Description of low energy πN scattering in nonlinear $SU(2)\times SU(2)$ chiral dynamics

M. I. Gaĭsak

Institute of Theoretical Physics, Academy of Sciences of the Ukrainian SSR, Kiev

V. I. Lend'el

Uzhgorod State University, Uzhgorod

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The present state of the theory of low energy πN scattering is reviewed. It is shown that the use of a nonlinear $SU(2)\times SU(2)$ chiral invariant Lagrangian in the exponential parametrization, the superpropagator method of regularizing the scattering amplitude, and the Padé approximation enables one to obtain a dynamical description of low energy πN scattering without the introduction of any arbitrary parameters. From the first two nonvanishing terms in the perturbation series, analytic expressions are obtained in the diagonal (1,1) Padé approximation for the s and p partial-wave amplitudes. The energy dependence and position of the s- and t-channel singularities of the πN scattering amplitude are in very satisfactory agreement with the experimental data.

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INTRODUCTION

In a great many recent investigations, unitary quantum field theory with nonpolynomial Lagrangians has been applied to the description of strong, weak, and electromagnetic interactions of elementary particles. One of the fruitful methods in this direction is the superpropagator method introduced for the first time by Volkov.[1] This method was successfully applied to the description of both low energy $\pi\pi$ scattering and the electromagnetic properties of π and K mesons and also some weak decay processes. The interesting review of Volkov and Pervushin[2] presents the results obtained, which show that this method can be successfully applied to the description of strong interactions. However, besides $\pi\pi$ scattering it would be interesting to obtain quantitative data for other strong interaction processes, in particular πN scattering.

The process of πN scattering has always been regarded as one of the most fundamental processes in the strong interactions of elementary particles. Therefore, the problem of theoretical interpretation of πN scattering has always been in the center of attention. Impressive progress in this direction in the region of low energies was achieved by the use of dispersion relations. Important results were obtained at Dubna by Shirkov, Isaev, Meshcheryakov, and others (see, for example, Refs. 3 and 4). The basic idea of this approach was to use backward dispersion relations. It was shown that the influence of the crossed annihilation cut is manifested most strongly in precisely these dispersion relations. The influence of the $\pi\pi$ interaction on πN scattering was determined, and it was shown in particular that for a satisfactory description of the lowest πN scattering phase shifts it is necessary to take into account the vector ρ meson and the scalar σ meson. The use of backward dispersion relations in combination with forward dispersion relations made it possible to find a mathematically noncontradictory system of equations for the partial-wave amplitudes of $\pi\pi$ and πN scattering. Investigation of the system obtained for $\pi\pi$ scattering showed that this system has resonance solutions corresponding to the ρ and σ mesons. These investigations were summarized by Shirkov *et al.* in Ref. 5. The Dubna approach was also used to find the resonance solutions of the πN system, the phase shifts of NN scattering, and the electromagnetic form factors of nucleons.[6]

However, because of the need to introduce a large number of variable parameters (such as the coupling constants, the masses of the bosons, and the parameters of the πN resonances) and also the need to particularize the high energy behavior of the amplitude, the dispersion approach unfortunately remains at the level of semi-phenomenological models. In the various studies based on the dispersion approach, the number of arbitrary parameters fluctuates between 5 and 10. A dynamical description by means of the method of dispersion relations has not been realized.

It is well known that a complete dynamical description can be based on the Lagrangian formalism. In this case, the number of arbitrary parameters is greatly reduced. However, the representation based on the Lagrangian formalism has itself suffered from serious shortcomings which have until very recently prevented implementation of the dynamical description. First, the perturbation series—the only amenable method of calculation—is divergent; second, some important types of interaction were found to be unrenormalizable, and even the type of interaction was not known to a sufficient extent.

Hadron physics of the last decades has been characterized by the conception of Yukawa type interactions as the basic interactions of pions and nucleons, although as early as 1960 Sakurai[7] noted that in the framework of a Lagrangian of the Yukawa type there do not exist simple schemes in which the pions could be linearly coupled to the nucleons and that calculations based on these Lagrangians are devoid of physical value.

