Influence of rotation on the collective properties of nuclei

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An attempt is made to give a fairly complete description of the theoretical and experimental investigations into the influence of rotation on the collective properties of nuclei. Attention is centered on the angular-momentum dependence of the correlation properties of the superconducting type, the deformation parameters, and the electrical properties of nuclei. Also considered are the influence of rotation on the volume, the momentum distribution of the nucleons, the spin polarization in the nuclei, and phenomena related to these effects.

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INTRODUCTION

Rotation of a free body and the effects due to it are among the simplest phenomena for physical interpretation. One can have considerable confidence in results obtained by studying the reaction of a system to rotation, i.e., the changes in its structure brought about by the rotation. One can therefore regard rotational motion as one of the means for studying structure. It is important that, in the sense of the strength of the disturbance of the system, rotation has the benefit of great universality: If there are available means for changing the angular frequency, decelerating or accelerating the rotation of the body, then one can regulate the perturbation introduced by the rotation into the structural parameters of the body in the range from a weak perturbation to a moderate one or even to one so strong that a further acceleration of the rotation causes the body to break up. Of course the possibilities for studying structure by means of rotation have certain limits, but for various questions the possibilities remain unique.

Experimental ways of obtaining rotational states of nuclei are very varied. The classical process of Coulomb excitation of nuclei[1] leads to states with finite angular momentum values. Heavy-ion reactions [(*HI*–*xn*) reactions][2] are well known and hardly any study of rotation of nuclei can dispense with them. Rotational states are also excited by the capture by nuclei of slow π^- mesons[3]; this effect was discovered comparatively recently and has not yet been widely used to study nuclear structure.

An immense number of experimental and theoretical investigations of the rotational properties of nuclei have been made. Among the many problems associated with rotation of nuclei, we concentrate our attention in this review on the influence of rotation on the collective parameters of nuclei, i.e., in our understanding, on the parameters that determine the self-consistent nuclear field.[1)] This will mean that the properties of odd nuclei, which above all reveal the coupling of the unpaired nucleon and the even–even core, will remain outside our purview. We restrict the general problem of describing the mass parameters of nuclei to studying the inertial properties of nuclei with respect to rotation. This also leads to a restriction on the class of problems studied: The lowest states for given angular momentum (the so-called yrast-band states) are analyzed in detail; the properties of states above the yrast band are mentioned only when we wish to obtain additional information about the yrast band or to test the conclusions obtained from studying it.

Even with all these restrictions, the number of questions considered remains very large, and we have attempted to give a fairly full description of these problems.

On the whole, we have attempted to avoid describing formal theoretical schemes and we have used comparatively simple methods which have also proved themselves in other branches of nuclear physics. Bohr's collective model and the Hartree–Fock–Bogolyubov self-consistent method with allowance for time dependence of the field provide a natural basis, and sometimes direct possibilities, for calculation in the solution of these problems, and we use these methods in the fol-

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[1)] We ask readers who are accustomed to the terminology of the theory of Green's functions to understand this expression as a synonym for mass operator. We shall consistently use the terminology of the variational approach and not both to "translate" it into other languages of nuclear theory because, it seems to us, the empirical material to be described below does not cast light on the details that distinguish these theories.

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lowing sections. However, the time is now surely ripe for at least a brief attempt to illuminate the numerous theoretical investigations into the accuracy and improvement of the nuclear cranking model and its relation to Bohr's model. This we do in Sec. 1. The bases of the following sections are: classification of nonadiabatic effects associated with rotation of nuclei; study of the influence of Coriolis antipairing on nuclear spectra; the derivation of simple relations that make it possible to extract information about the influence of rotation on the electrical properties of nuclei on the basis of Bohr's collective model; analysis of the change in the shape of nuclei at small, intermediate, and very large angular momenta. Strutinskiĭ's method for describing the properties of rotating nuclei is set forth, and we also discuss the properties of the states of nuclei in the transition regions between spherical and deformed nuclei.

1. THEORETICAL DESCRIPTION OF THE ROTATION OF NUCLEI

Our main interest in this review will be elucidating the structure of nuclei on the basis of data on the reaction of nuclei to rotation, i.e., on the basis of an analysis of the changes in the properties of nuclei produced by an increase in the angular momentum. The microscopic theory of the rotation of nuclei forms the basis of all the following exposition. Although the concept of rotation of a nucleus is very perspicuous from the physical point of view and many of the phenomena associated with it have a simple intuitive interpretation, the problem of constructing an exact theory of nuclear rotational motion proved very complicated and required numerous attempts. The difficulties in the way of constructing such a theory arise from the fact that rotational motion can be manifested only if the central symmetry in the internal coordinate system is broken (see Sec. 4.1 in Ref. 4). The central task of the theory is to construct a many-particle wave function that, on the one hand, describes the breaking of the symmetry but, on the other, is still fairly similar to the eigenstate wave function of the spherically symmetric nuclear Hamiltonian.[2)]

A. Bohr associated the deformation which is manifested in the existence of states of rotational type with a quadrupole deformation in the mass and charge distribution within the nucleus. The large value of the $B(E2)$ factors, which determine the probability of quadrupole transitions in rotational nuclei, was the first of many indications that such a connection exists.

The most direct path to a theoretical description of rotation based on A. Bohr's hypothesis would appear to be the construction of a theory that starts from a transformation of the coordinate system to some complete set of generalized variables including three Eulerian angles to determine the spatial orientation of the mass quadrupoloid of the nucleus. If Bohr's assumption corresponds fairly accurately to reality, such a coordinate transformation must lead to a separation of the nuclear Hamiltonian into rotational and internal parts coupled by an interaction term. This term is small and amenable to perturbation theory. Such a scheme is frequently used to derive phenomenological relationships between various nuclear parameters, and will be more fully described in this context in Sec. 3. However, it is only recently that a certain progress has been achieved in the development of a microscopic theory in this direction.[6-8] The introduction of generalized hyperspherical harmonics has in principle solved the problem of constructing a complete basis of many-particle states. In Refs. 6–8 there is a fairly complete analysis of the kinetic-energy operator of a many-fermion system described in terms of variables that include the orientation angles of the nuclear mass quadrupoloid. It has been shown necessary to take into account the terms of the Hamiltonian describing the coupling of the rotation to the internal motion if the inertia of nuclei is to be correctly described: The moments of inertia in the weak-coupling limit in this theory depend on the internal quadrupole moments as in the hydrodynamic model. The potential energy of the nucleus is reduced to a form amenable to systematic analysis. In the simplest approximations, calculations of the potential energy have proved possible for at least the lightest nuclei.[9] However, systematic calculations in this approach of the inertial parameters of nuclei and the change in the internal structure brought about by rotation have still to be made.

Thus, the approach based directly on A. Bohr's hypothesis, while being very consistent and promising, is as yet quite inadequate for dealing with the problem of the physical analysis of the huge empirical material on rotation of nuclei. Problems in the direct realization of Bohr's hypothesis arise in the construction of the internal many-particle wave functions. The internal wave function does not contain all the $3A$ coordinates of an A-nucleon nucleus among its collective variables (lacking nine[3)] in fact), so that the internal wave function cannot describe independent motion of the nucleon. But many data indicate that nucleons move more or less independently of one another in a common "average" field. It is now known that the energy spectra and magnetic moments of odd rotating nuclei can be explained on the basis of a model of independent quasiparticles. Reactions involving transfer (stripping or pickup) of one nucleon also confirm this. The statistical properties of nuclear states at high excitation energies can be understood by assuming the existence of quasiparticles weakly coupled to one another. However, the accumulation of all this information, to say nothing of the elucidation of the ac-

[2)]We recommend to the reader A. Bohr's lecture on the rotation of nuclei given when he received the 1976 Nobel Prize for physics.[5] In this fairly short lecture. Bohr considered a much wider circle of problems than we do here, and he also presented much interesting material on the history of the development of this direction in nuclear physics.

[3)]The three center-of-mass coordinates, the three Eulerian angles describing the orientation of the system in space, and three variables characterizing the linear dimensions of the system in three mutually perpendicular directions. In the problem of rotation, the last three variables can also be regarded as internal variables, and then the number of coordinates is only six greater than the number of internal variables.

tual concept of a quasiparticle, required much labor.[10,11]

It was found simpler in the construction of a microscopic theory of rotation to attempt a practical solution, taking as the point of departure, not exact determination of the collective variables, but a direct assumption about the nuclear wave functions. The key idea here was formulated by Rainwater[12] and developed by Bohr and Mottelson.[13] They conjectured that the nonsphericity of nuclei and the large quadupole moments are due to the fact that the average field in which the nucleons move loses its spherical symmetry in some nuclei. The gist of Rainwater and Bohr's idea is therefore that rotation of the nucleus is essentially rotation of the average deformed field of the nucleus.

This idea was implemented in concrete terms by Inglis,[14] who determined the inertial properties of a system of independent fermions moving in a field with the symmetry of an axial quadrupoloid. If such a system rotates with angular frequency ω about an axis which is not a symmetry axis, then it is natural to go over to a coordinate system in which the field is fixed. This then leads[15] to a transformation of the Hamiltonian of the system into the operator

$$\hat{R} = \hat{H} - \omega \hat{I}_x, \tag{1}$$

where $\hat{I}_x$ is the operator of projection of the angular momentum onto the rotation axis (the x axis). The energy of the system $E = \langle \omega | \hat{H} | \omega \rangle$ and the angular momentum I, which is given by

$$\sqrt{I(I+1)} = \langle \omega | \hat{I}_x | \omega \rangle, \tag{2}$$

increase together with angular frequency, and for small ω

$$E = E_0 + \mathcal{J}\omega^2/2 + \ldots \tag{3}$$

The moment of inertia determined in this manner is

$$\mathcal{J} = 2\hbar^2 \sum_{i \neq 0} |\langle i | I_x | 0 \rangle|^2 / (E_i - E_0), \tag{4}$$

where the summation is over all excited states i of the system, and E_i is the excitation energy; $\langle i | I_x | 0 \rangle$ is the matrix element of the operator of the x projection of the angular momentum taken between the ground state 0 and the excited state i. The moment of inertia (4) can be associated with an analogous parameter that determines the excitation energy of the rotational states of the nucleus:

$$E_I = E_0 + I(I+1)/2\mathcal{J} + \ldots; \tag{5}$$

where I is the angular momentum of the yrast-band state of the nucleus. After the effects of pairing correlations of superconducting type had been taken into account in the model,[16] i.e., the functions $|i\rangle$ and the energies E_i of the states describing the independent quasiparticles[11] had been modified, great progress was made in the quantitative description of moments of inertia. The empirical values of the parameter $\mathcal{J}$ in (5) for the nuclei of rare-earth and transuranium elements, which are very different from the values predicted by models of a rigid rotor or a perfect liquid drop with potential motion, were reproduced in Refs. 17–22 with an error of about 20–30% in a self-consistent manner from the data for the nonrotational properties of the nuclei.

The situation remained not entirely satisfactory from the point of view of someone who wishes to operate with a rigorously justified theory of nuclear rotation. Inglis's approach lacked a theoretical basis, and the results of calculations could not demonstrate convincingly that it would apply in new fields. An important step forward in this respect was the construction of a self-consistent cranking model.[23–26] The self-consistent cranking model uses the variational approach to find conditions of stationarity of the expectation value of the operator $\hat{R}$, which contains the many-particle nuclear Hamiltonian $H = H_0 + V$. The expectation value is found with respect to a many-particle state satisfying the stationarity condition:

$$\delta \langle \omega | \hat{R} | \omega \rangle = 0. \tag{6}$$

The many-particle functions used in the calculations are taken from the Hartree–Fock–Bogolyubov theory,[27,28] but the term $\lambda \hat{N}$ is subtracted from the kinetic energy in the Hamiltonian H_0 because the many-particle functions in Hartree–Fock–Bogolyubov theory are not eigenfunctions for the particle-number operator $\hat{N}$. The Hartree–Fock–Bogolyubov equations in the self-consistent cranking model have the form

$$\sum_j \begin{pmatrix} \varepsilon_{ij} + V_{ij} - \lambda\delta_{ij} - \omega j^x_{ij} & \Delta_{ij} \\ -\Delta^*_{ij} & -\varepsilon^*_{ij} - V^*_{ij} + \lambda\delta_{ij} + \omega j^x_{ij} \end{pmatrix} \begin{pmatrix} u^\mu_j \\ v^\mu_j \end{pmatrix} = E_\mu \begin{pmatrix} u^\mu_i \\ v^\mu_i \end{pmatrix}, \tag{7}$$

where ε_{ij} is the matrix element of the single-particle part of H_0; j^x_{ij} is the same for the operator $\hat{I}_x$; V_{ij} and Δ_{ij} are related to the antisymmetrized matrix elements of the effective two-particle interaction v as follows:

$$V_{ij} = \sum_{i'j'} v_{ii'jj'} \rho_{i'j'}; \quad \Delta_{ij} = (1/2) \sum_{i'j'} v_{iji'j'} t_{j'i'};$$
$$\rho_{ij} = \langle \omega | c^+_i c_j | \omega \rangle = \sum_\mu u^{\mu *}_i u^\mu_j;$$
$$t_{ij} = \langle \omega | c_i c_j | \omega \rangle = \sum_\mu v^{\mu *}_i u^\mu_j \tag{8}$$

[c^+_i and c_j are the operators of creation and annihilation of nucleons in the single-particle states i and j, respectively]. For small ω, the summation in (8) is over all values of μ to which negative values of E_μ correspond. At large ω, the many-particle state whose density matrices ρ and t are constructed from vectors corresponding to $E_\mu < 0$ may correspond to a nucleus with a different parity of the nucleon number than for ω near zero. In such a case, the eigenvector with the smallest positive E_μ must replace the vector with $E_\mu < 0$ in the sum (8). This and the symmetry properties of the solutions of Eqs. (7) and (8) are discussed in Refs. 29 and 30.

Associating the expectation value $\langle \omega | H | \omega \rangle$ with the en-

ergies E_I of the yrast-band states, one can, using (2), readily show that the parameter $\mathcal{J}$ in (5) is the moment of inertia defined by the relations of classical theory:

$$\mathcal{J} = \langle \omega | I_x | \omega \rangle / \omega = [2\, dE_I/dI\, (I+1)]^{-1}. \quad (9)$$

The equations of the self-consistent cranking model have so far been solved using an effective interaction that includes monopole pairing and factorizable quadrupole forces (Refs. 31–35), δ forces with density dependence (Migdal forces[36–39]), and the Bhargava–Thouless interaction.[40,41] The cranking model has also been used in conjunction with Strutinskiĭ's method (see Sec. 5)

The self-consistent cranking model does not lead to any changes in the expression for the moment of inertia as compared with Inglis's formula if only factorized forces of the pairing and the quadrupole interaction are taken into account in the interaction. For other forms of interaction, the moment of inertia is renormalized, though information about the terms of the Hamiltonian capable of such renormalization is very sparse. For example, in the variant of Migdal forces used in Refs. 37 and 38 there is no renormalization because of the compensation of the contributions from the particle–particle and particle–hole channels. All this gives a certain conviction to calculations of the moment of inertia by either Inglis's formula or the self-consistent cranking model. However, there do remain small unexplained discrepancies between the experimental moments of inertia and those found from Inglis's formula with allowance for pairing correlations. One of the explanations of the discrepancy[42,43] is the existence of quadrupole pairing (pairs with Y_{21} symmetry) having the same symmetry as I_x, and therefore contributing to $\mathcal{J}$. Although the coupling constant of the quadrupole pairing remains rather uncertain, it is known that the value needed to obtain the experimental $\mathcal{J}$ is not unreasonable and not very strongly different from the value obtained in Ref. 42 from the conditions of gauge invariance of the Hamiltonian and in Refs. 43 by an analysis of two-nucleon transfer reactions.

In the quasiclassical approximation, ω in Eqs. (6)–(9) has a clear physical meaning. In the conceptual framework employed, it remains unclear why the quasiclassical approximation can be used to describe the stationary states of nuclei at low angular momentum. In practice, problems arise because the many-particle wave functions in the cranking-model approach are not, as a rule, eigenfunctions of the angular momentum. This produces many difficulties: The connection between the rotational frequency and the angular momentum in (2) is introduced on the basis of the correspondence principle, but it does not follow from quantum theory; in the framework of what has been said, it is necessary to formulate additional hypotheses in order to obtain expressions for the nonenergy parameters of nuclei that are determined by the matrix elements of operators of physical observables between stationary states of a nucleus. In this connection, it would be of interest to develop different methods for describing rotation.

One such method was initiated by Hill and Wheeler,[44] who suggested that more complicated wave functions should be generated by means of the many-particle functions of the theory of independent particles (quasiparticles):

$$\Phi_\nu = \int d\alpha f_\nu(\alpha)\, \Psi_\nu(\alpha; \mathbf{x}_1, \ldots \mathbf{x}_A). \quad (10)$$

Here, the functions $\Psi_\nu(\alpha; \mathbf{x}_1, \ldots \mathbf{x}_A)$ describe in the general case the motion of nucleons (or nucleon quasiparticles) in a field of forces that depends on various parameters α; $f_\nu(\alpha)$ is a weighting function determined on the basis of the variational approach. In the description of rotational motion, the Eulerian angles determining the spatial orientation of the deformed (self-consistent) field are the parameters α, and the weighting function $f_\nu(\alpha)$ is a combination of the Wigner D functions[45–47]:

$$\Phi_{IM} = \int d\Omega \sum_K f_K D^{I*}_{MK}(\Omega)\, \hat{D}(\Omega)\, \Psi(\mathbf{x}, \ldots \mathbf{x}_A), \quad (11)$$

where

$$\hat{D}(\Omega) = \exp(-i\varphi \hat{I}_z) \exp(-i\theta \hat{I}_y) \exp(-i\psi \hat{I}_z) \quad (12)$$

is the operator of rotation with the Eulerian angles $\Omega = (\varphi, \theta, \psi)$.

The wave function (11) is an eigenfunction for the operators $I^2 = I_x^2 + I_y^2 + I_z^2$ and I_z. Thus, if the Hamiltonian H is known, the energy E_I of a state with angular momentum I can be determined as

$$E_I = \langle \Phi_{IM} | \hat{H} | \Phi_{IM} \rangle / \langle \Phi_{IM} | \Phi_{IM} \rangle = \frac{\int d\Omega \sum_{KK'} f^*_K f_{K'} D^{I*}_{KK'}(\Omega) \langle \Psi | \hat{H} \hat{D}(\Omega) | \Psi \rangle}{\int d\Omega \sum_{KK'} f^*_K f_{K'} D^{I*}_{KK'}(\Omega) \langle \Psi | \hat{D}(\Omega) | \Psi \rangle}. \quad (13)$$

The many-particle functions $|\Psi\rangle$ in (11)–(13) can be determined by requiring that the energies E_I in (13) satisfy the variational principle. The projection method has many advantages. The wave functions (11) are exact eigenfunctions of the angular momentum, and therefore the description of rotation in its framework is a quantum one, and not a quasiclassical one, as in the cranking model. At the same time, one can use it to study the reaction of the nuclear field to rotation in the same way as in the self-consistent cranking model. In contrast to the latter, the internal functions $|\Psi\rangle$ in the projection method cannot be found by solving the Schrödinger equation for the nucleons in some field; they are determined by the more complicated variational equations. Many investigations into the rotational properties of nuclei have been made in the framework of the projection method.[48,49] However, the complexity of the variational procedure in this method is so great that is leads to laborious and lengthy computer calculations under assumptions about the nature of the internal functions for which the analysis of the problem in the framework of the self-consistent cranking model is comparatively simple. In other words, the actual flexibility of the projection method with regard to the choice of the internal functions is very limited. At the same time, the conditions imposed on the choice of the internal functions in the framework of this method are very

stringent.

The need for a careful choice of the many-particle function $|\Psi\rangle$ becomes particularly clear when the projection method is applied to translational motion. In this case, the correct method must lead to a value of the mass parameter equal to the sum of the nucleon masses. When the projection method is used, this result is obtained if the wave functions Φ corresponding to different values of the total momentum are projected from functions Ψ related to one another by a Galileo transformation:

$$\Psi_{\mathbf{P}_2}(\mathbf{r}_1, \ldots, \mathbf{r}_A) = \exp[i(\mathbf{P}_2 - \mathbf{P}_1)(\mathbf{r}_1 + \ldots \mathbf{r}_A)/A]\,\Psi_{\mathbf{P}_1}(\mathbf{r}_1, \ldots, \mathbf{r}_A).$$

Similarly, for the correct description of rotational motion it is necessary that the functions $|\Psi\rangle$ contain an optimal fraction of the projected state.

A criterion that the selected class of functions $|\Psi\rangle$ in the projection method is sufficiently large can be found on the basis of certain additional assumptions. In this way, it has been possible to establish an intimate connection between the projection method and the self-consistent cranking model and thus obtain a basis for a quantum description of the rotation that requires many fewer calculations and pretends to a greater accuracy than when the projection method is used directly.

In Ref. 50, a method was proposed for the approximate calculation of the expression (13) based on the fact that for reasonable values of the deformation parameter of the averaǵe field the overlap integral

$$n(\Omega) = \langle\Psi|\hat{D}(\Omega)|\Psi\rangle \tag{14}$$

is a function with a sharp peak at $\Omega = 0$, and also at the values of Ω corresponding to rotations that leave the function $|\Psi\rangle$ unchanged. The overlap integral occurs directly in the denominator of (13). The matrix element $\langle\Psi|\hat{H}\hat{D}(\Omega)|\Psi\rangle$ can be approximated by the expression

$$h(\Omega) \equiv \langle\Psi|\hat{H}D(\Omega)|\Psi\rangle = (a_0 + \sum_i a_i\mathcal{I}_i + \sum_{ij} a_{ij}\mathcal{I}_i\mathcal{I}_j + \ldots)n(\Omega). \tag{15}$$

Here, we have introduced operators of the "collective angular momentum" $\mathcal{I}_i (i = x, y, z)$:

$$\left.\begin{aligned} &\mathcal{I}_x \pm i\mathcal{I}_y = \exp(\mp i\varphi)\left\{-\operatorname{ctg}\vartheta\frac{\partial}{\partial\varphi} \mp i\frac{\partial}{\partial\vartheta} + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\psi}\right\};\\ &\mathcal{I}_z = -i\frac{\partial}{\partial\varphi}\quad(\Omega = (\varphi, \vartheta, \psi)),\end{aligned}\right\} \tag{16}$$

whose action on the functions $n(\Omega)$ and $h(\Omega)$ is given by

$$\left.\begin{aligned} &\mathcal{I}_i n(\Omega) = \langle\Psi|\hat{I}_i\hat{D}(\Omega)|\Psi\rangle;\\ &\mathcal{I}_i h(\Omega) = \langle\Psi|\hat{H}\hat{I}_i\hat{D}(\Omega)|\Psi\rangle.\end{aligned}\right\} \tag{17}$$

In (15), the ellipsis stands for the terms of higher order in powers of $\mathcal{I}_i$, which decrease with increasing power if the ratio $h(\Omega)/n(\Omega)$ depends on the angles Ω much more weakly than each of the factors that determine it. Using (17), we can readily express a_0, a_i, etc., as the expectation values with respect to the state $|\Psi\rangle$ of the Hamiltonian multiplied by some combination of angular-momentum operators, and then find the approximate value of E_I in (13).

In Ref. 50, the following result is proved: If one sets all $g_K = 1$ in Eq. (11) and restricts oneself to the linear terms for $h(\Omega)$ in the expression (15), then one of the solutions of the variational problem of the projection theory simultaneously satisfies the equations of the self-consistent cranking model, i.e., Eqs. (1)–(5):

$$\delta E_I^{\text{Proj}} = \delta\langle H - \omega\hat{I}_x\rangle + \frac{\delta\langle I_x\rangle}{\mathcal{I}}\left(\sqrt{I(I+1)} - \langle\hat{I}_x\rangle\right) - \delta\mathcal{I}/\mathcal{I}^2\langle\hat{I}_x\rangle\left(\sqrt{I(I+1)} - \langle\hat{I}_x\rangle\right). \tag{18}$$

The uniqueness of the solution of Eq. (18) has not been established. However, it is shown in this way that the equation of the self-consistent cranking model also determines simultaneously an approximate solution for the internal function in the projection method.

