPARISON WITH THE RESULTS OF THE COLLECTIVE BOHR–MOTTELSON MODEL

In this section, we shall discuss in detail the situation with regard to collective surface excitations in a macroscopic system and show that the quantum hydrodynamic description goes over into the classical one for $L\gg1$. We recall first of all that with increasing L the contribution of the long-range components of the propagator $A(\mathbf{r},\mathbf{r}',\omega)$ to A_L decreases, and as a result so do the long-range components of the operator $\mathcal{R}_L$ (40), which determines the value ϕ_L of the quantum correction to the capon production amplitude $g_L=\alpha_L(\partial U/\partial r+\phi_L)$. The local components of $\mathcal{R}_L$, when integrated with the function $\partial U/\partial r$, make a small contribution $\sim L^2/(p_FR)^2$ since $\mathcal{R}_1(\partial U/\partial r)=0$ [see Eq. (44)]. Therefore, with increasing L, $g_L\to\alpha_L\Omega_1=\alpha_L\,\partial U/\partial r$, which agrees with the result of the Bohr–Mottelson model.[2] Hence, for the transition density $\rho_{tr}(r,\omega_L)=\int A_L(r,r';\omega_L)g_L\times(r',\omega_L)r'^2dr'$ we obtain $\rho_{tr}\to\alpha_L(A_1(0)\partial U/\partial r)\equiv\alpha_L\partial\rho/\partial r$. Thus, for large L, the entire change in the density occurs on the surface of the drop, which behaves as if it were incompressible. Incompressibility means that

$$d\rho/dt=\partial\rho/\partial t+(\mathbf{v}\nabla\rho)=0. \quad (100)$$

But $\delta\rho(r)=\alpha\,\partial\rho/\partial r$, i.e., $\partial(\delta\rho)/\partial t=-i\omega\alpha\,\partial\rho/\partial r$, and it then follows from (100) that

$$v_r(R)=\dot{\alpha}, \tag{101}$$

i.e., the velocity of the particles on the boundary is equal to the velocity with which the boundary moves. But this is the boundary condition of classical hydrodynamics.

With increasing L, the rigidity coefficient C_L (see Sec. 5) also approaches the classical limit. In the derivation given in Sec. 5 of the expression (65) for the coefficient of surface tension σ we have been somewhat inconsistent: The expression is derived, not from the static problem, but from the dynamical problem in an investigation by the adiabatic method of the equation for the frequencies ω_L of collective excitations. We show that in the static problem exactly the same result is obtained.[48] We impose on the system (assumed for simplicity neutral) a static ($\omega=0$) surface field: $V_0=\lambda_L(\partial U/\partial r)Y_{LM}(\mathbf{n})$, and we calculate the change in the internal energy. It is[48]

$$\delta E=-(V_0AV)/2. \tag{102}$$

For $L\gg1$, the effects of compressibility can be ignored and in the expression (51) for V_L we can replace the function Ω_L by $\partial U/\partial r$ and $\mathcal{V}_L$ by $V_0\sim\partial U/\partial r$. We obtain $V_L=\lambda_L/1-\varkappa_L/\varkappa_i\,\partial U/\partial r$. The corresponding change in the density is (to the same accuracy) $\delta\rho_L(r)=-\lambda_L/1-\varkappa_L/\varkappa_1\,\partial\rho/\partial r$, i.e., when the field is applied, only the radius of the system changes: $\delta R_L=\lambda_L/(1-\varkappa_L/\varkappa_1)$. Substituting the resulting expression for V_L in (102) and replacing λ_L by $\delta R_L(1-\varkappa_L/\varkappa_1)$, we find

$$\delta E=C_L\delta R_L^2/2,\quad \text{where } C_L=-(1-\varkappa_L/\varkappa_1)/\varkappa_1,$$

which is equal except for small quantum corrections to the determination (59) of the rigidity. For large L, to calculate κ_L, we can use the quasiclassical approximation, which [see (65)] gives $C_L=\sigma(L-1)(L+2)$. Substituting C_L in the expression for δE and making the standard transformation $\sum_L(L-1)(L+2)\delta R_L^2\equiv\delta S$, where δS is the change in the surface area of the drop, we obtain

$$\delta E=\sigma\cdot\delta S. \tag{103}$$

Thus, the change in the energy of the quantum drop for the L-th deformation of the surface is, if $L\gg1$, determined by the classical expression (103).

We now turn to the mass coefficient B_L and show that for sufficiently large L, for which $\omega_L\gg\varepsilon_F A^{-1/3}$, the value of B_L is inversely proportional to L, as must be in classical hydrodynamics. At such frequencies, to calculate the propagator A in a macroscopic system we can use the quasiclassical method, replacing summation over λ by integration. Therefore, for $L\gg1$, for the propagator $\tilde{A}$ averaged in this manner the point $\omega=0$ is not singular and

$$B_L=-\left(\frac{\partial U}{\partial r}\left(\frac{\partial\tilde{A}_L}{\partial\omega^2}\right)_{\omega=0}\frac{\partial U}{\partial r}\right) \tag{104}$$

(here we have used the fact that $\Omega_L\approx\partial U/\partial r$ at large L).

As we have already noted, the main contribution to B_L is made by the long-range part $(\partial A_L/\partial\omega^2)_0$, and the contribution of the local part is small. The form of this long-range interaction can be established in an infinite system. For this, we write the expansion (see, for example, Ref. 32):

$$A_\infty(k,\omega)=A_\infty(k,\omega=0)-\frac{\omega^2}{(kv)^2}\alpha(k^2),$$

i.e.,

$$-(\partial A_\infty/\partial\omega^2)_0=\alpha(k^2)/(kv)^2=\alpha_0/k^2+\alpha_1+\ldots.$$

On the transition to the coordinate representation, the first term of this expansion gives the long-range term

$$(\partial A_\infty(\mathbf{r}_1,\ \mathbf{r}_2;\ \omega)/\partial\omega^2)_0\sim1/|\mathbf{r}_1-\mathbf{r}_2|,$$

whose harmonic decreases with increasing L as $1/L$.

Similar arguments can also be advanced in a finite system directly in the coordinate representation. They also give $(\partial\tilde{A}_L/\partial\omega^2)_0\sim1/L$, i.e., the mass coefficient B_L at large L decreases, like the classical one: $B_L\sim1/L$.

We now discuss the general question of the connection between the quantum hydrodynamic description and the classical one. We have become accustomed to accepting that classical hydrodynamics describes collective motions in a system when real collisions establish thermodynamic equilibrium in every volume of the liquid that is small compared with the wavelength λ (i.e., $\omega\tau_{coll}\ll1$ or $\lambda_{coll}\ll\lambda$, where τ_{coll} is the characteristic collision time and λ_{coll} is the mean free path corresponding to it) and, therefore, the main source of dissipation of the excitation energy—its transfer to the single-particle degrees of freedom—is to a large extent shut off. At the same time, in the function $v_L(\mathbf{n})$, which determines the change in the distribution function accompanying the oscillation, there remain only the zeroth and first harmonics, and their substitution in the collision integral makes it vanish because of the laws of conservation of the particle number and the momentum. However, the hydrodynamic features of the collective motions are manifested not only when $\lambda_{coll}\ll\lambda$.

For example, in a neutral superconductor, because of the presence of the gap Δ in the spectrum of the single-particle excitations, transfer of energy from the collective degrees of freedom to the single-particle degrees is impossible as long as $\omega_L<2\Delta$. The velocity of these excitations is, as is well known, equal to the velocity of hydrodynamic sound $c^2=(\rho/m)\partial\mu/\partial\rho$, although the microscopic picture of the propagation of acoustic vibrations is much more complicated than the usual one.[49] This coincidence is possible because the pairing in the superfluid drop, like collisions in an ordinary drop, diagonalizes the momentum flux tensor, $\Pi_{ik}=p\delta_{ik}$, after which the relation $c^2=(\partial p/\partial\rho)=(\rho/m)\partial\mu/\partial\rho$ is a simple consequence of the basic hydrodynamic equations: the Euler and continuity equations.

Another example of hydrodynamic behavior of collective excitations in a quantum system is the quantum capillary waves considered here. The behavior of these excitations is determined by the general laws that con-

trol the properties of systems with spontaneously broken symmetry, and these are valid in both the quantum and the classical case. Of course, the situation in a quantum drop is more complicated because of the interaction of the capons with the zero-sound branch, and also with individual particle-hole excitations. This interference decreases with increasing angular momentum L, this characterizing the number of nodes of the wave function of the collective excitation. Thus, Bohr's correspondence principle is satisfied: The collective motion becomes classical when $L \gg 1$.

8. NUMERICAL CALCULATIONS

In the previous sections, we have established that in a Fermi-liquid drop there is a low-lying collective branch of quantum capillary waves. In this section, we gather together the results of calculations of the properties of low-lying collective states in nuclei, and these demonstrate convincingly that these states have all the characteristic features of capons.

Calculations in the framework of the quantum hydrodynamic approach have a number of differences from the standard calculations in the theory of finite Fermi systems, these being due primarily to the need to take into account the consistency condition (37). As is well known, the theory of finite Fermi systems is usually concerned with describing the behavior of systems in long-wavelength external fields. In such problems it is, first, sufficient to specify the value of the amplitude $\mathcal{F}$ at $q = 0$ and, second, it is convenient to use the λ representation since the matrix elements of the long-wavelength effective field are small for transitions between states far from the Fermi surface.

The equations that determine the properties of capons are different. They contain the derivatives $\partial U/\partial r$ and $\partial \rho/\partial r$, which have sharp peaks in the surface region of the system. Conversely, their Fourier transforms are smooth functions of the momentum, varying over an interval of order p_F. This means that when Eqs. (37) and (34) are solved we must in some manner specify the behavior of the amplitude $\mathcal{F}$ as a function of the momentum transfer q in a fairly large range $q \sim p_F$. In addition, we must give up the λ representation. The most natural thing to do is to solve the quantum hydrodynamic equations directly in coordinate space.[12] At the same time, we must also modify the form of the effective interaction $\mathcal{F}$ of the quasiparticles.