Recently, the situation in this field has been radically

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changed. First, Lagrangians (linear and nonlinear) were proposed[8,9] satisfying higher symmetries, and this was an important step forward compared with Lagrangians of the Yukawa type. Indeed, on the basis of a nonlinear chiral invariant Lagrangian, Weinberg[9] obtained the s-wave $\pi\pi$ and πN scattering lengths.

These results were also reproduced on the basis of a linear chiral Lagrangian (see, for example, Ref. 10). Thus, two approaches began to take shape: the use of linear and nonlinear chiral invariant Lagrangians. Investigations are currently being made in both directions. It is found that the linear and nonlinear models for $\pi\pi$ scattering are identical. However, bearing in mind the other processes—πN, NN, scattering, etc.—one cannot ignore the fact that the linear model suffers from considerable arbitrariness (apart from postulating the existence of the σ-meson field, it is here necessary to introduce interaction of nucleons with ρ, ω, and A_2 mesons, deuterons, etc.). In addition, the systematic development of a renormalizable theory with massive vector mesons encounters considerable difficulties.[11] In this sense, the nonlinear model is clearly more economic and promising. One may say that the linear model gives a semi-phenomenological description like the one by means of dispersion relations or by means of the single-boson model, while only the nonlinear model is capable of leading to a complete dynamical description.

Initially, chiral Lagrangians were intended to be used as phenomenological Lagrangians, in the three approximation, to reproduce the results of chiral algebra. To go beyond the threshold of the scattering process, i.e., to describe the energy dependence of the scattering amplitude, it would be necessary to know how to calculate loop diagrams. Linear Lagrangians without vector mesons lead to renormalizable theories, and in this case fundamental difficulties did not arise. But in the case of nonlinear Lagrangians horrendous divergences due to the nonpolynomial nature of the chiral Lagrangian seemed to present insuperable difficulties. However, Efimov and Fradkin[12] succeeded in solving successfully the problem of renormalizability of certain nonpolynomial interactions, and this opened up the possibility of using chiral Lagrangians for the above purposes.

The method for calculating the higher approximations of the scattering amplitudes in theories with nonpolynomial Lagrangians that is the most consistent and appropriate for practical use is the superpropagator method of regularization proposed by M. K. Volkov[1]; it is a method of analytic continuation of the amplitude as a function of certain coefficients $C(n)$, which characterize the extent to which the Lagrangian is nonpolynomial, to the complex n plane.

The known mathematical methods of approximate summation of divergent series can be successfully used to recover the scattering amplitude from the renormalized perturbation series. One of these methods is the method of Padé approximations, which, in conjunction with the simplest linear models, was used to describe strong interaction processes by Bessis *et al.*[13] Thus, one of the ways of avoiding the well known difficulties inherent in the perturbation method had been indicated. The method of Padé approximations can be regarded as one of the methods for extracting the physical information contained in the coefficients of the perturbation series when the Lagrangian formalism is used in the theory of strong interactions. It is therefore of undoubted interest to apply these methods to the description of low energy πN scattering—a process with physically richer experimental information.

1. USE OF LINEAR LAGRANGIANS

Mignaco, Pusterla, and Remiddi[14] and also Gammel, Mentzel, and Kubis[15] were the first to apply the method of Padé approximation to the problem of πN scattering. They used a simple Lagrangian of the Yukawa type:

$$\mathcal{L}(x) = -ig\bar{\psi}(x)\gamma_5\tau\psi(x)\pi(x) - \lambda(\pi(x)\pi(x))^2/4. \tag{1}$$

The results they obtained are an illustration of the inadequacy mentioned above of Yukawa-type Lagrangians; the partial waves with isospin $I=1/2$ completely contradict the experimental data. No great improvement resulted from the use of the σ model[16]:

$$\mathcal{L}(x) = g\bar{\psi}(x)(\sigma(x) + i\gamma_5\tau\pi(x))\psi(x) - \lambda(\sigma^2(x) + \pi^2(x))^2/4 + c\sigma(x). \tag{2}$$