The use of the projection method is based on the idea that there exists a fairly simple many-particle function Ψ whose expansion with respect to eigenfunctions of the angular momentum operators,

$$\Psi = \sum_{IM} g_{IM}\Phi_{IM}, \tag{19}$$

has at least one term which approximates an eigenfunction of the Hamiltonian. In the approach described in Refs. 32 and 51, this assumption is strengthened: It is required that all terms in (19) be approximate eigenfunctions of the Hamiltonian. In practical realizations of this approach, the function Ψ is assumed to be the vacuum with respect to the quasiparticles of a general Bogolyubov transformation.[28] Usually, the solution of the cranking model does not change appreciably when $\langle I_x\rangle$ is increased by an amount of order $[\langle\hat{I}^2\rangle - \langle\hat{I}\rangle^2]^{1/2}$. This circumstance makes the original assumption fairly plausible. If it were satisfied with high accuracy, the eigenstates of the Hamiltonian could be found by projecting them from the wave function Ψ in (19). However, this approach is rejected in Refs. 31 and 51 for two reasons: 1) Even small admixtures of excited nonrotational states can lead to gross errors in the calculation of the energies of yrast-band states, and 2) calculations in the projection theory are too complicated. Ways of overcoming these difficulties are opened up if the variational principle is formulated in such a way that its exact solutions can be expressed in the form (19) with functions Φ_{IM} corresponding to the yrast-band eigenstates of the nucleus.

We write the variation of the expectation value of the energy in the state (19) in the form[52]

$$\delta\langle\Psi|\hat{H}|\Psi\rangle = \sum_I E_I\delta\left(\sum_M |g_{IM}|^2\right) + \sum_{IM}|g_{IM}|^2\delta\langle\Phi_{IM}|\hat{H}|\Phi_{IM}\rangle. \tag{20}$$

If, as proposed above, the unvaried functions Φ_{IM} are eigenfunctions of the Hamiltonian $\hat{H}$, then a small norm-conserving variation of them does not contribute to the right-hand side of (20) in the linear approximation. It follows that

$$\delta\langle\Psi|(\hat{H} - h(\hat{I}^2))|\Psi\rangle = 0, \tag{21}$$

where the functional dependence of h on I^2 is the same as in the expression for the yrast-state energies as a function of $I(I+1)$:

$$h(I(I+1)) = E_I. \tag{22}$$

If the energies E_I of the states that occur with large weight in (17) differ sufficiently little from the energies determined by the simple formula for a rigid rotor:

$$E_I \approx E_{I_0} + [I(I+1) - I_0(I_0+1)]/2\mathcal{I},$$

and (or) if the states Ψ are such that $\langle \hat{I}^2 \rangle$ completely determines the higher moments of the operator $\hat{I}^2$, the variational principle (21) can be simplified by making the substitution

$$h(\hat{I}^2) = h_0 + \mu \hat{I}^2. \tag{23}$$

Equations (21), (22), or (23) contain two unknown elements, which must be determined if the problem is to be solved: 1) For given microscopic, $\hat{H}$, and model, h, Hamiltonians, it is necessary to find function Ψ satisfying the stationarity condition (21); 2) knowing the solution for Ψ and the microscopic Hamiltonian $\hat{H}$, it is necessary to determine the parameters of the model Hamiltonian h. At the same time, the condition of self-consistency with respect to the model Hamiltonian must be satisfied, i.e., the input values of the parameters of h for which the function Ψ is determined must be equal to the values calculated in accordance with the data on this function.

Assuming Ψ to be the Bogolyubov vacuum, the variational equation (21) can be written in terms of the density matrices ρ and t in Eq. (8) by standard methods. We then obtain the ordinary self-consistency conditions of the Hartree–Fock–Bogolyubov theory in addition to the condition discussed above.

To fix the parameters in h, we must use the fact that any superposition of yrast states satisfies Eq. (21), so that this equation determines an entire family of solutions. From this there follows the existence of operators with the properties of angular variables $\hat{\varphi}$ or spherical harmonics which depend on the angles $\hat{R}^+_{1m}$ and commute with the operator $\hat{H} - h$. If h is approximated by Eq. (23), the equations for $\hat{\varphi}$ have the form

$$\left[H - \frac{I^2}{2\mathcal{I}},\ \hat{\varphi}\right] = 0;\ [\hat{I}_x,\ \hat{\varphi}] = -i. \tag{24}$$

Equations (24) can be solved approximately, using the procedure and approximations of the random-phase method.[52] For the moment of inertia we then obtain once more the expression of the self-consistent cranking model. Actually, the equation that determines the state Ψ in this approach differs somewhat from the equation of that model (see, for example, Ref. 52).[4)] One can establish relations between the matrix elements $\hat{\varphi}$ in the random-phase approximation and spherical harmonics which depend on angular coordinates, whose action on the state Ψ leaves it a superposition of yrast-band states. These relations are convenient for calculating the matrix elements of multipole operators between stationary states of the yrast-band, and also for taking into account the quantum-mechanical corrections to the moment of inertia and the other parameters in the model Hamiltonian.[51]

The discussion of the approach of Refs. 32, 51, and 52 given above reveals the role of collective variables in microscopic theories of nuclear rotation. In the theory of Filippov and Vanagas,[6-9] the collective variables are fixed on the basis of physical considerations as definite functions of the spatial coordinates of the nucleons. In the models associated with the cranking model, such variables are determined from dynamical equations in such a way as to ensure optimal splitting of the Hamiltonian into two weakly coupled parts. In this case physical arguments are put forward concerning the class of many-particle states within which the collective variables operate.

In the earlier approaches based on the variational principle it was possible to solve problems arising in the cranking-model approach because of its quasiclassical nature. One of the general problems of nuclear physics, which touches the theory of rotation as well, relates to the applicability of the Hartree–Fock–Bogolyubov variational method. But there are methods—those of Klein, Kerman, *et al*[55] and Belyaev and Zelevinskiĭ[56,57]—which enable one to justify the cranking model without recourse to the variational method.

Both variants of the theory eschew the introduction in explicit form of any approximate expressions for the many-particle functions of the nucleus. Instead, they analyze the coefficients of fractional parentage:

$$\left.\begin{aligned} &\psi_i(x, IM) = \langle i | \psi(x) | IM \rangle \\ &\varphi_{i'}(x, IM) = \langle i' | \psi^+(x) | IM \rangle \end{aligned}\right\} \text{— in method of Klein } et\ al.$$
$$\mathcal{R} = \langle IM | \begin{pmatrix} \psi^+(x')\psi(x) & \psi(\tilde{x}')\psi(x) \\ \psi^+(x')\psi^+(\tilde{x}) & \psi(\tilde{x}')\psi^+(\tilde{x}) \end{pmatrix} | I'M' \rangle —$$
$$\text{in method of Belyaev and Zelevinskiĭ.[56,57]} \tag{25}$$

In the approach of Klein *et al*,[55] the coefficients of fractional parentage couple rotational states of even nuclei, $|IM\rangle$, to states of their odd neighbors $|i\rangle$ obtained by applying the operators $\psi(x)$ and $\psi^+(x)$ of annihilation or creation of a nucleon at the point x with definite spin and isospin projections. In the approach of Belyaev and Zelevinskiĭ only states of even nuclei are considered. The basic hypothesis, which makes it possible to construct a model of the nuclear spectrum, consists of a generalization of the Hartree–Fock–Bogolyubov theory for the rule for decoupling the fermion operators. Thus, the method of Klein *et al* uses the approximate relation

$$\begin{aligned} &\langle i | \psi^+(1)\psi(3)\psi(2) | IM \rangle \\ &\approx \sum_{I'M'} \{\langle i | \psi(2) | I'M' \rangle \langle I'M' | \psi^+(1)\psi(3) | IM \rangle \\ &\quad - \langle i | \psi(3) | I'M' \rangle \langle I'M' | \psi^+(1)\psi(2) | IM \rangle \\ &\quad + \langle i | \psi^+(1) | I'M' \rangle \langle I'M' | \psi(3)\psi(2) | IM \rangle\}. \end{aligned} \tag{26}$$

[4)] Our approach gives an exact value for the total mass of the nucleus. Modifying somewhat the second of Eqs. (24), one can obtain the exact value for the moment of inertia in Elliot's model[53] (see, for example, Ref. 54).

This assumption is equivalent to the basic assumption in the theory of Belyaev and Zelevinskiĭ to the effect that the generalized density matrix $\mathscr{R}$ is approximately closed in the state space of one rotational band of an even–even nucleus. Such assumptions are used in conjunction with the equations of motion for the operators in the derivation of a self-consistent system of equations for the coefficients of fractional parentage. In the adiabatic limit, the sum over the intermediate states in (26) is restricted to the diagonal term, which leads to the results of the Hartree–Fock–Bogolyubov theory. Approximate allowance for noncommutativity of the different components of the angular momentum leads to the self-consistent cranking model.[5)]

As yet, comparisons of the results of different methods of description of rotation are very fragmentary. The self-consistent cranking model does not need to be made more accurate in the calculation of the inertial parameters in two cases when the problem is solved analytically: in the calculation of the total mass in the case of translational motion and in the calculation of the moment of inertia in Elliot's model.[53] Effects arising from the difference between the variational equation in the approach of Refs. 31, 51, and 52 and the equations of the self-consistent cranking model were studied in Ref. 60, and an opinion about them can be obtained on the basis of the results published in Ref. 61. Zelevinskiĭ and Shtokman have made a systematic analysis of the corrections to the self-consistent cranking model that arise in their approach.[57] They obtained a very good description of the moments of inertia and a reasonable description of the nonadiabaticity parameter (see the following section) by taking into account a 10–15% correction of quantum-mechanical nature to the moment of inertia of the cranking model. However, the correction terms to the results of the self-consistent cranking model that follow from the different approaches do not fully agree. Nevertheless, in the majority of approaches the self-consistent cranking model appears as a first approximation, and the corrections to it obtained in different calculations are small. This is the justification for using the self-consistent cranking model to study the influence of rotation on the structure of nuclei.

2. NONADIABATIC EFFECTS IN DEFORMED NUCLEI AT SMALL AND INTERMEDIATE ANGULAR MOMENTA

Classification of nonadiabatic effects. The concept of nonadiabatic effects itself can be defined by expanding the energy of the system obtained by solving Eq. (7) in powers of the angular frequency ω. The quadratic term of the expansion is interpreted as the adiabatic energy of rotation. By definition, the terms of higher order are the nonadiabatic corrections.

To classify the corrections to the energy, it is helpful to use the proposition proved in Ref. 18 to the effect that the corrections of n-th order to Δ and V in the expansion in powers of ω are generated by the correction of order $n-1$ to the amplitudes u_i^μ and v_i^μ. The parts of Δ and V which have the same symmetry properties as the operator I_x (in the axisymmetric case, $K^\pi = 1^+$) are an exception. These parts of the self-consistent field contain corrections of the same order as the amplitudes $(u_i^\mu,\ v_i^\mu)$. One can distinguish three types of correction[31,62]:

1) the direct influence of the term $\omega\hat{I}_x$ on the quasiparticle amplitudes u_i^μ and v_i^μ and the energy E_μ for fixed characteristics of the self-consistent field Δ_{ij} and V_{ij}; in what follows, we shall refer to this as the influence of rotation on the single-particle motion;

2) the change in the pairing potential Δ, which is usually called the Coriolis antipairing effect (CAP);

3) the change in the average potential V, in particular, the change of the parameters characterizing the shape of the potential.

Instead of the angular-momentum dependence of the energy, it is frequently instructive to consider the moment of inertia as a function of the frequency. The moment of inertia is defined, as in classical mechanics, by $\mathscr{I} = I/\omega$, and the rotational frequency which occurs in the cranking model satisfies another classical relation, namely, $\omega = dE(I)/dI$ [see also Eq. (9)]. However, the deduction of the function $\mathscr{I}(\omega^2)$ from experimental information about the distances between the yrast-band levels is not entirely unique because the ratio of the differentials must be replaced by the ratio of finite differences. We adopt here the procedure of Ref. 63, which leads to an exact linear dependence of $\mathscr{I}$ on ω^2 at small angular momenta.

To describe states with sufficiently small angular momentum, we can approximate the function $\mathscr{I}(\omega^2)$ by the expression

$$\mathscr{I} = \mathscr{I}_0 + \omega^2 d\mathscr{I}/d\omega^2. \tag{27}$$

The energies of the states are then determined by the well known expressions of Harris (see, for example, Eqs. (11) and (12) in Ref. 63):

$$\left.\begin{aligned} E_I &= E_0 + \mathscr{I}_0\omega^2\left[1+\frac{3}{2}\frac{1}{\mathscr{I}_0}\left(\frac{d\mathscr{I}}{d\omega^2}\right)\omega^2\right];\\ I &= \omega\mathscr{I} = \mathscr{I}_0\omega\left[1+\frac{1}{\mathscr{I}_0}\left(\frac{d\mathscr{I}}{d\omega^2}\right)\omega^2\right], \end{aligned}\right\} \tag{28}$$

which, as a rule, reproduce the experimental values of the energy for stably deformed nuclei right up $I = (12-14)\hbar$. The slope $d\mathscr{I}/d\omega^2$ of the linear function $\mathscr{I}(\omega^2)$ is a convenient measure of the nonadiabaticity, and more convenient than the coefficient

$$B = -\frac{1}{4}\frac{1}{\mathscr{I}_0^3}\left(\frac{d\mathscr{I}}{d\omega^2}\right), \tag{29}$$

which occurs in the expansion of the energy in powers of the angular momentum I.

We shall restrict our discussion of the nonadiabatic effects at small angular momentum to the influence of

[5)] A more detailed description of the theory of Klein *et al.* in Russian can be found in the Appendix added to the Russian translation of Ref. 58; the method of Belyaev *et al.* is reviewed in Refs. 59.

rotation on the single-particle motion and the degrees of freedom associated with the shape and monopole pairing since this is the most important influence. In this case, all the nonadiabatic effects make an independent contribution to the fourth-order term because the effects 2) and 3) are due to the contribution of adiabatic rotation to the pairing tensor t_{ij} and the density matrix ρ_{ij} [see (7)]. It is only in the higher orders that the different nonadiabatic effects influence each other.

In Fig. 1, the experimental values of $d\mathcal{I}/d\omega^2$ are compared with the theoretical values calculated in Ref. 65. The agreement between theory and experiment is good in the center of the region of rare earth elements. Appreciable deviations are observed at the edges, i.e., in the transition nuclei. Let us consider separately the contributions of all three types of nonadiabatic effect.

1. The largest contribution is introduced into $d\mathcal{I}/d\omega^2$ by the influence of rotation on the single-particle motion. The corresponding expression can be found in Refs. 31 and 62. The main part of this correction is determined by the shell structure: The Coriolis force tends to polarize the angular momenta of the individual particles along the rotation axis, i.e., to disrupt the coupling scheme in deformed nuclei, in accordance with which the particles are aligned along the symmetry axis of the core.[66]

For large ω the solution of Eq. (7) without the use of perturbation theory shows that the angular momenta of several particles occupying states with large angular momentum can be almost completely aligned along the rotation axis (see Sec. 4). This effect is responsible for the experimentally observed phenomenon that the plot of $\mathcal{I}$ against ω^2 has a characteristic S-shaped form, which is known as back-bending, over an angular momentum range up to $20\hbar$. This phenomenon is direct evidence of a change in the properties of the nucleus if there is a sufficiently large increase in the angular momentum, which leads to a sudden increase in the moment of inertia and a corresponding decrease in the rotational frequency.

Back-bending has been observed in a number of well deformed nuclei[34,40,63,64,66] and has been discussed in detail in several reviews (see, for example, Ref. 67).

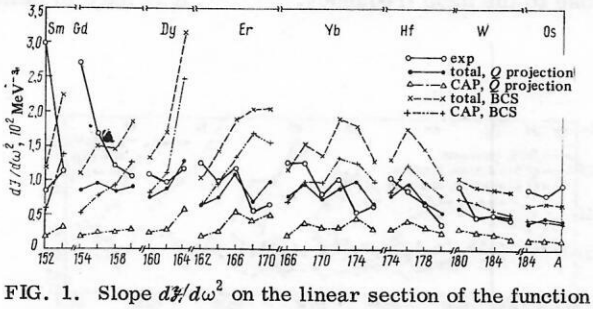

FIG. 1. Slope $d\mathcal{I}/d\omega^2$ on the linear section of the function (ω^2) in nuclei of rare earth elements. The contribution from the CAP effect was calculated in Ref. 65. The contributions from the change in the quadrupole and hexadecapole deformation, as also the fourth-order correction for rotation, are taken from Tables 5 and 6 of Ref. 22. (In which c_{VMI}^{-1} is equal to $2d\mathcal{I}/d\omega^2$.)

The alignment of two particles carrying a large angular momentum (for example, neutrons in the $i_{13/2}$ subshell in rare earth elements) leads to such a large gain in the total angular momentum that the core can rotate more slowly although the total angular momentum is larger.

2. The next most important effect is due to the Coriolis antipairing. In the calculations presented in Sec. 1 it is assumed that the pairing changes in the same way for particles in all single-particle levels.

In order to estimate quantitatively the CAP effect, it is necessary to use a description of pairing with projection onto a given number of particles, which we shall here call the Q projection. As can be seen in Fig. 1, the CAP effect is significantly overestimated in the BCS approximation. This can be understood on the basis of the expression[31,62]

$$(d\mathcal{I}/d\omega^2)_{\text{CAP}} \equiv \frac{\partial \mathcal{I}}{\partial \Delta}\frac{\partial \Delta}{\partial \omega^2} = \frac{1}{2}\left(\frac{\partial \mathcal{I}}{\partial \Delta}\right)^2 \Big/ \frac{\partial^2 E_0}{\partial \Delta^2}, \tag{30}$$

where

$$\frac{\partial \Delta}{\partial \omega^2} = \frac{1}{2}\frac{\partial \mathcal{I}}{\partial \Delta}\Big/\frac{\partial^2 E_0}{\partial \Delta^2}. \tag{31}$$

Here, Δ is the gap parameter; E_0 is the total energy with allowance for pairing; $\partial^2 E_0/\partial\Delta^2$ is the rigidity against variation of the pairing field. The main difference between the approach with conservation of particle number and BCS theory arises through the different estimates for $\partial^2 E_0/\partial\Delta^2$. In BCS theory, E_0 and $\partial^2 E_0/\partial\Delta^2$ are underestimated by a factor of about 2. The reason, as is readily seen, is that in the case of Q projection the wave function of a state for $\Delta \neq 0$ is closer to the exact solution of the pairing problem, and the energy is accordingly lower. At the same time, as is shown in Ref. 65, the quasiparticle energies, i.e., the even–odd mass differences, and the parameter of the pairing interaction determined on the basis of them are hardly changed. For $\Delta = 0$ there is a normal solution, which conserves the particle number. Therefore, the total pairing-correlation energy, defined as the difference $E_0(\Delta) - E_0(\Delta = 0)$, must increase in modulus.

In the middle of the region of the rare earth elements (stably deformed nuclei) the contribution to $d\mathcal{I}/d\omega^2$ of the degree of freedom associated with the shape is negligibly small, and the two corrections discussed above together make up the main part of the contribution. For these nuclei, the calculated value of $d\mathcal{I}/d\omega^2$ agrees fairly well with the experimental value, which gives one confidence that the contribution of the two corrections has been correctly estimated in order of magnitude. We emphasize that the previously noted enhanced stability of the pairing is necessary to avoid the too large theoretical values of $d\mathcal{I}/d\omega^2$ obtained, for example, in Ref. 31. In Ref. 22, Ma and Tsang, who used an approximate variant of Q projection, arrived at a similar conclusion.

It should be mentioned that the CAP contribution to $d\mathcal{I}/d\omega^2$ calculated in the BCS approximation is close to the experimental value. This has led various authors to conclude that the CAP contribution (and the degree of freedom associated with the shape) is the only source of

nonadiabatic corrections.[21,68,69] However, the coincidence appears to us fortuitous.

3. Change in the shape of the nucleus leads to the following contribution to $d\mathcal{I}/d\omega^2$ (Refs. 31 and 62):

$$\left(\frac{d\mathcal{I}}{d\omega^2}\right)_{\text{shape}} = \frac{\partial \mathcal{I}}{\partial \varepsilon}\frac{d\varepsilon}{d\omega^2};$$
$$\frac{d\varepsilon}{d\omega^2} = \frac{1}{2}\frac{\partial \mathcal{I}}{\partial \varepsilon}C_\varepsilon^{-1}, \qquad (32)$$

where the rigidity C_ε is the second derivative of the energy with respect to the deformation parameter ε and can be used to determine the shape of the average potential V. The degree of freedom associated with the shape makes a very small contribution to the total value of $d\mathcal{I}/d\omega^2$, as is shown by calculations in which the rigidity is calculated by the shell-correction method.[22] The degrees of freedom associated with the shape are discussed in more detail in Sec. 4, in which, in particular, we shall discuss the discrepancies between the calculated and empirical values of $d\mathcal{I}/d\omega^2$ that can be seen in Fig. 1 in soft deformed nuclei.

The CAP *effect in higher orders in* ω^2. A detailed investigation of the CAP effect without the use of perturbation theory has been made on several occasions,[34-36,39-41,43,70-75] and this has made it possible to study in some detail the transition from the superconducting to the normal state. The calculations revealed a fairly complicated picture of this transition, this being due to the strong interaction between the Coriolis antipairing and the nonadiabatic effects in the single-particle motion.

The critical angular momentum at which the transition from the superconducting to the normal state occurs can be estimated by equating the total energy of the pairing correlations E_0 to the gain in the rotational energy resulting from this transition:

$$E_0 = I_{\text{cr}}^2(1/2\mathcal{I}_1 - 1/2\mathcal{I}_2);$$
$$1/\mathcal{I}_{\text{cr}} = (1/2)(1/\mathcal{I}_1 + 1/\mathcal{I}_2) \approx 1/[0.4(\mathcal{I}_1 + \mathcal{I}_2)], \qquad (33)$$

where $\mathcal{I}_1$ is the moment of inertia in the ground state and $\mathcal{I}_2$ is the corresponding value for $\Delta = 0$. The second relation gives an estimate of the average moment of inertia in the region of the transition, an approximate expression for which is obtained using the typical value for the ratio of the geometric and arithmetic means of $\mathcal{I}_1$ and $\mathcal{I}_2$ in the nuclei of rare earth elements.

Denoting by ω_{cr} the ratio $I_{\text{cr}}/\mathcal{I}_{\text{cr}}$, we arrive at the estimate

$$\omega_{\text{cr}}^2 \approx 2.5E_0/(\mathcal{I}_2 - \mathcal{I}_1). \qquad (34)$$

The value of the critical angular momentum obtained on the basis of the expression (33) using in it the quantities calculated in the BCS approximation is $12\hbar$ and $18\hbar$ for nuclei of the rare-earth elements and actinides, respectively.[70] The results of more detailed calculations made under the same assumptions are shown in Fig. 2 (the BCS curve, neutrons). Similar results were obtained in Refs. 68-74.

As we have already noted, allowance for conservation of particle number leads to an enhanced stability of the superfluid phase, this being expressed by an increase in E_0 by a factor of about 2. At the same time, the moment of inertia hardly changes.[65] Then, in accordance with (9), I_{cr} increases by a factor 1.4.

The results of such a calculation are shown in Fig. 2. It can be seen that the critical frequencies for the neutron system lie at $\omega \approx 0.36$ MeV, and for the proton system at $\omega \approx 0.45$ MeV. The corresponding values of the angular momentum are $20\hbar$ and $35\hbar$. The CAP calculations were made without the use of perturbation theory in the determination of Δ, but the nonadiabatic effects in the single-particle motion were not taken into account. In such an approximation, one cannot obtain a detailed description of the transition region, although the critical frequency defined above is helpful for obtaining a general idea of when the transition occurs and for discussing the stability of pairing.

In Ref. 75, the CAP effect was investigated with allowance for the influence of rotation on the single-particle motion. Using the quasiclassical single-particle wave functions and a description of pairing equivalent to the BCS approximation, Grin' and Larkin found that $I_{\text{cr}} \approx (20-22)\hbar$ in the region of the rare earth elements and $(36-40)\hbar$ in the actinides, which exceeds the estimates given above.

