Relativistic three-dimensional description of the interaction of two fermions

N. B. Skachkov and I. L. Solovtsov

Joint Institute for Nuclear Research, Dubna
Fiz. Elem. Chastits At. Yadra **9**, 5–47 (January-February 1978)

In the framework of the quasipotential approach, a three-dimensional relativistic formalism is developed for the interaction of two spin-1/2 particles. The Feynman matrix elements of one-boson exchange, which are used as quasipotentials, and the quasipotential equations in the momentum representation are transformed in such a way that they become a direct geometrical generalization of the corresponding potentials and equations of nonrelativistic quantum mechanics. This similarity to the three-dimensional formalism of quantum mechanics is also retained after the transition to the relativistic configuration representation.

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INTRODUCTION

In the present review, we present a new mathematical formalism for describing a system of two spin-$\frac{1}{2}$ particles that is a relativistic generalization of the nonrelativistic formalism based on the Lippmann–Schwinger equation, the Schrödinger equation, and the system of partial-wave equations. Such a nonrelativistic formalism is widely used to describe NN interactions at low energies, bound states of the positronium type, and composite models of particles, including the spectrum of the recently discovered J/ψ mesons. The Breit–Fermi potentials, which contain relativistic terms of order v^2/c^2, are usually taken as the potentials.

However, now that the new meson factories are operating at intermediate energies at which relativistic effects make an appreciable contribution to the NN interaction, it is necessary to develop a formalism that would enable one to treat the nucleon–nucleon interaction in the relativistic region in a manner which approximates closely to the three-dimensional quantum-mechanical formalism that has been so well developed for low energies. Moreover, the intense development of high energy physics and relativistic nuclear physics makes it urgent to find a convenient and perspicuous mathematical formalism for describing the interaction of relativistic particles.

In the present review, taking the practically important example of the interaction of two spin-$\frac{1}{2}$ particles, we shall show that systematic use of Lobachevskiĭ geometry in the momentum representation makes it possible to give a description in the framework of a single three-dimensional formalism in which it is not necessary to go over from three to four dimensions in the relativistic region. In this formalism, the relativistic expressions do not change their form from their nonrelativistic analogs except in the geometrical nature of the quantities they contain. In other words, we shall show how the formalism of relativistic theory can be cast into a form in which it appears as a direct geometrical generalization of the formalism of quantum mechanics.

The main tools for describing the interaction of two elementary particles are the equations for the scattering amplitude and the wave function of the system. In quantum field theory, wide use is made of the covariant Bethe–Salpeter equation. However, this equation, which is based on the four-dimensional formalism, has a number of shortcomings. The most serious is that the wave function in them depends on two times.

To overcome this difficulty, Logunov and Tavkhelidze proposed the quasipotential approach[1] to the relativistic problem of two bodies. In their approach, the wave function of the system depends on one time variable, so that it has a clear physical meaning and the ordinary probability interpretation of quantum mechanics. The quasipotential approach has been successfully used to calculate relativistic corrections to the levels of the hydrogen atom and positronium, and also to describe scattering processes.[2,3]

The quasipotential equations for the relativistic scattering amplitude and the wave function are written down in the center-of-mass system of the two particles. They have a three-dimensional form, and in the momentum representation appear as relativistic generalizations of the Schrödinger and Lippmann–Schwinger equations. By analogy with these equations, the quasipotential wave function and the scattering amplitude are defined off the energy shell but, in contrast to the Bethe–Salpeter equation, on the mass shell.

In Refs. 4 and 5, a different variant of the quasipotential description was proposed for describing a system of two bodies; it is based on the covariant Hamiltonian formulation of quantum field theory developed by Kadyshevskiĭ.[6] A distinctive feature of the Hamiltonian approach is that all the integrations in the momentum space are performed over the three-dimensional hyperboloid surface of the mass shell[1)]:

$$p_0^2-\mathbf{p}^2=M^2. \tag{1}$$

[1)]We shall here use a system of units in which $\hbar=c=1$, but sometimes for convenience we will retain c.

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The total S matrix is determined by the Lagrangian $\mathcal{L}(x)$ of the theory:

$$S = T\exp\left[\mathrm{i} \int \mathscr{L}(x)\, dx \right].$$

We define the τ matrix by the relation $S = 1 + iT$. For it, we obviously have the representation

$$\mathrm{T} = \sum_{n=1}^{\infty} \mathrm{T}_n = \sum_{n=1}^{\infty} \frac{\mathrm{i}^{n-1}}{n!} \int T[\mathscr{L}(x_1)\ldots\mathscr{L}(x_n)]\, dx_1 \ldots dx_n$$
$$= \sum_{n=1}^{\infty} \mathrm{i}^{n-1} \int \theta(x_1^0 - x_2^0) \ldots \theta(x_{n-1}^0 - x_n^0)\, \mathscr{L}(x_1) \ldots \mathscr{L}(x_n)\, dx_1 \ldots dx_n.$$

In accordance with Ref. 6, in the last expression the θ function can be given a completely covariant form by introducing a unit timelike vector λ: $\lambda^2 = 1, \lambda_0 > 0$. Indeed, in the timelike direction $\theta(x_0 - y_0) = \theta(\lambda(x - y))$, and for spacelike intervals, for which $(x - y)^2 < 0$, the θ function does not contribute by virtue of the condition of local commutativity: $[\mathcal{L}(x), \mathcal{L}(y)] = 0$ if $(x - y)^2 < 0$.

The covariant θ function can be represented as a Fourier integral[6]:

$$\theta(\lambda x) = (1/2\pi \mathrm{i}) \int_{-\infty}^{\infty} d\tau \exp[\mathrm{i}\tau(\lambda x)]/(\tau - \mathrm{i}\varepsilon).$$

Going over to the momentum representation

$$\tilde{\mathscr{L}}(p) = \int \exp(-\mathrm{i}px)\, \mathscr{L}(x)\, dx,$$

we obtain

$$\mathrm{T}_n = \frac{1}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \tilde{\mathscr{L}}(-\lambda\tau_1) \frac{d\tau_1}{\tau_1 - \mathrm{i}\varepsilon}$$
$$\times \tilde{\mathscr{L}}(\lambda\tau_1 - \lambda\tau_2) \frac{d\tau_2}{\tau_2 - \mathrm{i}\varepsilon} \cdots \frac{d\tau_n}{\tau_n - \mathrm{i}\varepsilon} \tilde{\mathscr{L}}(\lambda\tau_n).$$

This last equation can be regarded as the result of iteration of some linear integral equation[6]

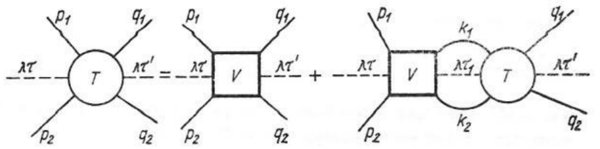
$$R(\lambda\tau) = \tilde{\mathscr{L}}(\lambda\tau) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\mathscr{L}}(\lambda\tau - \lambda\tau') \frac{d\tau'}{\tau' - \mathrm{i}\varepsilon} R(\lambda\tau')$$

under the condition

$$\mathrm{T} = \sum_{n=1}^{\infty} \mathrm{T}_n = R(0).$$

The integral equation we have obtained leads to a spurion diagram technique which differs from the Feynman technique in that all particle momenta, even those corresponding to internal lines, are on the mass shell (1).

The equation for the scattering amplitude of two fermions is represented graphically as follows:



The continuous lines correspond to fermions. The dashed lines correspond to spurions, which carry an additional momentum. The collection of irreducible diagrams corresponding to the square is called the quasipotential.

In the center-of-mass system, the equations for the wave function and the scattering amplitude describing the interaction of two spin-$\frac{1}{2}$ particles in the momentum space has the simple form[5]

$$E_p(E_p - E_q)\psi_q(p)_{\sigma_1\sigma_2}$$
$$= \frac{1}{(4\pi)^3} \sum_{\sigma_1'\sigma_2'} \frac{1}{M} \int d\Omega_{\mathbf{k}} V_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{p}, \mathbf{k}; E_q)\, \psi_q(k)_{\sigma_1'\sigma_2'}; \tag{2}$$

$$T_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{p}, \mathbf{q}) = V_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{p}, \mathbf{q}; E_q)$$
$$+ \frac{1}{(4\pi)^3} \sum_{\sigma_1''\sigma_2''} \frac{1}{M} \int d\Omega_{\mathbf{k}} \frac{V_{\sigma_1\sigma_2}^{\sigma_1''\sigma_2''}(\mathbf{p}, \mathbf{k}; E_q)\, T_{\sigma_1''\sigma_2''}^{\sigma_1'\sigma_2'}(\mathbf{k}, \mathbf{q})}{E_k(E_k - E_q - \mathrm{i}\varepsilon)}, \tag{3}$$

where $V(\mathbf{p}, \mathbf{k}; E_q)$, the quasipotential, depends in the general case on the energy $2E_q = 2\sqrt{q^2 + M^2}$ of the system, and the volume element

$$d\Omega_{\mathbf{k}} = d\mathbf{k}/\sqrt{1 + \mathbf{k}^2/M^2} \tag{4}$$

is an invariant measure on the hyperboloid (1). If the quasipotential is real, then on the energy shell $E_p = E_k = E_q$ the relativistic scattering amplitude in the Kadyshevskiĭ quasipotential approach, as in the Logunov–Tavkhelidze approach, satisfies the relativistic two-particle unitarity condition:

$$\mathrm{Im}\, T_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{p}, \mathbf{q}) = \frac{1}{(8\pi)^3} \sqrt{(E_q^2 - M^2)/E_q^2}$$
$$\times \sum_{\sigma_1''\sigma_2''} \int d\omega_k T_{\sigma_1\sigma_2}^{*\sigma_1''\sigma_2''}(\mathbf{p}, \mathbf{k})\, T_{\sigma_1''\sigma_2''}^{\sigma_1'\sigma_2'}(\mathbf{k}, \mathbf{q}); \tag{5}$$
$$d\omega_k = \sin\theta_k\, d\theta_k\, d\varphi_k$$

and is related to the elastic differential cross section by

$$\left.\begin{array}{c} \dfrac{d\sigma}{d\omega}_{\sigma_1\sigma_2 \to \sigma_1'\sigma_2'} = \dfrac{|T_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(s, t)|^2}{64\pi^2 s}; \\ s = (p_1 + p_2)^2 = 4E_q^2. \end{array}\right\} \tag{6}$$

Our aim is, on the basis of the quasipotential equations (2) and (3), to construct a three-dimensional formalism for spin-$\frac{1}{2}$ particles similar to the one proposed for the spinless case in Refs. 7 and 8.

1. RELATIVISTIC INTERACTION POTENTIALS OF TWO FERMIONS IN THE MOMENTUM REPRESENTATION

In Eqs. (2) and (3), we take the Feynman matrix elements corresponding to one-boson exchange as quasipotentials. This approximation, which is called the one-boson exchange model, is widely used in elementary particle physics, especially to describe the NN interaction. In the nonrelativistic limit, these matrix elements go over into the well known potentials of quantum mechanics, and they are therefore called the relativistic potentials of one-boson exchange (OBEP).[9] However the relativistic OBEP's bear little resemblance to the ordinary potentials of quantum mechanics in the form in which they are given in four-dimensional quantum field theory.

We shall show below that the use of Lobachevskii geometry makes it possible to go over to a three-dimensional expression of the relativistic OBEP's, in which they have the form of a direct geometrical generalization of the corresponding potentials of quantum mechanics taken in the momentum representation.[10] The spin of particles will be described, not in the language of the Dirac γ matrices, but by means of the Pauli σ matricies. To clarify the meaning of the quantities we use, we give some results on Lobachevskiĭ geometry.

Lobachevskiĭ space

As we have already said in the Introduction, the momenta of all particles in the quasipotential equations (2) and (3) are on the mass shell, i.e., their components are related by Eq. (1). Equation (1) defines in momentum space a three-surface whose geometry is not Euclidean but Lobachevskian. As coordinates on the surface (1), we choose the components of the momentum vector $\mathbf{p}$. These are the Cartesian coordinates on the hyperplane $p_0=0$ onto which the hyperboloid is mapped as a result of projection from the point $(\infty, 0)$. Our model of Lobachevskiĭ space will now be the complete three-dimensional p space with metric

$$ds^2=\frac{M^2\,d\mathbf{p}^2+[\mathbf{p}\times d\mathbf{p}]^2}{M^2+\mathbf{p}^2}=g_{ik}(\mathbf{p})\,dp^i\,dp^k. \tag{7}$$

In the nonrelativistic limit $c\to\infty$, the curvature of the hyperboloid tends to zero, and Lobachevskiĭ space goes over into the three-dimensional Euclidean momentum space. At the same time,

$$d\Omega_{\mathbf{k}}=d\mathbf{k}/\sqrt{1+\mathbf{k}^2/M^2}\to d\mathbf{k};\quad ds^2\to d\mathbf{p}^2.$$

The group of motions of the Lobachevskiĭ space realized on the hyperboloid (1) is the Lorentz group. The pure Lorentz transformations $\Lambda_{\mathbf{p}}$ (boosts), i.e., such that $\Lambda_{\mathbf{p}}(M,0)=(p^0,\mathbf{p})$,

$$\Lambda_{\mathbf{p}}^{-1}\mathbf{k}\equiv\mathbf{k}(-)\mathbf{p}=\mathbf{k}-(\mathbf{p}/M)\,[k_0-\mathbf{k}\cdot\mathbf{p}/(M+p_0)]=\Delta; \tag{8}$$

$$(\Lambda_{\mathbf{p}}^{-1}k)^0\equiv(k(-)p)^0=(k^0p^0-\mathbf{k}\cdot\mathbf{p})/M=\sqrt{M^2+(\mathbf{k}(-)\mathbf{p})^2}=\Delta^0, \tag{9}$$

go over in the nonrelativistic limit into the translation transformation in flat Euclidean space: $\mathbf{k}(-)\mathbf{p}\to\mathbf{k}-\mathbf{p}$.

By analogy with (8), we define the vector $\mathbf{k}(+)\mathbf{p}\equiv\Lambda_{\mathbf{p}}\mathbf{k}$, which in the nonrelativistic limit gives $\mathbf{k}+\mathbf{p}$. In spherical coordinates

$$\left.\begin{aligned}&p_0=M\,\mathrm{ch}\,\chi_p;\ \mathbf{p}=M\,\mathrm{sh}\,\chi_p\mathbf{n}_{\mathbf{p}};\ \mathbf{n}_{\mathbf{p}}=\mathbf{p}/|\mathbf{p}|;\\&k_0=M\,\mathrm{ch}\,\chi_k;\ \mathbf{k}=M\,\mathrm{sh}\,\chi_k\mathbf{n}_{\mathbf{k}};\ \mathbf{n}_{\mathbf{k}}=\mathbf{k}/|\mathbf{k}|\end{aligned}\right\} \tag{10}$$

Eq. (9) takes the form of the cosine theorem for a composite angle in Lobachevskiĭ trigonometry:

$$\mathrm{ch}\,\chi_{pk}=\sqrt{1+(\mathbf{k}(-)\mathbf{p})^2/M^2}=\mathrm{ch}\,\chi_p\,\mathrm{ch}\,\chi_k-\mathrm{sh}\,\chi_p\,\mathrm{sh}\,\chi_k\mathbf{n}_{\mathbf{p}}\cdot\mathbf{n}_{\mathbf{k}}. \tag{11}$$

The vector $\mathbf{k}(-)\mathbf{p}$ can be regarded as a relativistic geometrical generalization of the momentum-transfer vector $\mathbf{k}-\mathbf{p}$. By means of (8) and (9), we can readily verify that the square of the momentum-transfer four-vector can be expressed in terms of the vector $\Delta=\mathbf{k}(-)\mathbf{p}$ as follows[7]:

$$t=(k-p)^2=2M^2-2pk=2M^2-2M\sqrt{M^2+(\mathbf{k}(-)\mathbf{p})^2}. \tag{12}$$

In the velocity space, which is a Lobachevskiĭ space,[11-13] an important role is played by the concept of the half-velocity of a particle, proposed in Ref. 14. In Ref. 10, one of the present authors proposed the analogous half-momentum of a particle $\pi_{\mathbf{p}}=(\pi_{\mathbf{p}}^0,\boldsymbol{\pi}_{\mathbf{p}})=M(\mathrm{ch}\chi_p/2,\mathbf{n}_{\mathbf{p}}\mathrm{sh}\chi_p/2)$ and momentum half-transfer $\varkappa$, which is expressed in terms of the momentum transfer in Lobachevskiĭ space

$$\Delta_0=M\,\mathrm{ch}\,\chi_\Delta;\ \Delta=M\,\mathrm{sh}\,\chi_\Delta\mathbf{n}_\Delta;\ \mathbf{n}_\Delta=\Delta/|\Delta| \tag{13}$$

as follows:

$$\begin{aligned}&\varkappa_0=M\,\mathrm{ch}\,\chi_\Delta/2=M\sqrt{(\Delta_0+M)/2M};\\&\varkappa=M\mathbf{n}_\Delta\,\mathrm{sh}\,\chi_\Delta/2=\Delta\sqrt{M/[2(\Delta_0+M)]}.\end{aligned} \tag{14}$$

