Quantum field theory in asymptotically flat spacetime

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A systematic exposition is given for the quantum theory of massless fields in curved asymptotically flat spacetime, for which the concept itself and the basic properties are discussed in detail. On the basis of Schwinger's dynamical principle and Penrose's conformal technique a quantization scheme is developed and an explicit expression obtained for the S matrix and energy–momentum fluxes of particles. The formalism developed is applied to the creation of elementary particles in the gravitational field of black holes.

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INTRODUCTION

The problem of constructing a quantum field theory in curved spacetime, which long interested only a narrow group of specialists, has become extremely topical in recent years. Investigations of the quantum creation of particles in a gravitational field during the early stages of evolution of the Universe and, in particular, the recent discovery of the possible evaporation of small black holes have attracted the attention to this problem of not only gravitational specialists but also a large number of physicists occupied with field theory, elementary-particle physics, and quantum statistics.

In the present review, we set forth systematically the quantum theory in a case of importance for practical application when the gravitational field generated by a bounded distribution of matter decreases at infinity (spacetime is asymptotically flat). This theory for massless fields has now achieved a certain degree of completeness, and the use of geometrical methods (Penrose's conformal transformations) has made it possible to formulate the theory in an elegant and simple form. It should be emphasized, however, that this simplicity is the outcome of a prolonged and deep analysis of the problem in which such noted theoreticians as Fock, Sachs, Arnowitt, Deser, Misner, Penrose, Hawking, and others participated.

In writing this review we have attempted (sometimes at the cost of mathematical rigor) to elucidate above all the physical essence of the problems under consideration. To this end, in the first two sections we illustrate for the simplest example of a scalar field in Minkowski space the various concepts (perhaps not very familiar) used in what follows, such as the null infinities $\mathscr{I}^+$ and $\mathscr{I}^-$, the Penrose space, etc. We hope that the reader of this review will be able to appreciate the simplification that is achieved by consistent use of the conformal technique in the study of both the classical and the quantum theory of massless fields.

Essentially, the problem considered in the review can be elucidated by the simple example of the theory of a field φ with quadratic action:

$$W[\varphi]=\int_M d^4x\sqrt{-g}\,g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi;$$

here, $g^{\mu\nu}(x)$ are given functions (the metric). The task is to quantize this theory, i.e., to find canonical commutation relations and define the vacuum state and concept of a particle for this field. In the general case when the spacetime manifold M differs topologically from the ordinary Minkowski space, difficulties associated with this difference arise on quantization. However, even when M is topologically equivalent to Minkowski space it is impossible to avoid a considerable number of problems in the construction of a quantum field theory.

At the first glance in a "naive" approach, it would seem natural to approach this theory like an ordinary theory in flat space with coordinates x^μ chosen in some manner. The Hamiltonian is then constructed in accordance with the usual rules as the generator of translations along the coordinate x^0 and the vacuum would naturally be defined as the lowest state of this Hamiltonian. However, if we choose the coordinates x^μ differently we should obtain a different, and in general not equivalent, theory with different physical consequences. This is why we have to use an invariant geometrical language.

Usually, the vacuum state and the concept of a particle can be uniquely determined in a region in which the external field is absent. For this, it is sufficient that particles, moving in this field, enter after some time a region where this external field can be ignored. This means that in the considered case of the action W the metric g must tend to the metric of flat space along the trajectories of the particles, i.e., the null geodesics. A spacetime with metric satisfying this condition is

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called an asymptotically flat space. This important concept is considered in detail in the first four sections of the review. Since the gravitational field is "switched off" as the particles go away to infinity, one would expect that the symmetry of the Poincaré group would then be recovered. But this does not happen. It turns out that, because of the slow decrease of the gravitational field at infinity, a larger group of asymptotic symmetries operates at infinity. The reasons for the appearance of this group and the properties of the asymptotic symmetries are discussed in Secs. 5 and 6.

To quantize the gravitational field, we have used Schwinger's dynamical principle, which enables us to obtain commutation relations and dynamical operators on not only spacelike but also on null surfaces and, in particular, on the asymptotic null infinities $\mathcal{J}^{\pm}$ (Sec. 8). The definition of the asymptotic vacuum (Sec. 9) and the general expression for the S matrix that we obtain in asymptotically flat spacetime (Sec. 10) are used in Sec. 11 in a concrete situation to calculate the energy spectrum and flux of particles created by the gravitational field of black holes. Some of the results presented here are based on investigations of the present authors and are published for the first time.

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1. ASYMPTOTIC PROPERTIES OF CLASSICAL FIELDS IN FLAT SPACETIME

We begin by considering properties of the solutions of the equation

$$\Box \varphi(x)=0 \tag{1}$$

in ordinary spacetime but using methods which admit generalization to the case of curved spacetime. This section can be regarded as providing the skeleton of a theory that will then be developed for fields with various spins in asymptotically flat spacetime.

Radiation Problem. For Eq. (1), one can pose various problems. For example, one can specify the values of the field φ and its derivative on some spacelike surface (*Cauchy problem*) or specify φ on a characteristic surface (*Goursat problem*). We shall find it convenient to characterize the field φ in the spirit of scattering theory by its asymptotic behavior at infinity.

Instead of Cartesian coordinates x^{μ}, we introduce $r=|\mathbf{x}|$ and x^A, where x^A $(A=2,3)$ are coordinates on the surface of the unit sphere, for example, the spherical θ and φ angles, and we set $u=x^0-r$ (the retarded time). The metric $ds^2=(dx^0)^2-(d\mathbf{x})^2$ takes the form $ds^2=du^2+2dudr-r^2dl^2$, where dl is the element of length on the surface of the sphere; in spherical coordinates, $dl^2=d\theta^2+\sin^2\theta d\varphi^2$.

We shall be interested in the behavior of the solution φ of Eq. (1) for fixed u, x^A, and $r\to\infty$ (in this case, we shall say that the point x tends to $\mathcal{J}^+$). In Appendix 1 it is shown that if the field φ at the initial time is localized in space, then there exists

$$\lim_{\substack{r\to\infty\\ u,\,x^A=\mathrm{const}}} r\varphi(u, r, x^A)=\Phi_{\mathrm{out}}(u, x^A). \tag{2}$$

We shall call Φ_{out} the image of φ on $\mathcal{J}^+$ (see Appendix 1); φ is determined by its image uniquely. If instead of u we introduce the coordinate $v=x^0+r$ (advanced time), then there will exist

$$\lim_{\substack{r\to\infty\\ v,\,x^A=\mathrm{const}}} r\varphi(v, r, x^A)=\Phi_{\mathrm{in}}(v, x^A) \tag{3}$$

the image of φ on $\mathcal{J}^-$, which also uniquely characterizes the corresponding solution φ.

Note that the point x tends to $\mathcal{J}^+$ along a generator of the cone, i.e., along a null geodesic, and that r is an affine parameter along this geodesic. Thus, the assertions made above can be reformulated in geometrical language as follows: A massless scalar field in flat spacetime is uniquely determined by its asymptotic behaviors along null geodesics as the affine parameter r tends to infinity.

It follows from this in particular that any quantity which depends on φ, for example, the energy–momentum vector, can be expressed in terms of Φ_{out}.

For massive fields, it is evidently necessary to consider the asymptotic behavior along timelike curves.[1,2]

Penrose Space. It is convenient to make a conformal transformation and go over from the Minkowski spacetime M with metric $g_{\mu\nu}$ [we denote this pair by (M, g)] to a new "unphysical" spacetime (we shall call it the *Penrose space*) $\hat{M}$ with metric $\hat{g}_{\mu\nu}$ such that $\hat{M}$ is a compact manifold with a boundary that is the "infinity" of M. Then instead of considering the limits $r\to\infty$, we shall consider the behavior of a field in the neighborhood of the boundary.

We introduce the coordinate $\hat{r}=1/r$. It is obvious that $r\to\infty$ corresponds to $\hat{r}\to 0$. We denote by $\mathcal{J}^+$ the hypersurface defined by the equation $\hat{r}=0$ (u and x^A are the coordinates on $\mathcal{J}^+$). Since x^A range over the two-dimensional sphere S^2, and u over the real straight line R^1, it is clear that $\mathcal{J}^+$ has the topology $R^1\times S^2$. In the original coordinates, this hypersurface corresponds to fixed u and x^A and the limit $r\to\infty$, i.e., it corresponds to tending to infinity in M along future-directed light rays, and is therefore called *future null infinity*. The metric $ds^2=\hat{r}^{-2}[\hat{r}^2du^2-2dud\hat{r}-dl^2]$ is singular on $\mathcal{J}^+$, but if we make a conformal transformation and go over to the metric $d\hat{s}=\Omega ds$, where $\Omega=\hat{r}$, then $d\hat{s}^2=\hat{r}^2du^2-2dud\hat{r}-dl^2$ is regular at the point $\hat{r}=0$. In particular, on $\mathcal{J}^+$

$$\hat{g}_{00}=\hat{g}_{0A}=0;\ \hat{g}_{01}=1;\ \partial_{\hat{r}}\hat{g}_{00}=\partial_{\hat{r}}\hat{g}_{0A}=\partial_{\hat{r}}\hat{g}_{01}=0. \tag{4}$$

Similarly, using the advanced time $v=x^0+r$, we introduce past null infinity $\mathcal{J}^-$. In order to describe $\mathcal{J}^+$ and $\mathcal{J}^-$ simultaneously, it is convenient to go over to the coordinates u, v, x^A. The metric takes the form $ds^2=dudv-(u-v)^2dl^2/4$, $u\leqslant v$.

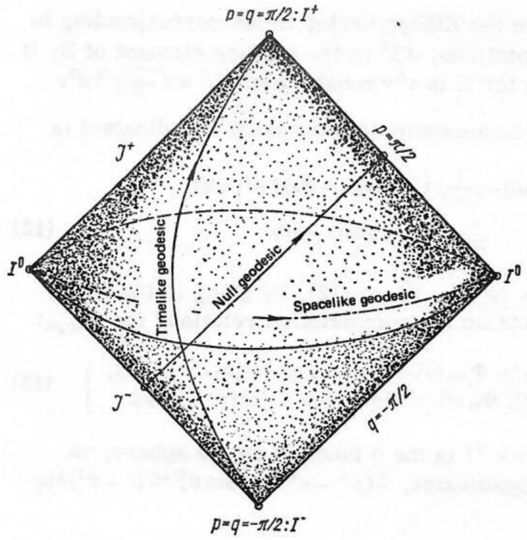

FIG. 1.

We introduce new coordinates p and q such that $v=\tan p$, $u=\tan q$ $(-\pi/2 \leqslant q \leqslant p \leqslant \pi/2)$. Then points at infinity have the finite coordinates $p=\pi/2$ and $q=-\pi/2$ (Fig. 1). Naturally, at these values of p and q the metric ds becomes meaningless, but if we go over to the conformal metric $d\hat{s}=\Omega ds$, where $\Omega=[(1+u)(1+v)]^{-1/2}$, then ds will be completely regular: $d\hat{s}^2=dpdq-\sin^2(p-q)\,dl^2/4$. Thus, we have obtained a manifold $\hat{M}$ parametrized by the coordinates p, q, x^A with boundary $q=-\pi/2$ ($\mathcal{J}^-$) and $p=\pi/2$ ($\mathcal{J}^+$). The interior $\hat{M}\setminus(\mathcal{J}^+\cup\mathcal{J}^-)$ is conformal to M. The space $\hat{M}$ has the following properties: 1) $\Omega=0$ on the boundary $\partial\hat{M}=\mathcal{J}^+\cup\mathcal{J}^-$ and $\Omega>0$ inside it; 2) every null geodesic in $\hat{M}$ has two end points on $\partial\hat{M}$. In what follows, for curved spacetime, the existence of a space $\hat{M}$ conformal to the given spacetime M and having the properties 1 and 2 will be taken as the definition of an asymptotically flat space.

Spinor Formalism in Flat Spacetime. It is well known that spinors are needed to describe fields with half-integral spin. In fact, it is also helpful to introduce a spinor representation when one is working with tensor fields.[3,7] The correspondence between spinor and tensor quantities is established on the basis of the local isomorphism of the Lorentz group $L_+^\uparrow$ and the group $SL(2,C)$ and is achieved by means of the Infeld–van der Waerden translating symbols $\sigma_\mu^{A\dot{B}}$, which coincide with the Pauli matrices $\sigma_\mu^{a\dot{b}}$ for a definite choice of the bases in the spinor and tensor spaces.[8] For example, the electromagnetic field tensor $F_{\mu\nu}$, which has the properties $F_{\mu\nu}=-F_{\nu\mu}$ and $\overline{F}_{\mu\nu}=F_{\mu\nu}$ (the bar denotes the complex conjugate), is associated with a symmetric spinor Φ_{AB} as follows:

$$F_{\mu\nu}\sigma^\mu_{A\dot{B}}\sigma^\nu_{C\dot{D}}=\Phi_{AC}\varepsilon_{\dot{B}\dot{D}}+\varepsilon_{AC}\overline{\Phi}_{\dot{B}\dot{D}},\quad \|\varepsilon_{AB}\|=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Maxwell's equations

$$\partial^\mu F_{\mu\nu}=0,\quad \partial_{[\mu}F_{\nu\lambda]}=0$$

(the square brackets denote antisymmetrization) are written in the form

$$\partial^{A\dot{B}}\Phi_{AC}=0,\quad \partial^{A\dot{B}}=\sigma_\mu^{A\dot{B}}\partial^\mu.$$

We recall that, quite generally, any massless field of spin s $(s>0)$ is described by a symmetric spinor $\varphi_{A_1\ldots A_{2s}}$ satisfying the equation

$$\partial^{A_1\dot{B}}\varphi_{A_1\ldots A_{2s}}=0.$$

Asymptotic Degeneracy. Taking the electromagnetic field as an example, we illustrate the important property of asymptotic degeneracy of massless fields. Consider the electromagnetic field produced by an isolated source and let us describe the rate of decrease of the strength of this field as one moves away from the source along the null geodesic γ. It is convenient to introduce a spinor basis o^A, ι^A which is related to γ as follows. Let l^μ be the tangent vector to γ; then o^A is its corresponding spinor, i.e., $l^\mu\sigma_\mu^{A\dot{B}}=o^A\bar{o}^{\dot{B}}$, and ι^A is the "orthogonal" spinor normalized by the condition $\iota_A o^A=1$. We decompose the electromagnetic field Φ_{AB} with respect to this basis:

$$\Phi_{AB}=\Phi_0\iota_A\iota_B+\Phi_1 o_{(A}\iota_{B)}+\Phi_2 o_A o_B. \tag{5}$$

Here $\Phi_0=\Phi_{AB}o^Ao^B$; $\Phi_1=\Phi_{AB}\iota^Ao^B$; $\Phi_2=\Phi_{AB}\iota^A\iota^B$; the round brackets denote symmetrization. If r is an affine parameter along the geodesic γ (one can assume that this is simply the length of the radius vector in spherical coordinates), then in the limit $r\to\infty$ we have the following asymptotic behavior: $\Phi_0\sim r^{-3}$, $\Phi_1\sim r^{-2}$ (Coulomb part), $\Phi_2\sim r^{-1}$ (radiation). In the tensor notation, we have the so-called property of asymptotic degeneracy[58]

$$F_{\mu\nu}=F^{(2)}_{\mu\nu}/r+F^{(1)}_{\mu\nu}/r^2+F^{(0)}_{\mu\nu}/r^3+O(1/r^4).$$

Note that Φ_0 and Φ_2 is the expansion (5) are the coefficients in front of pairs of identical spinors, while Φ_1 is the coefficient in front of a pair of different spinors. The $F^{(i)}_{\mu\nu}$ have corresponding algebraic properties (for more details on this, see Sec. 3).

Action of the Poincaré Group on $\mathcal{J}^\pm$. On $\mathcal{J}^\pm$, a certain nonlinear representation of the Poincaré group which acts in M arises. Moreover, the representation of the Poincaré group on $\mathcal{J}^\pm$ is isomorphic to the group itself. We shall describe this isomorphism in more detail.

To the transformation $x^\mu\to x'^\mu=\Lambda^\mu_\nu x^\nu+a^\mu$ of the Cartesian coordinates there corresponds a transformation of the coordinates u, r, θ, φ into $u', r', \theta', \varphi'$, this transformation depending in a complicated manner on u, r, θ, and φ. For $r=\infty$, the dependence simplifies and one can show[9] that

$$\left.\begin{array}{l}\lim\theta'=H(\theta,\varphi);\ \lim\varphi'=I(\theta,\varphi);\\ \lim u'=K^{-1}(u+a_0+a_1\sin\theta\cos\varphi+a_2\sin\theta\sin\varphi+a_3\cos\theta).\end{array}\right\} \tag{6}$$

Here, the limit is taken for $r\to\infty$ and fixed u, θ, φ. The functions H, I, and K in (6) are such that the transformation $\theta'=H(\theta,\varphi)$, $\varphi'=I(\theta,\varphi)$ is a conformal transformation on the sphere:

$$dl^2=d\theta^2+\sin^2\theta\,d\varphi^2=K^2(\theta',\varphi')(d\theta'^2+\sin^2\theta'\,d\varphi'^2). \tag{7}$$

This conformal transformation can be represented in the form

$$\zeta' = (a\zeta + b)/(c\zeta + d), \quad (8)$$

where $\zeta = \exp(i\varphi)\cot\theta/2$; $\zeta' = \exp(i\varphi')\cot\theta'/2$; a, b, c, d are complex parameters satisfying the condition $ad - bc = \pm 1$. The law of composition under conformal transformation coincides with the rules of matrix multiplication. We have therefore established a connection to the group $SL(2, C)$, from which there follows the isomorphism of the Poincaré group acting in M and its representation on $\mathcal{I}^{\pm}$.

2. QUANTUM THEORY IN MINKOWSKI SPACE

The procedure for quantizing a classical field φ consists of two parts: 1) construction of canonical commutation relations (operator algebra) and 2) the choice and interpretation of the states in the Hilbert space on which this operator algebra is realized (in particular, the choice of the vacuum).

Canonical Quantization. Let us recall briefly the basic formulas of the canonical quantization of a scalar field φ with action

$$W = \int d^4x \mathcal{L}(x), \quad \mathcal{L}(x) = \frac{1}{2}\sqrt{-g}\,(g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + \alpha R\varphi^2). \quad (9)$$

Here, R is the curvature and α some number. Although $R = 0$ in the flat case, the term $\alpha R\varphi^2$ gives a nonzero contribution on variation of $g_{\mu\nu}$; for example, the metric energy–momentum tensor $T_{\mu\nu} = \delta W/\delta g^{\mu\nu}$ coincides with the so-called improved energy–momentum tensor.[10,11] We shall use covariant notation. The metric $g_{\mu\nu}$ is flat but the coordinates may be curvilinear. In Cartesian coordinates, we obviously have $\sqrt{-g} = 1$, and the covariant derivatives coincide with the ordinary ones.

We introduce a spacelike Cauchy surface Σ defined by the equation $x^0 = \text{const}$ and define on it the momentum $\pi(x) = \partial\mathcal{L}/\partial\dot{\varphi} = \sqrt{-g}g^{0\mu}\varphi_{,\mu}$. The Hamiltonian has the form

$$H = \int_{x^0=\text{const}} (\pi\dot{\varphi} - \mathcal{L})\, d^3x = \int_{x^0=\text{const}} d^3x \Big[\frac{1}{g^{00}}\Big(\frac{\pi^2}{\sqrt{-g}} - \pi g^{0k}\varphi_{,k}\Big) + \frac{\sqrt{-g}}{(g^{00})^2} g^{0k}g^{0l}\varphi_{,k}\varphi_{,l} - \frac{1}{2}\alpha R\varphi^2\Big], \quad k, l = 1, 2, 3.$$

We specify the canonical commutation relations on Σ: $[\varphi(x), \pi(y)] = i\delta^{(3)}(x, y)$; the remaining commutators are zero. Here, the δ function is defined by

$$\int_{x^0=y^0=\text{const}} d^3y\,\delta^{(3)}(x, y) f(y) = f(x).$$

The energy–momentum tensor has the form

$$T_{\mu\nu} = \frac{\delta W}{\delta g^{\mu\nu}} = (2\alpha + 1)\varphi_{,\mu}\varphi_{,\nu} - (1 + 4\alpha) g_{\mu\nu}\varphi_{,\rho}g^{\rho\sigma}\varphi_{,\sigma}/2 + 2\alpha\varphi\varphi_{,\mu\nu}. \quad (10)$$

The energy–momentum operators $\mathbf{P}_\mu = \int_\Sigma \mathbf{T}_{\mu\nu}d\Sigma^\nu$ and the angular momentum tensor (in Cartesian coordinates) $\mathbf{M}_{\mu\nu} = \int_\Sigma (x_\mu \mathbf{T}_{\lambda\nu} - x_\nu \mathbf{T}_{\lambda\mu})d\Sigma^\lambda$ can be written in the form

$$\mathbf{P}[\xi] = \int_\Sigma \mathbf{T}_{\mu\nu}\xi^\mu\, d\Sigma^\nu, \quad (11)$$

where ξ^μ are the Killing vector fields corresponding to shifts and rotations; $d\Sigma^\nu$ is the surface element of Σ; if the equation for Σ is $x^0 = \text{const}$, then $d\Sigma^\nu = \sqrt{-g}g^{0\nu}d^3x$.

The field commutator (in Cartesian coordinates) is

$$[\varphi(x), \varphi(y)] = \frac{1}{(2\pi)^3}\int d^4k \exp(-ikx)\,\delta(k^2)\,\varepsilon(k^0) = \frac{1}{2\pi i}\varepsilon(x^0 - y^0)\,\delta((x - y)^2). \quad (12)$$

Operators on $\mathcal{I}^{\pm}$. From (12), by going to the limit, we readily obtain the commutation relations for $\Phi_{\text{out,in}}$:

$$\left.\begin{array}{l}[\Phi_{\text{out}}(u, x^A), \Phi_{\text{out}}(u', x'^A)] = -i\varepsilon(u - u')\,\delta(x^A - x'^A)/2;\\ [\Phi_{\text{in}}(v, x^A), \Phi_{\text{in}}(v', x'^A)] = -i\varepsilon(v - v')\,\delta(x^A - x'^A)/2.\end{array}\right\} \quad (13)$$

Here, $\delta(x^A = x'^A)$ is the δ function on the sphere; in spherical coordinates, $\delta(x^A - x'^A) = [\sin\theta]^{-1}\delta(\theta - \theta')\delta(\varphi - \varphi')$.