The determination of the critical frequency is based on the idea of crossing of two bands, whose nature is not particularized. In particular, it can be used to determine the frequency corresponding to the back-bending observed in the ground-state band of a number of deformed even-even nuclei. These values, which are called the experimental values, are also shown in Fig. 2. It can be seen that the bend occurs at precisely the critical frequency of the transition $\Delta \to 0$ in the neutron scheme predicted in the BCS approximation. This agreement was regarded as an argument in favor of the interpretation of back-bending as due to the disappearance of pairing.[72-75] However, the stabilization of pairing in the approach with conservation of the particle number makes such an interpretation improbable since the critical frequency lies systematically higher. The angular frequency at which alignment of two neutrons in the $i_{13/2}$ subshell is energetically advantageous is very close to the bend frequency.[63,64,66] It is hard to believe

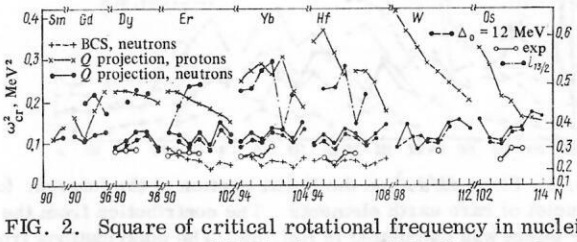

FIG. 2. Square of critical rotational frequency in nuclei of rare earth elements.[65] The line $\Delta_0 = 12$ MeV gives the results of calculations with reduced intensity of pairing and with allowance for the Q projection. The $i_{13/2}$ line corresponds to complete alignment of two neutrons in the $i_{13/2}$ subshell.

that allowance for conservation of particle number will strongly change this value since the two-quasiparticle energies and matrix elements of the Coriolis interaction are not very sensitive to projection onto a given number of particles (see Refs. 65 and 19). After the breaking of the $i_{13/2}$ neutron pair, strong pairing correlations still remain in the system.[34,35,39,65] In Fig. 2, we have also plotted the critical frequency of the transition $\Delta \to 0$ in the neutron system in which the spin of two neutrons in the $i_{13/2}$ subshell is completely aligned along the rotation axis. This transition frequency is higher than the frequency obtained in the case when the $i_{13/2}$ neutrons rotate together with all the remaining particles. Thus, the following qualitative picture of the transition $\Delta \to 0$ in the neutron system takes shape: At a frequency of order 0.27 MeV the two $i_{13/2}$ neutrons become aligned, this being manifested experimentally as back-bending, and it is accompanied by a sudden but not very strong reduction in the energy gap (two levels are blocked). After the jump, the gap decreases smoothly, as in nuclei without levels with large j near the Fermi surface. In principle, one could expect alignment of the following pair, accompanied by a new jump of the gap. However, such phenomena have not yet been reported (see the discussion in Ref. 76).

If one ignores the irregularities associated with alignment, the gap decreases fairly smoothly. Therefore, the critical frequencies in Fig. 2 can be interpreted as centers of a fairly wide transition region, and a certain fraction of pairing correlations must remain higher than ω_{cr}. This is confirmed by the experimental results of Ref. 77, which indicate that the moment of inertia at angular frequency between 0.3 and 0.5 MeV is approximately 15% smaller than the rigid-body value. (For the investigated nucleus ^{162}Yb, $\omega_{cr} \approx 0.5$ MeV in the proton system.)

The higher multipoles in the particle–particle channel may be important in a rotating nucleus, leading to coupled pairs that carry a finite angular momentum. In Ref. 43, the influence of quadrupole pairing was investigated. At small angular momenta, the quadrupole pairing field increases with the frequency, giving rise to the increase in the moment of inertia noted above. At large angular momenta the growth is halted and, in accordance with the predictions, the quadrupole pairing disappears together with the monopole part. The quadrupole pairing is only a relatively small correction to the monopole field. Calculations with effective forces[40,41] and without using a multipole expansion lead to the same qualitative conclusions as are obtained in the investigation of monopole pairing.

3. ELECTRICAL PROPERTIES OF ROTATING NUCLEI

The manifestation of nonadiabatic effects due to rotation in the electrical properties of nuclei requires an additional theoretical analysis, which we present in this section.

For the following discussion of the empirical data on the changes in the shapes of nuclei associated with angular momentum, it is convenient to introduce a definition of quadrupole deformation that uses directly measured experimental quantities. Thus, we use the following definition for the components α_μ of the quadrupole-deformation tensor, expressing them in terms of the quadrupole moment $\mathcal{M}(E_2, \mu)$ of the charge distribution[13]:

$$\mathcal{M}(E_2, \mu) \equiv (3/4\pi)\, ZeR^2\alpha_\mu, \tag{35}$$

where R is a typical radius of the nucleus, for example

$$R = 1.2A^{1/3}\ \text{F}; \tag{36}$$

Ze is the total charge of the nucleus and A is its mass number.

In accordance with the definition (35), the α_μ are functions of the spatial coordinates of the nucleons, so that the wave function Ψ of the nucleus can be written down as a function of these variables and also additional ("internal") variables ξ (see Sec. 1 and Refs. 7–9). The five components of the tensor α_μ can be associated with the three Eulerian angles $\Omega \equiv (\varphi, \nu, \psi)$, which characterize the orientation of the nucleus in space, and the two rotationally invariant deformation parameters β and γ by writing

$$\alpha_\mu = \beta\{\cos\gamma D^2_{\mu 0}(\Omega) + \sin\gamma\,(D^2_{\mu 2}(\Omega) + D^2_{\mu -2}(\Omega))/\sqrt{2}\}, \quad \beta \geqslant 0,\ 0 \leqslant \gamma \leqslant \pi/3, \tag{37}$$

where $D^\lambda_{\mu\nu}(\Omega) \equiv D^\lambda_{\mu\nu}(\varphi, \vartheta, \psi)$ are the generalized spherical functions of Wigner. To one set of tensor components α_μ there correspond different sets of Eulerian angles, which are related to one another by rotations belonging to the point group D_2, under whose action the orientation of the principal axes of the quadrupole-deformation tensor does not change. The wave function Ψ must obviously satisfy the conditions

$$R_e\Psi = R_i\Psi;\ R \in D_2, \tag{38}$$

in which the operators R_e and R_i describe the variations of the function Ψ due to, respectively, the change of the Eulerian angles and the internal variables under rotations which do not change α_μ. Accordingly, the nuclear Hamiltonian $\hat{H}$ must have the properties

$$R_e H R_e^{-1} = R_i H_i R_i^{-1},\ R \in D_2. \tag{39}$$

In the nonrelativistic approximation, the Hamiltonian H is a polynomial of second degree in the momentum operators. Further, it follows from the rotational invariance that the Hamiltonian does not depend on the Eulerian angles explicitly.[6)] In this approximation, we can write

[6)] It is necessary to bear in mind that the interaction of the nucleons includes nonlocal terms,[78] which lead to a lifting of the restriction on the degree of the polynomial dependence of H on the momenta, and, in addition, contains a tensor component which is noninvariant under rotations of only the spatial coordinates. This last circumstance has the consequence that the Hamiltonian can contain the Eulerian angles or operators which do not reduce to angular momentum operators and act in the space of the Eulerian angles.

$$H = h + A_1 I_1^2 + A_2 I_2^2 + A_3 I_3^2 + B_1 I_1 + B_2 I_2 + B_3 I_3 + C_1 (I_2 I_3 + I_3 I_2) + C_2 (I_3 I_1 + I_1 I_3) + C_3 (I_1 I_2 + I_2 I_1). \quad (40)$$

Here, the operators h, A_ν, B_ν, C_ν depend on β, γ, ξ and the generalized momenta canonically conjugate to these coordinates. Because of the conditions (39), the last six terms in the expression (40) do not contribute in the lowest order to the change in the shape, and on the basis of this we shall ignore them when discussing states with small angular momenta. Note, however, that these terms determine the alignment of the internal angular momenta in the direction of rotation. The sum of the first four terms can be represented in the form[7)]

$$H = H_0 + H_c, \quad (41)$$

where

$$H_0 = h + A (I_1^2 + I_2^2) + A_3 I_3^2; \quad (42)$$

$$H_c = h_0 (I_1^2 + I_2^2) + 2h_2 (I_1^2 - I_2^2); \quad (43)$$

$$A = \langle A_1 + A_2 \rangle_0 / 2; \quad (44)$$

$$h_0 = (A_1 + A_2)/2 - A; \quad (45)$$

$$h_2 = (A_1 - A_2)/4, \quad (46)$$

and the expectation value in (44) is with respect to the ground state of the Hamiltonian h.

The normalized eigenvalues of H_0 satisfying the conditions (38) are given by the expressions

$$\psi_{IMrKn} (\Omega, \beta, \gamma, \xi) = \sqrt{\frac{2I+1}{16\pi^2 (1+\delta_{K,0})}} \{D^I_{MK} (\Omega) + r (-1)^{I+K} D^I_{M,-K} (\Omega)\} \Phi_{rKn} (\beta, \gamma, \xi), \quad K \geqslant 0, \quad (47)$$

in which $\Phi_{rKn}(\beta, \gamma, \xi)$ are normalized solutions of the eigenvalue problem

$$(h + K^2 A_3) \Phi_{rKn} = (E_{rKn} + K^2 A) \Phi_{rKn} \quad (48)$$

with the symmetry properties

$$\hat{D}_i^{(3)} \Phi_{rKn} = (-1)^K \Phi_{rKn};$$
$$\hat{D}_i^{(2)} \Phi_{rKn} = r \Phi_{rKn}. \quad (49)$$

In Eqs. (49), $\hat{D}^{(\nu)}$ denotes a rotation through the angle π about the ν axis.

The eigenvalues corresponding to the wave functions Ψ_{IMrKn} are

$$E_{IrKn} = E_{rKn} + AI (I+1). \quad (50)$$

Thus, the operator H_0 has the properties of the Hamiltonian of an axial nucleus in the limit of strong binding of the nucleons to the core. The smallest of the eigenvalues E_{rKn} is the start of the ground-state rotational band, and therefore, for an even–even nucleus it corresponds to the values $K = 0$, $r = 1$. In what follows, we shall use $|0\rangle$ to label the ground-state function Φ_{rKn} and $|Kn\rangle$ for the remaining such functions with $r = 1$. We shall denote the corresponding excitation energies by E_{Kn}.

Since the strong binding scheme approximately describes the experimental data in nuclei in the middle of the region of rare earth elements, the term H_c can be regarded as a weak perturbation and taken into account by means of perturbation theory. In the first order, we then obtain the following expressions for the changes in the parameters β and γ due to the rotation:

$$\langle \beta_0 \rangle_I - \beta^0 = I (I+1) \beta^0 X/2; \quad (51)$$

$$\left\langle \frac{I_1^2 - I_2^2}{I (I+1)} \beta_2 \right\rangle_I = \sqrt{\frac{1}{2}} (I-1) (I+2) \beta^0 Y, \quad (52)$$

where

$$X = -(4/\beta^0) \sum_n \langle 0 | \beta_0 | 0n \rangle \langle 0 | h_0 | 0n \rangle / E_{0n}; \quad (53)$$

$$Y = -(\sqrt{8}/\beta^0) \sum_{2n} \langle 0 | \beta_2 | | 2n \rangle \langle 0 | h_2 | 2n \rangle / E_{2n} \quad (54)$$

$$\beta_0 \equiv \beta \cos \gamma; \ \beta_2 \equiv \beta \sin \gamma; \ \beta^0 = \langle \beta_0 \rangle_0. \quad (55)$$

In the same approximation, let us consider the matrix elements of the electric-quadrupole tensor. The matrix elements of transitions within the ground-state band ("intraband" transitions) can be expressed in terms of the quantities X and Y already introduced:

$$\frac{\langle I_0 \| \mathcal{M} (E_2) \| I_0 \rangle}{\sqrt{2I+1}} = \frac{3}{4\pi} ZeR^2 \beta^0 (I020 | I0) \times [1 + \sqrt{6} Y + (X/2 - \sqrt{3/2} Y) I (I+1)]; \quad (56)$$

$$\frac{\langle I-2, 0 \| \mathcal{M} (E2) \| I0 \rangle}{\sqrt{2I+1}} = \frac{3}{4\pi} ZeR^2 \beta^0 (I020 | I-2, 0) \times [1 + X/2 + \sqrt{8/3} Y + (X/2 + Y/\sqrt{6}) (I-1) I]. \quad (57)$$

For transitions between states of excited bands and states of the ground-state band ("interband" transitions) we use the additional simplifying assumptions that were first introduced by Hansen *et al.*[79] to analyze transitions between a γ-vibrational band and the ground-state band. This scheme was developed further in Refs. 80, 81, and other papers. In accordance with these papers, we make two assumptions: 1) Significant corrections to the transition matrix elements in the strong-binding limit are due solely to mixing of excited bands with the ground-state band. This assumption is justified by the fact that the effects of this mixing are determined by the large diagonal matrix element β^0; 2) the matrix elements $\beta^0 = \langle 0 | \beta_0 | 0 \rangle$ and $\langle Kn | \beta_0 | Kn \rangle$ are equal. Using these assumptions, we obtain in the first perturbation order in H_c:

$$\frac{\langle I'0 \| \mathcal{M} (E2) \| I0n \rangle}{\sqrt{2I+1}} = (3/4\pi) ZeR^2 (I020 | I'0) \langle 0 | \beta_0 | 0n \rangle [1 - z_0 f_0 (I, I')]; \quad (58)$$

$$\frac{\langle I'0 \| \mathcal{M} (E2) \| I2n \rangle}{\sqrt{2I+1}} = (3/4\pi) ZeR^2 (I22 (-2) | I'0) \langle 0 | \beta_2 | 2n \rangle [1 - z_2 f_2 (I, I')], \quad (59)$$

where

$$z_{0n} = -\frac{\beta^0}{\langle 0 | \beta_0 | 0n \rangle} \frac{\langle 0 | h_0 | 0n \rangle}{E_{0n}}; \quad (60)$$

$$z_{2n} = -\sqrt{48} \frac{\beta^0}{\langle 0 | \beta_2 | 2n \rangle} \frac{\langle 0 | h_2 | 2n \rangle}{E_{2n}}; \quad (61)$$

[7)] The discussion now following coincides in its broad features with the discussion in the monograph of Bohr and Mottelson.[4] Note, however, that the separation of the variables into internal and rotational variables made here differs somewhat from the one made in Ref. 4.

$$f_0(I,\ I') = I(I+1) - I'(I'+1); \tag{62}$$

$$f_2(I,\ I') = [I(I+1) - 4 - I'(I'+1)]/2. \tag{63}$$

As a special case of the relations (58) and (59), we have

$$B(E2;\ 00 \to 20n) = [(1 - 6z_{0n})\frac{3}{4\pi} ZeR^2 \langle 0 | \beta_0 | 0n\rangle]^2; \tag{64}$$

$$B(E2;\ 00 \to 22n) = [(1 - z_{2n})\frac{3}{4\pi} ZeR^2 \langle 0 | \beta_2 | 2n\rangle]^2. \tag{65}$$

Combining Eqs. (53)–(55) and (64), (65), we arrive at the relations

$$X = 4\sum_n \left(\frac{\langle 0 | \beta_0 | 0n\rangle}{\beta^0}\right)^2 z_{0n}; \tag{66}$$

$$Y = \frac{1}{\sqrt{6}} \sum_n \left(\frac{\langle 0 | \beta_2 | 2n\rangle}{\beta^0}\right)^2 z_{2n}. \tag{67}$$

We also give an expression for the coefficient B in the expression for the energies of the states of the ground-state band:

$$E_I = AI(I+1) + B[I(I+1)]^2 + \cdots \tag{68}$$

We have

$$B = -\sum_n \frac{\langle 0 | h_0 | 0n\rangle^2}{E_{0n}} - 2\sum_n \frac{\langle 0 | h_2 | 2n\rangle^2}{E_{2n}}$$
$$= -\sum_n E_{0n}\left(\frac{\langle 0 | \beta_0 | 0n\rangle}{\beta^0} z_{0n}\right)^2 - \frac{1}{24}\sum_n E_{2n}\left(\frac{\langle 0 | \beta_2 | 2n\rangle}{\beta^0} z_{2n}\right)^2. \tag{69}$$

The corrections to the limit expressions obtained in the strong-binding scheme for the energy of the states [see (68) and (69)] and the matrix elements of the transitions [see (56)–(65)] increase with the angular momentum I. Therefore, for sufficiently large I the perturbation theory described above becomes inapplicable in the analysis of the Hamiltonian (40). However, for $I \gg 1$ the states in the neighborhood of the yrast band can be described quasiclassically under the assumption that the angular momentum is oriented sufficiently accurately in the internal coordinated system.[8)] It is natural to include the kinetic energy that a nucleus would have in the case of fixed orientation of the angular momentum in the internal coordinate system in the definition of the internal energy in the limit of large angular momenta. Formally, this can be achieved by averaging the Hamiltonian (40) over the coherent states described in Ref. 83:

$$u_{IM}(\tilde{\vartheta},\ \tilde{\varphi};\ \Omega) = \sqrt{\frac{2I+1}{8\pi^2}} \sum_{K=-I}^{I} c_K^I(\tilde{\vartheta},\ \tilde{\varphi})\, D_{MK}^I(\Omega), \tag{70}$$

where

$$c_K^I(\tilde{\vartheta},\ \tilde{\varphi}) = D_{KI}^I(\tilde{\varphi},\ \tilde{\vartheta},\ 0)$$
$$\equiv (-1)^{I-K}\exp(-i\tilde{\varphi}K)(\cos\tilde{\vartheta}/2)^{I+K}\sin(\tilde{\vartheta}/2)^{I-K}. \tag{71}$$

For the mean values with respect to the states (70) and (71), we obtain

$$\left.\begin{aligned} \langle I_1\rangle &= I\cos\tilde{\varphi}\sin\tilde{\vartheta};\\ \langle I_2\rangle &= I\sin\tilde{\varphi}\sin\tilde{\vartheta};\\ \langle I_3\rangle &= I\cos\tilde{\vartheta}; \end{aligned}\right\} \tag{72}$$

$$\left.\begin{aligned} \langle I_1^2\rangle &= I(2I-1)\sin^2\tilde{\vartheta}\cos^2\tilde{\varphi}/2 + I/2;\\ \langle I_2^2\rangle &= I(2I-1)\sin^2\tilde{\vartheta}\sin^2\tilde{\varphi}/2 + I/2;\\ \langle I_3^2\rangle &= I(2I-1)\cos^2\tilde{\vartheta}/2 + I/2;\\ \langle I_1I_2 + I_2I_1\rangle &= I(2I-1)\sin^2\tilde{\vartheta}\sin 2\tilde{\varphi}/2;\\ \langle I_1I_3 + I_3I_1\rangle &= I(2I-1)\sin 2\tilde{\vartheta}\cos\tilde{\varphi}/2;\\ \langle I_2I_3 + I_3I_2\rangle &= I(2I-1)\sin 2\tilde{\vartheta}\sin\tilde{\varphi}/2. \end{aligned}\right\} \tag{73}$$

As a result of the averaging over the coherent states, the angular-momentum operators in the Hamiltonian (40) become C numbers, which depend on the two continuous parameters $\tilde{\vartheta}$ and $\tilde{\varphi}$ ($0 \leq \tilde{\vartheta} \leq \pi$, $0 \leq \tilde{\varphi} \leq 2\pi$). The functions $u_{IM}(\tilde{\vartheta}, \tilde{\varphi}; \Omega)$ do not contain the internal variables β, γ, ξ or the momenta conjugate to them, and therefore the averaging with respect to them of the Hamiltonian does not change the elements of the Hamiltonian that are not given explicitly in (72) and (73). The operator $H_{\langle I\rangle}$ obtained as a result of the averaging has the form

$$H_{\langle I\rangle} = h + A_1\langle I_1^2\rangle + A_2\langle I_2^2\rangle + A_3\langle I_3^2\rangle + B_1\langle I_1\rangle + B_2\langle I_2\rangle + B_3\langle I_3\rangle$$
$$+ C_1\langle I_2I_3 + I_3I_2\rangle + C_2\langle I_3I_1 + I_1I_3\rangle + C_3\langle I_1I_2 + I_2I_1\rangle \tag{74}$$

and acts only in the space of the internal variables. It depends on the angular momentum I and the angles $\tilde{\vartheta}$ and $\tilde{\varphi}$ as on parameters, and therefore its normalized eigenfunctions Φ_n and energy eigenvalues E_n depend on the absolute magnitude and direction of the vector $\langle I\rangle$. Without loss of generality, the angular variables $\tilde{\vartheta}$ and $\tilde{\varphi}$ can be chosen in such a way as to minimize the energy of the ground state for the Hamiltonian $H_{\langle I\rangle}$.

Using (74), we write

$$H = H_{\langle I\rangle} + \tilde{H}_c, \tag{75}$$

where the form of the operator $\tilde{H}_c$, which depends on all variables of the system, is determined in an obvious manner by Eq. (40).

It is convenient to go over to new angular variables $\Omega_{\langle I\rangle} = (\varphi', \vartheta', \psi')$ of the system, defining them as the Eulerian angles of a rotation equal to the sum of rotations through the angles $\Omega_{\langle I\rangle} = (\tilde{\varphi}, \tilde{\vartheta}, 0)$ and $\Omega = (\varphi, \vartheta, \psi)$. The physical meaning of this transition is clear if one considers the mean values of the operators of the projections of the internal angular momenta transformed to the new angles. In accordance with the general definition for the multipole operators of physical observables in the internal coordinate system,[4] we write

$$\mathcal{M}(\lambda m) = \frac{1}{2}\sum_{\varkappa=-\lambda}^{\lambda} \{D_{m\varkappa}^{\lambda}(\Omega_{\langle I\rangle})\,\tilde{\mathcal{M}}(\lambda,\ \varkappa)$$
$$+ \tilde{\mathcal{M}}(\lambda,\ \varkappa)\, D_{m\varkappa}^{\lambda}(\Omega_{\langle I\rangle})\}. \tag{76}$$

Here, $\mathcal{M}(\lambda m)$ and $\tilde{\mathcal{M}}(\lambda\varkappa)$ are the operators in the laboratory and internal systems, respectively. The definition of the angular variables $\Omega_{\langle I\rangle}$ given above means that

$$D_{m\varkappa}^{\lambda}(\Omega_{\langle I\rangle}) = \sum_{\mu} D_{m\mu}^{\lambda}(\Omega)\, D_{\mu\varkappa}^{\lambda}(\tilde{\varphi},\ \tilde{\vartheta},\ 0), \tag{77}$$

where $\Omega = (\varphi, \vartheta, \psi)$ are the Eulerian angles which deter-

[8)] Below, we follow basically the derivation of the quasiclassical limit for the matrix elements of the miltipole operators given in Ref. 82, using for this the formalism of Refs. 83 and 84.

mine the orientation of the mass quadrupoloid in accordance with (37). On the basis of (72), (76), and (77) one can show that the Cartesian components of the angular-momentum operator satisfy the relations

$$\langle \tilde{I}_1\rangle = \langle \tilde{I}_2\rangle = 0;\ \langle \tilde{I}_3\rangle = I. \tag{78}$$

In other words, the angles $\Omega_{\langle I\rangle}$ describe the orientation of a system of axes associated with the nucleus with one of them directed along the angular momentum.

The products $[(8\pi^2)^{-1}(2I+1)]^{1/2} D^I_{MK}(\Omega_{\langle I\rangle})$ of normalized Wigner functions and eigenfunctions of the Hamiltonian $\hat{H}_{\langle I\rangle}$, which we denote by $\Phi_n^{\langle I\rangle}(\xi)$, form a complete set with respect to which an eigenfunction of the Hamiltonian (40), (75) can be expanded:

$$|\alpha;\ IM\rangle = \sqrt{\frac{2I+1}{8\pi^2}} \sum_{n\varkappa} f_{n\varkappa}(\alpha I)\, D^I_{M,\,I-\varkappa}(\Omega_{\langle I\rangle})\, \Phi_n^{\langle I\rangle}(\xi). \tag{79}$$

The basis functions in (79) are eigenfunctions for the operator $\tilde{I}_3$ corresponding to the eigenvalues $I-\varkappa$.

It is natural to expect that for $I \gg 1$ in the states of the yrast band the angular momentum has a well defined direction in the internal coordinate system of the nucleus. Since coherent states minimize the uncertainty relation for the projections of the angular momenta onto the different axes, and the angular parameters $\tilde{\vartheta}$ and $\tilde{\varphi}$ are chosen in such a way as to minimize the energy of the ground state of $H_{\langle I\rangle}$, it is natural to assume that the orientation of the angular momentum in the lowest state of the total Hamiltonian $\hat{H}$ for $I = [\langle \hat{I}^2\rangle]^{1/2}$ does not differ strongly from the orientation in the coherent state, i.e., that in (79) the only $f_{n\varkappa}$ which are strongly nonzero are those for which

$$\varkappa/I \ll 1. \tag{80}$$

The last assumption makes it possible to analyze the dependence on I, $\tilde{\vartheta}$, $\tilde{\varphi}$ in the matrix elements of the multipole operator between eigenstates of the total Hamiltonian (40) for $I \gg 1$.