Here, it also proved impossible to find the scattering length and a whole series of partial waves. Various authors ascribed the failure to the circumstance that the Lagrangian (2) yields an incorrect relationship between the masses of the σ meson and the nucleon: $m_\sigma > m$ (Ref. 17). It should be emphasized that the disaster does not derive from this alone. As Shirkov and Serebryakov noted,[10] the Lagrangian (2) in this case must be augmented by terms responsible for exchange of the ρ meson. Allowance for just these terms can lead to the correct value of the isotopically odd scattering length and, therefore, to the correct behavior of the corresponding phase shifts. In addition, it is important to make additional allowance for the short-wavelength interaction.

Considered as a whole, this means that the number of parameters used in the approach based on a linear Lagrangian is rather large. Inclusion in the interaction of couplings to vector mesons catastrophically increases the number of invariants.

Nevertheless, the results obtained, together with the results on $\pi\pi$ scattering, demonstrated comvincingly that in principle the method of Padé approximation (provided the interaction Lagrangian is correctly chosen) can become the long sought for tool for investigating strong interactions. And indeed, much better numerical results were obtained by Fil'kov and Palyushev,[18] who proceeded from "bare" Born terms in the πN scattering amplitudes:

$$\operatorname{Re} A^+(s,t,u) = \alpha + \beta t/(m_\sigma^2 - t); \quad \operatorname{Re} A^-(s,t,u) = 0;$$
$$\operatorname{Re} B^\pm(s,t,u) = g_\pi^2[1/(m^2 - s) \mp 1/(m^2 - u)] \tag{3}$$

(here α and β are adjustable parameters determined experimentally) which nearly coincide with the first approximation of the nonlinear Lagrangian (see Ref. 10; see also Eq. (9) of the present paper). The coincidence would have been even more complete if they had included in B^- the additional constant corresponding to contact interaction or ρ-meson exchange in the linear approach. Nonlinearity was introduced into the Fil'kov–Palyushev model by means of the unitarity condition for calculating the amplitudes in higher order in the coupling constant. They obtained very reasonable agreement of the phase shifts calculated in the (1, 1) diagonal Padé approximation with experiment, although, it seems to us, the wave p_{11} requires special treatment since it is here necessary to find simultaneously a negative scattering "length" and approach of the phase shift to resonance, which is difficult on the basis of direct use of the (1, 1) Padé approximation.

However, the use of the arbitrary adjustable parameters α and β greatly reduces the value of the Fil'kov–Palyushev model. In addition, the use of dispersion relations in this model also requires particularization of the high energy behavior of the invariant amplitudes.

Thus, the approach based on the use of linear Lagrangians in fact reduces to the single-boson model since it requires the introduction of fields corresponding to each interacting particle, including an exchange particle. At the present time, the number of such particles is large and continues to grow as experimental information is accumulated. It is clear that the introduction of an ever greater number of fields contradicts quantum field theory.

As pointed out above, a different approach based on the use of nonpolynomial Lagrangians is more attractive. This is equivalent to assuming that the number of fundamental particles is bounded and that all the remaining particles must be bound states or resonances of them, i.e., they must have a dynamical origin. It is just such a dynamical description of the strong interactions of elementary particles that now occupies a number of investigators.

The results obtained by Lehmann, Volkov, Pervushin, and others[19,20] for low energy $\pi\pi$ scattering and the electromagnetic properties of pions and K mesons by using nonpolynomial Lagrangians and the superpropagator method of regularization indicate that a dynamical description of these processes is obtained and clearly suggest that it is this approach which will make it possible to advance further in our understanding of the interaction processes, in particular, the strong interactions of elementary particles.

Below, we set forth the application of this method to the description of low energy πN scattering in the framework of a nonlinear $SU(2)\times SU(2)$ chiral invariant Lagrangian using the Padé approximation.