The dynamical variables on $\mathcal{I}^{\pm}$ have the same form as in the classical theory, but their definition must be augmented since they become operators.

Data on $\mathcal{I}^{\pm}$ completely determine the solution of Eq. (1) in the whole of spacetime, so that we can quantize by specifying the commutation relations (13) on $\mathcal{I}^{+}$ or $\mathcal{I}^{-}$. Using the equations of motion, we can verify that this will be equivalent to canonical quantization. Note that on the null hypersurfaces $\mathcal{I}^{\pm}$, in contrast to the case of a spacelike hypersurface, the field values at different points are not dynamically independent and the commutator (13) is nonzero if the arguments of the fields lie on one null generator, i.e., can be related by a causal (light) signal.

Since there is an essentially one-to-one correspondence between φ and Φ_{out}, all quantities that depend on φ can be expressed in terms of Φ_{out}. In particular, using the fact that the expression $\mathbf{P}[\xi] = \int_\Sigma \mathbf{T}_{\mu\nu}\xi^\mu\, d\Sigma^\nu$ does not depend on Σ in order to go over to integration over the null hypersurface and taking into account the asymptotic behavior

$$\mathbf{T}_{\mu\nu} \sim [(2\alpha + 1)\dot{\Phi}^2 + 2\alpha\Phi\ddot{\Phi}]\, l_\mu l_\nu/r^2 + O(1/r^3) \quad \text{as} \quad r \to \infty,$$

where l_μ is the tangent vector to the null geodesic with affine parameter r, and $\dot{\Phi} = \partial_u \Phi$, we obtain

$$\mathbf{P}_\mu = \int du \sin\theta\, d\theta\, d\varphi\, n_\mu(\theta, \varphi)\,[(2\alpha + 1)\dot{\Phi}^2 + 2\alpha\Phi\ddot{\Phi}],$$
$$n_\mu(\theta, \varphi) = (1, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \quad (14)$$

Definition of the Vacuum. The vacuum, like any other state, is defined on the Cauchy hypersurface Σ and will be denoted by $|0, \Sigma\rangle$. In the flat case for the free field, the vacuum does not of course depend on Σ and is usually defined by the condition $\mathbf{P}_\mu|0\rangle = 0$, where $\mathbf{P}_\mu$ are the shift generators of the Poincaré group, or, equivalently, by the condition $\mathbf{a}_\alpha|0\rangle = 0$, where $\mathbf{a}_\alpha$ is the annihilation operator corresponding to negative-frequency solutions with respect to the Cartesian time x^0.

Note that if on $\mathcal{I}$ we introduce operators of creation and annihilation in accordance with positive and negative frequency with respect to the retarded (respectively, advanced) time u (respectively v):

$$\Phi_{\text{out}}(u, x^A) = \int [\exp(-i\omega u)\, a(\omega, x^A) + \exp(i\omega u)\, a^*(\omega, x^A)]\, d\omega,$$

then the vacuum $|0, \text{out}\rangle$ (similarly $|0, \text{in}\rangle$) can also be determined by the relation $a_{\text{out}}(\omega, x^A)\,|0, \text{out}\rangle = 0$. One can show that $|0\rangle = |0, \text{in}\rangle = |0, \text{out}\rangle$. Therefore, instead of the operation of normal ordering with respect to $|0\rangle$, which is usually used to give an operator meaning to expressions of the type (14), one can use the equivalent operation of normal ordering with respect to the in and out vacuums.

For the generalization to the case of curved spacetime, it is more instructive to recall the definition of the vacuum for a field φ that interacts with an external classical source $g(x)$ concentrated in a bounded region of spacetime and described by the equation $\Box\varphi = g$. Suppose $g(x) = 0$ for $x^0 < x_1^0$ (region Θ_1) and for $x^0 > x_2^0$ (region Θ_2). If the hypersurface Σ lies in the region Θ_1 or Θ_2, then there φ satisfies the equation $\Box\varphi = 0$ and coincides with the corresponding free fields φ_{in} and φ_{out}, by means of which the in and out vacuums can be defined in the usual manner. In general, these vacuums differ from one another, which corresponds to particle production by the source g.

3. THE CONCEPT OF ASYMPTOTICALLY FLAT SPACETIME AND ITS PROPERTIES

The nearest analog of flat spacetime as regards geometrical and physical properties is the class of spaces whose gravitational field is produced by isolated sources and, moreover, in such a way that the field decreases as one moves away from the source and the metric becomes "almost flat". Such spaces are said to be asymptotically flat (a precise definition will be given below).

At the first glance, it would seem that asymptotically flat spacetime is a fairly clear concept, but the precise definition and analysis of the problems that here arise are by no means simple. Clarification of these questions required the efforts of many physicists, for example, Fock,[12] Trautman,[13] Bondi,[14] Sachs,[9,15] Arnowitt, Deser, and Misner,[16] Newman and Penrose,[3,17] and even now not all problems have been completely solved. As usual in the general theory of relativity, the difficulties arise from the general coordinate invariance and the nonlinearity (i.e., essentially the equivalence principle) of the theory.

What is Asymptotically Flat Spacetime? Let us attempt to make more precise the intuitive definition given above. Above all, what do we mean by "isolated sources" and in what sense must we understand "moving away" from them? Putting it more precisely, to describe in what sense the metric $g_{\mu\nu}$ tends to the flat $\eta_{\mu\nu}$ we must specify:

1) in what sense does one move away from the source at infinity (for example, one could move away along null, timelike, or spacelike geodesics or in some other way);

2) what does it mean to say that the tensor $g_{\mu\nu}$ differs little from $\eta_{\mu\nu}$ (a "small" difference in one coordinate system may be "large" in another);

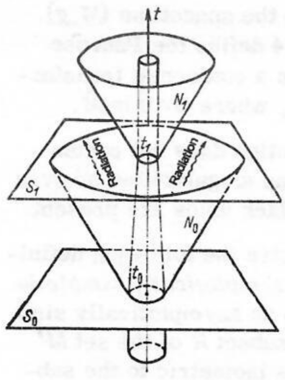

FIG. 2.

3) how rapidly must the metric $g_{\mu\nu}$ approach the flat metric (see the discussion of this question in Sec. 6).

The definition of asymptotically flat space is essentially a matter of convention and convenience. If the definition is to be reasonable, it must include the gravitational field produced by an isolated source. This problem was analyzed by Bondi[14] and Sachs.[9,19] In their work, particular null frames of reference play an important role. We shall show how these frames of reference arise by the following example. Suppose we wish to measure some integrated characteristics of a source of the gravitational field (for example, the mass) before radiation (at time t_0) and after radiation (after t_1) by means of measurements at infinity. Then for this purpose the spacelike hypersurfaces S_0 and S_1 (Fig. 2) are unsuitable, since all the radiation intersects them and a distant observer will merely measure the total mass of the source plus the radiation. It is necessary to use the null hypersurfaces N_0 and N_1.

It was shown in Refs. 3, 9, 14, and 19 that the metric of the gravitational field produced by an isolated source can be reduced to the form [cf. (28)]:

$$ds^2 = r^{-2} A\, dr^2 - 2B_k\, dx^k\, dr + r^2 C_{kl}\, dx^k\, dx^l \quad (r > r_0), \tag{15}$$

where A, B_k, C_{kl} are functions of x^k and r which are differentiable sufficiently often in the neighborhood of the hypersurfaces $\mathscr{J}^{\pm}$ (as $r \to \infty$). If it is assumed that the determinant of this metric does not vanish, then it can obviously be represented in the form $ds^2 = \Omega^2 d\hat{s}^2$, where $\Omega = r^{-1}$ and $d\hat{s}$ is regular on $\mathscr{J}^{\pm}$ (see the flat case above).

Definition of Asymptotically Flat Spacetime. We first formulate a precise definition of *asymptotically simple* (according to Penrose) spacetime.[3,6,57]

A spacetime M with metric $g_{\mu\nu}$ is said to be asymptotically simple if there exists another "unphysical" spacetime $\hat{M}$ with boundary $\partial\hat{M} = \mathscr{J}$ such that; 1) M is conformal to $\hat{M} \setminus \partial\hat{M}$; 2) there exist a smooth real function $\Omega \geq 0$ on $\hat{M}$ and a smooth pseudo-Riemannian metric $\hat{g}_{\mu\nu}$ on $\hat{M}$ such that $\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ on M; 3) $\Omega > 0$ in $\hat{M} \setminus \partial\hat{M}$ and $\Omega = 0$ on $\partial\hat{M}$, $\nabla_\mu \Omega|_{\partial\hat{M}} \neq 0$; 4) every null geodesic in M has two end points on $\partial\hat{M}$. It follows from this[6] that $\hat{M}$ has the topology of R^4. We shall call $(\hat{M}, \hat{g}, \Omega)$ the

Penrose space corresponding to the spacetime (M,g). One can show that conditions 1–4 define the Penrose space $(\hat{M},\hat{g},\Omega)$ uniquely to within a conformal transformation: $(\hat{M},\hat{g},\Omega)\to(\hat{M},\omega^2\hat{g},\omega\Omega)$, where $\omega>0$ in $\hat{M}$.

Generally speaking, our definition does not encompass the case when spacetime has singularities or even horizons, in particular, when black holes are present.

To encompass this case, we give the following definition. A spacetime M is called *asymptotically simple in the weak sense*[6] if there exists an asymptotically simple $\hat{M}'$ such that for some open subset K of the set $\hat{M}'$ (with $\partial\hat{M}\subset K$) the region $\hat{M}'\cap K$ is isometric to the subset M. This means that a spacetime which is asymptotically simple in the weak sense contains a conformal "exterior" infinity, but there may also be "interior" infinities.

We emphasize that the definition of an asymptotically simple spacetime is tailored to massless fields and is not convenient for working with massive fields. Attempts to introduce an analogous concept for massive fields were made in Refs. 1 and 2.

In what follows we shall restrict ourselves to the case when in a certain neighborhood of $\mathscr{J}$ the energy-momentum tensor of the matter vanishes. We shall then say that the corresponding asymptotically simple (in the weak sense) spacetime is an *asymptotically flat space*. In fact, many of the assertions given below are also true in the general case of a space which is asymptotically simple in the weak sense. The surface $\mathscr{J}$ will be timelike, spacelike, or null depending on whether the cosmological constant Λ is positive, negative, or zero. In what follows, we assume $\Lambda=0$.

Properties of Asymptotically Flat Space. Asymptotic Degeneracy. We note the following properties of asymptotically flat space: 1) $\mathscr{J}$ is an isotropic hypersurface; 2) the hypersurface $\mathscr{J}$ consists of two nonintersecting parts $\mathscr{J}^+$ (future null infinity) and $\mathscr{J}^-$ (past null infinity), each of which has the topology $R^1\times S^2$; 3) the curvature tensor vanishes, $R_{\mu\nu\lambda\rho}=0$, on $\mathscr{J}$, i.e., the spacetime (M,g) really is flat at infinity; 4) asymptotic degeneracy. We shall now describe in more detail the important property of asymptotic degeneracy of the gravitational field. A free gravitational field is described by the Weyl tensor $C_{\mu\nu\lambda\rho}$, which in vacuum coincides with the tensor $R_{\mu\nu\lambda\rho}$ and has the properties $C_{\mu\nu\lambda\rho}=C_{[\lambda\rho][\mu\nu]}$; $C_{\mu[\nu\lambda\rho]}=0$; $C^{\mu}_{\nu\mu\lambda}=0$. As in the electromagnetic case (see Sec. 1) the spinor form of this tensor can be written as follows:

$$C_{\mu\nu\lambda\rho}\sigma^{\mu}_{A\dot{A}}\sigma^{\nu}_{B\dot{B}}\sigma^{\lambda}_{C\dot{C}}\sigma^{\rho}_{D\dot{D}}=\Psi_{ABCD}\varepsilon_{\dot{A}\dot{B}}\varepsilon_{\dot{C}\dot{D}}+\varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}},$$

where Ψ_{ABCD} is a symmetric spinor called the Weyl spinor. Because of the symmetry of Ψ_{ABCD}, it can be represented in the form $\Psi_{ABCD}=\alpha_{(A}\beta_B\gamma_C\delta_{D)}$, where α, β,γ,δ are, in general, different spinors which are unique to within a factor; they may coincide. Listing all possible cases, we arrive at the Petrov classification[3,20]:

$$\Psi_{ABCD}=\begin{cases}\alpha_{(A}\beta_B\gamma_C\delta_{D)}, & \text{type I;}\\ \alpha_{(A}\alpha_B\gamma_C\delta_{D)}, & \text{type II;}\\ \alpha_{(A}\alpha_B\beta_C\beta_{D)}, & \text{type } D;\\ \alpha_{(A}\alpha_B\alpha_C\beta_{D)}, & \text{type III;}\\ \alpha_A\alpha_B\alpha_C\alpha_D, & \text{type } N.\end{cases}$$

None of the spinors $\alpha,\beta,\gamma,\delta$ defines a vector since they are defined only to within a factor; rather, each defines a certain null direction at the given point of spacetime. These directions are called the principal null directions of the gravitational field.

If we have chosen a basis in the spin space in the form of the spinors o^A and ι^A, then the components of Ψ_{ABCD} with respect to this base are usually denoted as follows:

$$\Psi_0=\Psi_{ABCD}o^Ao^Bo^Co^D;$$
$$\Psi_1=\Psi_{ABCD}o^Ao^Bo^C\iota^D;\quad \Psi_2=\Psi_{ABCD}o^Ao^B\iota^C\iota^D;$$
$$\Psi_3=\Psi_{ABCD}o^A\iota^B\iota^C\iota^D;\quad \Psi_4=\Psi_{ABCD}\iota^A\iota^B\iota^C\iota^D.$$

The five complex quantities Ψ_i correspond to the ten real components of $C_{\mu\nu\lambda\rho}$. The spinor Ψ_{ABCD} can now be represented in the form

$$\Psi_{ABCD}=\Psi_0\iota_A\iota_B\iota_C\iota_D-4\Psi_1\iota_{(A}\iota_B\iota_Co_{D)}$$
$$+6\Psi_2\iota_{(A}\iota_Bo_Co_{D)}-4\Psi_3\iota_{(A}o_Bo_Co_{D)}+\Psi_4o_Ao_Bo_Co_D.$$

Suppose we are given a null geodesic γ in M. We choose o^A in such a way that the corresponding vector is the tangent to γ, and let r be an affine parameter along γ [it can be chosen in such a way that $r\sim\Omega^{-1}$ as $\mathscr{J}$ is approached (see Sec. 4)]. One can then show that $\Psi_i\sim r^{-(5-i)}$, $i=0,\ldots,4$, as $r\to\infty$. Therefore, for the Weyl tensor we have

$$C_{\mu\nu\lambda\rho}=N_{\mu\nu\lambda\rho}/r+\mathrm{III}_{\mu\nu\lambda\rho}/r^2+\mathrm{II}_{\mu\nu\lambda\rho}/r^3+I_{\mu\nu\lambda\rho}/r^4+O(1/r^5),$$

where $N_{\mu\nu\lambda\rho},\ldots$ denotes a tensor having the corresponding Petrov type.[19] It is this relation that determines the property of asymptotic degeneracy of the gravitational field. If there is no gravitational radiation, the first two terms of the expansion are zero.

Examples of Asymptotically Flat Spaces. The class of metrics that come under the definition of asymptotically flat is very large and includes well known examples such as the Schwarzschild, Reissner–Nordström, Vaidya, and Kerr metrics, among others. An an example, in Fig. 3 we show a Penrose diagram for the maximally extended Schwarzschild metric

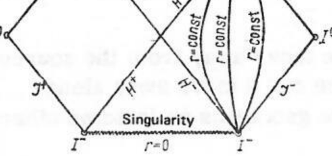

FIG. 3.

$$ds^2=(1-2m/r)\,dt^2-(1-2m/r)^{-1}\,dr^2-r^2\,dl^2 \quad (16)$$

in the coordinates $u=\tan^{-1} U$, $v=\tan^{-1} V$, where the Kruskal coordinates U and V are related to r and t (for $r>2m$) by $V/U=-\exp(-t/2m)$, $UV=\exp(r/2m)\times(1-r/2m)$.

In the coordinates U and V the metric (16) takes the form

$$ds^2=f^2\,dU\,dV-r^2\,dl^2,$$

where $f^2=(32m^3/r)\exp(-r/2m)$.

4. GEOMETRY OF THE NULL INFINITIES

In this section, we consider the choice of coordinates in the neighborhood of the null infinities $\mathcal{J}^{\pm}$. Although the definition of asymptotically flat space given in the preceding section was formulated in a manner that is independent of the choice of the coordinates, in concrete calculations it is convenient to use a definite coordinate system. Such calculations are significantly simplified if one chooses *Bondi conformal coordinates* as coordinates in the neighborhood of $\mathcal{J}^{\pm}$; in these coordinates, the geometrical characteristics of the surfaces $\mathcal{J}^{\pm}$ are described in the simplest way.[24]

Geometry of the Null Surfaces. The surfaces $\mathcal{J}^{\pm}$ in asymptotically flat space are null surfaces. We give some results relating to the geometrical properties of null surfaces that will be continually used in what follows.

Suppose we have a family of surfaces Σ_c, described by the equation $U(x)=c$. Such surfaces are said to be null if the vector field $l^\mu=g^{\mu\nu}U_{,\nu}$ of normals to it is null, $l^\mu l_\mu=0$. Since the tangent vectors ξ^μ to the surface Σ_c, are determined by the condition $l_\mu\xi^\mu=0$, the vector of the normal l^μ is obviously simultaneously a vector tangent to Σ_c. Therefore, on a null surface there is always a nonzero vector l^μ such that $l^\mu\xi_\mu=0$ for arbitrary tangent to Σ_c, so that the metric on the null surface Σ_c induced by its imbedding in the spacetime is degenerate. Many important differences between the geometrical properties of null surfaces and those of spacelike surfaces arise from this circumstance.

The integral curves $\gamma: x^\mu=x^\mu(r)$ of the vector field $l^\mu=g^{\mu\nu}U_{,\nu}$, which are determined by the equation

$$dx^\mu(r)/dr=l^\mu, \quad (17)$$

are null goedesics, and r is an affine parameters along these geodesics.[1)] An important property of the integral curves γ is that if such a curve passes through a point of the surface Σ_c then it lies entirely on this surface. The null curves γ are usually called *generators* of the surface Σ_c. Through every nonsingular point $p\,(l^\mu(p)\neq 0)$ of the surface Σ_c there passes one and only one generator. Important geometrical characteristics of a null surface Σ_c are the ***expansion*** ρ and ***shear*** σ of its generators[2)]:

$$\rho=-l^\mu_{;\mu}/2;\quad \sigma=[l_{(\alpha;\beta)}l^{\alpha;\beta}/2-(l^\alpha_{;\alpha})^2/4]^{1/2}. \quad (18)$$

Although the expressions given here for ρ and σ contain differentiation along a direction out of the surface Σ_c, one can show that the values of the invariants ρ and σ do not depend on the choice of the extension of the null vector field l^μ off Σ_c, i.e., they are completely determined by the field l^μ on Σ_c. The simplest examples of null surfaces in Minkowski space are the light cone and null hyperplane. They are both shear free ($\sigma=0$) and $\rho=-1/r\neq 0$ for the cone and $\rho=0$ for the hyperplane.

Coordinates on $\mathcal{J}^{\pm}$. If in a space or on a surface we are given a congruence[3)] of (null) geodesics, then it is possible to introduce in the following manner a coordinate system associated with this congruence. Suppose the coordinates x^A label curves of a congruence (i.e., x^A are constant along a geodesic in the congruence and the coordinates x^A are different for two different geodesics). The coordinates (u,x^A) of an arbitrary point p are determined by the coordinates x^A of the geodesic passing through the point p and the value u at the point p of the affine parameter along this geodesic.[4)] It is easy to see that the coordinates associated with a congruence are not defined uniquely. The general freedom in the choice of such coordinates is described as follows:

$$\left.\begin{array}{ll} x^A\to\tilde{x}^A=f^A(x^A) & \text{(relabeling of the geodesics);} \\ u\to\tilde{u}=B(x^A)\,(u+A(x^A)) & \text{(change of affine parameter along geodesics).} \end{array}\right\} \quad (19)$$

In the Penrose space $(\hat{M},\hat{g},\Omega)$ the surfaces $\mathcal{J}^{\pm}$, which are defined by the equation $\Omega=0$, are null. The vector field $\hat{n}^\mu=\hat{g}^{\mu\nu}\Omega_{,\nu}$ is tangent to the generators of the surfaces $\mathcal{J}^{\pm}$. If we denote by u an affine parameter along the generators, then on $\mathcal{J}^{\pm}$ we can always introduce coordinates (u,x^A) $(A=2,3)$ associated with the congruence of generators. Since $\mathcal{J}^{\pm}$ has the topology of $R^1\times S^2$, the sections of these surfaces $u=\text{const}$ are diffeomorphic to the sphere S^2 and as coordinates x^A one could therefore choose any coordinates on the sphere, for example, spherical θ and φ or complex stereographic coordinates $(\zeta,\bar{\zeta})$, $\zeta=\cot(\theta/2)\exp[(i\varphi)/2]$. The freedom in the choice of these coordinates (u,x^2,x^3) on $\mathcal{J}^{\pm}$ is characterized by (19).

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1) We recall that a curve γ: $x^\mu=x^\mu(r)$ is called a geodesic if the tangent vector dx^μ/dr is transported parallelly along it, i.e., $D(dx^\mu/dr)/Dr=\alpha(r)dx^\mu/dr$. The parameter r on the curve γ is said to be affine if the factor α vanishes when this parameter is chosen. An affine parameter along a curve is determined to within the transformation $r\to\tilde{r}=ar+b$.

2) A more detailed discussion of the properties of null surfaces and the geometrical meaning of the optical scalars ρ and σ, and also the proof of these assertions, can be found, for example, in Refs. 19, and 21–23.

3) I.e., a family of curves such that through every point of the space or the surface there passes only one curve.

4) Note that, in general, it is not always possible to introduce smooth coordinates associated with a congruence on the complete manifold. However, it can always be done locally in the neighborhood of any point. Thus, a complete set of coordinate systems (charts) is associated with a given congruence.