Combining (76) and (79), we write

$$\begin{aligned}\langle \alpha_2 I_2 M_2 |\, \mathscr{M}(\lambda,\ \mu = M_2 - M_1)\,|\, \alpha_1 I_1 M_1\rangle \\ = \sum_{\substack{n_1 n_2,\\ \varkappa_1,\varkappa_2,\varkappa}} f^*_{n_2\varkappa_2}(\alpha_2 I_2)\, f_{n_1\varkappa_1}(\alpha_1 I_1)\,(-1)^{M_2 - I_2 + \varkappa_2}\,[(2I_1+1)(2I_2+1)]^{1/2} \\ \times \begin{pmatrix} \lambda & I_1 & I_2 \\ \mu & M_2-\mu-M_2 & \end{pmatrix} \begin{pmatrix} \lambda & I_1 & I_2 \\ \varkappa & I_1-\varkappa_1-I_2+\varkappa_2 & \end{pmatrix} \\ \times \int d\xi\, \Phi^{*\langle I\rangle_2}_{n_2}(\xi)\, \mathscr{M}(\lambda\varkappa)\, \Phi^{\langle I\rangle_1}_{n_1}(\xi).\end{aligned} \tag{81}$$

If the inequality (80) is satisfied, and also $I \gg \lambda$, then the $3j$ symbols in the expression (80) can be replaced by their asymptotic values, using the formula[84]

$$(\lambda\mu\ (l-\tau)\ (m-\mu)\,|\, lm) \approx (-1)^{\lambda-\tau}\ D^{\lambda}_{\mu\tau}(0,\ \cos^{-1} s\, m/l,\ 0). \tag{82}$$

As a result, we obtain

$$\begin{aligned}\langle \alpha_2 I_2 M_2 |\, \mathscr{M}(\lambda\mu = M_2 - M_1)\,|\, \alpha_1 I_1 M_1\rangle \\ \approx D^{\lambda}_{\mu,\ I_2 - I_1}(0,\ \cos^{-1} M_1/I_1,\ 0) \\ \times \langle \alpha_2 I_2 |\, \tilde{\mathscr{M}}(\lambda,\ \varkappa = I_2 - I_1)\,|\, \alpha_1 I_1\rangle,\end{aligned} \tag{83}$$

where we have introduced the notation for the matrix element

$$\begin{aligned}\langle \alpha_2 I_2 |\, \tilde{\mathscr{M}}(\lambda\varkappa)\,|\, \alpha_1 I_1\rangle = \sum_{n_1 n_2} \sum_{\varkappa'} f^*_{n_2\varkappa'}(\alpha_2 I_2) \\ \times f_{n_1\varkappa'}(\alpha_1 I_1) \int d\xi \Phi^{*(I)_2}_{n_2}(\xi)\, \tilde{\mathscr{M}}(\lambda,\ \varkappa)\, \Phi^{(I)_1}_{n_1}(\xi).\end{aligned} \tag{84}$$

The existence of the rotational bands means that there is approximate coincidence of the internal part of the wave function (79) for several states with similar values I belonging to one band. Accordingly, we shall assume that the functions $\Phi_n^{\langle I\rangle}$ and the coefficients $f_{n\varkappa}(\alpha I)$ coincide for several nearly equal values of I for one α. Then the internal matrix elements in Eqs. (83) and (84) depend smoothly on $I = (I_1 + I_2)/2$ and can be regarded as collective parameters of the nuclei. In the zeroth approximation in the operator $\tilde{H}_c$ in (75) the sums over n in (79) and (84) are absent, which leads to an estimate of the internal matrix element for a transition within the yrast band:

$$\langle I\,|\, \tilde{\mathscr{M}}(\lambda\varkappa)\,|\, I\rangle = \int d\xi \Phi^{*\langle I\rangle}_0(\xi)\, \tilde{\mathscr{M}}(\lambda\varkappa)\, \Phi^{\langle I\rangle}_0(\xi), \tag{85}$$

where $\Phi_n^{\langle I\rangle}(\xi)$ is the wave function of the ground state of the Hamiltonian $H_{\langle I\rangle}$ for $I = (I_1 + I_2)/2$.

Note that for the values of the parameters $\tilde{\vartheta}$ and $\tilde{\varphi}$ which correspond to nonzero mean values in the last six terms of the sum (74) the invariance of the Hamiltonian $H_{\langle I\rangle}$ under transformations of the group D_2 is broken. In this situation, the direction of the angular momentum need not coincide with any of the principal axes of the tensor of the quadrupole moment of the charge of the nucleus.

The classification of the states $\Phi_n^{\langle I\rangle}$ with respect to the quantum number r given above [see (47) and (49)] is preserved if $R(\tilde{\varphi},\ \tilde{\vartheta},\ 0) \in O$, where O is the symmetry group of the cube having the same center as the quadrupole charge and sides perpendicular to the principal axes of the quadrupole charge, and if the terms $B_\nu \langle I_\nu\rangle$ in (74) are negligibly small. In this case, for the intraband matrix elements of the electric-quadrupole moment we obtain the well known parametrization[4]

$$\begin{aligned}\langle I+\tau M+\mu\,|\, \mathscr{M}(E2,\ \mu)\,|\, IM\rangle \approx -(5/16\pi)^{1/2}\, eQ(I)\, d^2_{\mu\tau} \\ \times (\cos^{-1} M/I) \begin{cases} \sin(\pi/6-\gamma) & \text{for } \tau = 0; \\ 0\,(1/I) & \text{for } \tau = \pm 1; \\ (1/\sqrt{2})\cos(\pi/6-\gamma) & \text{for } \tau = \pm 2 \end{cases}\end{aligned} \tag{86}$$

($d^2_{\mu\tau}$ is the nontrivial part of the Wigner function).

The electrical properties of rotating nuclei depend not only on the distribution of the charge within the nucleus but also on the orientation of the angular momentum in the coordinate system associated with the nucleus. To describe the possible orientations of the angular momentum in a system without breaking of the r symmetry when Eq. (86) holds, it is convenient to use the diagram in Fig. 3. The region of physically different γ values is contained in three sectors with opening angle $\pi/3$; the distance to the center in Fig. 3 determines the internal quadrupole moment $Q(I)$, or the deformation parameter β. Rotation occurs around the first axis of the mass quadrupoloid. For γ values which are multiples of $\pi/3$,

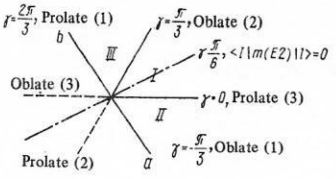

FIG. 3. Chart of quadrupole deformations of a body rotating about the first internal axis.

the body has axial symmetry. The axis and the symmetry type (prolate or oblate) are shown in Fig. 3. Sector I corresponds to yrast levels in the collective model of the nucleus with the distribution of the nucleon velocities and moments of inertia of the hydrodynamic model; sector II plays the same role for a rigid-body rotor.

As an illustration of what we have said, consider Fig. 4, in which we represent calculations of a two-phase model of the nucleus with the Hamiltonian[84]

$$\hat{\mathcal{H}}=\begin{pmatrix}\sum_{\nu=1}^{3}A_\nu^{(1)}I_\nu^2+V_1 & \zeta\\ \zeta & \sum_{\nu=1}^{3}A_\nu^{(2)}I_\nu^2+V_2\end{pmatrix}. \tag{86a}$$

The model with the Hamiltonian (86a) describes a rotor that can be in the superfluid phase with moments of inertia that depend on the deformation parameters in accordance with the expressions of the hydrodynamic model:

$$\mathcal{I}_\nu^{(1)}=\frac{1}{2A_\nu^{(1)}}=4D\beta^2\sin^2\left(\gamma-\frac{2\pi}{3}\nu\right)(\nu=1,\ 2,\ 3), \tag{87}$$

and also in the normal phase with rigid-body moments of inertia $A_\nu^{(2)}/2$. The constant ζ describes the coupling of the two phases. The method of approximate diagonalization of the Hamiltonian (86a) used in Ref. 84 has the same basic features as the procedure described above for introducing the Hamiltonian $H_{\langle I\rangle}$ in (74). In the variant of the calculation[9)] given in Fig. 4, the superfluid component of the wave function is predominant in the states $I=0,\ 2,\ \ldots,\ 10$. The parameter γ of departure from axial symmetry increases rapidly in this region of I, approaching the limit $\gamma_0=\pi/6$ for the collective model.[86] At angular momenta $I\geq 14$, the superfluid phase virtually disappears. The amount of nonaxial shape rapidly decreases, and the sign of γ changes. At $I=12$, the direction of the angular momentum in the internal coordinate system does not coincide with any of the principal axes of the inertia tensors of the superfluid and normal phases.

To conclude this section, we note that the problem of studying the dependence of the nuclear shape on the angular momentum has been found to be related to the problem of determining the dependence on the angular momentum in the distribution of the nucleons with respect to velocities. Unfortunately, direct empirical data on this last problem are not sufficiently numerous.

4. CHANGE IN THE SHAPE AND SIZES OF NUCLEI DUE TO ROTATION AT SMALL AND INTERMEDIATE ANGULAR MOMENTA

Stably deformed nuclei. It follows from (54) that the measurement of the relative probabilities of transitions between the lowest states of the ground-state band make it possible to determine the combination

$$\alpha=X/2+Y/\sqrt{6}. \tag{88}$$

The experimental values of the parameter α obtained from analysis of the factors $B(E2;\ I0\to I\pm 2,\ 0)$ are given in column 9 of Table I.

Measurement of the static quadrupole moments, which could in principle fix the second linear combination of the parameters X and Y given in (56), cannot at the present time be made with sufficient accuracy. Less direct information about nonadiabatic effects can be obtained on the basis of the relations (66) and (67). To arrive at definite conclusions about X and Y using these relations, it is necessary to assume a predominant contribution to the sum of (66) and (67) from low-lying excited bands for which z_{Kn} and $B(E2;\ 00\to 2Kn)$ are known. This assumption is confirmed by the following facts: 1) The bands under discussion usually correspond to large values of the factors $B(E2;\ 00\to 2Kn)$, so that for them the ratio $\langle 0|\beta_K|Kn\rangle/\beta^0$ is fairly large; 2) the contribution of these bands amounts to an appreciable fraction of the parameter α for the nuclei for which the empirical information makes it possible to carry out such a comparison.

Data on $B(E2;\ 00\to 2Kn)$ values from Coulomb excitation for the lowest excited bands $K=0$ and $K=2$ are very complete for nuclei of the rare earth elements. Detailed investigations have also been made to determine the coefficients z_{Kn}, which describe the deviations from Alaga's rules in the angular-momentum dependence of the reduced probabilities of quadrupole transitions. Fairly often, the simple single-parameter formulas (58) and (59) do not give a sufficiently accurate description of all the measured relations between the different $B(E2)$ factors of the transitions coupling a definite excited band to the ground-state band. Somewhat more complicated schemes have been developed for describing such exper-

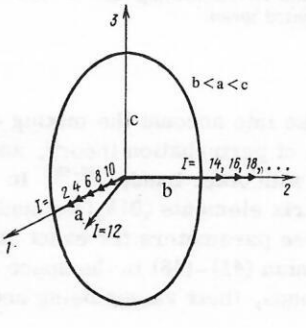

FIG. 4. Orientation of angular momentum in the internal coordinate system in two-phase model.[84] The arrows indicate the direction of the angular momentum, which is determined by Eqs. (72) with appropriate value of the quantum number I.

[9)] The parameters of the model in Ref. 84 are chosen in such a way as to describe the yrast band in the isotopes ^{156}Er (see Fig. 4) and ^{166}Er. However, the possibility of such an interpretation of the spectra of these nuclei, in particular, the value of the angular momentum which is critical for pairing, has not been established.

TABLE I. Experimental data on the changes in shape of nuclei resulting from rotation at small angular momenta.

Nucleus	E^*_{20}, keV	K^*	E^*_{2Kn}, keV	$\langle 0 \mid \beta_h \mid Kn\rangle^{*3}/\beta^0$	$10^3 z_{Kn}$*3	$10^3\delta X$	$10^3\delta Y$	$10^3\alpha$*24	$-\delta B_{X+Y}$*4, eV	$-\delta B_{CAP+Sp}$	$-B_{exp}$*5, eV
^{150}Nd	132	—	—	—	—	—	—	4 (2)	—	—	183
^{152}Sm	122	0	810	0.131 (6)	46 (7) *6	3.2	—	2.0 (6)	25	50	141
		2	1084	0.164 (5)	68 (4) *7	—	0,75		5.0		
^{154}Sm	82	—	—	—	—	—	—	0.6 (6)	—	15.2	23.5
^{154}Gd	123	0	816	0.152 (10)	47 (5) *6	4.3	—		35	48	141
		2	996	0.204 (18)	76 (3) *7		1.29	2.1 (7)	8.7		
^{156}Gd	89	0	1129	0.077 (8)	6 (5) *6	0.14		0.46 (53)	0.2	16,5	29,9
		2	1154	0.166 (4)	39 (4) *7	—	0,44		1.9		
^{158}Gd	80	2	1187	0.132 (5)	31 (11) *8	—	0.23	—	0.8	9.6	13
^{156}Dy	138	2	891	0.265 (27)	92 (4) *9	—	2.64	—	18.6	—	196
^{160}Dy	87	2	966	0,156 (6)	32 (4) *7	—	0,31	—	0.9	12.7	19.6
^{162}Dy	81	2	888	0,183 (9)	52 (8) *10	—	0.71	—	3,0	11.4	11.3
^{158}Er	193	—	—	—	—	—	—	4 (2)	—	—	—
^{160}Er	126	—	—	—	—	—	—	−1,2 (20)	—	—	—
^{166}Er	81	2	786	0.164 (2)	57 (10) *11	—	0,62	—	2.5	14.7	12.8
^{168}Er	80	2	821	0.159 (8)	36 (4) *11	—	0,37	—	1.0	10.0	7.1
^{170}Er	79	2	932	0,138 (7)	28 (6) *10	—	0,21	—	0.5	12.3	11.9
^{160}Yb	243	—	—	—	—	—	—	1.9 (6)	—	—	—
^{164}Yb	122	—	—	—	—	—	—	0,3 (2)	—	—	—
^{166}Yb	102	—	—	—	—	—	—	0.2 (3)	—	—	—
^{170}Yb	80	0	1146	0.059 (4)	−47 (3) *12	−0.66	—	—	8.2	9.0	12.4
		0	1306	0.077 (7)	20 (2) *12	0.48	—	—	2.9	—	—
		2	1139	0.110 (6)	54 (2) *13	—	0.27	—	1,6	—	—
^{174}Yb	76	—	—	—	—	—	—	−1.4 (13) *25	—	10.6	6.6
^{176}Yb	82	—	—	—	—	—	—	−2.4 (14) *25	—	9.2	7.4
^{174}Hf	91	0	901	0.123 (20)	24 (1) *14	1.46	—	—	7.0	16.0	21.3
^{176}Hf	88	0	1226	0.079 (4)	13 (5) *15	0.32	—	—	1.2	18.0	15.7
		2	1341	0.151 (17)	82 (4) *16	—	0.76	—	8.0	—	—
^{178}Hf	93	0	1276	0.021 (6)	32 (3) *17	0.06	—	—	0.5	15,7	16.3
		0	1496	0.937	−45 (3) *17	> -0.24 < 0	—	—	3.9	—	—
		2	1175	0.154 (5)	20 (10) *17	—	0,20	—	0.4	—	—
^{182}W	100	0	1257	0.024 (1)	121 (3) *18	0.28	—	—	9.8	14,5	15,2
		2	1221	0.097 (2)	20 (4) *18	—	0.07	—	0.2	—	—
^{184}W	111	0	1120	0.076 (8)	42 (3) *18	0.98	—	—	10.3	24.4	22.8
		2	904	0.094 (2)	50 (4) *18	—	0.18	—	0.7	—	—
		2	1386	0.078 (4)	30 (11) *19	—	0,07	—	0.3	—	—
^{186}W	123	0	1284	0.057 (2)	23 (8) *20	0.30	—	—	2.0	32.8	44,0
		2	737	0.216 (3)	75 (11) *21	—	1.43	—	6,7	—	—
^{186}Os	137	2	767	0.282 (18)	54 (13) *22	—	1,76	—	7.3	51,3	80.9
^{188}Os	155	2	633	0.347 (19)	134 (10) *22	—	6.6	—	43	75	139
^{190}Os	186	2	558	0.404 (12)	250 (8) *23	—	16.7	—	158	—	265
^{192}Os	206	2	489	0,445 (14)	322 (11) *23	—	26,0	—	242	—	369

*According to the data of Ref. 92.

*2 Empirical data on $[B(E2,00\to 3Kn)/B(E2,\ 00\to 2)]^{1/2}$ with allowance for band mixing in the approximation in which the parameters z_K were determined. The $B(E2;00\to 20)$ values are taken from Ref. 93; $B(E2, 00\to 2Kn)$ are the weighted means of the data on the Coulomb excitation and inelastic scattering of deuterons published up to January 1, 1976. The errors indicated in the table include only the experimental uncertainty in the determination of the corresponding values.

*3 The error includes only the experimental uncertainty in the branching ratios that were used to obtain the estimates.

*4 Equation (63); $E_{Kn} = E_{2Kn} - E_{20}$.

*5 The data of Ref. 22 calculated in accordance with the data on E_{20} and E_{40} in Ref. 92.

*6 Reference 87; the branching ratios of transitions from the state $I = 2$ were obtained in accordance with the scheme adopted in Ref. 88.

*7 Reference 94; branching ratio $3\to 4/3\to 2$.

*8 Reference 89; branching ratio $3\to 4/3\to 2$.

*9 Reference 95; branching ratio $3\to 4/3\to 2$.

*10 Reference 96; branching ratio $2\to 2/2\to 0$.

*11 Reference 97; branching ratio $3\to 4/3\to 2$.

*12 Reference 98; the branching ratios of transitions from the state $I = 2$ were obtained in accordance with the scheme adopted in Ref. 88.

*13 Reference 98; branching ratio $3\to 4/3\to 2$.

*14 Weighted mean of Refs. 99 and 100.

*15 Weighted mean of Refs. 101 and 102.

*16 Reference 102; branching ratio $3\to 4/3\to 2$.

*17 Reference 88.

*18 Reference 90.

*19 Reference 103; branching ratio $2\to 4/2\to 0$.

*20 Reference 104; branching ratio $2\to 2/2\to 0$.

*21 Weighted mean of Refs. 104 and 105; branching ratio $2\to 2/2\to 0$.

*22 Reference 106; branching ratio $3\to 4/3\to 2$.

*23 Weighted mean of Refs. 104 and 107; branching ratio $2\to 2/2\to 0$.

*24 References 108 and 109; weighted mean.

*25 Coulomb excitation.

iments; these schemes take into account the mixing of both bands without the use of perturbation theory, and also the effects of mixing with other bands.[87,88] In some cases,[89,90] the matrix elements $\langle 0 \mid h_k \mid Kn\rangle$ and $\langle 0 \mid \beta_K \mid Kn\rangle$ were used as free parameters for exact diagonalization of the Hamiltonian (41)–(46) in the space of three or four rotational bands, their values being chosen on the basis of the best description of the experimental data.[10] Common to all such schemes is that, on the

[10] In such calculations, the parameter A for each of the bands is also regarded as an adjustable parameter, and the second term of the expression (62) was included in the expression for H_0.

basis of experimental data, they make it possible to determine the values of the matrix elements $\langle 0|\beta_K|Kn\rangle$ and $\langle 0|h_k|Kn\rangle$ in addition to the branching rules in the cases when the factor $B(E2;\ 00\to 2Kn)$ is known.

In Table I we have collected empirical data on $\langle 0|\beta_K|Kn\rangle/\beta^0$ and z_{Kn} for a number of nuclei. Here, we also present the corresponding contributions to the expressions (60), (67), and (69). They are denoted, respectively, by δX, δY, and δB_{X+Y}. Note that δX and δY are of the order 10^{-3}. This means that when $I\approx(20-40)\hbar$ appreciable changes are to be expected in the form of stably deformed nuclei.[11)]

There is one further interesting feature, namely, that in virtually all cases the parameters z_{Kn}, and with them δX and δY, are positive. The positive sign of X means that the deformation of the nuclei increases with increasing angular momentum. The positive sign of Y can be interpreted as the occurrence of nonaxial deformation due to the presence of the collective angular momentum, whose direction coincides with the direction of the axis of intermediate length of the mass quadrupoloid of the nucleus.

A positive sign of both parameters X and Y follows from the simple model[13] of an oscillating quadrupoloid with velocity distribution corresponding to irrotational motion of an ideal incompressible fluid. In order to apply this Bohr model to the problem under consideration, it is necessary to assume that the potential energy and the mass parameters of the motion associated with the change in the coordinates β_0 and β_2 differ from the predictions of the hydrodynamic theory made in Ref. 13 (see the discussion in Ref. 91). These quantities must be parametrized in such a way as to reproduce the experimental values of the equilibrium deformation, the positions of the β- and γ-vibrational levels, and also the values of their corresponding $B(E2;\ 00\to 2Kn)$ factors. However, in Ref. 13, the formula of the hydrodynamic model (87) was used.[13] The Hamiltonian of the model can be expressed in the form (40)–(46), where H_0 is a function of the variables β and γ and the canonically conjugate momenta, and the inertial parameters A_ν are determined by (87).

Assuming that the internal motion can be described as small oscillations about the position with equilibrium deformation, we obtain

$$A=1/2\mathscr{Y}_1(\beta^0,\ 0)\equiv 1/2\,\mathscr{Y}_0; \tag{89}$$

$$\langle 0|h_0|0n\rangle=-\frac{1}{2\mathscr{Y}_0^{3/2}}\frac{\partial\mathscr{Y}_1(\beta^0,\ 0)}{\partial\beta}\langle 0|\beta_0|0n\rangle; \tag{90}$$

$$\langle 0|h_2|2n\rangle=-\frac{1}{2\mathscr{Y}_0^{3/2}}\frac{1}{\beta^0}\frac{\partial\mathscr{Y}_1(\beta^0,\ 0)}{\partial\gamma}\langle 0|\beta_2|2n\rangle. \tag{91}$$

From (87) it is easy to conclude that

$$\partial\mathscr{Y}_1(\beta^0,\ 0)/\partial\beta=6D(\beta^0)^2>0; \tag{92}$$

$$\partial\mathscr{Y}_1(\beta^0,\ 0)/\partial\gamma=\sqrt{3}\,D(\beta^0)^2>0. \tag{93}$$

Therefore, substituting (90) and (91) into (53)–(55), we obtain positive signs for both X and Y.

Note that the estimates of the matrix elements $\langle 0|h_0\times|0n\rangle$ and $\langle 0|h_2|2n\rangle$ in accordance with (90) and (91) do not agree with the empirical data if the numerical value of the parameter D in (87), (92), and (93) is determined in such a way that the moment of inertia $\mathscr{Y}_0$ in (89) correctly describes the position of the first rotational level 2^+. This may be an indication that the function $\mathscr{Y}_1(\beta,\gamma)$ does not act on the states of the lowest bands $K=0$ and $K=2$ in the way that is postulated in the hydrodynamic model.

Usually, the size of a nucleus is characterized by the rms radius. Measurement of isomeric shifts in the energy of gammas emitted by nuclei in different chemical compounds, which can be observed by means of the Mössbauer effect, is one of the sources of information about the variations of the rms radius in different nuclear states. Another source of such information is provided by the spectra of μ-mesic atoms, in which one can also identify nuclear transitions whose energies differ somewhat from the corresponding values for normal atoms because of the electromagnetic interaction of the meson with the nucleus.[110] The experimentally observed changes in the rms radius in the first excited state 2^+ of even–even deformed nuclei from the rms radius of the ground state are of the order 10^{-5}. Such changes can be due to not only a change in the shape of the nucleus but also of the linear dimensions of the system, which are not related to the change in the shape. For example, for a homogeneous ellipsoid and a nearly axisymmetric distribution of charge, the following expression was obtained in Ref. 110:

$$\frac{\delta\langle R^2\rangle_{2^+}}{\langle R^2\rangle}=\frac{8}{3}\delta^2\left(1-\frac{2}{3}\delta\right)X+\frac{2}{3}\left(1-\frac{4}{9}\delta^2\right)\frac{\delta V_{2^+}}{V}; \tag{94}$$

$$\delta=(3/2)(5/4\pi)^{1/2}\beta^0. \tag{95}$$

The second term on the right-hand side of (94) corresponds to the change in the volume, i.e., the change in the linear dimensions of the system.

In individual cases there exists experimental information about the parameters X and $\delta\langle R^2\rangle_{2^+}$ that makes it possible to estimate (in a very preliminary way) the influence of rotation on the linear dimensions of nuclei. For example, in ^{152}Sm and ^{154}Gd the experimental data on the parameter α agree well with the value

$$\alpha=\delta X/2+\delta Y/\sqrt{6}, \tag{96}$$

calculated in accordance with the data on the z_0 and z_2 values for these nuclei. This is a ground for assuming that in the expression (53) for X only the term describing the influence of the lowest excited state of the $K=0$ band is important, i.e., that $\delta X=X$. Although such arguments cannot be obtained for other nuclei, it is still instructive to use the expression (94) to determine the contribution of the lowest $K=0$ bands to the change in the rms radius. In Table II, the experimental values[110] of $\delta\langle R^2\rangle_{2^+}$ are compared with the values calculated in accordance with the formula

[11)] Note that it is precisely in this range of angular momenta that the disappearance of pairing correlations of superconducting type in nuclei is expected (see Sec. 2). As follows from the arguments given in Sec. 3, this can have a significant influence on the shape of nuclei.