The systematic use of the superpropagator method of regularization is an important step forward since this method makes it possible to obtain a theoretical description without the introduction of arbitrary parameters.

2. USE OF NONLINEAR LAGRANGIANS

Obviously, great interest attaches to the use of a Lagrangian by means of which one can, for the same parameter values, describe all processes in strong interactions. Therefore, on the basis of the fact that the description of $\pi\pi$ scattering and the electromagnetic properties of mesons was obtained in Refs. 19 and 20 by means of a nonlinear Lagrangian in Gürsey's parametrization, we shall also choose here this variant of $SU(2)\times SU(2)$ chiral invariant Lagrangian[21]:

$$\mathcal{L}(x)=g_{ij}(\pi)\,\partial_\mu\pi_i(x)\,\partial^\mu\pi_j(x)/2+i\overline{N}(x)\,\gamma_\mu\overleftrightarrow{\partial^\mu}N(x)/2 \\ -m\overline{N}(x)\,U(\pi)\,N(x)+ig'\overline{N}(x)\,\gamma_\mu U^+(\pi)\,\partial^\mu U(\pi)\,N(x)/2, \tag{4}$$

where $g_{ij}(\pi)$ is the metric tensor of the curved isotopic space with constant curvature, which for the exponential parametrization has the form

$$\left.\begin{array}{l} g_{ij}(\pi)=f^2(\pi^2)\,\delta_{ij}-[f^2(\pi^2)-1]\,\pi_i(x)\,\pi_j(x)/\pi^2(x); \\ f(\pi^2)=\sin Z/Z;\quad Z\equiv\sqrt{\pi^2(x)/F_\pi^2};\quad U(\pi)=\exp(i\gamma_5\tau\pi(x)/F_\pi); \end{array}\right\} \tag{5}$$

here F_π is the weak decay constant of the π meson ($F_\pi \approx 92$ MeV).

In this model, the pion fields are a nonlinear realization of the $SU(2)\times SU(2)$ chiral group, and the nucleon fields transform linearly with respect to transformations of the given group. As was noted by Lehmann and Trute,[19] this parametrization has the advantage that it leads to a localizable theory in the sense of Meĭman and Jaffe.

In order to eliminate pseudovector coupling, we subject the nucleon field to a Foldy–Dyson transformation:

$$N(x)=\exp(ia\gamma_5\pi(x)\,\tau)\,\psi(x). \tag{6}$$

Choosing the constant a appropriately, we obtain the following form of the Lagrangians that contribute to the πN scattering amplitude in the approximation in which we are interested:

$$\mathcal{L}_{\text{int}}(x)=\mathcal{L}_0(x)+\mathcal{L}_1(x)+\mathcal{L}_2(x)+\mathcal{L}_3(x), \tag{7}$$

where

$$\begin{aligned} \mathcal{L}_0(x)&=[f^2(\pi^2)-1]\,\{\partial_\mu\pi(x)\,\partial^\mu\pi(x)-[\pi(x)\,\partial_\mu\pi(x)]^2/\pi^2(x)\}/2; \\ \mathcal{L}_1(x)&=-(img_A/F_\pi)\,\overline{\psi}(x)\,\gamma_5\tau\psi(x)\,\pi(x)\sin Z_1/Z_1; \\ \mathcal{L}_2(x)&=m\overline{\psi}(x)\,\psi(x)\,(1-\cos Z_1); \\ \mathcal{L}_3(x)&=(1/2)\,\overline{\psi}(x)\,\gamma_\mu\tau\psi(x)\,\pi(x)\,x\partial^\mu\pi(x) \\ &\times[(g_A-1)\sin^2 Z_2-(g_A+1)\sin^2 Z_3]/\pi^2(x); \\ Z_1&\equiv\sqrt{g_A^2\pi^2/F_\pi^2};\quad Z_2\equiv\sqrt{(g_A+1)^2\pi^2/4F_\pi^2}; \\ Z_3&\equiv\sqrt{(g_A-1)^2\pi^2/4F_\pi^2};\quad g_A=1-g'. \end{aligned}$$

The validity of such a chiral quantized Lagrangian was justified by Pervushin in Ref. 22.