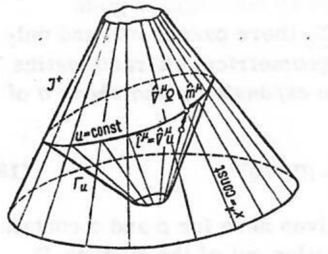

FIG. 4.

Metric of an Asymptotically Flat Space in the Neighborhood of $\mathcal{J}^\pm$. Bondi Coordinates. We now use the coordinates (u, x^2, x^3) introduced above on $\mathcal{J}^\pm$ in order to construct coordinates in the neighborhood of $\mathcal{J}^\pm$ in the Penrose space. For this, we first extend the coordinates (u, x^2, x^3) from the surface $\mathcal{J}^\pm$ to some neighborhood of it and, second, augment these coordinates by a fourth coordinate function $\hat{r}$, choosing it in such a way that the coefficients of the metric form $\hat{g}$ have the simplest possible form in the neighborhood of $\mathcal{J}^\pm$.

We choose an arbitrary section $u = u_0$ on $\mathcal{J}^\pm$. Through every point of this two-dimensional spacelike section there pass two null geodesics intersecting it orthogonally, one of which is a generator of $\mathcal{J}^\pm$, while the other does not lie on $\mathcal{J}^\pm$ (Fig. 4). The null geodesics that are orthogonal to the section $u = u_0$ and do not lie in $\mathcal{J}^\pm$ are the generators of some null surface, which we shall denote by Γ_{u_0}. To determine the coordinates of a point p in the neighborhood of $\mathcal{J}^\pm$, we find a null surface Γ_u passing through this point and let γ_u be the generator of Γ_u passing through p. We assign to the point p the coordinates (u, x^2, x^3) if this generator intersects $\mathcal{J}^\pm$ at the point with the coordinates (u, x^2, x^3). As the fourth coordinate $\hat{r}$, we choose the Penrose conformal factor[5)]

$$\hat{r} = \Omega. \tag{20}$$

From the procedure for the construction of the coordinates $(\hat{r}, u, x^2, x^3)$ it follows that these coordinates are uniquely determined provided the coordinates (u, x^2, x^3) at the surface $\mathcal{J}^\pm$ are chosen.

The condition of constancy of the coordinates x^A along the generators of the surface Γ_u and the null nature of Γ_u enable us to conclude that

$$\hat{g}^{0A} \equiv \hat{g}^{\mu\nu} U,_\mu x^A,_\nu = 0; \quad \hat{g}^{00} \equiv \hat{g}^{\mu\nu} u,_\mu u,_\nu = 0,$$

and therefore in the coordinates $(\hat{r}, u, x^2, x^3)$ the metric $\hat{g}$ has the form

$$\|\hat{g}^{\mu\nu}\| = \begin{pmatrix} 0 & \hat{g}^{01} & 0 \\ \hat{g}^{01} & \hat{g}^{11} & \hat{g}^{1A} \\ 0 & \hat{g}^{1A} & \hat{g}^{AB} \end{pmatrix}; \quad \|\hat{g}_{\mu\nu}\| = \begin{pmatrix} \hat{g}_{00} & \hat{g}_{01} & \hat{g}_{0A} \\ \hat{g}_{01} & 0 & 0 \\ \hat{g}_{0A} & 0 & \hat{g}_{AB} \end{pmatrix}, \tag{21}$$

where $\hat{g}^{AB}\hat{g}_{BC} = \delta^A_C$; $\hat{g}_{01} = (\hat{g}^{01})$; $\det\hat{g}_{\mu\nu} = -\hat{g}^2_{01}\det\hat{g}_{AB}$.

The expression for the metric (21) can be simplified if we recall that there is a freedom in the choice of the Penrose space corresponding to the original physical spacetime; for it is readily verified that to a given spacetime with Penrose space $(\hat{M}, \hat{g}, \Omega)$ there also corresponds the entire class of spaces $(\hat{M}, \omega^2\hat{g}, \omega\Omega)$, where $\omega > 0$ is an arbitrary bounded smooth function on $\hat{M}$. In particular, by the choice of ω (i.e., by the choice of a definite representative in the given class of Penrose spaces) one can achieve that

$$\det\|\hat{g}_{AB}\| = h(x^A), \tag{22}$$

where $h(x^A)$ is a definite fixed function whose choice is dictated by convenience.

On the surfaces $J^\pm$, which in the coordinate system we are using are determined by the equation $\hat{r} = 0$:

$$0 = \hat{g}^{\mu\nu}\Omega,_\mu\Omega,_\nu|_{\mathcal{J}^\pm} = \hat{g}^{11}|_{\mathcal{J}^\pm};$$
$$\hat{g}^{1A}|_{\mathcal{J}^\pm} = \hat{g}^{\mu\nu}\Omega,_\mu x^A,_\nu = 0,$$

and therefore on $\mathcal{J}^\pm$

$$\|\hat{g}^{\mu\nu}\||_{\mathcal{J}^\pm} = \begin{pmatrix} 0 & \hat{g}^{01} & 0 \\ \hat{g}^{01} & 0 & 0 \\ 0 & 0 & \hat{g}^{AB} \end{pmatrix};$$
$$\|\hat{g}_{\mu\nu}\||_{\mathcal{J}^\pm} = \begin{pmatrix} 0 & \hat{g}_{01} & 0 \\ \hat{g}_{01} & 0 & 0 \\ 0 & 0 & \hat{g}_{AB} \end{pmatrix}. \tag{23}$$

More complete information about the behavior of the coefficients of the metric form $\hat{g}$ in the neighborhood of $\mathcal{J}^\pm$ can be obtained by noting that the metric g satisfies Einstein's equations in vacuum in this neighborhood. This means that the metric $\hat{g}$ satisfies in the neighborhood of $\mathcal{J}^\pm$ the equation (see Appendix 2)

$$\hat{r}^2\hat{G}_{\alpha\beta} - 2\hat{r}\hat{\nabla}_\alpha\hat{\nabla}_\beta\hat{r} + (2\hat{r}\hat{\nabla}^\rho\hat{\nabla}_\rho\hat{r} - 3\hat{\nabla}_\rho\hat{r}\hat{\nabla}^\rho\hat{r})\,\hat{g}_{\alpha\beta} = 0.$$

Under the condition that the metric $\hat{g}$ is regular and with our choice $\Omega = \hat{r}$ of the Penrose conformal factor, the following equations hold[24]:

$$\hat{\nabla}_\alpha\hat{\nabla}_\beta\hat{r}|_{\mathcal{J}^\pm} = 0; \quad \hat{\nabla}_\alpha\hat{r}\hat{\nabla}^\beta\hat{r}/\hat{r}|_{\mathcal{J}^\pm} = 0. \tag{24}$$

The first equation means that the surface $\mathcal{J}^\pm$ is shear free ($\sigma = 0$) and that the expansion of its generators is zero ($\rho = 0$). Both these properties are a consequence of the relation (22), or, which is the same thing, of the choice of the conformal factor Ω made above. Equations (24) also enable us to show that, using the coordinate transformations (19), we can arrange for the following equations to hold on $\mathcal{J}^\pm$ [cf. Eq. (4)]:

$$\hat{g}_{00} = \hat{g}_{0A} = 0; \quad \hat{g}_{01} = 1;$$
$$\partial_{\hat{r}}\hat{g}_{00} = \partial_{\hat{r}}\hat{g}_{0A} = \partial_{\hat{r}}\hat{g}_{01} = 0;$$
$$\hat{g}_{AB} = g_{AB}; \quad g_{AB}\,dx^A\,dx^B = -dl^2, \tag{25}$$

where dl^2 is the interval on the surface of the sphere of unit radius. In complex stereographic coordinates

$$dl^2 = d\zeta\,d\bar{\zeta}/P_0^2; \quad P_0 = (1 + \zeta\bar{\zeta})/2. \tag{26}$$

The above coordinate system is now called the Bondi conformal coordinates.[24] The group of simultaneous

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[5)]Since $\Omega,_u \neq 0$ and $\hat{g}^{\mu\nu}\Omega,_\mu U,_\nu \neq 0$, on $\mathcal{J}^\pm$, one can show that in at least a certain neighborhood of $\mathcal{J}^\pm$ the coordinate functions $\hat{r}$, u, x^2, x^3 are independent.

transformations of the coordinates and the conformal factor Ω that leaves the conditions (25) and (26) on $\mathscr{I}^{\pm}$ invariant has the form [cf. Eqs. (6)–(8)]

$$\left.\begin{aligned} \bar{u}' &= K(u+\alpha(\zeta,\bar{\zeta}));\\ \zeta' &= (a\zeta+b)/(c\zeta+d), \quad ab-bc=\pm 1;\\ \hat{r}' &= K\hat{r}; \quad \Omega' = K\Omega, \end{aligned}\right\} \tag{27}$$

where $\alpha(\zeta,\bar{\zeta})$ is an arbitrary smooth function on the sphere and

$$K = K(\zeta,\bar{\zeta}) = (1+\zeta\bar{\zeta})\,[(a\zeta+b)\,(\overline{a\zeta}+\bar{b})+(c\zeta+d)\,(\overline{c\zeta}+\bar{d})]^{-1}.$$

The additional geometrical structure that arises on the surfaces $\mathscr{I}^{\pm}$ when their generators do not have shear or expansion has been called *strong geometry*.[5] Therefore, the group of transformations from one set of Bondi conformal coordinates to another set coincides with the group of transformations that preserve the strong geometry on $\mathscr{I}^{\pm}$.

In Bondi conformal coordinates, a regular metric $\hat{g}$ in the neighborhood of $\mathscr{I}^{\pm}$ can be represented in the form

$$d\hat{s}^2 = V\hat{r}^2\,du^2 + 2\,(1+B\hat{r}^2)\,du\,d\hat{r} + 2\hat{r}^2 U_A\,du\,dx^A + g_{AB}\,dx^A\,dx^B,$$

where $V=1-2M\hat{r}+O(\hat{r}^2)$; $B=O(1)$; $U_A=O(1)$; $h_{AB}=g_{AB}+O(\hat{r})$. Therefore, for large values of $r=\hat{r}^{-1}$ the metric g of asymptotically flat space has the form

$$ds^2 = r^2\,d\hat{s}^2 = V\,du^2 - 2(1+B/r^2)\,du\,dr + 2U_A\,du\,dx^A + r^2 h_{AB}\,dx^A\,dx^B, \tag{28}$$

where

$$V = 1-2M/r + O(r^{-2}); \quad B = O(1); \quad U_A = O(1); \quad h_{AB} = g_{AB} + O(r^{-1}). \tag{29}$$

The components of the metric in this expression satisfy a condition of uniform smoothness, i.e., the expansion (29) can be differentiated, and

$$\partial_r O(r^{-N}) = O(r^{-(N+1)}); \quad \partial_u O(r^{-N}) = \partial_A O(r^{-N}) = O(r^{-N}). \tag{30}$$

The requirement that there exist a coordinate system (u,r,x^2,x^3) in which the asymptotic behavior of the metric has the form (28)–(29) can be taken as the basis for the definition of asymptotically flat spacetime.[9]

5. THE GROUP OF ASYMPTOTIC SYMMETRIES

In asymptotically flat space, the gravitational field decreases with increasing distance from the sources, and the metric of the spacetime tends to the metric of Minkowski space. It is therefore natural to expect an asymptotically flat space to have a certain group of approximate symmetries which are such that as one moves away from the sources the transformations of the group differ less and less from exact symmetries. Such a group is called a *group of asymptotic symmetries*. An important and, at the first encounter, unexpected circumstance is that in the general case (in the presence of gravitational radiation) this group does not coincide with the Poincaré group but is appreciably larger, although it does contain the Poincaré group as a subgroup. In this section, we give a precise definition of the group of asymptotic symmetries. The structure, properties, and representations of this group, and also the reasons why it differs from the Poincaré group will be considered in the following section.

Group of Isometries of Spacetime and Conservation Laws. A group of transformations $x^\mu \to \tilde{x}^\mu = \tilde{x}^\mu(x^\mu)$ that acts on a spacetime manifold (M,g) and conserves the distance between arbitrary pairs of points is called an isometry group of the spacetime. The vector fields $\xi^\mu(x)$ that generate an infinitesimally small isometry transformation $x \overset{\mu}{\to} \tilde{x}^\mu = x^\mu + \varepsilon\xi^\mu(x)$ satisfy the following equation (the Killing equation):

$$2\xi_{(\mu;\nu)} \equiv (\xi_{\mu;\nu} + \xi_{\nu;\mu}) = 0; \quad \mathscr{L}_\xi g_{\mu\nu} = \lim_{\varepsilon\to 0}\,(\tilde{g}_{\mu\nu} - g_{\mu\nu})/\varepsilon = 0, \tag{31}$$

where $\mathfrak{L}_\xi g_{\mu\nu}$ is the Lie derivative of the field $g_{\mu\nu}$ along the vector field ξ^μ. If the metric g is given, then the set of Killing vector fields ξ^α [the solutions of Eq. (31)] form a Lie algebra corresponding to the isometry group of the spacetime. If instead of (31) the equation

$$\xi_{(\mu;\nu)} = \xi(x)\,g_{\mu\nu} \tag{32}$$

holds, then one says that the spacetime admits a group of *conformal isometries*. The transformations of this group conserve the angles between an arbitrary pair of vectors at a point. The function $\xi(x)$ in (32) can be found by multiplying both sides of (32) by $g^{\mu\nu}$:

$$\xi(x) = \xi^{\mu}_{;\mu}/4.$$

If ξ^μ is a Killing vector field in the space (M,g), then in the space $(M,\hat{g}=\Omega^2 g)$ it is a conformal Killing vector field, i.e., it satisfies an equation of the form (32):

$$\hat{\nabla}^{(\nu}\xi^{\mu)} = \Omega_{,\alpha}\xi^\alpha\hat{g}^{\mu\nu}/\Omega. \tag{33}$$

This assertion can be readily used by using the expression (A.11) for $\hat{\nabla}_\nu\xi^\mu$.

If a spacetime has an m-parameter isometry group, then in accordance with Noether's theorem there exist m conserved quantities (*invariants*):

$$P_a = P[\xi_{(a)}] = \int_\Sigma T_{\mu\nu}\xi^\mu_{(a)}\,d\Sigma^\nu, \tag{34}$$

where $\xi^\mu_{(a)}$ $(a=1,\ldots,m)$ are linearly independent Killing vector fields; $T_{\mu\nu}$ is the metric energy-momentum tensor. The quantities P_a are still conserved (do not depend on the choice of the global Cauchy surface Σ) if the $\xi^\mu_{(a)}$ are conformal Killing vector fields and $T^\mu_\mu = 0$.

Approximate Symmetry, Weak Gravitational Field, and the Poincaré Group. The Poincaré group is the maximal (ten-parameter) isometry group of flat spacetime. The presence of even a weak gravitational field (which is always present) has the consequence that the Poincaré group, which plays such an important role in elementary-particle physics, ceases strictly to be a group of exact isometries. In order to estimate the corresponding degree of symmetry breaking, we can proceed as follows. If a test particle moves in Minkowski space, its momentum $P=\xi_\mu u^\mu$, where $u^\mu = dx^\mu/d\tau$ and ξ^μ is the corresponding Killing vector field, is conserved. In the presence of a weak gravitational field, the rate of change of P as the particle moves along a geodesic is $(u^\nu u^\mu_{;\nu}=0)$

$$dP/d\tau = u^\nu(\xi_\mu u^\mu)_{;\nu} = u^\mu u^\nu \xi_{\mu;\nu}.$$

Therefore, the characteristic proper time during which the presence of the gravitational field leads to a finite change ΔP of the momentum is of the order of the momentum itself if $T \sim u^\mu \xi_\mu / u^\mu u^\nu \xi_{(\mu;\nu)}$. Let L be the distance traversed by the particle during this time. If L is much greater than the typical dimension l of the laboratory, the observable symmetry breaking is small: $l/L \ll 1$. In asymptotically flat space, L increases as one moves away from the gravitational field source, and therefore the breaking of the exact symmetry inherent in flat spacetime becomes ever less and less. For a precise definition of the concept of asymptotic symmetry it is, as usual, convenient to begin with the corresponding examination in the Penrose space.

Asymptotic Symmetries and the Group of Conformal Isometries on $\mathcal{J}^\pm$. The existence of asymptotic symmetries means that in the physical spacetime there must exist vector fields ξ^μ which are such that as one moves further away from the gravitational field sources the relations (31) are satisfied with ever greater accuracy. Therefore, in the corresponding Penrose space $(\hat{M}, \hat{g}, \Omega)$ these vector fields must satisfy Eqs. (33) in the neighborhood of $\mathcal{J}^\pm$, and the following definition can be given. A diffeomorphism of the Penrose space generated by the vector field ξ^μ that on $\mathcal{J}^\pm$ satisfies the equation[24]

$$[\hat{\nabla}^{(\nu}\xi^{\mu)} - \Omega_{,\alpha}\xi^\alpha \hat{g}^{\mu\nu}/\Omega]\,|_{\mathcal{J}^\pm} = 0, \tag{35}$$

is called an *asymptotic symmetry* transformation. Comparing (35) and (33), we see that such transformations are *conformal isometries* on $\mathcal{J}^\pm$. A direct consequence of Eq. (35) and the regularity of ξ^μ and $\hat{g}^{\mu\nu}$ on $^\pm$ is the vanishing on $\mathcal{J}^\pm$ of $\Omega_{,\alpha}$ and ξ^α, and, therefore, the vector field ξ^α is tangent to the surface $\mathcal{J}^\pm$. In the Bondi conformal coordinates $\Omega = \hat{r}$ and $0 = \Omega_{,\alpha}\xi^\alpha\,|\mathcal{J}^+$. Evaluating the indeterminate form $\Omega_{,\alpha}\xi^\alpha/\Omega$ on $\mathcal{J}^+$, we obtain $\Omega_{,\alpha}\xi^\alpha/\Omega = \xi^1/\hat{r} = \partial_{\hat{r}}\xi^1$, and therefore Eq. (35) can be written in the form

$$(\hat{g}^{\mu\rho}\xi^\nu_{,\rho} + \hat{g}^{\nu\rho}\xi^\mu_{,\rho} - \hat{g}^{\mu\nu}_{,\rho}\xi^\rho - 2\hat{g}^{\mu\nu}\partial_{\hat{r}}\xi^1)\,|_{\mathcal{J}^\pm} = 0.$$

The system of equations satisfied by the vector field ξ^α on $\mathcal{J}^\pm$ can be written in component form as follows:

$$\left.\begin{array}{l} \text{a) } \partial_{\hat{r}}\xi^0 = 0;\ \text{b) } \partial_u\xi^0 = \partial_{\hat{r}}\xi^1;\ \text{c) } \partial_{\hat{r}}\xi^A = -\hat{g}^{AB}\partial_B\xi^0; \\ \text{d) } \partial_u\xi^1 = 0;\ \text{e) } \partial_u\xi^A = 0;\ \text{f) } \hat{g}^{AB}\partial_{\hat{r}}\xi^1 = \xi^{(A:B)}. \end{array}\right\} \tag{36}$$

Here, $\partial_A \equiv \partial/\partial x^A$ and the colon denotes the covariant derivative on the two-dimensional surface with the metric $\hat{g}_{AB}$.

This system can be integrated as follows. Differentiating equation b) with respect to u and using d), we obtain $\partial_u^2\xi^0 = 0$. Therefore, with allowance for a) we have $\xi^0 = F(x^A)(u + \alpha(x^A))$. The relations b) and f) give $F = \xi^A_{:A}/2$ and $\xi^{(A:B)} = \hat{g}^{AB}\xi^C_{:C}/2$. It follows from e) that ξ^A does not depend on u. Thus, on $\mathcal{J}^\pm$ the vector field ξ^μ generating the asymptotic symmetry transformations has the form

$$\xi^1 = 0;\ \xi^0 = 1/2\,\xi^A_{:A}(u + \alpha(x^A));\ \xi^A = \overset{0}{\xi}{}^A(x^B), \tag{37}$$

the functions ξ^{0A} satisfying the equation

$$\overset{0}{\xi}{}^{(A:B)} = \hat{g}^{AB}\overset{0}{\xi}{}^C_{:C}/2. \tag{38}$$

The remaining unused equations of the system (36) enable us to determine the quantities $(\partial_{\hat{r}}\xi^\alpha)\,|\mathcal{J}^\pm$:

$$\begin{array}{l} \partial_{\hat{r}}\xi^0 = 0;\ \partial_{\hat{r}}\xi^1 = \xi^A_{:A}/2; \\ \partial_{\hat{r}}\xi^B = -(\xi^A_{:A}(u+\alpha)/2)^{:B}. \end{array} \tag{39}$$

The second and higher derivatives with respect to $\hat{r}$ of ξ^μ are not fixed by Eq. (35), so that the dependence of ξ^μ on $\hat{r}$ off $\mathcal{J}^\pm$ is in general arbitrary. The classes of asymptotic symmetry transformations that coincide on $\mathcal{J}^\pm$ are elements of the *group of asymptotic symmetries*.

We now establish what are the parameters on which the transformations of this group depend. In the Bondi conformal coordinates we are using, the metric $\hat{g}_{AB}\,|\mathcal{J}^\pm$ coincides with the metric on the surface of the unit sphere, and Eq. (38), like (32), determines the generators of the conformal transformations on this sphere [see (58)]. It is well known[9, 25] that the group of conformal isometries of the sphere is locally isomorphic to the (six-parameter) proper Lorentz group. Besides the functions $\xi^A(x^B)$, the complete group of asymptotic symmetries contains, as parameters, the arbitrary function $\alpha(x^A)$ on the sphere, and is therefore an infinite-dimensional group. This group has been called the Bondi–Metzner–Sachs group (abbreviated BMS group).

It follows from a comparison of (37) with (27) that the BMS group is isomorphic to the group of transformations from one system of Bondi conformal coordinates to another.

