Nuclear vertex constants

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The properties of the nuclear vertex constants, the relations between them and the nuclear wave functions and other quantities, and also their values for some light nuclei are reviewed.

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INTRODUCTION

In this review, we consider the properties and methods for determination of the nuclear vertex constants. Apart from kinematic factors, these are equal to the amplitudes of virtual or real decay (or fusion) of a nucleus a into two fragments b and c ($a \rightleftarrows b + c$) on the mass shell. The concept of nuclear vertex constants has taken a firm hold in theoretical nuclear physics during the last decade as this field has been penetrated by dispersion-theoretical methods based on the principles of unitarity and analyticity. These methods, which were first developed in particle physics, are now widely used to describe nuclear reactions, especially direct reactions and reactions in few-nucleon systems. In dispersion theory, a nuclear reaction is described as a sequence of certain intermediate virtual processes, an important role among these being played by the decay of nuclei into two fragments and the fusion of these fragments into the final reaction products.

The nuclear vertex constants are fundamental, model-independent quantities; in their physical meaning, they are analogous to the renormalized coupling constants in particle physics, and they have the same status as other primary characteristics of nuclei such as mass, spin, parity, and electromagnetic moments. To describe the properties of, say, a nucleon we require not only its mass, spin, parity, electric charge, and magnetic moment but also the constants of its coupling to mesons, which characterize the virtual emission of mesons by a nucleon. Similarly, a nucleus in a definite state is characterized by not only its mass, spin, etc., but also by nuclear vertex constants (one or several, depending on the spin kinematics of the decay).

In the framework of the traditional approach which regards the nucleus as a system of interacting nucleons described by a wave function that is a solution of a nonrelativistic Schrödinger equation, the nuclear vertex constant for the decay $a \to b + c$ is proportional to the coefficient in the asymptotic behavior of the relative-motion wave function of fragments b and c. Therefore, the nuclear vertex constant can be calculated in the framework of different nuclear models. On the other hand, there are now many methods available for estimating nuclear vertex constants on the basis of experimental data on elastic nuclear scattering and nuclear reactions. Comparison of the empirical and theoretical values of the nuclear vertex constants can give valuable information about the nuclear models employed, in particular about the form of nucleon–nucleon forces.

In the first section of the review, we discuss general properties of the amplitudes of virtual decays and the nuclear vertex constants and we establish the connection between the latter and other quantities that can be deduced from data on nuclear reactions, and we also establish the connection between the vertex constants and the nuclear wave functions. The discussion is based on the nonrelativistic formalism, which makes it possible to obtain general expressions that hold for arbitrary spins of the particles. The second section is devoted to describing different methods for finding the nuclear vertex constants. Finally, in the third section we gather together the available information about the constants, mainly for few-nucleon systems.

1. NUCLEAR VERTEX PARTS IN NONRELATIVISTIC THEORY

General properties of nuclear vertex parts and constants. In this subsection, we discuss the general properties of the three-leg vertex parts—the amplitudes of real or virtual decays of a nucleus into two nuclei—and we define the nuclear vertex constants. Our discussion is based on the nonrelativistic formalism, and we employ only the general principles of nonrelativistic invariance and the conservation laws. However, we assume that all particles interact through nuclear forces of finite range. The necessary modification of some of our assertions and expressions when allowance is made for the long-range Coulomb forces is discussed at the end of the section.

In what follows, we use this notation: m_i, $\mathbf{p}_i$, E_i, J_i, M_i, and $\mathbf{r}_i$ are the mass, momentum, kinetic energy, spin, spin projection, and radius vector of the center of mass of nucleus i;

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$$\left.\begin{aligned} &m_{ij}=m_i+m_j;\ \mu_{ij}=m_im_j/m_{ij};\\ &\varepsilon^i_{jk}=m_j+m_k-m_i=(\varkappa^i_{jk})^2/2\mu_{jk};\\ &\mathbf{q}_{ij}=(m_j\mathbf{p}_i-m_i\mathbf{p}_j)/m_{ij};\ \mathbf{v}_{ij}=\mathbf{q}_{ij}/q_{ij};\\ &E_{ij}=E_i+E_j-(\mathbf{p}_i+\mathbf{p}_j)^2/2m_{ij};\ \sigma_i=E_i-\mathbf{p}_i^2/2m_i;\ \hat{J}=2J+1; \end{aligned}\right\} \quad (1)$$

$\mathbf{q}_{ij}$ and E_{ij} are the momentum and kinetic energy of the relative motion of particles i and j. The quantities $\mathbf{q}_{ij}$, E_{ij}, and σ_i are invariants of the Galileo transformations. For real particles (on the mass shell) $\sigma_i=0$ and $E_{ij}=\mathbf{q}_{ij}^2/2\mu_{ij}$.

The state vector $|\mathbf{p}M\rangle$ of a free particle with momentum $\mathbf{p}$ and spin projection M is normalized by the condition

$$\langle \mathbf{p}'M'|\mathbf{p}M\rangle=(2\pi)^3\delta(\mathbf{p}'-\mathbf{p})\,\delta_{M'M}. \quad (2)$$

The amplitude $\mathscr{M}_{fi}$ of any process $i\to f$ is related to the element S_{fi} of the S matrix between states normalized in accordance with (2) by

$$S_{fi}=\delta_{fi}-i(2\pi)^4\mathscr{M}_{fi}\delta(\mathscr{P}_f-\mathscr{P}_i)\,\delta(\mathscr{E}_f-\mathscr{E}_i-Q), \quad (3)$$

where $\mathscr{P}_i$ and $\mathscr{E}_i$ (respectively, $\mathscr{P}_f$ and $\mathscr{E}_f$) are the sums of the momenta and kinetic energies of the initial (respectively, final) particles; Q is the difference between the masses of the initial and final particles. Apart from individual expressions in which $\hbar$ and c are inserted explicitly, $\hbar=c=1$ everywhere.

We consider a decay (real or virtual) that takes place as a result of strong interactions:

$$a(\alpha_aJ_aM_a)\to b(\alpha_bJ_bM_b)+c(\alpha_cJ_cM_c), \quad (4)$$

where α_i are the additional quantum numbers needed for the complete specification of the state of nucleus i. On the basis of the principles of nonrelativistic invariance, the general expression for the decay amplitude can be written in the form

$$\mathscr{M}^{M_a}_{M_bM_c}=\sqrt{4\pi}\sum_{lsm_lm_s}G_{abc}(ls;\sigma_a,\sigma_b,\sigma_c)\,(lm_lsm_s|J_aM_a)\times(J_bM_bJ_cM_c|sm_s)\,Y_{lm_l}(\mathbf{v}_{bc}). \quad (5)$$

Here, $(j_1m_1j_2m_2|j_3m_3)$ is a Clebsch–Gordan coefficient; Y_{lm_l} is a normalized spherical function satisfying the condition $Y^*_{lm_l}(\nu)=(-1)^{m_l}Y_{l-m_l}(\nu)$; $G_{abc}(ls;\sigma_i)$ are invariant amplitudes—the nuclear vertex form factors, whose dependence on the quantum numbers α_iJ_i is not specified explicitly. By virtue of the invariance of the S matrix in nonrelativistic theory under Galileo transformations, rotations, and inversion of the spatial coordinates, the nuclear vertex form factors depend only on the invariant combinations of the energies and momenta of particles a, b, and c. If all three particles are off the mass shell, then the form factors depend on three invariants, which we have taken to be σ_a, σ_b, and σ_c.

The summation in (5) is extended to the values of the spin s and the orbital angular momentum l of the decay channel restricted by the laws of conservation of angular momentum and spatial parity:

$$\left.\begin{aligned} &|J_b-J_c|\leqslant s\leqslant J_b+J_c;\\ &|J_a-s|\leqslant l\leqslant J_a+s;\\ &i^l(-1)^l=\xi_{abc}, \end{aligned}\right\} \quad (6)$$

where $\xi_{abc}=\xi_a\xi_b\xi_c$, and ξ_i is the intrinsic parity of particle i. Counting (see, for example, Ref. 1) the number of s and l values satisfying the conditions (6), we find the number N of independent nuclear vertex form factors $G_{abc}(ls;\sigma_i)$ in (5):

$$N=\begin{cases}{}^1/_2N(J_aJ_bJ_c) & \text{for even}\quad N(J_aJ_bJ_c);\\ {}^1/_2[N(J_aJ_bJ_c)+\xi_{abc}(-1)^{J_a+J_b+J_c}] & \\ \qquad\text{for odd} & N(J_aJ_bJ_c),\end{cases} \quad (7)$$

where

$$N(J_aJ_bJ_c)=\begin{cases}(2J_a+1)(J_b+J_c-|J_b-J_c|+1) & \\ \qquad\text{for } J_a\leqslant|J_b-J_c|;\\ 1+J_a+J_b+J_c+2(J_aJ_b+J_aJ_c+J_bJ_c) & \\ -(J_a^2+J_b^2+J_c^2)\ \text{ for } |J_b-J_c|\leqslant J_a\leqslant J_b+J_c;\\ (2J_b+1)(2J_c+1)\ \text{ for } J_b+J_c\leqslant J_a.\end{cases} \quad (8)$$

If particles b and c are identical, then the amplitude (5) must be symmetric or antisymmetric under permutations of these particles if they are bosons or fermions, respectively. This requirement leads to the additional restriction

$$(-1)^s=\xi_a, \quad (9)$$

which, in general, reduces the number of independent form factors compared with the number given by (7).

Using the relation

$$\langle \mathbf{p}_aM_a|\mathscr{M}|\mathbf{p}_bM_b,\mathbf{p}_cM_c\rangle=(-1)^{J_a+M_a+J_b+M_b+J_c+M_c}\times\langle-\mathbf{p}_b-M_b,-\mathbf{p}_c-M_c|\mathscr{M}|-\mathbf{p}_a-M_a\rangle, \quad (10)$$

which follows from the invariance of the S matrix under time inversion,[2] for the amplitude of the fusion process

$$b(\alpha_bJ_bM_b)+c(\alpha_cJ_cM_c)\to a(\alpha_aJ_aM_a) \quad (11)$$

we obtain from (5) the expression

$$\mathscr{M}^{M_bM_c}_{M_a}=\sqrt{4\pi}\sum_{lsm_lm_s}G_{abc}(ls;\sigma_a,\sigma_b,\sigma_c)\times(lm_lsm_s|J_aM_a)\,(J_bM_bJ_cM_c|sm_s)\,Y^*_{lm_l}(\mathbf{v}_{bc}), \quad (12)$$

which differs from (5) only in the replacement of $Y_{lm_l}(\boldsymbol{\nu}_{bc})$ by $Y^*_{lm_l}(\boldsymbol{\nu}_{bc})$.

Sometimes, for example when the exchange pole diagram is considered (Fig. 3), it is more convenient to use for the decay amplitude (4) the expression

$$\mathscr{M}^{M_a}_{M_bM_c}=\sqrt{4\pi}\sum_{ljm_lm_j}G_{abc}(lj;\sigma_a,\sigma_b,\sigma_c)\,(jm_jJ_bM_b|J_aM_a)\times(lm_lJ_cM_c|jm_j)\,Y_{lm_l}(\mathbf{v}_{bc}), \quad (13)$$

which differs from (5) by the scheme for coupling the angular momenta. The form factors $G_{abc}(ls;\sigma_i)$ and $G_{abc}(lj;\sigma_i)$ are related by

$$G_{abc}(lj;\sigma_i)=\sum_s w_{sj}G_{abc}(ls;\sigma_i);\quad G_{abc}(ls;\sigma_i)=\sum_j w_{sj}G_{abc}(lj;\sigma_i),\qquad(14)$$

in which

$$w_{sj}=(-1)^{J_b+J_c-s}(\hat{s}\hat{j})^{1/2}W(J_bJ_cJ_al;sj);\qquad(15)$$

W is a Racah coefficient. From (14) and (15),

$$\sum_s[G_{abc}(ls;\sigma_i)]^2=\sum_j[G_{abc}(lj;\sigma_i)]^2;\quad\sum_s|G_{abc}(ls;\sigma_i)|^2=\sum_j|G_{abc}(lj;\sigma_i)|^2.\qquad(16)$$

If the effects that break the isotopic invariance of the nuclear forces are ignored, we can introduce explicitly in (5) the isospin Clebsch–Gordan coefficient $(T_bM_{T_b}T_cM_{T_c}|\,T_aM_{T_a})$, where T_i and M_{T_i} are the isospin of particle i and its projection.

In their physical meaning, the on-shell values of the nuclear vertex form factors

$$G_{abc}(ls)\equiv G_{abc}(ls;\sigma_a,\sigma_b,\sigma_c)|_{\sigma_a=\sigma_b=\sigma_c=0}\qquad(17)$$

are equivalent to the renormalized coupling constants of the "fields" a, b, and c.[3,4] We shall therefore call the $G_{abc}(ls)$ the nuclear vertex constants. They have dimensions of (length)$^{1/2}$ and will be measured in units of $F^{1/2}$.

Using the general principles of the S-matrix theory of strong interactions[4,5] and assuming that in the neighborhood of the pole on the physical sheet corresponding to a single-particle intermediate state all the scattering or reaction amplitudes are analytic and univalent (i.e., the pole is not on a cut), we can show that for the virtual decay of the nuclear-stable particle a into two nuclear-stable particles b and c the phase of the nuclear vertex constant is determined by the relation

$$G_{abc}(ls)=\pm i^l|G_{abc}(ls)|.\qquad(18)$$

In what follows, we shall say that the "normal" case holds if (18) is satisfied.

Connection between the nuclear vertex constants and other quantities deduced from data on nuclear reactions. In Sec. 2, we shall consider various dispersion methods for finding the nuclear vertex constants from experimental data on elastic nuclear scattering and nuclear reactions. These methods are based on the fact that the amplitudes and differential cross sections have poles with respect to the energy and the cosine of the scattering angle corresponding to single-particle intermediate states in the direct or cross channel. For unity and convenience of the exposition, we obtain here expressions in terms of the nuclear vertex constants for quantities determined from experiments by means of different dispersion methods. Note that the derivation of these expressions does not use any model assumptions about the reaction mechanism or the structure of the nuclei. The only restriction is that the nonrelativistic formalism is used.

First, we give some general expressions and definitions. The differential cross section of the reaction $1+2\to3+4$ for nonpolarized particles in the center-of-mass system can be expressed as follows in terms of the amplitude $\mathcal{M}$ normalized in accordance with (3):

$$d\sigma/d\Omega\equiv\sigma(E,z)=(\mu_{12}\mu_{34}/4\pi^2)(q_{34}/q_{12})\,\overline{|\mathcal{M}|^2};\qquad(19)$$

$$\overline{|\mathcal{M}|^2}\equiv(\hat{J}_1\hat{J}_2)^{-1}\sum_{M_1M_2M_3M_4}|\mathcal{M}_{M_1M_2M_3M_4}(\mathbf{q}_{12},\mathbf{q}_{34})|^2,\qquad(20)$$

where

$$E\equiv E_{12}=q_{12}^2/2\mu_{12};\quad z\equiv\cos\theta=\mathbf{v}_{12}\cdot\mathbf{v}_{34};\qquad(21)$$

E and θ are the kinetic energy of the colliding particles and the scattering angle in the center-of-mass system (θ is the angle between the momenta of particles 1 and 3). In particular, for elastic scattering $1+2\to1+2$,

$$d\sigma/d\Omega=(\mu_{12}^2/4\pi^2)\,\overline{|\mathcal{M}|^2},\qquad(22)$$

and the amplitude $\mathcal{M}$ and the scattering amplitude f^C in the center-of-mass system in the ordinary normalization are connected by

$$f^C_{M_1M_2M_1'M_2'}(\mathbf{q}_{12},\mathbf{q}_{12}')=-(\mu_{12}/2\pi)\,\mathcal{M}_{M_1M_2M_1'M_2'}(\mathbf{q}_{12},\mathbf{q}_{12}'),\qquad(23)$$

where $\mathbf{q}_{12}$ and $\mathbf{q}_{12}'$ are the relative momenta before and after scattering.

The amplitudes of forward scattering in the center-of-mass system, f^C, and in the laboratory system L_2, in which particle 1 with momentum $\mathbf{p}_1$ strikes the target particle 2 at rest, (this amplitude is denoted by f^L) are related by

$$f^L_{M_1M_2M_1'M_2'}(\mathbf{p}_1,\mathbf{p}_1')|_{\mathbf{p}_1'=\mathbf{p}_1}=(m_{12}/m_2)\,f^C_{M_1M_2M_1'M_2'}(\mathbf{q}_{12},\mathbf{q}_{12}')|_{\mathbf{q}_{12}'=\mathbf{q}_{12}},\qquad(24)$$

where $\mathbf{p}_1'$ is the momentum of particle 1 after the scattering.

We introduce the amplitudes of forward elastic scattering in the center-of-mass system and the system L_2 averaged over the spin projections; these occur in the dispersion relations for the forward scattering amplitudes[6] (see Sec. 2):

$$f^C_{12}(E_{12})=(\hat{J}_1\hat{J}_2)^{-1}\sum_{M_1M_2}f^C_{M_1M_2M_1M_2}(\mathbf{q}_{12},\mathbf{q}_{12});\qquad(25a)$$

$$f^L_{12}(E_1)=(\hat{J}_1\hat{J}_2)^{-1}\sum_{M_1M_2}f^L_{M_1M_2M_1M_2}(\mathbf{p}_1,\mathbf{p}_1),\qquad(25b)$$

where

$$E_1=(m_{12}/m_2)\,E_{12}=p_1^2/2m_1\qquad(26)$$

is the kinetic energy of particle 1 in the system L_2. The spin-averaged amplitudes $f^L_{12}(E_1)$ and $f^C_{12}(E_{12})$ are also related by Eq. (24). From (23), (24), and (25b) we obtain an expression for $f^L_{12}(E_1)$ in terms of the amplitude $\mathcal{M}$ in the center-of-mass system:

$$f^L_{12}(E_1)=-(m_1/2\pi\hat{J}_1\hat{J}_2)\sum_{M_1M_2}\mathcal{M}_{M_1M_2M_1M_2}(\mathbf{q}_{12},\mathbf{q}_{12}).\qquad(27)$$

From the condition of unitarity of the S matrix the well

known optical theorem follows:

$$\operatorname{Im} f^L_{12}(E_1) = (p_1/4\pi)\,\sigma_t(E_1), \tag{28}$$

where $\delta_t(E_1)$ is the total cross section for the interaction of particles 1 and 2 at the energy E_1.