Equation (12) in terms of the momentum half-transfer vector $\varkappa$ takes the "absolute" form

$$t=(k-p)^2=-4\varkappa^2, \tag{15}$$

since in the nonrelativistic limit, when $\pi_{\mathbf{p}}\to\pi_{\mathbf{p}E}=\mathbf{p}/2$, and $\varkappa\to\varkappa_E=(\mathbf{k}-\mathbf{p})/2$, it goes over into the nonrelativistic relation

$$t=(k-p)^2=-4\varkappa^2\to-(\mathbf{k}-\mathbf{p})^2=-4\varkappa_E^2$$

without changing its form. In what follows, it will be shown that if precisely these quantities, the half-momentum and momentum half-transfer [rather than $\Delta=\mathbf{k}(-)\mathbf{p}$], are used, the relativistic expressions can be transformed to "absolute" form. The reason for this is as follows. The relativistic energy of a particle does not go over directly into the energy of a nonrelativistic particle, since it contains not only the kinetic energy but also the rest energy Mc^2:

$$p_0=\sqrt{\mathbf{p}^2+M^2}\xrightarrow[\frac{v^2}{c^2}\ll1]{}M+\mathbf{p}^2/(2M)=M+2\pi_{\mathbf{p}E}^2/M.$$

The exact expression for the energy of a relativistic particle in terms of the half-momentum has the same form:

$$p_0=\sqrt{\mathbf{p}^2+M^2}=M+2\pi_{\mathbf{p}}^2/M. \tag{16}$$

For the momentum transfer there is a similar expression:

$$\Delta_0=\sqrt{\Delta^2+M^2}=M+2\varkappa^2/M. \tag{17}$$

Thus, since the rest mass is not included in the energy of a particle in the nonrelativistic theory, one can generalize geometrically only the kinetic part $W_{\mathrm{kin}}=p_0-M=2\pi_{\mathbf{p}}^2/M$ of the energy. And it is this role that the half-momentum $\pi_{\mathbf{p}}$ of the particle serves.

Absolute form of relativistic one-boson exchange amplitudes

We shall show here how the relativistic one-boson exchange amplitudes can be expressed in terms of elements of Lobachevskiĭ space. We shall consider here the basic forms of interaction and, taking them as examples, demonstrate the transition to the construction of spin structures by means of vectors belonging to Lobachevskiĭ space. We shall subsequently need the expressions to find the form of the relativistic OBEP's in coordinate space.

Exchange of Pseudoscalar Meson. In the second approximation in the coupling constant, the relativistic fermion–fermion scattering amplitude corresponding to the exchange of a pseudoscalar meson is given by the expression

$$\langle \mathbf{p}_1\sigma_1;\ \mathbf{p}_2\sigma_2 | T^{(2)}_{PS} | \mathbf{k}_1\sigma'_1;\ \mathbf{k}_2\sigma'_2\rangle = -g^2 \frac{\bar u^{\sigma_1}(\mathbf{p}_1)\gamma_5 u^{\sigma'_1}(\mathbf{k}_1)\,\bar u^{\sigma_2}(\mathbf{p}_2)\gamma_5 u^{\sigma'_2}(\mathbf{k}_2)}{\mu^2-(p_1-k_1)^2}. \tag{18}$$

In (18), we go over to bispinors defined in the rest frames of the particles. The four-dimensional transformation matrices of the bispinors $u^\sigma(\mathbf{p}) = S_{\mathbf{p}} u^\sigma(0)$,[2)] corresponding to boosts are parametrized by the particle half-velocity $\omega = \mathbf{n}_{\mathbf{p}} \tanh \chi_p/2$:

$$S_{\mathbf{p}} = \sqrt{(p_0+M)/2M}\,(1+\boldsymbol{\alpha}\cdot\mathbf{p}/(p_0+M)) = \mathrm{ch}\chi_p/2 + \boldsymbol{\alpha}\cdot\mathbf{n}_{\mathbf{p}}\,\mathrm{sh}\,\chi_p/2; \tag{19}$$

where $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $p_0 = M\cosh\chi_p$.

The boosts $\Lambda_{\mathbf{p}}$ do not form a group. Their product is not in general a boost by the resultant vector but contains an additional rotation $V(\Lambda_{\mathbf{p}}, \mathbf{k})$ describing Thomas precession of the spin (Wigner rotation):

$$S_{\mathbf{p}}^{-1} S_{\mathbf{k}} = S_{\Lambda_{\mathbf{p}}^{-1}\mathbf{k}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\}. \tag{20}$$

Since the matrix γ_5 commutes with the matrices $\boldsymbol{\alpha}$, we obtain, using (20), the parametrization (18) in terms of the vectors $\boldsymbol{\Delta} = \mathbf{k}(-)\mathbf{p} = (\Lambda_{\mathbf{p}}^{-1}\mathbf{k})$. As a result and with allowance for (17), the expression (18) is transformed to

$$-g^2 \frac{\bar u^{\sigma_1}(\mathbf{p}_1)\gamma_5 u^{\sigma'_1}(\mathbf{k}_1)\,\bar u^{\sigma_2}(\mathbf{p}_2)\gamma_5 u^{\sigma'_2}(\mathbf{k}_2)}{\mu^2-(p_1-k_1)^2} = g^2 \sum_{\sigma_{1p}\sigma_{2p}} \frac{(\boldsymbol{\sigma}_1\boldsymbol{\varkappa}_1)_{\sigma_1\sigma_{2p}}(\boldsymbol{\sigma}_2\boldsymbol{\varkappa}_2)_{\sigma_2\sigma_{2p}}}{\mu^2+4\varkappa_1^2} \cdot$$
$$\times \mathscr{D}^{1/2}_{\sigma_{1p}\sigma'_1}\{V^{-1}(\Lambda_{\mathbf{p}_1}, \mathbf{k}_1)\}\,\mathscr{D}^{1/2}_{\sigma_{2p}\sigma'_2}\{V^{-1}(\Lambda_{\mathbf{p}_2}\mathbf{k}_2)\}. \tag{21}$$

where $(\boldsymbol{\sigma}\boldsymbol{\varkappa})_{\sigma_1\sigma_2} = \xi^*_{\sigma_1}(\boldsymbol{\sigma}\boldsymbol{\varkappa})\xi^{\sigma_2}$.

As can be seen from (21), after separation of the Wigner rotation[15]

$$\mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\} = \frac{(k_0+M)(p_0+M)-(\boldsymbol{\sigma}\mathbf{k})(\boldsymbol{\sigma}\mathbf{p})}{\sqrt{2(k_0+M)(p_0+M)(k_0p_0-\mathbf{k}\cdot\mathbf{p}+M^2)}}, \tag{22}$$

which has a kinematic origin, the remaining part of the amplitude

$$\langle \mathbf{p}_1\sigma_1;\ \mathbf{p}_2\sigma_2 | T^{(2)}_{PS} | \mathbf{k}_1\sigma_{1\mathbf{p}};\ \mathbf{k}_2\sigma_{2\mathbf{p}}\rangle = g^2 \frac{4(\boldsymbol{\sigma}_1\boldsymbol{\varkappa}_1)_{\sigma_1\sigma_{1p}}(\boldsymbol{\sigma}_2\boldsymbol{\varkappa}_2)_{\sigma_2\sigma_{2p}}}{\mu^2+4\varkappa_1^2} \tag{23}$$

depends on the vectors $\boldsymbol{\Delta} = \mathbf{k}(-)\mathbf{p}$ or, which is the same thing, on the half-transfer vectors $\boldsymbol{\varkappa}$ and is local in Lobachevskiĭ space.[10] Note that in the description of the *NN* interaction one usually uses the one-boson exchange potential, which is obtained from (18) by going to the nonrelativistic limit[9]:

$$V^{(2)}_{PS}(\mathbf{p}, \mathbf{k}) = g^2 \frac{(\boldsymbol{\sigma}_1\boldsymbol{\Delta}_{1E})(\boldsymbol{\sigma}_2\boldsymbol{\Delta}_{2E})}{\mu^2+\Delta_{1E}^2} = g^2 \frac{4(\boldsymbol{\sigma}_1\boldsymbol{\varkappa}_{1E})(\boldsymbol{\sigma}_2\boldsymbol{\varkappa}_{2E})}{\mu^2+4\varkappa_{1E}^2}, \tag{24}$$

where $\boldsymbol{\Delta}_{1E} = 2\boldsymbol{\varkappa}_{1E} = \mathbf{k}_1 - \mathbf{p}_1$.

It is readily seen that the relativistic expression (23) obtained from the Feynman matrix element (18) is a direct geometrical generalization of the nonrelativistic potential (24) expressed by means of the vector $\boldsymbol{\varkappa}_E$. The expression (23) has an "absolute" form, and the only difference from (24) resides in the geometrical nature of the vectors $\boldsymbol{\varkappa}$ and $\boldsymbol{\varkappa}_E$. The relativistic expression (23) can be obtained from (24) by replacing the vector $\boldsymbol{\varkappa}_E$ by its analog $\boldsymbol{\varkappa}$ (14) in Lobachevskiĭ space.

The Wigner rotation occurs in (21) in connection with the following fact. As follows from the law of transformation of the state vectors

$$U(\Lambda_{\mathbf{p}}^{-1})|\mathbf{k}, \sigma\rangle = \sum_{\sigma'} \mathscr{D}^{1/2}_{\sigma\sigma'}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\}|\mathbf{k}(-)\mathbf{p}, \sigma'\rangle,$$

the matrices describing the rotation of the spin under Lorentz transformations depend on the momentum of the state itself. As a result, they are different for the spin indices in the bra and ket part of the matrix element of the $T^{(2)}$ amplitude (18). In the terminology used by Cheshkov and Shirokov,[16] each spin index "sits" on its own momentum. The Wigner rotation in (21) puts them onto the single momentum $\mathbf{p}$, as a result of which they transform in accordance with the little group of this single vector under Lorentz transformations. The Wigner rotation derives from the relativistic spin kinematics. It must appear in (21) because the projections of the spins onto the Z axis in the definition of the state vectors $|\mathbf{k}, \sigma\rangle$ are specified in the rest frames of the particles. And with each of the momenta one can in general associate corresponding coordinate systems, whose axes need not coincide. It is the Wigner rotation which makes the axes coincide.[17]

From the point of view of Lobachevskiĭ geometry, the

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2)In the standard representation in which

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$

the bispinor in the rest frame has the form

$$u^\sigma(0) = \sqrt{2M}\begin{pmatrix} \xi^\sigma \\ 0 \end{pmatrix}.$$

In the spinor representation with

$$\gamma^\mu = \begin{pmatrix} 0 & g^{\mu\mu}\sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix},$$

the spinor has the form

$$u^\sigma(0) = \sqrt{M}\begin{pmatrix} \xi^\sigma \\ \xi^\sigma \end{pmatrix}$$

(ξ_0 are two-component Pauli spinors normalized by the condition $\xi^*_\sigma \xi^{\sigma'} = \delta^{\sigma'}_\sigma$).

Wigner rotation has a perspicuous meaning. It describes rotation through the angle between the old direction of the spin and the new one obtained as a result of parallel transport of the spin vector in the Lobachevskiĭ space (as a space of negative curvature) around the triangle formed by the vectors **p**, **k**, and **Δ**.[13] In the nonrelativistic limit, there is no Wigner rotation: $V(\Lambda_{\mathbf{p}}, \mathbf{k}) = 1$; $v^2/c^2 \to \infty$.

Exchange of a Vector Meson or Photon. The Born approximation for the scattering amplitude in this case is given by

$$\langle \mathbf{p}_1\sigma_1;\ \mathbf{p}_2\sigma_2 | T_V^{(2)} | \mathbf{k}_1\sigma_1';\ \mathbf{k}_2\sigma_2' \rangle = g_V^2 \frac{\bar{u}^{\sigma_1}(p_1)\gamma_\mu u^{\sigma_1'}(k_1)\cdot \bar{u}^{\sigma_2}(p_2)\gamma^\mu u^{\sigma_2'}(k_2)}{\mu^2 - (p_1 - k_1)^2}. \tag{25}$$

As before, we go over in (25) from bispinors to two-component spinors. Using (2) and the equation obtained in Ref. 18:

$$S_p^{-1}\gamma^\mu S_p = (\Lambda_p)^\mu_\nu\,\gamma^\nu = \gamma_0\,[p^\mu + 2\gamma_5 W^\mu(p)]/M, \tag{26}$$

where W^μ is the four-vector of the relativistic spin (the Pauli–Lubański vector) with components[19]

$$\left.\begin{aligned} &W^0(\mathbf{p}) = \boldsymbol{\sigma}\cdot\mathbf{p}/2;\quad \mathbf{W}(\mathbf{p}) = M\boldsymbol{\sigma}/2 + \mathbf{p}(\boldsymbol{\sigma}\cdot\mathbf{p})/[2(p_0 + M)];\\ &W^\mu(p)W_\mu(p) = -M^2 s(s+1) = -M^2(1/2)(1/2+1),\end{aligned}\right\} \tag{27}$$

we obtain the following expression (Fig. 1) for (25) in the center-of-mass system ($\boldsymbol{\varkappa}_1 = -\boldsymbol{\varkappa}_2 = \boldsymbol{\varkappa}$):

$$\begin{aligned}\langle \mathbf{p};\ \sigma_1\sigma_2 | T_V^{(2)} | \mathbf{k};\ \sigma_1'\sigma_2' \rangle = &\sum_{\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}} \langle \mathbf{p};\ \sigma_1\sigma_2 | T_V^{(2)} | \mathbf{k};\ \sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}} \rangle\\ &\times \mathscr{D}^{1/2}_{\sigma_{1\mathbf{p}}\sigma_1'}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\}\,\mathscr{D}^{1/2}_{\sigma_{2\mathbf{p}}\sigma_2'}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\},\end{aligned} \tag{28}$$

where

$$\begin{aligned} &\langle \mathbf{p};\ \sigma_1\ \sigma_2 | T_V^{(2)} | \mathbf{k};\ \sigma_{1p}, \sigma_{2p} \rangle = \xi^*_{\sigma_1}\xi^*_{\sigma_2} T_V^{(2)}(\mathbf{k}(-)\mathbf{p};\ \mathbf{p})\,\xi^{\sigma_{1p}}\xi^{\sigma_{2p}};\\ &T_V^{(2)}(\mathbf{k}(-)\mathbf{p};\ \mathbf{p}) = V_V^{(2)}(\mathbf{k}(-)\mathbf{p};\ \mathbf{p}) = -g_V^2\,[4M^2/(\mu^2 + 4\varkappa^2)]\\ &\quad - g_V^2 4\,\frac{[(\boldsymbol{\sigma}_1\boldsymbol{\varkappa})\cdot(\boldsymbol{\sigma}_2\boldsymbol{\varkappa}) - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\boldsymbol{\varkappa}^2]}{\mu^2 + 4\varkappa^2} - g_V^2\,\frac{8p_0\varkappa_0}{M^2}\,\frac{i(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)[\mathbf{p}\times\boldsymbol{\varkappa}]}{\mu^2 + 4\varkappa^2}\\ &\quad - g_V^2\,\frac{8}{M^2}\,\frac{p_0^2\varkappa_0^2 + 2p_0\varkappa_0(\mathbf{p}\cdot\boldsymbol{\varkappa}) - M^4}{\mu^2 + 4\varkappa^2}\\ &\quad - g_V^2\,\frac{8}{M^2}\,\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{p})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\varkappa})(\boldsymbol{\sigma}_2\cdot\mathbf{p})(\boldsymbol{\sigma}_2\cdot\boldsymbol{\varkappa})}{\mu^2 + 4\varkappa^2}.\end{aligned} \tag{29}$$

The first term in (29) is the relativistic generalization of the Yukawa potential; the second corresponds to tensor forces and the spin–spin interaction; the third contains the spin–orbit interaction; the fourth contributes to the orbital motion. The last term in (29) can be decomposed with respect to the spin structures:

$$\begin{aligned}(\boldsymbol{\sigma}_1\mathbf{p})(\boldsymbol{\sigma}_1\boldsymbol{\varkappa})\cdot(\boldsymbol{\sigma}_2\mathbf{p})(\boldsymbol{\sigma}_2\boldsymbol{\varkappa}) = (\mathbf{p}\boldsymbol{\varkappa})^2 + [\mathbf{p}\times\boldsymbol{\varkappa}]^2\\ + 8i\,\frac{\varkappa_0}{M^3}(\mathbf{p}\boldsymbol{\varkappa})^2(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)[\mathbf{p}\times\boldsymbol{\varkappa}] + 4(i\boldsymbol{\sigma}_1[\mathbf{p}\times\boldsymbol{\varkappa}] + i\boldsymbol{\sigma}_2[\mathbf{p}\times\boldsymbol{\varkappa}])^2,\end{aligned}$$

from which it can be seen that it contains contributions to the orbital motion and the spin–orbit interaction.