Asymptotic Symmetries in Physical Spacetime. Equations (37)–(39) show that the regular vector fields ξ^α which generate the asymptotic symmetries satisfy in the neighborhood of $\mathcal{J}^\pm$ the equations

$$\begin{array}{l} \xi^{\hat{r}} = \overset{0}{\xi}{}^A_{:A}\hat{r}/2 + O(\hat{r}^2);\ \xi^0 = \overset{0}{\xi}{}^A_{:A}(u + \alpha(x_B))/2 + O(\hat{r}^2); \\ \xi^B = \overset{0}{\xi}{}^B(x^A) - [\overset{0}{\xi}{}^A_{:A}(u + \alpha(x^B))]^{:B}\hat{r}/2 + O(\hat{r}^2). \end{array}$$

In the coordinates (u, r, x^2, x^3), where $r = 1/\hat{r}$ in physical spacetime in the limit $r \to \infty$

$$\begin{array}{l} \xi^r = -\overset{0}{\xi}{}^A_{:A}r/2 + O(1);\ \xi^0 = \overset{0}{\xi}{}^A_{:A}(u + \alpha(x^B))/2 + O(r^{-2}); \\ \xi^B = \overset{0}{\xi}{}^B(x^A) - [\overset{0}{\xi}{}^A_{:A}(u + \alpha(x^C))]^{:B}r^{-1} + O(r^{-2})/2. \end{array}$$

If we now use the expression (28) for the asymptotic behavior of the coefficients of the metric, we can show that in the coordinates (u, r, x^2, x^3) the quantities $\bar{\delta}g_{\mu\nu} \equiv -\xi_{(\mu;\nu)}$, which characterize the departure from exact symmetry, have the form

$$\left.\begin{array}{l} \bar{\delta}g_{11} = \bar{\delta}g_{1A} = g^{AB}\bar{\delta}g_{AB} = 0;\ \bar{\delta}g_{00} = O(r^{-1}); \\ \bar{\delta}g_{0A} = O(1);\ \bar{\delta}g_{01} = O(r^{-2});\ \bar{\delta}g_{AB} = O(r). \end{array}\right\} \tag{40}$$

The fulfillment of these asymptotic Killing equations can be taken as the basis for the definition of asymptotic symmetries (see, for example, Ref. 9). Transformations ξ^μ satisfying the conditions (40) preserve the asymptotic form of the metric (28). One can also show that the parameter $L \sim T \sim u^\mu \xi_\mu / u^\mu u^\nu \bar{\delta} g_{\mu\nu}$, which characterizes the size of the region in which the breaking of the exact symmetry becomes of order unity, increases unboundedly as $r \to \infty$ for vector fields ξ^μ satisfying Eqs. (40).

6. STRUCTURE AND REPRESENTATIONS OF THE BONDI-METZNER-SACHS GROUP

It was shown in the preceding sections that an asymptotically flat space admits approximate symmetries whose characteristic property is conservation of the form of the metric in the neighborhood of the null infinities $\mathcal{J}^\pm$. It is easy to verify that these transformations on $\mathcal{J}^+$ (for $\mathcal{J}^-$ all the results are analogous), namely, $(u, \zeta, \bar{\zeta}) \xrightarrow{B} (u', \zeta', \bar{\zeta}')$, form the group of asymptotic BMS symmetries[9,14]:

$$\left.\begin{array}{l} \zeta' = (a\zeta + b)/(c\zeta + d); \;\; u' = K_g(\zeta, \bar{\zeta})\,(u + \alpha(\zeta, \bar{\zeta})); \\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, C); \\ K_g(\zeta, \bar{\zeta}) = \dfrac{1 + |\zeta|^2}{(|a\zeta + b|^2 + |c\zeta + d|^2)}. \end{array}\right\} \tag{41}$$

Here, $\alpha(\zeta, \bar{\zeta})$ is a twice differentiable real function of $\zeta = \exp(i\varphi)\cot\theta/2$, where θ and φ are angular coordinates on S^2. For comparison, we recall that the Poincaré group $\mathcal{P}$, which describes the symmetries of Minkowski space, induces on $\mathcal{J}^+$ the transformation (see Sec. 1)

$$\zeta' = \frac{a\zeta + b}{c\zeta + d}; \;\; u' = K_g(\zeta, \bar{\zeta})\Big(u + a^0 + a^1 \frac{\zeta + \bar{\zeta}}{(1 + |\zeta|^2)} + a^2 \frac{\zeta - \bar{\zeta}}{i(1 + |\zeta|^2)} + a^3 \frac{|\zeta|^2 - 1}{(1 + |\zeta|^2)}\Big), \tag{42}$$

where a^μ is a translation vector, and $\zeta \to \zeta'$ is induced by Lorentz rotations.

The transformation (42) is obviously a special case of the transformation (41), i.e., the BMS group contains the transformations induced on $\mathcal{J}^+$ by the Poincaré group.

Why is the BMS group larger than the Poincaré Group? This question arises because Minkowski space and an asymptotically flat space are "constructed in the same way" at infinity: on $\mathcal{J}^\pm$, where the BMS group and the group $\mathcal{P}$ are compared, they generate one and the same degenerate metric:

$$dl^2 = d\zeta\, d\bar{\zeta}/(1 + |\zeta|^2). \tag{43}$$

This apparent contradiction can be resolved if one recalls that the group $\mathcal{P}$ is determined by the stringent requirement that it preserve the form of the metric $\eta^{\mu\nu}$ in the whole of Minkowski space, while the BMS group is determined by the requirement that it preserve (with a certain given accuracy) the form of the metric $g^{\mu\nu}$ of the asymptotically flat space only in the neighborhood of the null infinities $\mathcal{J}^\pm$. It is natural that a group which preserves the form of the metric only on $\mathcal{J}^\pm$ [see (43)] should be an even larger group. Indeed, it is readily verified that this group is determined by the transformation (the Newman–Unti group[18,26]) $(u, \xi, \bar{\xi}) \to (u', \xi', \bar{\xi}')$:

$$\zeta' = (a\zeta + b)/(c\zeta + d); \quad u' = G(u, \zeta, \bar{\zeta}),$$

where, as before, the matrices $\binom{a\ b}{c\ d} \in SL(2, C)$, and G is an arbitrary smooth real function. The arguments given above can be tested on a simple example[27] that simultaneously demonstrates the extent to which a concept such as asymptotically flat space is nontrivial. We consider first the isometry group of the two-dimensional plane E^2 with ordinary positive-definite metric $g^{ij}(x)$, which in polar coordinates $(x^1 = r, x^2 = \theta)$ has the form $g^{ij}(r, \theta) = \binom{1\ \ 0}{0\ r^{-2}}$. In this case, the Killing equations (31) have the solution

$$\begin{aligned} \xi^1 &= \omega^{(a,b)}(\theta) = a\cos\theta + b\sin\theta; \\ \xi^2 &= c + r^{-1}\omega^{(a,b)}_{,2}(\theta), \end{aligned} \tag{44}$$

where a, b, and c are constants of integration. The three linearly independent fields $\xi^i(a)$, $\xi^i(b)$, and $\xi^i(c)$ determine an infinitesimal group of isometric motions of the plane E^2, the parameters a and b determining translations and c rotations.

We now consider a two-dimensional Riemannian manifold R^2 with positive-definite metric $g^{ij}(x)$. Suppose $r(x)$ and $\theta(x)$ are real differentiable functions on R^2 which define new coordinates that satisfy "polarity" conditions:

$$g^{ij} r_{,i} r_{,j} = 1; \quad g^{ij} r_{,i} \theta_{,j} = 0. \tag{45}$$

In these coordinates $(\bar{x}^1 = r, \bar{x}^2 = \theta)$ the metric tensor satisfies $g^{ij}(r, \theta) = \binom{1\ \ \ 0}{0\ f(\theta, r)}$, where $f(\theta, r)$ is a differentiable function. On the metric, we now impose conditions that would simulate for us an asymptotically flat space [cf (28)]:

1) the chart $\{r, \theta\}$ covers the complete manifold R^2 except for a bounded region $0 < r_0 \leq r < \infty, 0 \leq \theta < 2\pi$;

2) for $r_0 \leq r$, we have $f(r, \theta) = r^{-2} + O(r^{-\lambda})$; $\partial_\theta f(r, \theta) = O(r^{-q})$, with $\lambda > 2$;

3) for $r_0 \leq r$, a condition of uniform smoothness is satisfied [see Eq. (30)]: $\partial_r[f(r, \theta) - r^{-2}] = O(r^{-q})$; $\partial_\theta f(r, \theta) = O(r^{-q})$, $q \geq \lambda$. In Cartesian coordinates, $g^{ij}(x) = \binom{1\ 0}{0\ 1} = \eta^{ij}$.

We now note that in general R^2 has no nontrivial isometries. This follows directly from the fact that the Killing equations in this case do not have nontrivial solutions. However, if we are interested in *asymptotic isometries*, for which the metric is preserved only asymptotically $(r \geq r_0)$[6]:

$$\mathcal{L}_\xi g^{11} = 0; \quad \mathcal{L}_\xi g^{12} = \mathcal{L}_\xi g^{21} = 0; \quad \mathcal{L}_\xi g^{22} = Q_\lambda(r, \theta), \tag{46}$$

then nontrivial solutions (46) exist, and for large r they

[6]Here, $Q_\lambda(r, \theta)$ is an arbitrary function that in the limit $r \to \infty$ is $O(r^{-\lambda})$.

do not depend on the concrete form of the term $Q_\lambda(r,\theta)$ but are determined solely by the "rate" at which the metric $g^{ij}(r,\theta)$ tends to the flat metric; for it follows from the first two equations of (46) that for $r_0<r$

$$\xi^1=\omega(\theta);\quad \xi^2=\pi(\theta)+r^{-1}\omega_{,2}(\theta)+O(r^{-\mu}),\quad \mu>1, \tag{47}$$

where $\omega(\theta)$ and $\pi(\theta)$ are arbitrary differentiable functions. From the third equation of (46), we obtain

$$2r^{-2}\pi_{,2}(\theta)+2r^{-3}[\omega_{,22}(\theta)+\omega(\theta)]+O(r^{-q})=Q_\lambda(r,\theta),\quad q\geqslant\lambda. \tag{48}$$

It follows from (48) in the limit $r\to\infty$ that $\pi(\theta)=c$. At the same time, if $2<\lambda\leqslant 3$, then, in contrast to the "flat" case E^2, we do not obtain any conditions on the function $\omega(\theta)$:

$$\xi^1=\omega(\theta);\quad \xi^2=c+r^{-1}\omega_{,2}(\theta)+O(r^{-\mu}),\quad 1<\mu\leqslant 2. \tag{49}$$

Comparing (49) with (44), we see that the two-parameter family of "translations" $\omega^{(a,b)}(\theta)$ in the case of asymptotic isometries of asymptotically flat space R^2: $2<\lambda\leqslant 3$, *can be extended* to the infinite-parameter family of "supertranslations" $\omega(\theta)$. The same thing happens in the case of asymptotically flat spacetime [cf Eqs. (41) and (42)].

We now note that if $\lambda>3$ in conditions 2) and 3) the solution of the Killing equations (46) for large $r>r_0$ has the form [cf Eq. (44)]:

$$\xi^1=\omega^{(a,b)}(\theta);\quad \xi^2=K+r^{-1}\omega^{(a,b)}_{,2}(\theta)+O(r^{-\mu}),\quad \mu>2. \tag{50}$$

Therefore, if the metric becomes flat "too fast" with increasing r (in the present example $\lambda>3$, and for the physical spacetime this corresponds to the absence of gravitational radiation at infinity[27,28]), then the group of asymptotic isometries cannot be extended but coincides with the asymptotic isometries of flat space. Thus, in the case of asymptotically flat spacetime, as it was defined above (Sec. 3), the group of asymptotic symmetries is extended to the BMS group because of the "slow" tending of the metric $g^{\mu\nu}(x)$ to the flat-space metric $\eta^{\mu\nu}$. The transformations (41) of the form $(u,\zeta,\bar\zeta)\to(u'=u+\alpha(\zeta,\bar\zeta),\zeta,\bar\zeta)$ form an Abelian subgroup of the supertranslations A, and the supertranslations, for which

$$\alpha(\zeta,\bar\zeta)=a^0+a^1(\zeta+\bar\zeta)/(1+|\zeta|^2)+a^2(\zeta-\bar\zeta)/i(1+|\zeta|^2)+a^3(|\zeta|^2-1)/(1+|\zeta|^2), \tag{51}$$

form the four-parameter Abelian subgroup V of translations in the BMS group.

Structure of the BMS Group (Refs. 9 and 27). 1. Semidirect product. Absence of local compactness. It follows from the definition (41) of the BMS group that it has the structure of the semidirect product of the supertranslations A and the group $G=SL(2,C)$:

$$\mathscr{B}=A\otimes_T G. \tag{52}$$

Here, T indicates that the semidirect product is taken with respect to the automorphism $A\xrightarrow{T(G)}A$, i.e., on A there is defined a representation $T(G)$ of the group G [cf Eq. (41)]:

$$\left.\begin{array}{l}(T(g)\alpha)(\zeta,\bar\zeta)=K_g(\zeta,\bar\zeta)\,\alpha(\zeta g,\overline{\zeta g});\\ g\in G;\quad \zeta g\equiv(a\zeta+b)/(c\zeta+d).\end{array}\right\} \tag{53}$$

The group $\mathscr{B}$ consists of the elements of $A\times G$ with the following law of composition:

$$\left.\begin{array}{l}(\alpha_1,g_1)(\alpha_2 g_2)=(\alpha_1+T(g_1)\alpha_2,\ g_1g_2);\\ \alpha_1,\ \alpha_2\in A;\quad g_1,\ g_2\in G.\end{array}\right\} \tag{54}$$

If $g=\pm I$, i.e., if g belongs to the center Z_2 of G, then $T(\pm I)=I$. The factor group $L_+^\uparrow=G/Z_2$ is isomorphic to the proper orthochronous Lorentz group $L_+^\uparrow$, so that besides the universal covering group $\mathscr{B}$ (52) one can introduce the proper group $\mathscr{B}_+^\uparrow=A\otimes_T L_+^\uparrow$. Finally, the Poincaré group $\mathscr{P}$ (42) also has the semidirect product structure:

$$\mathscr{P}=V\otimes_T G, \tag{55}$$

where V is the four-parameter subgroup of translations $V\subset A$, and therefore $\mathscr{P}\subset\mathscr{B}$. We now note that, in contrast to the ten-parameter group $\mathscr{P}$, the group $\mathscr{B}$ is an infinite-parameter group since the elements α of the group A belong to the Hilbert space of functions that are square integrable on the two-dimensional sphere S^2. Therefore, the subgroup $A\otimes_T I$ (in the natural topology) is locally isomorphic to the Hilbert space, so that the group $\mathscr{B}$ is not locally compact. That the BMS group is not locally compact means that the problem of finding its irreducible unitary representations is harder than in the case of the Poincaré group.

2. Lie algebra. The algebra of infinitesimal operators for the group $\mathscr{B}$ can be constructed by means of the explicit expressions (41) for the BMS transformations in accordance with the usual rules.[9] We recall only that one can choose the normalized spherical harmonics $Y_{lm}(\theta,\varphi)$ as a complete orthonormal basis in the Hilbert space of the supertranslations A. Then an arbitrary element $\alpha(\theta,\varphi)\in A$ can be represented in the form of the expansion[7)]

$$\left.\begin{array}{l}\alpha(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}a_{lm}Y_{lm}(\theta,\varphi);\\ \bar a_{lm}=a_{l-m}.\end{array}\right\} \tag{56}$$

For the group A, the linearly independent set of infinitestimal operators has the form

$$\left.\begin{array}{l}P_{lm}=Y_{lm}(\theta,\varphi)\,\partial/\partial u;\\ P_{lm}=P^*_{l-m};\quad P_{lm}=0;\quad |m|>l,\end{array}\right\} \tag{57}$$

and the general linear combination is

$$P=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}b_{lm}P_{lm}=\beta(\theta,\varphi)\frac{\partial}{\partial u}. \tag{58}$$

For the six-parameter proper conformal subgroup the generators L^{ab} (corresponding to infinitesimal rotations

[7)]We recall that $\alpha(\theta,\varphi)$ is a twice differentiable function on S^2.

in the $\{x^a, x^b\}$ plane of Minkowski space) can be conveniently represented in the form

$$\left.\begin{aligned}&L^{12}=\partial/\partial\varphi\equiv L_z;\quad L^{30}=\sin\theta\,(\partial/\partial\theta)+u\cos\theta\,(\partial/\partial u)\equiv R_z;\\&L^{\pm}=\pm\,\mathrm{i}L^{23}+L^{13}=\exp(\pm\,\mathrm{i}\varphi)\,[\partial/\partial\theta\pm\mathrm{i}\,\mathrm{ctg}\,(\partial/\partial\varphi)];\\&R^{\pm}=\mp\,\mathrm{i}L^{20}+L^{10}\\&=-\exp(\pm\,\mathrm{i}\varphi)\,[\cos\theta\,(\partial/\partial\theta)\pm\mathrm{i}\,\mathrm{cosec}\,\theta\,(\partial/\partial\varphi)-u\sin\theta\,(\partial/\partial u)].\end{aligned}\right\}\quad(59)$$

From (58) and (59) we can find the commutation relations for the Lie algebra of the BMS group (or $\mathcal{B}$):

$$\left.\begin{aligned}&[L^{ab}, L^{cd}]=-\eta^{ad}L^{bc}-\eta^{bc}L^{ad}+\eta^{ac}L^{bd}+\eta^{bd}L^{ac};\\&[L^{ab}, \alpha\partial/\partial u]=\{(L^{ab}\alpha)-\alpha(\theta,\varphi)\,W(L^{ab})\}\,\partial/\partial u,\end{aligned}\right\}\quad(60)$$

where $W(L^{ab})$ is determined by the relation

$$\partial(L^{ab}f)/\partial u=L^{ab}\,\partial f/\partial u+W\,\partial f/\partial u;\quad(61)$$

here, $f(u)$ is an arbitrary smooth function.

3. Normal subgroups. For any group Γ, an important property which determines its representations (and sometimes also a way of finding them) is its possession of normal subgroups (normal divisors). A set $N\subset\Gamma$ is a normal subgroup if for arbitrary $g\in\Gamma$ in $n\in N$ it follows that $g^{-1}ng\in N$. In the case of the BMS subgroup it follows from the definition (53)–(54) that the supertranslations $A\otimes_T I$ form a normal Abelian subgroup of $\mathcal{B}$, and the factor group $\mathcal{B}/A\otimes_T Z_2$ is isomorphic to the proper orthochronous Lorentz group. Further, it follows from the representation (56) that the space of the supertranslations A is the orthogonal sum of the translations V and the genuine supertranslations Σ:

$$A\approx V\oplus\Sigma;\quad(62)$$

$$V: v(\theta,\varphi)=\sum_{l=0}^{1}\sum_{m=-l}^{l}a_{lm}Y_{lm}(\theta,\varphi)$$
$$=a^0+a^1\sin\theta\cos\varphi+a^2\sin\theta\sin\varphi+a^3\cos\theta;$$
$$\Sigma: \sigma(\theta,\varphi)=\sum_{l=2}^{\infty}\sum_{m=-l}^{l}a_{lm}Y_{lm}(\theta,\varphi).$$

The translations V commute with the supertranslations Σ and, as can be readily verified by means of the commutation relations (59), are invariant under the action of the representation $T(G)$ of the conformal group G (53). Therefore, the translations $V\otimes_T I$ also form a normal Abelian subgroup, which is a four-parameter group. It is, however, remarkable[9, 27] that $V\otimes_T I$ is the only four-parameter normal Abelian subgroup of the group $\mathcal{B}$ (or $\mathcal{B}_+^\uparrow$). This property of $V\otimes_T I$ uniquely distinguishes the subgroup of translations in the BMS group, and this is very important for constructing quantum theory in asymptotically flat spaces (see, for example, Refs. 34 and 35).

Unitary Representations of the BMS Group and their Properties. 1. The method of induced representations.[36-38] The method of induced representations is an effective tool for constructing unitary representations of groups that are a semidirect product $N\otimes_T H$, with N a normal Abelian subgroup. As we have seen, the Poincaré group and the BMS group, whose unitary representations will be compared below, are of this kind. Let us recall how induced representations are constructed.

First, one constructs continuous irreducible representations of the subgroup N; these representations are one-dimensional and form the Abelian group $\hat{N}$ (*characters*) dual to N. The action of the group H on N determines the action of H on $\hat{N}$ as well. For given character $\chi\in\hat{N}$ the largest subgroup $H_\chi\subset H$ which leaves χ fixed is called the little group for χ, and the set $\mathcal{O}_\chi=H\chi$ of characters which can be obtained from χ by applying H is called the orbit of χ. The main assertion[36-37] is that a given character χ and representation U of the little group H_χ uniquely (to within unitary equivalence) determine a representation of the group $N\otimes_T H$, which is unitary if there is defined on the orbit $\mathcal{O}_\chi$ a measure which is quasi-invariant under the action of the group H. Such a representation of $N\otimes_T H$ is called the *representation induced* by U and χ. If $N\otimes_T H$ is locally compact and U is irreducible, then the induced representation is irreducible. If the semidirect product $N\otimes_T H$ is *regular* in the sense of Mackey,[36] then any irreducible representation of $N\otimes_T H$ is an induced representation. Thus, the irreducible representations of $N\otimes_T H$ can be classified in accordance with the irreducible representations U_s of the little group H_χ and the corresponding orbit $\mathcal{O}_\chi$.

2. Representations of the Poincaré Group. The Poincaré group $\mathcal{P}$ is a regular semidirect product of V and G (55), and therefore all irreducible representations of $\mathcal{P}$ are induced representations. The characters $\hat{V}$ are determined by the points of the vector space dual to V (the momentum space): $V'\ni p=(p^0,p^1,p^2,p^3)$: $\chi_p=\exp[i(pv)]$. The action of G on χ_p conserves $p^2=m^2$, i.e., in the given case there are only three little groups G_χ corresponding to the three types of characters: $\chi_{p^2>0}, \chi_{p^2=0}, \chi_{p^2<0}$, and, accordingly, three orbits $\mathcal{O}_\chi$, into which the space of characters $\hat{V}$ is fibered. The irreducible unitary representations of the little groups, $G_\chi-U_s$, together with the type of orbit determine all the irreducible unitary representations of the Poincaré group $\mathcal{P}$, the index s specifying the spin of the representation (Table I).

In Table I, Δ consists of triangle matrices whose diagonal elements are in modulus equal to unity, and $SU(1,1)$ is the noncompact group of pseudounitary matrices. The representations U_s with continuous spin are usually rejected as unphysical.