We now turn to the derivation of expressions for the residues of the scattering amplitudes f^L at the direct and the exchange poles. We consider the process of elastic scattering $b+c \to b+c$ (b is the projectile and c the target), and we assume that the forward scattering amplitude averaged over the spins in the system L_c, i.e., $f^L_{bc}(E_b)$, has a "direct" pole in the E_b plane corresponding to the Feynman pole diagram (Fig. 1). This diagram describes scattering through the intermediate state a, which is a bound nuclear-stable state of the system $b+c$. Obviously, to find the residue of the amplitude $f^L_{bc}(E_b)$ at the direct pole it is sufficient to calculate the contribution to this amplitude from only the pole diagram. In accordance with the general rules[7,8] for expressing the amplitudes of nonrelativistic Feynman diagrams, the amplitude $\mathscr{M}$ of the diagram shown in Fig. 1 has the following form in the center-of-mass system:

$$\mathscr{M}_{M_b M_c M_b' M_c'}(\mathbf{q}_{bc}, \mathbf{q}_{bc}') = \sum_{M_a} \mathscr{M}^{M_b M_c}_{M_a} (\sigma_a + i\delta)^{-1} \mathscr{M}^{M_a}_{M_b' M_c'}, \quad \delta \to +0, \tag{29}$$

where σ_a is defined in (1), and $\mathscr{M}^{M_b M_c}_{M_a}$ and $\mathscr{M}^{M_a}_{M_b' M_c'}$ are the amplitudes of the virtual processes $b+c \to a$ and $a \to b'+c'$ ($b' \equiv b$, $c' \equiv c$) at the vertices of the diagram given by Eqs. (12) and (5), in which we must set $\sigma_b = \sigma_c = \sigma_{b'} = \sigma_{c'} = 0$ since the initial and final particles are on the mass shell. We can express σ_a in terms of the kinetic energy E_b of the incident particle b in the system L_c:

$$\sigma_a = (m_c/m_a)(E_b - E_b^{\mathrm{dir}}), \tag{30}$$

where

$$E_b^{\mathrm{dir}} = -(m_a/m_c)\,\varepsilon^a_{bc} \tag{31}$$

is the position of the direct pole of the amplitude $f^L_{bc}(E_b)$ in the E_b plane. Determining the residue at the direct pole by the usual relation

$$r^{\mathrm{dir}}_{abc} = \lim_{E_b \to E_b^{\mathrm{dir}}} (E_b - E_b^{\mathrm{dir}})\, f^L_{bc}(E_b) \tag{32}$$

and using Eqs. (29), (30), (12), (5), (27), and the orthogonality relations of the Clebsch–Gordan coefficients, we obtain

$$r^{\mathrm{dir}}_{abc} = -\frac{m_a m_b}{2\pi m_c} \frac{\hat{J}_a}{\hat{J}_b \hat{J}_c} \sum_{ls} G^2_{abc}(ls), \tag{33}$$

where the summation is over l and s values restricted

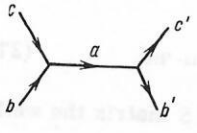

FIG. 1.

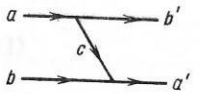

FIG. 2.

by the conditions (6). In the normal case, the sign of the residue is determined in accordance with (6) and (18) by the intrinsic parities of particles a, b, and c:

$$\operatorname{sign} r^{\mathrm{dir}}_{abc} = -\operatorname{sign} \xi_{abc}. \tag{34}$$

We now consider the elastic scattering process $b+a \to b+a$ (b is the projectile and a the target) and find the residue of the spin-averaged amplitude of forward scattering $f^L_{ba}(E_b)$ in the system L_a at the exchange pole corresponding to the Feynman pole diagram in Fig. 2. This diagram describes the scattering of particles b and a by the exchange of the nuclear-stable particle c. In the center-of-mass system, the amplitude $\mathscr{M}$ of the diagram in Fig. 2 has the form

$$\mathscr{M}_{M_b M_a M_b' M_a'}(\mathbf{q}_{ba}, \mathbf{q}_{ba}') = (-1)^{2J_b} \sum_{M_c} \mathscr{M}^{M_a}_{M_b' M_c} (\sigma_c + i\delta)^{-1} \mathscr{M}^{M_b M_c}_{M_a'}, \tag{35}$$

where $\mathscr{M}^{M_a}_{M_b' M_c}$ and $\mathscr{M}^{M_b M_c}_{M_a'}$ are given as before by Eqs. (5) and (12). Compared with the amplitude of the nonexchange pole diagram (see Fig. 1) the amplitude of the exchange diagram (see Fig. 2) contains the additional phase factor $(-1)^{2J_b}$, which arises because the particles are identical. The presence of this factor can be proved in different ways, for example, by using the standard formalism of multiparticle scattering theory[9] with explicit allowance for the indistinguishability of the nucleons (see, for example, Ref. 10 and also Ref. 11, which gives the final working expressions for taking into account these effects) or by regarding first the exchange pole as a direct pole in the cross channel $\bar{b}+a \to \bar{b}+a$ and noting that the intrinsic parities of particle b and its antiparticle $\bar{b}$ are the same (respectively opposite) if b is a boson (respectively fermion).[12] For forward scattering, σ_c in (35) can be expressed in terms of the kinetic energy E_b of particle b in the system L_a:

$$\sigma_c = -(m_a/m_c)(E_b - E_b^{\mathrm{exch}}), \tag{36}$$

where

$$E_b^{\mathrm{exch}} = -(m_c/m_a)\,\varepsilon^a_{bc} \tag{37}$$

is the position of the exchange pole of the amplitude $f^L_{ba}(E_b)$ in the E_b plane. Determining the residue of $f^L_{ba}(E_b)$ by

$$r^{\mathrm{exch}}_{abc} = \lim_{E_b \to E_b^{\mathrm{exch}}} (E_b - E_b^{\mathrm{exch}})\, f^L_{ba}(E_b) \tag{38}$$

and using Eqs. (35), (36), (5), (12), and (27), we obtain

$$r^{\mathrm{exch}}_{abc} = (-1)^{2J_b} \frac{\mu_{bc}}{2\pi \hat{J}_b} \sum_{ls} G^2_{abc}(ls), \tag{39}$$

where the summation is over the l and s values restricted by the conditions (6). In the normal case (18),

$$\operatorname{sign} r^{\mathrm{exch}}_{abc} = \operatorname{sign}[(-1)^{2J_b} \xi_{abc}]. \tag{40}$$

Note that by virtue of (16) the right-hand sides of Eqs. (33) and (39) can be expressed in terms of the nuclear vertex constants $G_{abc}(lj)$. From (33) and (39), we obtain the ratio of the residues r^{exch}_{abc} and r^{dir}_{abc} of the spin-averaged amplitudes $f^{L}_{ba}(E_b)$ and $f^{L}_{bc}(E_b)$ of forward elastic ba and bc scattering:

$$r^{\text{exch}}_{abc}/r^{\text{dir}}_{abc}=(-1)^{2J_b+1}\, m_c^2\hat{J}_c/m_a^2\hat{J}_a. \tag{41}$$

This ratio is determined by purely kinematic factors.

The residues ρ^{dir}_{abc} and ρ^{exch}_{abc} of the spin-averaged amplitudes $f^{C}_{bc}(E_{bc})$ and $f^{C}_{ba}(E_{ba})$ of forward elastic bc and ba scattering in the center-of-mass systems at the direct pole (see Fig. 1) and the exchange pole (see Fig. 2), respectively, which are defined by

$$\left.\begin{aligned}
&\rho^{\text{dir}}_{abc}=\lim_{E_{bc}\to E^{\text{dir}}_{bc}}(E_{bc}-E^{\text{dir}}_{bc})\, f^{C}_{bc}(E_{bc}),\ E^{\text{dir}}_{bc}=-\varepsilon^{a}_{bc};\\
&\rho^{\text{exch}}_{abc}=\lim_{E_{ba}\to E^{\text{exch}}_{ba}}(E_{ba}-E^{\text{exch}}_{ba})\, f^{C}_{ba}(E_{ba});\\
&E^{\text{exch}}_{ba}=-(m_c/m_{ba})\,\varepsilon^{a}_{bc},
\end{aligned}\right\} \tag{42}$$

can be expressed by means of (24) and (26) in terms of the residues of the corresponding amplitudes in the L systems:

$$\left.\begin{aligned}
&\rho^{\text{dir}}_{abc}=(m_c/m_{bc})^2\, r^{\text{dir}}_{abc};\\
&\rho^{\text{exch}}_{abc}=(m_{ba}/m_a)^2\, r^{\text{exch}}_{abc}.
\end{aligned}\right\} \tag{43}$$

We find an expression in terms of the nuclear vertex constants for

$$R^{\text{exch}}_{abc,\,ABc}=\lim_{z\to z_p}(z-z_p)^2\,\sigma(E,z), \tag{44}$$

where $\sigma(E,z)$ is the differential cross section of the reaction $a+B\to b+A$ in the center-of-mass system and $z=z_p$ is the position of the exchange pole of the reaction amplitude with respect to the variable $z=\cos\theta$, which corresponds to the mechanism of transfer of particle c from nucleus a to nucleus B. This mechanism is described by the pole diagram shown in Fig. 3. The quantity (44) can be determined from experimental data on $\sigma(E,z)$ in the method of analytic continuation to the pole[13,14] and in the singularity-subtraction method[15] (see Sec. 2).

Besides the pole term corresponding to the diagram in Fig. 3, the reaction amplitude contains a "background" term, which is finite at $z=z_p$ and does not contribute to (44). Therefore, to find $R^{\text{exch}}_{abc,\,ABc}$ it is sufficient to find the cross section $\sigma^{\text{pole}}(E,z)$ corresponding to only the pole reaction amplitude. Here, it is more convenient to use expressions of the type (13), and not (5), for the three-leg vertex parts $a\to b+c$ and $B+c\to A$, a more compact expression being obtained (the expression for $\sigma^{\text{pole}}(E,z)$ in the ls coupling scheme at the vertices is obtained in Ref. 16). The amplitude of the diagram in Fig. 3 has the form

$$\begin{aligned}
&\mathcal{M}^{M_aM_BM_bM_A}(\mathbf{q}_{aB},\mathbf{q}_{bA})\\
&=\sum_{M_c}\mathcal{M}^{M_a}_{M_bM_c}(\mathbf{q}_{bc})\,(\sigma_c+i\delta)^{-1}\,\mathcal{M}^{M_BM_c}_{M_A}(\mathbf{q}_{Bc}),
\end{aligned} \tag{45}$$

where $\mathcal{M}^{M_a}_{M_bM_c}$ is given by Eq. (13) and $\mathcal{M}^{M_BM_c}_{M_A}$ by a similar expression, and σ_c can be expressed in terms of the cosine $z=\cos\theta$ of the scattering angle:

$$\sigma_c=q_{aB}q_{bA}(z-z_p)/m_c. \tag{46}$$

The position of the exchange pole in the z plane is given by

$$z_p=[(1+g^2)\,E+\varepsilon^{A}_{Bc}-g^2\varepsilon^{a}_{bc}]/2g\sqrt{EE'}, \tag{47}$$

where

$$\begin{aligned}
&g^2=m_bm_B/m_am_A;\ E=q^2_{aB}/2\mu_{aB};\\
&E'=q^2_{bA}/2\mu_{bA}=E+Q;\ Q=m_a+m_B-m_b-m_A=\varepsilon^{A}_{Bc}-\varepsilon^{a}_{bc}.
\end{aligned} \tag{48}$$

From (45), (46), (19), and (20), we obtain

$$\begin{aligned}
\sigma^{\text{pole}}(E,z)&=\frac{m_c^2}{16\pi^2EE'}\,\frac{q_{bA}}{q_{aB}}\,\frac{\hat{J}_A}{\hat{J}_B}\,\frac{1}{(z-z_p)^2}\\
&\times\sum_{n=0}^{n_{\max}}S^{n}_{abc}(q_{bc})\,S^{n}_{ABc}(q_{Bc})\,P_n(z_{aA}).
\end{aligned} \tag{49}$$

Here

$$\begin{aligned}
S^{n}_{abc}(q_{bc})&=\sum_{l_al'_aj_a}(-1)^{J_c-j_a}(\hat{l}_a\hat{l}'_a)^{1/2}\,(l_a0l'_a0\,|\,n0)\\
&\times W(l_al'_aJ_cJ_c;nj_a)\,G_{abc}(l_aj_a;q_{bc})\,G^{*}_{abc}(l'_aj_a;q_{bc});
\end{aligned} \tag{50}$$

$S^{n}_{ABc}(q_{Bc})$ is given by the expression obtained from (50) by the index interchange $a\to A$, $b\to B$;

$$\begin{aligned}
&z_{aA}\equiv\mathbf{v}_{bc}\mathbf{v}_{Bc}=\frac{g(E+E')-(1+g^2)\sqrt{EE'}\,z}{2g\sqrt{EE'}\sqrt{z_a-z}\sqrt{z_A-z}};\\
&z_a=\frac{g^2E+E'}{2g\sqrt{EE'}};\ z_A=\frac{E+g^2E'}{2g\sqrt{EE'}};
\end{aligned} \tag{51}$$

$P_n(z_{aA})$ is a Legendre polynomial; the summation over n in (49) is with respect to even values from $n=0$ to $n=n_{\max}$, where

$$n_{\max}=\min\{2l^{\max}_a;\,2l^{\max}_A;\,2J_c-[1-(-1)^{2J_c}]/2\}; \tag{52}$$

$l^{\max}_a$ and $l^{\max}_A$ are the maximal values of the orbital angular momenta l_a and l_A at the vertices $a\to b+c$ and $B+c\to A$ allowed by the conditions (6).

Since the particles a, b and A, B are on the mass shell in the diagram of Fig. 3, the nuclear vertex form factors which occur in (45) depend only on the single variable σ_c. By virtue of the identities

$$\sigma_c=-q^2_{bc}/2\mu_{bc}-\varepsilon^{a}_{bc}=-q^2_{Bc}/2\mu_{Bc}-\varepsilon^{A}_{Bc} \tag{53}$$

the nuclear vertex form factors of the vertices $a\to b+c$ and $B+c\to A$ can be assumed to be functions of the relative momenta q_{bc} and q_{Bc}, respectively, as is assumed

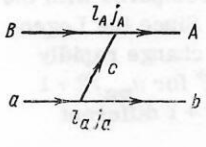

FIG. 3.

in (50). We can represent $G_{abc}(l_aj_a;q_{bc})$ in the form[17]

$$G_{abc}(l_aj_a;\,q_{bc}) = (q_{bc}/i\varkappa^a_{bc})^{l_a}\, g_{abc}(l_aj_a;\,q^2_{bc})\, G_{abc}(l_aj_a), \tag{54}$$

where the factor $(q_{bc}/i\varkappa^a_{bc})^{l_a}$ takes into account the threshold behavior of the form factor as $q_{bc}\to 0$, and "reduced" nuclear vertex form factor $g_{abc}(l_aj_a;q^2_{bc})$ is analytic in the q^2_{bc} plane with cut along the negative real half-axis from $q^2_{bc}=-(q^0_{bc})^2$ to $q^2_{bc}=-\infty$ and, by definition, is equal to 1 at $q_{bc}=i\varkappa^a_{bc}$ (on the mass shell). The nuclear vertex form factor for the vertex $B+c\to A$ can be represented in similar form. By virtue of (46) and (53) for fixed value of the energy E in the physical region the form factors $G_{abc}(l_aj_a;q_{bc})$ and $G_{ABc}(l_Aj_A;q_{Bc})$ are analytic functions of z. However, the expression for $\sigma^{\text{pole}}(E,z)$ given by (49)–(51) is not an analytic function of z because it contains complex-conjugate nuclear vertex form factors. If in (50) we make the substitutions

$$\begin{aligned} G^*_{abc}(l_aj_a;\,q_{bc}) &\to G^*_{abc}(l_aj_a;\,q^*_{bc});\\ G^*_{ABc}(l_Aj_A;\,q_{Bc}) &\to G^*_{ABc}(l_Aj_A;\,q^*_{Bc}), \end{aligned} \tag{55}$$

then the resulting expression $\tilde\sigma^{\text{pole}}(E,z)$ becomes an analytic function of z, coinciding for physical values of z $(-1\leqslant z\leqslant 1)$ with the function $\sigma^{\text{pole}}(E,z)$. In the methods of analytic continuation to the pole and the singularity-subtraction method mentioned above, it is assumed that the experimentally measured values of the differential cross section $\sigma(E,z)$ in the physical region are equal to the values of some analytic functions; therefore, to calculate (44) one should use, not $\sigma^{\text{pole}}(E,z)$, but $\tilde\sigma^{\text{pole}}(E,z)$. Assuming that the reduced nuclear vertex form factor $g_{abc}(l_aj_a;q^2_{bc})$ is a real-definite analytic function on the first sheet of the q^2_{bc} plane ($|\arg q^2_{bc}|<\pi$), i.e., that

$$g^*_{abc}(l_aj_a;\,(q^2_{bc})^*) = g_{abc}(l_aj_a;\,q^2_{bc}), \tag{56}$$

we obtain from (54)

$$G^*_{abc}(l_aj_a;\,q^*_{bc})\,|_{q_{bc}=i\varkappa^a_{bc}} = (-1)^{l_a} G^*_{abc}(l_aj_a), \tag{57}$$

and similarly for the vertex $B+c\to A$. Finally, from (44), (49)–(51), (55), and (57) we find

$$\begin{aligned} R^{\text{exch}}_{adc,\,ABc} &\equiv \lim_{z\to z_p}(z-z_p)^2\,\tilde\sigma^{\text{pole}}(E,z)\\ &= \frac{m_c^2}{16\pi^2 EE'}\,\frac{q_{bA}}{q_{aB}}\,\frac{\hat J_A}{\hat J_B}\,\xi_{abAB}\sum_{n=0}^{n_{\max}} S^n_{abc}S^n_{ABc}P_n(z_0), \end{aligned} \tag{58}$$

where $\xi_{abAB}=\xi_a\xi_b\xi_A\xi_B$ is the product of the intrinsic parities of the initial and final particles:

$$\begin{aligned} S^n_{abc} = \xi_{abc}S^n_{abc}(q_{bc})\,|_{q_{bc}=i\varkappa^a_{bc}} = \sum_{l_a l'_a j_a}(-1)^{J_c-j_a}(\hat l_a\hat l'_a)^{1/2}(l_a0l'_a0\,|\,n0)\\ \times W(l_al'_aJ_cJ_c;\,nj_a)\,G_{abc}(l_aj_a)\,G^*_{abc}(l'_aj_a); \end{aligned} \tag{59}$$

S^n_{ABc} is given by a similar expression;

$$z_0 \equiv z_{aA}\,|_{z=z_p} = \frac{(1-g^2)\,E+\varepsilon^A_{Bc}+g^2\varepsilon^a_{bc}}{2g\sqrt{\varepsilon^A_{Bc}\varepsilon^a_{bc}}}. \tag{60}$$

Because of the symmetry properties of the Clebsch–Gordan and Racah coefficients, S^n_{abc} and S^n_{ABc} are real.

It follows from (52) that if the spin of the exchange particle is $J_c=0$ or $\frac12$, then $n_{\max}=0$. In this case, the expression (58) simplifies for a purely kinematic reason, and it takes the form

$$\begin{aligned} R^{\text{exch}}_{abc,\,ABc} &= \frac{m_c^2}{16\pi^2EE'}\,\frac{q_{bA}}{q_{aB}}\,\frac{\hat J_A}{\hat J_B\hat J_c}\,\xi_{abAB}\\ &\times\sum_{l_aj_a}|G_{abc}(l_aj_a)|^2\sum_{l_Aj_A}|G_{ABc}(l_Aj_A)|^2,\quad J_c=0\ \text{ or }\ 1/2. \end{aligned} \tag{61}$$

Equation (58) takes the same form for $J_c\geqslant 1$ as well if for any dynamical reason related to the structure of the nuclear states a nuclear vertex constant with zero orbital angular momentum of the decay channel is predominant at either the vertex $a\to b+c$ or $B+c\to A$, i.e., if one can assume $l_a=0$ or $l_A=0$, and the contribution from the terms with $l_a\neq 0$ or $l_A\neq 0$ to S^n_{abc} or S^n_{ABc} can be ignored. In the normal case (18), the expression (61) can be written in the form

$$\begin{aligned} R^{\text{exch}}_{abc,\,ABc} &= \frac{m_c^2}{16\pi^2EE'}\,\frac{q_{bA}}{q_{aB}}\,\frac{\hat J_A}{\hat J_B\hat J_c}\\ &\times\sum_{l_aj_a}G^2_{abc}(l_aj_a)\sum_{l_Aj_A}G^2_{ABc}(l_Aj_A),\quad J_c=0\ \text{ or }\ 1/2. \end{aligned} \tag{62}$$

From (58) for the reaction $a+B\to b+A$ we find an expression for the exchange elastic scattering $a+b\to b+a$ (see Fig. 2) by the substitution $B\to b$, $A\to a$, $\theta\to\pi-\theta$:

$$R^{\text{exch}}_{abc} \equiv R^{\text{exch}}_{abc,\,abc} = \frac{m_c^2}{16\pi^2E^2}\,\frac{\hat J_a}{\hat J_b}\sum_{n=0}^{n_{\max}}(S^n_{abc})^2P_n(z_0), \tag{63}$$

where now

$$\left.\begin{aligned} z_p &= -[(1+g^2)+(1-g^2)\,(\varepsilon^a_{bc}/E)]/2g;\\ z_0 &= [(1+g^2)+(1-g^2)\,(E/\varepsilon^a_{bc})]/2g;\\ g &= m_b/m_a;\ E=E_{ba};\ z=\cos\theta; \end{aligned}\right\} \tag{64}$$

here θ is the angle between the momenta of particle b in the center-of-mass system before and after the scattering.