Choice of the Effective Interaction. Since there does not yet exist a reliable method for calculating the amplitude $\mathcal{F}$ in terms of the vacuum interaction, we proceed in the manner adopted in the theory of finite Fermi systems: We specify the functional form of $\mathcal{F}$, and find the values of the parameters by confronting theory and experiment. We recall that the phenomena we consider, which are related to the properties of capons—low-lying collective states of "normal" parity ($2^+, 3^-, 4^+$), are determined basically by the spin-independent components of $\mathcal{F}$, which are also responsible for the basic (Woods–Saxon) component U_0 of the self-consistent field in the nuclei. Therefore, we restrict ourselves for the time being to these terms. We shall also ignore the nonzero harmonics of $\mathcal{F}$ (velocity forces), since they are small in nuclei. We write $\mathcal{F}$ in the form

$$\mathcal{F}(\mathbf{r}, \mathbf{r}') = c_0 v(|\mathbf{r}-\mathbf{r}'|)\{\hat{f}_{ex} + (\hat{f}_{in} - \hat{f}_{ex})\tilde{\xi}(\mathbf{r}, \mathbf{r}')\} + \mathcal{F}^c(|\mathbf{r}-\mathbf{r}'|), \quad (105)$$

where $c_0 = (dn/d\varepsilon_F)^{-1} = 360$ MeV.F^3; $\hat{f} = f + f'\tau_1\tau_2$; $\tilde{\xi}(\mathbf{r}, \mathbf{r}')$ is the function that determines the form of the "interpolation"—the transition from the internal constants (f_{in}, f'_{in}) to the external (f_{ex}, f'_{ex}); $\mathcal{F}^c(|\mathbf{r}-\mathbf{r}'|)$ is the ordinary Coulomb interaction acting between only the protons. We do not take into account the exchange Coulomb interaction, nor the electromagnetic interaction of the neutrons.

For the function $\tilde{\xi}(\mathbf{r}, \mathbf{r}')$, we have used the following two forms:

$$\tilde{\xi}_1(\mathbf{r}, \mathbf{r}') = \xi[(\mathbf{r}+\mathbf{r}')/2]; \quad (106)$$

$$\tilde{\xi}_2(\mathbf{r}, \mathbf{r}') = [\xi(\mathbf{r})\xi(\mathbf{r}')]^{\alpha}, \quad (106')$$

where $\xi(x) = \rho(x)/\rho_0$ ($\rho_0 = 0.145$ F^{-3} is the real nuclear density corresponding to $r_0 = 1.18$ F). In the calculations of the properties of definite nuclei, we have assumed that $\rho(r)$ is a Fermi function: $\xi(x) = [1 + \exp((x - R - \Delta R)/d]^{-1}$. Here, R is the range of the self-consistent potential; d is the diffuseness parameter. The interpolation radius is determined by ΔR. For $\Delta R = 0.5$–0.6 F, the function $\rho_0\xi(r)$ almost repeats the real nuclear density $\rho(r)$. Converting $v(r)$ in (105) to the momentum representation and making an expansion with respect to q^2, we obtain

$$v = v_0(1 - r_{eff}^2 q^2/2 + r_{eff}^4 q^4/8 + \ldots) \quad (107)$$

Restricting ourselves to the first two terms of the expansion (107), we obtain forces of Skyrme type.[42] If the terms $\sim q^4$ do not play an important role, then any $v(r)$ corresponding to the same r_{eff} as the Skyrme forces, for example, the Gaussian

$$v_G = (\sqrt{2\pi}\, r_{eff})^{-3} \exp[-(\mathbf{r}-\mathbf{r}')^2/2r_{eff}^2] \quad (107')$$

or the Yukawa interaction

$$v_U = [\sqrt{2}/(4\pi r_{eff}^3)] \exp[-\sqrt{2}|\mathbf{r}-\mathbf{r}'|/r_{eff}] \quad (107'')$$

must lead to the same results. The calculations show that this is the situation in the description of the properties of capons with small L. This can be readily understood on the basis of the equations of Sec. 4. As a result of transformations that use the consistency condition, the interaction $\mathcal{F}$ occurs only in expressions of the type $(\partial\rho/\partial r)(\mathcal{F}_L - \mathcal{F}_1)A_L\,\partial U/\partial r)$, and such differences contain the small parameter $L^2(p_F R)^2$. In a heavy nucleus like ^{208}Pb, the effective-range approximation (for $r_{eff} \lesssim 1$ F) operates fairly well right up to $L = 5$. At the same time, this parameter is near unity in ^{40}Ca already for $L = 3$, and therefore the Yukawa and Gaussian interactions give strongly different results (see below).

More detailed information about the behavior of $\mathcal{F}(q, \rho)$ is also required for a number of other problems, for example, in an investigation of the possibility of phase transitions in nuclear matter. The parametrization of the vacuum part of $\mathcal{F}$ can be made much more

precise by solving the equation for the interaction amplitude of two particles near the edge of the nucleus. It can be solved in the gas approximation by using the known experimental data on the nucleon scattering amplitude in vacuum. With regard to the constants that characterize the behavior of $\mathcal{F}$ within the nucleus, their number can be reduced by using the condition of antisymmetry of the total amplitude Γ and the equations which relate it to $\mathcal{F}$ (for more details about this, see Ref. 50).

Coordinate Representation for the Propagator A. In the solution of Eq. (34) for $g_L(r)$ or Eq. (37), which determines the consistency conditions, the standard methods based on a representation of the eigenfunctions φ_λ of the quasiparticles are inconvenient since the problem contains eigenfunctions $g_L \sim \partial U/\partial r$ with strong coordinate dependence. Therefore, an adequate method of solving these equations is one based on the coordinate representation in which the propagator $A(\mathbf{r}, \mathbf{r}'; \omega)$ is determined by Eq. (32). We write

$$G^q(\mathbf{r}, \mathbf{r}'; \varepsilon) = \sum_\lambda \left(\frac{n_\lambda}{\varepsilon - \varepsilon_\lambda - i\delta} + \frac{1-n_\lambda}{\varepsilon - \varepsilon_\lambda + i\delta}\right) \varphi_\lambda^*(\mathbf{r})\, \varphi_\lambda(\mathbf{r}'). \quad (108)$$

Here, n_λ are the population numbers; ϵ_λ are the energies, and $\varphi_\lambda = R_{nlj}(r)\Phi_{jlm}(\mathbf{n}, s)$ are the quasiparticle eigenfunctions. Substituting (108) for one of the Green's functions in the expression (32) for A and closing the contour of integration in the upper half-plane, we readily obtain

$$A(\mathbf{r}, \mathbf{r}'; \omega) = \sum_\lambda n_\lambda \varphi_\lambda^*(\mathbf{r})\, \varphi_\lambda(\mathbf{r}') \times [G^q(\mathbf{r}, \mathbf{r}'; \varepsilon_\lambda + \omega) + G^q(\mathbf{r}, \mathbf{r}'; \varepsilon_\lambda - \omega)]. \quad (109)$$

Further, introducing the expansion

$$G^q(\mathbf{r}, \mathbf{r}'; \varepsilon) = \sum_{jlm} \Phi_{jlm}^*(\mathbf{n}, s)\, \Phi_{jlm}(\mathbf{n}', s')\, G_{lj}^q(r, r'; \varepsilon)$$

and separating the angular variables in (109) in the standard manner, we obtain from (36) for $A_L(r, r'; \omega)$

$$A_L(r, r'; \omega) = \sum_{jlj'l'} B_{jlj'l'}^L \sum_n k_{nlj} R_{nlj}(r) \times R_{nlj}(r')\, [G_{l'j'}(r, r'; \varepsilon_\lambda + \omega) + G_{l'j'}(r, r'; \varepsilon_\lambda - \omega)]. \quad (110)$$

Here, k_{nlj} is the population factor of the single-particle level (nlj): $k_{nlj} = N_{nlj}/(2j+1)$, where N_{nlj} is the total number of particles on this level. The angular factor is

$$B_{jlj'l'}^L = \frac{1}{4\pi}(2l+1)(2l'+1)(2j+1)(2j'+1) \times \begin{pmatrix} l & l' & L \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{Bmatrix} l & j & 1/2 \\ j' & l' & L \end{Bmatrix}^2.$$

Note that in (109) and (110) the summation is only over the filled levels.

To calculate $G_{lj}^q(r, r'; \varepsilon)$, we write $G_{lj}^q = g_{lj}/(r \cdot r')$, where g_{lj} is the Green's function of the one-dimensional Schrödinger equation

$$(\varepsilon - H_{lj})\, g_{lj}(r, r'; \varepsilon) = \delta(r - r'). \quad (111)$$

Here $H_{lj} = H_0 + l(l+1)/(2mr^2)$, and H_0 is the quasiparticle Hamiltonian (it contains a central potential U_0 of Woods-Saxon type, the Coulomb potential—for the protons—and spin—orbit terms $U_{lj}^{sl} \sim \langle \sigma l \rangle_{lj}$).

We introduce two independent solutions $y_1(r, \varepsilon)$ and $y_2(r, \varepsilon)$ of Eq. (111) without a right-hand side, satisfying the boundary conditions $y_1(0) = 0$ and $y_2(\infty)$ bounded. Then

$$g_{lj}(r, r'; \varepsilon) = y_1(r_<, \varepsilon)\, y_2(r_>, \varepsilon)/W(\varepsilon), \quad (112)$$

where the Wronskian $W(\varepsilon) = y_1 y_2' - y_1' y_2$ depends only on the energy. Substitution of (112) in (110) solves the problem of the exact calculation of $A_L(r, r'; \omega)$.

A detailed description of the scheme of solution of the equations for the effective field in the coordinate representation can be found in Refs. 12 and 51. An analogous procedure for solving the equation for the effective fields in the coordinate representation was independently developed in Ref. 52.