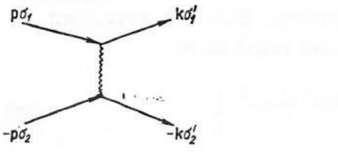

FIG. 1. Diagram of two-fermion scattering in the approximation of vector-particle exchange.

Now the Breit interaction potentials of two fermions can be obtained[20] from the Feynman matrix element corresponding to exchange of a vector particle by going over to the nonrelativistic limit or by using a Foldy–Wouthuysen transformation. We give them for convenience in the same gauge in which the photon propagator in (25) is taken. Rewritten in terms of the nonrelativistic momentum half-transfer $\boldsymbol{\varkappa}_E = (\mathbf{k} - \mathbf{p})/2$, they have the form

$$\begin{aligned} V^{(2)}_{\text{Breit}}(\mathbf{k}(-)\mathbf{p};\ \mathbf{p}) = &-g_V^2\,\frac{4M^2}{\mu^2 + 4\varkappa_E^2} - g_V^2\,\frac{4}{c^2}\,\frac{(\boldsymbol{\sigma}_1\boldsymbol{\varkappa}_E)\cdot(\boldsymbol{\sigma}_2\boldsymbol{\varkappa}_E) - (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)\boldsymbol{\varkappa}_E^2}{(c\mu)^2 + 4\varkappa_E^2}\\ &- g_V^2\,\frac{8}{c^2}\,\frac{i(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)[\mathbf{p}\times\boldsymbol{\varkappa}_E]}{\mu^2 + 4\varkappa_E^2} - g_V^2\,\frac{8}{c^2}\,\frac{(\mathbf{p} + \boldsymbol{\varkappa}_E)^2}{(c\mu)^2 + 4\varkappa_E^2}.\end{aligned} \tag{30}$$

Comparing the expressions (29) and (30), we see that the first three terms in (29) can be regarded as a direct geometrical generalization of the nonrelativistic terms in (30). An important difference between (29) and (30) is that in the exact relativistic expression the spin–orbit interaction occurs with the factor $p_0\varkappa_0/M^2$, which contains a dependence on the energy. This means that the contribution of the spin–orbit interaction increases with increasing energy. In the nonrelativistic limit, the factor $p_0\varkappa_0/M^2 \to 1$. The last term in (29), which is proportional to $1/c^4$, does not contribute to (30), where we have retained only the terms of order $1/c^2$.

Thus, the transition to the three-dimensional expressions in terms of Lobachevskiĭ space makes it possible to represent the relativistic OBEP's as direct geometrical generalizations of the potentials of quantum mechanics. Note also that our use of Lobachevskiĭ geometry makes it possible to go over from the four-dimensional description of spin by means of the Dirac γ matrices to a three-dimensional description by means of the Pauli α matrices. Let us consider this important point in more detail.

The relativistic spin vector $W^\mu(\mathbf{p})$ (27) can be obtained by a boost from its value in the rest frame:

$$W^\mu(\mathbf{p}) = (\Lambda_{\mathbf{p}})^\mu_\nu\, W^\nu(0),$$

in which it has only three necessary components, for example, for spin $\frac{1}{2}$

$$W_0(0) = 0;\quad \mathbf{W}(0) = M\boldsymbol{\sigma}/2.$$

Because the condition

$$p_\mu W^\mu(\mathbf{p}) = 0 \tag{31}$$

holds in any coordinate system, only three of its components are independent. Taking into account these relations and the definition of the momentum-transfer vector $\mathbf{k}(-)\mathbf{p} = (\Lambda_{\mathbf{p}}^{-1}\mathbf{k})$ in Lobachevskiĭ space (8), we can readily prove the important equation

$$(p - k)_\mu W^\mu(\mathbf{p}) = M\boldsymbol{\sigma}(\mathbf{k}(-)\mathbf{p})/2, \tag{32}$$

which establishes the equivalence of the four- and three-dimensional descriptions.

Equation (32) enables us to establish the connection between our three-dimensional parametrization and the general method of parametrizing currents by means of the relativistic spin four-vector developed by Cheshkov and Shirokov.[16] Thus, in accordance with Ref. 16, the matrix element of a local current operator $j(0)$ (scalar or pseudoscalar) can be written as

$$\langle \mathbf{p};\ \sigma | j(0) | \mathbf{k};\ \sigma' \rangle = \frac{1}{(2\pi)^3} \sum_{\sigma_{\mathbf{p}}=-s}^{s} \sum_{n=0}^{2s} \langle \sigma | \{ik_\mu W^\mu(\mathbf{p})\}^n | \sigma_{\mathbf{p}} \rangle \mathscr{D}^s_{\sigma_{\mathbf{p}}\sigma'} \{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\} f_n(t). \quad (33)$$

The invariant functions $f_n(t)$ (form factors) are obtained as coefficients in the expansion of the matrix element with respect to the linearly independent scalars of the rotation group—the scalar products $\{k_\mu W^\mu(\mathbf{p})\}^n$. For a scalar operator, the summation is over even n; for pseudoscalar, over odd n. By virtue of (32) and with allowance for (33), we can cast this expression into the three-dimensional form

$$\langle p;\ \sigma | j(0) | k;\ \sigma' \rangle = \frac{1}{(2\pi)^3} \sum_{\sigma_p=-s}^{s} \sum_{n=0}^{2s} \langle \sigma | \{i\mathbf{W}(0)(\mathbf{k}(-)\mathbf{p})\}^n | \sigma_p \rangle \times \mathscr{D}^s_{\sigma_p\sigma'} \{V^{-1}(\Lambda_p, k)\} f_n(t), \quad (34)$$

in which the relativistic spin vector in the rest frame, $\mathbf{W}(0)$, no longer has redundant components.

Thus, the three-dimensional formulation in Lobachevskiĭ space plays the same role as the Foldy–Wouthuysen transformation, namely, the elimination of the redundant components of the spin vector in the description of relativistic particles. But, in contrast to the Foldy–Wouthuysen transformation, which is associated with an expansion of the interaction terms in a series in powers of v^2/c^2, our transformation is exact. The formulation given here makes it possible to construct a three-dimensional relativistic formalism similar to that of quantum mechanics and develop a phenomenological approach to the relativistic problem of NN interactions. Indeed, the method we have described for constructing three-dimensional relativistic potentials can now be regarded as a method of generalization to the relativistic case of the phenomenological potentials of quantum mechanics taken in the momentum representation. As can be seen from the examples considered above, for such a generalization we need to express the nonrelativistic potentials in momentum space in terms of half momenta. The relativistic expressions are obtained by replacing $\varkappa_E = (\mathbf{k}-\mathbf{p})/2$ by $\varkappa = \sqrt{M/[2(\Delta_0+M)]}(\mathbf{k}(-)\mathbf{p})$ and adding the kinematic Wigner rotation. Below, for the example of the NN interaction, we demonstrate the method of constructing spin structures by means of a triplet of vectors in Lobachevskiĭ space.

Construction of *NN* spin structures in Lobachevskiĭ space

To describe the scattering of two spin-$\frac{1}{2}$ particles one constructs five independent invariant spin structures by means of bispinors and γ matrices:

$$\left.\begin{aligned} &R_\alpha = (\bar{u}(\mathbf{p})\Gamma_\alpha u(\mathbf{k}))(\bar{u}(-\mathbf{p})\Gamma_\alpha u(-\mathbf{k}));\\ &\Gamma_\alpha = (I,\ \gamma_5,\ \gamma_\mu,\ \gamma_\mu\gamma_5,\ (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2). \end{aligned}\right\} \quad (35)$$

In the general case, i.e., not restricting ourselves to the OBEP approximation, the quasipotential $V(\mathbf{p},\mathbf{k};E_q)$ in Eq. (2) is decomposed with respect to the spin structures (35). Repeating the arguments used to express (21) and (25) in terms of elements of Lobachevskiĭ space [(23) and (29)], and using also Eqs. (26) and (27), we can cast the structures (35) into three-dimensional form once we have separated the Wigner rotation. However, in a phenomenological description it is convenient to avoid the four-dimensional formalism and from the very start use the three-dimensional analogy with the formalism of quantum mechanics. For this, we construct a triplet of independent vectors in Lobachevskiĭ space and, by means of them, the requisite number of spin structures.

We recall that usually, off the energy shell ($\mathbf{p}^2 \neq \mathbf{k}^2$), these spin structures are constructed by means of a triplet of orthogonal vectors[5]:

$$\mathbf{l} = (\mathbf{k}+\mathbf{p})/N_l;\quad \mathbf{n} = [\mathbf{k}\times\mathbf{p}]/N_{\mathbf{n}};\quad \mathbf{m} = [\mathbf{l}\times\mathbf{n}]/N_{\mathbf{m}}. \quad (36)$$

On the energy shell, they go over into the ordinary basis vectors

$$\mathbf{l} = (\mathbf{k}+\mathbf{p})/N_l;\quad \mathbf{n} = [\mathbf{k}\times\mathbf{p}]/N_{\mathbf{n}};\quad \mathbf{m} = (\mathbf{k}-\mathbf{p})/N_{\mathbf{m}}. \quad (37)$$

Lobachevskiĭ space has constant negative curvature. This has the consequence that, in contrast to nonrelativistic theory in which the vectors $\mathbf{l}$ and $\mathbf{m}$ are orthogonal, $(\mathbf{k}+\mathbf{p})\cdot(\mathbf{k}-\mathbf{p}) = 0$, on the energy shell, the vectors $\Delta_+ = \mathbf{k}(+)\mathbf{p}$ and $\Delta_- = \mathbf{k}(-)\mathbf{p}$ are not orthogonal in the sense of the ordinary scalar product:

$$\Delta_+\cdot\Delta_- = (\mathbf{k}(+)\mathbf{p})\cdot(\mathbf{k}(-)\mathbf{p})|_{\mathbf{k}^2=\mathbf{p}^2} = -[\mathbf{k}\times\mathbf{p}]^2/M^2.$$

But, as vectors in Lobachevskiĭ space, they are orthogonal in the sense of the scalar product in Lobachevskiĭ space. It is readily verified that if the metric tensor in (7) is used then

$$\Delta_+ \odot \Delta_- \equiv (\mathbf{k}(+)\mathbf{p})^i g_{ik} (\mathbf{k}(-)\mathbf{p})^k = 0.$$

Since the quasipotential equation is written down off the energy shell, we need a triplet of orthogonal vectors off the energy shell. We now show that if the kinematic Wigner rotation is separated from the quasipotential, i.e., if we go over to a parametrization of the form (28), then the requirement of a definite T parity of the spin structures leads to a unique determination of these structures.

For this, we represent the quasipotential $V(\mathbf{p},\mathbf{k};E_q)$ in (2) and (3) in the form

$$V(\mathbf{p},\mathbf{k};E_q) = \sum_j V_j(\mathbf{k}(-)\mathbf{p},\mathbf{p};E_q) Q_1^{(j)}(\mathbf{p},\mathbf{k}) Q_2^{(j)}(\mathbf{p},\mathbf{k}) \times \mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}},\mathbf{k})\} \mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}},\mathbf{k})\}, \quad (38)$$

where $V_j(\mathbf{k}(-)\mathbf{p},\mathbf{p};E_q)$ are invariant functions; $Q_{1,2}^{(j)}(\mathbf{p},\mathbf{k})$ are the required spin structures. For a T-invariant amplitude or quasipotential we must have

$$\left.\begin{aligned} &T(\mathbf{p},\mathbf{k}) = (U_{1T}^{-1}U_{2T}^{-1}T(-\mathbf{k},-\mathbf{p})U_{1T}U_{2T})^T;\\ &U_{iT} = \alpha(\sigma_2)_i;\quad |\alpha| = 1, \end{aligned}\right\} \quad (39)$$

where U_{iT} is a unitary operator on the spinor space of particle i. Remembering that $V^{-1}(\Lambda_{-k}, -p) = V(\Lambda_p, k)$, we obtain from (38) and (39) the condition of T invariance for the spin structures:

$$Q(\mathbf{p}, \mathbf{k}) = \pm \mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\} \times [U_T^{-1} Q(-\mathbf{k}, -\mathbf{p}) U_T]^T [\mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\}]^+, \quad (40)$$

which differs from the nonrelativistic condition only by the presence of the Wigner rotations.

We have at our disposal the two vectors **p** and **k**. The Wigner rotation takes place in the plane passing through these two vectors and around the vector $\mathbf{n} = [\mathbf{k} \times \mathbf{p}]/N_n$. Since this rotation does not change the vector **n**, the spin structure $(\sigma \cdot \mathbf{n})$ has positive T parity.

The remaining two unit vectors must lie in the plane formed by the vectors **p** and **k**. We require that the structure $(\sigma \cdot \mathbf{m})$, where we write **m** in the form $(\mathbf{k}+x\mathbf{p})/|\mathbf{k}+x\mathbf{p}|$, have negative T parity. In accordance with (40), this means

$$\sigma \cdot (\mathbf{k} + x\mathbf{p}) = \mathscr{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\} \sigma \cdot (-\mathbf{p} - x\mathbf{k}) \mathscr{D}^{+1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\}, \quad (41)$$

whence with allowance for the explicit expression (22) for the $\mathfrak{D}$ function we can readily determine the required factor $x(p_0, k_0, \mathbf{p} \cdot \mathbf{k})$:

$$x = -(k_0 - \mathbf{k} \cdot \mathbf{p}/(p_0 + M))/M.$$

The vector $\mathbf{k} + x\mathbf{p}$ which we have obtained is none other than the momentum-transfer vector in Lobachevskiĭ space already introduced: $\Delta = \mathbf{k}(-)\mathbf{p}(8)$. Thus,

$$m = (k(-)\mathbf{p})/|\mathbf{k}(-)\mathbf{p}|. \quad (42)$$

Similarly, we can find the vector **l**. However, it is simpler to take **l** in the form

$$\mathbf{l} = [\mathbf{n} \times \mathbf{m}], \quad (43)$$

and verify the T parity of $(\sigma \mathbf{l})$. For this, we use the relation $(\sigma \mathbf{l}) = -i(\sigma \cdot \mathbf{n})(\sigma \cdot \mathbf{m})$. Placing it between two $\mathfrak{D}$ functions and using (41), we readily establish that the condition of positive T parity is satisfied with the **l** vector (43).

It is now easy to write down a general parametrization of the P- and T-invariant scattering amplitude of two fermions off the energy shell:

$$T(\mathbf{p}, \mathbf{k}) = \{V_1 + V_2(\sigma_1\mathbf{n} + \sigma_2\mathbf{n}) + V_3(\sigma_1\mathbf{n})(\sigma_2\mathbf{n}) + V_4(\sigma_1\mathbf{l})(\sigma_2\mathbf{l}) + V_5(\sigma_1\mathbf{m})(\sigma_2\mathbf{m}) + V_6[(\sigma_1\mathbf{l})(\sigma_2\mathbf{m}) + (\sigma_1\mathbf{m})(\sigma_2\mathbf{l})]\} \times \mathscr{D}_1^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\} \mathscr{D}_2^{1/2}\{V^{-1}(\Lambda_p, \mathbf{k})\}. \quad (44)$$

The function $V_6(p_0, k_0, \mathbf{p} \cdot \mathbf{k})$ is odd under transposition of the momenta p and k. On the energy shell, when $p_0 = k_0$, the sixth term in (44) vanishes: $V_6(p_0, k_0, \mathbf{p} \cdot \mathbf{k}) = -V_6(k_0, p_0, \mathbf{p} \cdot \mathbf{k}) = 0$.

The advantages of the spin structures introduced for parametrizing the relativistic amplitudes given by quantum field theory come out most clearly in the example of the γ_5 interaction; for in accordance with (21) the amplitude $T_{PS}^{(2)}(\mathbf{p}, \mathbf{k})$ contains only the single spin structure $(\sigma_1\mathbf{m})(\sigma_2\mathbf{m})$, whereas three constructions[5] corresponding to the fourth, fifth, and sixth terms in (44) occur if the ordinary triplet $\mathbf{l}_E$, $\mathbf{m}_E$, and $\mathbf{n}_E$ of unit vectors is used.