3. Induced representations of the BMS group. The BMS group is also the semidirect product (52), but nothing is known about its regularity. Therefore, the induced representations need not exhaust all possible irreducible unitary representations of the

TABLE I.

$\mathcal{O}_\chi$	G_χ	s
$p^2=m^2>0$; sign $p^0=\pm1$	$SU(2)$	Discrete
$p^2=m^2=0$; sign $p^0=\pm1$	Δ	Continuous, but there is a discrete subclass
$p^2=m^2<0$	$SU(1,1)$	Continuous

TABLE II.

Φ_0	G_χ	Θ_χ
$\Phi_0(\zeta, \bar{\zeta})$	Z_2	$G\Phi_0 \approx G/Z_2 \approx L_+^\uparrow$
$\Phi_0 = \xi(\lvert\zeta\rvert)$	Γ	$G\Phi_0 \approx G/\Gamma \approx L_+^\uparrow/SO(2)$
$\Phi_0 = K$ (const)	$SU(2)$	$G\Phi_0 \approx G/SU(2) \approx L_+^\uparrow/SU(2)$

group $\mathfrak{B}$. In addition, $\mathfrak{B}$ is not *locally compact*, and therefore the induced representations constructed in accordance with the above method need not necessarily lead to irreducible representations. However, McCarthy[30-33] has constructed all induced unitary representations of $\mathfrak{B}$ and shown that they are irreducible.

For the Abelian group of supertranslations A, the irreducible representations, as for V, are one-dimensional and form the group of characters $\hat{A} \ni \chi\Phi = \exp(i\langle\Phi, \alpha\rangle)$, where $\Phi \in A$ defines a functional in the dual space A' with respect to the scalar product $\langle\Phi, \alpha\rangle$ on the sphere S^2: $\langle\Phi, \alpha\rangle = \int_{S^2} d\mu(x)\Phi(x)\alpha(x)$; $x \in S^2$; $d\mu(x) = 1/4\pi \sin\theta\, d\theta\, d\varphi$. McCarthy has shown that for $\mathfrak{B}$ the little groups G_χ are determined by the condition [see Eqs. (53) and (41)]

$$(T_g\Phi_0)(\zeta, \bar{\zeta}) = K_g^{-3}(\zeta, \bar{\zeta})\,\Phi_0(\zeta g, \overline{\zeta g}), \tag{63}$$

and that they are all compact. Therefore, none of the irreducible unitary representations of $\mathfrak{B}$ contain unphysical representations with *continuous spin*. From the point of view of physics, this property of the BMS group renders it more suitable for describing elementary particles than the Poincaré group.[30] It may seem strange that although $\mathcal{P} \subset \mathfrak{B}$ the little groups corresponding to $\mathfrak{B}$ do not contain the Poincaré little groups, which include a noncompact group. The point is that for the group $\mathcal{P}$ the momentum space V' dual to the translations V (with respect to the scalar product in Minkowski space) is invariant under the action of the group G. For the BMS group in the space A' dual to the supertranslations A with respect to the scalar product on the sphere S^2 the momentum subspace $V' \subset A'$ is not G-invariant. It occurs only as the factor space A'/V^0, where V^0 is the G-invariant annihilator of V corresponding to the orthogonal decomposition $A' \approx \Sigma^0 \oplus V^0$:

$$\Phi(\theta, \varphi) = (p^0 + p^1 \sin\theta\cos\varphi + p^2 \sin\theta\sin\varphi + p^3\cos\theta) + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} p_{lm} Y_{lm}(\theta, \varphi).$$

As McCarthy has shown,[27] the BMS group is the smallest extension of the Poincaré group that leads to the disappearance of the irreducible unitary representations with continuous spin.

In Table II, we give the orbits and little groups for $\mathfrak{B}$ analogous to the table given for $\mathcal{P}$.[8)]

In Table II, to characterize the orbits, we have used the fact that every $\Theta_{\chi(\Phi_0)}$ is a homogeneous space with respect to the action of G, and therefore it is homeomorphic to the corresponding factor space $G/G_{\chi(\Phi_0)}$ with respect to the little group. The little groups and the orbits uniquely classify the induced unitary representations of the BMS group (for details, we refer the reader to Refs. 29–33).

8)For the nonconnected little groups the analogous table can be found in Refs. 31–33.

7. MASSLESS FIELDS IN ASYMPTOTICALLY FLAT SPACETIME

The concept of an asymptotically flat space introduced above is very convenient for discussing scattering problems for massless fields. The universal nature of the behavior of these fields in the asymptotic region enables one, by means of a conformal transformation, to go over from the scattering problem in the physical spacetime to a problem with initial data on the null surface $\mathcal{J}$ in the Penrose space. It can then be shown that the important property of asymptotic degeneracy (cf. Sec. 1) follows from the regularity of the behavior of the conformally transformed field on $\mathcal{J}$. As we have already said, the spinor formalism is the most convenient for describing massless fields. Spinor analysis in curved spacetime and its application to describe the properties of massless fields in asymptotically flat space are discussed in Refs. 3, 7, 22, and 23.

Spinor Formalism in Curved Space. Spinors are introduced at each point of spacetime in the same way as in the flat case. The curvature of spacetime is manifested as one goes from point to point. Tensors and spinors are connected by means of the Hermitian quantities $\sigma^\mu_{A\dot{B}}$, but now they depend on x. They have the following basic properties:

$$\sigma^\mu_{A\dot{B}}\sigma^\nu_{C\dot{D}}g_{\mu\nu} = \varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}};$$

$$\sigma^{A\dot{B}}_\mu\sigma^{C\dot{D}}_\nu g^{\mu\nu} = \varepsilon^{AC}\varepsilon^{\dot{B}\dot{D}};$$

$$g_{\mu\nu}(x) = \sigma^{A\dot{B}}_\mu(x)\,\sigma^{C\dot{D}}_\nu(x)\,\varepsilon_{AC}\varepsilon_{\dot{B}\dot{D}}.$$

Here $\|\varepsilon_{AB}\| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. One defines the derivative $\nabla^{A\dot{B}} = \sigma^{A\dot{B}}_\mu\nabla^\mu$ and postulates that the covariant derivative acts on the matrices $\varepsilon_{A\dot{B}}$ and $\sigma^{A\dot{B}}_\mu$ as follows: $\nabla_\mu\varepsilon_{AB} = 0$, $\nabla_\mu\sigma^\nu_{A\dot{B}} = 0$.

Massless Fields of Spin s. A massless field of arbitrary spin s is described, as in the flat case, by a symmetric spinor $\varphi_{A_1 \ldots A_s}$ which satisfies the equation

$$\nabla^{A_1\dot{B}}\varphi_{A_1 \ldots A_{2s}} = 0 \tag{64}$$

(for higher spins, the well known difficulties in the interpretation of this equation in curved spacetime arise). The scalar massless field φ satisfies the equation

$$(\Box - \alpha R)\,\varphi \equiv (\nabla_\mu\nabla^\mu - \alpha R)\,\varphi = 0, \tag{65}$$

where α is some constant; R is the curvature. It is well known that Eqs. (64) are conformally invariant [Eq. (65) is conformally invariant for $\alpha = -1/6$]. The properties of various quantities under the conformal transformation $g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ are discussed in Appendix 2.

Asymptotic Degeneracy. If one requires that the field $\varphi_{A_1\cdots A_{2s}}$ be *asymptotically regular*, i.e., that the conformally transformed field $\hat{\varphi}_{A_1\cdots A_{2s}}=\Omega^{-1}\varphi_{A_1\cdots A_{2s}}$ be continuous on $\mathcal{J}$ [this corresponds to the behavior $\varphi\sim\Phi r^{-1}$, $r\to\infty$ (cf. Sec. 1)], then one can show that for any null geodesic γ and spin basis o^A, ι^A transported parallel along γ with spinor o^A tangent to γ

$$\lim_{r\to\infty}(r^k\varphi_{A_1\ldots A_{2s}}o^{A_1}\ldots o^{A_k})=0; \tag{66}$$

here, r is an affine parameter on γ. The condition (66) means that at least $2s-k+1$ of the principal null directions of $\varphi_{A_1\cdots A_{2s}}$ coincide to order $O(r^{-(k+1)})$.

For an asymptotically regular field $\varphi_{A_1\cdots A_{2s}}$ there exists a limit along the null geodesic γ:

$$\Phi=\lim_{r\to\infty}(r\varphi_{A_1\ldots A_{2s}}\iota^{A_1}\ldots\iota^{A_{2s}}), \tag{67}$$

and the value of this limit is the same for all null geodesics γ which terminate at the same point on $\mathcal{J}$. The Φ are called the image of the field $\varphi_{A_1\cdots A_{2s}}$ on $\mathcal{J}$. In particular, for a scalar field

$$\Phi=\lim_{r\to\infty}(r\varphi)=\hat{\varphi}|_{\mathcal{J}}. \tag{68}$$

Remarks. 1. Note that for the theory presented here (in particular, for the application of the Penrose-space techniques) it is not so much conformal invariance (which is absent when $\alpha\neq 1/6$) which is important as rather universality of the field behavior as $r\to\infty$.

2. If one considers the field φ in the Schwarzschild metric (16), then $\varphi(r,u,\theta,\varphi)\sim\Phi(u,\theta,\varphi)/r$, where $u=t-r-2m\ln|r-2m|$, as $r\to\infty$. The logarithm is the analog of the Coulomb phase shift for scattering on a Coulomb potential in quantum mechanics and it arises because of the slow decrease of the gravitational field in the asymptotic region $[g_{\mu\nu}\simeq\eta_{\mu\nu}+O(r^{-1})]$.

3. It is easy to verify the invariance of the Yang-Mills equations (see Appendix 2):

$$\nabla^\mu F^a_{\mu\nu}=gf^{abc}F^b_{\nu\lambda}A^{c,\lambda};$$
$$F^a_{\mu\nu}=\nabla_\mu A^a_\nu-\nabla_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu,$$

under the conformal transformation $g\to\hat{g}$, $A^a_\mu\to\hat{A}^a_\mu=A^a_\mu$. It would be interesting to analyze Yang-Mills theory more fully in the spirit of the ideas presented in the present review.

Asymptotic Invariants. If $\xi^\mu_{(i)}$ are the generators of symmetry transformations of the spacetime, then the invariants associated with them can be written in the form (35):

$$P_i=\int_\Sigma T_{\mu\nu}\xi^\mu_{(i)}\,d\Sigma^\nu.$$

This expression can be rewritten identically as follows:

$$P_i=\int_\Sigma \hat{T}_{\mu\nu}\xi^\mu_{(i)}\,d\hat{\Sigma}^\nu, \tag{69}$$

where $\hat{T}_{\mu\nu}=\Omega^{-2}T_{\mu\nu}$ and $d\hat{\Sigma}^\mu=\Omega^2 d\Sigma^\mu$ is the volume element on Σ in the metric $\hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$.

We now consider the Penrose space $(\hat{M},\hat{g},\Omega)$ corresponding to the given asymptotically flat space (M,g), $\hat{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$. As Σ, we choose the null infinities $\mathcal{J}$, and assume $T^{\mathcal{J}}_{\mu\nu}$ is the limiting value of $\Omega^{-2}T_{\mu\nu}$ on $\mathcal{J}$. In this case, the P_i defined by Eq. (69), where $\xi^\mu_{(i)}$ are the generators of the BMS group of asymptotic symmetries on $\mathcal{J}$, will be called *asymptotic invariants*.

In the simplest case of a scalar massless field, for which the action is

$$W[\varphi]=\frac{1}{2}\int(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\alpha R\varphi^2)\sqrt{-g}\,d^4x \tag{70}$$

and the corresponding metric energy-momentum tensor is

$$T_{\mu\nu}=(2\alpha+1)\,\partial_\mu\varphi\partial_\nu\varphi-\frac{1+4\alpha}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi+2\alpha\varphi\nabla_\mu\nabla_\nu\varphi$$
$$+\alpha/2\left(R_{\mu\nu}-\frac{1}{6}g_{\mu\nu}R\right)\varphi^2, \tag{71}$$

the calculations made in Appendix 3 enable us to obtain the following expression for the asymptotic invariants (cf. Sec. 2):

$$P_i=P[\xi_{(i)}]=\int_{\mathcal{J}}[(2\alpha+1)\,\partial_u\Phi\xi^u_{(i)}\partial_u\Phi+2\alpha\Phi\xi^u_{(i)}\partial_u\partial_u\Phi]\,du\,d\sigma. \tag{72}$$

Here, Φ is the image of the field φ on $\mathcal{J}$; $\xi^\mu_{(i)}$ are the generators of the asymptotic symmetry transformations, and $d\sigma$ is the element of surface area of the unit sphere.

8. QUANTUM THEORY OF MASSLESS FIELDS IN ASYMPTOTICALLY FLAT SPACETIME

As we have already noted in Sec. 1, the quantization problem consists of two parts: First, it is necessary to find the commutation relations of the field operators (construct the operator algebra); second, it is necessary to find a realization of the operator algebra in a definite state space, defining thereby, in particular, the vacuum state. We begin by considering the first problem, deferring the discussion of the choice of the vacuum to the following section.

Schwinger's Dynamical Principle. The quantum theory of a free field in curved spacetime. For simplicity,[9] we shall, as before, consider the theory of a scalar field φ for which the action is

$$W[\varphi]=\frac{1}{2}\int(K^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+\alpha R\varphi^2\sqrt{-g})\,d^4x, \tag{73}$$

where $K^{\mu\nu}=\sqrt{-g}g^{\mu\nu}$. The corresponding equation of motion obtained by varying this action with respect to φ is

$$D[\varphi]=\Box\varphi-\alpha R\varphi\equiv(1/\sqrt{-g})\,\partial_\mu(K^{\mu\nu}\partial_\nu\varphi)-\alpha R\varphi=0, \tag{74}$$

and for $\alpha=-1/6$ it is conformally invariant.

To find the commutation relations, it is convenient to use the formalism developed by Schwinger.[39] For

[9] An analogous treatment for the electromagnetic field and gravitational perturbations can be found in Ref. 60.

this, we note that, introducing additional variables φ_μ, we can go over from (73) to an action that is linear in the derivatives of the field variables (to the *first-order formalism*). Direct verification shows that the action

$$W[\varphi, \varphi_\mu] = \frac{1}{2}\int d^4x\,[K^{\mu\nu}\varphi_\mu\partial_\nu\varphi - K^{\mu\nu}\varphi_\mu\varphi_\nu - \varphi\partial_\mu(K^{\mu\nu}\varphi_\nu) + \alpha R\varphi^2\sqrt{-g}] \quad (75)$$

when varied with respect to φ_μ and φ leads to the system of equations

$$\varphi_\mu = \partial_\mu\varphi;\quad \partial_\mu(K^{\mu\nu}\varphi_\nu) - \alpha R\sqrt{-g}\,\varphi = 0, \quad (76)$$

which is equivalent to Eq. (74) for the field φ. The variation of the action $\delta W[\varphi, \varphi_\mu]$ for nonzero variations of the field variables $\delta\varphi$ and $\delta\varphi_\mu$ on the Cauchy hypersurfaces Σ_0 and Σ_1 which bound the four-dimensional volume V^4 has, with allowance for the equations of motion (76), the form

$$\delta W[\varphi, \varphi_\mu] = G_{\Sigma_1} - G_{\Sigma_0},$$

where

$$G_\Sigma = \frac{1}{2}\int_\Sigma K^{\mu\nu}(\varphi_\mu\delta\varphi - \varphi\delta\varphi_\mu)\,d\Sigma_\nu \quad (77)$$

are the generators of infinitesimally small transformations of the field variables. Therefore, the commutation relations on the surface Σ are determined from the equations

$$[\varphi, G_\Sigma] = i\delta\varphi/2;\quad [\varphi_\mu, G_\Sigma] = i\delta\varphi_\mu/2, \quad (78)$$

where the fields φ and φ_μ and the variations $\delta\varphi$ and $\delta\varphi_\mu$ are considered on the surface Σ. Note that to obtain the commutation relations for the fields from Eqs. (78) it is necessary to express the variations $\delta\varphi$ and $\delta\varphi_\mu$ in (77) in terms of independent variations on Σ.

Quantization on Spacelike Hypersurfaces. We consider first the case when Σ is *spacelike*. As Cauchy data on Σ one can choose two independent functions $\varphi|_\Sigma$ and $\partial_0\varphi|_\Sigma$ or, which is equivalent, $\varphi|_\Sigma$ and $\pi|_\Sigma = K^{0\mu}\partial_\mu\varphi|_\Sigma$. Choosing the coordinates in such a way that $x^0 = \text{const}$ on Σ, we obtain ($d\Sigma_\mu = \delta^0_\mu d^3x$)

$$G_\Sigma = \int_{x^0=\text{const}} \frac{1}{2}(\pi\delta\varphi - \varphi\delta\pi)\,d^3x.$$

Rewriting Eqs. (78) in the form

$$[\varphi, G_\Sigma] = i\delta\varphi/2;\quad [\pi, G_\Sigma] = i\delta\pi/2$$

and remembering that the variations $\delta\varphi$ and $\delta\pi$ on Σ are independent, we can readily obtain the following *canonical commutation relations*:

$$\left.\begin{aligned}&[\varphi(x), \varphi(y)]_{x^0=y^0} = [\pi(x), \pi(y)]_{x^0=y^0} = 0;\\ &[\varphi(x), \pi(y)]_{x^0=y^0} = i\delta^{(3)}(\mathbf{x}-\mathbf{y}).\end{aligned}\right\} \quad (79)$$

Constructing in the usual manner the Hamiltonian

$$H = \int_\Sigma d^3x\,(\pi(x)\,\partial_0\varphi(x) - \mathscr{L}(x))$$

$$= \frac{1}{2}\int_{x^0=\text{const}} d^3x\Big[\frac{\pi^2}{\sqrt{-g}\,g^{00}} - 2\frac{\pi g^{0i}}{g^{00}}\varphi_{,i} + \sqrt{-g}\Big(\frac{g^{0i}\varphi_{,i}}{g^{00}}\frac{g^{0j}\varphi_{,j}}{g^{00}} - g^{ij}\varphi_{,i}\varphi_{,j} - \alpha R\varphi^2\Big)\Big], \quad (80)$$

one can see that the Hamiltonian equations of motion

$$\partial_0\pi = i[H, \pi];\quad \partial_0\varphi = i[H, \varphi] \quad (81)$$

are completely equivalent to the Lagrangian equation (74). The initial data $\varphi(x^0, x)$ and $\pi(x^0, x)$ for this equation must satisfy the commutation relations (79). To find a solution of the operator equations (81) [or, equivalently, of the Heisenberg equation (74)] one usually proceeds as follows. One considers *complex c*-number solutions f of Eq. (74). In the space $\mathfrak{F}$ of these solutions one can introduce the indefinite scalar product

$$\langle f_1, f_2\rangle = i\int_\Sigma \bar{f}_1\overleftrightarrow{\partial_\mu}f_2\,d\Sigma^\mu \equiv i\int_\Sigma(\bar{f}_1\partial_\mu f_2 - f_2\partial_\mu\bar{f}_1)\,d\Sigma^\mu. \quad (82)$$

That the scalar product of the solutions f_1 and f_2 is independent of the choice of the Cauchy surface Σ is established as follows:

$$i\Big[\int_{\Sigma_1} - \int_{\Sigma_0}\Big]\bar{f}_1\overleftrightarrow{\partial_\mu}f_2\,d\Sigma^\mu = i\int_{\Sigma_0}^{\Sigma_1} d^4x\,\partial_\nu[K^{\nu\mu}(\bar{f}_1\overleftrightarrow{\partial_\mu}f_2)]$$

$$= \int_{\Sigma_0}^{\Sigma_1}\sqrt{-g}\,d^4x\,\{\bar{f}_1 D[f_2] - D[\bar{f}_1]f_2\} = 0. \quad (83)$$

The scalar produce introduced here satisfies the relations

$$\left.\begin{aligned}&\langle\alpha f_1, \beta f_2\rangle = \bar{\alpha}\beta\langle f_1, f_2\rangle;\\ &\langle\bar{f}_1, \bar{f}_2\rangle = -\langle f_1, f_2\rangle;\\ &\overline{\langle f_1, f_2\rangle} = \langle f_2, f_1\rangle.\end{aligned}\right\} \quad (84)$$

We assume that in the solution $\mathfrak{F} = \{f\}$ there is a normalized *basis*, i.e., a set of solutions $\{f_\alpha, \bar{f}_\alpha\}$ such that

$$\langle f_\alpha, f_\beta\rangle = \delta_{\alpha\beta},\ \langle f_\alpha, \bar{f}_\beta\rangle = 0, \quad (85)$$

and an arbitrary function f in $\mathfrak{F}$ can be represented in the form

$$f = \sum_\alpha (a_\alpha f_\alpha + b_\alpha\bar{f}_\alpha). \quad (86)$$

In this case, an operator solution of Eq. (74) is

$$\varphi = \sum_\alpha (a_\alpha f_\alpha + a^*_\alpha\bar{f}_\alpha), \quad (87)$$

where

$$a_\alpha = \langle f_\alpha, \varphi\rangle,\ a^*_\alpha = -\langle\bar{f}_\alpha, \varphi\rangle \quad (88)$$

are constant operators. In order to find the commutation relations of these operators, we note that the commutation relations (79) can be written in the equivalent form

$$[\varphi(f), \varphi(g)] = \langle g, \bar{f}\rangle, \quad (89)$$

where $\varphi(f) = \langle f, \varphi\rangle$; f and g are arbitrary complex c-number solutions of Eq. (74).[10] Therefore, if we take f and g to be basis solutions, we readily obtain

[10]The commutation relations (89) are a special case of the general expression for the commutators of Heisenberg operators that satisfy linear equations (see, for example, Ref. 61).

$$[a_\alpha, a_\beta] = [a^*_\alpha, a^*_\beta] = 0; \quad [a_\alpha, a^*_\beta] = \delta_{\alpha\beta}. \tag{90}$$

Thus, the general solution of the operator equation (74) with the initial data (79) has the form (87), where the operators $\mathbf{a}_\alpha$ and $\mathbf{a}^*_\alpha$ satisfy the commutation relations (90).