Analysis of Consistency Conditions. As we have noted in Sec. 1, the consistency conditions make it possible to establish fairly stringent restrictions on the effective interaction $\mathcal{F}$, in particular, to reject a δ-function interaction.[12] These restrictions apply mainly to the amplitude $\mathcal{F}^+ = c_0 \cdot 2f(\mathbf{r}, \mathbf{r}')$, which does not depend on the isospin since it forms the main characteristics of the nuclei: the depth of the potential well U_0 and the density ρ_0. For nuclei with $N = Z$, if the Coulomb interaction is ignored, the consistency conditions (37) contain only this component $\mathcal{F}(U^- = 0, \rho^- = 0)$:

$$\frac{\partial U^+}{\partial r} = \int \mathcal{F}_1^+(r, r') \frac{\partial \rho^+}{\partial r'} r'^2\, dr'. \quad (113)$$

We begin our analysis of the consistency conditions with a "large nucleus" ($A^{1/3} \gg 1$, $N = Z$, Coulomb interaction absent). The solution of Eq. (113) for such a system[53] enables us to find the chemical potential μ_∞ of nuclear matter—the main coefficient of the Weizsäcker formula:

$$E_0(A) = \mu_\infty A + \gamma A^{2/3} + \beta(N-Z)^2/A + \ldots, \quad (114)$$

and to calculate the depth of the well U_0 and the equilibrium density ρ_0. Equation (113) was solved iteratively. We started with the Fermi function $\rho^{(0)}(r)$ with $R^{(0)} = r_0^{(0)} \cdot A^{1/3}$. We then found the self-consistent potential $U^{(0)}(r) = \mathcal{F}\partial\rho^{(0)}/\partial r$ and the density $\rho^{(1)}(r) = \sum_\lambda n_\lambda |\varphi_\lambda^{(0)}(r)|^2$, and repeated the procedure: $U^{(1)}(r) = \mathcal{F}\partial\rho^{(1)}/\partial r$, etc. The calculation was made in a nucleus with $A = 7000$ for the interation (106). The initial value of $r_0^{(0)}$ was varied from 1 to 1.3 F. In all cases, the iterative procedure converged. For the parameter values $f_{in} = 0.2$, $f_{ex} = -2$, $r_{eff} = 0.9$ F, which, as we shall see below, lead to a good description of the experimental data on the low-lying collective states, we obtain $r_0 = 1.175$ F, $\mu = -14.3$ MeV, and $U_0 = -49.3$ MeV (r_0 is determined from the condition $R = r_0 A^{1/3}$, where R is the half-density radius; μ is the energy of the last filled level; U_0 is the depth of the well averaged over the interior region; the scale of the fluctuation of U_0 within the nucleus is ~0.5 MeV). The value calculated from this $\varepsilon_F = p_F^2/2m = (9\pi)^{3/2}/8r_0^2 m = 35.1$ MeV agrees to a good accuracy with $\varepsilon_F = \mu - U_0 = 35.0$ MeV. The coefficient $\gamma = 4\pi r_0^2 \sigma$ in Eq. (114) for these values of the parameters of $\mathcal{F}$ is found to be 19.3 MeV.[46]

Using the obtained results, we can find μ_∞. From Eq. (114), we have $\mu(A) = \mu_\infty + (2/3)\gamma A^{-1/3}$, whence μ_∞

TABLE VII. Properties of nuclear matter.

Parameters of the Weizsäcker formula	Results of evaluation of experimental data		Theoretical calculation	
	Ref. 55	Ref. 36	Hartree-Fock (Ref. 55)	hours
μ_∞, MeV	−15.25	−15.98	−16.0	−14.9
γ, MeV	17.07	20,76	19.8	19
β, MeV	33.16	36.5	33.8	30.3
r_0, F	1.22	1,175	1,15	1.17

$=-14.9$ MeV follows for $\gamma \approx 19$ MeV. Substituting the value $\varepsilon_F = 35$ MeV in Migdal's formula

$$\beta = \varepsilon_F (1+2f')/3 \qquad (115)$$

and using the value $f' = 0.8$ found in Ref. 54, we obtain $\beta = 30.3$ MeV.

The calculated characteristics of nuclear matter (the coefficients of the Weizsäcker formula and r_0) are given in Table VII.

An analogous calculation was made for the magic nuclei ^{208}Pb and ^{40}Ca. Instead of (113), we now have the system of equations (28), which it is more convenient to rewrite in the form

$$\frac{\partial U^i}{\partial r} = \int \mathcal{F}_1^{ik}(r, r') \frac{\partial \rho^k}{\partial r'} r'^2 dr'. \qquad (116)$$

We recall that the indices $i, k = 1, 2$ refer to neutrons and protons, respectively. Here, by $\mathcal{F}$ it is now necessary to understand the total amplitude (105). In addition, to obtain a realistic spectrum of single-particle levels it is necessary to take into account the spin-orbit interaction. A direct iterative solution of (116) is impossible since the diemensionless constant $f^- = 2f' > 1 (f' \approx 0.8)$ and the iterations diverge. This divergence can be eliminated by means of the "collapse" procedure usually employed in the Hartree-Fock method. It is as follows. Suppose that as a result of the n−th iteration in (116) we have obtained the potential U_n^i; then for the subsequent iteration one takes as original potential, not U_n^i, but the combination $\tilde{U}_n^i = \alpha U_n^i + (1-\alpha) U_{n-1}^i$, where $0 < \alpha < 1$. This method of iteration was used, in particular, in Ref. 56 to reconcile the "small" components of the amplitude $\mathcal{F}$ and the potential U (Coulomb, isovector, and spin-orbit), the main isoscalar component of the potential U being chosen phenomenologically in the Woods-Saxon form; it proved possible to reproduce the single-particle spectra of magic nuclei fairly well.

We used the coefficient $\alpha = 0.5$ in this procedure. The iterations converage fairly rapidly: after 15–16 iterations, the position of the levels can be fixed with accuracy ~1 keV. The spin-orbit interaction was chosen in the form proposed in Ref. 9, which leads to the potential[56]

$$U_{LS} = c_0 r_0^2 \left\{ \frac{1}{r} \frac{d}{dr} (\varkappa \rho^+ - 2t_3 \varkappa' \rho^{-1}) \sigma l - \frac{1}{r^2} \frac{d}{dr} r (\varkappa \rho_1^+ - 2t_3 \varkappa' \rho_1^-) \right\}. \qquad (117)$$

Here, $r_0 = 1.2$ F, and $\rho_1^{(n,p)}$ are the densities for the filled states of the spin-orbit doublets in which one level lies above the Fermi surface; the constants $\varkappa = 0.18$ and $\varkappa' = -0.10$ were taken from Ref. 56. In addition, allowance was made for the exchange Coulomb interaction, which leads in the Thomas-Fermi approximation to a term in the potential of the form $U_{\text{exc}}^c = -e^2 [3/\pi \rho p(r)]^{1/3} \approx -0.5$ MeV.

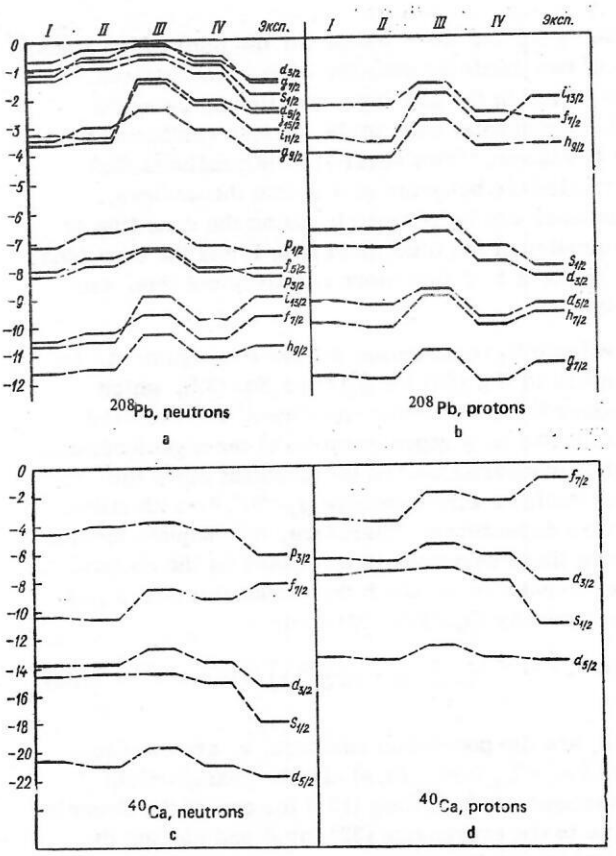

FIG. 4. Single-particle spectra of the nuclei ^{208}Pb and ^{40}Ca.

In Figs. 4a and 4b we show the calculated single-particle spectra of the nucleus ^{208}Pb for several sets of the interaction parameters for fixed values of the constants $f_{in} = 0.2$, $f'_{in} = f'_{ex} = 0.8$. Variant I corresponds to Gaussian interaction ($f_{ex} = 2.1$) and II to the Yukawa interaction ($f_{ex} = -2.1$). It can be seen that the calculation reproduces quite well the experimental level schemes, especially for the neutrons. However, the levels with large and small l are shifted relative to the experimental levels in opposite directions. This indicates that U contains a small term proportional to l^2. The existence of such a term follows from general arguments[9] and may be due to velocity forces. We took it in the simplest form, the same for neutrons and protons:

$$U_{l^2} = c_0 r^{-2} \lambda (d\rho^+/dr)^2,$$

with constant $\lambda = 5$ F^5. Variant III differs from II by the addition of U_{l^2}. It can be seen that the relative position of the levels is significantly improved, though the picture as a whole is shifted upward by about 1–2 MeV. This shift is eliminated by a small variation of the constant f_{ex}: If instead of $f_{ex} = -2.10$ we take $f_{ex} = -2.15$, then the experimental levels are reproduced fairly well (variant IV). In Figs. 4c and 4d, we show the results of analogous calculations for ^{40}Ca. Here, the agreement with the experiments is not so good.