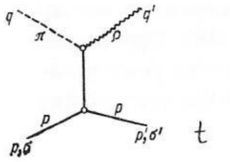

FIG. 2. Diagram of pion exchange for the process $\pi^+ + p \to \rho^+ + p$.

We demonstrate the convenience of parametrizing the amplitudes in terms of elements of Lobachevskiĭ space for one further example involving particles of different spins. We consider the interaction of pseudoscalar, spinor, and vector particles[21]: $\pi^+ + p \to \rho^+ + p$. The amplitude of such a process in the pion-exchange approximation (Fig. 2) has the following form in the normalization adopted in Ref. 22:

$$T^{(2)}(\mathbf{p}, \mathbf{p}') = \frac{2igf_{\rho\pi\pi}}{(2\pi)^6} \frac{\bar{u}^{\sigma'}(\mathbf{p}')\gamma^5 u^{\sigma}(\mathbf{p})}{\mu_\pi^2 - (p - p')^2} \mathbf{e}^{\mu}(\lambda, \mathbf{q}') \mathbf{q}_{\mu},$$

where $\mathbf{e}^{\mu}(\lambda, \mathbf{q}')$ is the polarization four-vector of the vector meson. The polarization index λ takes the values -1, 0, 1. In accordance with Ref. 21, the four-dimensional scalar product $\mathbf{e}^{\mu}(\lambda, \mathbf{q}')q_{\mu}$ can be reduced to a three-dimensional scalar product as follows:

$$\mathbf{e}^{\mu}(\lambda, \mathbf{q}') q_{\mu} = e^{\nu}(\lambda, 0)(\Lambda_{\mathbf{q}'}^{-1})_{\nu}^{\mu} q_{\mu} = \mathbf{e}(\lambda, 0) \cdot \Delta_q,$$

where we have chosen $e^0(\lambda, 0) = 0$. Using the method set forth above, we can also readily transform $\bar{u}^{\sigma'}(\mathbf{p}')\gamma^5 u^{\sigma}(\mathbf{p})$ to three-dimensional form. As a result of separation of the Wigner rotation, we obtain a simple three-dimensional parametrization of the amplitude of this process[21]:

$$T^{(2)}(\mathbf{p}, \mathbf{p}') = -\frac{2igf_{\rho\pi\pi}}{(2\pi)^6} \frac{[\sigma \times \varkappa_p]}{\mu_\pi^2 + 4\varkappa_p^2} \mathbf{e}(\lambda, 0) \cdot \Delta_q.$$

It is interesting to note that the momentum half-transfer $\varkappa_p$ corresponds to spinor particles, i.e., fermions. The momentum transfer Δ_q corresponds to bosons.

Local form of the quasipotential equation

As we have established above, once the kinematic Wigner rotation has been separated out, the remaining part of the quasipotential

$$V_{\sigma_1\sigma_2}^{(2)\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}}(\mathbf{k}(-)\mathbf{p}; \mathbf{p}) \equiv T_{\sigma_1\sigma_2}^{(2)\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}}(\mathbf{k}(-)\mathbf{p}; \mathbf{p})$$

is local in the Lobachevskiĭ space. The spin indices of this part "sit" on a single momentum. Note that on the left-hand side of Eq. (2) the spin indices of the wave function $\Psi_q(\mathbf{p})_{\sigma_1\sigma_2}$ sit on the momentum **p**, while on the right-hand side in $\Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}$ they sit on **k**. We now "reseat" the spin indices of $\Psi_q^{12}(\mathbf{k})$ on **p** as well. This transformation has the form

$$\Psi_q(\mathbf{k})_{\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}} = \sum_{\sigma_1'\sigma_2'} \mathscr{D}_{\sigma_{1\mathbf{p}}\sigma_1'}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}}, \mathbf{k})\} \mathscr{D}_{\sigma_{2\mathbf{p}}\sigma_2'}^{1/2}\{V^{-1}(\Lambda_p, \mathbf{k})\} \Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}. \quad (45)$$

Since the $\mathfrak{D}$ functions needed for this operation are already contained in the quasipotential (38), they do this job automatically. As a result, we arrive at an equation for the wave function in which all the spin indices sit on the single momentum p:

$$E_p(E_p-E_q)\Psi_q(\mathbf{p})_{\sigma_1\sigma_2}$$
$$=\frac{1}{(4\pi)^3}\sum_{\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}}\int\frac{d^3k}{E_k}V^{\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}}_{\sigma_1\sigma_2}(\mathbf{k}(-)\mathbf{p};\mathbf{p};E_q)\Psi_q(\mathbf{k})_{\sigma_{1\mathbf{p}}\sigma_{2\mathbf{p}}}. \qquad (46)$$

With this form of the quasipotential equation, the interaction is described by a quasipotential $V(\mathbf{k}(-)\mathbf{p};\mathbf{p},E_q)$ which is local in Lobachevskiĭ space, and the right-hand side of (46) is a contraction in Lobachevskiĭ space. The naturalness of going over to this form of the equation becomes obvious when it is solved perturbatively; for if we use perturbation theory to solve the equation for the wave function describing the scattering of two particles[5]:

$$\Psi_q(\mathbf{p})_{\sigma_1\sigma_2}=\frac{(2\pi)^3}{M}\delta(\mathbf{p}-\mathbf{k})\sqrt{\mathbf{p}^2+M^2}\,\xi_{\sigma_1}\xi_{\sigma_2}$$
$$+\frac{1}{E_q(E_p-E_q-i\varepsilon)}\frac{1}{(4\pi)^3}\sum_{\sigma_1'\sigma_2'}\int\frac{d^3k}{E_k}V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)\Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}, \qquad (47)$$

then when $\Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}$ on the right-hand side of (47) is replaced in the first approximation by the expression $(2\pi)^3/M\delta(\mathbf{p}-\mathbf{k})\sqrt{\mathbf{p}^2+M^2}\xi_{\sigma_1'}\xi_{\sigma_2'}/M$, which describes free motion, the spin indices $\sigma_1'\sigma_2'$ sitting on the momentum p must be reseated on k. But if we work with the equations in the form (46), there is no need for this additional operation since it is already taken into account by the transformation (45). Clearly, in the equation for the relativistic scattering amplitude (3), by virtue of the unitarity of the matrix $\mathfrak{D}^{1/2}\{V^{-1}(\Lambda_p,\mathbf{k})\}$, one can also reseat similarly all the spin indices on a single momentum, for example, p.

In order to achieve complete analogy with the nonrelativistic Lipmann–Schwinger equation, we go over to a Green's function linear in E_p. For this, following Ref. 7, we define a new amplitude $A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}$ off the energy shell in the following manner:

$$A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{q})=T^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{q})/(8\pi\sqrt{4E_pE_q}), \qquad (48)$$

and introduce the quasipotential

$$\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)=-V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)/(2M\sqrt{4E_pE_q}). \qquad (49)$$

Equations (2) and (3) in terms of the new quantities take the form

$$A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{q})=-\frac{M}{4\pi}\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{q};E_q)$$
$$+\frac{1}{(2\pi)^3}\sum_{\sigma_1''\sigma_2''}\int\tilde{V}^{\sigma_1''\sigma_2''}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_0)\frac{d\Omega_\mathbf{k}}{2E_q-2E_k+i\varepsilon}A^{\sigma_1'\sigma_2'}_{\sigma_1''\sigma_2''}(\mathbf{k},\mathbf{q}); \qquad (50)$$

$$(2E_q-2E_p)\Psi_q(\mathbf{p})_{\sigma_1\sigma_2}=\frac{1}{(2\pi)^3}\sum_{\sigma_1'\sigma_2'}\int\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)\Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}d\Omega_\mathbf{k} \qquad (51)$$

and formally are a direct geometrical generalization of the Lipmann–Schwinger and Schrödinger equations for particles with spin. We take (50) and (51) as the basic equations for the unknown scattering amplitude $A^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}$ and wave function $\Psi_{q\sigma_1\sigma_2}$. We shall assume that the quasipotential $\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)$ is given and, on the energy shell $E_p=E_k=E_q$, is related to the quasipotential $V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)$, i.e., the set of Feynman matrix elements, by

$$\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)=-V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{k};E_q)/(4ME_q). \qquad (52)$$

In order to preserve the locality of the quasipotential in the Lobachevskiĭ momentum space off the energy shell as well, we define its off-energy-shell extension by means of (52) and not (49). Thus, in the second order in the coupling constant the term $\tilde{V}(\mathbf{p},\mathbf{k};E_q)$ in (52) differs from $V(\mathbf{p},\mathbf{k};E_q)$ defined by Eqs. (18) and (25) only by the factor $-1/(4ME_q)$. The quasipotential equations (2) and (3) after reseating of all the spin indices on a single momentum and transformation to the form (50) and (51) appear as the direct geometrical generalization of the analogous nonrelativistic equations. The quasipotentials in them are defined in accordance with Eqs. (52), (23), and (29) and have the form of geometrical generalizations of their nonrelativistic analogs (24) and (30).

2. RELATIVISTIC POTENTIALS IN COORDINATE SPACE

In this section, we shall find the form of the relativistic potentials (23) and (29) in the relativistic configuration representation. The relativistic configuration representation was introduced earlier in Ref. 7. It is well known that the transition to the ordinary coordinate space is made by means of a Fourier transformation, i.e., an expansion with respect to the functions $\exp(i\mathbf{q}\cdot\mathbf{r})$ that realize unitary irreducible representations of the Galileo group. In the relativistic region, the momentum space is no longer Euclidean but rather the Lobachevskiĭ space realized on the upper sheet of the hyperboloid (1). The group of motions of Lobachevskiĭ space is the Lorentz group. Therefore, if the quasipotentials (23) and (29), which are local in Lobachevskiĭ space, are associated with local expressions in the configuration representation, the transition to it must be made by means of expansions on the group of motions of Lobachevskiĭ space—the Lorentz group.

An expansion with respect to unitary irreducible representations of the Lorentz group has already been used in the theory of elementary particles in Refs. 23 and 24. However, in all these cases the group parameter, which in our approach plays the role of relative coordinate, was not interpreted as a relativistic generalization of the modulus of a radius vector. This prevented the authors of Refs. 23 and 24 from interpreting the expansion on the Lorentz group as a relativistic generalization of the Fourier transformation to the coordinate representation. We give briefly the main results concerning the relativistic configuration representation needed in the subsequent constructions.

Relativistic configuration representation

Mathematically, the formalism of harmonic analysis on the Lorentz group is well known. In Refs. 7 and 8, this formalism is used in the form given it in Refs. 23 and 24.

A complete and orthogonal system of functions on the hyperboloid (1) was introduced in Ref. 23:

$$\left.\begin{array}{c}\xi(\mathbf{p};\,\mathbf{n},\,r)=[(p_0-\mathbf{p}\cdot\mathbf{n})/M]^{-1-irM};\\ p_0=\sqrt{\mathbf{p}^2+M^2};\quad \mathbf{n}=(\sin\theta\cos\varphi,\,\sin\theta\sin\varphi,\,\cos\theta);\quad \mathbf{n}^2=1.\end{array}\right\}\tag{53}$$

The completeness and orthogonality relations for the functions (53) are given in the Appendix. The functions $\xi(\mathbf{p},\mathbf{n},r)$ realize infinite-dimensional, unitary, irreducible representations of the Lorentz group, and the parameter r in (53) is related to the eigenvalues of the Lorentz-group Casimir operator $\hat{C}_L=M_{\mu\nu}M^{\mu\nu}/4$ ($M_{\mu\nu}$ are the generators of the Lorentz group) by

$$\hat{C}_L\xi(\mathbf{p};\,\mathbf{n},\,r)=\left(\frac{1}{M^2}+r^2\right)\xi(\mathbf{p};\,\mathbf{n},\,r),\quad 0\leqslant r<\infty.\tag{54}$$

In Ref. 7, it was suggested that the parameter r should be regarded as the relativistic generalization of a relative coordinate; for in the nonrelativistic limit, in which the Lobachevskiĭ space goes over into Euclidean space:

$$\xi(\mathbf{p};\,\mathbf{n},\,r)\to\exp(i\mathbf{p}\cdot\mathbf{r}),\tag{55}$$

the Lorentz-group Casimir operator $\hat{C}_L$ goes over into the Casimir operator $\hat{C}_E$ of the group of motions of Euclidean space:

$$\hat{C}_L\to\left(i\frac{\partial}{\partial\mathbf{p}}\right)^2=\hat{C}_E.\tag{56}$$

Note that ordinary plane waves $\exp(i\mathbf{p}\cdot\mathbf{r})$ realize unitary irreducible representations of the group of translations of flat Euclidean space. On them, the square of the nonrelativistic coordinate is an eigenvalue of the Casimir operator $\hat{C}_E$ of the Euclidean group:

$$\hat{C}_E\exp(i\mathbf{p}\cdot\mathbf{r})=r^2\exp(i\mathbf{p}\cdot\mathbf{r}).\tag{57}$$

Therefore, the relativistic generalization of the relative coordinate proposed in Ref. 7 preserves its group-theoretical meaning.

In Ref. 7, it was shown that the operator of the free Hamiltonian for the plane waves (53)

$$\hat{H}_0\xi(\mathbf{p};\,\mathbf{n},\,r)=p_0\xi(\mathbf{p};\,\mathbf{n},\,r)\tag{58}$$

is the differential–difference operator

$$\hat{H}_0=M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right)+\frac{i}{r}\,\mathrm{sh}\left(\frac{i}{M}\frac{\partial}{\partial r}\right)-\frac{\Delta_{\theta,\varphi}}{2Mr^2}\exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right)\tag{59}$$

with a step proportional to the Compton wavelength $1/M$ of the particle. In the nonrelativistic limit,

$$\exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right)-1\to\frac{i}{M}\frac{\partial}{\partial r}$$

and $\hat{H}_0$ goes over into the free Hamiltonian of the Schrödinger equation. Similarly, in Ref. 27 a momentum operator was found that satisfies the condition

$$\hat{\mathbf{P}}\xi(\mathbf{p};\,\mathbf{n},\,r)=\mathbf{p}\xi(\mathbf{p};\,\mathbf{n},\,r).\tag{60}$$

The explicit form of the operator $\hat{\mathbf{P}}$ is given in the Appendix.

In Refs. 7 and 8, the image of the meson propagator $1/(\mu^2-t)$ in the new coordinate space was found:

$$V^{\mathrm{rel}}_{\mathrm{Yuk}}(r)=\begin{cases}\dfrac{1}{4\pi r}\dfrac{\mathrm{ch}\,rMa}{\mathrm{sh}\,rM\pi}\ \text{for}\ \ \mu^2<4M^2;\ a=\arccos[(\mu^2-2M^2)/2M^2]; & (61a)\\ \dfrac{1}{4\pi r}\dfrac{\cos rMb}{\mathrm{sh}\,rM\pi}\ \text{for}\ \ \mu^2>4M^2;\ b=\mathrm{Arch}\,[(\mu^2-2M^2)/2M^2], & (61b)\end{cases}$$

which can be regarded as a relativistic generalization of the Yukawa potential. For $\mu^2<4M^2$, the expression (61a) goes over into the ordinary Yukawa potential $\exp(-\mu r)/4\pi r$. For $\mu=0$, Eq. (61) is the image of the propagator for exchange of a massless particle, $1/(p-k)^2$, and appears as the Coulomb potential modified at distances of the order of the Compton wavelength:

$$V^{\mathrm{rel}}_{\mathrm{Yuk}}(r)=(\mathrm{cth}\,rM\pi)/4\pi r.\tag{62}$$

To conclude this section, we note some important properties of the new relativistic coordinate. The first is obvious—the modulus of the new relativistic coordinate r is a relativistic invariant since it parametrizes the eigenvalues of the invariant Casimir operator $\hat{C}_L$ of the Lorentz group [see (54)]. As a result, the images of the meson propagators—the relativistic potentials (61) and (62)—are also relativistically invariant quantities.