We further consider scattering in the normal case (18). Then

$$(S^0_{abc})^2 = (4\pi^2(\hat J_b)^2/\mu^3_{bc}\hat J_c)\,(r^{\text{exch}}_{abc})^2, \tag{65}$$

where r^{exch}_{abc} is determined by (38) and (39), and from (63) and (65) we obtain

$$R^{\text{exch}}_{abc} = (\hat J_a\hat J_b/4\hat J_c)(m_a/m_bE)^2\,(r^{\text{exch}}_{abc})^2\,[1+\sum_{n=2}^{n_{\max}}(S^n_{abc}/S^0_{abc})^2P_n(z_0)]. \tag{66}$$

It follows from (66) that if the spin in the exchange elastic scattering of the exchanged particle satisfies $J_c=0$ or $\frac12$ $(n_{\max}=0)$, then the combinations of nuclear vertex constants in r^{exch}_{abc} and R^{exch}_{abc} are the same. But if $J_c\geqslant 1$, then these combinations are in general different, i.e., the value of R^{exch}_{abc} gives additional information about the nuclear vertex constants as compared with the value of only r^{exch}_{abc} [or r^{dir}_{abc}, see (33)]. Since the Legendre polynomials $P_n(z_0)$ for $n=2,4,\ldots$ change rapidly with the energy E, by determining R^{exch}_{abc} for $n_{\max}/2+1$ different energies we can obtain $n_{\max}/2+1$ different

combinations of the nuclear vertex constants $|S^n_{abc}|$, $n = 0, 2, \ldots, n_{max}$.

As examples, let us consider the processes of elastic scattering of a neutron on a triton and of a deuteron on an α particle. These will be discussed in what follows. In both cases, the scattering amplitudes contain contributions from the exchange diagrams shown in Fig. 4 (deuteron exchange). At both vertices $t \rightleftarrows d+n$ and $\alpha \rightleftarrows d+d$ the kinematically allowed values are $(l, j) = (0, 1)$ and $(2, 1)$. By means of Eqs. (39) and (63), in which we must set $a=t$, $b=n$, and $c=d$, we obtain for nt scattering

$$r^{exch}_{tnd} = -(6\pi\lambda\!\!\!^{-}_N)^{-1}(G^2_{t0}+G^2_{t2}); \tag{67}$$

$$R^{exch}_{tnd} = \frac{16}{3}\left(\frac{\hbar c}{E_n}\right)^2 (r^{exch}_{tnd})^2\left[1+4\rho_t^2\left(\frac{1-\rho_t/\sqrt{8}}{1+\rho_t^2}\right)^2 P_2(z_0)\right], \tag{68}$$

where $\lambda\!\!\!^{-}_N = \hbar/m_N c = 0.2103$ F is the Compton wavelength of the nucleon; $\hbar c = 197.3$ MeV · F;

$$G_{t0} \equiv G_{tnd}\,(l=0,\ j=1);\ G_{t2} \equiv G_{tnd}\ (l=2,\ j=1);$$
$$\rho_t = G_{t2}/G_{t0};\ z_0 = 5/3 + E_n/\varepsilon^t_{dn};\ \varepsilon^t_{dn} = 6{,}2575 \text{ MeV}; \tag{69}$$

E_d is the laboratory energy of the incident neutrons in MeV; G^2_{tl} is measured in fermis.

For $d\alpha$ scattering, we obtain similarly

$$r^{exch}_{add} = (6\pi\lambda\!\!\!^{-}_N)^{-1}(G^2_{dd0}+G^2_{dd2}); \tag{70}$$

$$R^{exch}_{\alpha dd} = \frac{9}{4}\left(\frac{\hbar c}{E_d}\right)^2 (r^{exch}_{add})^2\left[1+4\rho^2_{dd}\left(\frac{1-\rho_{dd}/\sqrt{8}}{1+\rho^2_{dd}}\right)^2 P_2(z_0)\right]^2, \tag{71}$$

where

$$G_{dd0} \equiv G_{\alpha dd}\ (l=0,\ j=1);\ G_{dd2} \equiv G_{\alpha dd}\ (l=2,\ j=1);$$
$$\rho_{dd} = G_{dd2}/G_{dd0};\ z_0 = 5/4 + E_d/2\varepsilon^\alpha_{dd};\ \varepsilon^\alpha_{dd} = 23{,}847 \text{ MeV}; \tag{72}$$

E_d is the laboratory energy of the incident deuterons in MeV.

It can be seen from (68) and (71) that simultaneous knowledge of r^{exch} and R^{exch} or knowledge of the energy dependence of R^{exch} enables us to estimate the ratio ρ, i.e., to obtain information about the admixture of the D wave at the vertex. It is important that even a small D-wave admixture can strongly change the values of R^{exch}. For example, in the case of nt scattering at $E_n = 10$ MeV, the value of R^{exch}_{tnd} increases by about 6, 30, and 60% as a result of a D-wave admixture corresponding to $\rho_t^2 = 10^{-3}$, $5 \cdot 10^{-3}$, and 10^{-2}, respectively, although the change in r^{exch}_{tnd} can be ignored.

We have dwelt in such detail on the connections between r^{dir}, r^{exch}, and R^{exch} with one another and with the nuclear vertex constants not only on account of their importance but also because this question has not been investigated systematically in any of the theoretical studies devoted to the determination of these quantities from experiments. This seems to explain why some

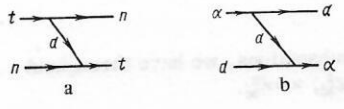

FIG. 4.

papers contain incorrect general assertions or incorrect interpretations of the analyses of experimental data. For example, Dubnička and Dumbrais[18] in fact assert in their review that r^{dir}, r^{exch}, and R^{exch} contain the same combinations of nuclear vertex constants (see Eqs. (5.15), (5.18) and the text on p. 164 of Ref. 18). On p. 164 of Ref. 18 it is asserted categorically without any justification that "... the vertex constants G^2 are not related uniquely to these fundamental quantities [by which Dubnička and Dumbrais mean quantities of the type r^{dir}, r^{exch}, or R^{exch}] and depend on specific assumptions and models." The incorrectness of these assertions is obvious in the light of what we have said above. In Ref. 14, which is devoted to the analysis of nt scattering data by the method of continuation of the differential cross section to the deuteron exchange pole, the results are incorrectly interpreted. First, the actually deduced quantity $(\frac{3}{16})(E_n/\hbar c)^2 R^{exch}_{tnd}$ is identified with $(r^{exch}_{tnd})^2$, which is strictly true only for $\rho_t = 0$. Second, r^{exch}_{tnd} is erroneously identified with r^{dir}_{tnd}, which is determined in Ref. 19 from analysis of nd scattering data by means of the dispersion relation for the forward scattering amplitude $f^L_{nd}(E_n)$, although in this case $r^{exch}_{tnd} = (\frac{2}{3}) r^{dir}_{tnd}$. Therefore, the coincidence of the numerical value $r^{exch}_{tnd} \approx 0.38$ found in Ref. 14 (with allowance for the correction of the error in Ref. 20) and the value $r^{dir}_{tnd} = 0.382$ in Ref. 19 in fact indicates that the result of Ref. 14 is false since it differs by about 1.5 times from all other reliable estimates of the nuclear vertex constant G^2_{t0} (see Sec. 3 of this review). Note also that the residues r^{exch} given in Refs. 14, 20, and 21 have incorrect signs that disagree with the rule (40) in a number of cases.

The cause of such errors in Refs. 14, 18, 20, 21 is that the derivation in them of the relations between quantities of the type r^{exch} and R^{exch} was based on the following rather arbitrary procedure. For the vertex $a \rightleftarrows b + c$, a certain relativistic effective Lagrangian was constructed and used to find the corresponding Feynman amplitudes and quantities of the type r^{exch} and R^{exch}. In the effective Lagrangian, allowance was made for some one type of coupling of the field operators corresponding to the particles a, b, c and only one vertex coupling constant was accordingly introduced. Other possible types of couplings allowed by the laws of conservation of parity, angular momentum, etc., and the requirements of relativistic invariance were not considered, although in relativistic theory the number of possible types of field couplings and coupling constants corresponding to them is in general actually larger than in nonrelativistic theory. Thus, the relativistic amplitude of the virtual decay $d \to p + n$ contains four invariant form factors,[22] of which only two, corresponding to the S and D deuteron states, survive in the nonrelativistic limit. In some cases, the effective Lagrangian employed had a definitely relativistic nature and did not correspond to any possible type of coupling in the nonrelativistic limit (see, for example, Ref. 21), since in this limit the Lagrangian was of order $\sim(v/c)^2$, for example, $\sim(\varepsilon^a_{bc}/m_a)$. Quite generally, the use of the relativistic formalism to describe nonrelativistic nuclear processes is unjustified since, on the one hand, the calculations are made much more complicated by the com-

plicated relativistic kinematics and the complexity of the relativistic description of processes in which particles with large spins participate and, on the other, the final expressions are too cumbersome because they contain in addition to terms ~1 numerous negligibly small terms $\sim(v/c)^2$. It was for these reasons that the dispersion theory of direct nuclear reactions[23,24] was developed from the very start with a nonrelativistic diagram formalism,[25-27] with subsequent consideration in a number of investigations of the theory of nonrelativistic vertex parts, nonrelativistic methods of calculation of the contributions from different singularities, etc.[8,28,29]

Finally, we give the expressions that relate the nuclear vertex constant to the residue of the partial-wave amplitude of the S matrix of elastic scattering of particles b and c at the pole corresponding to the bound state a of the system $b+c$, and to the decay width of the resonance state.

We define the partial-wave amplitude of the S matrix for elastic bc scattering by the relation[1)]

$$\langle l's' | S^I(q_{bc}) | ls\rangle = \delta_{ll'}\delta_{ss'} - (i/\pi)\, q_{bc}\mu_{bc} \langle l's' | \mathcal{M}^I(q_{bc}) | ls\rangle, \quad (73)$$

where s and l are the spin and orbital angular momentum of the channel; I is the total angular momentum. The partial-wave amplitudes of the $\mathcal{M}$ matrix are determined by the expansion

$$\mathcal{M}_{M_b M_c M_b' M_c'}(\mathbf{q}_{bc}, \mathbf{q}_{bc}') = 4\pi \sum_{\substack{lsl's' \\ Mm_l m_s m_l' m_s'}} (l'm_l' s'm_s' | IM)\, (J_b M_b' J_c M_c' | s'm_s')\, Y_{l'm_l'}(\nu_{bc}') \times \langle l's' | \mathcal{M}^I(q_{bc}) | ls\rangle\, (lm_l sm_s | IM)\, (J_b M_b J_c M_c | sm_s)\, Y^*_{lm_l}(\nu_{bc}). \quad (74)$$

The differential scattering cross section can be expressed in terms of the amplitude $\mathcal{M}$ by means of Eqs. (22) and (20). The partial-wave amplitudes with $I=J_a$ have a pole at $q_{bc}=i\varkappa_{bc}^a$, and the residue at this pole is equal to

$$\operatorname{res}\langle l's' | S^{J_a}(q_{bc}) | ls\rangle|_{q_{bc}=i\varkappa_{bc}^a} = \lim_{q_{bc}\to i\varkappa_{bc}^a} (q_{bc} - i\varkappa_{bc}^a)\langle l's' | S^{J_a}(q_{bc}) | ls\rangle = -(i/\pi)\,(\mu_{bc}c/\hbar)^2\, G_{abc}(l's')\, G_{abc}(ls). \quad (75)$$

Equation (75) follows directly from Eqs. (29), (5), (74), and (73). In the normal case (18), for the residue of the diagonal matrix element we obtain from (75)

$$\operatorname{res}\langle ls | S^{J_a}(q_{bc}) | ls\rangle|_{q_{bc}=i\varkappa_{bc}^a} = (-1)^l\, (i/\pi)\, (\mu_{bc}c/\hbar)^2\, |G_{abc}(ls)|^2. \quad (76)$$

If the decay $a\to b+c$ is energetically possible, i.e., a is a resonance state with mass $m_a = m_b + m_c + E_0 - i\Gamma/2$, then the nuclear vertex constant $G_{abc}(ls)$ can be related to the partial width $\Gamma_{abc}(ls)$ of the decay of the resonance through the channel $(bcls)$. For a narrow resonance ($\Gamma \ll E_0$), we have[17]

$$|G_{abc}(ls)|^2 \approx (\pi/q_{bc}\mu_{bc}c^2)\, \Gamma_{abc}(ls), \quad q_{bc} = (2\mu_{bc}E_0)^{1/2}/\hbar. \quad (77)$$

Relationship of the nuclear vertex constants and vertex form factors to the nuclear wave functions. In nonrelativistic theory, in which a complex nucleus is regarded as a bound state of a system of nucleons, the nuclear vertex form factors for the decay of a stable nucleus a into the stable nuclei b and c can be expressed in terms of the wave functions of the nuclei a, b, and c. The corresponding expressions can be most readily obtained by calculating the residue of the amplitude of elastic bc scattering off the mass shell with respect to the energy at the pole corresponding to the nuclear-stable particle a. This residue is equal to the product of the amplitudes of the virtual processes $b+c\to a$ and $a\to b+c$:

$$\operatorname{res}\mathcal{M} = \sum_{M_a} \mathcal{M}_{M_a}^{M_b M_c} \cdot \mathcal{M}_{M_b M_c'}^{M_a}. \quad (78)$$

In accordance with Refs. 10 and 11, the amplitude $\mathcal{M}$ of elastic bc scattering has the form[2)]

$$\langle \mathbf{q}'b'c' | \mathcal{M}(E) | \mathbf{q}bc\rangle = \langle \mathbf{q}'b'c' | V_{bc} P_{bc} [1 + (E - H + i\delta)^{-1} V_{bc}] | \mathbf{q}bc\rangle, \quad \delta \to +0. \quad (79)$$

Here, $H = H_b + H_c + T_{bc} + V_{bc}$; H_b and H_c are the Hamiltonians of the nuclei b and c; $T_{bc} = -\nabla_r^2/2\mu$ is the operator of the relative kinetic energy of b and c; $\mathbf{r} = \mathbf{r}_b - \mathbf{r}_c$; V_{bc} is the operator of the interaction of b and c; P_{bc} is the operator of antisymmetrization with respect to the variables of the nucleons in the different nuclei b and c; $|b\rangle = |\alpha_b J_b M_b\rangle (|b'\rangle = |\alpha_b J_b M_b'\rangle)$ and $|c\rangle = |\alpha_c J_c M_c\rangle (|c'\rangle = |\alpha_c J_c M_c'\rangle)$ are the antisymmetric normalized internal wave functions of the nuclei b and c;

$$H_b|b\rangle = -\varepsilon_b|b\rangle, \quad H_c|c\rangle = -\varepsilon_c|c\rangle, \quad (80)$$

ε_b and ε_c are the total binding energies of the nuclei b and c; $|\mathbf{q}\rangle$ and $|\mathbf{q}'\rangle$ are the plane waves that describe the relative motion of b and c before and after scattering, normalized in accordance with (2). In (79),

$$E = -\varepsilon_b - \varepsilon_c + E_{bc}, \quad E_{bc} = q^2/2\mu + \sigma_b + \sigma_c = q'^2/2\mu + \sigma_b' + \sigma_c'. \quad (81)$$

If $q' \neq q$, $E_{bc} \neq q^2/2\mu$, $E_{bc} \neq q'^2/2\mu$, Eq. (79) gives the most general expression for the scattering amplitude off the mass shell in the nonrelativistic Schrödinger formalism.

In the isospin formalism[11] we have

$$P_{bc} = 1 - \sum_{i_b i_c} P_{i_b i_c} + \sum_{\substack{i_b<j_b \\ i_c<j_c}} P_{i_b i_c} P_{j_b j_c} - \ldots, \quad (82)$$

where $P_{i_b i_c}$ is the transposition operator that interchanges the spatial and spin–isospin coordinates of nucleon i_b in b and nucleon i_c in c. The total number of transposition operators on the right-hand side of (82) is

[1)] For spinless particles, $\langle l | S^I(q_{bc}) | l\rangle \equiv S_l \equiv \exp(2i\delta_l)$, where δ_l is the phase shift.

[2)] In this and the following subsections, we have throughout $\mathbf{q} = \mathbf{q}_{bc}$, $\mathbf{r} = \mathbf{r}_{bc}$, $\mu = \mu_{bc}$, $\varepsilon = \varepsilon_{bc}^a$, $\varkappa = \varkappa_{bc}^a$.

$$N_{bc}=(A_b+A_c)!/A_b!\,A_c!, \tag{83a}$$

where A_b and A_c are the mass numbers of the nuclei b and c. For antisymmetrization separately with respect to the protons and the neutrons, the operator P_{bc} is equal to the product of two operators of the type (82) acting on the proton and neutron variables, respectively. In this case,

$$N_{bc}=(Z_b+Z_c)!\,(N_b+N_c)!/Z_b!\,Z_c!\,N_b!\,N_c!, \tag{83b}$$

where Z_b, Z_c and N_b, N_c are the numbers of protons and neutrons in the nuclei b and c, respectively.

Using the identity $P_{bc}P_{bc}/N_{bc}=P_{bc}$, we write (79) in the form

$$\begin{aligned}\langle \mathbf{q}'b'c'|\mathcal{M}(E)|\mathbf{q}bc\rangle &= \langle \mathbf{q}'b'c'|V_{bc}P_{bc}|\mathbf{q}bc\rangle\\ &+\sum_{a_n}\langle \mathbf{q}'b'c'|V_{bc}P_{bc}N_{bc}^{-1/2}|a_n\rangle\,(E-E_{a_n}+i\delta)^{-1}\\ &\times\langle a_n|N_{bc}^{-1/2}P_{bc}V_{bc}|\mathbf{q}bc\rangle,\end{aligned} \tag{84}$$

where $|a_n\rangle=|\alpha_{a_n}J_{a_n}M_{a_n}\rangle$ is a complete system of antisymmetric and normalized eigenfunctions of the Hamiltonian H: $H|a_n\rangle=E_{a_n}|a_n\rangle$.