TABLE II. Comparison of experimental values of $\delta\langle R^2\rangle_{2^+}$ with the values found from (94). The values of the parameter X are taken from Table I, and the experimental values from the review Ref. 110.

Nucleus	δ	$10^3\langle R^2\rangle_{2^+}^X$	$10^3\langle R^2\rangle_{2^+}^{exp}$	Nucleus	δ	$10^3\langle R^2\rangle_{2^+}^X$	$10^3\langle R^2\rangle_{2^+}^{exp}$
^{152}Sm	0.230	9.4	20 12	^{174}Hf	0.233	2.09	—
^{154}Gd	0.245	14.3	25 (5) 16 (3)	^{176}Hf	0.243	0.50	1.6
^{156}Gd	0.274	0.57	3.6 2.0	^{178}Hf	0.226	0.08 $\left.\begin{matrix}>\\<\end{matrix}\right\}$ −0.33 0	0.7
^{170}Yb	0.261	−2.62 1.90	2.0 1.1 1.5	^{182}W	0.206	0.33	−0.44
				^{184}W	0.195	1.04	0.36
				^{186}W	0.189	0.30	0.33

$$\delta\langle R^2\rangle_{2^+}^{(X)} = (8/3)\,\delta^2(1-2\delta/3)\,\delta X. \tag{97}$$

If the difference between these quantities is interpreted as due to change in the volume of the nucleus, we arrive at the conclusion that the relative change of the volume in the 2^+ rotational state as compared with the ground state is of the order $\delta V_{2^+}/V \sim 10^{-5}$. The ratio of $(\delta R^2)_{2^+}^{exp}$ to $(\delta R^2)_{2^+}^{X}$ can be regarded as an indication that the volume of nuclei at the start of the region of rare earth elements increases as a result of rotation, but rotation is accompanied by a certain decrease in the volume for heavier nuclei (^{182,184}W).

Turning to a description of calculations of nonadiabatic effects due to rotation that can be made in the framework of the microscopic theory, we note that on the basis of Eqs. (83) and (86) we can write

$$\left.\begin{aligned}&\frac{\delta\langle R^2\rangle_{2^+}}{\langle R^2\rangle} = 3\,(d^2\langle R^2\rangle/dM^2)/\langle R^2\rangle;\\ &X = (d^2\langle Q_{20}\rangle/dM^2)/\langle Q_{20}\rangle;\\ &Y = 4\,(d^2\langle Q_{22}\rangle/dM^2)/\langle Q_{20}\rangle,\end{aligned}\right\} \tag{98}$$

where

$$M = \sqrt{I(I+1)}, \tag{99}$$

and the averaging is with respect to the internal wave functions. The applicability of the quasiclassical relations (98) and (99) for small values of I can be justified in microscopic approaches (see Sec. 1 or Ref. 51). The microscopic theory leads to analogous expressions with a many-particle function with respect to which averaging is performed which depends on all $3A$ coordinates of the nucleons in the nucleus and satisfies the variational equations of the Hartree–Fock–Bogolyubov theory. Equations (98), in which the mean value of the single-particle operator is written in the form

$$\langle\hat{O}\rangle = \sum_{ij} o_{ij}\rho_{ij} = \mathrm{tr}o\rho, \tag{100}$$

and the density matrix ρ is determined by Eq. (7), approximate the values of the parameters in which we are interested.

To calculate $\delta\langle R^2\rangle_{2^+}$, X, and Y, we must calculate the first two derivatives of the density matrix ρ with respect to the angular momentum M or with respect to the rotational frequency $\omega = M/\mathcal{I}$. Equations that determine $d^2\rho/d\omega^2$ and are valid for arbitrary form of the effective interaction between the nucleons are given, for example, in Ref. 110. If the interaction is approximated by factorized quadrupole and pairing forces, the solution of the equations of the cranking model for $d\rho/d\omega$ and $d^2\rho/d\omega^2$ are known.[31] The second derivative of the single-particle density matrix consists in this case of parts associated, respectively, with the influence of rotation on the single-particle degrees of freedom and also with the change in the shape parameters $\langle Q_{20}\rangle$, $\langle Q_{22}\rangle$ and the parameters Δ and λ (Δ and λ are the gap and the chemical potential) (the CAP effect). Thus, in a model with quadrupole and pairing forces, the quantities X and Y in conjunction with $d^2\Delta/dM^2$ and $d^2\lambda/dM^2$ completely determine the reaction of the nucleus to rotation in the second order in M and ω.

For $I \gg 1$, when the use of the parameters X and Y becomes meaningless, one can use the expression of the microscopic theory[112] that is the analog of Eqs. (83) and (84):

$$\begin{aligned}\langle I'M'|\mathcal{M}(\lambda,\mu)|IM\rangle &= (-1)^{I'-M'}\begin{pmatrix} I' & \lambda & I\\ -M' & \mu & M\end{pmatrix}\\ &\times\left[\frac{(2I+1)(2I'+1)}{I+I'+1}\right]^{1/2}\mathrm{tr}\,m(\lambda,\mu'=I'-I)\\ &\times\hat{D}^{(y)}_{\pi/2}\rho^{(I+I')/2}\hat{D}^{(y)+}_{\pi/2}.\end{aligned} \tag{101}$$

Here, $\hat{D}^{(y)}_{\pi/2}$ is the matrix of rotation about the y axis through angle $\pi/2$. In (101), the notation for the sum over the two-quasiparticle states introduced in (100) is used. The density matrix ρ in (101) is defined in accordance with the rules of the cranking model, so that

$$\mathrm{tr}\,j_x\rho^{(I)} = I. \tag{102}$$

Equation (101) is obtained under the assumption that I, $I' \gg 1$, but it gives reasonable results even for small values of I comparable with λ. In particular, from Eqs. (101) and (102) there follows the exact relation

$$M = \langle IM|I_z|IM\rangle \tag{103}$$

for all values of I.

Calculations of $\delta\langle R^2\rangle_{2^+}$ in the framework of the microscopic model have been made in Refs. 31, 32, 38, and 110. The results of the calculations depend strongly on the parameters of the microscopic Hamiltonian, in particular on the form of the residual interaction. The best description of the experimental data for all deformed nuclei of rare earth elements was obtained in calculations using the Migdal interaction.[38,110] In particular, these calculations reproduce the reduction in the rms radii of the nuclei $^{188-192}$Os. In the isotopes ^{174,176}Yb and ^{182}W, calculations with Migdal forces do not, it is true, lead to negative values of $\delta\langle R^2\rangle_{2^+}$, but they do show that a well defined minimum of $\delta\langle R^2\rangle_{2^+}$ corresponds to these isotopes. The calculations also predict a decrease in the rms radii in the Dy nuclei and the light Er and Yb isotopes. The agreement between experiments and the results of calculation of $\delta\langle R^2\rangle_{2^+}$ using a pairing and a quadrupole interaction[31,32] (Fig. 5) is somewhat less good. In particular, in calculations with factorized $P+Q$ interaction the decrease in the rms radii associated

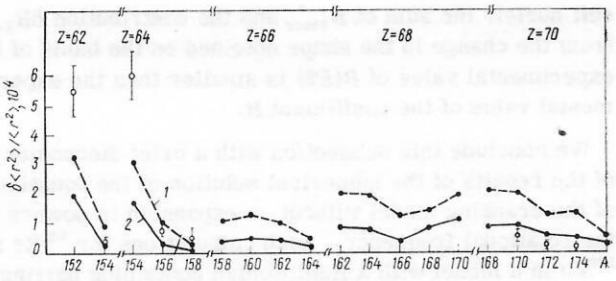

FIG. 5. Changes in the rms radius in nuclei of rare earth elements calculated in Ref. 32 with the Woods–Saxon potential $P+Q$ forces. The coupling constants were chosen for each nucleus in such a way as to reproduce the experimental even–even mass differences and the energies of the β- and γ-vibrational levels; 1) moment of inertia calculated in accordance with the cranking model; 2) experimental moment of inertia.

with rotation is not reproduced.

The influence of rotation on the single-particle motion (see Sec. 2) leads in virtually all cases to a positive contribution to $\delta\langle R^2\rangle_{2^+}$. In this respect, nonadiabatic effects of this kind are similar to those produced by centrifugal forces. Note that the influence of centrifugal forces on the rms radius in the model of a Fermi gas is given by

$$\delta\langle R^2\rangle_{2^+}/\langle R^2\rangle = (54/7)\,(\gamma_0 A\varepsilon_F)^{-1}. \qquad (104)$$

For nuclei of rare earth elements, an estimate in accordance with this formula gives $\delta\langle R^2\rangle_2/\langle R^2\rangle \approx 4\cdot 10^{-5}$.

The CAP effect has a tendency to reduce the rms radius as the angular momentum increases. This can be understood by writing down the expression for the change in the rms radius produced by the CAP effect:

$$\delta\langle R^2\rangle_{\text{CAP}} = \sum_i (r^2)_{ii}\,\delta\,(v_i^2)_{\text{CAP}}. \qquad (105)$$

In a system without pairing correlations, the probabilities of population of the single-particle levels v_i^2 decrease abruptly from 1 to 0 at the Fermi limit, whereas in the presence of pairing correlations the v_i^2 depend smoothly on the single-particle energies ε_i. Thus, the pairing correlations have the consequence that the levels below the Fermi surface are only partly occupied, while the levels above the Fermi surface are not completely vacant. Since the matrix elements r_{ii}^2 have a tendency to increase with increasing single-particle energy ε_i, the presence of the pairing correlations leads to an increase in the rms radius as compared with its value in a state without correlations. The CAP effect leads to a reduction of the pairing correlations with increasing angular momentum, and this gives rise to a certain decrease in the rms radius and volume.

The contribution of nonadiabatic effects associated with deformation of the self-consistent field is predominant in nuclei near the boundary of the deformed region. In the nuclei ^{152}Sm and ^{154}Gd, which are soft with respect to β deformation, the elongation produced by the rotation gives a large positive contribution to $\delta\langle R^2\rangle_{2^+}$. As one moves away from the boundary of the region of stable deformation, the importance of changes in the shape decreases. This has the consequence that the CAP effect becomes predominant in some nuclei. It should be noted that the value of $\delta\langle R^2\rangle_{2^+}$ is determined by a very small number of single-particle levels situated near the Fermi surface. Therefore, the tendency for the volume to decrease as a consequence of the CAP effect is not always manifested, and the sign of the change in the rms radius in the nuclei of rare earth elements depends in practice on the position of the proton levels $(7/2)^-[523]$ and the neutron levels $(11/2)^-[505]$.

The shape parameters of nuclei in the microscopic model were calculated in Refs. 31–33, in which pairing and quadrupole interaction forces were used, and also in the second of the papers in Ref. 38, in which Migdal forces were used. Here too there is a strong dependence of the results of the calculations on the input parameters. The results of the calculation of α and Y made in Ref. 32 with pairing and quadrupole interaction forces are given in Fig. 6. The existing data on α are too sparse to draw reliable conclusions on the basis of a comparison with theory (in Fig. 6, we show the existing experimental material for the parameter α). One can still see that the main tendencies of the experimental data are correctly reproduced by the theoretical estimates. This applies equally to calculations with both types of interaction. Incidentally, the calculation made with pairing and quadrupole interaction forces does not enable us to explain the negative values of the parameter α observed, with large error, in ^{174,176}Yb. In this respect, the microscopic model with factorized forces exhibits features that make it similar to the phenomenological model considered above, in which allowance is made for the coupling between the ground-state band and the β- and γ-vibrational bands. It seems that the nonadiabatic effects associated with the mixing of these bands are taken into account correctly in the model with pairing and quadrupole forces (this is confirmed by the calculations of z_0 and z_2), but the other nonadiabatic effects are underestimated.

The microscopic calculations of Y, like the estimates based on the mixing of the ground-state and the γ-vibrational bands, lead to positive values of this quantity. The possibility of relating the sign of Y and certain similarities in the properties of nuclei to the properties of a superfluid liquid drop were investigated in Ref. 32. Since the superfluid properties are due to pairing correlations of superconducting type, it is natural to expect

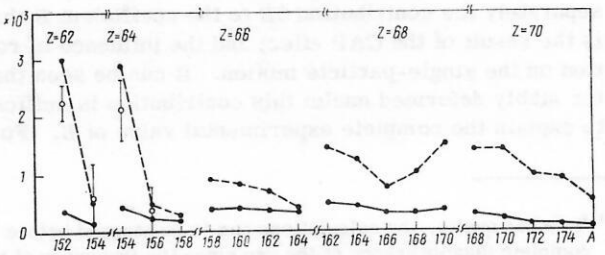

FIG. 6. Values of α calculated in Ref. 32 (dashed lines) and the experimental values; values of y calculated in Ref. 32 (continuous lines).

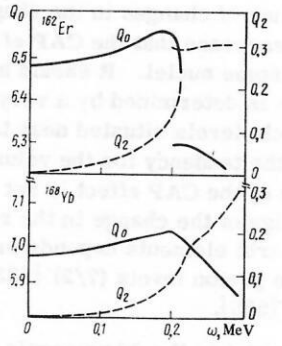

FIG. 7. Intrinsic quadrupole moment of ^{168}Er and ^{168}Yb as a function of the angular frequency of rotation.[34]

that the disappearance of the correlation properties will lead to a change in the sign of Y if the connection really does exist. The calculation made in the last of the papers of Ref. 32 for several nuclei of the rare earth elements showed that a decrease in the gap really is accompanied by a decrease in the parameter Y, and also of the parameter X. In ^{168}Er, the value of Y becomes negative when the gap disappears.[12)]

In Refs. 22 and 62, the deformability of nuclei was calculated by means of the shell-correction method, the calculations leading in the whole region of the rare earth elements to values much lower than the experimental. However, it must be borne in mind that the relative change in the deformation is of order 10^{-3}, i.e., in the experiments the deformation energy was investigated in a very small neighborhood of the equilibrium point. The aim of the approach based on the calculation of the shell correction is to describe the behavior of the deformation energy over a much wider range of variation of the deformation.[113,114] In particular, when one is calculating the liquid-drop part of the deformation energy, it is reasonable to assume that for comparatively large variations of the deformation the nucleus is deformed like an incompressible drop. For a change in the deformation of order 0.1%, this assumption may be invalid. Other degrees of freedom, for example, a change in the radial distribution of the nucleons, may also make a contribution of the same order to the increase in the quadrupole tensor.

As was noted in Sec. 2, the calculated value of $d\mathcal{I}/d\omega^2$ for comparatively soft nuclei of rare earth elements is much lower than the experimental value, whereas for stably deformed nuclei the two quantities agree well. Since the shell-correction method was used in these calculations, the deviations are related to the very small value of X predicted by this method. In Table I we give separately the contribution δB to the coefficient B that is the result of the CAP effect and the influence of rotation on the single-particle motion. It can be seen that for stably deformed nuclei this contribution is sufficient to explain the complete experimental value of B. For soft nuclei, the sum of B_{theor} and the contribution δB_{X+Y} from the change in the shape obtained on the basis of the experimental value of $B(E2)$ is smaller than the experimental value of the coefficient B.

We conclude this subsection with a brief discussion of the results of the numerical solution of the equations of the cranking model without an expansion in powers of the rotational frequency. Such calculations for ^{162}Er and ^{168}Yb in a model with a Hamiltonian containing pairing and quadrupole interaction and a spherical potential of the shell model were made in Ref. 29. The calculations took into account only nucleons above the closed shells $Z=40$, $N=70$. Figure 7 shows the quadrupole moments of these nuclei as functions of the angular frequency. At small ω, the calculations lead to the already familiar results: Rotation leads to the lenghening of the nuclei and the nonaxial shape characteristic of the hydrodynamic model. However, the lengthening is replaced by the opposite tendency for $\omega \geq 0.2$ MeV. At these values of the rotational frequency there are also irregularities in the ω dependence of the parameter Q (or the angle γ of departure from axial shape). The resulting picture cannot be interpreted in either the hydrodynamic model or the model of a rigid rotor. Similar results have been obtained in Ref. 72. The changes in the shape of the nuclei found in these calculations at sufficiently high frequencies are to be attributed to shell effects, which are described in Sec. 5.

Coexistence of shapes in nuclei of the transition region. It is natural that in soft transition nuclei the degrees of freedom describing the change in shape are the ones most strongly coupled to rotation. It is therefore sensible to solve the Hartree–Fock–Bogolyubov equations with constraints, restricting oneself to only the nonadiabatic effects associated with the change in the shape (see the classification in Sec. 2). In the framework of such an approximation, the total energy is

$$E_I(\varepsilon) = E_0(\varepsilon) + I(I+1)/2\mathcal{I}(\varepsilon), \qquad (106)$$

where ε is the set of parameters that fix the (axisymmetric) shape of the nucleus; $\mathcal{I}(\varepsilon)$ is the moment of inertia in the cranking model; $E_0(\varepsilon)$ is the deformation energy of the nonrotating nucleus. The shape of the nucleus for spin I is obtained by minimizing $E_I(\varepsilon)$ with respect to ε.

In such an approach, an investigation was made in Ref. 115 of the shape of the lightest mercury isotopes accessible to experimental investigation. The energy $E_0(\varepsilon)$ was calculated by Strutinskii's shell-correction method. The energy was found to have two minima, one of them corresponding to a spherical shell in the proton system, and the other to a deformed neutron shell. The deformation energy of the nucleus ^{184}Hg, which is slightly oblate in the ground state, $\varepsilon = -0.12$, in accordance with the almost closed proton shell ($Z=80$), is shown in Fig. 8. At $\varepsilon = 0.23$, one can see a second minimum, which lies 0.29 MeV above the ground state and arises because the neutron component in $E_0(\varepsilon)$ has a deep minimum for $N=104$ at this deformation, as is well known from the study of stably deformed nuclei of rare earth

12)In other nuclei, the calculations had to be stopped before the complete disappearance of the gap since the frequency of the β- and (or) γ-vibrational excitations found in the random-phase approximation vanished and then became imaginary at $\Delta = 0$ when the pairing was reduced.

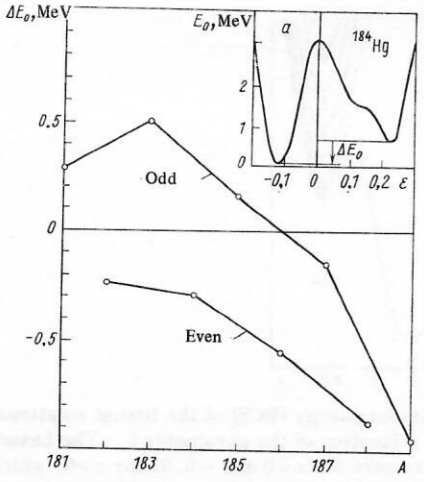

FIG. 8. Difference ΔE_0 of energies in the oblate and prolate minima in mercury isotopes (without rotation). a) Energy of ^{184}Hg nucleus as function of the deformation parameter ε (both curves are taken from Ref. 115).

elements. The other isotopes of mercury in the region $181 \leqslant A \leqslant 189$ have deformation energy of similar kind. The difference of the energies between the two minima, ΔE_0, in these nuclei is shown in Fig. 8. Similar results for the deformation energy were obtained in Refs. 116–120, in some of which the Hartree–Fock method with constraints and various functionals of the energy density in finite nuclei was used instead of the shell-correction method. It can be seen from Fig. 8 that the odd isotopes in the ground state are oblate for $A \geqslant 187$ and prolate for $A \leqslant 185$. This transition with decreasing number of neutrons from a slight oblate deformation to a large prolate deformation explains the experimentally observed abrupt increase in the rms radius between ^{187}Hg and ^{185}Hg (see, for example, Ref. 116 and the references quoted there).

All the even isotopes in this region are oblate in the ground state (see Fig. 8; the reasons for the different values of ΔE_0 in odd and even isotopes are explained in Ref. 115). The moment of inertia $\mathcal{J}(\varepsilon)$ in the "oblate" minimum is smaller than in the "prolate" minimum, this being due to the different deformations. This is illustrated in Fig. 9, which shows the sequence of rotational levels at the two minima (the slope of the curves is $\mathcal{J}/2$). It can be seen that the oblate form is preferable at high spins. In the case of ^{184,186}Hg, the yrast states 0^+ and 2^+ are oblate, whereas the yrast states with higher spin are prolate. In ^{188}Hg, the transition occurs between 4^+ and 6^+, in agreement with the large value of ΔE_0. The transition with the change in the shape leads to a very specific change in the distances between the levels of the yrast band, and this is observed experimentally and is reproduced in Fig. 9. The experimental lifetimes against $E2$ transitions, which are a direct measure of the quadrupole moment [see Eq. (52)], also confirm the large deformation of the prolate branch of the yrast band (see, for example, Refs. 121 and 122).

Of very great interest is the behavior of the deformation energy in the case of nonaxial forms ($\gamma \neq 0$), through which a smooth transition may possibly be made from one axial minimum to another. In accordance with Ref. 117, there is a broad and flat barrier separating the two minima. The hypothesis of a separation of two minima found support in the experiment of Refs. 123–126, in which ^{184,186,188}Hg nuclei were observed to have states with small spin belonging to the prolate minimum, and also two states with large spin belonging to the oblate minimum. These states are shown in Fig. 9 on the continuation of the two branches of the yrast line. The energies of these non-yrast states are well reproduced in static calculations,[115] in which only a single parameter (the pairing constant) is varied in a very small range. The discrepancy in the energy of the higher terms of the oblate band is evidence of nonadiabatic effects of other types (see the classification in Sec. 2) not taken into account in the approach adopted here.

As we have already pointed out, the $B(E2)$ values in ^{184,186}Hg in the transitions $2^+ \to 0^+$ and $6^+ \to 4^+$ can be understood if one bears in mind the different deformations of the corresponding bands. In the transition $4^+ \to 2^+$, $B(E2)$ has an intermediate value, which shows that this transition is not forbidden and, therefore, that there is a coupling between the oblate and prolate states. On the other hand, the branching ratio of the $E2$ transitions in ^{188}Hg (Ref. 123) and the energies of the bands clearly indicate a rather good separation of these states. In Ref. 119, the $E2$ transitions in ^{184,186}Hg are interpreted in the framework of a simple phenomenological model. The prolate states are idealized as the rotational states of a rigid rotor, and the weakly deformed oblate states as the states of an harmonic vibrator of spherical shape. The parameters are chosen on the basis of energy relations. In addition, a mixing of states is introduced by adding an interaction between the rotor and the vibrator in the 2^+ and 4^+ states. Two matrix elements of the interaction are selected by reproducing the $B(E2)$ value in the yrast transitions. Once the parameters have been fixed, the branching ratio of the transitions from the prolate states to the prolate and oblate states are calculated, and these do not contradict the experimental estimates.[124]

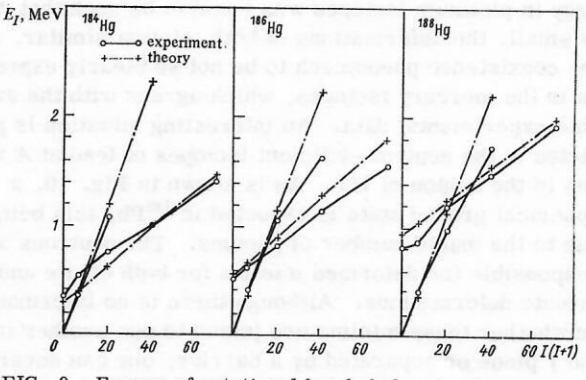

FIG. 9. Energy of rotational bands belonging to mercury isotopes of two different shapes. The experimental points are taken from Refs. 123–125. The theoretical values correspond to two different minima of the deformation energy[115] [see Eq. (101)]. The rotational states corresponding to one and the same minimum are joined by segments of a straight line. The 0^+ state in the oblate minimum corresponds to the ground state and is taken as the origin. The prolate band begins at a finite energy.

The mixing of states gives rise to a certain deviation of the energies from the unperturbed values, but the structure of the bands remains typical of a rotational band in the region of crossing of the bands. The mixed states $(2^+, 4^+)$ always "repel" one another. For ^{186}Hg, the experimental energies have the predicted property, whereas in ^{184}Hg the model requires improvement since the 4^+ level lies above the unperturbed state.

A development of the phenomenological approach to coupling between bands that belong to two minima of different deformations is presented in Ref. 120, in which the nonaxial degree of freedom is treated dynamically. The dependence of the deformation energy on γ is approximated by a curve composed of parts of two parabolas whose point of intersection corresponds to the apex of the barrier separating the two minima. The parameters of the model are the mass coefficients in the γ degree of freedom, the two moments of inertia at the two minima, the curvature of the parabolas, and the distance between their minima, i.e., the quantity ΔE_0 defined above. Choosing these parameters, one can reproduce the spectrum of the nuclei well. The wave functions of the states are fairly well localized in one of the minima, except for the immediate region near the point of crossing of the bands, at which the wave functions contain components of both minima. Unfortunately, the quadrupole matrix elements were not calculated.