Another important property of the relativistic coordinate follows from its connection with the invariant mean square radius of the system:

$$\langle r_0^2\rangle=\frac{6}{F(0)}\frac{\partial F(t)}{\partial t}\bigg|_{t=0},\tag{63}$$

where $F(t)$ is the invariant form factor of the system. It is well known that in the Breit coordinate system [in which $p=-k$ and $t=(p-k)^2=-\mathbf{q}^2=-(\mathbf{p}-\mathbf{k})^2$] the expression for the invariant mean square radius takes the nonrelativistic form

$$\langle r_0^2\rangle\big|_{\mathrm{Br.\,syst}}=-\frac{6}{F_0}\frac{\partial F(t)}{\partial\mathbf{q}^2}\bigg|_{\mathbf{q}^2=0}=\frac{1}{F(0)}\left[\left(i\frac{\partial}{\partial\mathbf{q}}\right)^2F(t)\right]_{\mathbf{q}^2=0}$$

and is an eigenvalue of the Casimir operator $\hat{C}_E$ of the group of motions of the three-dimensional Euclidean momentum space. It was shown in Ref. 28 that the group-theoretical meaning of eigenvalue of the Lorentz-group Casimir operator can be given to the formal definition of the invariant mean square radius of the system (63):

$$\langle r_0^2\rangle=\frac{6}{F(0)}\frac{\partial F(t)}{\partial t}\bigg|_{t=0}=\frac{1}{F(0)}[\hat{C}_LF(t)]\big|_{t=0}.\tag{64}$$

The transformation to the relativistic coordinate space for the form factor $F(t)$ has by virtue of the spherical symmetry the form[28]

$$F(t)=4\pi\int\frac{\sin rMy}{rM\,\mathrm{sh}\,y}F(r)r^2dr;$$
$$y=\mathrm{Arch}(1-t/2M^2)$$

and holds in any coordinate system (and not only in the Breit system, as in the usual approach). It follows from the last equation and from (64) and (54) that the expression for the invariant mean square radius in terms of the invariant distribution $F(r)$ has the form

$$\langle r_0^2\rangle=\frac{\int(1/M^2+r^2)F(r)\,dr}{\int F(r)\,dr}=\frac{1}{M^2}+\frac{\int r^2F(r)\,dr}{\int F(r)\,dr}. \qquad (65)$$

This equation, which relates the new coordinate r to the radius of the system through $\langle r_0^2\rangle$, will be important in what follows in the discussion of the form of the relativistic tensor forces. It should be noted that in the case when the distribution $F(r)$ in the new coordinate space is a function of constant sign, the new coordinate, as follows from (65), describes only distances greater than the Compton wavelength $1/M$.

Relativistic spin-orbit interaction

Let us consider the part of the quasipotential (29) containing the spin–orbit interaction:

$$V_{SL}(\Delta,\mathbf{p})=-ig_V^2\frac{4p_0}{M}\frac{\mathbf{S}\cdot[\mathbf{p}\times\Delta]}{\mu^2+4\varkappa^2};\quad \mathbf{S}=\frac{\sigma_1+\sigma_2}{2}. \qquad (66)$$

By means of the transformations with the functions (53) in r space, we obtain[29]

$$V_{SL}(\mathbf{r},\mathbf{p})=\frac{g_V^2}{(2\pi)^3}\frac{p_0}{M}(\mathbf{S}\cdot[\mathbf{p}\times\nabla_{\mathrm{diff}}])V_{\mathrm{Yuk}}(r),$$

where ∇_{diff} is the differential-difference operator related to the momentum operator by $\nabla_{\mathrm{diff}}=i\hat{\mathbf{P}}$.

We use the explicit form of the operator ∇_{diff} (see the Appendix) and represent (66) in the form

$$V_{SL}(\mathbf{r},\mathbf{p})$$
$$=\frac{g_V^2}{(2\pi)^3}\frac{p_0}{M}\frac{1}{r+i/M}\frac{1}{2i/M}\left[1-\exp\left(-2\frac{i}{M}\frac{\partial}{\partial r}\right)\right](\mathbf{S}\cdot\mathbf{L})V_{\mathrm{Yuk}}(r). \qquad (67)$$

The orbital angular momentum operator $\mathbf{L}$ can be expressed in terms of the vector product of $\mathbf{r}$ and $\mathbf{p}$ in accordance with the formula[29]

$$\mathbf{L}=[\mathbf{r}\times\mathbf{p}]\exp\left(-\frac{i}{M}\frac{\partial}{\partial r}\right), \qquad (68)$$

which obviously goes over into the nonrelativistic expression. However, the explicit expression for the operator $\hat{\mathbf{P}}$ in spherical coordinates (1) contains the operations of differentiation with respect to the angular variables in the usual manner, whereas the operation of finite-difference differentiation affects only the modulus of r and does not occur in the expression (68) for the operator $\mathbf{L}$ in spherical coordinates:

$$L_x=i\left(\sin\varphi\frac{\partial}{\partial\theta}+\cos\varphi\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi}\right);$$
$$L_y=-i\left(-\cos\varphi\frac{\partial}{\partial\theta}+\sin\varphi\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi}\right);$$
$$L_z=-i\frac{\partial}{\partial\varphi},$$

so that $\mathbf{L}_2=-\Delta_{\theta,\varphi}$, where $\Delta_{\theta,\varphi}$ is the Laplacian on a sphere. This remark makes it possible to use the nonrelativistic formalism in a partial-wave expansion, making modifications only in the radial part.

In the nonrelativistic limit, the operator

$$\frac{1}{2i/M}\left[1-\exp\left(-2\frac{i}{M}\frac{\partial}{\partial r}\right)\right]$$

goes over into the ordinary operator $\partial/\partial r$ of differentiation, and the potential (67) goes over into the potential of the spin-orbit interaction of quantum mechanics:

$$V_{SL}^{\mathrm{nrel}}(r)\sim\frac{1}{c^2r}\frac{\partial V}{\partial r}(\mathbf{S}\cdot\mathbf{L}).$$

It should be noted especially that in the relativistic potential (67) there are no terms containing higher orders of a singularity than occur in the Coulomb or Yukawa potentials; for in the relativistic case the role of differentiation is played by the finite-shift operations, as a result of which the potential acquires a singularity at a complex point. The shift is made along the imaginary axis with a step proportional to the Compton wavelength of the particle. Therefore, in the nonrelativistic limit these singularities are superimposed on the singularity at the origin of the potential itself. Thus, one can say that the semirelativistic description associated with expansion in powers of v^2/c^2 of the interaction terms leads to a merging of the singularities, whereas use of a systematic relativistic formalism removes this degeneracy.

Relativistic tensor forces

In the nonrelativistic meson theory of nuclear forces, the tensor potential arises as the Fourier transform of the expression $(\sigma_1\cdot\mathbf{q})(\sigma_2\cdot\mathbf{q})/(\mu^2+\mathbf{q}^2)$, and it can be represented as the result of application of the operator $(\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)$ to the well known Yukawa potential:

$$(\sigma_1\cdot\nabla)(\sigma_2\cdot\nabla)_{\mathrm{Yuk}}^{\mathrm{nrel}}(r)=\frac{1}{3}[\mu^2V_{\mathrm{Yuk}}^{\mathrm{nrel}}(r)-4\pi\delta^{(3)}(\mathbf{r})](\sigma_1\cdot\sigma_2)$$
$$+\frac{1}{3}\left[\mu^2+\frac{3\mu}{r}+\frac{3}{r^2}\right]V_{\mathrm{Yuk}}^{\mathrm{nrel}}(r)S_{1,2}, \qquad (69)$$

where

$$\left.\begin{aligned}&S_{1,2}=3(\sigma_1\cdot\mathbf{n})(\sigma_2\cdot\mathbf{n})-(\sigma_1\cdot\sigma_2);\\&V_{\mathrm{Yuk}}^{\mathrm{nrel}}(r)=\exp(-\mu r)/4\pi r.\end{aligned}\right\} \qquad (70)$$

In the relativistic case, the tensor forces arise from considering the analogous part of the quasipotential in the momentum space:

$$\frac{4(\sigma_1\cdot\varkappa)(\sigma_2\cdot\varkappa)}{\mu^2+4\varkappa^2}=\frac{(\sigma_1\cdot\Delta)(\sigma_2\cdot\Delta)}{\mu^2-2M^2+2M\Delta_0}\frac{2M}{\Delta_0+M}. \qquad (71)$$

After the transition to the r space with the functions (53), we find that

$$V(\mathbf{r})=\frac{4M^2}{4M^2-\mu^2}(\sigma_1\cdot\nabla_{\mathrm{diff}})(\sigma_2\cdot\nabla_{\mathrm{diff}})\left(1-\frac{1}{2\operatorname{ch} rMa}\right)V_{\mathrm{Yuk}}(r). \tag{72}$$

Let us consider the case of the Yukawa potential (61a) for $\mu^2<4M^2$. Performing the difference differentiation, we obtain, as in the nonrelativistic case, scalar and tensor parts[29]:

$$V(\mathbf{r})=V_S(r)(\sigma_1\cdot\sigma_2)+V_T(r)S_{1,2}, \tag{73}$$

where

$$V_S(r)=\frac{1}{3}\left[\mu^2V_{\mathrm{Yuk}}(r)-8\pi\frac{\delta(1/M^2+r^2)\,\delta(\mathbf{n})}{r}\right] \tag{74}$$

and

$$V_T(r)=\frac{1}{3}\frac{r^2}{(r+i/M)(r+2i/M)}\left[\mu^2+\frac{3\mu}{r}\left(1-\frac{\mu^2}{2M^2}\right)\frac{\operatorname{th} rMa}{\sqrt{1-\mu^2/4M^2}}\right.$$
$$\left.+\frac{1}{r^2}\frac{3-2\mu^2/M^2(1-\mu^2/4M^2)-3/(2\operatorname{ch} rMa)}{1-\mu^2/4M^2}\right]V_{\mathrm{Yuk}}(r). \tag{75}$$

The potential (73) with such V_S and V_T goes over in the nonrelativistic limit into the expression (69). We see that the relativistic expression (74) contains a δ function of the argument $1/M^2+r^2$ instead of $\delta(r)$ in (69). As we noted earlier [see (65)], it is the combination $1/M^2+r^2$ that measures the mean square radius of the system in the relativistic case, and the center of the system corresponds to the point $X^2=1/M^2+r^2=0$ in the relativistic case, in complete analogy with the nonrelativistic term $\delta(r)$ in (69). As in the case of the spin-orbit interaction, the potential V_T (75), in contrast to the nonrelativistic expression (69), does not contain singularities at the origin higher than the Yukawa potential (61) itself.

3. QUASIPOTENTIAL EQUATION FOR PARTICLES WITH SPIN IN THE RELATIVISTIC CONFIGURATION REPRESENTATION

In the quasipotential equation (2), the transition to the configuration representation can be made by means of two different sets of functions on the hyperboloid (1). One of them consists of the functions (53), and the expansion itself, in the notation adopted in Ref. 7, is

$$\Psi(\mathbf{r})=\frac{1}{(2\pi)^3}\int\frac{d^3p}{p_0}\xi(\mathbf{p};\mathbf{n},r)\,\Psi(\mathbf{p}). \tag{76}$$

The other complete set, which was obtained in Ref. 30 (see also Refs. 31 and 32), contains a dependence on the spin variables. The difference between these two complete sets of functions arises from the different laws of transformation of spinless and spin wave functions.[30]

However, if we work with the quasipotential equation in the form (46), expansions with respect to the spinless scalar plane waves (53) are sufficient; for in the covariant formulation of the equation for the wave function

$$\sqrt{s_p}(\sqrt{s_p}-\sqrt{s_q})\,\Psi_q(\mathbf{p})_{\sigma_1\sigma_2}$$
$$=\frac{4}{(4\pi^3)}\sum_{\sigma_1'\sigma_2'}\frac{1}{M}\int d\Omega_{\mathbf{k}}V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{k}(-)\mathbf{p},\mathbf{p};E_q)\,\Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}$$

the Green's function $(\sqrt{s_p}-\sqrt{s_q}-i\varepsilon)^{-1}$, which goes over into $(2E_p-2E_q-i\varepsilon)^{-1}$ in the center-of-mass system [see (50) and (51)], is a scalar in the spin space, and the entire dependence of the Hamiltonian on the spin is concentrated in the quasipotential.[3)] Therefore, after all the spin indices have been seated on the single momentum p they transform in accordance with the little group of this vector (the indices σ_p take the numerical values $\pm\frac{1}{2}$), i.e., they undergo the same Wigner rotation under Lorentz transformations. By virtue of the unitarity of the matrix $\mathfrak{D}^{1/2}\{V^{-1}(\Lambda_{\mathbf{p}},\mathbf{k})\}$, the Wigner rotations are separated out on the left- and right-hand sides of Eqs. (50) and (51) and do not change the form of the equations or the potentials. Thus, for our purposes it is sufficient to have a complete and orthogonal system of functions in Lobachevskiĭ space, and these are provided by the spinless functions (53).

In what follows, we shall work with the equation for the wave function in a fixed coordinate system—the center-of-mass system. After the transformation (76) has been applied, Eq. (51) takes the form

$$(2E_q-2\hat{H}_0)\,\Psi_q(\mathbf{r})_{\sigma_1\sigma_2}$$
$$=\frac{1}{(2\pi)^6}\int d\Omega_{\mathbf{p}}\xi(\mathbf{p};\mathbf{n},r)\sum_{\sigma_1'\sigma_2'}\int d\Omega_k\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\Delta,\mathbf{p};E_q)\,\Psi_q(\mathbf{k})_{\sigma_1'\sigma_2'}. \tag{77}$$

Our aim is to transform the right-hand side of Eq. (77), which contains the interaction, to a local form in the new coordinate space. For this, in the integral part of (77) we go over to the wave function in the r space:

$$\Psi_q(\mathbf{k})_{\sigma_1\sigma_2}=\int d^3r_1\xi^*(\mathbf{k};\mathbf{n}_1,r_1)\,\Psi_q(\mathbf{r}_1)_{\sigma_1\sigma_2}$$

and apply the equation

$$\xi(\mathbf{k};\mathbf{n},r)=\xi(\mathbf{k}(-)\mathbf{p};\mathbf{n}_{\Lambda_{\mathbf{p}}},r)\,\xi(\mathbf{p};\mathbf{n},r),$$

where we have the unit vector[31,32]

$$\mathbf{n}_{\Lambda_{\mathbf{p}}}=\frac{M\mathbf{n}-\mathbf{p}[1-(\mathbf{p}\cdot\mathbf{n})/(p_0+M)]}{p_0-\mathbf{p}\cdot\mathbf{n}}. \tag{78}$$

In (77), we use the invariance of the volume element $d\Omega_{\mathbf{k}}=d\Omega_{\mathbf{k}(-)\mathbf{p}}$ and go over to the image of the potential in the relativistic configuration representation:

$$V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(r,\mathbf{n};\mathbf{p};E_q)=\frac{1}{(2\pi)^6}\int d\Omega_{\Delta}\xi^*(\Delta;\mathbf{n},r)\,\tilde{V}^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\Delta,\mathbf{p};E_q). \tag{79}$$

After substitution of (79) in (77), the right-hand side takes the form

$$\int d^3r_1\sum_{\sigma_1'\sigma_2'}\int d\Omega_{\mathbf{p}}\xi(\mathbf{p};\mathbf{n},r)\,\xi^*(\mathbf{p};\mathbf{n}_1,r_1)V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(r_1,\mathbf{n}_{\Lambda_{\mathbf{p}}};\mathbf{p};E_q)\,\Psi_q(r)_{\sigma_1'\sigma_2'}. \tag{80}$$

We know from the previous subsection that the dependence of the potential $V(r_1,\mathbf{n}_{\Lambda_{\mathbf{p}}};\mathbf{p};E_q)$ on the unit vector $\mathbf{n}_{\Lambda_{\mathbf{p}}}$ is concentrated in the spin structures $S^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}$. As a result, the potential can be represented in the form

$$V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(r,\mathbf{n}_{\Lambda_{\mathbf{p}}};\mathbf{p};E_q)=V(r;\mathbf{p};E_q)\,S^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\mathbf{p},\mathbf{n}_{\Lambda_{\mathbf{p}}}). \tag{81}$$

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3)Such a formalism is similar to the nonrelativistic formalism (Pauli equation), in which only the terms of the interaction depend on the spin.