It can be seen from (84) that the scattering amplitude has a pole at $E=E_a=-\varepsilon_a$, where $|a\rangle=|\alpha_a J_a M_a\rangle$ is a bound state of the system $b+c$. Comparing the residue at this pole with (78), we find the amplitude of the virtual decay $a\to b+c$:

$$\mathcal{M}^{M_a}_{M_bM_c}(\mathbf{q})=\langle \mathbf{q}bc|V_{bc}P_{bc}N_{bc}^{-1/2}|a\rangle. \tag{85}$$

Since $P_{bc}|a\rangle=N_{bc}|a\rangle$,

$$\begin{aligned}\mathcal{M}^{M_a}_{M_bM_c}(\mathbf{q}) &= N_{bc}^{1/2}\langle \mathbf{q}\alpha_bJ_bM_b\alpha_cJ_cM_c|V_{bc}|\alpha_aJ_aM_a\rangle\\ &=N_{bc}^{1/2}\int\exp(-i\mathbf{q}\cdot\mathbf{r})\,\Psi^+(\alpha_bJ_bM_b;\tau_b)\,\Psi^+(\alpha_cJ_cM_c;\tau_c)\\ &\times V_{bc}(\tau_b,\tau_c,\mathbf{r})\,\Psi(\alpha_aJ_aM_a;\tau_b,\tau_c,\mathbf{r})\,d\tau_b\,d\tau_c\,d\mathbf{r},\end{aligned} \tag{86}$$

where τ_b and τ_c are the sets of internal coordinates (including the spin or spin–isospin coordinates) of the nuclei b and c.[3] From (86) and (5) we find an expression for the nuclear vertex form factor:

$$\begin{aligned}G_{abc}(ls;q) &= (4\pi N_{bc})^{1/2}\int_0^\infty j_l(qr)\,r^2\,dr\\ &\times\int\Psi^+(\alpha_bJ_b\alpha_cJ_clsJ_aM_a;\tau_b,\tau_c,\mathbf{n})\,V_{bc}(\tau_b,\tau_c,\mathbf{r})\\ &\times\Psi(\alpha_aJ_aM_a;\tau_b,\tau_c,\mathbf{r})\,d\tau_b\,d\tau_c\,d\Omega_n,\quad \mathbf{n}\equiv\mathbf{r}/r,\end{aligned} \tag{87}$$

where j_l is a spherical Bessel function;

$$\begin{aligned}&\Psi(\alpha_bJ_b\alpha_cJ_clsJ_aM_a;\tau_b,\tau_c,\mathbf{n})\\ &=\sum_{m_lm_sM_bM_c}(lm_lsm_s|J_aM_a)\,(J_bM_bJ_cM_c|sm_s)\\ &\times i^lY_{lm_l}(\mathbf{n})\,\Psi(\alpha_bJ_bM_b;\tau_b)\,\Psi(\alpha_cJ_cM_c;\tau_c).\end{aligned} \tag{88}$$

Substituting $V_{bc}=H-H_b-H_c-T_{bc}$ in (86) and taking into account (80), we find a second expression for the decay amplitude:

$$\begin{aligned}\mathcal{M}^{M_a}_{M_bM_c}(\mathbf{q}) &= -N_{bc}^{1/2}\left(\frac{q^2}{2\mu}+\varepsilon\right)\langle \mathbf{q}\alpha_bJ_bM_b\alpha_cJ_cM_c|\alpha_aJ_aM_a\rangle\\ &=-N_{bc}^{1/2}\left(\frac{q^2}{2\mu}+\varepsilon\right)\int\exp(-i\mathbf{q}\cdot\mathbf{r})\,I_{abc}(\mathbf{r})\,d\mathbf{r},\end{aligned} \tag{89}$$

where

$$\begin{aligned}I_{abc}(\mathbf{r}) &= \int\Psi^+(\alpha_bJ_bM_b;\tau_b)\,\Psi^+(\alpha_cJ_cM_c;\tau_c)\\ &\times\Psi(\alpha_aJ_aM_a;\tau_b,\tau_c,\mathbf{r})\,d\tau_b\,d\tau_c\end{aligned} \tag{90}$$

is the integral of the overlapping of the internal wave functions of the nuclei a, b, and c, which can be represented in the form

$$\begin{aligned}I_{abc}(\mathbf{r}) &= \sum_{lsm_lm_s}(lm_lsm_s|J_aM_a)\\ &\times(J_bM_bJ_cM_c|sm_s)\,i^lY_{lm_l}(\mathbf{n})\,I_{abc}(ls;\mathbf{r}).\end{aligned} \tag{91}$$

The summation with respect to l and s in (91) is over the values restricted by the conditions (6). From (89) and (91) we obtain the expression (5) for the decay amplitude, in which

$$G_{abc}(ls;q)=-(\pi N_{bc})^{1/2}\mu^{-1}(q^2+\varkappa^2)\int_0^\infty j_l(qr)\,I_{abc}(ls;r)\,r^2\,dr. \tag{92}$$

In (92), $\sigma_a=E_{bc}+\varepsilon=0$ by definition, and the invariant q is related to the invariants σ_b and σ_c by

$$q^2=-\varkappa^2-2\mu(\sigma_b+\sigma_c). \tag{93}$$

The radial parts of the overlap integrals $I_{abc}(ls;r)$ are solutions of a system of equations (see, for example, Refs. 30, 31) that are obtained by the standard method from the Schrödinger equation for $\Psi(\alpha_aJ_aM_a;\tau_a)$ by substitution into it of the expansion

$$\Psi(\alpha_aJ_aM_a;\tau_a)=\sum_{bcls}I_{abc}(ls;r)\,\Psi(\alpha_bJ_b\alpha_cJ_clsJ_aM_a;\tau_b,\tau_c,\mathbf{n}) \tag{94}$$

with respect to the complete system of orthonormal functions (88), where $\sum_{bc}$ denotes summation over all states of the subsystems b and c for which the transition $a\to b+c$ is allowed by the selection rules for the strong interactions.

By virtue of the invariance of the nuclear Hamiltonians under spatial rotations and time reversal, the radial functions $I_{abc}(ls;r)$ can always be chosen to be real (see, for example, Sec. 2, of Ch. 1 and Appendix 2 in the monograph Ref. 32). We shall assume, as usual (see, for example, Ref. 30) that the asymptotic behavior of the function $I_{abc}(ls;r)$ at large r is completely determined by the binding energy ε:

$$I_{abc}(ls;r)\approx C_{abc}(ls)\,r^{-1}\exp(-\varkappa r),\quad r\to\infty, \tag{95}$$

where $C_{abc}(ls)$ is a real constant.

It should be said that this assumption, which is obvious for the virtual decay of a two-nucleon nucleus, requires a special proof in the case of decay of a many-nucleon nucleus into two fragments. So far as we know,

[3] It is assumed that all the internal wave functions of the nuclei are translationally invariant, i.e., the wave function of a nucleus containing A nucleons depends on $3(A-1)$ relative spatial coordinates.

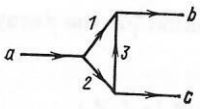

FIG. 5.

no rigorous proof of the relation (95) has been given in the literature. Moreover, there are indications that in individual cases the asymptotic behavior may differ from (95), namely, the damping at large r may be weaker than predicted by Eq. (95) (see Ref. 33). This conclusion follows, for example, from investigation of the singularities of the triangle vertex Feynman diagrams—of the type shown in Fig. 5—with respect to the variable σ_c at $\sigma_a=\sigma_b=0$ or with respect to the variable σ_b at $\sigma_a=\sigma_c=0$. In some cases, for example, for the vertices $^{16}O\to{}^{13}N$ (^{13}C) + ^{3}H (^{3}He) or $^{20}Ne\to{}^{17}F$ (^{17}O) + ^{3}H (^{3}He), the singularity of the triangle diagram (the anomalous threshold) $q^2=-q_\Delta^2$ ($q_\Delta>0$) of the reduced nuclear vertex form factor $g_{abc}(ls;q^2)$ [see (54)] on the physical sheet of the q^2 plane is to the right of the point $q^2=-\varkappa^2$ corresponding to the mass shell: $q_\Delta<\varkappa$. Since in accordance with (92)

$$I_{abc}(ls;r)=-2\pi^{-3/2}N_{bc}^{-1/2}\mu^{-1}\int_0^\infty j_l(qr)\,G_{abc}(ls;q)\,\frac{q^2\,dq}{(q^2+\varkappa^2)}, \qquad (96)$$

it follows that for $q_\Delta<\varkappa$ the anomalous threshold makes a contribution to the asymptotic behavior of the form[33]

$$I_{abc}^{\Delta}(ls;r)\propto r^{-2}\exp(-q_\Delta r),\quad r\to\infty, \qquad (97)$$

which decreases more slowly than a term of the form (95) which results from the pole denominator $(q^2+\varkappa^2)^{-1}$. For a definitive conclusion about the possibility of anomalous asymptotic behavior of the type (97), we undoubtedly need a more complete analysis of the equations for the functions $I_{abc}(ls;r)$.

In the case of the normal asymptotic behavior, we obtain from (95) and (92) an expression for the nuclear vertex constant in terms of the asymptotic normalization constant:

$$G_{abc}(ls)\equiv\lim_{q\to i\varkappa}G_{abc}(ls;q)=-i^l(\pi N_{bc})^{1/2}(\hbar/\mu c)\,C_{abc}(ls). \qquad (98)$$

The phase of the vertex constant agrees with (18). Note that the problem of determining the nuclear vertex constant and its phase in the case of anomalous asymptotic behavior requires further investigation.

Substituting (98) into (75), we find an expression for the residue of the S matrix of elastic scattering in terms of the asymptotic normalization constants[4)]:

$$\operatorname{res}\langle l's'|S^{J_a}(q)|ls\rangle|_{q=i\varkappa}=-i^{l'+l+1}N_{bc}C_{abc}(l's')\,C_{abc}(ls). \qquad (99)$$

For practical calculations of the nuclear vertex constants on the basis of model nuclear wave functions Eq. (87) may be helpful; here for this purpose one must set $q=i\varkappa$. By virtue of (98), Eq. (87) for $q=i\varkappa$ gives an integral expression for the asymptotic normalization constant $C_{abc}(ls)$. Such an expression for the asymptotic normalization of the overlap integral of the triton and deuteron wave functions was derived recently by Lehman and Gibson[37] who note its convenience for practical calculation, especially when wave functions in the momentum representation are used.

To analyze data on direct nuclear reactions in the framework of traditional theories that use a Schrödinger formalism of wave functions and interparticle interaction potentials (Butler's theory, [38] the distorted-wave Born approximation, [39] etc.), information about nuclear structure is usually extracted in the form of reduced widths and spectroscopic factors. We give the expressions that relate these quantities to the nuclear vertex constants. We define a generalized spectroscopic factor $S_{abc}(ls)$ by

$$S_{abc}(ls)=N_{bc}Z_{abc}(ls), \qquad (100)$$

where

$$Z_{abc}(ls)=\int_0^\infty I_{abc}^2(ls;r)\,r^2\,dr. \qquad (101)$$

From the condition of normalization of $\Psi(\alpha_a J_a M_a;\tau_a)$ and (94) it follows that

$$\sum_{bcls}Z_{abc}(ls)=1. \qquad (102)$$

If the states of nuclei are described by the shell model, then for the virtual decay $a\to b+N$ (N is a nucleon) summed over the spin of channel s the generalized spectroscopic factor is equal to the spectroscopic factor $S(l)$ defined by Eq. (II.15) of Ref. 40. We introduce the normalized radial function

$$R_{abc}(ls;r)=Z_{abc}^{-1/2}(ls)\,I_{abc}(ls;r). \qquad (103)$$

Remembering that for sufficiently large $r\geq r_0$ (r_0 is the channel radius)

$$R_{abc}(ls;r)\approx R_{abc}(ls;r_0)\,h_l^{(1)}(i\varkappa r)/h_l^{(1)}(i\varkappa r_0), \qquad (104)$$

where $h_l^{(1)}(x)$ is a spherical Hankel function of the first kind, and determining the partial reduced width $\theta_{abc}^2(ls)$ by the relation[5)]

$$\theta_{abc}^2(ls)=r_0^3R_{abc}^2(ls;r_0)/3, \qquad (105)$$

we find from (95), (98), (100), and (103)–(105) the required expression:

$$G_{abc}(ls)=(3\pi/r_0)^{1/2}(\hbar/\mu c)\,S_{abc}^{1/2}(ls)\,\theta_{abc}(ls)\,[\varkappa r_0h_l^{(1)}(i\varkappa r_0)]^{-1}. \qquad (106)$$

[4)]For the single-channel spinless case, Eq. (99) was obtained for the first time in Refs. 34 and 35. For the many-channel spinless case, this equation is derived in the book Ref. 36.

[5)]For the virtual emission of a nucleon ($a\to b+N$), $\theta_{abN}^2(ls)$ is equal to the single-particle reduced width θ_0^2 given by Eq. (II.14) in Ref. 40.

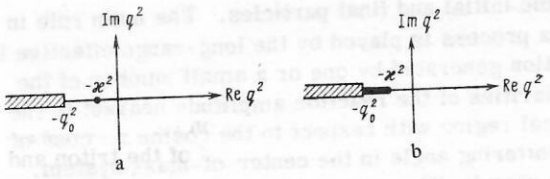

FIG. 6.

The reduced widths $\theta^2_{abc}(ls)$ depend exponentially on the channel radius r_0. In contrast, the nuclear vertex constants calculated in accordance with (106) do not depend on r_0 to the accuracy with which Eq. (104) holds. Of course, this remark is of a purely practical nature since theoretically the nuclear vertex constants do not depend on r_0 at all.

Coulomb effects at the three-leg vertices. Hitherto, it has been assumed that the virtual decay $a \to b+c$ is due to strong interactions. If particles b and c are charged, then they can also have a Coulomb interaction. Allowance for this interaction leads to qualitatively new effects. In particular, the analytic properties of the reduced nuclear vertex form factors $g_{abc}(ls;q^2)$ in the q^2 plane are changed. In the presence of only strong interactions, the function $g_{abc}(ls;q^2)$ is analytic in the q^2 plane with cut $(-q_0^2, -\infty)$, and in the normal case[6)] the point $q^2=-q_0^2$ is always to the left of the point $q^2=-\varkappa^2$ (Fig. 6a). The situation changes when the electromagnetic interaction (in particular, the Coulomb interaction) between the particles becomes important. This is evident already from examination of the simplest vertex graph (Fig. 7) describing the elastic interaction of b and c through the exchange of a photon (in the static limit for spinless particles, this graph describes the Coulomb interaction of b and c in the first order).

To the graph in Fig. 7 there corresponds the branch point (Ref. 41) $q^2=-q_\Delta^2=-\varkappa^2$, at which the "elastic" Coulomb cut begins (Fig. 6b). This additional cut is important in the case when the Coulomb parameter η for the bound state $a=b+c$ defined by Eq. (109) is sufficiently large.

For a rigorous estimate of the Coulomb effects in the nonrelativistic nuclear vertex form factors it is necessary to solve the system of coupled equations for the radial parts of the overlap integrals $I_{abc}(ls;r)$. Since this problem has not hitherto been considered in the literature, we use the results of the examination of the influence of Coulomb effects on the nuclear vertex constants and form factors in the two-particle potential model (Refs. 42 and 43). This approach makes it possible to separate fairly accurately the contribution to the vertex form factors from the "elastic" Coulomb cut.

In accordance with Ref. 43, we represent the radial part of the overlap integral in the form

$$I_{abc}(ls;\,r)=I^C_{abc}(ls;\,r)+I^N_{abc}(ls;\,r), \tag{107}$$

where

$$I^C_{abc}(ls;\,r)=C_{abc}(ls)\,r^{-1}W_{-\eta,\,l+\frac12}(2\varkappa r); \tag{108}$$

$C_{abc}(ls)$ is a real constant;

$$\eta=\mu Z_b Z_c e^2/\hbar\varkappa \tag{109}$$

is the Coulomb parameter for the bound state $a=b+c$; Z_ie is the charge of particle i; $W_{-\eta,\,l+1/2}(2\varkappa r)$ is the Whittaker function.[44] In the limit $r\to\infty$,

$$\left.\begin{aligned} I^C_{abc}(ls;\,r)&\approx C_{abc}(ls)\,r^{-1}\exp\left[-\varkappa r-\eta\ln(2\varkappa r)\right];\\ I^N_{abc}(ls;\,r)&\sim\exp\left[-(\varkappa+\gamma)\,r\right],\end{aligned}\right\} \tag{110}$$

where $\gamma>0$, and in the two-particle potential model it determines the falloff of the tail of the nuclear potential: $V^N(r)\sim\exp(-\gamma r)$. In accordance with (107), the nuclear vertex form factor given by (92) can be written in the form[43]

$$G_{abc}(ls;\,q)=(q/i\varkappa)^l\,g_{abc}(ls;\,q^2)\,G^r_{abc}(ls), \tag{111}$$

where

$$G^r_{abc}(ls)=-\exp\left[i\pi(l+\eta)/2\right](\pi N_{bc})^{1/2}(\hbar/\mu c)\,C_{abc}(ls) \tag{112}$$

is the so-called renormalized nuclear vertex constant, and the reduced nuclear vertex form factor is a sum of two terms—the purely Coulomb term and the nuclear–Coulomb term:

$$g_{abc}(ls;\,q^2)=g^C_{abc}(l;\,q^2)+g^N_{abc}(ls;\,q^2), \tag{113}$$

with

$$g^C_{abc}(l;\,q^2)=\exp(-i\pi\eta/2)\,(q^2+\varkappa^2)\,(\varkappa/q)^l \times\int_0^\infty j_l(qr)\,W_{-\eta,\,l+\frac12}(2\varkappa r)\,r\,dr, \tag{114}$$

and $g^N_{abc}(ls;q^2)$ is given by an analogous expression in which the Whittaker function is replaced by the function $C^{-1}_{abc}(ls)\varkappa I^N_{abc}(ls;r)$. The spectral representation for the Coulomb reduced nuclear vertex form factor has the form[43]

$$g^C_{abc}(l;\,q^2)\equiv g^C_{abc}(l;\,t)=\exp(-i\pi\eta/2)\,[(l+1)!/\Gamma(l+1+\eta)]\,t^l \times\int_0^\infty\tau^{l+\eta}\left(1+\sqrt{1+\tau}\right)^{-2\eta}(\tau+t)^{-l-2}\,d\tau,\quad t=1+(q^2/\varkappa^2). \tag{115}$$

In the limit $q^2\to-\varkappa^2$,

$$\left.\begin{aligned} g^C_{abc}(l;\,q^2)&=\exp(-i\pi\eta/2)\,\Gamma(1-\eta)\,(t/4)^\eta\,[1+O(t)];\\ g^N_{abc}(ls;\,q^2)&=O(t).\end{aligned}\right\} \tag{116}$$

For $\eta=0$,

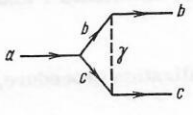

FIG. 7.

[6)] It is assumed below that the normal asymptotic behavior (95) of the overlap integrals holds.

$$g^C_{abc}(l;q^2)|_{\eta=0}=1. \qquad (117)$$

The function $g^C_{abc}(l;q^2)$ has a branch point of power type at $q^2=-\varkappa^2$ and takes into account fully the contribution to the reduced vertex form factor (113) from the elastic Coulomb cut $(-\nu^2,-\infty)$. The nearest singularity of the function $g^N_{abc}(ls;q^2)$ is at $q^2=-(\varkappa+\gamma^2)$, and at the point $q^2=-\varkappa^2$ it is regular.