Characteristics of Low-Lying Collective States in Even–Even Nuclei. In Sec. 4, Eq. (34) of Migdal's theory for $g_L(r;\omega)$, which determines the frequencies ω_L and the amplitudes g_L for production of the collective state, was transformed by means of the consistency condition (37). The reduced probabilities B(EL)

TABLE VIII. Characteristics of low-lying collective states as functions of the effective range of the interaction*.

Nucleus	J^π	Frequency and transition probability	Experimental data[57-59]	$r_{eff}F$ [interaction of Gaussian type (107')]		$r_{eff}F$ [interaction of Yukawa type (107")]	
				1	0.8	1	0.8
^{208}Pb	3⁻	ω	2.614	2.74	2.44	2.60	2.32
		$B(E3)\cdot10^{-5}$	5.4±0.3 (7.1±0.4)	7.02	9.0	6.94	8.62
	5⁻	ω	3.197	3.33	3.21	3.26	3.16
		$B(E5)\cdot10^{-8}$	4.62±0.55	2.84	4.44	3.68	4.42
	2⁺	ω	4.086	4.62	4.58	4.61	4.50
		$B(E2)\cdot10^{-3}$	2.96±0.18	2.57	2.77	2.73	2.87
	4⁺	ω	4.323	5.03	4.91	4.96	4.85
		$B(E4)\cdot10^{-6}$	7.57	8.00	6.76	9.37	13.0
	6⁺	ω	4.425	4.33	4.26	4.27	4.22
		$B(E6)\cdot10^{-10}$	2.1	1.86	2.05	2.05	1.98
^{40}Ca	3⁻	ω	3.73	5.09	4.4	4.48	3.92
		$B(E3)\cdot10^{-4}$	2.11	0.92	1.48	1.44	1.77
	5⁻	ω	4.48	6.05	5.83	5.73	5.53
		$B(E5)\cdot10^{-6}$	—	2.7	2.4	2.36	2.50

*Here and in the following tables, the excitation energies ω_L are given in MeV, and $B(EL)$ in $e^2\cdot F^{2L}$.

and the transition densities $\rho^L_{tr}(r)$ can be expressed in terms of the amplitude g_L in accordance with Eqs. (45), (54), and (55).

In systematic calculations of the characteristics of the low-lying collective states, we used $U(r)$ and $\mathcal{F}$ that are not completely self-consistent with one another. As $U(r)$ we took the potential from Ref. 45 fitted fairly well to the experimental single-level levels (as a rule, the discrepancy with the experiments does not exceed 300–500 keV). The amplitude $\mathcal{F}$ was chosen in the form (105), and we then carried out a partial "matching" of $\mathcal{F}$ and U: The interaction parameters f_{in}, f_{ex}, and ΔR were determined in such a way that $\varkappa_1$ found from (42) was equal to (43). This in conjunction with Eq. (50) for ϕ_L ($g_L=\alpha_L\Omega_L$, $\Omega_L=\partial U/\partial r+\phi_L$) ensures automatic fulfillment of the conditions $\omega_1=0$ and $g_1\sim\partial U/\partial r$. In this way, one can correctly specify the initial point (the ghost state $L^\pi=1^-$) for the capon band.

With this method of calculation, the results depend very weakly on all of the interaction parameters except the effective range r_{eff}, the dependence on which really is manifested in ^{208}Pb from $L=3$ onward. In Table VIII, we compare the results of calculations of the characteristics of the low-lying collective states in ^{208}Pb and ^{40}Ca for Gaussian and Yukawa interactions with two values $r_{eff}=0.8$ and 1 F (such a change of r_{eff} is fairly large since the actual parameter r^2_{eff} varies by more than 30%). It can be seen that in ^{208}Pb the greatest variations occur in the case of the 3⁻ state. This is not surprising since this is the most collective of the low-lying collective states in ^{208}Pb, and therefore the energy shift relative to the single-particle differences is maximal and so are the changes in this shift due to the modification of the interaction. With regard to the calculation of the low-lying collective states in ^{40}Ca, here $L(L+1)/A^{2/3}\sim1$ already for $L=3$. In this case, the effective-range approximation does not work well and there are appreciable differences when the Gaussian and Yukawa interactions are used. As can be seen from the table, the best results are obtained for the Yukawa interaction, which evidently gives a better description of the q^2 dependence of $\mathcal{F}$ for large q. Therefore, in the following calculations we used the Yukawa interaction. We should in any case

TABLE IX. Characteristics of low-lying collective states as functions of the form of interpolation.

Nucleus	π	Frequency and transition probability	Experimental data[57-59]	Form of interpolation		
				(106)	(106'), $\alpha=1/2$	(106'), $\alpha=1/3$
^{208}Pb	3⁻	ω	2.614	2.32	2,34	2.32
		$B(E3)\cdot10^{-5}$	5.4±0.3 (7.1±0.4)	8.62	8,24	8.63
	2⁺	ω	4.086	4.50	4.57	4.57
		$B(E2)\cdot10^{-3}$	2.96±0.18	2.87	2.84	2.84
^{40}Ca	3⁻	ω	3.73	3.92	3.93	3.90
		$B(E3)\cdot10^{-4}$	2.11	1.77	1.71	1.76
	5⁻	ω	4.48	5.53	5.55	5.52
		$B(E5)\cdot10^{-6}$	—	2.50	2.80	2.13

mention that a more detailed study of the low-lying collective states with large L in principle enables one to determine more accurately the q^2 dependence of $\mathcal{F}$ at large q.

The characteristics of the low-lying collective states depend very weakly on the remaining parameters of the interaction (105). The analysis made in Ref. 12 showed that for the computational scheme described above the characteristics depend very weakly on the parameters f_{in}, f_{ex}, and ΔR; for a change of f_{in} from 0.25 to 0.65 and of ΔR from 0.5 to 1 F the frequencies ω_L changed by not more than 100 keV; the B(EL) also changed very little. The characteristics hardly depend on the constant f' since even in heavy nuclei the low-lying collective states are isoscalar excitations to a high accuracy. It follows from the data given in Table IX that the results of the calculations depend little on the chosen form of the interpolation. In the following calculations, we used the interpolation (106') with $\alpha=0.5$. The other parameters were chosen as follows: Yukawa interaction with $r_{eff}=0.9$; $f'_{in}=f'_{ex}=0.8$; $f_{in}=0.2$; $f_{ex}=-2$. The parameter ΔR was determined in accordance with prescription described above. In various nuclei, it varied in the range 0.5–0.6 F. This variant of the interaction is the preferred one on the basis of the complete set of data. The results of the calculations of ω_L and B(EL) for low-lying collective states in various magic nuclei for this variant of the interaction are given in Table X. Where possible, we compare the result with an experimental result and with the results of Refs. 18 and 23. In Ref. 23, the calculation was made in the framework of the theory of finite Fermi systems, and in Ref. 18 in the framework of an approach based on Hartree–Fock theory with effective forces (Skyrme forces and modifications of them). As can be seen from the table, our calculations taken as a whole give a fairly good description of the experiments.[10] However, there are a number of serious discrepancies. In the first place, this applies to the 2⁺ and 4⁺ states in ^{208}Pb and to the ^{40}Ca nucleus. In the case of ^{40}Ca, the main cause of the discrepancy is, as we have noted above, that the effective-range approximation does not work well. For the discrepancies in

[10]We should point out the discrepancy in the experimental values of the probability $B(E3)$ for excitation of the 3⁻ level in ^{208}Pb deduced from data on inelastic scattering of electrons (~32 single-particle units) and from Coulomb excitation (~40 single-particle units). In Tables VIII–X we give both values (the latter, in parentheses). The electron data should be regarded as more reliable, though a recent electron experiment[60] gave an intermediate value (~36 single-particle units).

TABLE X. Characteristics of low-lying collective states in magic nuclei.

Nucleus	J^π	Frequency and transition probability	Experimental data[57-59]	Theoretical calculation		
				hours	Ref. 23	Ref. 18
^{40}Ca	3⁻	ω	3.73	4.20		3.5
		$B(E3)\cdot10^{-4}$	2.11	1.53		1.77
	5⁻	ω	4.48	5.6		5.0
		$B(E5)\cdot10^{-6}$	—	2.62		
^{48}Ca	2⁺	ω		3.19		
		$B(E2)\cdot10^{-4}$	—	2.89		
	3⁻	ω		4.02		
		$B(E3)\cdot10^{-3}$		6.8		
^{88}Sr	2⁺	ω	1.84	1.37		
		$B(E2)\cdot10^{-2}$	8.22	7.30		
	3⁻	ω	2.74	2.16		
		$B(E3)\cdot10^{-4}$	6.2	6.8		
^{132}Sn	2⁺	ω	—	4.0		
		$B(E2)\cdot10^{-2}$		6.77		
	3⁻	ω	—	4.18		
		$B(E3)\cdot10^{-5}$		1.15		
^{208}Pb	3⁻	ω	2.614	2.48	2.63	2.8
		$B(E3)\cdot10^{-5}$	5.4±0.3 (7.1±0.4)	7.56	5.46	6.4
	5⁻	ω	3.197	3.21	3.39	3.4
		$B(E5)\cdot10^{-8}$	4.62±0.55	4.50	2.85	4.6
	2⁺	ω	4.086	4.58	4.49	5.6
		$B(E2)\cdot10^{-3}$	2.96±0.18	2.90	3.07	3.33
	4⁺	ω	4.323	4.9	4.69	6.4
		$B(E4)\cdot10^{-7}$	1.287	1.09	0.757	1.4
	6⁺	ω	4.425	4.24	4.776	
		$B(E6)\cdot10^{-10}$	2.3	1.87	2.1	

^{208}Pb, various factors can also be recognized. The most obvious of them are: 1) the inaccurate reproduction of the single-particle levels by the potential $U(r)$ which we used; 2) the adiabatic corrections (the role of spin forces); 3) the mixing of one- and two-phonon states. To analyze the first, we made a calculation for the 3⁻ and 2⁺ states in ^{208}Pb with nonlocal potential U^{nonloc} which depends on l and j: For every (l,j) the depth of the well was chosen in such a way that the level nearest the Fermi surface had an energy equal to the experimental energy (this method is used, for example, in Ref. 47). The results of the calculations are as follows: $\omega_3 = 2.68$ MeV; $B(E3) = 6.11\cdot10^5\ e^2\cdot F^6$; $\omega_2 = 4.6$ MeV; $B(E2) = 1.89\cdot10^3\ e^2\cdot F^4$. We see that in the case of the 2⁺ level the situation is not improved. The most important cause of the remaining discrepancy is the mixing of the collective 2_1^+ level with the two-phonon level $\{3^-\times3^-\}_{2^+}$. A calculation of the mixing made without new parameters by A. P. Platonov by the method proposed in Ref. 25 showed that allowance for this effect reduces the energy of the 2_1^+ level by $\Delta\omega \approx 0.5$ MeV, which leads to agreement with experiment.