In Refs. 23, calculations were made of the low energy πN scattering on the basis of a nonlinear chiral Lagrangian of the Weinberg type in the single-loop approximation. In Refs. 23, the Lagrangian was used only to calculate the imaginary part of the scattering amplitude. With regard to the real part, it was recovered by means of one-dimensional dispersion relations for the invariant amplitudes $A^{\pm}(s, t, u)$ and $B^{\pm}(s, t, u)$. Good agreement between theory and experiment was obtained. A shortcoming of this method is the need to introduce a number of arbitrary parameters as subtraction constants. Therefore, let us consider the contribution to the total amplitude of πN scattering of tree-type diagrams together with single-loop diagrams using a $SU(2)\times SU(2)$ chiral invariant Lagrangian in the exponential parametrization (7), which can be regularized by means of the superpropagator method. On the basis of the results obtained, we recover the amplitude of πN scattering in the (1, 1) diagonal Padé approximation.

The expression for the S matrix and the rules for ordering operators by means of the Hori operator are given in Ref. 24 and are a generalization to the case of isotopic spin of the expressions given in Refs. 25–27.

3. RENORMALIZATION OF THE πN SCATTERING AMPLITUDE. CONTRIBUTIONS OF THE DIAGRAMS TO THE INVARIANT AMPLITUDES

We shall consider the process $\pi_{\alpha}(q_1) + N_{\beta}(p_1) \to \pi_{\gamma}(q_2) + N_{\delta}(p_2)$. We shall proceed from the definition of the scattering amplitude adopted in Ref. 5:

$$\langle f|S-1|i\rangle = \frac{im}{(2\pi)^2\sqrt{4q_1^0 q_2^0 p_1^0 p_2^0}}\,\bar{u}(p_2)\,T_{\gamma\alpha}^{\delta\beta}u(p_1)\,\delta^4(p_1+q_1-p_2-q_2), \tag{8}$$

where

$$\begin{aligned} T_{\gamma\alpha}^{\delta\beta} &= \delta_{\gamma\alpha}\delta_{\delta\beta}[A^+(s,t,u)+(\hat{q}_1+\hat{q}_2)B^+(s,t,u)/2] \\ &+ [\tau_\gamma,\tau_\alpha]^{\delta\beta}[A^-(s,t,u)+(\hat{q}_1+\hat{q}_2)B^-(s,t,u)/2]/2; \\ s &= (p_1+q_1)^2=(p_2+q_2)^2; \quad t=(q_1-q_2)^2=(p_1-p_2)^2; \\ u &= (p_2-q_1)^2=(p_1-q_2)^2; \end{aligned}$$

p_1, p_2, q_1, and q_2 are the four-momenta of the nucleons and mesons, respectively.

In order to obtain the amplitude of πN scattering in the first and the second order in the main coupling constant on the basis of the interaction Lagrangian (7), we must calculate the contribution of tree diagrams.

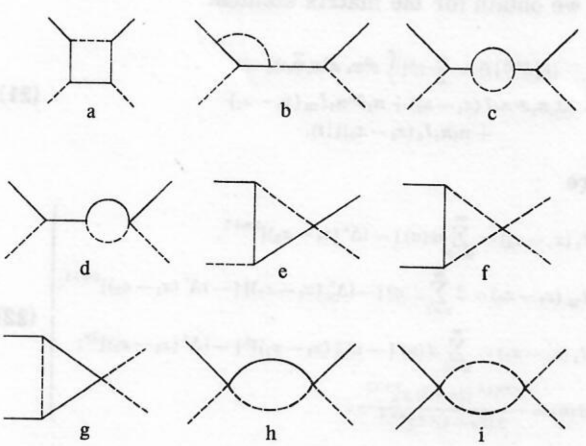

FIG. 1. Diagrams of third and fourth order in the main coupling constant that contribute to the πN scattering amplitude on the basis of the Lagrangian (7).