Quantization on Null Hypersurfaces. If Σ is a null surface, then the initial data $\pi|_\Sigma = K^{0\mu}\partial_\mu\varphi|_\Sigma$ and $\varphi|_\Sigma$ are not independent. This is most readily seen by choosing on Σ the coordinates (r, x^A) associated with the congruence of the generators of Σ.[11] In these coordinates, $\pi|_\Sigma = \sqrt{-g}\partial_r\varphi|_\Sigma$. Therefore, the variations $\delta\pi$ and $\delta\varphi$ on Σ are not dependent. In order to express the generators G_Σ solely in terms of the independent variations $\delta\varphi$, we transform the expression (77) as follows:

$$G_\Sigma = \frac{1}{2}\int_\Sigma \sqrt{-g}\,(\partial_r\varphi\delta\varphi - \varphi\delta\partial_r\varphi)\,dr\,d^2x$$
$$= \int_\Sigma \partial_r\tilde{\varphi}\delta\tilde{\varphi}\,dr\,d^2x - G_b(r_0, r_1), \tag{91}$$

where $\tilde{\varphi} = \sqrt[4]{-g}\,\varphi$;

$$G_b(r_0, r_1) = \frac{1}{2}\int_\Sigma [\tilde{\varphi}(r_1, x^A)\,\delta\tilde{\varphi}(r_1, x^A) - \tilde{\varphi}(r_0, x^A)\,\delta\tilde{\varphi}(r_0, x^A)]\,d^2x, \tag{92}$$

and $r_0 = r_0(x^A)$ and $r_1 = r_1(x^A)$ are the (finite or infinite) values of the affine parameter r corresponding to the beginning and end of the generators of Σ.

Considering variations $\delta\tilde{\varphi}$ such that $\delta\tilde{\varphi}(r_1) = \delta\varphi(r_0) = 0$, we obtain from the relations (78)

$$[\tilde{\varphi}(r, x^A),\ \partial_{r'}\tilde{\varphi}(r', x'^A)] = (i/2)\,\delta(r - r')\,\delta(x^A - x'^A).$$

Integrating this equation with respect to r' and determining the "constant" of integration from the antisymmetry condition of the commutator, we find

$$[\tilde{\varphi}(r, x^A),\ \tilde{\varphi}(r', x'^A)] = (-i/2)\,\varepsilon(r - r')\,\delta(x^A - x'^A), \tag{93}$$

where $\varepsilon(x) = [\theta(x) - \theta(-x)]/2$, i.e., the commutation relations of the fields φ on the null surface Σ have the form

$$[\varphi(r, x^A),\ \varphi(r', x'^A)]$$
$$= (-i/2)\,\varepsilon(r - r')\,\delta(x^A - x'^A)/\sqrt[4]{-g}\,\sqrt[4]{-g'}. \tag{94}$$

Using the commutation relations we have obtained, we can show that for variations which do not vanish at the ends of the generators of Σ the following condition must hold:

$$[\tilde{\varphi},\ G_b(r_0, r_1)] = 0,$$

i.e.,

$$\delta\tilde{\varphi}(r_0, x^A) - \delta\tilde{\varphi}(r_1, x^A) = 0,$$

and that the variations at the ends are not independent.

Note that the commutation relations on null surfaces in curved spacetime obtained here by means of Schwinger's method are consistent, in particular, with the commutation relations on the light cone in Minkowski space derived in Refs. 9 and 40.

A simple calculation shows that, as in the case of a spacelike surface Σ, the commutation relations (93) can be represented in the form (95):

$$[\varphi(f),\ \varphi(g)] = \langle g, \bar{f}\rangle, \tag{95}$$

where f and g are arbitrary functions on Σ, and $\langle g, \bar{f}\rangle$ is the scalar product of the functions g and $\bar{f}$ on the null surface Σ defined by Eq. (82).

Conformal Transformations and Commutation Relations on $\mathcal{J}^\pm$. In quantum field theory, as in the classical theory, there is in addition to the Cauchy problem the interesting problem of determining a field from its image on $\mathcal{J}^\pm$ (scattering problem). As usual, it is convenient to consider, not the asymptotic behavior, but local characteristics of the field on $\mathcal{J}^\pm$ in the Penrose space. To this end, we first establish how the relations obtained in the previous section are modified if we make a conformal transformation of the metric: $g_{\mu\nu} \to \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$.

It is readily verified that if the scalar field f transforms under the conformal transformation in accordance with the law

$$f \to \hat{f} = \Omega^{-1} f, \tag{96}$$

then the scalar product (83) remains invariant, i.e.,

$$\langle \hat{f}_1, \hat{f}_2\rangle = i\int_\Sigma \bar{\hat{f}}_1 \overleftrightarrow{\partial_\mu} \hat{f}_2\, d\hat{\Sigma}^\mu = \langle f_1, f_2\rangle, \tag{97}$$

where $d\hat{\Sigma}^\mu = \Omega^2 d\Sigma^\mu$. Therefore, the commutation relations (95) in the space (M, g) have the consequence that in the space $(M, \hat{g})$ analogous relations are satisfied:

$$[\hat{\varphi}(\hat{f}),\ \hat{\varphi}(\hat{g})] = \langle \hat{g}, \bar{\hat{f}}\rangle, \tag{98}$$

where $\hat{\varphi} = \Omega^{-1}\varphi$ and $\hat{\varphi}(\hat{f}) = \langle \hat{f}, \hat{\varphi}\rangle$.

We now consider the *Penrose space* $(\hat{M}, \hat{g}, \Omega)$ and take the null infinity $\mathcal{J}^\pm$ as the surface Σ. Denoting, as usual, the images of functions on $\mathcal{J}^\pm$ by corresponding capital letters, we obtain[12]

$$\Phi = (\Omega^{-1}\varphi)|_{\mathcal{J}}, \quad F = (\Omega^{-1} f)|_{\mathcal{J}}.$$

From the commutation relations (98),

$$[\Phi(F),\ \Phi(G)] = \langle G, \bar{F}\rangle_{\mathcal{J}}, \tag{99}$$

where $\Phi(F) = \langle F, \Phi\rangle$ and

$$\langle A, B\rangle_{\mathcal{J}} = i\int_{\mathcal{J}} \hat{g}^{\mu\nu} \bar{A}\, \overleftrightarrow{\partial_\mu} B\, d\hat{\Sigma}_\nu. \tag{100}$$

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[11] See Sec. 4 for the definition of the coordinates associated with a congruence.

[12] In cases when no confusion is possible we shall sometimes omit the indices *in* and *out* of the images on $\mathcal{J}^\pm$, simultaneously omitting the corresponding indices + and − of $\mathcal{J}^\pm$. Therefore, for example, $\Phi = (\Omega^{-1}\varphi)|\mathcal{J}$ denotes the following two equations: $\Phi_{out} = (\Omega^{-1}\varphi)|\mathcal{J}^+$ and $\Phi_{in} = (\Omega^{-1}\varphi)|\mathcal{J}^-$.

In the Bondi conformal coordinates $(u, \hat{r}, x^2, x^3)$ (see Sec. 4), the expression for the scalar product on $\mathcal{J}$ has the simple form

$$\langle A, B\rangle_{\mathcal{J}} = i\int_{-\infty}^{+\infty} du \int d\sigma \bar{A}\overleftrightarrow{\partial_u}B. \tag{101}$$

If $\{F_\alpha, \bar{F}_\alpha\}$ is the image on $\mathcal{J}$ of the normalized basis $\{f_\alpha, \bar{f}_\alpha\}$ in the space (M,g), then the image of the field φ on $\mathcal{J}$:

$$\Phi = \sum_\alpha (F_\alpha \mathbf{a}_\alpha + \bar{F}_\alpha \mathbf{a}_\alpha^*), \tag{102}$$

and the operators $\mathbf{a}_\alpha$ and $\mathbf{a}_\alpha^*$ satisfy the commutation relations (90). The expression (102) in this case represents the "initial data" for the scattering problem.

We now establish the explicit form of the commutator of the operator Φ on $\mathcal{J}$. For this we note that the field $\tilde{\varphi}$ does not change under a conformal transformation:

$$\tilde{\varphi} \equiv \sqrt[4]{-g}\,\varphi = \sqrt[4]{-\hat{g}}\,\hat{\varphi} \equiv \hat{\tilde{\varphi}},$$

and therefore from the relation (93) on the surface $\mathcal{J}$ in the Penrose space in Bondi coordinates we obtain

$$[\Phi(u, x^A),\ \Phi(u', x'^A)] = (-i/2)\,\varepsilon(u-u')\,\delta(x^A - x'^A)/\sqrt{\det\|g_{AB}\|}, \tag{103}$$

where $\sqrt{\det\|g_{AB}\|}\,dx^2dx^3$ is the element of surface area on the unit sphere in the coordinates x^2 and x^3.

Generators of the Group of Asymptotic Symmetries in the Second Quantization Representation. In the construction of quantum field theory in flat space an important role is played (in particular, in the choice of the vacuum) by the operators of the energy, the momentum, and the angular momentum. These operators realize a representation of the generator algebra of the Poincaré group (the group of isometries of Minkowski space) (see Sec. 1). In this section, we construct the corresponding representation for the generators of the group of asymptotic symmetries. Let ξ^μ be the generators of the asymptotic symmetries on $\mathcal{J}$. We denote by L_ξ the differential operator

$$L_\xi = i\,[\xi^\mu\partial_\mu + \hat{\nabla}_\mu\xi^\mu/4]\,|_{\mathcal{J}} \equiv i\,[\xi^\mu\partial_\mu + \xi^A_{,A}/2]\,|_{\mathcal{J}}. \tag{104}$$

The operators L_ξ (in contrast to $i\xi^\mu\partial_\mu$) are Hermitian with respect to the scalar product (100) on $\mathcal{J}$:

$$\langle F, L_\xi G\rangle_{\mathcal{J}} = \langle L_\xi F, G\rangle_{\mathcal{J}}. \tag{105}$$

A simple verification shows that

$$[L_\xi, L_\eta] \equiv L_\xi L_\eta - L_\eta L_\xi = iL_{[\xi,\eta]}, \tag{106}$$

where $[\xi,\eta]^\mu = \xi^\nu\eta^\mu_{,\nu} - \eta^\nu\xi^\mu_{,\nu}$, and therefore a representation of the Lie algebra corresponding to the BMS group is realized in the Hermitian operators L_ξ.

We now consider the operators[13)]

$$\mathbf{P}_\xi = \langle\Phi, L_\xi\Phi\rangle_{\mathcal{J}}/2. \tag{107}$$

By direct calculations, we can show that

$$[\mathbf{P}_\xi, \Phi] = L_\xi\Phi. \tag{108}$$

This equation enables us to show that

$$[\mathbf{P}_\xi, \mathbf{P}_\eta] = \mathbf{P}_{[\xi,\eta]}/i, \tag{109}$$

i.e., the constructed operators $\mathbf{P}_\xi$ are the required expressions for the generators of the asymptotic symmetries in the second-quantization representation. We prove (109) as follows. Applying (108) successively, we obtain

$$[\mathbf{P}_\xi, [\mathbf{P}_\eta, \Phi]] = L_\eta L_\xi\Phi; \quad [\mathbf{P}_\eta, [\mathbf{P}_\xi, \Phi]] = L_\xi L_\eta\Phi.$$

Subtracting these two equations and using the Jacobi identity $[[A,B],C] = [A,[B,C]] - [B,[A,C]]$, we find

$$[[\mathbf{P}_\xi, \mathbf{P}_\eta], \Phi] = [L_\eta, L_\xi]\,\Phi = L_{[\xi,\eta]}\Phi/i = [\mathbf{P}_{[\xi,\eta]}/i, \Phi].$$

Thus, the operator $[\mathbf{P}_\xi, \mathbf{P}_\eta] - \mathbf{P}_{[\xi,\eta]}/i$ commutes with Φ and with any operators constructed from Φ. From the completeness of the operator algebra Eq. (105) then follows.

To conclude this section, we note that the operators of the asymptotic invariants (72),

$$\mathbf{P}[\xi] = \int_{\mathcal{J}} du\, d\sigma\, [(2\alpha+1)\,\partial_u\Phi\xi^\mu\partial_\mu\Phi + 2\alpha\Phi\xi^\mu\partial_\mu\partial_u\Phi], \tag{110}$$

can be reduced by means of the operation of integration by parts with respect to the coordinate u to a form that coincides with the expression for $\mathbf{P}_\xi$:

$$\mathbf{P}[\xi] = \mathbf{P}_\xi = \frac{1}{2}\int_{\mathcal{J}} du\, d\sigma\, (\partial_u\Phi\xi^\mu\partial_\mu\Phi - \Phi\xi^\mu\partial_\mu\partial_u\Phi). \tag{111}$$

9. DEFINITION OF THE VACUUM

We now turn to the second part of the quantization problem—the definition and interpretation of the vacuum and n-particle states in curved spacetime.

Concept of the Vacuum in Curved Spacetime. Usually, the definition of the vacuum is tied to the existence of a symmetry group. In particular, if the spacetime admits a global timelike Killing field (or, in the case of massless particles, a conformal timelike Killing field), then the vacuum is defined as the lowest eigenstate of the generator of translations (the Hamiltonians) along this field. At the same time, there exists an invariant decomposition into positive- and negative-frequency solutions of Eq. (74) and the vacuum is annihilated

13)We defer for the moment the question of augmenting the definition of the operators. We merely point out that it is achieved in the cases considered below by means of the operation of normal ordering (see Sec. 10) and reduces, as usual, to subtraction of zero-point vibrations.

under application of the annihilation operator corresponding to negative frequencies. In the general case, there is no such global Killing field and the definition of the vacuum becomes nonunique. This lack of uniqueness corresponds to the physical lack of uniqueness in distinguishing virtual particles from real particles and is manifested in the lack of uniqueness of the decomposition into positive and negative frequencies.

Above, we have spoken of a global vacuum. We recall, however, (see Sec. 2) that the vacuum, like all other states, is defined on the Cauchy surfaces Σ. In Ref. 41 it was shown that on null Cauchy hypersurfaces and for a fairly large class of spaces one can define a vacuum in a natural manner by using the affine properties of such hypersurfaces.

To define the vacuum on Σ (see Sec. 2) it is not necessary to require that the system under consideration be globally invariant with respect to the Poincaré group (in the flat case); it is sufficient that in a certain neighborhood of Σ there be approximate invariance (see the discussion of this concept in Sec. 5), and the smaller is the departure of the symmetry from exact symmetry, the closer are the properties of the vacuum $|0;\Sigma\rangle$ on Σ to the properties of the ordinary flat vacuum $|0\rangle$. Therefore, in curved spacetime one can, naturally, define a vacuum on hypersurfaces for which spacetime in their neighborhood can with a good accuracy be regarded as flat.

Vacuum on $\mathcal{J}$. In asymptotically flat space, such hypersurfaces are, as we know, $\mathcal{J}^-$ and $\mathcal{J}^+$, on which there are defined the two representations described above of the Bondi-Metzner-Sachs algebra with uniquely distinguished translation generators $\mathbf{P}_{\text{in},\mu}$ and $\mathbf{P}_{\text{out},\mu}$. The vacuum on $\mathcal{J}^+|_{0,\text{out}}\rangle$ (respectively on $\mathcal{J}^-|0;\text{in}\rangle$) can now be defined by the condition

$$\mathbf{P}_{\text{out},\mu}|0;\text{out}\rangle=0\ \ (\mathbf{P}_{\text{in},\mu}|0;\text{in}\rangle=0) \tag{112}$$

or, equivalently, by the condition

$$\mathbf{a}_{\text{out},\alpha}|0;\text{out}\rangle=0\ \ (\mathbf{a}_{\text{in},\alpha}|0;\text{in}\rangle=0),$$

where $\mathbf{a}_{\text{out},\alpha}=\langle f_\alpha,\varphi\rangle$ and f_α is any solution of the wave equation (74) having image on $\mathcal{J}^+$ equal to

$$F_{\text{out},\alpha}=\frac{1}{\sqrt{2\pi}}\int_0^\infty d\omega\exp(-i\omega u)\,\mathcal{F}_k(\omega)\,Y_{lm}(\theta,\varphi),\ \alpha=(k,l,m), \tag{113}$$

i.e., containing only negative frequencies with respect to the retarded time u. The operators $\mathbf{a}_{\text{out},\alpha}$ and $\mathbf{a}^*_{\text{out},\alpha}=-\langle\bar f_\alpha,\varphi\rangle$ satisfy the canonical commutation relations

$$\left.\begin{aligned}&[\mathbf{a}_{\text{out},\alpha},\mathbf{a}^*_{\text{out},\beta}]=\delta_{\alpha\beta};\\&[\mathbf{a}_{\text{out},\alpha},\mathbf{a}_{\text{out},\beta}]=[\mathbf{a}^*_{\text{out},\alpha},\mathbf{a}^*_{\text{out},\beta}]=0.\end{aligned}\right\} \tag{114}$$

Similarly, we can define the operators $\mathbf{a}_{\text{in},\alpha}$, $\mathbf{a}^*_{\text{in},\alpha}$, which also satisfy canonical commutation relations. The n-particle *in* and *out* states are defined as follows:

$$\left.\begin{aligned}&|\alpha_1,\ldots,\alpha_n;\text{in}\rangle=\mathbf{a}^*_{\text{in},\alpha_1}\ldots\mathbf{a}^*_{\text{in},\alpha_n}|0;\text{in}\rangle;\\&|\alpha_1,\ldots,\alpha_n;\text{out}\rangle=\mathbf{a}^*_{\text{out},\alpha_1}\ldots\mathbf{a}^*_{\text{out},\alpha_n}|0;\text{out}\rangle,\end{aligned}\right\} \tag{115}$$

and correspond to particles that come in from the null infinity $\mathcal{J}^-$ and go out to the null infinity $\mathcal{J}^+$. We denote the corresponding Hilbert spaces by $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$.

The operators $\mathbf{a}_{\text{in},\alpha}$ and $\mathbf{a}_{\text{out},\alpha}$ are related by a linear canonical transformation, since they can be expressed in terms of the solutions of one and the same linear equation (74):

$$\left.\begin{aligned}&\mathbf{a}_{\text{out},\alpha}=\sum_\beta(A_{\alpha\beta}\mathbf{a}_{\text{in},\beta}+B_{\alpha\beta}\mathbf{a}^*_{\text{in},\beta});\\&\mathbf{a}^*_{\text{out},\alpha}=\sum_\beta(\bar A_{\alpha\beta}\mathbf{a}^*_{\text{in},\beta}+\bar B_{\alpha\beta}\mathbf{a}_{\text{in},\beta}).\end{aligned}\right\} \tag{116}$$

Here, $A_{\alpha\beta}=\langle f_\alpha,p_\beta\rangle$ and $B_{\alpha\beta}=\langle f_\alpha,\bar p_\beta\rangle$, where p_β are solutions of (74) with negative-frequency images on $\mathcal{J}^-$ with respect to the advanced time v.

10. *S* MATRIX IN ASYMPTOTICALLY FLAT SPACETIME

Using the *in* and *out* states introduced above, we obtain in this section expressions for the *S* matrix and for the fluxes on $\mathcal{J}$.

Functional Expression for the S Matrix. If we assume that the operators $\mathbf{a}_{\text{in},\alpha}$ and $\mathbf{a}_{\text{out},\alpha}$ form a complete set and that the spaces $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ coincide, then there must exist a unitary operator **S**, called the *S* matrix, which has the property $\mathbf{S}^*\mathbf{a}_{\text{in},\alpha}\mathbf{S}=\mathbf{a}_{\text{out},\alpha}$. Obviously, $|0;\text{in}\rangle=\mathbf{S}|0;\text{out}\rangle$ and

$$\begin{aligned}&S_{\beta_1\ldots\beta_n;\alpha_1\ldots\alpha_m}\equiv\langle\beta_1,\ldots,\beta_n;\text{out}|\alpha_1,\ldots,\alpha_m;\text{in}\rangle\\&=\langle 0;\text{out}|\mathbf{a}_{\text{out},\beta_1}\ldots\mathbf{a}_{\text{out},\beta_n}\mathbf{S}\mathbf{a}^*_{\text{out}\,\alpha_1}\ldots\mathbf{a}^*_{\text{out},\alpha_n}|0;\text{out}\rangle.\end{aligned} \tag{117}$$

The operator **S** realizes a linear canonical transformation, and it is easy to obtain an expression for it in terms of the quantities $A_{\alpha\beta}$ and $B_{\alpha\beta}$ (116). This expression can be most readily written down if one uses the functional representation[42] of second-quantization operators.