In Sec. 5 we have given the unnormalized amplitudes Ω_L for the production of capons with $L^\pi = 2^+$, 3⁻, and 5⁻ in ^{208}Pb calculated by the described manner (see Fig. 3). In classical hydrodynamics, all the low-lying collective states are capillary waves, with $\Omega_L^{cl} = \partial U/\partial r$ corresponding to them. As can be seen from Fig. 3, in all cases the Ω_L contain large surface components and quantum volume terms. The same applies to the transition densities $\rho_{tr}^L(r)$ (Figs. 5 and 6): Here, the classical hydrodynamic result is $\rho_{tr} \sim \partial\rho/\partial r$. For all the remaining low-lying collective states in ^{208}Pb and other nuclei a similar picture is obtained. This shows

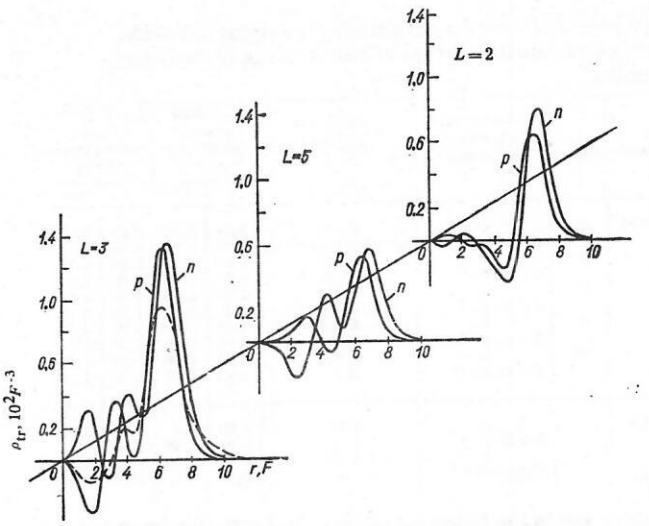

FIG. 5. Radial distributions of transition densities for first collective states in ^{208}Pb. The charged (p) and neutral (n) components are given; the dashed line is the fit of Ref. 19 to the experimental data.

that the low-lying collective states with the considered selection rules have the characteristic features of capons. However, it must not be assumed that they are pure capons. Analysis shows that some of these states are a superposition of a capon and individual particle–hole configurations (for example, 2⁺ and 5⁻ states in ^{208}Pb). Excitations whose density matrix does not contain a distinguished contribution of some one-particle–hole configuration must be regarded as purely collective, and this assertion also applies, of course, to capons. From this point of view, the purest of the capons among the low-lying collective states are the first 3⁻ states in even–even nuclei.

With regard to states of positive parity, capon features are most clearly manifested in giant isoscalar resonances with energies ~10 MeV, some of which have already been observed experimentally (for example, the 0⁺ and 2⁺ resonances in ^{208}Pb). This is indicated, in particular, by the calculations of Ref. 18. The low-lying collective 2⁺ states with energy 4 MeV in ^{208}Pb are associated with fragmentation of the 2⁺ capon resonance over particle-hole configurations, among which several configurations corresponding to lower excitation energies (~5 MeV) arise because of the spin-orbit term in the self-consistent potential. Nevertheless, here too

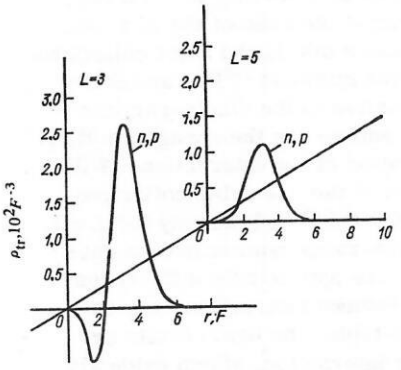

FIG. 6. Transition densities for collective 3⁻ and 5⁻ states in ^{40}Ca. In the figure, the charged and the neutral components cannot be distinguished.

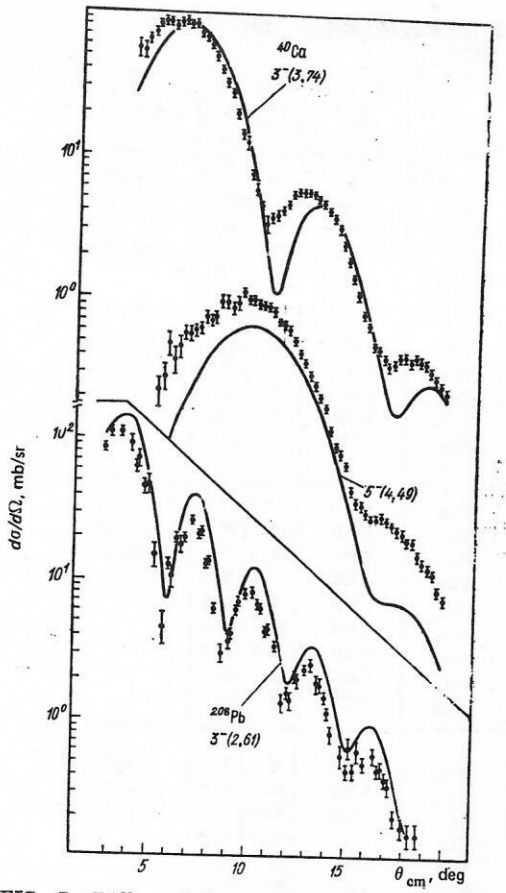

FIG. 7. Differential cross sections for elastic scattering of 1.044-GeV protons on ^{40}Ca and ^{208}Pb with excitation of the first low-lying collective states. The calculation is taken from Ref. 61 and the experimental data from Ref. 20. For the 5⁻ (4.49 MeV) state in ^{40}Ca the scale is increased by a factor 10.

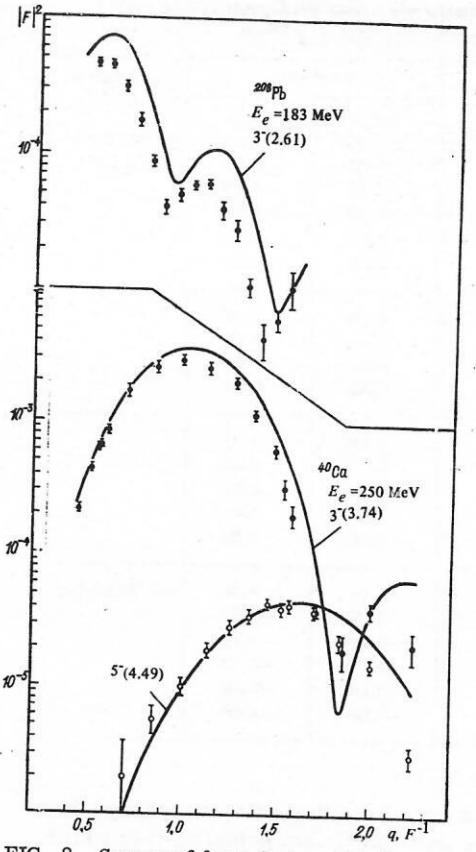

FIG. 8. Square of form factor of inelastic scattering of fast electrons on ^{40}Ca and ^{208}Pb with excitation of the first low-lying collective states. The calculation is taken from Ref. 61, and the experimental data from Refs. 19 and 20.

the surface part of the form factor plays the principal role in the excitation of the state.

In Fig. 5, the dashed line is the experimental transition density[19] for the excitation of a 3⁻ state by fast electrons. It can be seen that the calculated $\rho^{3^-}_{tr}$ curve basically repeats the coordinate dependence of the experimental transition density. However, at the maximum it is somewhat greater. This is due to the defects of the potential that we have used,[45] which does not accurately reproduce the single-particle levels and gives a too large value of $B(E3)$. The calculation described above with a nonlocal potential, which reproduces the levels exactly, gives a value of $B(E3)$ about 20% smaller and agrees much better with the experiment for $\rho^{3^-}_{tr}$.

Figure 7 shows the differential cross sections $d\sigma/d\Omega$ of inelastic scattering of 1.044-GeV protons on ^{208}Pb (with excitation of 3⁻ level) and ^{40}Ca (with excitation of 3⁻ and 5⁻ levels).[20] The calculation was made by V. E. Starodubskiĭ using Glauber's method[61] and our transition densities.

In Fig. 8 we also give the calculations of Starodubskiĭ for the excitation of the same states by fast electrons. It can be seen that in all cases the agreement with experiment is fairly good.

Single-Particle Characteristics of Odd Nuclei. The method developed for solving the equation for $g_L(r)$ can be automatically transferred to the effective field $V_L(r)$ as well; this determines the static moments and the probabilities of single-particle transitions in odd nuclei. The equations for V_L are given in Sec. 4 [Eqs. (51) and (52)]. In the nuclei neighboring ^{208}Pb we calculated the quadrupole moments Q, $B(E2)$, and the isomeric shifts ΔE of the mesic-atom lines. In the calculation of ΔE, we used the rough method of Ref. 60, in which ΔE is expressed in terms of $\Delta\langle r^2\rangle$, the change of the mean square charge radius on excitation. The results are given in Tables XI-XIII. It can be seen that the greatest discrepancies with experiment are obtained in the case of the isomeric shift. This is a very subtle nuclear characteristic, the difference of the matrix elements $V_\lambda[f(r)] - V_{\lambda_0}[f(r)]$ ($f(r)$ is the electric field produced by the meson; λ_0 corresponds to the ground state and λ to the excited state), and the result is much less than each matrix element separately. Therefore, any small corrections can here be important-in particular, it is possible that in the isomeric shifts admixtures of low-lying phonons, whose contribution we have ignored, may be important. In Table XIV, we give the quadrupole moments of near-magic nuclei. The agreement with the experimental data is satisfactory.