Estimates based on the overlap integrals give for the branching ratio the value $B(E2,\ 4_1^+ \to 2_2^+)/B(E2,\ 4_1^+ \to 2_2^+) \approx 0.01$, which does not contradict the experimental estimate $\leqslant 0.03$ of Ref. 123. The model in Ref. 119 leads to the value 0.003, which also does not contradict the experiment. With regard to the energies, the experimental deviations from the uncoupled bands, i.e., from straight lines, are reproduced in this model. In particular, the shift upward of the 4^+ yrast level in ^{184}Hg is correctly reproduced.

The region around the light isotopes of mercury was investigated in Ref. 127 by means of static calculations, which were discussed above.[115] The deformation energy in platinum isotopes was found to be such that ΔE_0 is small, the deformations of both minima similar, and the coexistence phenomena to be not so clearly expressed as in the mercury isotopes, which agrees with the available experimental data. An interesting situation is predicted in the neutron-deficient isotopes of lead at A values in the region of 190. As is shown in Fig. 10, a spherical ground state is expected in ^{188}Pb, this being due to the magic number of protons. The neutrons are responsible for deformed minima for both oblate and prolate deformations. Although there is no information on whether these minima are joined to one another in the γ plane or separated by a barrier, one can nevertheless predict that the characteristic rotational states of the deformed band must belong to an yrast band for $I = 4\hbar$. In ^{184,186}Pb, the states of the deformed bands have a prolate shape with a high probability since the prolate minimum has a lower energy. In ^{192}Pb, in contrast, the oblate minimum lies lower. A preliminary interpretation of the ^{192}Pb spectrum in terms of the coexistence of spherical and deformed shapes is given in Ref. 127.

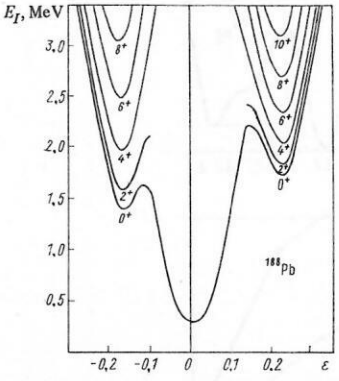

FIG. 10. Deformation energy (BCS) of the lowest rotational states in ^{188}Pb as a function of the parameter ε. The hexadecapole deformation is zero for $\varepsilon < 0$ and -0.01 for $\varepsilon > 0$, which is close to the equilibrium values. The results obtained by means of Q projection are close to the results shown in the figure, the deviation not exceeding 0.1 MeV.

The coexistence phenomenon considered in this subsection is characterized by a difference of deformations of a comparatively small amount, so that the expected accuracy of the determination by the shell-correction method of the relative position of the minima of the deformation energy is equal to several hundred keV. In the calculation of the deformation of the mercury isotopes,[115] the difference of the energies at the two minima could be varied in a certain range by varying the strength of the pairing interaction in the proton system. An increase in the corresponding proton parameter by 10% made it possible to explain successfully a large number of experimental data on the even–even and odd mercury isotopes.

5. DESCRIPTION OF RAPIDLY ROTATING NUCLEI

As we have already noted in Sec. 2, with increasing rotational frequency the role of pairing decreases, and at a certain critical angular momentum the energy gap becomes zero. In this section, we describe rapidly rotating nuclei in which the influence of pairing can be ignored.

In the case of rapid rotation, the nonadiabatic effects in the single-particle motion are large, and perturbation theory cannot be used to describe them. The change in the shape of the nucleus (see the classification of the effects in Sec. 2) plays the main role and is so great that one must have recourse to the methods developed in the theory of fission to describe it. The connection between the description of rapidly rotating nuclei and fission theory is dictated by the very nature of the phenomenon since the limit to the acceleration of the rotation of nuclei is largely imposed by their fission.

For general orientation about the processes that take place in a nucleus when the rotation is speeded up to the maximal amount possible it is helpful in a first approximation to ignore the quantum nature of the motion of the individual nucleons and consider a model in which the nucleus is represented as a classical rotating charged liquid drop. In the next approximation one introduces cor-

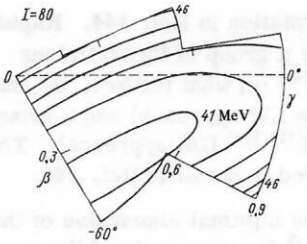

FIG. 11. Deformation energy of the nucleus ^{160}Dy in the liquid drop model at angular momentum $I = 80$ as a function of the parameters β and γ of ellipsoidal deformation.[114] The constant-level contours are drawn every 1 MeV. The energy of the nonrotating sphere is taken as the origin.

rections associated with the quantum nature of the single-particle motion (called the shell corrections), which are extremely important for obtaining quantitative results.

Rotation of nuclei in the liquid drop model. We consider a system of charged particles of uniform density in a finite volume. We shall describe the effects of the change in the shape of the nucleus by introducing a surface energy proportional to the surface area. We restrict ourselves to considering the rotation of such a system as a whole, ignoring the change in the density due to the centrifugal distension. The moment of inertia in such an approximation is equal to the moment of inertia of a rigid body of the same shape and mass. We shall study the total energy of such a system as a functional of the shape of the surface; in particular, we shall consider its extremal points. The absolute minimum of the total energy corresponds to the ground state of the system. The difference between the energies at the extremal points at the absolute minimum and at the saddle point gives the value of the fission barrier.

There is an intimate connection between this problem and the astrophysical problem of the shape of gravitating uniform masses, this being based on the fact that the Newtonian potential of gravitational attraction has the same spatial dependence as the repulsive Coulomb potential. The efforts of many eminent mathematicians have been directed toward the solution of the astrophysical problem (see, for example, the review, Ref. 128).

The problem of the form of a charged liquid drop, which has, as we said above, a direct bearing on nuclear physics, was considered in Refs. 128–138. A detailed review of the evolution of the stationary shapes when the angular momentum of the drop is varied was given in Ref. 128. We mention here the results that will be important in what follows in the discussion of the extremal shapes calculated with allowance for the quantum corrections.

Without rotation, the absolute minimum is attained for a spherical shape. As the rotation is speeded up, an ever more oblate axisymmetric shape becomes energetically advantageous, this continuing until the angular momentum reaches the first critical value (of order $80\hbar$ in the nuclei of the rare earths), at which the axial symmetry is lost. At the same time, the classical energy begins to depend weakly on the total lengthening of the nucleus and on the departure of its shape from axial form in a certain fairly large range of variation of the parameters that characterize the shape (as compared with the characteristic interval in which the quantum correction changes appreciably). In other words, there is a comparatively large range of nuclear shapes for which approximate equilibrium of the classical Coulomb, surface, and centrifugal forces is attained. We shall refer to this as the pre-barrier region since a slight increase of the angular momentum above the first critical value results in an appreciable displacement of the absolute minimum, this corresponding to a change of the equilibrium shape from oblate through nonaxial to a nearly prolate form. Simultaneously, the position of the absolute minimum approaches the saddle point.

When the absolute minimum and the saddle point merge, which occurs at the second critical value of the angular momentum, the nucleus becomes ("classically") unstable against fission into two fragments. At larger values of the angular momentum, there are no equilibrium forms. The heaviest nuclei ($A > 230$) become "classically" unstable against fission already when the first critical angular momentum is reached.

Note that the shapes for which stable equilibrium is attained are, in contrast to the shapes corresponding to the saddle point, nearly ellipsoidal (except for the lightest rapidly rotating nuclei), which justifies the study of the role of the quantum corrections made in Refs. 113, 114, 139, and 140 under a restriction to ellipsoid shapes (see below). In the same papers, examples are given that illustrate the sequence of equilibrium shapes in the liquid drop model (Fig. 11). Figure 11 shows the energy surface for ellipsoidal deformations characterized by the parameters β and γ at an angular momentum near the first critical value. One can easily see the broad region of smooth variation of the deformation energy along the straight line joining the points with the coordinates $(\beta, \gamma) = (0.45, -60°)$ and $(0.9, 0°)$.

Estimates for the first and second critical values of the angular momentum for nuclei along the valley of β stability are given in the review, Ref. 128. The second critical value attains its largest value, $97\hbar$, in nuclei with $A \approx 130$. We give more detailed information for the nucleus ^{127}La, in which the first critical value is $68\hbar$, and the second $85\hbar$. The rotational energy for the second critical value and external shape is 60 MeV, which is not more than 6% of the binding energy of this nucleus. The corresponding angular frequency of rotation is $\hbar\omega = 1.4$ MeV.

The above pronounced change in the shape of the liquid drop as the angular momentum varies from the first to the second critical value is well manifested as a giant back-bending effect[140] (Fig. 12). It is natural that in this region any slight change in the energy surface, in particular corresponding to allowance for the quantum shell correction, leads to an appreciable change in the position of the minimum.

Shell effects in rapidly rotating nuclei. It follows from the estimates made above that even the most rapid

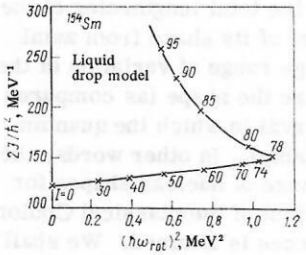

FIG. 12. Twice the moment of inertia of the yrast band in the nucleus ^{154}Sm as a function of the square of the rotational frequency in the liquid drop model.[140]

rotation, almost sufficient to break the nucleus apart, decreases the total binding energy of the nucleons by only a few percent. In this sense, rotation only slightly changes the conditions of existence of nuclear matter. The large effect of this perturbation expressed in the strong change in the shape is explained by the fact that rotation "interferes" with the delicate balance of the comparatively large Coulomb forces and the surface tension forces. (In the ^{127}La nucleus, for which the above estimates were made, the Coulomb and surface energies for spherical shape are 456 and 445 MeV, respectively.)

Therefore, if there is a rotation, the local properties of the nucleus change little and one can assume that the mean free path of the nucleons and the correlations in the single-particle motion are not changed so much that it becomes meaningless to use the concept of an average field to describe the motion of the nucleons in the rotating coordinate system. (We are not here referring to the pairing correlations of superconducting type; the role of these forces has been discussed above and it has been shown that they can be ignored in the case of rapid rotation.)

However, the average field can change not only its shape, and estimates of the corresponding effects are given below. Even more important is the direct influence of the rotation on the single-particle motion, as is shown by the estimates of ωj^x_{ij} [see Eq. (6) for $v_{iji'j'} = 0$]. This quantity can be estimated for the nucleus ^{127}La for maximally fast rotation using the value $\hbar\omega = 1.4$ MeV and assuming that the matrix elements of the single-particle angular momentum in this nucleus do not exceed $6\hbar$. Then $|\omega j^x_{ij}| \lesssim 8.5$ MeV, which, on the one hand, is comparable with the distance between the shells, and, on the other, is less than the Fermi energy (≈ 40 MeV). On the basis of the estimates made here, it must be expected that the shell effects, which are strongly related to the level density at the Fermi surface, undergo appreciable variation as a result of rapid rotation of the nuclei, and in this sense the perturbation introduced by the rotation is not small. However, numerical estimates of such changes can be made in the same approximations based on the average field concept that have proved themselves well in the calculation of the deformation energy of nonrotating nuclei.

To calculate the total energy of a rotating nucleus as a functional of its shape, it is therefore natural to use Strutinskiĭ's method, which has been very effective in the calculation of the energy of a nonrotating nucleus.[141-143] The modification of the method due to rotation is described for slow rotation in Ref. 144. Rapid rotation has been studied by a group of theoreticians from Lund and Warsaw[139,140] (in what follows, we shall refer to these studies as the LW approach) and a group from Dubna and Rossendorf[113,114] (DR approach). The two approaches are compared in detail in Ref. 145.

The problem of finding the minimal eigenvalue of the many-particle Hamiltonian $\hat{H}$ for given spin of the nucleus, i.e., the energy of the yrast state, can be solved by Lagrange's method, which leads directly to the operator $\hat{R}$ [see the definition (1)], which, by analogy with the corresponding function in classical mechanics,[15] we shall call the quantum Routh operator, or the Routhian.[113,114] In a rigorous formulation of the problem, when the commutativity of $\hat{I}$ and $\hat{H}$ is not violated by the approximations which are introduced, $\hat{R}$ commutes with $\hat{H}$, and therefore has common eigenstates with $\hat{H}$. In such a case, the transition from $\hat{H}$ to $\hat{R}$ does not introduce additional approximations.

One of the possible approximate methods of solving the problem of the minimal eigenvalue of $\hat{H}$ is to find the expectation value $\langle I|\hat{H}|I\rangle$, where $|I\rangle$ is a many-particle function corresponding to definite spin I of the nucleus. Varying $|I\rangle$, we find the minimal value of $\langle I|\hat{H}|I\rangle$, which we denote by $E(I)$. To avoid constructing functions $|I\rangle$ with definite spin I, we go over to the investigation of the minimal expectation value $R(\beta,\omega) = \langle\beta,\omega|\hat{R}|\beta,\omega\rangle|_{\min}$ on a certain class of functions $|\beta,\omega\rangle$. The parameters β determine the shape of the nucleus. In Refs. 113 and 114 it is noted that if the dependence of $E(\beta,\mathbf{I})$ on $\mathbf{I} = \langle\beta,\omega|\hat{\mathbf{I}}|\beta,\omega\rangle$ is convex the functions $E(\beta,\mathbf{I})$ and $R(\beta,\omega)$ are uniquely related by an equation that follows from (1):

$$E(\beta,\mathbf{I}) = \langle\beta,\omega(\mathbf{I})|\hat{H}|\beta,\omega(\mathbf{I})\rangle = R(\beta,\omega) + \mathbf{I}\cdot\omega, \tag{107}$$

where ω is determined by the condition

$$\mathbf{I} = -\partial R/\partial\omega \equiv \mathbf{I}(\beta,\omega). \tag{108}$$

It follows from (107) and (108) that

$$\omega = \partial E/\partial\mathbf{I}. \tag{109}$$

In what follows, we shall use the word Routhian for both the operator $\hat{R}$ and its expectation value $R(\beta,\omega)$, which will not cause confusion.

If the dependence of $E(\beta,\mathbf{I})$ on $\mathbf{I}$ is not convex, no value of ω corresponds to a certain value of $\mathbf{I}$ [Eq. (108) has no solution].

As we have already noted in Sec. 1, the introduction of the Routhian $R(\beta,\omega)$ into the problem of rotation about an axis which is not a symmetry axis can be interpreted as a transition to a coordinate system rotating together with the average field of the nucleus with frequency ω. In other words, the introduction of the Routhian constitutes the essence of the cranking model without perturbation theory. The study of $R(\beta,\omega)$ is no less justified in a description of the yrast states with angular momentum directed along the symmetry axis of the field, al-

though the interpretation of such a state as rotational may meet objections. Formally, this case differs from rotation in which the direction of the symmetry axis of the potential changes with the time, in that for each I there is a corresponding configuration in an unchanged potential. But there are numerous physical arguments suggesting that one should extend the term rotation to this case as well, as A. Bohr does.[5]

In the model of independent particles, the Routhian R_{ip} is equal to the sum of the lowest A (A is the number of particles) eigenvalues ε_i^ω of the single-particle Routhian [see Eqs. (6) and (7) for $v_{ijj'j'}=0$]:

$$r=t-\omega \mathbf{j}+V(\beta), \tag{110}$$

where t is the kinetic energy; j is the angular momentum; and V is the average field of the nuclear forces in the rotating nucleus, this depending, in general, on the rotational frequency ω. This dependence is comparatively weak, as was shown in Ref. 114, and in practical calculations it has not hitherto been taken into account (see below). Thus,

$$R_{ip}(\beta, \omega)=\sum_{\varepsilon_i^\omega<\lambda_{ip}} \varepsilon_i^\omega. \tag{111}$$

The derivative $-\partial R_{ip}/\partial\omega$, which is denoted below by $I_{ip}(\beta, \omega)$, is represented as a sum of A expectation values of the operator j. These values are calculated from the eigenfunctions corresponding to ε_i^ω:

$$\partial R_{ip}/\partial\omega \equiv \mathbf{I}_{ip}(\beta, \omega)=\sum_{\varepsilon_i^\omega<\lambda_{ip}} \mathbf{j}_i. \tag{112}$$

Here and in the preceding equation, λ_{ip} is the Fermi level in the model of independent particles.

For small ω, when perturbation theory can be used, ε_i^ω and $\mathbf{j}_i$ have the form

$$\varepsilon_i^\omega=\varepsilon_i^0-\boldsymbol{\omega}\cdot\langle i|\mathbf{j}|i\rangle+\sum_{k\neq i}\frac{|\boldsymbol{\omega}\cdot\langle k|\mathbf{j}|i\rangle|^2}{\varepsilon_i^0-\varepsilon_k^0}; \tag{113}$$

$$\mathbf{j}_i=\langle i|\mathbf{j}|i\rangle-2\sum_{k\neq i}\frac{(\boldsymbol{\omega}\cdot\langle i|\mathbf{j}|k\rangle)\langle k|\mathbf{j}|i\rangle}{\varepsilon_i^0-\varepsilon_k^0}, \tag{114}$$

where $\varepsilon_i^{(0)}$ and $|i\rangle$ are the eigenvalues and eigenfunctions of the single-particle Hamiltonian [see the expression (110) for $\omega=0$].

In the case of rotation about a symmetry axis of the nucleus, which is discussed in detail in Ref. 140, the sums over k are zero, and the remaining terms represent an exact solution for all ω. In the case of rotation about an axis perpendicular to the symmetry axis, the diagonal elements are not present. (The case $\Omega=\frac{1}{2}$ requires special treatment.[30]) For large ω, the corresponding quantities are calculated numerically.

It is known that in the absence of rotation the model of independent particles is inapplicable for the calculation of the total deformation energy. In Refs. 113 and 114 it is noted that for the Routhian $R(\beta, \omega)$ one can prove Strutinskiĭ's energy theorem[146] in exactly the same way as the theorem is proved for the total energy of a nonrotating nucleus. Therefore, it can be calculated by Strutinskiĭ's method. Essentially, the proof is based on the stationarity of $R(\beta, \omega)$ for a small variation of the many-particle wave function $|\beta, \omega\rangle$, which is a Slater determinant constructed from single-particle wave functions. Since $E(\beta, \mathrm{I})$ is determined by the condition of stationarity in the presence of the constraint $\langle\omega|\mathbf{I}|\omega\rangle=\mathrm{I}$ [see Eq. (2)], Strutinskiĭ's theorem is also valid for it. The stationarity of $E(\beta, \mathrm{I})$ for constant I can be readily deduced directly from the stationarity of $E(\beta, \omega)$ for constant ω. Indeed, we obtain the first variation of $E(\beta, \mathrm{I})$ for a small variation of $|\omega\rangle$ on the basis of Eq. (107):

$$(\delta E)_{\mathrm{I}}=(\delta R)_\omega+\partial R/\partial\omega\cdot\delta\omega+\mathbf{I}\cdot\delta\omega.$$

There must be a concomitant change of ω if the spin I is to remain constant when $|\omega\rangle$ varies. Using (108), we obtain $(\delta E)_{\mathrm{I}}=(\delta R)_\omega$, i.e., if $(\delta R)_\omega=0$, then $(\delta E)_{\mathrm{I}}=0$.

Thus, we have shown that to within variations of the density of second order, which are usually not considered in Strutinskiĭ's method, one can obtain a splitting of $E(\beta, \mathrm{I})$ into a "smooth" classical part and a shell correction on the basis of the corresponding splitting for $R(\beta, \omega)$ and $I(\beta, \omega)$:

$$R(\beta, \omega)=R_{\mathrm{cl}}(\beta, \omega)+\delta R(\beta, \omega); \tag{115}$$

$$\mathrm{I}(\beta, \omega)=\mathrm{I}_{\mathrm{cl}}^{\mathrm{s}}(\beta, \omega)+\delta\mathrm{I}(\beta, \omega), \tag{116}$$

where R_{cl} and I_{cl} are the classical expressions for a rotating charged liquid drop:

$$R_{\mathrm{cl}}(\beta, \omega)=U_{LD}-\tfrac{1}{2}\omega\cdot\mathcal{I}_{\mathrm{cl}}(\beta)\,\omega; \tag{117}$$

$$\mathrm{I}_{\mathrm{cl}}(\beta, \omega)=-\partial R_{\mathrm{cl}}/\partial\omega=\mathcal{I}_{\mathrm{cl}}(\beta)\cdot\omega. \tag{118}$$

Here, U_{LD} is the sum of the classical surface and Coulomb energies of the nucleus in the liquid drop model. Since the quasiclassical rotational motion of the nucleus is nearly that of a rigid body, the rigid-body inertia tensor is usually used for $\mathcal{I}_{\mathrm{cl}}$.

The shell corrections $\delta R(\beta, \omega)$ and $\delta\mathrm{I}(\beta, \omega)$ have somewhat different forms in the two approaches mentioned above. In what follows, we shall adopt the formalism developed in the Dubna-Rossendorf approach:

$$\delta R(\beta, \omega)=R_{ip}(\beta, \omega)-\tilde{R}(\beta, \omega); \tag{119}$$

$$\delta\mathrm{I}(\beta, \omega)\equiv-\partial(\delta R(\beta, \omega))/\partial\omega=\mathrm{I}_{ip}(\beta, \omega)-\tilde{\mathrm{I}}(\beta, \omega), \tag{120}$$

where $\tilde{R}(\beta, \omega)$ is the result of averaging R_{ip} defined in the usual manner in Strutinskiĭ's method.

In the case of slow rotation, the expressions for R_{ip} and I_{ip} are well known[15] [see also Eq. (2)], and the expressions for $\tilde{R}$ and $\tilde{\mathrm{I}}$ are obtained by smoothing the expressions (113) and (114) and using approximate expressions for the variation of the smooth quantities in the case of a small variation of the single-particle spectrum.[143] Details of the derivation in the case of rotation about an axis perpendicular to the symmetry axis are given in Ref. 144, and the results have the form

$$\tilde{R}(\beta, \omega)=\tilde{R}(\beta, 0)-(\omega^2/2)\mathcal{I}_\perp; \tag{121}$$

$$\tilde{I}(\beta, \omega)=\omega\mathcal{I}_\perp, \tag{122}$$

where

$$\tilde{\mathcal{J}}^{\perp} = \sum_{i \neq k} \frac{\tilde{n}(\varepsilon_i^{(0)}) - \tilde{n}(\varepsilon_k^{(0)})}{\varepsilon_k^{(0)} - \varepsilon_i^{(0)}} |\langle i|j^x|k\rangle|^2; \qquad (123)$$

ω and $\tilde{I}$ are the moduli of the corresponding vectors, which in the given special case are directed along the rotation axis. Here, $\tilde{n}$ are smooth population numbers in Strutinskiĭ's method.[143]

To obtain the corresponding expressions in the case of slow rotation around the symmetry axis, it is necessary to use a higher order in the expansion of the smooth quantities with respect to the perturbation in the single-particle spectrum. The results have the same form as the expressions (121) and (122), in which the moment of inertia is represented by

$$\mathcal{J}^{\parallel} = \sum_k |\langle k|j_z|k\rangle|^2 S(\tilde{\lambda} - \varepsilon_i^{\omega}), \qquad (124)$$

where $\tilde{\lambda}$ is the Fermi level in the smooth system, which is fixed by specifying a definite number of particles; $S(z) = \xi(z/\gamma)\gamma$; $\xi(x)$ is the function introduced in the review of Brack *et al.*[143] by means of which the single-particle spectrum is smoothed; γ is the averaging interval. Without going into discussion about the possibility of rotation of a symmetric body about a symmetry axis, we note that $\mathcal{J}^{\parallel}$ characterizes the energy as a function of the angular momentum for changes in the configuration of the nucleons that alter the projection of the angular momentum onto the symmetry axis.

In Ref. 144, by analogy with perturbation theory at finite temperature, it is noted that the expressions (121) and (122) can have a wider range of applicability than the expansion (113) and (114), on which they are formally based; for in calculations with different potentials[114,140,145] it has been established that the numerical estimates of $\tilde{I}(\beta, \omega)$ as a function of ω lie on a straight line with remarkable accuracy.

Thus, the problem of finding the energy $E(\beta, \mathbf{I})$ of the nucleus as a function of angular momentum I at deformation β reduces to finding the classical quantities (117) and (118) and the shell corrections (119) and (120), substituting the resulting expressions in (115) and (116), and solving the system (107)–(108).