It can be seen from (111), (113), and (116) that in the presence of a repulsive Coulomb interaction the definition of the nuclear vertex constants as the values of the nuclear vertex form factors on the mass shell [see (17)] becomes meaningless since $G_{abc}(ls;q)|_{q=i\varkappa}=0$ (in the case of an attractive Coulomb interaction ($\eta<0$) this definition is also meaningless since then $G_{abc}(ls;q)|_{q=i\varkappa}=\infty$). One can, however, introduce a renormalized nuclear vertex constant, defining it by[43]

$$G^r_{abc}(ls)=\lim_{q\to i\varkappa}[\mathcal{P}G_{abc}(ls;q)/g^C_{abc}(l;q^2)], \qquad (118)$$

where $\mathcal{P}G_{abc}(ls;q)$ is the principal term in the expansion of the irregular part of $G_{abc}(ls;q)$ in the neighborhood of the singular point $q=i\varkappa$ on the physical sheet. If $\eta<1$, then (118) contains not $\mathcal{P}G_{abc}(ls;q)$ but simply $G_{abc}(ls;q)$.[7)]

The residue of the partial-wave S-matrix element of elastic bc scattering at the pole $q=i\varkappa$ can be expressed in terms of the renormalized nuclear vertex constant by means of the expression

$$\operatorname{res}\langle l's'|S^{J_a}(q)|ls\rangle|_{q=i\varkappa}=-(i/\pi)(\mu c/\hbar)^2 G^r_{abc}(l's')G^r_{abc}(ls), \qquad (119)$$

which is analogous to Eq. (75) in the absence of the Coulomb interaction. From (119) and (112) we obtain an expression for this residue in terms of the asymptotic normalization constants:

$$\operatorname{res}\langle l's'|S^{J_a}(q)|ls\rangle|_{q=i\varkappa}=-i^{l'+l+1}\exp(i\pi\eta)C_{abc}(l's')C_{abc}(ls), \qquad (120)$$

which for the single-channel spinless case coincides with the well known result of Refs. 36 and 47.

Note finally that Eqs. (104)–(106), which relate the spectroscopic factor and the reduced partial width to the nuclear vertex constant, are modified in an obvious manner when allowance is made for the Coulomb interaction: The nuclear vertex constant goes over into the renormalized nuclear vertex constant, and the function $\varkappa r h_l^{(1)}(i\varkappa r)$ must be replaced by the function $-i^{-l-\eta}W_{-\eta,\,l+1/2}(2\varkappa r)$.

2. METHODS OF DETERMINING THE NUCLEAR VERTEX CONSTANTS

Peripheral model. The peripheral model for describing direct two-body nuclear reactions was proposed in Ref. 48 (see also Ref. 49). The model is based on the fairly reliably established fact that a direct reaction is predominantly a peripheral, surface process that takes place for large relative orbital angular momenta l and l' of the initial and final particles. The main role in such a process is played by the long-range effective interaction generated by one or a small number of the singularities of the reaction amplitude nearest to the physical region with respect to the cosine $z=\cos\theta$ of the scattering angle in the center-of-mass system. These singularities make the dominant contribution to the peripheral partial-wave amplitudes with $l\geq L$ and $l'\geq L'$, which can either be calculated fairly accurately or expressed in a simple form in terms of known factors and a small number of parameters that have a clear physical meaning (for example, in terms of the nuclear vertex constants in the case of the pole mechanism; see below). In contrast, the lower partial-wave amplitudes (small l and l') are determined by many singularities, both near and far, of the amplitude with respect to z and E (E is the relative kinetic energy of the colliding particles), and an accurate calculation or simple parametrization of them is in practice impossible. It is assumed, however, that their total contribution to the cross section at small or large angles θ (depending on the reaction mechanism) is small because of destructive interference, and also because of their relative suppression when there is a large number of open reaction channels. (This assumption is confirmed by calculations made in a three-body model[50] and the coupled-channel method.[51]) Therefore, in a calculation of the differential cross section in the peripheral model allowance is made for only the peripheral partial-wave amplitudes, and for their calculation one allows for the contribution of one or a few of the nearest singularities in z, these corresponding to the simplest Feynman diagrams.

In the part relating to the calculation of the peripheral partial-wave amplitudes, the peripheral model makes essential use of a theorem which states that the asymptotic behavior of the coefficients f_l in the expansion of the analytic function $f(z)$ with respect to the Legendre polynomials $P_l(z)$ is completely determined in the limit $l\to\infty$ by the position and type of the singularity of $f(z)$ nearest to the physical region $[-1,1]$ (Refs. 5 and 52). In particular, if $f(z)$ is the amplitude of a Feynman diagram with n internal lines and v vertices, then in the neighborhood of the singularity $z=z_0$ of this diagram[41,53]

$$f(z)\approx C\times\begin{cases}(z_0-z)^\gamma, & \gamma\neq 0,\ 1,\ 2,\ \ldots;\\ -(z_0-z)^\gamma\ln(z_0-z), & \gamma=0,\ 1,\ 2,\ \ldots,\end{cases} \qquad (121)$$

where $\gamma=(3n-4v+3)/2$ and, in accordance with Ref. 52, the contribution to f_l from this singularity as $l\to\infty$ is given by

$$f_l\approx C(\pi/2)^{1/2}\varepsilon_\gamma(\sinh\beta_0)^{\gamma/2}l^{-(\gamma+3/2)}\exp[-\beta_0(l+1/2)],$$
$$\varepsilon_\gamma=\begin{cases}1/\Gamma(-\gamma), & \gamma\neq 0,\ 1,\ 2,\ \ldots;\\ (-1)^\gamma\gamma!, & \gamma=0,\ 1,\ 2,\ \ldots,\end{cases} \qquad (122)$$

where $z_0=\cosh\beta_0$; $\beta_0>0$ (we assume that $z_0>1$; if $z_0<-1$, then $z_0=-\cosh\beta_0$).

Of course, this theorem cannot guarantee that the asymptotic regime that commences in the limit of very large l is preserved down to the physically interesting

[7)] For the physical meaning of the renormalization procedure, see Refs. 43, 45, and 46.

"surface" values $l \gtrsim kR$ that make the main contribution to the reaction cross section. The particular situation depends on both the position of the singularities and their strengths. Whereas the former can be established by investigating the singularities of the Feynman diagrams,[25-27,54] with regard to the latter little is known *a priori*. Therefore, the assumptions of the peripheral model are willy-nilly hypothetical and must be tested experimentally. This has now been done[17,43,49,55-63] for direct transfer reactions $B(a,b)A$, for which the nearest singularity is the pole $z=z_p$ [see (47)] corresponding to the diagram in Fig. 3. These reactions usually include (d,p) and $t,d)$ stripping reactions and the inverse reactions, and also a number of other reactions involving the transfer of a nucleon or a cluster. Comparison with experiment showed that for good quantitative description of the cross sections of the stripping reactions at small angles it is sufficient to take into account the contribution from only the pole $z=z_p$ and the effects of elastic rescattering of the particles in the entrance and exit channels,[60] while the contribution from the other more distant singularities in z, in particular the singularities due to the dependence of the reduced nuclear vertex form factors [see (54)] on the momentum transfer, can be ignored.[64]

Reactions of "pole" type are especially convenient for obtaining information about the nuclear vertex constants. In the general case, the differential cross sections of such reactions in the peripheral model have the form[17,48,65]

$$\sigma(E,z)=\sum_{j_a l_a j_A}\Big|\sum_{l_A l' J M} G_{abc}(l_a j_a)\,G_{ABc}(l_A j_A) \times W(l_a j_a l_A j_A;\,J_c J)\,T_{LL'}(l_a l_A J M;\,E,\,z)\Big|^2, \tag{123}$$

where G_{abc} and G_{ABc} are the nuclear vertex constants for the vertices $a\to b+c$ and $B+c\to A$; $T_{LL'}$ is a known function of E and θ which depends on the orbital angular momenta l_a and l_A at the vertices of the pole diagram in Fig. 3, the angular momentum J transferred in the reaction, its projection $M(-J\leqslant M\leqslant J)$, and the cutoff parameters L and L' for the orbital angular momenta l and l' of the entrance and exit channels $(l\geqslant L, l'\geqslant L')$. With allowance for the effects of elastic rescattering in the diffraction approximation,[66] each pole partial-wave amplitude $M_{ll'}(E)$ in the expansion of $T_{LL'}$ with respect to the partial waves is additionally multiplied by the factor $\exp[i(\delta_l+\varphi_{l'})]$, where δ_l and $\varphi_{l'}$ are the phase shifts of elastic aB and bA scattering.[60] The summation over j_a, l_a and j_A, l_A in (123) is extended to values restricted by conditions of the type (6). The structure of Eq. (123) simplifies in important special cases. For example, if the spin of the transferred particle is $J_c=0$ or $\frac{1}{2}$, then to each value of $j_a(j_A)$ there corresponds only one value of $l_a(l_A)$ and Eq. (123) takes the form

$$\sigma(E,z)=\sum_{j_a j_A}|G_{abc}(l_a j_a)\cdot G_{ABc}(l_A j_A)|^2\sigma_{LL'}(j_a j_A;\,E,\,z). \tag{124}$$

If $l_a=0$, then

$$\sigma(E,z)=|G_{abc}(0J_c)|^2\sum_{l_A}\Big(\sum_{j_A}|G_{ABc}(l_A j_A)|^2\Big)\sigma_{LL'}(l_A;\,E,\,z). \tag{125}$$

Thus, the combination in the peripheral model of the idea of the dominant role of the nearest singularities in z (Refs. 23, 24, and 67) and the idea of a peripheral nature of the direct processes in the l space leads to a simple parametrization of the cross sections of reactions involving transfer of a nucleon or a cluster in terms of the nuclear vertex constants. Fitting the cross sections calculated in accordance with Eqs. (123)–(125) to the experimental cross sections at small angles [for reactions of the type (d,p)] or large angles (for exchange elastic scattering), we can find the nuclear vertex constants. This method was used in Refs. 56 and 57 to obtain for the first time an estimate of the $^4\text{He}\to{}^3\text{He}+n$, $t\to d+n$, and $^3\text{He}\to d+p$ nuclear vertex constants, these agreeing with their modern estimates. Subsequently, the peripheral model was used to estimate the nuclear vertex constants for $^4\text{He}\to d+d$ (Ref. 62), $^6\text{Li}\to\alpha+d$ (Ref. 68), and other reactions and also to make numerous estimates of the vertex constants for the virtual or real (from a resonance state) emission of a nucleon by a complex nucleus.[17,43,55,57-60]

Dispersion K-matrix approach. The idea that the nearest singularities in z play an important role in direct reactions is also used in the dispersion K-matrix approach,[69-71] in which it is combined with the unitarity principle for the S matrix. It uses the formalism of the reaction K matrix, which is related to the S and T matrices by

$$S=(1-iK/2)(1+iK/2)^{-1};\quad T=K(1+iK/2)^{-1}, \tag{126}$$

with $S=1-iT$, i.e., $T=(2\pi)^4\mathcal{M}$, where $\mathcal{M}$ is the matrix defined in (3). In the dispersion K-matrix approach, the elements $K_{fi}(E,z)$ of the K matrix are regarded as analytic functions of z and it is assumed that the nearest singularities in z make the main contribution to them. In the z plane, the matrix elements $K_{fi}(E,z)$ have only those singularities of $T_{fi}(E,z)$ that are intrinsic singularities of the irreducible Feynman diagrams for the process $i\to f$ (Refs. 70 and 71). Diagrams are said to be irreducible or reducible, respectively, if they cannot or if they can be represented in the form of two or more blocks which are joined by internal lines and are such that each block describes a physical process permitted at the given energy. It follows from this in particular that if $T_{fi}(E,z)$ has a pole at the point $z=z_p$ then so does $K_{fi}(E,z)$ have a pole with respect to z at the same point and with the same residue. Similarly, if $T_{fi}(E,z)$ has an anomalous logarithmic singularity in the z plane, then so does $K_{fi}(E,z)$ have this singularity, and the discontinuities of T_{fi} and K_{fi} across the corresponding cuts are equal.

Accordingly, to describe direct reactions in the dispersion K-matrix approach one takes $K_{fi}(E,z)$ as the sum of the simplest Feynman diagrams—pole, triangle, etc. At the same time, in the T matrix one takes into account an infinite chain of these diagrams joined by lines corresponding to particles on the mass shell. The main advantage of the dispersion K-matrix approach is that the S matrix is unitary for any approximate but Hermitian K matrix.

The dispersion K-matrix approach has been used on a number of occasions to analyze experimental differential cross sections of elastic scattering processes in which the pole mechanisms of nucleon or cluster exchange corresponding to the diagram in Fig. 2 play an important role at large scattering angles θ. The corresponding nuclear vertex constants can be found by analyzing these processes.

In Refs. 69 and 70, the approach was used to analyze elastic $N\alpha$ scattering through large angles at energies below the threshold for the disintegration of α particles. The amplitude of the pole diagram corresponding to triton or ^{3}He exchange (for $p\alpha$ and $n\alpha$ scattering, respectively) was taken as the K matrix, and it proved possible to give a good description of the measured cross section in the backward hemisphere and to find the $\alpha \to t+p$ and $\alpha \to {}^3\mathrm{He}+n$ nuclear vertex constants. However, the s-wave phase shifts, for which the contribution from more distant singularities is the most important, had to be regarded as a disposable parameter. It was also shown that the contribution of double triton or ^{3}He exchange to the K matrix hardly affects the results. However, inclusion in the K matrix of the triangle diagram describing scattering of the incident nucleon on one of the nucleons of the α particle made it possible to describe the experiment satisfactorily in the complete range of scattering angles.

In Ref. 72, the approach was used to analyze elastic αt scattering at $E_\alpha = 8.25$ MeV. As in Refs. 69 and 70, the s-wave phase shift was fitted. Besides the purely pole approximation for the K matrix (proton exchange), allowance was also made (by solving the corresponding integral equation) for the contribution to the K matrix from multiple proton exchange, which was found to be important, noticeably improving the agreement between theory and experiment. We should like to take the opportunity of saying that the sign of the pole contribution to the K matrix in Ref. 72 should be reversed. This does not affect the value $G^2_{\alpha tp} = 10.0$ F for the $\alpha \to t+p$ nuclear vertex constant extracted in the purely pole approximation for the K matrix, but the value of $G^2_{\alpha tp}$ obtained in the calculations with the integral equation should be changed from 12.2 to 7.1 F.

In Ref. 73, finally, the dispersion K-matrix approach in the pole approximation (deuteron exchange) was used to analyze large-angle elastic $\alpha {}^6\mathrm{Li}$ scattering at $E_\alpha = 3.0$ MeV, and the ${}^6\mathrm{Li} \to \alpha + d$ nuclear vertex constant was estimated.

The calculations made in this approach showed that when the K matrix is parametrized in terms of the nearest singularities in z the differential cross sections of elastic scattering can be described in a wider range of angles than by the analogous parametrization of the T matrix in conjunction with the purely peripheral approximation based on the peripheral model. This is because the unitarization of the contributions from the nearest singularities in the dispersion K-matrix approach makes it possible to approximate the partial-wave scattering amplitudes in a range of l values which is extended by the inclusion of smaller l. It was found, however, that allowance for only the nearest exchange pole singularity or even the set of singularities corresponding to the infinite series of K-matrix ladder diagrams[72] does not yet ensure a good description of the lowest partial-wave amplitudes—in the cases considered, the s-wave scattering phase shifts. Compared with the peripheral model, the calculations by the approach are more sensitive to the values of the vertex constants because these occur nonlinearly in the unitarized partial-wave amplitudes of the T matrix. But at the same time the values of the vertex constants obtained from the few cases when the scattering cross section has been analyzed in the region of the exchange peak ($\theta \sim 180°$) in the framework of the peripheral model and in the dispersion K-matrix approach with allowance for the same singularities in z have differed little.

Hitherto, the dispersion K-matrix approach has been used to estimate the vertex constants from elastic scattering data in cases when all the inelastic channels have been closed or almost closed. As the inelastic channels are opened, the analysis is of course complicated by the rapid increase in the number of elements of the K matrix which must be parametrized in terms of the nearest singularities in the z plane. This restricts the practical application of the approach, at least at the present stage of development of the dispersion theory of direct reactions.

Distorted-wave Born approximation (DWBA). The DWBA is the most widely used method for analyzing data on direct reactions in which a nucleon or a cluster is transferred (see, for example, Ref. 39). The framework is the traditional formalism of wave functions and interparticle interaction potentials. In principle, the DWBA is suitable for obtaining information about the nuclear vertex constants, but the parametrization of the reaction amplitude usually employed in this method is inconvenient for such a purpose.

As an example, let us consider the classical deuteron stripping reaction $B(d,p)A$ at energies above the Coulomb barrier (see Fig. 3). In this case, the differential cross section in the DWBA is represented in the form[39]

$$\sigma(E, z) = S_{ABn}(lj)\sigma^{\mathrm{DWBA}}(E, z), \tag{127}$$

where $S_{ABn}(lj)$, the spectroscopic factor for separation of a neutron, is determined by Eqs. (100) and (101), and σ^{DWBA} is the so-called reduced, or single-particle, cross section (for simplicity, we assume that l and j each take one value). To calculate σ^{DWBA}, it is necessary to find the radial part of the overlap integral $R_{ABn}(lj;r)$ [see (103)], which is normalized to unity and occurs in the corresponding reaction amplitude $\mathscr{M}^{\mathrm{DWBA}}$; this radial part is usually approximated by the single-particle wave function $\psi_{lj}(r)$ calculated for the Woods–Saxon potential with fixed geometrical parameters r_0 (the radius) and a (the diffuseness) and a well depth fitted by means of the experimental neutron-separation energy, [8] so that

[8] In foreign literature, this is called the well-depth procedure.

$$\psi_{lj}(r) \approx -i^l b_{lj} \varkappa h_l^{(1)}(i\varkappa r), \quad r > R, \tag{128}$$

where R is of the order of the radius of the nucleus. The spectroscopic factor $S_{ABn}(lj)$ is estimated from the ratio of $\sigma(E, z)$ to $\sigma^{\text{DWBA}}(E, z)$ in the region of the main peak of the angular distribution at small angles.