9. INTERACTION OF PARTICLES WITH LOW-LYING COLLECTIVE EXCITATIONS

In the collective model,[2] it is assumed that the change in the nuclear density ρ due to a collective ex-

TABLE XI. Quadrupole moments Q of ground and excited states in the nuclei neighboring ^{208}Pb.

Nucleus	State λ_0	Q, b (calculated)		Q, b (experiment of Ref. 23)
		Ref. 23	hours	
^{209}Bi	$1h_{9/2}$	−0.425	−0.376	−0.379
	$2f_{7/2}$	−0.407	−0.379	
	$1i_{13/2}$	−0.623	−0.581	
	$2f_{5/2}$	−0.356	−0.315	
	$3p_{3/2}$	−0.231	−0.220	
^{207}Tl	$2d_{3/2}$	0.214	0.185	
	$1h_{11/2}$	0.528	0.490	
	$2d_{5/2}$	0.311	0.278	
	$1g_{7/2}$	0.331	0.298	
^{207}Pb	$2f_{5/2}$	0.198	0.193	
	$3p_{3/2}$	0.119	0.123	
	$1i_{13/2}$	0.368	0.366	
	$2f_{7/2}$	0.231	0.225	
	$1h_{9/2}$	0.225	0.251	
^{209}Pb	$2g_{9/2}$	−0.241	−0.251	−0.280±0.022
	$1i_{11/2}$	−0.284	−0.303	
	$1j_{15/2}$	−0.418	−0.409	
	$3d_{5/2}$	−0.139	−0.151	
	$2g_{7/2}$	−0.235	−0.232	
	$3d_{3/2}$	−0.099	−0.099	

citation leads to a corresponding adiabatic change $\delta U = (\delta U/\delta\rho)\delta\rho$ in the self-consistent potential U in which the particles move, and the interaction H^{int} of the particles with the vibrations is determined from the single-particle Schrödinger equation

$$(p^2/2m + U + \delta U)\,\tilde{\varphi}_\lambda = \varepsilon_\lambda \tilde{\varphi}_\lambda, \qquad (118)$$

For a drop of incompressible liquid, it is assumed that $U = U(\mathbf{r} - \mathbf{R}_0 - \delta\mathbf{R}(\theta, \varphi))$ and for small vibrations of the shape about a sphere of radius R_0 only the term linear in the small deformation δR in δU is retained: $\delta U = (\partial U/\partial R)\,\delta R = -R_0(\partial U/\partial r)\sum_{LM} Y^*_{LM}\alpha_{LM}$, and after

TABLE XII. Isomer shifts ΔE of mesic-atom lines in nuclei neighboring ^{208}Pb.

Nucleus	State λ_0	ΔE, keV		
		hour calculation	calculation of Ref. 23	Experiment of Ref. 23
^{209}Bi	$2f_{7/2}$	2.71	1.115	—
	$1i_{13/2}$	7.14	8.109	3.6±0.7
	$1f_{5/2}$	1.55	—	—
	$3p_{3/2}$	3.12	—	—
^{209}Pb	$1i_{11/2}$	−0.48	—	—
	$3d_{5/2}$	−1.21	—	—
	$4s_{1/2}$	−2.13	—	—
	$2g_{7/2}$	+0.05	—	—
	$3d_{3/2}$	−1.33	—	—
^{207}Pb	$2f_{5/2}$	+0.19	0.675	0.11±0.50
	$3p_{3/2}$	−0.07	0.412	1.52±0.50
	$1i_{13/2}$	+1.82	—	—
	$2f_{7/2}$	+0.34	0.442	6.3±0.8
	$1h_{9/2}$	−0.53	—	—
^{207}Tl	$2d_{3/2}$	−0.68	—	—
	$1h_{11/2}$	5.77	—	—
	$2d_{5/2}$	0.38	—	—

TABLE XIII. $B(E2)$ for single-particle transitions in nuclei neighboring ^{208}Pb.

Nucleus	$\|i\rangle$	$\|f\rangle$	Transition energy, MeV	$B(E2)$, $e^2\cdot F^4$ (calculated)		$B(E2)$, $e^2\cdot F^4$ (experiment of Ref. 23)
				Ref. 23	hours	
^{209}Bi	$2f_{7/2}$	$1h_{9/2}$	0.897	18	11.2	30±3
	$2f_{5/2}$	$1h_{9/2}$	2.822	672	296	480±170
	$2f_{5/2}$	$2f_{7/2}$	1.295	66	46.8	
	$3p_{3/2}$	$2f_{7/2}$	1.219	725	560	900±600
	$3p_{3/2}$	$2f_{5/2}$	0.294	99	83	
^{207}Tl	$2d_{3/2}$	$3s_{1/2}$	0.351	199	156	280±40
	$2d_{5/2}$	$3s_{1/2}$	1.003	230	174	
	$2d_{5/2}$	$2d_{3/2}$	1.797	70	50	
	$1g_{7/2}$	$2d_{5/2}$	1.800	24	12.9	
	$1g_{7/2}$	$2d_{3/2}$	3.123	335	126	
^{209}Pb	$1i_{11/2}$	$2g_{9/2}$	0.778	2.5	2.5	
	$3d_{5/2}$	$2g_{9/2}$	1.566	145	132	185±45
	$4s_{1/2}$	$3d_{5/2}$	0.466	128	123	156±8
	$2g_{7/2}$	$2g_{9/2}$	2.493	17.4	10.2	
	$2g_{7/2}$	$3d_{5/2}$	1.127	7.9	7.2	
	$3d_{3/2}$	$3d_{5/2}$	0.973	23.4	22.1	
	$3d_{3/2}$	$4s_{1/2}$	0.507	42.9	37.7	
	$3d_{3/2}$	$2g_{7/2}$	0.046	148	130	
	$2g_{7/2}$	$1i_{11/2}$	1.577	206	128	
^{207}Pb	$2f_{5/2}$	$3p_{1/2}$	0.570	85.7	71.2	71±3
	$3p_{3/2}$	$3p_{1/2}$	0.898	82.3	76.6	62±3
	$3p_{3/2}$	$2f_{5/2}$	0.328	33.8	28.9	
	$2f_{7/2}$	$2f_{5/2}$	1.170	17.7	12.8	
	$2f_{7/2}$	$3p_{3/2}$	1.442	119	93	
	$1h_{9/2}$	$2f_{5/2}$	2.839	196	79	
	$1h_{9/2}$	$2f_{7/2}$	1.069	7.1	4.1	

the quantization procedure this is taken to be H^{int}, while $g_{LM} \sim \alpha_{LM}(\partial U/\partial r)\,Y_{LM}$ is treated as the production amplitude of a phonon with quantum numbers LM. Analogously, in this model one obtains $\delta\rho = (-\partial\rho/\partial r)\delta R$, which corresponds to the transition density ρ_{tr} for classical capillary waves.

When considering second-order effects, for example, in a calculation of level shifts in "particle plus phonon" multiplets, only the simplest diagrams of the type shown in Figs. 9a and 9b are usually taken into account in the collective model. Recently, attempts have been made[47] to calculate more complicated diagrams, but agreement with experiment was not achieved, which is

TABLE XIV. Quadrupole moments Q of the ground states of near-magic nuclei.

Nucleus	State λ_0	Q, F^2	
		Calculation	Experiment
^{209}Bi	$1h_{9/2}$ (p)	−37.6	−37.9
^{209}Pb	$2g_{9/2}$ (n)	−25.1	−28.0±2.2
^{89}Sr	$2d_{5/2}$ (n)	−30.6	—
^{87}Sr	$1g_{9/2}$ (n^{-1})	49.1	—
^{87}Rb	$2p_{3/2}$ (p^{-1})	20.7	12.9±0.4
^{41}Sc	$1f_{7/2}$ (p)	−14.1	—
^{41}Ca	$1f_{7/2}$ (n)	−6.3	—
^{39}Ca	$1d_{3/1}$ (n^{-1})	+3.1	—
^{39}K	$1d_{3/2}$ (p^{-1})	6.3	6.16±0.35

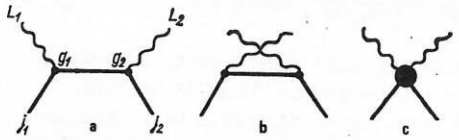

FIG. 9. Second-order diagrams contributing to the energy shift of members of the "quasiparticle + phonon" multiplet. Diagram c does not include the first two diagrams a and b.

not surprising since the problem does not in fact contain a small parameter and perturbation theory cannot be used. This means that the renormalization of the "bare" interaction H^{int} is of order unity, and therefore all diagrams must be taken into account in order to obtain the correct result. Let us illustrate this assertion by a simple example.

Suppose that in the system a dipole phonon whose amplitude satisfies $g_1=(\partial U/\partial r_k)\,\delta R_k\,(L=1$ and $\omega=0)$ appears. Then obviously the energy ε_λ of the particle from the interaction with this phonon must not change in all orders in $\delta\mathbf{R}$ since such a situation corresponds to a shift of the nucleus as a whole. But if one considers the contribution of the first two diagrams in Fig. 9, using as vertices $H^{\text{int}}=\delta U=g_1$, then the change $\delta\varepsilon_\lambda$ in the energy will not be zero. To eliminate this contradiction, it is necessary to add the contribution of the local diagram c. In the considered example, it can be readily calculated: To the diagram c there corresponds a second-order term in the expansion of δU with respect to $\delta\mathbf{R}$ equal to $\delta^2U=(\partial^2U/\partial R_i\partial R_k)\,\delta R_i\delta R_k/2$. The complete answer can be found from (118). We have

$$\delta\varepsilon_\lambda=\delta R_i\delta R_k\left\{\sum_{\lambda'}\left(\frac{\partial U}{\partial r_i}\right)_{\lambda\lambda'}\frac{1}{\varepsilon_\lambda-\varepsilon_{\lambda'}}\left(\frac{\partial U}{\partial r_k}\right)_{\lambda'\lambda}+\frac{1}{2}\left(\frac{\partial^2U}{\partial r_i\,\partial r_k}\right)_{\lambda\lambda'}\right\}. \tag{119}$$

Now, using the relation

$$(\partial U/\partial r_i)_{\lambda\lambda'}=(\varepsilon_{\lambda'}-\varepsilon_\lambda)\left(\frac{\partial}{\partial r_i}\right)_{\lambda\lambda'},$$

we arrive at the correct result $\delta\varepsilon_\lambda=0$. Thus, the correct result is obtained only after summation of all diagrams. For real phonons, the situation remains the same—as is shown here, they are in many respects similar to a dipole phonon, which belongs to the same branch of surface excitations.