For small ω, the rotational energy in the Dubna–Rossendorf approach goes over into the expression given in Ref. 144:

$$E_{\mathrm{DR}}(\beta, I) - E_{\mathrm{DR}}(\beta, 0) = (\omega^2/2)[(\tilde{\mathcal{J}}_{1p} - \tilde{\mathcal{J}}) + \mathcal{J}_{c1}] = I^2/2[(\mathcal{J}_{1p} - \mathcal{J}) + \mathcal{J}_{c1}], \qquad (125)$$

whereas the same limiting case in the Lund–Warsaw approach has the form

$$E_{\mathrm{LW}}(\beta, I) - E_{\mathrm{LW}}(\beta, 0) = I^2[(1/\mathcal{J}_{1p} - (\tilde{1}/\tilde{\mathcal{J}})) + 1/\mathcal{J}_{c1}]/2, \qquad (126)$$

i.e., it is not the moment of inertia but its reciprocal that is renormalized.

The expressions (125) and (126) coincide in two cases: when there is no "renormalization" ($\tilde{\mathcal{J}} = \mathcal{J}_{c1}$) and when the shell correction is zero ($\mathcal{J}_{1p} = \tilde{\mathcal{J}}$); the second case is an exception rather than the rule.[144]

Shape of the nucleus at large angular momenta. Basic theoretical predictions. Actual calculations of the change in the shape of a nucleus under the influence of rapid rotation on the basis of the theoretical assumptions discussed above were made in Refs. 113, 114 and Refs. 139, 140 using the Nilsson single-particle potential and in Ref. 145 using the Woods–Saxon potential. The total energy of the nucleus was calculated as a function of the parameters that determine the shape of the nucleus [see Eqs. (107), (108) and (115), (116)]. In these investigations, only ellipsoidal shapes were considered in the study of the shell corrections. An absolute minimum on the energy surface corresponds to an yrast state, i.e., to a rotating "ground" state, whereas local minima, if such exist, are responsible for shape isomers. The calculations show that both the classical component in the total energy as well as the shell correction are important for determining the shape of the nucleus. The classical part describes the general behavior of the energy surface that was characterized above (see Fig. 11).

The local behavior of the energy surface is determined by the shell correction, which is particularly important in the region of smooth variation of the classical energy surface, i.e., for small deformations (ground states of nuclei in the absence of rotation) and in the pre-barrier region. As an example of the modulation of the classical energy by the shell correction, Fig. 13 shows the total deformation energy of the same nucleus in the same range of variation of the deformation parameters and for the same angular momentum as the classical energy in Fig. 11.

Let us first discuss the change in the shape of the ground state. Deformed nuclei in the middle of the region of the rare earths have energy in the "prolate" minimum which is 2–3 MeV lower than the stationary energy for "oblate" deformation.[147] Calculations show that even rapid rotation cannot significantly alter this relation.[113,114] However, there is a change in the position of the absolute minimum. It is shifted in the majority of well deformed nuclei of the rare earths (except for some nuclei in the very middle of the region such as, for example, ^{174}Hf and ^{176}Hf) in the direction of smaller deformation β and, in some cases, in the direction of positive values of the parameter γ of the nonaxial shape. It should be noted that the results obtained in Ref. 29, with allowance for the pairing forces, at the largest angular momenta considered in Ref. 29 exhibit many

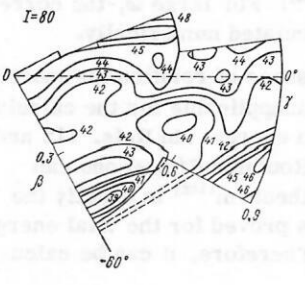

FIG. 13. Deformation energy with allowance for the shell correction. The notation is the same as in Fig. 11.

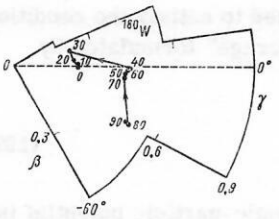

FIG. 14. Position of the minimum in the total deformation energy of the nucleus ^{166}Er in the (β,γ) plane for different total angular momenta. The points are joined by arrows that indicate the direction of the change in the shape of the nucleus when a cascade of gamma rays is emitted.[114]

features in common with those found in Ref. 114. With increasing spin, the importance of the classical forces increases, γ takes a negative value, and β increases. As a result, the position of the minimum in some nuclei (especially in nuclei with $N=98$ and 102) describes a spiral-like figure.[114] As an example, Figs. 14 and 15 show the positions of the minima in the total deformation energy in the nuclei ^{166}Er and ^{180}W. For large angular momenta, for which points are not shown in the figures, the nuclei undergo fission.

The changes in the single-particle level scheme that are responsible for this behavior of the total energy are discussed in detail in Ref. 140.

The situation is quite different in neutron-deficient nuclei and in nuclei in the transition region, in which, as is well known from calculations without allowance for rotation, the potential energy surface has two minima, oblate and prolate, at approximately the same deformation β (Ref. 148). In such nuclei, rotation deepens the oblate minimum, which remains the ground state for I up to approximately $(50-60)\hbar$, changing its position slightly. Oblate rotating nuclei are of particular interest from the experimental point of view, since γ transitions in the case of rotation about a symmetry axis do not have collective enhancement because the charge density does not change in such transitions to within A^{-1}, and it is therefore theoretically possible to have rapidly rotating metastable states; these are the so-called yrast traps.[149,140]

The study of possible traps in oblate nuclei rotating around the symmetry axis is facilitated by the obvious circumstance that the projection of the single-particle angular momentum onto the rotation axis commutes with the single-particle Hamiltonian and the eigenvalues of the single-particle Routhian (110) can be obtained directly on the basis of the eigenvalues for $\omega=0$. Of particular interest are the sections in which the dependence of $E(\beta,I)$ on I is not convex. In Ref. 140 it is noted that the state with lowest energy for given I may be attained, not for the configuration in which all the lowest eigenvalues of the Routhian (110) are occupied, but in a certain configuration with a vacant state below the Fermi level and a filled one above it (a particle-hole state). One then sometimes has a situation in which states with spins $I-1$ and $I-2$ have greater energy than the original state with spin I. In such a case, γ decay must take place with $\Delta I>2$, which, generally speaking, causes it to be hindered. In addition, hindrance is also possible in the case when the transition to the lower state is accompanied by a change in the state of more than one particle. All such cases have been investigated in detail in the nuclei ^{150}Gd and ^{158}Yb in Ref. 140. But traps are predicted in a very small number of cases.

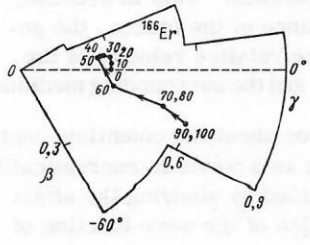

FIG. 15. The same as in Fig. 13 for the nucleus ^{180}W.

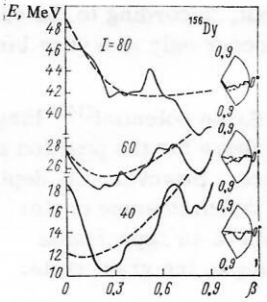

FIG. 16. Section of surface representing the deformation energy taken approximately along the "fission valley" for three different angular momenta. The dashed curves are the classical deformation energy.[113]

In Ref. 150 it is noted that residual (for example, quadrupole–quadrupole) forces may facilitate the alignment of the spins of the nucleons and, therefore, the formation of metastable states with very high spin.

In the pre-barrier region, as we have already noted above, the range of smooth variation of the classical energy is not small compared with the characteristic interval over which the shell correction varies appreciably (see, for example, Figs. 11 and 13). Therefore, in this region there is a high probability of finding a minimum of the shell correction and, hence, of the total deformation energy, which leads to an increase in the limiting angular momentum which a nucleus can have without fissioning. Indeed, it is shown in Ref. 114 by a numerical calculation that the raising of the limiting angular momentum may be as much as $(30-40)\hbar$. It should be noted that it is assumed in the calculations that the nucleus is in the yrast band, i.e., it is "cold." An internal excitation (a raising of the temperature) smooths the irregularities associated with the shell structure, and the classical description becomes more and more exact.

A particularly deep minimum in the case of a large prolate deformation arises according to the calculations of Ref. 114 in the nucleus ^{154}Dy ("deformed shell" in Strutinskiĭ's terminology[141,142]), this becoming absolute at angular momentum near the limiting value. A similar prediction is made for the neighboring isotope ^{156}Dy, as can be seen in Fig. 16.

In nuclei of rare earth elements, one also encounters a local minimum at a large oblate deformation, and this

sometimes becomes absolute, but, according to the calculations of Ref. 114, this can occur only at a very large angular momentum.

Calculations with the Woods–Saxon potential[145] largely confirm the results obtained above for the position of the extrema of the energy surface. However, the depth of the minima may be changed, which in some nuclei leads at certain angular momenta to an appreciable change in the shape corresponding to the yrast state.

In Ref. 114 it is noted that the changes in the shape of a rapidly rotating nucleus described above have important consequences for the development of a cascade of gamma rays.

The application of Strutinskiĭ's method to the rotation of a light nucleus such as ^{20}Ne is discussed in Ref. 151.

It should be noted that the deformation energy of rapidly rotating nuclei has also been calculated in a simplified model[152,153] in an approach of the Hartree–Fock–Bogolyubov type. However, as the authors of this investigation themselves point out, the restricted single-particle space and schematic treatment of the nucleon–nucleon interactions used in the calculations must cast doubt on the adequacy of the model for the problem posed.

Reaction to rotation of nuclear characteristics besides the shape and the superfluidity. Historically, it so happened that the degrees of freedom associated with the shape and superfluidity were at the center of interest in the discussion of the reaction of nuclei to rotation. Accordingly, in the present review we have hitherto dealt almost exclusively with these properties. It must be borne in mind that the shape and superfluidity are not the only characteristics of a nuclear system which are affected by a finite angular momentum imparted to the system. Other properties such as the density distribution in the internal coordinate system, the nature of the mass flow produced by the orbital angular momentum, and also the degree of polarization of the spins of the nucleons, are of interest.

Correct understanding of such properties of nuclear rotational motion is particularly important in attempts to predict the behavior of a nucleus in high-spin yrast states on the basis of Strutinskiĭ's method of shell corrections, since this determines the choice of the classical Routhian and the single-particle potential as functions of the rotational frequency ω, these being used as a substitution instead of the corresponding smoothed quantities (see above).

Unfortunately, the empirical data so far accumulated give almost no help for the solution of this problem, in stark contrast to the situation for zero angular velocity, in which the freedom in the choice of the mean energy and the single-particle potential is strongly restricted by a large number of empirical data such as the mass, the lifetimes for spontaneous fission, the spectra of odd nuclei, etc.

A theoretical treatment of the problem is given in Ref. 114, in which a study is made of the averaged density matrix $\tilde{\rho}$, which is assumed to satisfy the condition of self-consistency "on the average" formulated by Strutinskiĭ[143,145]:

$$\tilde{\rho} \to \tilde{V} \to \rho_{1p} \xrightarrow{\text{"Smoothing"}} \tilde{\rho}, \tag{127}$$

or, to put it in words: The single-particle potential is the result of convolution of the density matrix $\tilde{\rho}$ with the two-nucleon forces. The matrix $\tilde{\rho}$, in its turn, is obtained by smoothing the matrix ρ_{1p}, which corresponds to the lowest stationary filled levels in the potential $\tilde{V}$. In principle, the density matrix $\tilde{\rho}$ should also be used to calculate the average Routhian $\tilde{R}$. Note that in the case of rotation the eigenstates of the operator r defined by Eq. (110) are stationary single-particle states.

Following Ref. 114, we consider the following characteristics of a nucleus at large angular momenta, these being described by the average density matrix $\tilde{\rho}$:

1) the nature of the rotational flow;

2) the density distribution in the internal coordinate system;

3) the spin polarization;

4) the structure of the spin–orbit potential in the rotating nucleus.

The nature of the rotational flow. Under fairly general assumptions, one can show that the average flow generated by a single-particle Routhian of the form (110) corresponds to rotation of the system as a rigid body (see, for example, the discussion in Ref. 4, Vol. II, pp. 79, 80 of the Russian translation). To understand how this result is obtained, let us consider as a simplest example a system of spinless "fermions" moving in a local potential well. In this case, the single-particle Routhian is given by

$$r = \mathbf{p}^2/2M + \tilde{V}(\mathbf{r}) - \boldsymbol{\omega}\cdot(\mathbf{r}\times\mathbf{p}) = (\mathbf{p} - M(\boldsymbol{\omega}\times\mathbf{r}))^2/2M + \tilde{V}(\mathbf{r}) - M(\boldsymbol{\omega}\times\mathbf{r})^2/2. \tag{128}$$

The second form of the expression (128) shows that in the local approximation the Routhian has a minimum if at every point $\mathbf{r}$ the particles have momenta $\mathbf{p}$ distributed isotropically around the mean value

$$\langle \mathbf{p} \rangle_{\mathbf{r}} = M(\boldsymbol{\omega}\times\mathbf{r}), \tag{129}$$

which, as it happens, corresponds to rotation of the system as a rigid body.

Migdal has shown[154] that the result (129) is also obtained in the case when the convolution of the density matrix with the two-nucleon forces leads to a velocity-dependent single-particle potential. This is because, on account of Galileo invariance of the forces, the potential can depend only on the relative velocity of the particle under consideration and the surrounding medium.

Certain general information about the conditions under which rotation of the system as a whole is energetically disadvantageous can be obtained by studying the effect of the following transformation of the wave function of

the system on its physical properties:

$$|\phi\rangle = \exp\left\{-\frac{iM}{\hbar}\sum_{\nu=1}^{A}\phi(\mathbf{r}_\nu)\right\}|u_0\rangle.$$

Here, $|\mathbf{u}_0\rangle$ is a state corresponding to some local velocity distribution $\mathbf{u}_0(\mathbf{r})$ given by (129); M is the nucleon mass; $\phi(\mathbf{r})$ is an arbitrary function of the spatial coordinate of the nucleon. The state $|\phi\rangle$ corresponds to a local velocity distribution $\mathbf{u}(\mathbf{r}) = \nabla\phi(\mathbf{r}) + \mathbf{u}_0$, i.e., this transformation of the wave function corresponds to excitation in the system of irrotational motion. It is obvious that such a transformation leaves unchanged all the characteristics of the charge and mass distribution of the nucleons and, in particular, does not affect the volume or multipole moments of the charge. On the basis of the invariance of $\phi(\mathbf{r})$ under time reversal, one can also show that the monopole pairing is unaffected by the transformation. Therefore, the gap parameter Δ does not depend on ϕ:

$$\Delta = g\langle\phi|\sum_i a_i^+ a_{\tilde{i}}^+|\phi\rangle = g\langle\mathbf{u}_0|\sum_i a_i^+ a_{\tilde{i}}^+|\mathbf{u}_0\rangle,$$

if the state $|\mathbf{u}_0\rangle$ contains only pairing correlations of monopole type, when $\langle\mathbf{u}_0|a_i^+ a_j^+|\mathbf{u}_0\rangle \neq 0$ for $i = \tilde{j}$ (the states i and $\tilde{j}$ are conjugate with respect to the time).

Excitation of the irrotational motion increases the mean kinetic energy compared with the estimate obtained above [see (128)] for the rotation of the system as a whole. However, the distribution (129) corresponds to the appearance of centrifugal forces which change the mass distribution in the nuclei, and also nondiagonal matrix elements of the Routhian (128), which affect the monopole pairing. If the changes in the potential energy associated with this are large, then the irrotational motion is energetically advantageous. At small angular momenta, pairing plays the dominant role (see Sec. 2), and the velocity distribution of the nucleons may be very different from (129). As we shall show below, the influence of the centrifugal forces is very small in virtually the entire region of angular momenta for which the nuclei exist as a whole, so that in the absence of pairing (129) must be satisfied with good accuracy.

It follows from (129) that the orbital part of $\tilde{\mathbf{L}}$ of the smoothed angular momentum is given by

$$\tilde{\mathbf{L}} = \mathcal{I}_{\text{rig}}\cdot\boldsymbol{\omega}, \tag{130}$$

where $\mathcal{I}_{\text{rig}}(\rho)$ is the classical inertial tensor, to which there corresponds the spatial density given by

$$\rho(\mathbf{r}) \equiv \langle\mathbf{r}|\tilde{\rho}|\mathbf{r}\rangle. \tag{131}$$

This approximation for the classical moment of inertia has been used hitherto in practical calculations by Strutinskiĭ's method.

Experimentally, it has been found[155,77] that for the nucleus ^{162}Yb the moment of inertia when $I \gtrsim 20$ is close to the spherical rigid-body value, although the experimental errors are large. This is because the spin is determined by estimating the number of γ transitions that take place from the investigated level.

The density distribution in the internal coordinate system. The last term in the expression (128):

$$V_{\text{cf}}(\mathbf{r}) = -M(\boldsymbol{\omega}\times\mathbf{r})^2/2 \tag{132}$$

is the centrifugal potential. We have already discussed above the importance of the centrifugal forces in the determination of the shape of the nucleus. Within the nuclear surface, the centrifugal forces tend to change the density distribution that would be present in the nucleus in the absence of rotation, shifting the nucleons to the side from the rotation axis.

Ignoring Coulomb and surface effects, which compensate one another to a certain extent (see, for example, Ref. 4, Vol. I, p. 254 of the Russian translation), one can describe the interior of the nonrotating nucleus by means of a constant density ρ_0 equal to the equilibrium density of nuclear matter.

The equilibrium density $\rho(\mathbf{r})$ of the nucleus in the field of the centrifugal forces is given in the first order as

$$(\rho(\mathbf{r}) - \rho_0)/\rho_0 = (\lambda_{\text{cf}} - V_{\text{cf}}(\mathbf{r}))/b; \qquad b \equiv K/9, \tag{133}$$

where K is the compressibility modulus of nuclear matter; λ_{cf} is a constant which depends on the boundary conditions. Assuming for simplicity that the surface is a sphere of radius R, and assuming as boundary conditions that the generalized forces corresponding to the coordinate R are zero, we obtain

$$\lambda_{\text{cf}} = -M\omega^2R^2/3. \tag{134}$$

Thus, the density is decreased on the axis of rotation and increased on the "equator". The maximal relative change in the density occurs on the rotation axis and is $|\lambda_{\text{cf}}|/b$. The mean density is reduced by the relative amount $\frac{2}{5}|\lambda_{\text{cf}}|/b$. (Note that this is due to the increase in the radius R.)

The relative change of the classical moment of inertia, which is due to the change in the density, can be readily calculated; the result takes the form

$$(\mathcal{I}_{\text{rig}} - (\mathcal{I}_{\text{rig}})_0)/\mathcal{I}_{\text{rig}} = 0{,}52\,|\lambda_{\text{cf}}|\,b. \tag{135}$$

The constant λ_{cf} is the change in the Fermi level in the centrifugal field. Adding all the contributions to this change in accordance with Eq. (128) and using the Thomas–Fermi expression for the kinetic energy:

$$\varepsilon_{\text{F}} \equiv ((\mathbf{p} - M(\boldsymbol{\omega}\times\mathbf{r}))^2/2M)_{\text{F}} = (\rho/\rho_0)^{2/3}\varepsilon_{\text{F}}^0 \tag{136}$$

($\varepsilon_{\text{F}}^0 \approx 38$ MeV), we obtain

$$\lambda_{\text{cf}} = \tfrac{2}{3}\frac{(\rho-\rho_0)}{\rho_0}\varepsilon_{\text{F}}^0 + (\tilde{V}(\mathbf{r}) - \tilde{V}_0) + V_{\text{cf}}(\mathbf{r}), \tag{137}$$

whence, using (133), we arrive at the result

$$\tilde{V}(\mathbf{r}) - \tilde{V}_0 = \left(1 - \frac{2}{3}\frac{\varepsilon_F^0}{b}\right)(\lambda_{cf} - V_{cf}(\mathbf{r})). \quad (138)$$

For a nucleus with $A = 170$ and $I = 80\hbar$, the value of $|\lambda_{cf}|$ is 1 MeV. Taking $K = 250$ MeV, we find that $|\lambda_{cf}|/b$ is 4%. For any reasonable value of K, the value of $|\tilde{V}(\mathbf{r}) - \tilde{V}_0|$ determined by (138) does not exceed some fractional part of an MeV. It should also be noted that λ_{cf} and $V_{cf}(\mathbf{r})$ are proportional to I^2.

Thus, all the effects of the change of the density induced by the centrifugal forces are relatively small and they can be ignored in practical calculations. Only the very largest possible spin values may be an exception.

The spin polarization. Taking into account the spin of the nucleus, we note that the operator r contains, in addition to the contribution taken into account in Eq. (128), the term

$$V_s^{cor} \equiv -\omega \cdot \mathbf{s}. \quad (139)$$

This term leads to spin polarization. For a free fermion, the polarization vector is given by

$$\langle \mathbf{s} \rangle = \frac{3}{8\varepsilon_F^0}\omega. \quad (140)$$

The corresponding contribution to the moment of inertia is

$$\mathcal{I}_s = \mathcal{I}_s^{F\cdot g} \equiv \frac{3A}{8\varepsilon_F^0} \approx 0.68A^{-2/3}\mathcal{I}_{rig}\ (\beta = 0). \quad (141)$$

Pik-Pichak showed for the first time[156] that the two-nucleon spin-orbit forces for a nonzero rotor of the velocity field (129) lead to an exchange term in the single-particle potential of the rotating nucleus. This term has the same structure as the field V_s^{cor}, differing by a numerical factor[114] of order 1.2. As a result, the expression (141) for the moment of inertia must be multiplied by a factor of order 2.2.

One further correction to the result (141) arises when allowance is made for the following term in the two-nucleon forces:

$$-\beta(\mathbf{s}_1 \cdot \mathbf{s}_2). \quad (142)$$

Taking into account both these terms and the spin–orbit interaction, we obtain

$$\mathcal{I}_s \approx 2.2\mathcal{I}_s^{F\cdot g}/(1 + 3\beta/8\varepsilon_F^0). \quad (143)$$

The value of the constant β, which must be substituted in (143), is very uncertain. In Ref. 114, the result $\mathcal{I}_s \approx 3.3\mathcal{I}_s^{F\cdot g}$ is given, this being based on the value of β taken from Skyrme's effective interactions[157] (variant III). However, the estimates made in Ref. 114 are strongly simplified since they do not take into account the Pauli principle. If it is properly taken into account and β is estimated on the basis of the widely used two-nucleon interactions, we obtain for $\mathcal{I}_s/\mathcal{I}_s^{F\cdot g}$ a value in the range

$$0.4 \leqslant \mathcal{I}_s/\mathcal{I}_s^{F\cdot g} \leqslant 2. \quad (144)$$

The contribution to the spin–orbit potential from the two-nucleon spin–orbit and spin–spin interactions can be represented as

$$\delta\tilde{V}_s = -(\mathcal{I}_s/\mathcal{I}_s^{F\cdot g} - 1)\,\omega \cdot \mathbf{s}. \quad (145)$$

In connection with the result (142), it should be said that if a quasiclassical approximation of higher order[158] is used for the density $\tilde{\rho}$ of a noninteracting Fermi gas, then one can obtain a correction to the rigid-body moment of inertia that has a similar functional form, but with a different numerical factor:

$$\mathcal{I}_{dia} = -A/(2\varepsilon_F^0) = -0.9A^{-2/3}\mathcal{I}_{rig}\ (\beta = 0). \quad (146)$$

This contribution is analogous[159] to the diamagnetic susceptibility of an electron gas introduced by Landau[160] and is due to the curving of the classical orbits in the magnetic field. Since the correction term $\mathcal{I}_{dia}$ arises as a result of the orbital part of the motion of the nucleons, the spin-dependent part of the two-nucleon forces does not affect the estimate of it.

The structure of the spin-orbit potential in the rotating nucleus. Repeating the usual derivations of the single-particle spin–orbit potential on the basis of the two-nucleon spin–orbit forces, but taking into account the finiteness of the mean flow of nucleons given by (129), and also the spin polarization

$$\langle \mathbf{s} \rangle = \mathcal{I}_s \omega / A, \quad (147)$$

we can readily conclude that the ordinary expression for the spin–orbit potential in a spherical nucleus must be transformed to

$$\tilde{V}_{s0} = k\left(\frac{1}{r}\frac{d\rho(r)}{dr}\right)(\mathbf{l} - \mathbf{r} \times \langle \mathbf{p} \rangle_r)(\mathbf{s} + \langle \mathbf{s} \rangle), \quad (148)$$

where k is the ordinary coefficient of the spin–orbit potential.