The function $V(r; \mathbf{p}; E_q)$, which depends only on the modulus of the coordinate r, can be taken in front of the integration with respect to the momentum if the vector $\mathbf{p}$ in it is replaced by the operator $\hat{\mathbf{P}}$. As a result, (80) takes the form[29]

$$\int d^3r_1 V(r_1; \hat{\mathbf{P}}; E_q) \sum_{\sigma_1'\sigma_2'} Z_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{r}, \mathbf{r}_1) \Psi_q(\mathbf{r}_1)_{\sigma_1'\sigma_2'}, \tag{82}$$

where the function $Z_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}$ is determined by the spin structures of the potential,

$$Z_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{r}, \mathbf{r}_1) = \int d\Omega_{\mathbf{p}} \xi(\mathbf{p}; \mathbf{n}, r) S_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\mathbf{p}, \mathbf{n}_{\Lambda_{\mathbf{p}}}) \xi^*(\mathbf{p}; \mathbf{n}_1, r_1). \tag{83}$$

It is obvious from (83) that for the part of the potential that does not depend on the spin variables, i.e., for $S_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'} \sim \delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}$, the interaction is described in a local manner:

$$(2E_q - 2\hat{H}_0)\Psi_q(\mathbf{r})_{\sigma_1\sigma_2} = \sum_{\sigma_1'\sigma_2'} V_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(r; \mathbf{p}; E_q)\Psi_q(\mathbf{r})_{\sigma_1'\sigma_2'}, \tag{84}$$

since the function $Z(\mathbf{r}, \mathbf{r}_1)$ becomes a δ function: $Z_{\text{spinless}}(\mathbf{r}, \mathbf{r}_1) = \delta(\mathbf{r} - \mathbf{r}_1)$. Important interaction potentials such as the Yukawa (61) and Coulomb (62) belong to this case.

We shall show further that for other forms of interaction a similar localization can be achieved. Thus, for the spin–orbit interaction the Z function is proportional to the δ function, and for the tensor forces one can separate a term proportional to the δ function from $Z(\mathbf{r}, \mathbf{r}_1)$.

Local form of the equation for the spin-orbit and tensor forces

For the spin–orbit interaction, the spin structure of the potential has in accordance with (66) and (78) the form

$$S(\mathbf{p}, \mathbf{n}_{\Lambda_{\mathbf{p}}}) = \mathbf{S}\cdot[\mathbf{n}_{\Lambda_{\mathbf{p}}} \times \mathbf{p}] = (\mathbf{S}\cdot[\mathbf{n} \times \mathbf{p}])\, M/(p_0 - \mathbf{p}\cdot\mathbf{n}).$$

Then in accordance with the definition (83)

$$\hat{Z}(\mathbf{r}, \mathbf{r}_1) = \exp\left[-\frac{i}{M}\frac{\partial}{\partial r}\right] \int d\Omega_{\mathbf{p}} (\mathbf{S}\cdot[\mathbf{n}\times\mathbf{p}])\, \xi(\mathbf{p}; \mathbf{n}, r)\, \xi^*(\mathbf{p}; \mathbf{n}_1, r_1) = (\mathbf{S}\cdot\mathbf{L})\, \delta(\mathbf{r} - \mathbf{r}_1)/r.$$

Thus, the integral in (82) can be performed and the equation takes for form[29]

$$(2E_q - 2\hat{H}_0)\Psi_q(\mathbf{r}) = \hat{V}_{SL}(\mathbf{r})\Psi_q(\mathbf{r}),$$

where

$$V_{SL}(\mathbf{r}) = \frac{g_V^2}{(2\pi)^3}\frac{r + iM}{r - iM}\frac{1}{r}\frac{1}{2i/M}\left[\exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - \exp\left(-\frac{i}{M}\frac{\partial}{\partial r}\right)\right] \times V_{\text{Yuk}}(r)\frac{\hat{H}_0}{M}(\mathbf{S}\cdot\mathbf{L}). \tag{85}$$

An angular dependence of the tensor interaction is contained in the operator

$$(\sigma_1\cdot\mathbf{n}_{\Lambda_{\mathbf{p}}})(\sigma_2\cdot\mathbf{n}_{\Lambda_{\mathbf{p}}}), \tag{86}$$

where the unit vector $\mathbf{n}_{\Lambda_{\mathbf{p}}}$ is given by (78). In this case, we split the function $Z(\mathbf{r}, \mathbf{r}_1)$ into two parts:

$$Z(\mathbf{r}, \mathbf{r}_1) = \exp\left(-2\frac{i}{M}\frac{\partial}{\partial r}\right)(\sigma_1\cdot\mathbf{n})(\sigma_2\cdot\mathbf{n})\,\delta^{(3)}(\mathbf{r} - \mathbf{r}_1) + \tilde{Z}(\mathbf{r}, \mathbf{r}_1). \tag{87}$$

The first term obviously removes the integration in (82), while the second is, compared with the first, a relativistic correction of order $1/Mc$:

$$\tilde{Z}(\mathbf{r}, \mathbf{r}_1) = \exp\left(-2\frac{i}{M}\frac{\partial}{\partial r}\right)\frac{1}{M}\int d\Omega_{\mathbf{p}}\xi(\mathbf{p}; \mathbf{n}, r)\left\{-[(\sigma_1\cdot\mathbf{n})(\sigma_2\cdot\mathbf{p}) + (\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{n})]\left(1 - \frac{\mathbf{p}\cdot\mathbf{n}}{p_0 + M}\right) + \frac{1}{M}(\sigma_1\cdot\mathbf{p})(\sigma_2\cdot\mathbf{p})\left(1 - \frac{\mathbf{p}\cdot\mathbf{n}}{p_0 + M}\right)^2\right\}\xi^*(\mathbf{p}; \mathbf{n}_1, r_1). \tag{88}$$

Therefore, the right-hand side of the quasipotential equation has the form

$$\left[V_S(r)(\sigma_1\cdot\sigma_2) + V_T^*\left(r - \frac{2i}{M}\right)S_{1,2}\right]\Psi_q(\mathbf{r}) + \int d^3r_1 V_T^*(r_1)\tilde{Z}(\mathbf{r}, \mathbf{r}_1)\Psi_q(\mathbf{r}_1). \tag{89}$$

Thus, the first term on the right-hand side of (89) is a local relativistic tensor interaction, differing from the nonrelativistic one in the new expressions for the radial potential functions $V_S(r)$ and $V_T(r)$.

Note that if desired one can take the integral in the relativistic integral correction term (89) in any order in powers of p/M if the remaining part of the integrand in (88) is expanded in powers of p/M and the condition of orthogonality of the "plane waves" is used. However, in a phenomenological description one need not consider the correction terms to the local part of the tensor interaction, since the local part is itself completely relativisitic. At the same time, the equation with relativistic tensor potential can be written in a form analogous to the Schrödinger equation:

$$(2E_q - 2\hat{H}_0)\Psi_q(\mathbf{r}) = g_V^2\left[V_S(r)(\sigma_1\cdot\sigma_2) + V_T^*\left(r - \frac{2i}{M}\right)S_{1,2}\right]\Psi_q(\mathbf{r}). \tag{90}$$

System of partial-wave equations

We denote by $\chi^{(S)}(\sigma)$ the eigenfunctions of the operators of the square of the total spin $\mathbf{S} = (\sigma_1 + \sigma_2)/2$ and the projection of the spin onto the direction of the Z axis:

$$\left.\begin{aligned} &\mathbf{S}^2\chi^{(S)}(\sigma) = s(s+1)\chi^{(S)}(\sigma);\\ &S_Z\chi^{(S)}(\sigma) = \sigma\chi^{(S)}(\sigma). \end{aligned}\right\} \tag{91}$$

The spinors $\chi^{(S)}(\sigma)$ form a complete and orthogonal system of functions. For $S = 1$, the functions $\chi^{(1)}(\sigma)$ have the form

$$\chi^{(1)}(-1) = \begin{pmatrix}1\\0\\0\end{pmatrix};\quad \chi^{(1)}(0) = \begin{pmatrix}0\\1\\0\end{pmatrix};\quad \chi^{(1)}(1) = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{92}$$

Considering the partial-wave expansions of plane waves (see the Appendix) we write

$$\xi(\mathbf{q}; \mathbf{n}, r)\chi_\mu^{(S)}(\sigma) = 4\pi\sum_{jlM} i^l p_l(\operatorname{ch}\chi_q, r)\{\Omega_{jlM}^{*(S)}(\mathbf{n}_q)\}_\sigma\{\Omega_{jlM}^{(S)}(\mathbf{n})\}_\mu, \tag{93}$$

where $\mathbf{n}_q = \mathbf{q}/q$; the functions $p_l(\cosh\chi_q, r)$ are defined in accordance with (A.4) The spherical spinors $\Omega_{jlM}^{(S)}$ have the usual form

$$\{\Omega_{jlM}^{(S)}(\mathbf{n})\}_\alpha = \langle lM - \alpha s\alpha \mid jM\rangle Y_{l,M-\alpha}(\mathbf{n}). \tag{94}$$

Thus, the angular part of the partial-wave expansion for the plane wave with spin (93) in our approach has the same form as in the nonrelativistic formalism. This makes it convenient to use the proposed formalism since one can exploit the results obtained in nonrelativistic quantum mechanics. Let us demonstrate this for the example of the interaction of two fermions. Suppose their interaction is described by the quasipotential

$$\hat{V}(\mathbf{r}) = \hat{V}_S(r) + \hat{V}_{SL}(r)(\mathbf{S}\cdot\mathbf{L}) + \hat{V}_T(r) S_{1,2}, \tag{95}$$

where $\mathbf{S} = (\sigma_1 + \sigma_2)/2$ is the operator of the total spin and $S_{1,2} = 6(\mathbf{S}\cdot\mathbf{n})^2 - 2\mathbf{S}^2$ is the operator of the tensor forces. If the one-boson exchange model is used, it is necessary to take the relativistic Yukawa potential (61) as $\hat{V}_S$ and replace $\hat{V}_{SL}$ and V_T, respectively, by the expressions (85) and the tensor part of the potential in (89).[4]

We introduce the radial function $\omega^j_{l's';ls}(r)$ by means of an expansion analogous to the plane-wave expansion (93):

$$\Psi^{(S)}_{q\mu}(\mathbf{r};\sigma) = \frac{4\pi}{r\,\mathrm{sh}\,\chi_q} \sum_{\substack{jlM\\ s'l'}} \omega^j_{l's';ls}(r) \{\Omega^{*(S)}_{jlM}(\mathbf{n}_q)\}_\sigma \{\Omega^{(S)}_{jl'M}(\mathbf{n})\}_\mu. \tag{96}$$

Then the three-dimensional equation for the function $\Psi^{(S)}_q(\mathbf{r};\sigma)$ goes over into a system of radial equations, which in the general case has the form

$$(2E_q - 2\hat{H}^{\mathrm{rad}}_0)\,\omega^j_{l's';ls}(r) = \sum_{l''s''} \hat{v}^j_{l's';l''s''}(r;E_q)\,\omega^j_{l''s'';ls}(r), \tag{97}$$

where

$$H^{\mathrm{rad}}_0 = M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) + \frac{l'(l'+1)}{2Mr(r+i/M)}\exp\left(\frac{i}{M}\frac{\partial}{\partial r}\right); \tag{98}$$

$$\hat{v}^j_{l's';ls}(r;E_q) = \int d\omega_n \Omega^{*(S')}_{jl'M}(\mathbf{n})\,\hat{V}(\mathbf{r};E_q)\,\Omega^{(S)}_{jlM}(\mathbf{n}). \tag{99}$$

The matrix elements of the quasipotential $\hat{v}^j(r;E_q)$ can be readily calculated. Moreover, because of the complete correspondence between the relativistic spin structures of the quasipotential (95) and the nonrelativistic spin structures, the matrix elements (99) have formally the same form as in the nonrelativistic case. The only difference is in the explicit expressions for the functions $\hat{V}_S$, $\hat{V}_{SL}$, and $\hat{V}_T$.

The required system of partial-wave equations for two fermions interacting through the quasipotential (95) has the form[5][29]:

a) for $S=0, l'=j$

$$\left[2E_q - 2M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{Mr(r+i/M)}\right]\omega_{j,0} = \hat{V}_S\omega_{j,0}; \tag{100}$$

b) for $S=1, l'=j$

$$\left[2E_q - 2M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{Mr(r+i/M)}\right]\omega_{j,1} = (\hat{V}_S + 2\hat{V}_T)\,\omega_{j,1}; \tag{101}$$

c) for $S=1, l'=j-1$

$$\left[2E_q - 2M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - \frac{j(j-1)}{Mr(r+i/M)}\right]\omega_{j-1,1}$$
$$= \left[\hat{V}_S + (j-1)\hat{V}_{RL} - \frac{2(j-1)}{2j+1}\hat{V}_T\right]\omega_{j-1,1} - \frac{6\sqrt{j(j+1)}}{2j+1}V_T\omega_{j+1,1}; \tag{102}$$

d) for $S=1, l'=j+1$

$$\left[2E_q - 2M\,\mathrm{ch}\left(\frac{i}{M}\frac{\partial}{\partial r}\right) - \frac{(j+1)(j+2)}{Mr(r+i/M)}\right]\omega_{j+1,1}$$
$$= \left[\hat{V}_S - (j+2)\hat{V}_{SL} - \frac{2(j+1)}{2j+1}\hat{V}_T\right]\omega_{j+1,1} - \frac{6\sqrt{j(j+1)}}{2j+1}\hat{V}_T\omega_{j-1,1}. \tag{103}$$

Thus, Eqs. (100) and (101), which describe the interaction in the singlet state and in the triplet state with $l'=l=j$, are not coupled. Only Eqs. (102) and (103) for the states with $l'=j-1$ and $l'=j+1$, which have the same parity, are coupled.

We now give the solution of Eq. (100) for the Coulomb interaction. For this, in (29) we take only the first term for $\mu=0$:

$$V\hat{V}^{(2)}(\mathbf{k}(-)\,\mathbf{p};\,\mathbf{p};\,E_q) = -e^2\frac{4M^2}{4\varkappa^2} = e^2\frac{2M}{\Delta_0 - M} = e^2\frac{4M^2}{(p-k)^2},$$

which corresponds to the contribution of the scalar part of the photon propagator. In the relativistic configuration representation, its analog is the relativistic attractive Coulomb potential:

$$V(r) = 4M\alpha(\mathrm{cth}\,\pi rM)/r. \tag{104}$$

Going over to the second-order equation, we arrive in accordance with (52) at the potential $\tilde{V}(r) = -(\alpha/E_q)(\coth \pi rM/\hbar)/r$. At the same time, the quasipotential equation takes the form

$$\left(2\hat{H}_0 - \frac{\alpha}{E_q}\frac{\mathrm{cth}\,\pi rM}{r}\right)\Psi_q(\mathbf{r}) = 2E_q\Psi_q(\mathbf{r}). \tag{105}$$

Using the results of Ref. 8, we find that this equation has as solutions the functions

$$\Psi_q(\mathbf{r}) = \sum_{l=0}^{\infty}(2l+1)\,i^l\Psi_{ql}(r)\,p_l\left(\frac{\mathbf{q}\cdot\mathbf{r}}{q\cdot r}\right), \tag{106}$$

where the radial wave functions

$$\left.\begin{aligned}
&\Psi_{ql}(r) = \exp\left[-ix\left(l+1-\frac{\alpha\,\mathrm{cth}\,\pi rM}{\sin 2x}\right)\right]\\
&\times\exp(-xrM)\,\frac{\Gamma(irM+l+1)}{\Gamma(irM+1)}\,{}_2F_1\Big(l+1-irM,\\
&l+1-\frac{\alpha\,\mathrm{cth}\,\pi r M}{\sin 2x},\ 2l+2,\ 1-\exp(-2ix)\Big);\\
&E_q = M\cos x
\end{aligned}\right\} \tag{107}$$

are real because of the inclusion of the i-periodic factor $\exp[-ix(l+1-\alpha\coth\pi rM/\sin 2x)]$. It is important to note that this same factor ensures finiteness of the wave function at the point $X^2=0$ when one calculates the mean value of the spin–spin interaction (74) with respect to the functions (107).

The energy levels are found from the requirement that the hypergeometric function in (107) increase not faster than a polynomial in the limit $r\to\infty$. As a result, we arrive at the quantization rule

$$\alpha/\sin 2x_q = n. \tag{108}$$

[4]As was shown above, the integral term for the tensor forces in (89) can be ignored in a phenomenological description.

[5]For brevity, we omit the indices jlM of the function $\omega^j_{l's';ls}(r)$.

This rule coincides exactly with the formula that determines the energy spectrum obtained by solution of the Logunov–Tavkhelidze equation with the potential (104) (see Ref. 27):

$$|W| = 4M\sin^2\left(\frac{1}{4}\arcsin\frac{\alpha}{n}\right), \tag{109}$$

where, in accordance with $2E_q = 2M\cos x_q = 2M - |W|$, $|W|$ is the binding energy. From (109) in the nonrelativistic limit $|W| \ll M$ we obtain Bohr's formula for the case of equal masses:

$$W = -|W| = -\frac{\alpha^2}{2n^2}\frac{M}{2}\ (n = 1, 2, 3, \ldots).$$

4. QUASIPOTENTIAL DESCRIPTION OF A QUARK-ANTIQUARK SYSTEM AT SHORT DISTANCES AND THE SPECTRUM OF VECTOR MESONS

In this section, we use the formalism developed above to describe mesons as composite particles consisting of a bound state of a quark and antiquark with spins $\frac{1}{2}$.