We formulate a microscopic method for calculating the effect of interaction between the particles and phonons. The important feature of this method is that in it one does not introduce the Hamiltonian H^{int} of the collective model, but uses as the vertices in the diagrams of Figs. 9a and 9b the capon production amplitudes g_L, which are found by solving Eqs. (5) and (34). The diagrams shown in Fig. 9 can be interpreted as the change of the particle's Green's function in the second order in the external field produced by the phonons. The change of G in the field of a single phonon is

$$(\delta G)_1=Gg_{L_1}G.$$

In the field of another phonon, δG is then changed;

$$(\delta^2G)_{12}=(\delta G)_2\,g_{L_1}G+Gg_{L_1}(\delta G)_2+G(\delta g)_{12}G.$$

Using the fact that here $(\delta G)_2=Gg_{L_2}G$, we obtain

$$(\delta^2G)_{12}=Gg_{L_2}Gg_{L_1}G+Gg_{L_1}Gg_{L_2}G+G(\delta g)_{12}G. \tag{120}$$

From this we find the change of the mass operator of the particle:

TABLE XV. Shift ΔE_I (keV) of the levels of the $(h_{9/2}\times 3^-)I^\pi$ multiplet relative to the unperturbed value $\omega=2614$ keV.

I^π	Contribution of pole terms	Contribution of spin-orbit interaction	Contribution of nonpole diagrams	Total ΔE_I	Experiment
$3/2^+$	42.6	14.7	128.4	185.7	−120
$5/2^+$	−3.5	−5.5	57.2	48.2	4
$7/2^+$	9.4	4.5	15.2	29.1	−29
$9/2^+$	−93.4	−17.8	51.5	−59.7	−49
$11/2^+$	−44.9	−26.8	60.0	−11.7	−14
$13/2^+$	−62.3	72.8	14.9	25.4	−14
$15/2^+$	153.1	0	77.6	230.7	130

$$(\delta^2\Sigma)_{12}=g_{L_1}Gg_{L_2}+g_{L_2}Gg_{L_1}+(\delta g)_{12}. \tag{121}$$

The expectation value of this expression, taken with respect to the corresponding wave functions, gives the energy shift of the levels in the multiplets. The last term in (121) describes the change in the phonon production amplitude in the field of the other phonon. Varying (5), we arrive at the equation

$$(\delta g)_{12}=\tfrac{1}{2}\left[\delta\eta_2(\delta G)_1+\delta\eta_1(\delta G_2)\right]+\eta(\delta^2G)_{12}. \tag{122}$$

Note the presence in this equation of the term with the variation δU, which is the change of the local block of the two-particle interaction U as a result of the excitation of the phonon. This variation can be found from the relation (12), which contains the block of the three-particle interaction $\mathcal{K}^{(3)}$. Writing $\delta\mathfrak{u}=\mathcal{K}^{(3)}Gg_LG$, where $g_L=\alpha_L\Omega Y_{LM}\approx\alpha_L(\partial U/d\mathbf{r})Y_{LM}$, and using the fact that $\mathcal{K}^{(3)}$ is a local block, we find to a good accuracy $\delta\mathfrak{u}=\alpha_LY_{LM}\mathcal{K}^{(3)}\partial\rho/\partial r$. This with allowance for the consistency condition (13) gives $\delta\mathfrak{u}=\alpha_L(\partial\mathfrak{u}/\partial r)Y_{LM}=\alpha_L(\partial\mathfrak{u}/\partial\rho)(\partial\rho/\partial r)Y_{LM}$. After the renormalization, the derivative $\partial U/\partial\rho$ is replaced by $\partial\tau/\partial\rho$. As a result, we obtain a closed system of equations, and by solving it one can self-consistently calculate all effects of second order in the particle-phonon interaction.

In Table XV, we give the results of a calculation of the spectrum of the levels of the $(h_{9/2}\times3^-)$ multiplet of the nucleus ^{209}Bi—the traditional object for such calculations (see, for example, Ref. 47). In contrast to the analogous investigations of other authors, we take into account systematically the contribution of all nonpole phonon-scattering diagrams.

The calculation is made using the nonlocal single-particle potential introduced by Hamamoto (see Sec. 8), and also with allowance for spin-orbit forces. The theoretical and experimental values of ΔE_I can be compared by examining the final columns of the table. It should be borne in mind that to the energy of the level with $I=3/2$ it is necessary to add a negative term of order 200 keV, since in this case the particle + phonon state is strongly mixed with the ^{210}Po$(0^+)\,d_{3/2}^{-1}$ state.[47] A similar situation also obtains for the other state of the $(h_{9/2}\times3^-)$ multiplet with spin $I=13/2$: It is strongly mixed with the single-particle $i_{13/2}$ state. In this case also, the expressions of perturbation theory must be made more accurate. Analysis shows that inapplicability of standard perturbation theory is one of the principal reasons for discrepancy between theory and experiment.

APPENDIX

Renormalization of consistency conditions

The philosophy of the Fermi-liquid approach consists in the explicit separation in the equations of the theory of the contribution of the single-particle degrees of freedom, since it is these, or rather those of them whose energy adjoins the Fermi energy, that are responsible for the fluctuations in the properties of the nuclei. This means that all the irregularities arise as a result of the integration of the pole parts G^q of the Green's function G in the neighborhood of the point $\varepsilon = \mu$. Therefore, in the basic equations of the theory of finite Fermi systems describing the response of the system to an external field or the effective interaction of the quasiparticles, the Green's function product GG is split into two parts:

$$GG = A + B, \tag{A.1}$$

where B is the regular part of GG common to all nuclei while A contains the nonuniversal components. Since the product G^qG^q is a rapidly varying function in the neighborhood of the point $\varepsilon = \mu$, it can be written in the form of the series

$$G^q(\varepsilon)\,G^q(\varepsilon+\omega) = \delta(\varepsilon-\mu)\int G^qG^q\,\frac{d\varepsilon_1}{2\pi i} - \delta'(\varepsilon-\mu)\int(\varepsilon_1-\mu)\,G^qG^q\,\frac{d\varepsilon_1}{2\pi i}+\dots. \tag{A.2}$$

In the second and following terms of this expansion, the contribution of the region of integration adjoining the Fermi surface is small, and these terms can be included without qualms in B, i.e., one can assume that $A = A^{(0)}$:

$$A^{(0)} = \delta(\varepsilon-\mu)\int G^qG^q\,\frac{d\varepsilon_1}{2\pi i}. \tag{A.3}$$

This is the standard way of introducing $A(\mathbf{r}_1, \mathbf{r}_2; \omega)$ in Migdal's theory[9] if one uses the usual definition of $G^q(\mathbf{r}_1, \mathbf{r}_2; \varepsilon)$:

$$a^{-1}(\varepsilon - H^q)\,G^q = \delta(\mathbf{r}_1-\mathbf{r}_2), \tag{A.4}$$

which is valid when the renormalization factor $a = (1-\partial\Sigma/\partial\varepsilon)^{-1}$ can be assumed independent of $\mathbf{r}$. In reality, a always depends on $\mathbf{r}$, going over from the value a_{in} within the system to a_{ex} outside it. In real nuclei, $a_{in} \approx 0.9$ (Ref. 23), and therefore in many calculations allowance for the $\mathbf{r}$ dependence of a gives small corrections. But in some cases, in particular, in the analysis of consistency conditions, in which the surface region plays an important role, such allowance may be important. First of all we note that if $a \neq$ const, one cannot introduce an energy-independent quasiparticle Hamiltonian H^q. Indeed, expanding the operator $G^{-1} = \varepsilon - \varepsilon_p^0 - \Sigma$ near the point $\varepsilon = \mu$, we obtain the standard equation for G_q^q (Ref. 9):

$$[\varepsilon - \varepsilon_p^0 - \Sigma^q(\varepsilon)]\,G^q(\mathbf{r}_1, \mathbf{r}_2; \varepsilon) = \delta(\mathbf{r}_1-\mathbf{r}_2), \tag{A.5}$$

where

$$\Sigma^q(\varepsilon) = \Sigma(\mu) + \left(\frac{\partial\Sigma}{\partial\varepsilon}\right)_{\varepsilon=\mu}\cdot(\varepsilon-\mu)$$

If $a^{-1} = 1 - \partial\Sigma/\partial\varepsilon$ did not depend on $\mathbf{r}$, then it could be taken in front of the brackets and (A.5) thereby reduced to (A.5). Then $Hq = a[\varepsilon_p^0 + \Sigma(\mu) - \mu\partial\Sigma/\partial\varepsilon]$. Expanding $G^q(\mathbf{r}_1, \mathbf{r}_2, \varepsilon)$ in this case with respect to the eigenfunctions $\varphi_\lambda(\mathbf{r})$ of the Hamiltonian $H^q(\mathbf{r}, \mathbf{p})$, we would obtain the usual expression (108):

$$G^q = a\sum_\lambda\left(\frac{n_\lambda}{\varepsilon-\varepsilon_\lambda-i\delta} + \frac{1-n_\lambda}{\varepsilon-\varepsilon_\lambda+i\delta}\right)\varphi_\lambda^*(\mathbf{r}_1)\,\varphi_\lambda(\mathbf{r}_2). \tag{A.6}$$