It was pointed out in Refs. 60 and 74 that this method of parametrization and calculation of a nucleon-transfer-reaction cross section in the separate terms $S_{ABn}(lj)$ and $\psi_{lj}(r)$ does not adequately match the physical picture of a surface reaction. First, the low ($l < kR$) partial-wave reaction amplitudes in σ^{DWBA} cannot be accurately calculated in the DWBA since to do this one must take into account explicitly the coupling between the different reaction channels.[51] Second, in the surface reaction in the region of the main peak of the angular distribution the most important contribution is made by the peripheral partial-wave amplitudes ($l \gtrsim kR$), for whose calculation one can use the approximation (128), so that the absolute magnitude of these amplitudes is determined by the product $S_{ABn}^{1/2}(lj) b_{lj}$. Third, the b_{lj} are extremely sensitive to the values of the geometrical parameters r_0 and a (see, for example, Ref. 75), and no single well justified prescription exists *a priori* for their choice. For example, microscopic calculations of the overlap integrals[76] indicate that the geometrical parameters of the equivalent Woods–Saxon potential for which the function $\psi_{lj}(r)$ approximates best the microscopically calculated function $R_{ABn}(lj; r)$ depend strongly on the state and are very different from the usually employed "standard" values $r_0 \approx 1.25$ F and $a \approx 0.6$ F. This does not occasion surprise since the equivalent single-particle potential must take into account not only the influence of the average field but also the effects of the residual interactions between the nucleons in the nucleus. For this reason, the empirical values of $S_{ABn}(lj)$ obtained in the framework of the DWBA are in practice model-dependent. They depend strongly not only on the values of the geometrical parameters used in the well-depth procedure but also on the parameters of the optical potentials if the region within the nucleus ($l < kR$) makes a large contribution to the cross section σ^{DWBA}. If it does not, for example, if the contribution of the region within the nucleus is suppressed by the natural damping of the distorted waves and the oscillations of the integrand (for sufficiently heavy nuclei) or by the introduction of a cutoff to the radial integrals at the lower limit (as is frequently done in the case of light nuclei to match the calculated angular distributions to the measured distributions), then the absolute value of the cross section is in fact determined, not by $S_{ABn}(lj)$, but by the product $S_{ABn}(lj) \cdot b_{lj}^2$. It is this product that can be estimated in practice in a model-independent manner on the basis of the experimental data. But it follows from Eqs. (128), (95), (98), (100), (101), and (103) that this product is directly related to the $A \to B + n$ vertex constant:

$$|G_{ABn}(lj)|^2 = \pi (\hbar/\mu_{Bn} c)^2 S_{ABn}(lj) b_{lj}^2. \tag{129}$$

Therefore, if the DWBA is appropriately modified by eliminating the incorrectly calculated low partial-wave amplitudes, it can be used to estimate the nuclear vertex constants. It was shown in Refs. 60 and 74 for the example of the (d, p) reaction at $E_d = 12$ MeV on $1p$-shell nuclei (from ^{6}Li to ^{14}N) that the vertex constants for neutron separation from the final nucleus obtained by the peripheral model and by the modified DWBA with cutoff of the radial integrals at $R = 4$ F (such a cutoff was introduced in the original investigation, Ref. 77, in which the cross sections of these reactions were measured and then analyzed by the DWBA), and also by the singularity-subtraction method (see the following subsection), agree well with one another (see Sec. 3).

What we have said above applies not only to the (d, p) reaction at energies above the Coulomb barrier but also to the reactions (d, p) and (t, d) and the analogous reactions at energies below the Coulomb barrier, and also to nucleon transfer reactions initiated by heavy ions at any energies. In all these reactions, either because of the strong Coulomb repulsion or because of the combined influence of Coulomb repulsion and strong absorption (for heavy ions) the transfer process takes place at large distances. Therefore, making a DWBA analysis of the data, one can deduce Sb^2 or the product $(Sb^2)_1 (Sb^2)_2$ of such quantities in the case of a reaction between complex nuclei. This circumstance and also the fact that the combination Sb^2 depends weakly on the model parameters have been noted on several occasions.[78–82] But on none of these occasions was a clear physical interpretation of Sb^2 as a nuclear vertex constant given. Indeed, it was even asserted in Ref. 80 that Sb^2 does not carry new physical information and its introduction is merely an attempt to find a less model-dependent quantity than S.

Continuation to the pole and singularity subtraction. Conceptually, the methods based on these procedures derive from Chew's paper Ref. 83, in which he pointed out for the first time the possibility of determining the pion–nucleon coupling constants by extrapolating the differential cross section of nucleon–nucleon scattering to the exchange pion pole in the z plane. The possibility was actually realized in Refs. 84 and other papers. Subsequently,[85,86] there were developed the more effective mathematical methods of polynomial extrapolation of the cross section to the unphysical region based on the technique of optimal conformal mappings. These methods are reviewed with a detailed bibliography in Refs. 87 and 88.

The method of continuation to the exchange pole in the z plane was applied to the problem of determining the nuclear vertex constants from measured differential cross sections of elastic nuclear scattering in Ref. 13 (see also (Ref. 89), and it was then used for this purpose in Refs. 14, 20, 21, and 90. The singularity-subtraction method for determining vertex constants from measured differential cross sections of elastic scattering and nucleon or cluster transfer reactions was proposed in Ref. 15 and then used in Refs. 63, 91, and 92. As proposed in Ref. 15, this method is similar to the one used in Ref. 85, but differs from it in explicitly eschewing model arguments in the determination of the strength of the subtracted singularity and in using a different technical procedure. We shall outline the gist

of these two methods, without going into the technical details related to problems of optimal polynomial approximation, error estimation, etc., which have been discussed fairly fully in the cited reviews and the original papers.

We begin with the singularity-subtraction method. In it, as in the continuation method, it is assumed that the differential cross section $\sigma(E, z)$ of the reaction $B(a, b)A$ in the center-of-mass system is an analytic function of $z=\cos\theta$. Suppose the pole $z=z_p$ corresponding to the diagram in Fig. 3 is the singularity of the reaction amplitude nearest to the physical region $[-1, 1]$. Then, in accordance with what we have said in Sec. 1, $\sigma(E, z)$ can be written in the form

$$\sigma(E, z)=R^{\text{exch}}/(z_p-z)^2+P/(z_p-z)+\sigma_0(E, z). \tag{130}$$

Here, z_p is given by Eq. (47); $R^{\text{exch}}\equiv R^{\text{exch}}_{abc, ABc}$ is determined by Eqs. (58) and (59) in the general case and by Eqs. (61) and (62) in special cases; P is a constant; the function $\sigma_0(E, z)$ is regular for $z=z_p$ and, by hypothesis, its nearest singularity $z=z_0$ is further from the physical region than the pole $z=z_p$ (more precisely, the point $z=z_0$ lies outside the ellipse $\mathscr{E}(z_p)$ which passes through the point $z=z_p$, with foci at the points $z=\pm 1$, and semi-major axis $(|z_p+1|+|z_p-1|)/2$). We expand the functions $(z_p-z)\sigma(E, z)$ and $(z_p-z)^2\sigma(E, z)$ in series in Legendre polynomials $P_l(z)$ which converge within the ellipses $\mathscr{E}(z_p)$ and $\mathscr{E}(z_0)$ respectively:

$$(z_p-z)\sigma(E, z)=\frac{R^{\text{exch}}}{(z_p-z)}+[P+(z_p-z)\sigma_0(E, z)] \tag{131}$$

$$=R^{\text{exch}}\sum_{l=0}^{\infty}\hat{l}Q_l(z_p)P_l(z)+\sum_{l=0}^{\infty}\hat{l}a_lP_l(z) \tag{132}$$

$$=\sum_{l=0}^{\infty}\hat{l}A_lP_l(z), \tag{133}$$

$$(z_p-z)^2\sigma(E, z)=R^{\text{exch}}+[(z_p-z)P+(z_p-z)^2\sigma_0(E, z)] \tag{134}$$

$$=\sum_{l=0}^{\infty}\hat{l}B_lP_l(z), \tag{135}$$

where $Q_l(z_p)$ is a Legendre function of the second kind[44]; a_l is the coefficient in the expansion of the "background" term in the square brackets in (131). In accordance with (132) and (133), A_l can be represented as the sum of the contributions from the pole term A_l^p and the background term a_l:

$$A_l=A_l^p+a_l,\quad A_l^p=R^{\text{exch}}Q_l(z_p), \tag{136}$$

and A_l^p satisfy the well known recursion relation[44]

$$z_pA_l^p-(2l+1)^{-1}[lA_{l-1}^p+(l+1)A_{l+1}^p]=0. \tag{137}$$

Using the recursion relation for $P_l(z)$, from (133) and (135)–(137) we find the connection between B_l and the background coefficients a_l:

$$B_l=z_pa_l-(2l+1)^{-1}[la_{l-1}+(l+1)a_{l+1}]. \tag{138}$$

In accordance with (122), in the limit $l\to\infty$

$$A_l^p\sim\exp(-\beta_pl),\quad a_l\sim\exp(-\beta_0l), \tag{139}$$

where $\beta_0>\beta_p>0$. Therefore, for sufficiently large $l\geq l_0$ in (136) we can ignore a_l compared with A_l^p and determine R^{exch} by means of the relations

$$R^{\text{exch}}\approx A_l/Q_l(z_p),\quad l\geq l_0. \tag{140}$$

The main problem is to estimate l_0. For this, we can use the coefficients B_l in the expansion (135); for it follows from the asymptotic theorem formulated in the first subsection of this section, and also directly from (138) and (139), that

$$B_l=(z_p-z_0)a_l[1+O(1/|z_p-z_0|l)],\quad l\to\infty. \tag{141}$$

Therefore, the relation (140) is certainly valid for $l\geq l_0\gg 1$, for which

$$|B_l/(z_p-z_0)|\ll A_l. \tag{142}$$

Hitherto we have assumed that the cross section $\sigma(E, z)$ is defined in the complete interval $[-1, 1]$. But experimentally $\sigma(E, z)$ is known only at individual points of this interval and with errors. Therefore, for the functions (131) and (134) one can find only polynomial approximations in the form of finite sums over $P_l(z)$. In other words, only finite numbers N_A and N_B of the statistically significant coefficients A_l and B_l are known. Suppose the background coefficients a_l do not satisfy the recursion relation (137). Then in accordance with (138) the nonsignificance of B_l for $l\geq N_B$ entails the nonsignificance of a_l for $l\geq N_B-1$. If $N_A\geq N_B$, then from $N=N_A-N_B+1$ coefficients A_l with $l=N_B-1,\ N_B, \ldots, N_A-1$ one can find R^{exch} by means of (140). We obtain the most reliable result when $N\geq 3$ since then the fulfillment of the recursion relation (137) can be verified. For $N=2$, a control is provided by the self-consistency of the two values of R^{exch} found from A_l with $l=N_B-1$ and N_B.

In practice, R^{exch} is estimated in the singularity-subtraction method by the following procedure, which gives the method its name. One introduces the function

$$(z_p-z)\sigma(E, z)-\frac{R}{(z_p-z)}=\sum_{l=0}^{\infty}\hat{l}b_l(R)P_l(z), \tag{143}$$

where R is an adjustable parameter (the unknown strength of the pole in the cross section), and the coefficients $b_l(R)$ depend linearly on R:

$$b_l(R)=(R^{\text{exch}}-R)Q_l(z_p)+a_l. \tag{144}$$

For any two values of R, for example, $R=0$ and $R\neq 0$, one finds all the significant coefficients $b_l(R)$. If $R\neq R^{\text{exch}}$ then there are N_A such significant coefficients. One then determines the number N_B of the significant coefficients B_l. For $N_A\geq N_B$, one can find R^{exch} from the $N=N_A-N_B+1$ coefficients $b_l(0)$ and $b_l(R)$ with $l\geq N_B-1$ in accordance with the formula

$$R^{\text{exch}}=Rb_l(0)/[b_l(0)-b_l(R)]. \tag{145}$$

In the continuation method, R^{exch} is estimated by continuation to $z=z_p$ of the polynomial approximation of the experimental values of the function (134), i.e., for the

estimate one uses the expression

$$R^{\text{exch}} = \lim_{z \to z_p} (z_p - z)^2 \sigma(E, z) \approx \sum_{l=0}^{N_B - 1} \hat{l} B_l P_l(z_p). \quad (146)$$

Mathematically, the singularity-subtraction method and the continuation method are equivalent, i.e., they must give consistent values of R^{exch}, but in the former control on the authenticity of the R^{exch} values can be carried out more readily and perspicuously.

In actual calculations by both methods, two important improvements are introduced into the above scheme. First, a conformal mapping $w = w(z)$ of the original z plane is made. Usually[85,88] this is made onto the interior of the ellipse with foci at the points $w = \pm 1$. Under the mapping the interval $-1 \leq z \leq 1$ is mapped to the interval $-1 \leq w \leq 1$, the pole $w_p = w(z_p)$ remains inside the ellipse, and all the remaining singularities, including the cuts from the branch points, are mapped onto the ellipse. This accelerates the asymptotic convergence of the polynomial approximations for the function (134), or rather for the function $(w_p - w)^2 \sigma(E, z(w))$ (Refs. 87 and 88), since now the holomorphicity domain and the domain of convergence of the polynomial series coincide. One can therefore ensure better polynomial approximation of the function in the physical region with fewer adjustable parameters (expansion coefficients of the type B_l), which is important in view of the finite accuracy of the experiment. It is clear from relations of the type (138) or (141) that one simultaneously achieves a more rapid suppression of the background effects in the expansion coefficients of the function $(w_p - w)\sigma(E, z(w))$, which leads to a more accurate determination of R^{exch} from relations of the type (140) or (145).

Second, in a number of papers[15,90] polynomials orthonormalized on the set of experimental points by the weighting procedure of the least-squares method[93] were used. The coefficients of the expansions with respect to such polynomials are uncorrelated, and their standard deviations are equal to 1. This simplifies the calculation of the errors and makes the estimates of R^{exch} obtained in the singularity-subtraction method from different coefficients statistically independent (if, of course, the experimental errors are purely statistical). All the arguments adduced above remain in force; it is only necessary to replace the recursion relations for the Legendre polynomials by the well known Forsyth recursion relations[93] and modify Eqs. (137), (138), etc. accordingly.

In connection with the use of optimal conformal mappings, we note that rigorous theorems relating to the acceleration of the convergence apply to the asymptotic region, i.e., they hold for polynomial approximations of degree $n \to \infty$ (Ref. 87). In practice, because of the restricted experimental accuracy, one must deal with finite n, for example, $n = 3, 4$. However, the experience gained in particle physics shows that even for small n the expansions in the w plane converge more rapidly and are less sensitive to the influence of the background singularities, which are shifted to the boundary of the convergence ellipse. For all that, it is difficult to concur with the opinion which is sometimes expressed[14,88] to the effect that the results of calculations based on the conformal mapping technique are completely model-independent. Such calculations have a cryptomodellity in that, first, different conformal mappings take into account the analytic structure of the amplitude to different degrees and, second, the assertions of the asymptotic theorems are applied to polynomial approximations of finite order. We can find confirmation of this example in the results of Ref. 94, in which the pion–nucleon coupling constant f^2 was deduced from elastic np scattering data. Three different conformal mappings were used, each giving an equally good approximation of the function $(w_p - w)^2 \sigma(E, z(w))$ in the physical region for the same degree $n = 4$ of the approximating polynomials. Nevertheless, the estimates of f^2 obtained by continuing the different approximations to the exchange pion pole differ appreciably (up to 15%) because the various conformal mappings reflect the analytic structure of the np scattering amplitude differently.

One can count on success when using either the continuation method or the singularity-subtraction method only if the exchange peak corresponding to the pole mechanism in Fig. 3 is manifested sufficiently clearly in the experimental angular distribution. From this point of view, the physical condition of applicability of methods for analyzing experimental data such as the peripheral model, the DWBA, the continuation method, and the singularity-subtraction method is essentially the same. This was demonstrated in Refs. 60, 63, and 74, which show that the first two and the fourth of these methods give similar estimates for the vertex constants in those cases when the first two approaches give good descriptions of the angular distributions. Attempts to apply the continuation method to analyze elastic scattering data for which the exchange peak is not clearly manifested have little success and lead to unstable values of the vertex constants, these fluctuating strongly with the energy.[20,21] In this connection, we mention Ref. 95, in which it is pointed out that the method is unsuitable for analyzing data on reactions exhibiting resonances in the energy E, confirming from this point of view the doubt relating to the estimates of the vertex constants obtained in Refs. 20 and 21.

Finally, we mention Ref. 96, with its suggestion that rational functions rather than polynomial approximations should be used in the continuation method, and also Ref. 97, in which an attempt is made to take into account approximately in the continuation method the effects of the Coulomb distortions in the entrance and exit channels that lead to a modification of the pole singularity $z = z_p$ (the pole is transformed into a branch point).[9)]

[9)]We should like to take the opportunity to point out that the method proposed in Ref. 91 for taking into account Coulomb effects at the vertices of the pole diagram in Fig. 3 is incorrect since in this method the Coulomb effects eliminated from the pole term arise in the background terms. In addition, in Eq. (4) of Ref. 91 the exponents η_x and η_B must be replaced by $2\eta_x$ and $2\eta_B$.

Dispersion relation for the forward scattering amplitude. A dispersion relation for the amplitude of forward elastic scattering was first applied to scattering of a nucleon on the deuteron in Ref. 98. But it was only after Refs. 6 and 99 that forward dispersion relations were regularly used to determine vertex constants. We shall briefly outline this method of analyzing data, referring the reader to Ref. 6 and the original papers cited below for theoretical and practical details.

We consider elastic scattering $1+2\to 1+2$. Let $f(E_1)$ be the relativistic forward scattering amplitude in the system L_2 averaged over the spin projections of the particles; it depends on the relativistic kinetic energy

$$E_1=(m_1^2+p_1^2)^{1/2}-m_1 \tag{147}$$

of the projectile 1 in the system L_2. In the nonrelativistic limit, the amplitude $f(E_1)$ is equal to the amplitude $f_{12}^L(E_1)$ determined by Eqs. (25b), (27), and (28). In the relativistic case, Eq. (28), which determines the normalization and phase of $f(E_1)$, stands, but the connection between E_1 and the momentum p_1 is given by Eq. (147) and not (26). In accordance with Ref. 6, the once-subtracted forward dispersion relation for the amplitude $f(E_1)$ at $E_1=0$ can be written in the form

$$\begin{aligned}\operatorname{Re} f(E_1)=f(0)+\frac{p_1^2}{2\pi^2}\,\mathcal{P}\int_0^\infty\frac{\sigma_t(E_1')\,dp_1'}{(p_1'^2-p_1^2)}\\ +\sum_\alpha\frac{r_\alpha E_1}{(E_1-E_1^\alpha)\,E_1^\alpha}+\frac{E_1}{\pi}\int_{-2m_1}^{E_1^{\text{cut}}}\frac{\operatorname{Im} f(E_1')\,dE_1'}{(E_1'-E_1)\,E_1'}\\ -\frac{E_1}{4\pi^2}\int_0^\infty\frac{[\bar\sigma_t(E_1')-\sigma_t(E_1')]}{(E_1'+2m_1)\,(E_1'+E_1+2m_1)}\,p_1'\,dE_1'.\end{aligned} \tag{148}$$

Here, $\sigma_t(E_1)$ and $\bar\sigma_t(E_1)$ are the total cross sections for the interaction of particle 1 and its antiparticle $\bar 1$ with the target particle 2 at energy E_1. The symbol $\mathcal{P}$ denotes the principal value of the integral. The sum over α includes contributions from the direct and exchange poles of the amplitude $f(E_1)$, the residues r_α at these poles in the nonrelativistic limit being expressed in terms of the nuclear vertex constants by means of equations of the type (32), (33), (38), and (39). (Note that the r_α in (148) have the opposite sign to the corresponding r_β in Eq. (59) of Ref. 6, in which the residue of the amplitude $f(E_1)$ at the pole $E_1=E_1^\beta$ is defined as $-r_\beta$.)