Recently, to explain the negative results of numerous attempts to find quarks in the free state, a model with confined quarks has been proposed. In the formulation using a potential,[33-35] the quarks are confined within a particle by means of a part of the potential that increases with the distance (with regard to the solution of a quasipotential equation with a potential of this form in the relativistic configuration representation, see Ref. 36):

$$V_{\text{conf}}\ (r) = \lambda r, \tag{110}$$

which is added to the quasirelativistic Breit–Fermi potential (see Secs. 2 and 3) resulting from the exchange of a massless gluon between the quarks. Thus, the total interaction potential of the quark and antiquark has the form[33-35]

$$V_{q\bar{q}}(r) = V_{\text{Breit}}\ (r) + \lambda r + V_0. \tag{111}$$

However, since mesons have a finite radius, the notion of the quark-confining potential as increasing linearly in the whole of space is not necessary.

It was shown in Ref. 37 that the formalism of the relativistic configuration representation makes possible a group-theoretical introduction of a potential that confines quarks within a finite region with radius of the order of the Compton wavelength of the particle. In order to describe distances shorter than the particle Compton wavelength, it is necessary to modify the expansion of the wave function on the Lorentz group.

It is well known that, besides the principal series of unitary irreducible representations of the Lorentz group used here in Secs. 2 and 3, there are also the representations of the complementary series. For both series, the eigenvalues of the Lorentz-group Casimir operator, which in accordance with (64) and (65) are the square X^2 of the distance from the center of the system, are parametrized as follows:

$$C_L^{\wedge} \to X^2 = \begin{cases} 1/M^2 + r^2, & 0 \leqslant r < \infty \\ & \text{for the principal series;} \\ 1/M^2 - \rho^2, & 0 < \rho \leqslant 1/M \\ & \text{for the complementary series.} \end{cases} \tag{112}$$

Therefore, the transition to distances less than the Compton wavelength, $X^2 < 1/M^2$, can be achieved by including in the expansion of the wave function the complementary series of unitary irreducible representations of the Lorentz group. The group parameter ρ then acquires the meaning of the relative coordinate measured from the boundary of the sphere $X^2 = 1/M^2$ toward the center, so that $\rho = 1/M$ corresponds to the coordinate origin $X^2 = 0$.[28]

For the complementary series, the analog of the plane waves $\xi(\mathbf{p}, \mathbf{r})$ (53) of the principal series is the functions $\xi(\mathbf{p}, \rho) = [(p_0 - \mathbf{pn}/M]^{-1-\rho M}$. Formally, they can (like the eigenvalues of the Casimir operator $\hat{C}_L$) be obtained from $\xi(\mathbf{p}, \mathbf{r}) = [(p_0 - \mathbf{pn}/M]^{-1+irM}$ by the substitution $r \to i\rho$. The expansion of the wave function $\Psi(\mathbf{p})$ of the relative motion of the two quarks with allowance for the complementary series takes the form

$$\left.\begin{aligned} \Psi(\mathbf{p}) = \int_0^\infty dr \int d\omega_{\mathbf{n}}\xi(\mathbf{p}, \mathbf{r})\,\Psi(\mathbf{r}) \\ + \int_0^{1/M} d\rho \int d\omega_{\mathbf{n}}\zeta(\mathbf{p}, \rho)\,\Psi(\rho); \\ d\omega_{\mathbf{n}} = \sin\theta\, d\theta\, d\varphi. \end{aligned}\right\} \tag{113}$$

The partial-wave expansion for the "plane waves" of the complementary series is given by

$$\zeta(\mathbf{p}, \rho) = \sum_{l=0}^{\infty}(2l+1)\,\pi_l(\rho, \operatorname{ch}\chi_p)\,P_l(\cos\theta_{\mathbf{pn}}), \tag{114}$$

with the radial functions

$$\pi_0(\rho, \operatorname{ch}\chi_p) = \operatorname{sh}\rho M\chi_p/(\rho M \operatorname{sh}\chi_p); \tag{115}$$

$$\pi_l(\rho, \operatorname{ch}\chi_p) = \sqrt{\frac{\pi}{2\operatorname{sh}\chi_p}}\,\frac{\Gamma(\rho+l+1)}{\Gamma(\rho+1)}\,P^{-1/2-l}_{-1/2-\rho}(\operatorname{ch}\chi_p). \tag{116}$$

Formally, one can obtain the same results from the functions of the principal series $p_l(r, \cosh\chi_p)$ [see (A.3)–(A.5)] by the substitution $r \to i\rho$.

We now consider the analog of the relativistic attractive Coulomb potential $V(r) = (-1/4\pi r)\coth\pi r M$ at distances less than $1/M$. Going over in (62) to the complementary series by the substitution $r \to i\rho$, we obtain the potential

$$V(\rho) = (1/4\pi\rho)\operatorname{ctg}\pi\rho M, \tag{117}$$

which is shown in Fig. 3. Such a potential confines the quarks within a sphere with radius $R^2 = X^2 = 1/M^2$, but in contrast to (110) it acts only in the restricted volume $X^2 \leqslant 1/M^2$.

In Ref. 37, a free-Hamiltonian operator for the "plane waves" of the complementary series was found:

$$\begin{gathered} \hat{H}_0\zeta(\mathbf{p}, \rho) = 2E_p\zeta(\mathbf{p}, \rho); \\ 2E_p = 2M\operatorname{ch}\chi_p = 2\sqrt{M^2 + \mathbf{p}^2}. \end{gathered} \tag{118}$$

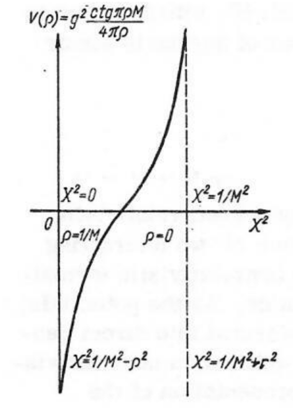


FIG. 3. Quark-confinement potential $V(\rho) = (4\pi\rho)^{-1}\cot\pi\rho M$.

As in the case of the principal series, it is a finite-difference operator:

$$\hat{H}_0 = 2M\,\mathrm{ch}\left(\frac{1}{M}\frac{\partial}{\partial\rho}\right) + \frac{2M}{\rho}\,\mathrm{sh}\left(\frac{1}{M}\frac{\partial}{\partial\rho}\right) + \frac{\Delta_{\theta,\varphi}}{M\rho^2}\exp\left(\frac{1}{M}\frac{\partial}{\partial\rho}\right) \tag{119}$$

and it can be obtained from (59) by the substitution $r \to i\rho$. Therefore, the formalism of partial-wave equations describing the relative motion of two particles with spin $\frac{1}{2}$ can be transferred from the r space to the ρ space by the substitution $r \to i\rho$.

Therefore, in a state with total spin $\mathbf{S} = (\sigma_1 + \sigma_2)/2 = 1$ the motion of two quarks with spin $\frac{1}{2}$ in the central field (117) is described by the equation

$$(\hat{H}_0 + V(\rho))\,\Psi_q(\rho) = 2E_q\Psi_q(\rho). \tag{120}$$

The energy $2E_q$ of the system represents the mass of the particle formed by the bound state of the quark and antiquark. The solution of Eq. (120) in the region $0 \leqslant X^2 \leqslant 1/(2M)^2$, where $\cot\pi\rho M < 0$ and $M_{bd} \equiv 2E_q = 2M\cos x$ for states with $l = 0$ has the form

$$\begin{aligned}&\Psi_{q,\,l=0}(\rho) = \\ &= (\exp(-ix)\sin x)\exp(-ix\rho M) \\ &\times \exp\left[x\frac{\mathrm{ctg}\,\pi\rho M}{2\sin x}\right]{}_2F_1\Big(1+\rho M,\ 1 \\ &+ i\,\frac{\mathrm{ctg}\,\pi\rho M}{2\sin x};\ 2;\ 2i\exp(-ix)\sin x\Big).\end{aligned} \tag{121}$$

The function $\cot\pi\rho M$ in (117) is a constant under operations of the finite-difference calculus occurring in the Hamiltonian (119), as the result of which it plays the role of an effective coupling constant in Eq. (120).

The requirement of regularity of the solution at $X^2 = 0$ ($\rho = 1/M$) leads to the quantization condition $\sin 2x = x$, which determines two energy levels: one with $M_{bd} \equiv 2E_q \approx 1.398M$, and the other with $M_{bd} \equiv 2E_q = 2M$. In the region $1/(2M)^2 \leqslant X^2 < 1/M^2$, where $\cot\pi\rho M > 0$ and $2E_q = 2M\cosh\chi \geqslant 2M$, the wave function is obtained from (121) by the substitution $x \to i\chi$. The requirement of regularity of the solution at the point $X^2 = 1/M^2 (\rho = 0)$ leads to the condition $2\sinh\chi\exp(-\chi) = \chi$, which determines one further level with $M_{bd} = 2E_q \approx 2.968M$.

Thus, in the quark–antiquark system in the field of the potential (117) in the state with $l = 0$ and $s = 1$ one can have three energy levels or three excited states of one particle with masses (M_q is the quark mass)

$$\left.\begin{aligned} M_{bd}^{\mathrm{I}} &\equiv 2E_q^{\mathrm{I}} = 1.39798M_q; \\ M_{bd}^{\mathrm{II}} &\equiv 2E_q^{\mathrm{II}} = 2M_q; \\ M_{bd}^{\mathrm{III}} &\equiv 2E_q^{\mathrm{III}} = 2.96750M_q. \end{aligned}\right\} \tag{122}$$

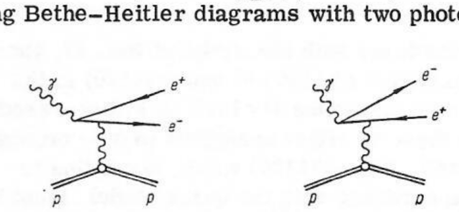
If we are interested in the masses of the excited states of the ρ meson, then, equating $M^{\mathrm{I}} = M_\rho = 773 \pm 3$ MeV, we obtain for the quark mass $M_Q = 553 \pm 2$ MeV. It follows from (122) that the masses of the second and third radial excitations of the ρ meson are, respectively, $M_{\rho'} = M^{\mathrm{II}} = 1106 \pm 4$ MeV and $M_{\rho''} = M^{\mathrm{III}} = 1645 \pm 6$ MeV.

This result was obtained for the first time in Ref. 37 at the time when only the third particle in this spectrum could be identified with the ρ'' meson with $M_{\rho''\,\mathrm{exp}} \approx 1650$ MeV, $\Gamma_{\rho''} > 200$ MeV and there was no experimentally discovered candidate at the position of the second radial excitation with mass 1106 ± 4 MeV. However, a few months later at DESY in the process $\gamma + p \to V + p$, $V \to e^+e^-$ a vector meson with $M \approx 1110$ MeV and $\Gamma \approx 20$–30 MeV was discovered experimentally.[39]

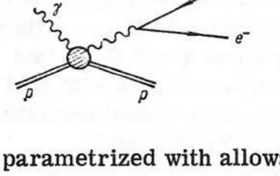
The amplitude of such a process can be represented in the form

$$A = A_{\mathrm{BH}}(2\gamma) + A_c + A(3\gamma) + \ldots, \tag{123}$$

where the amplitude $A_{\mathrm{BH}}(2\gamma)$ corresponds to the following Bethe–Heitler diagrams with two photons:



The amplitude of Coulomb scattering corresponding to the diagram



is parametrized with allowance for possible contributions of vector mesons in the intermediate state:

$$\left.\begin{aligned} A_c &\sim i\sum_V\left(\frac{m_V^2}{\mu^2}\frac{\exp(i\varphi_V)}{m_V^2 - \mu^2 - im_V\Gamma_V}\sqrt{\frac{d\sigma^V/dt}{\gamma_V^2/4\pi}}\right); \\ \frac{d\sigma^V}{dt} &= \frac{d\sigma(\gamma p \to V_p)}{dt}\bigg|_{t=0}\exp(b_V t), \end{aligned}\right\} \tag{124}$$

where m_V is the mass of the vector meson; b_V is the slope of the differential cross section of meson photoproduction on the nucleon. The corrections from the Bethe–Heitler processes with the participation of two or more gammas are small because

$$|A|^2 \approx |A_c|^2 + 2\,\mathrm{Re}\,(A_c A_{\mathrm{BH}}) + |A_{\mathrm{BH}}(2\gamma)|^2. \tag{125}$$

Since the Bethe–Heitler amplitude $A_{\mathrm{BH}}(2\gamma)$ greatly exceeds the amplitude of the Compton process, a signal indicating the existence of structure in the mass distribution in A_c can be more readily detected from the interference term $\mathrm{Re}(A_c A_{BH})$ than from A_c^2.

In order to get rid of the large contribution of the Bethe–Heitler term $A_{BH}^2(2\gamma)$, the authors of Ref. 39 used the fact that the interference term $\mathrm{Re}(A_c A_{BH})$ is antisymmetric under transposition of the momenta $\mathbf{p}_-$ and $\mathbf{p}_+$ of the electron and positron, respectively, and every event in the mass distribution occurs with weight $\mathrm{sign}(|\mathbf{p}_+|\theta_+ - |\mathbf{p}_-|\theta_-)$ [where θ_+ and θ_- are the polar angles of emission of the positron and electron, respectively, with respect to the proton beam], which makes it possible to take into account the contribution of only the interference term.

In the mass distribution separated in this way from the interference term, a peak was also found at $M \approx 1110$ MeV in addition to the contributions from the ρ, ω, and φ mesons. The width of this new vector meson was estimated from the resolution of the instrument itself, $\Gamma_{M1110} \leqslant 30$ MeV, and $BR(d\sigma/dt)|_{t=0} = 4.9 \cdot 10^{-5}$ $(\mathrm{GeV}/c)^2$. Note that for the ρ, ω, and φ mesons $BR(d\sigma/dt)|_{t=0} \approx (1-4) \cdot 10^{-3} \mu\mathrm{b}$ $(\mathrm{GeV}/c)^2$, i.e., two orders of magnitude greater than for the new meson.[39]

Equations (122) determine a spectrum of excited states of the ω meson analogous to the ρ-meson spectrum. Thus, setting $M^{\mathrm{I}} = M_\omega = 783$ MeV, we obtain $M_{q'} = 560$ MeV, whence $M_\omega^{\mathrm{II}} = M_{\omega'} = 1120$ MeV and $M_\omega^{\mathrm{III}} = M_{\omega'} = 1662$ MeV.

Thus, in accordance with the model of Ref. 37, there must be two particles $\rho'(1106 \pm 4)$ and $\omega'(1120)$ in the region of the new resonance $M = 1110$ MeV discovered at DESY and for these an effect analogous to $\rho - \omega$ mixing is to be expected. The $\omega'(1120)$ width, according to estimates in accordance with the quark model, must be appreciably less than the $\rho'(1106 \pm 4)$ width, which explains the relatively small width of the resonance structure observed at $M = 1110$ MeV.

Let us now consider briefly some of the consequences of extending our model of quark confinement to the case of the J/ψ particles. If we assume $M^{\mathrm{I}} = M_{\psi(3095)}$, then for the mass of the heavy charmed quark Q we find $M_Q = 2216$ MeV, which gives in accordance with (122) the following values of the masses of the radial excitations of ψ: $M_\psi^{\mathrm{II}} = 4432(\pm 4)$ MeV and $M_\psi^{\mathrm{III}} = 6576$ MeV.

Therefore, for our confining potential (111) the first radial excitation of the ψ meson is not $\psi'(3685)$ but $\psi(4414)$. This is not surprising since the interpretation of $\psi'(3685)$ as a radial excitation of $J/\psi(3095)$ usually leads to the conclusion that the motion of quarks is nonrelativistic,[35] whereas our description is essentially relativistic and occurs in the region $X^2 < 1/M^2$, which does not have a nonrelativistic analog. At the same time, $\psi'(3685)$ can be interpreted as a bound state of two other heavy quarks Q' with mass $M_{Q'} = 2635$ MeV, and then in accordance with (122) its possible radial excitations are $M_{\psi'}^{\mathrm{II}} = 5270$ MeV and $M_{\psi'}^{\mathrm{III}} = 7820$ MeV.

Thus, in our model of quark confinement by means of the potential (117) it is necessary to introduce more than one charmed quark with different masses in order to explain the spectrum of the J/ψ particles; this is in agreement with the results of Ref. 40, which discusses a possible increase in the number of quarks to six or more.