But the derivative $\partial\Sigma/\partial\varepsilon$ does depend on $\mathbf{r}$, and taking $a(\mathbf{r})$ in front of the brackets in (A.5) gives nothing. If we now attempt to obtain an expansion of G^q analogous to (A.6), we see that the $\varphi_\lambda(\mathbf{r})$ are eigenfunctions of the operator H' (see Ref. 9):

$$H'(\varepsilon)\,\varphi_\lambda(\mathbf{r}, \varepsilon) = [\varepsilon_p^0 + \Sigma(\mu) - \mu\partial\Sigma/\partial\varepsilon + \varepsilon\partial\Sigma/\partial\varepsilon]\,\varphi_\lambda = E_\lambda(\varepsilon)\,\varphi_\lambda(\mathbf{r}, \varepsilon), \tag{A.7}$$

which explicitly contains the variable ε. As a result, we obtain

$$G^q = \sum_\lambda \varphi_\lambda^*(\mathbf{r}_1, \varepsilon)\,\varphi_\lambda(\mathbf{r}_2, \varepsilon)/(\varepsilon - E_\lambda(\varepsilon) - i\delta\theta_\lambda). \tag{A.8}$$

The quasiparticle energies—the poles of the function G^q—are determined by the equation $\varepsilon_\lambda = E_\lambda(\varepsilon_\lambda)$, and the residues are $a_\lambda = [1 - \partial\Sigma/\partial\varepsilon]_\lambda^{-1}$.

The different wave functions φ_λ, as can be seen from (A.7), are now not orthogonal to one another. This nonorthogonality occurs because of the $\mathbf{r}$ dependence of the term $\partial\Sigma/\partial\varepsilon$, and even for strongly differing states is small ($\sim A^{-1/3}$).

The density matrix corresponding to the Green's function (A.8) has the form

$$\tilde{\rho}^q = \int G^q\,\frac{d\varepsilon}{2\pi i} = \sum_\lambda n_\lambda a_\lambda \varphi_\lambda^*(\mathbf{r}_1, \varepsilon_\lambda)\,\varphi_\lambda(\mathbf{r}_2, \varepsilon_\lambda). \tag{A.9}$$

In order to obtain results as close as possible to the expressions for the case $a =$ const, we introduce the mean value of the renormalization factor $a = \Sigma n_\lambda a_\lambda / \Sigma n_\lambda$ and the density matrix of the quasiparticles, which is equal to $\rho^q = \tilde{\rho}^q / a$:

$$\rho^q = \sum\frac{a_\lambda}{a}\,n_\lambda\varphi_\lambda^*(\mathbf{r}_1, \varepsilon_\lambda)\,\varphi_\lambda(\mathbf{r}_2, \varepsilon_\lambda). \tag{A.10}$$

It is readily seen that the density matrix ρ^q is normalized to the number of particles:

$$\int\rho^q(\mathbf{r}, \mathbf{r})\,d^3r = \sum a_\lambda n_\lambda / a = \sum_\lambda n_\lambda.$$

Introducing H^q by analogy with (A.4), we obtain

$$\varepsilon_\lambda = (a_\lambda/a)\,(H^q)_{\lambda\lambda}. \tag{A.11}$$

We see that the role of the self-consistent potential of the quasiparticles at the Fermi surface is played by

$$U^q = a(\Sigma - \mu\,\partial\Sigma/\partial\varepsilon). \tag{A.12}$$

We now turn to the renormalization of the consistency condition (7). Splitting GG into A and B in accordance with Eqs. (A.1) and (A.3), we obtain

$$\partial\Sigma(\varepsilon)/\partial\mathbf{R} = \Gamma^\omega(\varepsilon, \mu)\,A\,\partial\Sigma(\mu)/\partial\mathbf{R}', \tag{A.13}$$

where

$$\Gamma^\omega(\varepsilon, \varepsilon') = \mathcal{U}(\varepsilon, \varepsilon') + \int\mathcal{U}(\varepsilon, \varepsilon'')\,B(\varepsilon'')\,\Gamma^\omega(\varepsilon'', \varepsilon')\,\frac{d\varepsilon''}{2\pi i}. \tag{A.14}$$

We now attempt to transform (A.13) to a form as similar to (113) as possible. For this, we go over to the quasiparticle characteristics Σ^q and ρ^q. From (A.13),

$$\partial\Sigma^q(\mu)/\partial\mathbf{R} = \Gamma^\omega(\mu, \mu)\,A\,\partial\Sigma^q(\mu)/\partial\mathbf{R}' \tag{A.15}$$

and

$$\frac{\partial}{\partial\mathbf{R}}\left(\frac{\partial\Sigma^q}{\partial\varepsilon}\right)_{\varepsilon=\mu} = \left(\frac{\partial\Gamma^\omega(\varepsilon, \mu)}{\partial\varepsilon}\right)_{\varepsilon=\mu} A\,\frac{\partial\Sigma(\mu)}{\partial\mathbf{R}'}. \tag{A.16}$$

We now rewrite (A.13) in the form

$$\frac{\partial\Sigma^q(\varepsilon)}{\partial R}=\Gamma^\omega(\varepsilon,\mu)\int G^q(\varepsilon_1)\frac{\partial\Sigma^q(\varepsilon_1)}{\partial R'}G^q(\varepsilon_1)\frac{d\varepsilon_1}{2\pi i}$$
$$-\Gamma^\omega(\varepsilon,\mu)\int G^q(\varepsilon_1)\frac{\partial}{\partial R'}\left[\frac{\partial\Sigma^q(\varepsilon_1)}{\partial\varepsilon_1}\right]_{\varepsilon_1=\mu}(\varepsilon_1-\mu)\,G^q(\varepsilon_1)\frac{d\varepsilon_1}{2\pi i}.$$

Remembering that

$$\int G^q(\varepsilon)\frac{\partial\Sigma^q(\varepsilon)}{\partial R}G^q(\varepsilon)\frac{d\varepsilon}{2\pi i}=\int\frac{\partial G^q(\varepsilon)}{\partial R}\frac{d\varepsilon}{2\pi i}=\frac{\partial\tilde{\rho}^q}{\partial R},$$

and using also the relation (A.16), we obtain

$$\frac{\partial\Sigma^q(\varepsilon)}{\partial R}=\Gamma^\omega(\varepsilon,\mu)\frac{\partial\tilde{\rho}^q}{\partial R'}-\Gamma^\omega(\varepsilon,\mu)\int G^q(\varepsilon_1)G^q(\varepsilon_1)$$
$$\times\left(\frac{\partial\Gamma^\omega}{\partial\varepsilon_1}\right)_{\varepsilon_1=\mu}(\varepsilon_1-\mu)\frac{d\varepsilon_1}{2\pi i}\int G^q(\varepsilon_2)G^q(\varepsilon_2)\frac{d\varepsilon_2}{2\pi i}\frac{\partial\Sigma^q(\mu)}{\partial R''}.$$

Repeating this procedure, we arrive at the condition

$$\partial\Sigma^q(\varepsilon)/\partial R=\tilde{\Gamma}^\omega(\varepsilon,\mu)\,\partial\tilde{\rho}^q/\partial R', \tag{A.17}$$

where the amplitude $\tilde{\Gamma}^\omega(\varepsilon,\mu)$ satisfies the equation

$$\tilde{\Gamma}^\omega(r_1,r_2,r_3,r_4;\varepsilon,\mu)=\Gamma^\omega(r_1,r_2,r_3,r_4;\varepsilon,\mu)$$
$$-\int K(r_1,r_2,r_5,r_6)\,\tilde{\Gamma}^\omega(r_5,r_6,r_3,r_4;\mu,\mu)\,d^3r_5\,d^3r_6 \tag{A.18}$$

with kernel

$$K(r_1,r_2,r_5,r_6)=\int d^3r_7\,d^3r_8\left(\frac{\partial\Gamma^\omega}{\partial\varepsilon}(r_1,r_2,r_7,r_8)\right)_{\varepsilon=\mu}$$
$$\times\int\frac{d\varepsilon}{2\pi i}(\varepsilon-\mu)\,G^q(r_7,r_5;\varepsilon)\,G^q(r_6,r_8;\varepsilon). \tag{A.19}$$

As can be seen from (A.19) and the definition of a, the kernel K for small $(a-1)$ is proportional to $(a-1)$, but the kernel does not have the additional small factor $\sim A^{-1/3}$. In order to cast (A.17) into a form as close to (113) as possible, we go over to quasiparticles in the same way as in (A.10): by means of the average renormalization factor a. By analogy with the usual definition of $\mathcal{F}=a^2\Gamma^\omega(\mu,\mu)$, we introduce the amplitude $\tilde{\mathcal{F}}=a^2\tilde{\Gamma}^\omega(\mu,\mu)$. Then for the quasiparticle potential defined by (A.12) we have

$$\partial U^q/\partial R=(\tilde{\mathcal{F}}-\mu\partial\tilde{\mathcal{F}}/\partial\varepsilon)\,\partial\rho^q/\partial R', \tag{A.20}$$

where $\tilde{\mathcal{F}}$ is related to $\mathcal{F}$ by a relation of the same form as (A.18):

$$\tilde{\mathcal{F}}=\mathcal{F}+K\tilde{\mathcal{F}}. \tag{A.21}$$

We see that the final form of the consistency conditions (A.20) in the theory of finite Fermi systems is very similar to the ordinary consistency condition (113). However, there is a very important difference: The amplitude $\tilde{\mathcal{F}}-\mu\partial\tilde{\mathcal{F}}/\partial\varepsilon$, which plays the role of an effective interaction in this equation, is not the same as the amplitude $\mathcal{F}$ which determines the characteristics of the low-lying states. The satisfactory agreement between our calculations and the experiments indicates that the terms containing $\partial\mathcal{F}/\partial\varepsilon$ are small. However, in a number of problems it is important to take into account such terms. One of these problems is that of finding the total energy of the nucleus.

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