The problem of determining the nuclear vertex constants in the method using forward dispersion relations reduces to determining the residues r_α. Since the positions of the poles E_1^α are known (they are given by the masses of the particles), it follows that r_α can be found by calculating all the remaining terms in Eq. (148). We can express $\operatorname{Re} f(E_1)$ and $f(0)$ and the integral over the physical region [the second term on the right-hand side of (148)] in terms of measured quantities—the phase shifts and the total cross section σ_t.[10] However, direct calculation of the fourth and fifth terms on the right-hand side of Eq. (148), which include a contribution from the "unphysical" cut $-\infty\leqslant E_1\leqslant E_1^{\text{cut}}$ ($E_1^{\text{cut}}<0$) entails difficulties due to the absence of reliable information on $\operatorname{Im} f(E_1)$ on a segment of the unphysical cut $(-2m_1, E_1^{\text{cut}})$ and the absence or incompleteness of the experimental data on the cross section $\bar\sigma_t$. According to the estimates of Ref. 6, the contribution of the fifth term is small at not too large E_1. Thus, for nucleon scattering on the ^{4}He nucleus at $E_1\sim 100$ MeV the absolute magnitude of the fifth term is less than 2% of the absolute magnitude of the amplitude $f(E_1)$. Therefore, in practical calculations the fifth term is usually ignored. The contribution of the fourth term either is ignored, if the start of the cut $E_1=E_1^{\text{cut}}$ is relatively far from the low-energy part of the physical region compared with the positions of the poles E_1^α (Ref. 19), or this contribution is parametrized in one way or another, for example, by means of a small number of effective poles (with each such pole, there are associated two parameters—the position of the pole and the residue at it).[6,100]

In practical calculations, one usually introduces the difference function

$$\Delta(E_1)\equiv\operatorname{Re} f(E_1)-f(0)-\frac{p_1^2}{2\pi^2}\,\mathcal{P}\int_0^\infty\frac{\sigma_t(E_1')\,dp_1'}{p_1'^2-p_1^2}, \tag{149}$$

which is calculated on the basis of the experimental data in a certain range of physical E_1 values. In the most favorable situation, when the contributions of the fourth and fifth terms to the right-hand side of (148) are unimportant in this interval,

$$\Delta(E_1)\approx\sum_\alpha\frac{r_\alpha E_1}{(E_1-E_1^\alpha)\,E_1^\alpha} \tag{150}$$

and the residues r_α can be found by fitting (150) to the calculated values of $\Delta(E_1)$. This method was used, for example, to analyze *nd* scattering data in Ref. 19, in which the vertex constants [or rather combinations of them, see (33) and (39)] were found for the $t\to d+n$ and $d\to p+n$ vertices. A similar method but with the introduction of effective poles to approximate the contribution of the unphysical cuts was used to analyze $n\alpha$ scattering in Refs. 6 and 100, in which the $\alpha\to{}^3\text{He}+n$ vertex constant was estimated. It was found that the empirical values of this vertex constant depend strongly on the number of effective poles used (1 or 2 poles). The procedure of approximating the unphysical cuts by a set of effective poles was criticized on account of its lack of uniqueness in Ref. 101, in which a different method was proposed for finding the r_α on the basis of an expansion of the residue function

$$R(E_1)=\frac{\Delta(E_1)}{E_1}\prod_\alpha(E_1-E_1^\alpha)\,E_1^\alpha \tag{151}$$

in a series in powers of E_1 and extrapolation of the truncated power series to the pole $E_1=E_1^\alpha$. This method was significantly improved in Ref. 102, in which a conformal mapping of the original E_1 plane was made in order to reduce the influence of the nearest unphysical cuts and reduce the number of adjustable parameters in the polynomial approximation of the residue function $R(E_1)$.

[10] We can also find $\operatorname{Re} f(E_1)$ by analyzing the Coulomb–nuclear interference at small angles (for the scattering of charged particles) or from the differential forward-scattering cross section using the values of $\operatorname{Im} f(E_1)$ given by the optical theorem (for neutral particles).[6,18]

In conclusion, we mention the important papers Refs. 103—106, in which forward dispersion relations were extended to the scattering of charged particles. In this case, one is confronted by a number of specific problems due to the infinite range of the Coulomb forces. Allowance for the Coulomb interaction leads to a modification of the forward dispersion relations themselves[104] as well as the methods for finding the residues, and in particular the residues at the exchange poles[105] (strictly speaking, the exchange poles become branch points).

Other methods. We here mention some other methods used to estimate the nuclear vertex constants.

In calculations of the partial-wave scattering amplitudes by the dispersion N/D method, the vertex constants occur as parameters and can be determined by comparison with measurements. In this way, the $t-d+n$ (Refs. 107 and 108) and $^6\mathrm{Li}-\alpha+d$ (Ref. 109) vertex constants were determined (for the value of the last vertex constant, see Ref. 110).

In Ref. 110, it was pointed out that the values of the nuclear vertex constants could be deduced from data on the nuclear charge form factors. In this approach, it is assumed that the part of the structure form factor that varies rapidly at small momentum transfers q (the body form factor in foreign literature) gets its main contribution from the amplitudes of the simplest Feynman diagrams which have singularities in q nearest the point $q=0$. The upshot is that the vertex constants can be expressed in terms of the rms charge radii of the nuclei and known quantities (binding energies, masses). This method was used in Refs. 110 and 111 to find the $d-p+n$ and $^6\mathrm{Li}-\alpha+d$ vertex constants. In Ref. 112, the vertex constant G_{abp}, which characterizes the virtual emission of a proton by nucleus $a(a-b+p)$, was estimated by using the fact that the absolute normalization of the tail of the density $\rho(r)$ of the charge distribution (more precisely, the part of it that corresponds to the structure form factor) is proportional to $|G_{abp}|^2$ (see Ref. 110). The density $\rho(r)$ was constructed on the basis of data on the charge form factor obtained from elastic scattering of electrons on the nucleus. This led to a determination of $|G_{abp}|^2$ for a number of nuclei from $^4\mathrm{He}$ to $^{40}\mathrm{Ca}$, and also the $^6\mathrm{Li}-\alpha+d$ vertex constant. Data on the charge form factors were also used to estimate vertex constants in Ref. 113.

The nuclear vertex constants can also be determined from the reactions $(p,2p)$, (p,pd), $(\alpha,2\alpha)$, etc., of quasielastic knockout of nucleons or clusters from nuclei. To analyze such reactions in the region of the quasielastic scattering peak one usually uses the impulse approximation with plane or distorted waves, and recently the peripheral model too (for references to the corresponding theoretical papers, see Refs. 114-116). As yet, the accuracy of the estimates deduced from the knockout reactions is not as good as for the estimates based on binary reactions. The reasons for this are to be sought in the imperfection of the theory of three-particle reactions as well as in the generally inadequate accuracy of the experimental data (especially at high energies, where one can hope for better conditions of applicability of the impulse approximation or the peripheral model).

Theoretical calculations of the nuclear vertex constants. The vertex constants can be found, if one knows the wave functions of the nuclei participating in a virtual decay, by using Eq. (98), which relates the vertex constant to the coefficient in the asymptotic behavior of the radial part of the overlap integral.

For the simplest case of $d-p+n$ virtual decay, the asymptotic coefficients of the S and D states of the deuteron and the vertex constants corresponding to them were calculated with different nucleon–nucleon potentials (Sec. 3).

For the nucleus $^3\mathrm{H}$ (respectively $^3\mathrm{He}$) the vertex constant corresponding to the decay $t-d+n$ to the S state of the relative motion of the d and the n was calculated in a number of studies[117-119] by solving the three-nucleon integral Faddeev equation with different NN potentials. The results of these calculations are summarized in Sec. 3 and demonstrate the rather strong dependence of the vertex-constant values on the form of the NN potential. In Refs. 118 and 119, the vertex constant was also calculated for the decay of the triton into the singlet deuteron (a virtual state of the deuteron) and a neutron, and this was found to depend very strongly on the form of the NN potential.

For the $^4\mathrm{He}$ nucleus, the vertex constants corresponding to the decays $^4\mathrm{He}-{}^3\mathrm{He}+n$ and $^4\mathrm{He}-d+d$ were calculated in Ref. 120 by solving the four-nucleon Fadeev–Yakubovskiĭ integral equation with a separable S-wave Yamaguchi potential (see Sec. 3).

The vertex constant for the virtual decay $^6\mathrm{Li}-\alpha+d$ to the S state of relative motion of the α and the d was calculated in Ref. 121 in a three-body model $(\alpha+n+p)$ and also in various $(\alpha+d)$ cluster models (see Sec. 3).

For heavier nuclei, microscopic calculations of the vertex constants on the basis of given NN potentials are as yet unknown.

3. NUCLEAR VERTEX CONSTANTS FOR SPECIFIC NUCLEI

In this section, we gather together the existing values of the vertex constants for the lightest nuclei ($A \leq 6$); we also briefly discuss the vertex constants for heavier nuclei. Information on the vertex constants for the vertices we consider are also contained to some extent in Refs. 29 and 122–127. For all these vertices, an anomalous asymptotic behavior of the form (97) does not hold, and we shall assume that the phases of the vertex constants are determined by the relation (18).

The $d-p+n$ vertex. For this vertex, the conservation laws permit two l and s sets: $l=0$, $s=1$ [$G_{dpn}(01) \equiv G_{d0}$], and $l=2$, $s=1$ [$G_{dpn}(21) \equiv G_{d2}$]. The values of G_{d0}^2 and $\rho_d^2 \equiv (G_{d2}/G_{d0})^2$ are given in Table I. In the second column, we indicate the method used to find the given value. As can be seen in Table I, the theoretical values of G_{d0}^2 and ρ_d^2 calculated in accordance with (98) by solv-

TABLE I. Nuclear vertex constants for the $d \to p + n$ vertex.

No.	Method of determining the vertex constants	G_{d0}^2, F	$\rho_d^2 \times 10^4$	Literature
1	SSE*, Hamada–Johnston	0.440	7.24	[128]
2	SSE, modified Hamada–Johnston	0.433	6.99	[128]
3	SSE, Reid with hard core	0.429	6.71	[129]
4	SSE, Reid with soft core	0.426	6.87	[129]
5	SSE, Peltola with hard core	0.432	6.90	[130]
6	SSE Peltola with soft core	0.433	6.93	[130]
7	Forward dispersion relation, *nd* scattering	0.43 ± 0.01	—	[19]
8	Peripheral model, *pd* scattering	0.40 ± 0.01	—	[61]
9	DWBA, below-barrier reaction $^{208}Pb(d,p)^{209}Pb$ with polarized deuterons	—	5.40 ± 0.80	[131]
10	Forward dispersion relation, *pd* scattering	0.43	—	[105]
11	Continuation method, *nd* scattering	0.32–0.43	—	[20]
12	Continuation method, *pd* scattering	0.399 ± 0.005	—	[96]

*SSE stands for the solution of the Schrödinger equation for the given potential.

ing the Schrödinger equation for different realistic potentials differ little: $G_d^2 = (0.43 \pm 0.01)$ F, $\rho_d^2 = (6.7 - 7.2) \cdot 10^{-4}$. The same value $G_{d0}^2 = 0.43$ F is obtained by using a forward dispersion relation for *nd* scattering[19] and *pd* scattering,[105] and also by applying Eq. (76) to the amplitude of *np* scattering in the triplet *S* state written in the effective-range approximation:

$$\mathcal{M}_{np}(q_{np}) = 4\pi\lambda\!\!\!^{-}_N [\varkappa + iq_{np} - (\varkappa^2 + q_{np}^2) r_t/2]^{-1};\ G_{d0}^2 = 8\pi\varkappa (1 - \varkappa r_t)^{-1} \lambda\!\!\!^{-2}_N, \quad (152)$$

where r_t is the triplet effective range. The single phenomenological value $\rho_d^2 = (5.40 \pm 0.80) \cdot 10^{-4}$ (ninth row of Table I) was obtained by an analysis in the DWBA framework of data on the tensor analyzing power in the reaction $^{208}Pb(d,p)^{209}Pb$* at energy 9 MeV of polarized deuterons.

Note that the values of the vertex constants obtained by means of forward dispersion relations and the continuation method (rows 7 and 10–12 in Table I) refer strictly to the combination $G_{d0}^2 + G_{d2}^2$, but because ρ_d^2 is small we can assume that $G_{d0}^2 + G_{d2}^2 \approx G_{d0}^2$.

The value of G_{d0}^2 was also determined in Ref. 132 by the continuation method from low-energy *pd*-scattering data, but the neglect of Coulomb effects and, apparently, the low accuracy of the data led to a large spread in the deduced values of G_{d0}^2, which we do not give here.

Considered as a whole, we can conclude that the $d \to p + n$ vertex constants are known with good accuracy (especially G_{d0}), and that their values can be used to deduce vertex constants for more complicated nuclei.

The $t \to d + n$ and $\tau \to d + p$ ($t \equiv {}^3H$, $\tau \equiv {}^3He$) vertices. For these vertices, $l = 0$, $s = 1/2$ or $l = 2$, $s = 3/2$.[11] If effects that break isotopic invariance are ignored, $G_{tdn}(ls) = G_{\tau dp}(ls) = G_{Tl}$. The vertex constants G_{Tl}^2, expressed in F, are related as follows to $C^2(^3H)$, $D^2(^3H)$, and r_t, which are used in Refs. 117, 123, and 19:

$$\left.\begin{aligned} G_{T0}^2 &= 27\pi\varkappa_{dN}^T \lambda\!\!\!^{-2}_N C^2(^3H)/4 = 3 \cdot 10^4 D^2(^3H)/2\ (\hbar c)^2; \\ r_t &= -r_{TNd}^{dir} = (4\pi\lambda\!\!\!^{-}_N)^{-1} (G_{T0}^2 + G_{T2}^2), \end{aligned}\right\} \quad (153)$$

where $\lambda\!\!\!^{-}_N = \hbar/m_N c = 0.2103$ F; $(\hbar c)^2 = 3.89 \cdot 10^4$ MeV$^2 \cdot$ F^2; $C^2(^3H)$ and r_t are dimensionless; and $D^2(^3H)$ is expressed in the units 10^4 MeV$^2 \cdot$ F^3. Note also that for an arbitrary vertex $a \to b + c$ with $l = 0$ the vertex constants $G_{abs}^2(0s)$ can be expressed in terms of the dimensionless quantity $\tilde{g}_{bc;0}^a$ of Ref. 122:

$$G_{abc}^2(0s) = 2\pi^2 (\hbar/\mu_{bc}c) (\tilde{g}_{bc;0}^a)^2. \quad (154)$$

The available information on the G_{T0}^2 values are collected in Table II; the sparse information available about $\rho_T^2 \equiv G_{T2}^2/G_{T0}^2$ is given below in the text. The letters t and τ in the third column of Table II indicate the vertices to which the given value applies. In the first three rows, which correspond to the analysis of different reactions (and at different energies) in the framework of the peripheral model, averaged values including data on both vertices are given. The following four rows correspond to the dispersion methods: forward dispersion relations and the N/D method. In the former case, r_{tnd}^{dir} is the quantity actually determined, and the values given in the table refer to the combination $G_{T0}^2 + G_{T2}^2$, but because ρ_T^2 is small (see below) $G_{T0}^2 + G_{T2}^2 \approx G_{T0}^2$.

The value $G_{T0}^2 = (1.04 \pm 0.19)$ F (eighth row of Table II) corresponds to the mean value $D^2(^3H) = (2.7 \pm 0.5) \cdot 10^4$ MeV$^2 \cdot$ F^3 given in Ref. 123, which follows from analysis of the below- and above-barrier (t,d) reactions on the nuclei ^{12}C, ^{26}Mg, ^{40}Ca, ^{90}Zr, and ^{208}Pb in the DWBA.[134]

In rows 9–11 of Table II we give the values of G_{T0}^2 ob-

[11] Note that in both cases $j = 1$, where $\mathbf{j} = \mathbf{l} + \mathbf{J}_d$. Therefore $[G_{\tau dN}(l, j=1)]^2 = [G_{\tau dN}(l,s)]^2$; $T = t,\ \tau$; $N = n, p$.

TABLE II. Nuclear vertex constants for the $t \to d+n$ and $\tau \to d+p$ vertices.

No.	Method of determining the vertex constants	G^2_{T0}, F	Literature
1	Peripheral model, reactions $d(d,p)t$, $\tau(n,d)d$, $\tau(n,t)p$, $t(p,\tau)n$, $\tau(p,\tau)p$	$1.11 \pm 0.07(t,\tau)$	[58,61,133]
2	Peripheral model, reactions (d,t) and (τ,d) on complex nuclei	$1.11 \pm 0.07(t,\tau)$	[57]
3	Combined results of peripheral model	$1.11 \pm 0.05(t,\tau)$	[57,58,61,133]
4	Forward dispersion relation, nd scattering	$1.01 \pm 0.11(t)$	[19]
5	Forward dispersion relation with conformal mapping with respect to the energy variable, nd scattering	$0.92 \pm 0.13(t)$	[102]
6	N/D method, nd scattering	$1.43(t)$	[107]
7	N/D method, nd scattering	$1.06 \pm 0.07(t)$	[108]
8	DWBA	$1.04 \pm 0.19(t)$	[123,134]
9	Continuation method, $p\tau$ scattering (without allowance for Coulomb interaction)	$1.02(\tau)$	[90]
10	Continuation method, $p\tau$ scattering (with allowance for Coulomb interaction)	$1.12 \pm 0.03(\tau)$	[97]
11	Continuation method, nt scattering	$1.50(t)$	[14,20]
12	Singularity-subtraction method, reaction $d(d,p)t$	$1.15 \pm 0.03(t)$	[15]
13	Low-energy parametrization of nd scattering amplitude	$1.05(t)$	[135]
	Solution of Faddeev equation for NN potentials		
14	Reid with soft core	$1.02 \pm 0.01(t)$	[117]
15	Separable Yamaguchi	$1.9(t)$	[118]
16	Malfliet–Tjon	$1.6(t)$	[119]
17	Darewich–Green	$1.4(t)$	[119]
18	Bressel–Kerman–Ruben	$1.03(t)$	[119]
	Solution of equations for interpolation wave function with NN potentials		
19	Rectangular well	$0.6(t)$	[136]
20	Eikemaier–Hackenbroich	$0.4(t)$	[136]

tained by continuation to the exchange deuteron pole of the experimental differential cross sections of elastic $p\tau$ or nt scattering at different energies $E_{p,n} < 20$ MeV. The directly determined quantity was R^{exch}_{tnd} [see (68) and (67)]; the values given in Table II were obtained neglecting G^2_{T2}. In Ref. 97 (tenth row of Table II) allowance was made for the Coulomb interaction between the proton and the ^{3}He nucleus, though the procedure used for this purpose is not entirely clear; in addition, no allowance was made for the Coulomb effects at the $^3\text{He} \rightleftarrows d+p$ vertex. The value 1.15 ± 0.03 F (row 12) corresponds to the combination $G^2_{T0} + G^2_{T2}$. The value $G^2_{T0} = 1.05$ F (row 13) was obtained in Ref. 135 by a parametrization of the doublet S phase shift $^2\delta_0$ of low-energy nd scattering.

The remaining values of G^2_{T0} given in Table II were found from the ^{3}H wave functions obtained by solving the Faddeev equations or the equations for the interpolation wave functions[137,138]; in Ref. 117, an analytic approximation of the solution was used with subsequent continuation of the nuclear vertex form factor to the point $q_{nd} = i\varkappa_{nd}$ [see (92)], and in Refs. 118 and 119 the value of G_{T0} was found directly from the Faddeev equations by means of integral expressions analogous to the expressions (32) and (33) of Ref. 37. The values of G^2_{T0} obtained in the framework of the interpolation approach by means of Eq. (98) (rows 19 and 20 in Table II) are appreciably lower than the remaining values of G^2_{T0}, which may be due to the use of the lowest approximation of the method of K harmonics in the actual calculations in Ref. 136.

With regard to ρ^2_T, the only information available on it was obtained in the framework of the peripheral model.[57,58,61,133] The first three rows of Table II correspond to the following values of ρ^2_T, respectively: $(2.8 \pm 0.9) \cdot 10^{-2}$, $(6.0 \pm 1.0) \cdot 10^{-2}$, and $(4.5 \pm 0.8) \cdot 10^{-2}$.