We recall that with every operator written in the normal form

$$\mathbf{A}=\sum_{\substack{m,n\\(\alpha),(\beta)}}A_{mn}(\alpha_1,\ldots,\alpha_m|\beta_1,\ldots,\beta_n)\times\mathbf{a}^*_{\text{out},\alpha_1}\ldots\mathbf{a}^*_{\text{out},\alpha_m}\mathbf{a}_{\text{out},\beta_1}\ldots\mathbf{a}_{\text{out},\beta_n},$$

there is associated a functional

$$A(a^*,a)=\sum_{\substack{m,n\\(\alpha),(\beta)}}A_{mn}(\alpha_1,\ldots,\alpha_m|\beta_1,\ldots,\beta_n)\,a^*_{\alpha_1}\ldots a^*_{\alpha_m}a_{\beta_1}\ldots a_{\beta_n},$$

and with the vector

$$\Phi=\sum_{n,(\alpha)}\Phi_n(\alpha_1,\ldots,\alpha_n)\,\mathbf{a}^*_{\text{out},\alpha_1}\ldots\mathbf{a}^*_{\text{out},\alpha_n}|0;\text{out}\rangle$$

there is associated the functional

$$\Phi(a^*)=\sum_{n,(\alpha)}\Phi_n(\alpha_1,\ldots,\alpha_n)\,a^*_{\alpha_1}\ldots a^*_{\alpha_n},$$

and the scalar product can be expressed in the form of the functional integral

$$(\Phi_1,\Phi_2)=\int\overline{\Phi_1(a^*)}\,\Phi_2(a^*)\exp(-a^*a)\,Da^*Da. \tag{118}$$

Here and in what follows, we use the notation $a^*a = \Sigma_\alpha a^*_\alpha a_\alpha$, etc. The functional expression for the S matrix has the form (see Appendix 4):

$$S(a^*, a) = [\det(A^*A)]^{-1/4} \exp\{[a\overline{B}A^{-1}a + 2a^*(A^{-1}-I)a - a^*A^{-1}Ba^*]/2\}, \quad (119)$$

and the corresponding operator is

$$\mathbf{S} = [\det(A^*A)]^{-1/4} \times \exp(-\mathbf{a}^*A^{-1}B\mathbf{a}^*/2) : \exp[\mathbf{a}^*(A^{-1}-I)\mathbf{a}] : \exp(\mathbf{a}\overline{B}A^{-1}\mathbf{a}/2), \quad (120)$$

where the colons denote normal ordering with respect to the *out* vacuum. Therefore, the elements of the S matrix (117) can be expressed in terms of the variational derivatives of the generating functional (see Appendix 4):

$$S_{\beta_1, \ldots, \beta_n; \alpha_1, \ldots, \alpha_m} = (\delta^n/\delta j_{\beta_1} \ldots \delta j_{\beta_n})(\delta^m/\delta j^*_{\alpha_1} \ldots \delta j^*_{\alpha_m}) \exp[iw(j, j^*)]/|_{j=j^*=0}, \quad (121)$$

where

$$\exp[iw(j, j^*)] = [\det(A^*A)]^{-1/4} \exp[(-jA^{-1}Bj + j^*\overline{B}A^{-1}j^* + 2j^*A^{-1}j)/2]. \quad (122)$$

We give one further expression for the S matrix (which may be helpful in applications) based on the *background formalism*.[43,44]

In flat spacetime, the elements of the S matrix with the action $W[\varphi]$ can be represented in the form

$$S_{p_1, \ldots, p_n; k_1, \ldots, k_m} = [\delta^m/\delta a(k_1) \ldots \delta a(k_m)][\delta^n/\delta b(p_1) \ldots \delta b(p_n)] \times Z(a, b)|_{a=b=0}, \quad (123)$$

where $Z(a,b) = \int \exp(iW[\varphi]) D\varphi$ and the integration is over all functions with the asymptotic behaviors

$$\left.\begin{aligned} \varphi(x) &\sim \int a(k) \exp i[\omega(k)x^0 - kx]\, d^3k + \text{c.c.}, \quad x^0 \to -\infty; \\ \varphi(x) &\sim \int b(p) \exp\{-i[\omega(p)x^0 - px]\}\, d^3p + \text{c.c.}, \quad x^0 \to +\infty. \end{aligned}\right\} \quad (124)$$

If the action $W[\varphi]$ is quadratic in φ, then the integral can be readily calculated, and $Z(a,b) = c \exp iW[\varphi_{cl}]$, where φ_{cl} is a solution of the classical equations of motion with the asymptotic behaviors (124).

The generalization to the case of asymptotically flat spacetime is obvious:

$$S_{\beta_1, \ldots, \beta_n; \alpha_1, \ldots, \alpha_m} = (\delta^n/\delta F_{\text{out}, \beta_1} \ldots \delta F_{\text{out}, \beta_n}) \times (\delta^m/\delta \overline{F}_{\text{in}, \alpha_1} \ldots \delta \overline{F}_{\text{in}, \alpha_m}) c \exp iW[\varphi_{cl}]|_{F_{\text{in}}=0, F_{\text{out}}=0}. \quad (125)$$

where φ_{cl} is a solution of Eqs. (74) with the asymptotic behaviors $\varphi_{cl}(\nu, r, \theta, \varphi) \sim F_{\text{in}, \alpha}(\nu, \theta, \varphi) r^{-1} + \text{c.c.}$, $r \to \infty$, ν, θ, φ fixed; $\varphi_{cl}(u, r\theta, \varphi) \sim F_{\text{out}, \alpha}(u, \theta, \varphi) r^{-1} + \text{c.c.}$, $r \to \infty$, u, θ, φ fixed; $F_{\text{in}, \alpha}$ and $F_{\text{out}, \alpha}$ are positive-frequency bases on $\mathcal{J}^\pm$ (113).

Calculation of the fluxes on $\mathcal{J}$. In order to avoid the divergences associated with the zero-point vibrations, we introduce the operation of normal ordering for operators on $\mathcal{J}$:

$$\mathcal{N}_{\substack{\text{in} \\ \text{out}}}(\mathbf{A}) = \mathbf{A} - \langle 0; {}^{\text{in}}_{\text{out}} | \mathbf{A} | 0; {}^{\text{in}}_{\text{out}} \rangle. \quad (126)$$

The final operators $\mathcal{N}_{\text{out}}(\mathbf{P}_i)$, where $\mathbf{P}_i$ is given by the expression (110), are the operators of the energy-momentum flux on $\mathcal{J}^+$. Therefore, in particular, the energy-momentum flux of particles created from the vacuum in the gravitational field is

$$\begin{aligned} P_i &= \langle 0; \text{in} | \mathcal{N}_{\text{out}}(\mathbf{P}_i) | 0; \text{in} \rangle \\ &= \int_{-\infty}^{+\infty} du \int d\sigma \langle 0; \text{in} | \mathcal{N}_{\text{out}}(T_i(\Phi_{\text{out}}, \Phi_{\text{out}})) | 0; \text{in} \rangle, \end{aligned} \quad (127)$$

where

$$T_i(A, B) = (2\alpha+1)\,\partial_u A \xi^\mu_{(i)} \partial_\mu B + 2\alpha A \xi^\mu_{(i)} \partial_\mu \partial_u B \quad (128)$$

and $\xi^\mu_{(i)}$ are the generators of the subgroup of translations on $\mathcal{J}^+$.

If the instrument of a distant observer (an observer) on $\mathcal{J}^+$) detects emitted particles only in the range (u_1, u_2) of retarded times, then the measured energy-momentum flux during this interval is [35]

$$P_i^f = \int du d\sigma f(u) \langle 0; \text{in} | \mathcal{N}_{\text{out}}(\mathbf{P}_i) | 0; \text{in} \rangle, \quad (129)$$

where the resolution function $f(u)$ of the instrument is unity in the interval (u_1, u_2) and vanishes outside it. Substituting the expansion (102) in (127) and remembering that terms containing two operators of creation or two operators of annihilation vanish after integration with respect to u, we obtain

$$\left.\begin{aligned} P_i &= \int du d\sigma \sum_{\alpha, \beta} [T_i(\overline{F}_{\text{out}, \alpha}, F_{\text{out}, \beta}) + T_i(F_{\text{out}, \beta}, \overline{F}_{\text{out}, \alpha})] \langle 0; \text{in} | \mathbf{a}^*_{\text{out}, \alpha} \mathbf{a}_{\text{out}, \beta} | 0; \text{in} \rangle \\ &= \int du d\sigma \sum_{\alpha, \beta, \lambda} [T_i(\overline{F}_{\text{out}, \alpha} F_{\text{out}, \beta}) + T_i(F_{\text{out}, \beta}, \overline{F}_{\text{out}, \alpha})] \overline{B}_{\alpha\lambda} B_{\beta\lambda}. \end{aligned}\right\} \quad (130)$$

The operator $\mathbf{n}_\alpha$ of the number of particles in the state α on $\mathcal{J}^+$ is

$$\mathbf{n}_{\text{out}, \alpha} = \mathbf{a}^*_{\text{out}, \alpha} \mathbf{a}_{\text{out}, \alpha},$$

so that the expectation value of the number of particles created from the vacuum in this state is

$$n_{\text{out}, \alpha} = \langle 0; \text{in} | \mathbf{n}_{\text{out}, \alpha} | 0; \text{in} \rangle = \sum_\beta \overline{B}_{\alpha\beta} B_{\alpha\beta}. \quad (131)$$

11. QUANTUM PROCESSES IN BLACK HOLES

In this section we shall show how the formalism developed earlier can be used in the extremely important case when the source of the gravitational field is a *collapsing spherically symmetric body* (Fig. 5)[34] (see also Refs. 35, 45, 47–53).

Calculation of Observables on $\mathcal{J}^+$ in the Presence of an Event Horizon. Density Matrix. An important feature of this problem is the formation of an event hori-

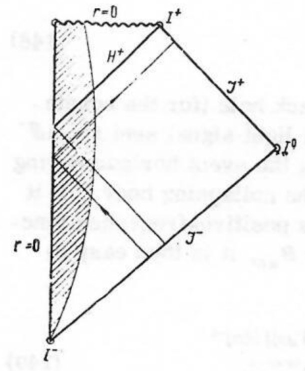

FIG. 5.

zon H^+ due to the collapse of the body.[14] This has the consequence that although the surface $\mathcal{J}^-$ is as before a global Cauchy surface for massless fields, a global Cauchy surface in the future no longer coincides with $\mathcal{J}^+$. As a global Cauchy surface in the future one can, for example, take $\mathcal{J}^+ \cup H^+$ (massless particles either escape to infinity or fall into the black hole). Accordingly, as a basis in the future one can choose the set $(f_\alpha, q_\alpha, \bar{f}_\alpha, \bar{q}_\alpha)$, where the basis functions f_α and q_α are defined by the conditions

$$\left.\begin{aligned} & f_\alpha|_{H^+} = 0;\ Q_{\text{out},\,\alpha} = 0; \\ & F_{\text{out},\,\alpha} = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \frac{\exp(-i\omega u)}{\sqrt{2\omega}} \mathcal{F}_k(\omega) Y_{lm}(\theta, \varphi). \end{aligned}\right\} \quad (132)$$

Here, u is an affine parameter on $\mathcal{J}^+$ (Bondi time). The particular choice of the positive-frequency basis q_α and the definition of the positive-frequency functions on H^+ associated with the choice of the vacuum $|0; H^+\rangle$ are, as we shall see, unimportant for the finding of the matrix elements of observables on $\mathcal{J}^+$.[15]

The expansion of the field φ with respect to the basis functions has the form

$$\varphi = \sum_\alpha (f_\alpha \mathbf{a}_{\text{out},\,\alpha} + \bar{f}_\alpha \mathbf{a}^*_{\text{out},\,\alpha} + q_\alpha \mathbf{b}_\alpha + \bar{q}_\alpha \mathbf{b}^*_\alpha), \quad (133)$$

where

$$\mathbf{a}_{\text{out},\,\alpha} = \langle f_\alpha, \varphi\rangle = \langle F_{\text{out},\,\alpha} \Phi_{\text{out}}\rangle \mathcal{J}^+;$$

$$\mathbf{b}_\beta = \langle q_\beta, \varphi\rangle = \langle q_\beta, \varphi\rangle_{H^+}$$

are operators of annihilation of particles in the state α on $\mathcal{J}^+$ and in the state β on H^+, respectively. The vacuum state in the future, $|0; \text{out}\rangle$, which is defined by the conditions

$$\mathbf{a}_{\text{out},\,\alpha}|0;\ \text{out}\rangle = \mathbf{b}_\beta|0;\ \text{out}\rangle = 0, \quad (134)$$

can be represented in the form

$$|0;\ \text{out}\rangle = |0;\ \mathcal{J}^+\rangle|0;\ H^+\rangle, \quad (135)$$

where the corresponding vacuums on $\mathcal{J}^+$ and H^+ satisfy the conditions

$$\mathbf{a}_{\text{out},\,\alpha}|0;\ \mathcal{J}^+\rangle = 0;\ \mathbf{b}_\beta|0;\ H^+\rangle = 0.$$

A vector of the complete Hilbert state space in the future, H_{out}, is expressed in the form

$$|\Psi\rangle = \sum_{m,\,n} \Psi_{m,\,n} |n;\ \mathcal{J}^+\rangle |m;\ H^+\rangle, \quad (136)$$

where

$$|n;\ \mathcal{J}^+\rangle = |\alpha_1, \ldots, \alpha_n;\ \mathcal{J}^+\rangle = \mathbf{a}^*_{\text{out},\,\alpha_1} \ldots \mathbf{a}^*_{\text{out},\,\alpha_n}|0;\ \mathcal{J}^+\rangle;$$

$$|m;\ H^+\rangle = |\beta_1, \ldots, \beta_m;\ H^+\rangle = \mathbf{b}^*_{\beta_1} \ldots \mathbf{b}^*_{\beta_m}|0;\ H^+\rangle.$$

The observables $\mathfrak{a}$ on $\mathcal{J}^+$ are functions of the operators $\mathbf{a}_{\text{out},\,\alpha}$ and $\mathbf{a}^*_{\text{out},\,\alpha}$: $\mathfrak{a} = \mathfrak{a}(\mathbf{a}_{\text{out}}, \mathbf{a}^*_{\text{out}})$. We now consider the calculation of the expectation value of this observable in an arbitrary *in*-state Ψ:

$$\mathfrak{a}_\Psi = \langle\Psi|\mathfrak{a}(\mathbf{a}_{\text{out}}, \mathbf{a}^*_{\text{out}})|\Psi\rangle. \quad (137)$$

Using the expansions (136) of this *in*-state Ψ with respect to the *out* basis, we obtain

$$\mathfrak{a}_\Psi = \sum_{\substack{m,\,n \\ m',\,n'}} \Psi_{mn} \bar{\Psi}_{m'n'} \langle H^+;\ m'|m;\ H^+\rangle \langle n';\ \mathcal{J}^+|\mathfrak{a}|n;\ \mathcal{J}^+\rangle$$

$$= \sum_{n,\,n} R_{n'n} \langle n';\ \mathcal{J}^+|\mathfrak{a}|n;\ \mathcal{J}^+\rangle, \quad (138)$$

where $R_{n'n} = \sum_m \bar{\Psi}_{mn'} \Psi_{mn}$.

If we introduce the density matrix

$$\rho = \sum_{n,\,n'} |n';\ \mathcal{J}^+\rangle R_{nn'} \langle n;\ \mathcal{J}^+|, \quad (139)$$

then Eq. (138) can be written as follows[48]:

$$\mathfrak{a}_\Psi = \mathrm{Sp}_{\mathcal{J}^+}(\rho\mathfrak{a}), \quad (140)$$

where $\mathrm{Sp}_{\mathcal{J}^+}(\mathbf{B})$ denotes the sum of the diagonal matrix elements of the operator $\mathbf{B}$ with respect to the complete system of states on $\mathcal{J}^+$. The density matrix arises because in this problem one completely ignores the information about the particles that fall into the black hole.

One can show that the density matrix ρ does not depend on the choice of the vacuum on H^+; for on the transition to a new vacuum vector $|0';\ H^+\rangle$ on H^+, the coefficients Ψ_{mn} in Eq. (136) are subjected to a unitary transformation: $\Psi_{mn} \to \Psi'_{m'n} = \sum_m C_{m'm} \Psi_{mn}$. However, the quantities $R_{n'n} = \sum_m \bar{\Psi}_{mn'} \Psi_{mn}$ do not change because of the unitarity of the matrix $C_{m',\,m}$, so that the density matrix ρ is unaffected by the transformation.

Hawking Effect. Thermal Nature of the Radiation from Black Holes. We now calculate the flux of radiation from a black hole resulting from the collapse of a

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[14] An event horizon is a null surface bounding a region of spacetime from which it is impossible for signals to escape to an observer in the neighborhood of the null infinity $\mathcal{J}^+$. The formal definition of the event horizon H^+ is as follows:[46] $H^+ = \partial(\mathcal{J}^-[\mathcal{J}^+])$ where $\mathcal{J}^-[A]$ is the "casual past" of the set of events A and ∂K is the boundary of the set K.

[15] It is shown in Refs. 41 and 47 that the positive-frequency functions on H^+ can be defined in a natural manner as functions that contain only positive frequencies with respect to the affine parameter along the generators of the event horizon H^+. The corresponding definition of the vacuum is a special case of the definition of the $\mathfrak{N}$ vacuum introduced in Ref. 41. The choice of the vacuum $|0;\ H^+\rangle$ is extremely important when one is discussing the back reaction of the created particles on the metric of spacetime.

nonrotating body. For the flux of the energy E of the particles created from the vacuum through a sphere of large radius (the flux on $\mathscr{I}^+$) in the interval of (retarded) time (u_1, u_2) we can use the expression (130):

$$E(u_1, u_2) = \int_{u_1}^{u_2} du d\sigma \langle 0;\ \text{in} \,|\, \mathscr{N}_{\text{out}} (\partial_u \Phi_{\text{out}} \partial_u \Phi_{\text{out}}) \,|\, 0; \text{in} \rangle$$

$$= 2 \int_{u_1}^{u_2} du d\sigma \sum_{\alpha, \alpha', \beta} \partial_u \overline{F}_{\text{out}, \alpha} \partial_u F_{\text{out}, \alpha'} \overline{B}_{\alpha\beta} B_{\alpha'\beta}. \tag{141}$$

Thus, to find $E(u_1, u_2)$ it is necessary to calculate the quantities

$$B_{\alpha\beta} = \langle f_\alpha, \overline{p}_\beta \rangle. \tag{142}$$

By virtue of the spherical symmetry of the problem, it is convenient to take as the basis functions f_α and p_α $(\alpha = \omega, l, m)$

$$\left. \begin{aligned} F_{\text{out}, \alpha} &\equiv F_{\text{out}; \omega, l, m} = (2\pi)^{-1/2} [\exp(-i\omega u)/\sqrt{2\omega}\,] \\ &\quad \times Y_{lm}(\theta, \varphi); \\ P_{\text{in}, \alpha} &\equiv P_{\text{in}; \omega, l, m} = (2\pi)^{-1/2} [\exp(-i\omega v)/\sqrt{2\omega}\,] \\ &\quad \times Y_{lm}(\theta, \varphi), \end{aligned} \right\} \tag{143}$$

where u and v are affine parameters on $\mathscr{I}^+$ and $\mathscr{I}^-$, respectively. The normalization conditions for these bases are

$$\begin{aligned} \langle f_{\omega lm}, f_{\omega' l' m'} \rangle &= \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}; \\ \langle p_{\omega lm}, p_{\omega' l' m'} \rangle &= \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}. \end{aligned} \tag{144}$$

Since the scalar product (142) does not depend on the choice of the Cauchy surface Σ on which it is calculated, we can take Σ to be $\mathscr{I}^-$. In this case

$$B_{\alpha, \alpha'} \equiv B_{\omega lm, \omega' l' m'}$$
$$= \frac{i}{\sqrt{2\pi}} \int dv d\sigma \overline{F}_{\text{in}, \omega lm} \overleftrightarrow{\partial_v} \frac{\exp(i\omega' v)}{\sqrt{2\omega'}} \overline{Y}_{l'm'} \tag{145}$$

and to determine $B_{\alpha\alpha'}$, we must find the image $F_{\text{in}, \alpha}$ on $\mathscr{I}^-$ of the solution of the equation whose image on $\mathscr{I}^+$ is expressed by (143). This image on $\mathscr{I}^-$ can be expressed in the form of the sum

$$F_{\text{in}, \alpha} = F^{(1)}_{\text{in}, \alpha} + F^{(2)}_{\text{in}, \alpha}, \tag{146}$$

where $F^{(1)}_{\text{in}, \alpha} = (4\pi\omega)^{-1/2} \alpha_{\omega l} \exp(-i\omega v) Y_{lm}(\theta, \varphi)$ corresponds to scattering of a monochromatic wave emitted from $\mathscr{I}^+$ on the potential barrier of the gravitational and centrifugal forces. To calculate $F^{(2)}_{\text{in}, \alpha}$, one can use the approximation of geometrical optics[34] since near the event horizon the effective wavelength tends to zero. Therefore,

$$F^{(2)}_{\text{in}, \alpha} = \beta_{\omega l} (2\pi)^{-1/2} \exp[-i\omega W(v)] (2\omega)^{-1/2} Y_{lm}(\theta, \varphi). \tag{147}$$

where $u = W(v)$ is the retarded time at which a light ray emitted from $\mathscr{I}^-$ at the advanced time v hits $\mathscr{I}^+$. The quantities $\alpha_{\omega l}$ and $\beta_{\omega l}$ are the reflection and transmission coefficients for the wave f_α in the gravitational field of the black hole.

A simple analysis[34] of the propagation of light rays in the gravitational field of a collapsing body shows that for large values of u the function $u = W(v)$ has the form

$$\left. \begin{aligned} W(v) &= -4m \ln(v_0 - v) + W_0; \\ F^{(2)}_{\text{in}, \alpha}(v) &= \theta(v_0 - v) F^{(2)}_{\text{in}, \alpha}, \end{aligned} \right\} \tag{148}$$

where m is the mass of the black hole (for the nonstationary case, see Ref. 35). (A light signal sent from $\mathscr{I}^-$ at the time v_0 propagates along the event horizon having passed through the center of the collapsing body.) If it is borne in mind that $F^{(1)}_{\text{in}, \alpha}$ is a positive-frequency function and does not contribute to $B_{\alpha\beta}$, it is then easy to obtain from (145)

$$\begin{aligned} B_{\alpha\alpha'} &= -\delta_{ll'} \delta_{mm'} \overline{\beta}_{\omega l} \exp[i(v\omega' + W_0 \omega)] (2\pi)^{-1} \\ &\quad \times (\omega'/\omega)^{1/2} \Gamma(1 - 4im\omega) (i\omega')^{4im\omega - 1}. \end{aligned} \tag{149}$$

Therefore

$$\sum_\beta \overline{B}_{\alpha\beta} B_{\alpha'\beta} = \delta_{ll'} \delta_{mm'} |\beta_{\omega l}|^2 \omega \, \delta(\omega - \omega') [\exp(8\pi m\omega) - 1]^{-1}. \tag{150}$$

In the derivation of this expression it is helpful to bear in mind that

$$\frac{1}{2\pi} \int_0^\infty dt \; t^{-(1 + 4im\omega)} = \frac{1}{4m} \delta(\omega);$$
$$\Gamma(1 + ix) \Gamma(1 - ix) = \pi x / \sinh \pi x.$$

Substituting the expression (150) in (141), we finally obtain

$$\frac{dE}{du} = \frac{E(u_1, u_2)}{(u_2 - u_1)} = \frac{1}{2\pi} \int_0^\infty d\omega \sum_{l, m} |\beta_{\omega l}|^2 \frac{\omega}{\exp(8\pi m\omega) - 1}, \tag{151}$$

i.e., after the formation of the black hole (for large u), the black hole becomes the source of a stationary flux of radiation, and the spectrum of this radiation (if scattering of the radiation on the gravitational field is ignored) has a thermal nature effective temperature $\theta = \hbar c^3 / 8\pi G m$. This agrees with the fact that in this problem the corresponding density matrix (139) is thermal, i.e.,

$$\rho = \exp\{-\theta^{-1} \sum_{\omega lm} (\omega a^*_{\text{out}, \omega lm} a_{\text{out}, \omega lm})\}. \tag{152}$$

This last expression for the density matrix in the problem of the formation of a black hole can also be obtained by direct calculation.[49-52,60]

APPENDIX 1

Asymptotic behavior of a massless field in Minkowski space

In this Appendix, we prove a theorem on the asymptotic behavior of solutions of the wave equation on $\mathscr{I}^{\pm}$. We use the method that in our opinion is the most convenient for discussing such problems—the Radon transformation[54] (other approaches are presented in Refs. 55 and 56).