CONCLUSIONS

We have proposed a three-dimensional relativistic formalism for describing a system of two interacting fermions that stays close to the nonrelativistic formalism of ordinary quantum mechanics. As the potentials, we have taken the OBEP's transformed into direct geometrical generalizations of the well known nonrelativistic potentials. Note that the representation of the images of the potentials in momentum space in an absolutely geometric form opens up the possibility for a phenomenological relativization of nonrelativistic models. For this purpose, in the nonrelativistic expression it is necessary to replace the vector of the Euclidean half-transfer $\boldsymbol{\varkappa}_E = (\mathbf{k} - \mathbf{p})/2$ by the vector $\boldsymbol{\varkappa}$ of the half-transfer in Lobachevskiĭ space, and then add to the amplitude a kinematic Wigner rotation.

The use of Lobachevskiĭ geometry to describe the interaction of two spin-$\frac{1}{2}$ particles enables one to introduce a triplet of orthogonal vectors $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ which are such that the most economic parametrization of the relativistic amplitude and quasipotential is obtained in terms of them. For example, for the exchange of a pseudoscalar meson the decomposition of the Born scattering amplitude of two nucleons off the energy shell with respect to the spin structures contains only a single term, whereas this decomposition includes three terms when the ordinary Euclidean triplet of unit vectors is used.[5]

The form found for the OBEP's is convenient for use in the Kadyshevskiĭ quasipotential equation of the relativistic configuration representation. In this formalism, the operators of the free Hamiltonian $\hat{H}_0$ and the momentum vector $\hat{\mathbf{P}}$ are differential-difference operators. The images of the OBEP's found in the coordinate space can be regarded as relativistic generalizations of the potential of the Coulomb, Yukawa, spin-orbit, and tensor forces.

In contrast to the nonrelativistic case, the relativistic spin–orbit and tensor interactions do not contain singular terms of order r^{-3}. This is due to the finite-difference nature of the operators $\hat{H}_0$ and $\hat{\mathbf{P}}$, which contain the shift along the imaginary axis with step $\hbar/Mc$. In contrast to pointwise differentiation, application of a difference operator to a singular function does not increase the order of the singularity. There arises merely a new singularity at a complex point separated by the Compton wavelength from the real axis. From the point of view of the nonrelativistic description, these two different poles "appear" as a single singularity of higher order. In this case, the semirelativistic formalism can be called degenerate. The systematic relativistic description has a higher "resolution" since it lifts this degeneracy.

It is interesting to note that potentials with singularities at complex points have already been introduced in a purely phenomenological manner[41] (see also Ref. 42) to describe a number of characteristic features of high-energy scattering. Thus, our formalism provides some kind of justification for the use of such potentials.

In many ways, the formalism of partial-wave equations constructed here is analogous to the nonrelativistic formalism. This is because the Lorentz group contains the rotation group as a subgroup, so that its generators include the orbital-angular-momentum operator **L**. We hope that the formalism developed here will be convenient for solving problems in relativistic nuclear physics and for constructing relativistic composite models of elementary particles.

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APPENDIX

The completeness and orthogonality relations for the functions (53) are

$$\left.\begin{aligned}&\frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}}\xi^*(\mathbf{p};\,\mathbf{n},\,r)\,\xi(\mathbf{p};\,\mathbf{n}_1,\,r_1)=\delta^{(3)}(\mathbf{r}-\mathbf{r}_1);\\ &\frac{1}{(2\pi)^3}\int d\mathbf{r}\xi^*(\mathbf{p};\,\mathbf{n},\,r)\,\xi(\mathbf{p}_1,\,\mathbf{n},\,r)=\delta^{(3)}(\mathbf{p}-\mathbf{p}_1)\sqrt{1+\mathbf{p}^2/M^2};\\ &\mathbf{r}=r\cdot\mathbf{n};\ d\mathbf{r}=r^2drd\omega_{\mathbf{n}}=r^2dr\sin\theta\,d\theta\,d\varphi.\end{aligned}\right\}\tag{A.1}$$

The partial-wave expansion for the plane waves (53) is

$$\xi(\mathbf{p};\,\mathbf{n},\,r)=\sum_{l=0}^{\infty}(2l+1)\,\mathrm{i}^l p_l(\operatorname{ch}\chi_p,\,r)\,P_l((\mathbf{p}\cdot\mathbf{n})/|\mathbf{p}|),\tag{A.2}$$

where the radial functions

$$p_l(\operatorname{ch}\chi_p,\,r)=(-\mathrm{i})^l\sqrt{\frac{\pi}{2\operatorname{sh}\chi_p}}\,\frac{\Gamma(\mathrm{i}rM+l+1)}{\Gamma(\mathrm{i}rM+1)}\,P^{-1/2-l}_{-1/2+\mathrm{i}rM}(\operatorname{ch}\chi_p);\tag{A.3}$$

$$p_l(\operatorname{ch}\chi_p,\,r)=\mathrm{i}^l\frac{\Gamma(\mathrm{i}rM+1)}{\Gamma(-\mathrm{i}rM+l+1)}\,(\operatorname{sh}\chi_p)^l\left(\frac{d}{d\operatorname{ch}\chi_p}\right)^l p_0(\operatorname{ch}\chi_p,\,r)\tag{A.4}$$

$$p_0(\operatorname{ch}\chi_p,\,r)=\sin rM\chi_p/rM\operatorname{sh}\chi_p\tag{A.5}$$

go over in the nonrelativistic limit $M\gg 1$, $\chi_p\ll 1$ into spherical Bessel functions:

$$p_l(\operatorname{ch}\chi_p,\,r)\to j_l(pr).\tag{A.6}$$

The components of the finite-difference momentum operator $\hat{\mathbf{p}}=-\mathrm{i}\nabla_{\mathrm{diff}}$, which satisfies (60), can be written in spherical coordinates as follows:

$$\left.\begin{aligned}&\hat{p}_1=-\sin\theta\cos\varphi\left[M\exp\left(\frac{\mathrm{i}}{M}\frac{\partial}{\partial r}\right)-\hat{H}_0\right]\\ &\quad-\mathrm{i}\left(\frac{\cos\theta\cdot\cos\varphi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\exp\left(\frac{\mathrm{i}}{M}\frac{\partial}{\partial r}\right);\\ &\hat{p}_2=-\sin\theta\cdot\sin\varphi\left[M\exp\left(\frac{\mathrm{i}}{M}\frac{\partial}{\partial r}\right)-\hat{H}_0\right]\\ &\quad-\mathrm{i}\left(\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\exp\left(\frac{\mathrm{i}}{M}\frac{\partial}{\partial r}\right);\\ &\hat{p}_3=-\cos\theta\left[M\exp\left(\frac{\mathrm{i}}{M}\frac{\partial}{\partial r}\right)-\hat{H}_0\right]+\mathrm{i}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\exp\left(\frac{\mathrm{i}}{M}\frac{\partial}{\partial r}\right).\end{aligned}\right\}\tag{A.7}$$

[1]A. A. Logunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963).

[2]V. A. Matveev, R. M. Muradyan, and A. N. Tavkhelidze, Preprints JINR E-3498 (in English) (1967), R2-3900 (in Russian) (1968); P. N. Bogolyubov, Fiz. Élem. Chastits At. Yadra **3**, No. 1, 144 (1972) [Sov. J. Part. Nucl. **3**, 71 (1972)]; R. N. Faustov, Dokl. Akad. Nauk SSSR **154**, 1329 (1964) [Sov. Phys. Dokl. **9**, 482 (1964)], Nucl. Phys. **75**, 669 (1966); Fiz. Élem. Chastits At. Yadra **3**, No. 3, 238 (1972) [Sov. J. Part. Nucl. **3**, 119 (1972)]; A. A. Logunov *et al*, Teor. Mat. Fiz. **6**, 157 (1971); V. I. Savrin, N. E. Tyurin, and O. A. Khrustalev, Fiz. Élem. Chastits At. Yadra **7**, No. 1, 21 (1976) [Sov. J. Part. Nucl. **7**, 9 (1976)]; V. R. Garsevanishvili *et al*, Teor. Mat. Fiz. **23**, 310 (1975); Teor. Mat. Fiz. **25**, 36 (1975); S. V. Goloskokov *et al*, Teor. Mat. Fiz. **24**, 24 (1975); A. N. Kvinikhidze *et al*, Fiz. Élem. Chastits At. Yadra **8**, No. 3, 478 (1977) [Sov. J. Part. Nucl. **8**, 196 (1977)].

[3]V. G. Kadyshevskiĭ and A. N. Tavkhelidze, in: Problemy Teoreticheskoĭ Fiziki (Problems of Theoretical Physics, Collection dedicated to N. N. Bogolyubov on his 60th birthday), Nauka (1969). In Ref. 3, there is a detailed bibliography on the quasipotential approach.

[4]V. G. Kadyshevsky, Nucl. Phys. B **6**, 125 (1968).

[5]V. G. Kadyshevsky and M. D. Mateev, Nuovo Cimento A **55**, 275 (1968).

[6]V. G. Kadyshevskiĭ, Zh. Éksp. Teor. Fiz. **46**, 654 (1964), **46** 872 (1964) [Sov. Phys. JETP **19**, 443 (1964), **19**, 597 (1964)].

[7]V. G. Kadyshevsky, R. M. Mir-Kasimov, and N. B. Skachkov, Nuovo Cimento A **55**, 233 (1968).

[8]V. G. Kadyshevskiĭ, R. M. Mir-Kasimov, and N. B. Skachkov, Fiz. Élem. Chastits At. Yadra **2**, 635 (1972) [Sov. J. Part. Nucl. **2**, Part 3, 69 (1972)].

[9]V. V. Babikov, Nucl. Phys. **76**, 665 (1966).

[10]N. B. Skachkov, Teor. Mat. Fiz. **22**, 213 (1975).

[11]V. A. Fock, Teoriya Prostranstva, Vremeni, Tyagoteniya (The Theory of Space, Time and Gravitation, Oxford (1964)), Gostekhizdat, Moscow (1955).

[12]N. A. Chernikov, Dokl. Akad. Nauk SSSR **114**, 530 (1957) [Sov. Phys. Dokl. **2**, 103 (1957)]; Nauchnye Doklady Vyssheĭ Shkoly **2**, 158 (1958), **3**, 151 (1959); Preprint JINR R2-7231 (in Russian), Dubna (1961); Fiz. Élem. Chastits At. Yadra **4**, No. 3, 773 (1973) [Sov. J. Part. Nucl. **4**, 315 (1973)].

[13]G. C. Wick, Ann. Phys. **18**, 65 (1962); Ya. A. Smorodinskiĭ, Zh. Éksp. Fiz. **43**, 2217 (1962) [Sov. Phys. JETP **16**, 1566 (1963)].

[14]N. A. Chernikov, Zh. Éksp. Teor. Fiz. **33**, 541 (1957) [Sov. Phys. JETP **6**, 422 (1958)]; in: Lektsii v Mezhdunarodnoĭ Shkole Teoreticheskoĭ Fiziki pri OIYaI (Lectures at the International School of Theoretical Physics at the JINR) Vol. 3, Dubna (1964), p. 151; Preprint ITF-68-44 (in Russian), Institute of Theoretical Physics, Kiev (1968).

[15]Yu. M. Shirokov, Dokl. Akad. Nauk SSSR **99**, 737 (1954).

[16]A. A. Cheshkov and Yu. M. Shirokov, Zh. Éksp. Teor. Fiz. **44**, 1982 (1963) [Sov. Phys. JETP **17**, 1333 (1963)].

[17]P. Moussa and R. Stora, Analysis of Scattering and Decay, Gordon and Breach, New York (1968).

[18]N. B. Skachkov, Preprint JINR E2-7333 (in English) (1973).

[19]Yu. M. Shirokov, Zh. Éksp. Teor. Fiz. **21**, 748 (1951), **33**, 1196 (1957) [Sov. Phys. JETP **6**, 919 (1958)].

[20]A. I. Akhiezer and V. B. Berestetskiĭ, Kvantovaya Élektrodinamika (Quantum Electrodynamics, New York (1965)), Fizmatgiz, Moscow (1959); A. A. Sokolov, Vvedenie v Kvantovuyu Élektrodinamiku (Introduction to Quantum Electrodynamics), Fizmatgiz, Moscow (1958).

[21]N. B. Skachkov, Preprint JINR E2-7159 (in English), (1973).

[22]N. N. Bogolyubov and D. V. Shirkov, Vvedenie v Teoriyu Kvantovykh Poleĭ (Introduction to the Theory of Quantized Fields, Interscience (1959)), Gostekhizdat, Moscow (1957).

[23]I. S. Shapiro, Dokl. Akad. Nauk SSSR **106**, 647 (1956); Zh. Éksp. Teor. Fiz. **43**, 1727 (1962) [Sov. Phys. JETP **16**, 1219 (1963)].

[24]N. Ya. Vilenkin and Ya. A. Smorodinskiĭ, Zh. Éksp. Teor. Fiz. **46**, 1793 (1964) [Sov. Phys. JETP **19**, 1209 (1964)].

[25]M. A. Naĭmark, Lineĭnye Predstavleniya Gruppy Lorentsa (Linear Representations of the Lorentz Group, Pergamon Press, London (1964)), Fizmatgiz, Moscow (1958).

[26]I. M. Gelfand, M. I. Graev, and N. Ya. Vilenkin, Obobshchennye Funktsii (Generalized Functions: Integral Geometry and Representation Theory, Academic Press, New York, 1966) Vol. 5, Fizmatgiz, Moscow (1962).
[27]V. G. Kadyshevskiĭ, R. M. Kir-Kasimov, and M. Freeman, Yad. Fiz. **9**, 646 (1969) [Sov. J. Nucl. Phys. **9**, 373 (1969)].
[28]N. B. Skachkov, Teor. Mat. Fiz. **25**, 313 (1975).
[29]N. B. Skachkov and I. L. Solovtsov, Preprints JINR E2-9763, E2-10260 (in English), R2-10320 (in Russian), Dubna (1976).
[30]Chou Kuang-Chao and L. G. Zastavenko, Zh. Éksp. Teor. Fiz. **35**, 1417 (1958), **38**, 134 (1960) [Sov. Phys. JETP **8**, 990 (1959), **11**, 97 (1960)].
[31]V. S. Popov, Zh. Éksp. Teor. Fiz. **37**, 1116 (1959) [Sov. Phys. JETP **10**, 794 (1960)]
[32]S. Shch. Mavrodiev and N. B. Skachkov, Teor. Mat. Fiz. **23**, 32 (1975).
[33]T. Appelquist and H. D. Politzer, Phys. Rev. Lett. **34**, 43 (1974); A. De Rújula and S. L. Glashow, Phys. Rev. Lett. **34**, 46 (1974); B. J. Harrington, S. Y. Park, and A. Yildiz, Phys. Rev. Lett. **34**, 168 (1975).
[34]H. J. Schnitzer, Phys. Rev. Lett. **35**, 1540 (1975); Phys. Rev. D **13**, 74 (1976); J. S. Kang and H. J. Schnitzer, Phys. Rev. D **12**, 841 2791 (1975).
[35]R. Barbieri, R. Kögerler, Z. Kunszt, and R. Catto, Ref. TH. CERN-2036, June, 1975; E. Eichten *et al*, Phys. Rev. Lett. **34**, 369 (1975); J. F. Gunion and R. S. Willey, Phys. Rev. D **12**, 174 (1975); J. G. Gunion and L. F. Li, Phys. Rev. D **12**, 3583 (1975).
[36]K. S. Jhung, K. H. Chung, and R. S. Willey, Phys. Rev. D **12**, 1999 (1975); I. V. Amirkhanov, G. V. Grusha, and R. M. Mir-Kasimov, Preprint JINR E2-9767; E2-9868, Dubna (1976).
[37]N. B. Skachkov, Pis'ma Zh. Éksp. Teor. Fiz. **23**, 713 (1976) [JETP Lett. **23**, 657 (1976)]; Preprint JINR E2-9748 (in English), Dubna (1976).
[38]S. Ström, Arkiv Fysik, **38**, No. 22, 373 (1968).
[39]S. Bartalucci *et al*, Preprint DESY-76/43 (1976); Preprint LNF-77/1 (P) Frascati (1976).
[40]H. Harari, Phys. Lett. B **57**, 265 (1975); A. De Rújula, H. Georgi, and S. L. Glashow, Harvard Preprint (1975).
[41]A. Paliov and S. Rosendorff, J. Math. Phys. **15**, 1045 (1974).
[42]A. T. Filippov, Preprints JINR E2-6933 (1973), E2-7563 (1974), E2-7929 (1974) (in English), Dubna.

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