It can be seen from Table II that the empirical values of G^2_{T0} obtained from data on nuclear reactions by means of different methods agree well with one another (except for the results of Refs. 107, 14, and 20[12]): G^2_{T0}

[12]As we have already noted, the claim made in Refs. 14 and 20 that the result obtained there agrees with Locher's[19] is incorrect. In addition, an evaluation by one of us (I. Borbely) by the same continuation method and of the same data as in Refs. 14 and 20 gave $G^2_{T0} = 2.66$ F and not $G^2_{T0} = 1.50$ F.

TABLE III. Nuclear vertex constants for the $\alpha \to t+p$ and $\alpha \to \tau+n$ vertices.

No.	Method of determining the vertex constants	G^2_{tp} or $G^2_{\tau n}$, F	Literature
1	Peripheral model, reactions $\tau(d,p)\alpha$, $\alpha(p,d)\tau$, $\alpha(p,\alpha)p$	$7.3 \pm 0.4(TN)$	[49, 58, 62]
2	Forward dispersion relation, $n\alpha$ scattering	$11.3 \pm 1.4(\tau n)$	[100]
3	Forward dispersion relation with conformal mapping with respect to the energy variable, $n\alpha$ scattering	$14.4 \pm 2.8(\tau n)$	[102]
4	Forward dispersion relation with allowance for Coulomb interaction, $p\alpha$ and $n\alpha$ scattering	$13.4 \pm 1.1(tp)$ $10.6 \pm 1.1(\tau n)$	[101]
5	Continuation method, $n\alpha$ scattering	$8.5 \pm 1.3(\tau n)$	[13]
6	Parametrization of wave function of relative motion of the t and the p or the τ and the n in ^{4}He	$6.7(tp)$ $7.1(\tau n)$	[113]
7	Dispersion K-matrix approach, $p\alpha$ and $n\alpha$ scattering	$7.0(tp)$ $8.0-8.5(\tau n)$	[69, 70]
8	Dispersion K-matrix approach, $t\alpha$ scattering	$7.1-10.0(tp)$	[72]
9	Exchange mechanism of $\tau\alpha$ scattering	$17.5(\tau n)$	[139]
10	Calculation by means of the Faddeev–Yakubovskiĭ equations with separable Yamaguchi NN potential	$17.9 \pm 1.7(TN)$	[120]

$\approx 1.0-1.1$ F. At the same time, the spread of the theoretical G^2_{T0} values obtained with different NN potentials is appreciably greater. It is interesting to note that potentials with the correct one-pion tail (Reid and Bressel–Kerman–Ruben) lead to nearly equal values of G^2_{T0} that agree with the empirical values. The value of ρ^2_T needs to be made more precise.

The $\alpha+t+p$ and $\alpha \to \tau+n$ ($\alpha \equiv {}^4$He) vertices. For the decays $\alpha \to t+p$ and $\alpha \to \tau+n$ we have $l=0$ and $s=0$ unambiguously; denoting $G_{tp} \equiv G_{\alpha tp}(00)$ and $G_{\tau n} \equiv G_{\alpha \tau n}(00)$, we have in the framework of isotopic invariance $G_{tp} = -G_{\tau n} \equiv G_{TN}$. The vertex constant G^2_{TN}, expressed in terms, is related to the dimensionless quantities r_0, R_N, and C^2 in Refs. 100, 101, and 113:

$$G^2_{TN} = (16\pi/3)\,\lambda_N r_0 = (16\pi/3)\,\lambda_N R_N = (32/9)\pi\varkappa^{\alpha}_{TN}\lambda^2_N C^2. \qquad (155)$$

The values of G^2_{TN} found by the different methods are given in Table III. The letters tp and τn in the third column indicate to which of the two vertices the value corresponds. The value of G^2_{TN} in the first row was obtained by averaging the results found in the peripheral model for different reactions and including both vertices. In rows 2–4 we give the values of G^2_{TN} found by means of forward dispersion relations for elastic $N\alpha$ scattering. In Ref. 101, in deriving the forward dispersion relation for $p\alpha$ scattering, Plattner separated out explicitly the amplitude of purely Rutherford $p\alpha$ scattering, and also the transmission factors corresponding to the Coulomb interaction of the p and the α in the initial and final states for the triton-transfer mechanism; the Coulomb effects at the $\alpha \to t+p$ vertex were ignored. Without going into detail about the remaining values of G^2_{TN} in Table III, let us merely mention that the overestimation of the single theoretical value $G^2_{TN} = 17.9 \pm 1.7$ F (row 10) compared with the empirical values may be due to appreciable overbinding of the α particle for the NN potential used in Ref. 120. It follows from Table III that the vertex constants G^2_{tp} and $G^2_{\tau n}$ are as yet known with lower accuracy than G^2_{T0}. As one would expect, $G^2_{tp} \approx G^2_{\tau n}$.

The $\alpha \to d+d$ vertex. For this vertex, $l=s=0$ or $l=s=2$. Data on the vertex constant $G^2_{dd0} \equiv [G_{\alpha dd}(00)]^2$ are given in Table IV. The value $G^2_{dd0} = 0.43$ F, which differs strongly from the remaining values of Table IV, follows [if $G^2_{\alpha dd}(22)$ is ignored] from the mean value $\gamma^{\text{exch}}_{\alpha dd} = 0.108$ given in Ref. 21; to the individual values of $\gamma^{\text{exch}}_{\alpha dd}$ found in Ref. 21 by means of the continuation method at different energies there correspond values of G^2_{dd0} in the interval 0.15–0.69 F. Note that this result could not be reproduced when one of us (I. Borbely) made a similar analysis in the framework of the continuation method of the same data; this reanalysis gave $G^2_{dd0} = 8-33$ F.

The vertex constant G^2_{dd0} is related to the quantity C^2_{obs} used in Ref. 127 by

$$G^2_{dd0} = 6\pi\varkappa^{\alpha}_{dd}\lambda^2_N C^2_{\text{obs}}. \qquad (156)$$

Apart from the quantity C_{obs}, which differs only by trivial factors from the asymptotic constant $C_{\alpha dd}(00)$ [see (95)], the quantity $C = C_{\text{obs}}/\sqrt{P_{dd}}$, where P_{dd} is the prob-

TABLE IV. Nuclear vertex constants for the $\alpha \to d+d$ vertex.

No.	Method of determining the vertex constants	G^2_{dd0}, F	Literature
1	Peripheral model, $d\alpha$ scattering and reaction (d,α) on complex nuclei	12–30	[62]
2	Calculation by means of Faddeev–Yakubovskiĭ equations with separable Yamaguchi NN potential	18.1 ± 1.3	[120]
3	Continuation method, $d\alpha$ scattering	0.43	[21]

TABLE V. Nuclear vertex constants for the $^6\text{Li} \to \alpha + d$ vertex.

No.	Method of determining the vertex constants	$G^2_{\alpha d}$, F	$(G^r_{\alpha d})^2$, F	Literature
1	Phenomenological cluster wave functions	0.19		[126]
2	Wave function of three-body model	0.35		[121]
3	Cluster wave function		0.72	[141]
4	Calculation of ^{6}Li charge form factor	0.13–0.20	0.3–0.4	[110, 111]
5	Calculation of charge distribution in ^{6}Li		0.27	[112]
6	N/D method, $d\alpha$ scattering	0.13		[109]
7	Analysis of elastic $d\alpha$ scattering	0.10		[122]
8	Forward dispersion relation, $d\alpha$ scattering		0.22 ± 0.01	[142]
9	Molecular wave function method, α^6Li scattering		0.58 ± 0.06	[143]
10	Peripheral model, d^6Li scattering	0.39 ± 0.09	0.83 ± 0.11	[68]
11	DWBA, reaction $^6\text{Li}(p, {}^3\text{He})^4\text{He}$		0.42	[144]
12	Impulse approximation with cutoff, reaction $^6\text{Li}(p, pd)^4\text{He}$	0.12–0.31		[145]
13	Impulse approximation with cutoff, reaction $^6\text{Li}(p, pd)^4\text{He}$		0.18–0.68	[68]
14	Impulse approximation with distorted waves, reaction $^6\text{Li}(p, pd)^4\text{He}$	0.27–0.33		[146]
15	Impulse approximation, reaction $^6\text{Li}(p, pd)^4\text{He}$	0.096 ± 0.05		[147]
16	Impulse approximation, reaction $^6\text{Li}(p, pd)^4\text{He}$	0.114		[148]
17	Pole approximation, reaction $^6\text{Li}(p, pd)^4\text{He}$	0.061		[149]
18	Impulse approximation with cutoff, reaction $^6\text{Li}(d, 2d)^4\text{He}$	0.37	0.54–1.41	[68, 150]
19	Impulse approximation with cutoff, reaction $^6\text{Li}(\alpha, 2\alpha)^2\text{H}$	0.12		[151]
20	Impulse approximation with cutoff, reaction $^6\text{Li}(\alpha, 2\alpha)^2\text{H}$	0.028 ± 0.013		[152]
21	Peripheral model, reaction $^6\text{Li}(\alpha, 2\alpha)^2\text{H}$	0.3–1.0	0.75–2.5	[115]
22	Impulse approximation with distorted waves, reaction $^6\text{Li}(\alpha, 2\alpha)^2\text{H}$	0.10 ± 0.04		[153]
23	Pole approximation, reaction $^6\text{Li}(\alpha, 2\alpha)^2\text{H}$	0.050 ± 0.006		[154]
24	Impulse approximation with distorted waves, reactions $^6\text{Li}(e, ed)^4\text{He}$ and $^6\text{Li}(e, e\alpha)^2\text{H}$	0.22–0.23		[155]

ability of the $(d+d)$ state in ^{4}He, was also considered in Ref. 127; a clear definition of P_{dd} is not given in Ref. 127. In this connection, we note that the value of C, in contrast to C_{obs}, cannot be found experimentally, and it is therefore unclear how the empirical values of C^2 collected in Ref. 127 were obtained. The meaning of the theoretical quantities C^2 given in Ref. 127 with reference to Ref. 140 is also unclear, because in Ref. 140 there is a calculation of only D_0^2, which occurs in calculations of the reaction cross sections in the DWBA and which is related to the vertex constant $G_{\alpha dd}(00; q = 0)$, and not to the vertex constant $G_{dd0} = G_{\alpha dd}(00; q = i\varkappa^{\alpha}_{dd})$; because of the large binding energy of ^{4}He, the values of the vertex constant for $q = 0$ and $q = i\varkappa^{\alpha}_{dd}$ may be very different. Moreover, in Ref. 140 a Gaussian wave function was used for ^{4}He, and this does not have the asymptotic behavior (95), so that C for it becomes meaningless.

Thus, G^2_{dd0} is as yet known only with poor accuracy. No information is available on the vertex constant $G^2_{\alpha dd}(22)$. In principle, as we have pointed out above, it could be determined by means of the continuation method from analysis of elastic $d\alpha$ scattering at different energies.

The $^6\text{Li} \to \alpha + d$ *vertex.* In this case $l = 0$, $s = 1$ or $l = 2$, $s = 1$. The main information about $G^2_{\alpha d} \equiv G^2_{^6\text{Li}\alpha d}(00)$ is collected in Table V. Because of the low threshold of disintegration of ^{6}Li into $\alpha + d$ ($\varepsilon^{^6\text{Li}}_{\alpha d} = 1.47$ MeV) the Coulomb parameter η for the $^6\text{Li} \to \alpha + d$ vertex is not small ($\eta = 0.30$), and the Coulomb effects at the vertex discussed at the end of Sec. 1 are important. Therefore, in Table V we give separately the values found without $(G^2_{\alpha d})$ and with $[(G^r_{\alpha d})^2]$ allowance for these effects.

In the first row of Table V we give the value obtained in accordance with the formula

$$G^2_{\alpha d} = P_{\alpha d} C^2_{^6\text{Li}-\alpha-d} (2\pi\varkappa^{^6\text{Li}}_{\alpha d}/\mu^2_{\alpha d}), \tag{157}$$

where $C_{^6\text{Li}-\alpha-d}$ is the coefficient in the asymptotic behavior of the ^{6}Li wave function in the $\alpha + d$ channel, and $P_{\alpha d}$ characterizes the probability of the $\alpha + d$ state in ^{6}Li (Ref. 126); to the value $G^2_{\alpha d} = 0.19$ F there corre-

spond the values $C^2_{^6\mathrm{Li}-\alpha-d}=5.7$ and $P_{\alpha d}=0.7$ given in Ref. 126, the value 5.7 being the mean found from various (basically cluster) wave functions of ^{6}Li that do not take into account the Coulomb interaction between the α and d clusters. In the second row, we give the value of $G^2_{\alpha d}$ obtained in accordance with the expressions (95) and (98) from the ^{6}Li wave function found in Ref. 121 by solving the Schrödinger equation in the model of three bodies (α, p, n) with separable NN and αN potentials without allowance for the Coulomb interaction. The value of $(G^r_{\alpha d})^2$ in the third row was obtained from the cluster wave function of Ref. 141; the αd interaction was described by the sum of a nuclear potential of Woods–Saxon type and the Coulomb potential.

In rows 4 and 5 we give the values found from the data on the ^{6}Li charge form factor (see Sec. 1). The values of $G^2_{\alpha d}$ obtained by analyzing the low-energy elastic $d\alpha$ scattering are given in rows 6–8. Note the value $(G^r_{\alpha d})^2 = (0.22 \pm 0.01)$ F found by a forward dispersion relation[142] with allowance for the Coulomb interaction of α and d. It should, however, be noted that in Ref. 142 allowance was made for only the contribution from the ^{6}Li pole in the direct channel; the pole diagram with deuteron exchange, whose contribution may be important because of the large value of the vertex constant G^2_{dd0}, was ignored. The next three rows contain the values of $G^2_{\alpha d}$ obtained by analyzing binary processes at low energies in the framework of various models under the assumption of the deuteron-exchange mechanism [α ^{6}Li scattering and the reaction ^{6}Li$(p, ^3\mathrm{He})$ ^{4}He] or α-particle exchange (d ^{6}Li scattering). In all three cases, the Coulomb $d\alpha$ interaction was taken into account.

The remaining values of $G^2_{\alpha d}$ given in Table V were determined by analyzing experimental data on quasielastic knockout of d or α clusters from ^{6}Li in the range 10–700 MeV of bombarding-particle energies. The theoretical analysis was based on either the impulse approximation (with plane or distorted waves, with or without cutoff of the radial integrals) or the pole peripheral model of the quasielastic knockout reaction.[68,115] The calculations were made for different wave functions of the relative motion of the α and d clusters in ^{6}Li both with and without allowance for the Coulomb interaction between the clusters.

It can be seen from Table V that allowance for the vertex Coulomb effects increases $G^2_{\alpha d}$ by more than a factor of 2. Unfortunately, the spread of $G^2_{\alpha d}$ and $(G^r_{\alpha d})^2$ values is very large, and the different accuracy in the methods of their determination prevents one deriving mean values of $G^2_{\alpha d}$ and $(G^r_{\alpha d})^2$. Therefore, Lim's conclusion[126] that $C^2_{^6\mathrm{Li}-\alpha-d}=5.7$ ($G^2_{\alpha d}=0.19$ F) appears premature. The observed spread in the estimates for $G^2_{\alpha d}$ and $(G^r_{\alpha d})^2$ call for further efforts to improve the values of these constants.

In the framework of the peripheral model for backward elastic d ^{6}Li scattering in Ref. 68, not only $G^2_{\alpha d}$ and $(G^r_{\alpha d})^2$ were estimated but also the D-state admixture: $\rho^2_{\alpha d}\equiv[G^r_{^6\mathrm{Li}\alpha d}(21)/G^r_{\alpha d}]^2$, and the value $\rho^2_{\alpha d}=1.7\cdot10^{-2}$ was obtained; this was found to be virtually unaffected by the neglect of Coulomb effects at the vertex. The value of $\rho^2_{\alpha d}$ can also be estimated by means of the analysis made in Ref. 144: $\rho^2_{\alpha d}=2\cdot10^{-3}$. These two estimates differ by an order of magnitude and are hardly reliable. There is, however, no doubt that $\rho^2_{\alpha d}\ll1$, which agrees with the small quadrupole moment of ^{6}Li.

Nuclear vertex constants for light nuclei. For a large number of light nuclei with $A>6$ the vertex constants G^2_{ABN} corresponding to separation of a nucleon from the nucleus were found by analyzing data on single-nucleon transfer reactions [(d,p), (d,t), $(d, ^3\mathrm{He})$, etc.] in the framework of the peripheral model[17,43,48,55,57–60]; both ground and excited states of the nuclei A and B were considered. The values of G^2_{ABN} were also found by means of the singularity-subtraction method in Ref. 60 and 63. In Ref. 20, the vertex constants for the vertices ^{12}C(ground) – ^{11}C(ground) + n and ^{14}N(ground) – ^{13}N(ground) + n were obtained in the framework of the continuation method. As was noted in Sec. 2, if the DWBA is suitably modified it can be used to determine vertex constants. We shall not give all the known vertex constants for light nuclei but limit ourselves, as an illustration, to a few vertices of the type $A \to B+n$, for each of which the vertex constants were found by three different methods: peripheral model, singularity-subtraction method, and the DWBA[74,60] (Table VI). All the vertex constants in Table VI refer to the ground states of the nuclei A and B and correspond to orbital angular momentum $l_A=1$ at the vertex; they were all obtained by analyzing (d,p) reactions at 12 MeV. We see that the vertex constants determined in different ways agree quite well with one another.

CONCLUSIONS

We now have at our disposal a variety of methods for estimating nuclear vertex constants on the basis of data on nuclear reactions. None can really pretend to an exceptional role, though the continuation, singularity-subtraction, and forward-dispersion-relation methods may turn out to be more accurate. This will be decided not so much theoretically as rather by the accumulation of experience in the application of the various methods and confrontation of the results they provide. At the present time, concrete information about the vertex constants is too sparse and does not permit definitive conclusions. Only one thing is clear—the prerequisite for successful application of each method is the fulfillment of definite physical conditions, and in this sense the possibilities of even powerful methods such as the

TABLE VI. Nuclear vertex constants G^2_{ABn} for some light nuclei.

№	Nucleus A	Nucleus B	$-G^2_{ABn}$, F		
			Periph. mod.	Sing.-subt.	DWBA
1	^{10}Be	^{9}Be	2,0	1,2	1,7
2	^{11}B	^{10}B	3,7	4,1	3.3
3	^{12}B	^{11}B	0,2	0,3	0.2
4	^{13}C	^{12}C	0,6	0,4	0,6
5	^{14}C	^{13}C	3.6	4,7	3.2

three mentioned above in conjunction with the technique of optimal conformal mappings are limited.

The fact that the vertex constants play an important role in the theory of nuclear reactions forces one to reconsider the connection between this theory and the theory of nuclear structure. The point is that structure calculations must give a characteristic such as the amplitude of the tail of the overlap integral of nuclear wave functions. As yet, we have only learnt to solve this problem sufficiently accurately for the simplest three- and four-nucleon systems in the case of the two-particle nucleon—nucleon interaction. This has shown that the vertex constants depend rather strongly on the form of the two-particle potential. The possibility cannot be precluded that these constants are just as sensitive to the effects of many-particle forces and meson-exchange effects. We can already assert with confidence that in calculations of the vertex constants for complex nuclei in the framework of nuclear models (for example, in the shell model) the constants will be extremely sensitive to the geometry of the average field and to configuration mixing.

All this indicates the need for systematic accumulation of empirical information on the vertex constants. For this, in particular, one requires precision (with error ~ 1%) experimental data on the differential cross sections of scattering and transfer reactions. Unfortunately, such data are as yet exceptionally sparse.

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