THEOREM. Let $\varphi(t, x)$ be a solution of the wave equation

$$(\partial_t^2 - \Delta) \varphi(t, x) = 0, \quad x \in R^3$$

with initial data $\varphi|_{t=0} = f_1(x)$, $\dot{\varphi}|_{t=0} = f_2(x)$. Suppose that

$f_1, f_2 \in C^2(R^3)$ and the energy is finite:

$$\int (|\nabla f_1|^2 + f_2^2)\, dx < \infty.$$

Suppose

$$\Phi(u, \sigma) = \frac{1}{4\pi}[\partial_u R_1(u, \sigma) - R_2(u, \sigma)],$$

where

$$R_i(u, \sigma) = \int_{S^2} f_i(x)\, \sigma(x\sigma - u)\, d\sigma; \quad \sigma \in S^2; \quad |\sigma| = 1;$$

$d\sigma$ is the surface element on S^2; $i = 1, 2$. Then if $\Phi(u, \sigma)$ has compact support, there exists

$$\lim_{r \to \infty} \{r\varphi(u + r, r\sigma) = \Phi(u, \sigma)\}.$$

The solution φ can be determined from the function Φ uniquely:

$$\left.\begin{aligned} f_1(x) &= -\frac{1}{2\pi}\int \Phi'(x\sigma, \sigma)\, d\sigma, \\ f_2(x) &= -\frac{1}{2\pi}\int \Phi''(x\sigma, \sigma)\, d\sigma, \end{aligned}\right\} \tag{A.1}$$

where the prime denotes the derivative of Φ with respect to the first argument.

The *proof* is based on the following representation of solutions of the wave equation:

$$\varphi(t, x) = \int K(x\sigma - t, \sigma)\, d\sigma,$$

where

$$K(u, \sigma) = -\frac{1}{2\pi}\partial_u \Phi(u, \sigma). \tag{A.2}$$

Hence $\varphi(u + r, r\sigma) = \int K[r(\sigma v - 1) - u, v]\, dv$. Since the function K has compact support, in the limit $r \to \infty$

$$r\int K(r(\sigma v - 1), v)\, dv \to r\int K(r(\sigma v - 1), \sigma)\, dv$$

$$= 2\pi r\int_{-1}^{+1} K(r(z - 1) - u, \sigma)\, dz \to 2\pi\int_0^{\infty} K(\rho - u, \sigma)\, d\rho = \Phi(u, \sigma).$$

To prove the representation (A.2), we use the well-known equation

$$\Delta^2 \int |(x, \sigma)|\, d\sigma = -16\pi^2 \delta^{(3)}(x),$$

by virtue of which for any $f \in C_0^\infty(R^3)$

$$f(x) = -\frac{1}{8\pi^2}\Delta\int R(x\sigma, \sigma)\, d\sigma,$$

where $R(u, \sigma) = \int f(x)\delta(x\sigma - u)dx$. It follows that the pair of functions $\{f_1(x), f_2(x)\}$ is in a one-to-one correspondence with the function

$$K(u, \sigma) = \frac{1}{8\pi^2}[\partial_u^2 R_1(u, \sigma) + \partial_u R_2(u, \sigma)],$$

i.e., we have an expansion of the initial data with respect to plane waves. Equation (A.2) is then obtained by a shift with respect to t.

APPENDIX 2

Conformal transformations

Conformal Mapping of Riemannian Spaces. Besides the original Riemannian space (M, g), consider the space $(M, \hat{g})$, where

$$\hat{g}_{\alpha\beta} = \Omega^2 g_{\alpha\beta}; \quad \hat{g}^{\alpha\beta} = \Omega^{-2} g^{\alpha\beta}. \tag{A.3}$$

We shall denote all quantities referring to the space $(M, \hat{g})$ by the same symbols as for the space (M, g), adding the cap. We denote $\Omega_\alpha = \hat{\nabla}_\alpha \Omega$;

$$\left.\begin{aligned} &\hat{\omega} = \hat{g}^{\alpha\beta}\Omega_\alpha\Omega_\beta/\Omega^2; \quad \hat{\sigma}_{\alpha\beta} = \hat{\nabla}_\alpha\hat{\nabla}_\beta\Omega/\Omega; \\ &\hat{\sigma} = \hat{g}^{\alpha\beta}\hat{\sigma}_{\alpha\beta} = \hat{\Box}\Omega/\Omega. \end{aligned}\right\} \tag{A.4}$$

We have the following connection between the corresponding Christoffel symbols, curvature tensors, and other analogous quantities in the spaces (M, g) and $(M, \hat{g})$ (see, for example, Ref. 20):

$$\Gamma^\alpha_{\beta\gamma} = \hat{\Gamma}^\alpha_{\beta\gamma} - \Omega^{-1}(\delta^\alpha_\beta\Omega_\gamma + \delta^\alpha_\gamma\Omega_\beta - \hat{g}_{\beta\gamma}\hat{g}^{\alpha\tau}\Omega_\tau); \tag{A.5}$$

$$\begin{aligned} R_{\alpha\beta\gamma\delta} = \Omega^{-2}[&\hat{R}_{\alpha\beta\gamma\delta} - \hat{g}_{\alpha\delta}\hat{\sigma}_{\beta\gamma} - \hat{g}_{\beta\gamma}\hat{\sigma}_{\alpha\delta} + \hat{g}_{\beta\gamma}\hat{\sigma}_{\beta\delta} \\ &+ \hat{g}_{\beta\delta}\hat{\sigma}_{\alpha\gamma} + \hat{\omega}(\hat{g}_{\alpha\delta}\hat{g}_{\beta\gamma} - \hat{g}_{\alpha\gamma}\hat{g}_{\beta\delta})]; \end{aligned} \tag{A.6}$$

$$R_{\beta\gamma} = \hat{R}_{\beta\gamma} - 2\hat{\sigma}_{\beta\gamma} + (3\hat{\omega} - \hat{\sigma})\hat{g}_{\beta\gamma}; \tag{A.7}$$

$$R = \Omega^2(\hat{R} - 6\hat{\sigma} + 12\hat{\omega}); \tag{A.8}$$

$$G_{\beta\gamma} \equiv R_{\beta\gamma} - g_{\beta\gamma}R/2 = \hat{G}_{\beta\gamma} - 2\hat{\sigma}_{\beta\gamma} + (2\hat{\sigma} - 3\hat{\omega})\hat{g}_{\beta\gamma}; \tag{A.9}$$

$$C^\alpha_{\cdot\beta\gamma\delta} = \hat{C}^\alpha_{\cdot\beta\gamma\delta}, \tag{A.10}$$

where $C^\alpha_{\cdot\beta\gamma\delta}$ is the Weyl tensor.

Using the relation (A.5), we can in particular readily show that

$$\nabla_\mu\xi^\nu = \hat{\nabla}_\mu\xi^\nu - \Omega^{-1}(\Omega_\mu\xi^\nu + \delta^\nu_\mu\Omega_\alpha\xi^\alpha - \xi_\mu\hat{g}^{\nu\tau}\Omega_\tau). \tag{A.11}$$

Suppose the curve $\gamma : x^\mu = x^\mu(r)$ is a null geodesic in (M, g); then it is simultaneously a null geodesic in the metric $\hat{g}$. If r is an affine parameter along γ in the metric g, then

$$\hat{r} = \int dr\,\Omega^2(x^\mu(r))$$

is an affine parameter along this curve in the metric $\hat{g}$. The proof of this assertion follows from (A.11) by noting that the tangent vector $l^\mu = dx^\mu/dr$ to the null geodesic satisfies the equation $l^\mu\nabla_\mu l^\nu = 0$.

Conformal Invariance of Equations. Suppose that a field (or set of fields) φ satisfies the equation $D_g[\varphi] = 0$, whose coefficients depend on the metric g. Consider a conformal transformation of the metric of the form (A.3). One says that the equation for φ is conformally invariant if there exists a transformation $\phi \to \hat{\varphi} = \hat{\varphi}(\varphi, \Omega)$ such that $\hat{\varphi}$ is a solution of the equation $D_{\hat{g}}[\hat{\varphi}] = 0$ if and only if φ satisfies the equation $D_g[\varphi] = 0$.

The equation

$$(\Box + R/6)\varphi = 0 \tag{A.12}$$

for a scalar massless field φ is conformally invariant, and $\hat{\varphi} = \Omega^{-1}\varphi$. Note that from Eqs. (A.8) and (A.11)

$$\Box\varphi = \Omega^3(\hat{\Box}\hat{\varphi} - 2\hat{\omega}\hat{\varphi} + \hat{\sigma}\hat{\varphi});\quad R\varphi = \Omega^3(\hat{R} - 6\hat{\sigma} + 12\hat{\omega})\hat{\varphi},$$

and therefore

$$\Box\varphi + R\varphi/6 = \Omega^3(\hat{\Box}\hat{\varphi} + \hat{R}\hat{\varphi}/6).$$

Maxwell's equations and the equations of the Yang–Mills field are conformally invariant. The conformal invariance of the Maxwell equations

$$F_{[\mu\nu,\lambda]} = 0,\quad F^{\mu\nu}_{\ ;\nu} = 0 \tag{A.13}$$

can be directly established if one sets $\hat{A}_\mu = A_\mu$ or, equivalently, $\hat{F}_{\mu\nu} = F_{\mu\nu}$. The first of Eqs. (A.13) under this transformation remains unchanged, and the second can be written in the form

$$F^{\mu\nu}_{\ ;\nu} \equiv (g)^{-1/2}\partial_\nu(g^{\mu\alpha}g^{\nu\beta}\sqrt{-g}\,F_{\alpha\beta}) = 0.$$

The invariance of this equation under conformal transformation follows from the equation $\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}\sqrt{-\hat{g}} = g^{\mu\alpha}g^{\nu\beta}\sqrt{-g}$.

The conformal invariance of the Yang–Mills equations is verified similarly by setting $\hat{A}^a_\mu = A^a_\mu$.

Conformal Transformations of Spinor Quantities. Assuming that the Infeld–van der Waerden symbols $\sigma^\mu_{A\dot{A}}$ which realize the isomorphism between the Minkowski space and the space of two-index Hermitian spinors do not change under the conformal transformation, the equation

$$g^{\mu\nu} = \sigma^\mu_{A\dot{A}}\sigma^\nu_{B\dot{B}}\varepsilon^{AB}\varepsilon^{\dot{A}\dot{B}}$$

yields

$$\hat{\varepsilon}^{AB} = \Omega^{-1}\varepsilon^{AB},\quad \hat{\varepsilon}_{AB} = \Omega\varepsilon_{AB}. \tag{A.14}$$

The spinor covariant derivative $\hat{\nabla}_{A\dot{A}} = \hat{\sigma}^\mu_{A\dot{A}}\hat{\nabla}_\mu$ in the space $(M, \hat{g})$ is related as follows to the spinor covariant derivative $\nabla_{A\dot{A}}$ in the space (M, g):

$$\begin{aligned}\hat{\nabla}_{A\dot{A}}T^{D\dot{D}}_{B\dot{B}} &= \nabla_{A\dot{A}}T^{D\dot{D}\cdots}_{B\dot{B}\cdots} - \Gamma_{B\dot{A}}T^{D\dot{D}\cdots}_{A\dot{B}\cdots}\\ &- \Gamma_{A\dot{B}}T^{D\dot{D}\cdots}_{B\dot{A}\cdots} + \varepsilon^D_A\Gamma_{F\dot{A}}T^{F\dot{D}\cdots}_{B\dot{B}\cdots} + \varepsilon^{\dot{D}}_{\dot{A}}\Gamma_{A\dot{F}}T^{D\dot{F}\cdots}_{B\dot{B}\cdots},\end{aligned} \tag{A.15}$$

where

$$\Gamma_{A\dot{A}} = \Omega^{-1}\nabla_{A\dot{A}}\Omega. \tag{A.16}$$

The equation of a massless field of spin s in spinor form is

$$\nabla^{A_1\dot{A}}\varphi_{A_1\ldots A_{2s}} = 0, \tag{A.17}$$

where $\varphi_{A_1\ldots A_{2s}}$ is a symmetric spinor. Using the relation (A.15), we can readily verify that if the field $\varphi_{A_1\ldots A_{2s}}$ under the conformal transformation (A.3) transforms in accordance with the law $\hat{\varphi}_{A_1\ldots A_{2s}} = \Omega^{-1}\varphi_{A_1\ldots A_{2s}}$, then Eq. (A.17) is conformally invariant.

APPENDIX 3

Calculation of asymptotic invariants

We calculate the asymptotic invariants in the case of a scalar massless field whose action is given by Eq. (70). For this, we first of all express $\hat{T}_{\mu\nu} = \Omega^{-2}T_{\mu\nu}$, where $T_{\mu\nu}$ has the form (71), in terms of $\hat{\varphi} = \Omega^{-1}\varphi$ and $\hat{g}$:

$$\hat{T}_{\mu\nu}(\hat{\varphi}, \hat{g}) = \Omega^{-2}T_{\mu\nu}(\varphi = \Omega\hat{\varphi},\ g = \Omega^{-2}\hat{g}). \tag{A.18}$$

Note that

$$\begin{aligned}\Omega^{-2}g_{\mu\nu}g^{\alpha\beta}\varphi_{,\alpha}\varphi_{,\beta} &= \hat{g}_{\mu\nu}[\Omega^{-2}\hat{g}^{\alpha\beta}\Omega_\alpha\Omega_\beta\hat{\varphi}^2\\ &+ 2\Omega^{-1}\hat{g}^{\alpha\beta}\hat{\varphi}_{,\alpha}\Omega_\beta\hat{\varphi} + \hat{g}^{\alpha\beta}\hat{\varphi}_{,\alpha}\hat{\varphi}_{,\beta}];\end{aligned} \tag{A.19}$$

$$\begin{aligned}\nabla_\beta\nabla_\alpha\varphi &= \Omega\hat{\nabla}_\beta\hat{\nabla}_\alpha\hat{\varphi} + \hat{\varphi}\hat{\nabla}_\beta\hat{\nabla}_\alpha\Omega + 2\hat{\nabla}_\alpha\hat{\varphi}\hat{\nabla}_\beta\Omega + 2\hat{\nabla}_\beta\hat{\varphi}\hat{\nabla}_\alpha\Omega\\ &+ 2\hat{\nabla}_\alpha\Omega\hat{\nabla}_\beta\Omega/\Omega + \hat{g}_{\alpha\beta}(\hat{\nabla}_\lambda\Omega\hat{\nabla}^\lambda\Omega\hat{\varphi}/\Omega + \hat{\nabla}^\lambda\Omega\hat{\nabla}_\lambda\hat{\varphi}).\end{aligned} \tag{A.20}$$

Therefore, for $\hat{T}_{\mu\nu}(\hat{\varphi}, \hat{g})$ in the region where $R_{\mu\nu} = 0$,

$$\begin{aligned}\hat{T}_{\mu\nu}(\hat{\varphi}, \hat{g}) &= (2\alpha + 1)[\Omega^{-2}\Omega_\mu\Omega_\nu\hat{\varphi}^2 + \Omega^{-1}\Omega_\mu\hat{\varphi}_{,\nu}\hat{\varphi} + \Omega^{-1}\Omega_\nu\hat{\varphi}_{,\mu}\hat{\varphi}\\ &+ \hat{\varphi}_{,\mu}\hat{\varphi}_{,\nu}] - [(1 + 4\alpha)/2]\hat{g}_{\mu\nu}[\hat{\omega}\hat{\varphi}^2 + 2\Omega^{-1}\hat{g}^{\alpha\beta}\hat{\varphi}_{,\alpha}\Omega_\beta\hat{\varphi}\\ &+ \hat{g}^{\alpha\beta}\hat{\varphi}_{,\alpha}\hat{\varphi}_{,\beta}] + 2\alpha\hat{\varphi}[\hat{\nabla}_\mu\hat{\nabla}_\nu\hat{\varphi} + \hat{\sigma}_{\mu\nu}\hat{\varphi} + 2(\Omega_\mu/\Omega)\hat{\nabla}_\nu\hat{\varphi}\\ &+ 2(\Omega_\nu/\Omega)\hat{\nabla}_\mu\hat{\varphi} + 2(\Omega_\mu\Omega_\nu/\Omega^2)\hat{\varphi} - \hat{g}_{\mu\nu}(\hat{\omega}\hat{\varphi} + \Omega_\gamma\hat{\nabla}^\lambda\hat{\varphi}/\Omega)].\end{aligned} \tag{A.21}$$

The asymptotic invariants

$$P[\xi] = \int_{\mathscr{J}}\hat{T}_{\mu\nu}\xi^\mu d\hat{\Sigma}^\nu \tag{A.22}$$

can be conveniently calculated in the Bondi conformal coordinates $(u, \hat{r}, x^2, x^3)$. In these coordinates $\Omega = \hat{r}$ and in the limit $\hat{r} \to 0$ (on the surface $\mathscr{J}$) we have (see Secs. 4 and 5)

$$\left.\begin{aligned}&\text{a)}\quad \hat{\nabla}_\alpha\hat{\nabla}_\beta\Omega = 0;\\ &\text{b)}\quad \hat{g}^{\alpha\beta}\Omega_\alpha\Omega_\beta/\Omega = 0;\\ &\text{c)}\quad \Omega_\alpha\xi^\alpha = \xi^1 = 0;\\ &\text{d)}\quad \Omega_\alpha\xi^\alpha/\Omega = \partial_{\hat{r}}\xi^{(1)} = \xi^A_{:A}/2.\end{aligned}\right\} \tag{A.23}$$

Since $d\hat{\Sigma}^\mu$ in the integral (A.22) is equal to $\hat{g}^{\mu\alpha}\Omega_\alpha du d\sigma$ ($d\sigma$ is the element of surface area on the unit sphere), with allowance for (A.23) we obtain

$$P[\xi] = \int_{\mathscr{J}}du d\sigma\,[(2\alpha + 1)\partial_u\hat{\varphi}\xi^\mu\partial_\mu\hat{\varphi} + 2\alpha\hat{\varphi}(A\hat{\varphi} + \hat{\nabla}^\mu\Omega\xi^\nu\hat{\nabla}_\mu\hat{\nabla}_\nu\hat{\varphi})].$$

where $A = \lim_{\hat{r}\to 0}[\Omega^{-1}\hat{\nabla}^\mu\Omega\xi^\nu\hat{\nabla}_\mu\hat{\nabla}_\nu\Omega]$.

We now show that A vanishes on $\mathscr{J}$. For this we note that Eq. (36) on $\mathscr{J}$ enables us to write

$$\hat{\nabla}^\mu\Omega\hat{\nabla}^\nu\Omega\hat{\nabla}_\mu\xi_\nu/\Omega = (\Omega_\alpha\xi^\alpha/\Omega)\hat{g}^{\lambda\nu}\Omega_\lambda\Omega_\nu/\Omega = 0.$$

Therefore

$$A = \hat{\nabla}^{\mu}\Omega\hat{\nabla}_{\mu}(\xi^{\nu}\Omega_{\nu})/\Omega = \hat{\nabla}^{\mu}\Omega\hat{\nabla}_{\mu}(\xi^{\nu}\Omega_{\nu}/\Omega) = \partial_u(\partial_{\hat{r}}\xi^1) = 0.$$

Using Eq. (A.23a), we can transform the expression $\hat{\nabla}^{\mu}\Omega\xi^{\nu}\hat{\nabla}_{\mu}\hat{\nabla}\nu\varphi$ to the form $\xi^{\nu}\hat{\nabla}_{\nu}(\Omega_{\mu}\hat{\nabla}^{\mu}\hat{\varphi}) = \xi^{\nu}\partial\nu\partial\mu\hat{\varphi}$. If, as usual, we denote the image of the function φ on $\mathscr{I}$ by Φ: $\Phi = \hat{\varphi}\big|_{\mathscr{I}}$, we finally obtain

$$P[\xi] = \int_{\mathscr{I}} du d\sigma\, [(2\alpha+1)\,\partial_u\Phi\xi^{\mu}\partial_{\mu}\Phi + 2\alpha\Phi\xi^{\mu}\dot{\partial}_{\mu}\partial_u\Phi]. \quad \text{(A.24)}$$

APPENDIX 4

Generating functional for the S matrix

We here derive the expression (122) for the generating functional of the S matrix. Alongside the functional $A(a^*, a)$ corresponding to the operator A it is convenient to use the functional $\tilde{A}(a^*, a) = A(a^*, a)\exp(a^*a)$.

If the vector $\Psi = A\Phi$, then the corresponding functional representations are related as follows:

$$\Psi(a^*) = \int \tilde{A}(a^*, b)\,\Phi(b^*)\exp(-b^*b)\,Db^*Db.$$

Therefore, the element of the S matrix is equal to

$$S_{\beta_1,\ldots,\beta_n;\,\alpha_1,\ldots,\alpha_m} = \int S(a^*, b)\exp(-b^*b - a^*a)\,a_{\beta_1}\ldots a_{\beta_n} \times b^*_{\alpha_1}\ldots b^*_{\alpha_m}\,Db^*Db\,Da^*Da$$
$$= (\delta^n/\delta j_{\beta_1}\ldots\delta j_{\beta_n})(\delta^m/\delta j^*_{\alpha_1}\ldots\delta j^*_{\alpha_m})\exp[iw(j, j^*)]\big|_{\substack{j=0\\ j^*=0}},$$

where

$$\exp[iw(j, j^*)] = \int S(a^*, b)\exp(-b^*b - a^*a + ja + j^*b^*)\,Da^*DaDb^*Db.$$

The last expression can be calculated using the formula

$$\int \exp[-(F, AF)/2 + (\varphi, F)]\,Df^*Df = (\det AA^*)^{-1/4}\exp[(\varphi, A^{-1}\varphi)/2],$$

where $F = \binom{f}{f^*}$, $\varphi = \binom{\varphi_1}{\varphi_2}$.

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