To obtain a general impression of the relative importance of the different correction terms that appear in (148), we note that on the equator of the nucleus we can give an estimate for the quantity

$$|\mathbf{r} \times \langle \mathbf{p} \rangle_r| = (5/2)\, I/A, \quad (149)$$

which has a numerical value of order $1\hbar$ for $A = 170$ and $I = 80\hbar$. For the same values of A and I and for $\mathcal{I}_s \approx \mathcal{I}_s^{F\cdot g}$, we have $|\langle s \rangle| \approx 10^{-2}\hbar$. Since the potential (148) vanishes after averaging over the local Fermi sphere, it does not contribute to the moment of inertia.

The investigations made above make it possible to estimate the accuracy of the results. Unfortunately, the experiments are still in a state in which it is too early to speak of a direct verification of the calculated corrections.

CONCLUSIONS

The theoretical estimates and also the few available experimental data (see Sec. 5) make it possible to de-

termine the limiting values of the angular momentum that quasistationary nuclear states can have. The critical angular momenta in the most strongly bound nuclei reach about $I_{cr} \approx 10^2\hbar$, and the exact values of I_{cr} are determined by quantum effects. Study of the reaction of nuclei to rotation in the region of angular momenta from zero to I_{cr} reveals very varied phenomena: In nuclei, the correlations of superconducting type disappear when the angular momentum of the state reaches certain values which are critical for pairing and approximately equal to $(20-30)\hbar$ (see Sec. 2). The spherical symmetry in the density distribution in the nuclei in which it is present for $I=0$ is lost at values of I which are sufficiently large to reduce the kinetic energy (see Secs. 4 and 5), and the same happens to axial symmetry (see Sec. 5). However, in some nuclei axial symmetry can appear once more when even larger angular momenta are reached, and it is possible that the collective rotation in such a case takes place around the axis of axial symmetry, and not perpendicular to it, as is the case at small angular momenta (see Sec. 5). Predictions of this kind are related to a further characteristic feature of rotating bodies: Their energy depends on not only the shape and the distribution of the mass over the volume, but also on the velocity distribution of the matter within the body. Judging from the results of theoretical investigations (see Secs. 3 and 5), the orientation of the angular momentum in the internal coordinate system of rotating nuclei is changed in a very complicated manner in the states of the yrast band when the angular momentum is varied. Experimental information on this question is very scanty, and theory too has made only the first steps toward studying the velocity distribution of the nucleons in rotating nuclei.

Rotation also influences the size of nuclei. Although in this respect the experimental information is extremely scanty, both theory and experiment indicate the existence of interesting effects associated with a change in the nuclear volume as a result of rotation: Nuclei with strong pairing correlations can decrease their volume as the angular momentum is increased from zero to the values critical for pairing, and then expand with a further increase in I (see Sec. 4).

A new form of nuclear isomerism and phase transitions with change of shape is associated with rotation. Centrifugal forces make states with large deformation advantageous since the moment of inertia increases with the deformation. Detailed calculations and numerous experimental investigations give a consistent picture indicating that in one and the same nucleus, properties of objects having different shapes coexist. Examples of such coexistence of shapes are fairly numerous in the region of small and intermediate spins (see Sec. 4). With regard to states with large angular momenta, the problem of nuclear isomerism still remains virtually uninvestigated by the experimentalists, while the theoretical predictions in this field are still not yet very definite (see Sec. 5).

An important feature of the theoretical calculations relating to all these effects is that they are based on models which are virtually free of adjustable parameters other than the ones used to analyze the internal excitations of nuclei. As a whole, the conclusions of theory and experiment about the reaction of nuclei to rotation are in agreement, and this a good confirmation that the existing nuclear models are valid. Certain "fine" details in the large body of experimental information relating to rotation of nuclei do not agree with the theoretical estimates. Further investigation of these details is very helpful for extracting additional information about the residual interaction of nucleons.

It is conceivable that in the near future it will become possible to make a much more detailed comparison of theory and the experimental data on nuclear rotation, in particular in the region of large angular momenta. We hope that the material in this review will be helpful for such a comparison.

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[1]K. Alder *et al.*, Rev. Mod. Phys. **28**, 432 (1956).
[2]J. O. Newton, in: Nucl. Spectroscopy and Reactions (Ed. G. Cerny), Part C, Academic Press, New York (1947), p. 185.
[3]D. Chultem, in: Izbrannye Voporsy Struktury Yadra (Selected Problems in Nuclear Structure), Vol. 2, JINR, Dubna (1976), p. 344.
[4]Å. Bohr and B. R. Mottelson, Nuclear Structure, Vols. I and II, Benjamin, New York (1969, 1975) [Russian translation of Vol. I published by Mir, Moscow, 1971].
[5]Å. Bohr, Rev. Mod. Phys. **48**, 365 (1976).
[6]A. Ya. Dzyublin *et al.*, Yad. Fiz. **15**, 869 (1972) [Sov. J. Nucl. Phys. **15**, 487 (1972)].
[7]G. F. Filippov, Fiz. Elem. Chastits At. Yadra **4**, 992 (1973) [Sov. J. Part. Nucl. **4**, 405 (1974)].
[8]V. V. Vanagas, Lektsii Shkoly MIFI (Lectures at the School Organized by the Moscow Engineering Physics Institute], MIFI, Moscow (1974).
[9]A. I. Baz' and G. F. Filippov, in: Izbrannye Voprosy Struktury Yadra (Selected Problems in Nuclear Structure), Vol. 2, JINR, Dubna (1976), p. 268.
[10]N. N. Bogolyubov, B. V. Tolmachev, and D. V. Shirkov, Novyĭ Metod v Teorii Sverkhprovodimosti (A New Method in the Theory of Superconductivity, New York, 1959), Izd-vo Akad. Nauk SSSR, Moscow (1958).
[11]V. G. Solov'ev, Teoriya Slozhnykh Yader (Theory of Complex Nuclei, Pergamon Press, Oxford, 1976), Nauka, Moscow (1971).
[12]J. Rainwater, Phys. Rev. **79**, 432 (1950).
[13]Å. Bohr, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. **26**, No. 14 (1952); Å. Bohr and B. R.Mottelson, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. **27**, No. 16 (1953).
[14]D. R. Inglis, Phys. Rev. **96**, 1059 (1954); **97**, 701 (1955); **103**, 1786 (1955).
[15]L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics, third edition, Pergamon Press, Oxford, 1976), Nauka, Moscow (1973).
[16]S. T. Belyaev, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. **31**, No. 11 (1959).
[17]S. G. Nilsson and O. Prior, K. Dan. Vidensk. Selsk., Mat.-Fys. Medd. **32**, No. 16 (1961).
[18]E. R. Marshalek, Phys. Rev. **139**, B770 (1965).
[19]M. Rich, Nucl. Phys. A **90**, 407 (1967).

[20]M. Brack *et al.*, Nucl. Phys. A **234**, 185 (1974).
[21]J. Krumlinde, Nucl. Phys. A **121**, 306 (1968); **160**, 471 (1971).
[22]C. W. Ma and C. F. Tsang, Phys. Rev. C **11**, 213 (1975).
[23]A. B. Migdal, Zh. Eksp. Teor. Fiz. **37**, 249 (1959) [Sov. Phys. JETP **10**, 176 (1960)].
[24]D. J. Thouless, Nucl. Phys. **21**, 225 (1960).
[25]S. T. Belyaev, Zh. Eksp. Teor. Fiz. **40**, 672 (1960) [Sov. Phys. JETP **13**, 470 (1961)].
[26]D. J. Thouless and J. G. Valatin, Nucl. Phys. **31**, 211 (1962)
[27]N. N. Bogolyubov, Zh. Eksp. Teor. Fiz. **34**, 58, 73 (1958) [Sov. Phys. JETP **7**, 41, 51 (1958)]; Dokl. Akad. Nauk SSSR **119**, 244 (1958) [Sov. Phys. Dokl. **3**, 292 (1958)].
[28]N. N. Bogolyubov, Usp. Fiz. Nauk **67**, 549 (1959) [Sov. Phys. Usp. **2**, 236 (1959)].
[29]B. Banerjee, P. Ring, and H. J. Mang, Nucl. Phys. A **221**, 564 (1974).
[30]A. L. Goodman, Nucl. Phys. A **230**, 466 (1974).
[31]E. R. Marshalek, Phys. Rev. **158**, 993 (1967); Phys. Rev. Lett. **20**, 214 (1968).
[32]D. Karadzhov *et al.*, Preprint [in Russian], JINR R4-8965, Dubna (1975); Yad. Fiz. **21**, 964 (1975); **24**, 888 (1976) [Sov. J. Nucl. Phys. **21**, 495 (1975); **24**, 464 (1976)].
[33]M. I. Shtokman, Preprint [in Russian], IYaF 74-43, Novosibirsk (1974).
[34]B. Banerjee, H. J. Mang, and P. Ring, Nucl. Phys. A **213**, 366 (1974).
[35]A. Faessler *et al.*, in: Proc. Intern. Symp. on Highly Excited States in Nuclei, Vol. 1, Jülich (1975), p. 58.
[36]B. L. Birbrair, Nucl. Phys. A **212**, 27 (1973).
[37]J. Meyer, J. Speth, and J. H. Vogeler, Nucl. Phys. A **193**, 60 (1972).
[38]J. Meyer and J. Speth, Nucl. Phys. A **203**, 17 (1973); J. Meyer and J. Speth, Yad. Fiz. **17**, 1197 (1973) [Sov. J. Nucl. Phys. **17**, 623 (1973)].
[39]Yu. T. Grin', Phys. Lett. B **52**, 135 (1974).
[40]P. C. Bhargava and D. J. Thouless, Nucl. Phys. A **215**, 515 (1972).
[41]A. L. Goodman, Nucl. Phys. A **265**, 113 (1976).
[42]S. T. Belyaev, in: Selected Topics in Nuclear Structure, Vienna (1962).
[43]I. Hamamoto, Nucl. Phys. A **232**, 445 (1974); Preprint, Nordita 76/11 (1976).
[44]D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).
[45]R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. A **70**, 381 (1957).
[46]J. Yoccoz, Proc. Phys. Soc. A **70**, 388 (1957).
[47]T. H. R. Skyrme, Proc. Phys. Soc. A **70**, 433 (1957).
[48]F. Gruemmer, K. W. Schmid, and A. Faessler, Nucl. Phys. A **239**, 289 (1975).
[49]S. Frauendorf, D. Janssen, and L. Münchow, Yad. Fiz. **12**, 939 (1970); **13**, 60 (1971) [Sov. J. Nucl. Phys. **12**, 512 (1971); **13**, 33 (1971)].
[50]A. Kamlah, Z. Phys. **216**, 52 (1968); P. Ring, R. Beck, and H. J. Mang, Z. Phys. **231**, 10 (1970); **231**, 26 (1970).
[51]I. N. Mikhaĭlov, É. Nadzhakov, and D. Karadzhov, Fiz. Elem. Chastits At. Yadra **4**, 311 (1973) [Sov. J. Part. Nucl. **4**, 129 (1973)].
[52]I. N. Mikhaĭlov and D. Janssen, Preprint [in Russian], JINR R4-10333, Dubna (1977); Izv. Akad. Nauk SSSR, Ser. Fiz. (1977).
[53]J. P. Elliott, Proc. Roy. Soc. A **245**, 128, 562 (1958).
[54]I. N. Mikhailov and E. Nadzhakov, Compt. Rend. Acad. Sci. Bulgar. **22**, 1221 (1969); Preprint [in Russian], JINR R4-4293, Dubna (1969).
[55]A. K. Kerman and A. Klein, Phys. Lett. **1**, 185 (1962); Phys. Rev. **132**, 1326 (1963); A. Klein and A. Kerman, Phys. Rev. B **138**, 1323 (1965); R. M. Dreizler and A. Klein, Phys. Lett. B **30**, 236 (1969); A. Klein, L. Celenza, and A. K. Kerman, Phys. Rev. B **140**, 245 (1966).
[56]S. T. Belyaev and V. G. Zelevinskiĭ, Yad. Fiz. **16**, 1195 (1972); **17**, 525 (1973) [Sov. J. Nucl. Phys. **16**, 657 (1973); **17**, 269 (1973)].
[57]V. G. Zelevinskiĭ and M. I. Shtokman, Izv. Akad. Nauk SSSR, Ser. Fiz. **36**, 2577 (1972).
[58]A. M. Lane, Nuclear Theory: Pairing Force Correlation and Collective Motion, Benjamin (1964) (Russian translation published by Atomizdat, Moscow, 1967).
[59]V. G. Zelevinskiĭ, Mikroskopicheskaya Teoriya Kollektivnykh Sostoyaniĭ (Microscopic Theory of Collective States), Moscow Engineering Physics Institute (1973); Yadernoe Vrashchenie i Vysokie Yadernye Sostoyaniya (Nuclear Rotation and High Nuclear States), Moscow Engineering Physics Institute (1974).
[60]S. Frauendorf, D. Janssen, and L. Münchow, Nucl. Phys. A **125**, 369 (1969).
[61]V. M. Kolomiets, Yad. Fiz. **22**, 269 (1975) [Sov. J. Nucl. Phys. **22**, 139 (1976)].
[62]C. W. Ma and J. O. Rasmussen, Phys. Rev. C **2**, 798 (1970).
[63]R. A. Sorenson, Rev. Mod. Phys. **45**, 353 (1973).
[64]A. Johnson and Z. Symański, Phys. Rep. **7C**, 181 (1973).
[65]S. Frauendorf, Nucl. Phys. A **263**, 150 (1976).
[66]F. S. Stephens *et al.*, Nucl. Phys. **222**, 235 (1974).
[67]Yu. T. Grin', Fiz. Elem. Chastits At. Yadra **6**, 1105 (1975) [Sov. J. Part. Nucl. **6**, 446 (1975)].
[68]D. R. Bes, S. Landowne, and M. A. J. Mariscotti, Phys. Rev. **166**, 1045 (1968).
[69]M. Sano and M. Wakai, Nucl. Phys. A **97**, 298 (1967).
[70]B. R. Mottelson and J. G. Valatin, Phys. Rev. Lett. **5**, 511 (1960).
[71]K. Y. Chan and J. G. Valatin, Nucl. Phys. **82**, 222 (1966); **85**, 261 (1966).
[72]A. Faessler, L. Lin, and F. Wittman, Phys. Lett. B **44**, 127 (1972); A. Faessler *et al.*, Phys. Lett. B **48**, 87 (1974).
[73]K. Kumar, Phys. Scripta **6**, 270 (1972); Phys. Rev. Lett. **30**, 1227 (1973).
[74]V. Yu. Keloglu, N. I. Pyatov, and M. I. Chernei, in: Yadernaya Spektroskopiya i Teoriya Yadra. Tezisy Dokladov XII Soveshchaniya po Yadernoĭ Spektroskopii (Nuclear Spectroscopy and Nuclear Theory. Abstracts of Papers at 12th Symposium on Nuclear Spectroscopy, Dubna, 1971), JINR, D6-5783 (1971).
[75]Yu. T. Grin' and A. I. Larkin, Yad. Fiz. **2**, 40 (1966) [Sov. J. Nucl. Phys. **2**, 27 (1964).
[76]J. Krumlinde and A. Szymański, Ann. Phys. **79**, 201 (1973).
[77]R. S. Simon *et al.*, Phys. Rev. Lett. **36**, 359 (1976).
[78]R. J. Blin-Stoyle, Fundamental Interactions and the Nucleus, North-Holland, Amsterdam (1973) (Russian translation published by Mir, Moscow, 1976).
[79]P. G. Hansen, O. B. Nielsen, and R. K. Sheline, Nucl. Phys. **12**, 389 (1959).
[80]P. O. Lipas, Nucl. Phys. **39**, 468 (1962).
[81]V. M. Mikhaĭlov, Izv. Akad. Nauk SSSR, Ser. Fiz. **30**, 1334 (1966).
[82]I. N. Mikhaĭlov, Soobshchenie (Communication), JINR R4-7862, Dubna (1974).
[83]D. Janssen, Yad. Fiz. **25**, 897 (1977) [Sov. J. Nucl. Phys. **25**, 479 (1977)].
[84]F. R. Mai *et al.*, Preprint JINR E4-10508 (1977).
[85]P. J. Brussaard and H. A. Tolhoek, Physica **23**, 955 (1957).
[86]R. J. Turner and T. Kishimoto, Nucl. Phys. A **217**, 317 (1973).
[87]N. Rud, H. L. Nielsen, and K. Wilsky, Nucl. Phys. A **167**, 401 (1971).
[88]T. A. Siddiqi and G. T. Emergy, Phys. Rev. C **6**, 1009 (1972).
[89]H. A. Baader, Dissertation, Technische Universität München (1970).
[90]C. Günther *et al.*, Nucl. Phys. A **172**, 273 (1972).
[91]K. Kumar and M. Baranger, Nucl. Phys. A **92**, 608 (1967).
[92]M. Sakai, At. Data **15**, 513 (1975).
[93]P. H. Stelson and L. Grodzins, Nucl. Data A **1**, 21 (1965).
[94]J. H. Hamilton, Izv. Akad. Nauk SSSR, Ser. Fiz. **36**, 17

(1972).
[95]D. Ward *et al.*, in: Proc. Intern. Conf. Properties of Nuclear States, Univ. of Montreal, Montreal (1969); Contribution No. 234.
[96]J. M. Domingos, G. D. Symons, and A. C. Douglas, Nucl. Phys. A **180**, 600 (1972).
[97]C. Gunther and D. R. Parsignault, Phys. Rev. **153**, 1297 (1967).
[98]D. C. Camp and F. M. Bernthal, Phys. Rev. C **6**, 1040 (1972).
[99]H. Ejiri and G. B. Hagemann, Nucl. Phys. A **161**, 449 (1971).
[100]W. W. Bowman and T. T. Sugihara, Nucl. Phys. A **178**, 263 (1971).
[101]T. L. Khoo *et al.*, Nucl. Phys. A **202**, 289 (1973).
[102]T. Hammer, H. Ejiri, and G. B. Hageman, Nucl. Phys. A **202**, 321 (1973).
[103]M. J. Canty *et al.*, Nucl. Phys. A **203**, 421 (1973).
[104]W. T. Milner *et al.*, Nucl. Phys. A **177**, 1 (1971).
[105]E. Monnand, J. Blachot, and A. Moussa, Nucl. Phys. A **134**, 321 (1969).
[106]B. Fogelberg, Nucl. Phys. A **197**, 497 (1972).
[107]R. F. Casten *et al.*, Phys. Rev. **187**, 1532 (1969).
[108]B. Bochev, L. Aleksandrov, and T. Kutsarova, Preprints [in Russian], JINR R5-7881, R5-8321, Dubna (1974).
[109]B. Bochev *et al.*, Yad. Fiz. **16**, 633 (1972) [Sov. J. Nucl. Phys. **16**, 355 (1973)]; Phys. Scr. **6**, 243 (1973).
[110]J. Speth *et al.*, "Isomer shifts and nuclear models," in: Mössbauer Isomer Shifts (Eds. G. K. Shenoy and F. W. Wagner), North-Holland, Amsterdam (1977).
[111]D. Karadjov, I. N. Mikhaĭlov, and J. Piperova, Phys. Lett. B **46**, 163 (1973).
[112]J. Meyer, Nucl. Phys. A **137**, 193 (1969).
[113]K. Neergård and V. V. Pashkevich, Phys. Lett. B **59**, 218 (1975).
[114]K. Neergård, V. V. Pashkevich, and S. Frauendorf, Nucl. Phys. A **262**, 61 (1976).
[115]S. Frauendorf and V. V. Pashkevich, Phys. Lett. B **55**, 365 (1975).
[116]J. Bonn *et al.*, Z. Phys. A **276**, 203 (1976).
[117]A. Faessler *et al.*, Phys. Lett. B **39**, 579 (1972).
[118]S. G. Nilsson, Nucl. Phys. A **222**, 221 (1974); M. Cailliau *et al.*, Phys. Lett. B **46**, 11 (1973).
[119]F. Dickmann and K. Dietrich, Z. Phys. **271**, 417 (1974).
[120]D. Kolb and C. Y. Wong, Nucl. Phys. A **245**, 205 (1975).
[121]N. Rud *et al.*, Phys. Rev. Lett. **31**, 1421 (1973).
[122]D. Proetel *et al.*, Phys. Rev. Lett. **31**, 896 (1973).
[123]J. H. Hamilton *et al.*, Phys. Rev. Lett. **35**, 562 (1975).
[124]R. Béraud *et al.*, J. Phys. (Paris) **36**, Colloque C5, 101 (1975).
[125]J. D. Cole *et al.*, Phys. Rev. Lett. **37**, 1185 (1976).
[126]J. H. Hamilton, in: Izbrannye Voprosy Struktury Yadra (Selected Problems in Nuclear Structure), Vol. 2, JINR, Dubna (1976), p. 303.
[127]F. R. May, V. V. Pashkevich, and S. Frauendorf, Preprint [in Russian], JINR R4-10173, Dubna (1976); Phys. Lett. B **68**, 113 (1977).
[128]S. Cohen, F. Plasil, and W. J. Swiatecki, Ann. Phys. **82**, 557(1974).
[129]G. A. Pik-Pichak, Zh. Eksp. Teor. Fiz. **34**, 341 (1958) [Sov. Phys. JETP **7**, 238 (1958).
[130]G. A. Pik-Pichak, Zh. Eksp. Teor. Fiz. **42**, 1294 (1962) [Sov. Phys. JETP **15**, 897 (1962)].
[131]G. A. Pik-Pichak, Zh. Eksp. Teor. Fiz. **43**, 1701 (1962) [Sov. Phys. JETP **16**, 1201 (1963)].
[132]R. Beringer and W. J. Knox, Phys. Rev. **121**, 1195 (1961).
[133]J. R. Hiskes, Report UCRL-9275 (1960).
[134]D. Sperber, Phys. Rev. **130**, 468 (1963).
[135]B. C. Carlson and Pao Lu, in: Proc. Rutherford Jubilee Intern. Conf. Manchester, 1961 (Ed. J. B. Birks), L. Heywood and Co. (1962), p. 291.
[136]S. Copen, F. Plasil, and W. J. Swiatecki, in: Proc. Third Conf. on Reactions between Complex Nuclei (Eds. A. Ghiorso, R. M. Diamond, and H. E. Conzett), Asilomar University of California Press (1963), p. 326.
[137]S. Chandrasekhar, Proc. R. Soc. London, Ser. B **286**, 1 (1964).
[138]C. E. Rosenkilde, J. Math. Phys. **8**, 98 (1967).
[139]R. Bengtsson *et al.*, Phys. Lett. B **57**, 301 (1975).
[140]G. Andersson *et al.*, Nucl. Phys. A **268**, 205 (1975).
[141]V. M. Strutinsky, Nucl. Phys. A **95**, 420 (1967).
[142]V. M. Strutinsky, Nucl. Phys. A **122**, 1 (1968).
[143]M. Brack *et al.*, Rev. Mod. Phys. **44**, 320 (1972).
[144]V. V. Pashkevich and S. Frauendorf, Yad. Fiz. **20**, 1122 (1974) [Sov. J. Nucl. Phys. **20**, 588 (1975)].
[145]K. Neergård *et al.*, Nucl. Phys. A **287**, 48 (1977).
[146]V. M. Strutinsky, Nucl. Phys. A **218**, 169 (1974).
[147]D. A. Arsen'ev, L. A. Malov, V. V. Pashkevich, and V. G. Solov'ev, Izv. Akad. Nauk SSSR, Ser. Fiz. **32**, 866 (1968).
[148]D. A. Arsen'ev *et al.*, Yad. Fiz. **8**, 883 (1968) [Sov. J. Nucl. Phys. **8**, 514 (1969)].
[149]Å. Bohr and B. R. Mottelson, Phys. Scr. A **10**, 13 (1974).
[150]A. Faessler, M. Ploszajczak, and K. R. Sandhya Devi, Phys. Rev. Lett. **36**, 1028 (1976).
[151]H. Chandra and U. Mosel, Phys. Lett. B **64**, 373 (1976).
[152]A. Faessler *et al.*, Nucl. Phys. A **256**, 106 (1976).
[153]A. Faessler, R. R. Hilton, and K. R. Sandhya Devi, Phys. Lett. B **61**, 133 (1976).
[154]A. V. Migdal, Nucl. Phys. **13**, 655 (1959).
[155]M. V. Banashik *et al.*, Phys. Rev. Lett. **34**, 892 (1975).
[156]G. A. Pik-Pichak, Yad. Fiz. **6**, 722 (1967) [Sov. J. Nucl. Phys. **6**, 525 (1968)].
[157]D. Vautherin and D. M. Brink, Phys. Rev. C **5**, 626 (1972).
[158]M. Brack and B. K. Jennings, Nucl. Phys. A **258**, 264 (1976).
[159]J. Dabrowski, Phys. Lett. B **59**, 132 (1975).
[160]L. D. Landau, Z. Phys. **64**, 629 (1930).

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