In the third and fourth order in the main coupling constant, we shall calculate the contribution of the diagrams shown in Fig. 1. These diagrams (together with the crossed diagrams and disconnected diagrams) exhaust the contribution to the πN scattering amplitude in the first and second perturbation approximations. The diagram in Fig. 1a gives a finite contribution, whereas the remaining diagrams diverge. The divergent diagrams of Figs. 1b–1e and 1g can be renormalized by means of renormalizations of the nucleon mass, δm, and the meson mass, δm_π^2, and renormalization of the two charges. The diagrams in Figs. 1h and 1i cannot be renormalized in the usual manner, and we shall therefore calculate them by means of the superpropagator method of regularization.

The contribution of the tree-type diagrams can be written in the form

$$\left.\begin{aligned} A^+(s,t,u) &= \frac{m\,g_A^2}{F_\pi^2}; \\ B^+(s,t,u) &= g_A^2 m^2[1/(m^2-s)-1/(m^2-u)]/F_\pi^2; \\ A^-(s,t,u) &= 0; \\ B^-(s,t,u) &= (1-g_A^2)/2F_\pi^2+g_A^2 m^2[1/(m^2-s)+1/(m^2-u)]/F_\pi^2. \end{aligned}\right\} \tag{9}$$

Since the Lagrangian (4) is invariant under transformations of the group $SU(2)\times SU(2)$, the pion mass is zero. The consistent introduction of a pion mass into (4), which entails breaking of the chiral symmetries, has hitherto remained an open problem, and discussion of it would go beyond our scope. Lehmann[28] notes that the assumption of a vanishing pion mass does not influence the numerical calculations of $\pi\pi$ scattering. This assumption is all the more justified for πN scattering, in which the natural parameter of the theory is the ratio ω/m, which is fairly small (ω is the cms pion energy). Therefore, in what follows we shall calculate the contribution of the higher diagrams under the assumption $m_\pi = 0$. In calculating the diagrams of third and fourth order in the main coupling constant, we shall follow Ref. 29.

The contributions of the diagrams in Figs. 1d–1g can be written as follows:

$$\left.\begin{aligned} &\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_2(z)\ \text{(see Fig. 1e)} \\ A^{\pm}(s,t,u) &= 32f^4m^3\left[\mp\frac{s-m^2}{s}\ln\frac{m^2}{m^2-s}\pm(s\leftrightarrow u)\right]; \\ B^{\pm}(s,t,u) &= \pm 32f^4m^2\left\{\frac{m^2}{s}\ln\frac{m^2}{m^2-s}+\frac{m^2}{s-m^2}\right. \\ &\left.\times\left[L\left(\frac{s}{m^2}\right)-L(1)\right]\mp(s\leftrightarrow u)\right\}, \end{aligned}\right\} \tag{10}$$

where $L(x)=\int_0^x[\ln(1-y)/y]\,dy$ is Euler's dilogarithm;

$$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_2(z)\ \text{(see Fig. 1g)}$$

$$\left.\begin{aligned} A^+(s,t,u) &= -96f^4m^3\sqrt{\frac{4m^2-t}{t}}\tan^{-1}\sqrt{\frac{t}{4m^2-t}}; \\ A^-(s,t,u) &= B^{\pm}(s,t,u)=0; \end{aligned}\right\} \tag{11}$$

$$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_2(z)\ \text{(see Fig. 1d)}$$

$$\left.\begin{aligned} A^{\pm}(s,t,u) &= 16f^4m^3\left[\frac{m^4-s^2}{s^2}\ln\frac{m^2}{m^2-s}+\frac{s-m^2}{s}\pm(s\leftrightarrow u)\right]; \\ B^{\pm}(s,t,u) &= 16f^4m^2\left[\frac{m^2(m^2-s)}{s^2}\ln\frac{m^2}{m^2-s}+\frac{s-m^2}{s}\mp(s\leftrightarrow u)\right]; \end{aligned}\right\} \tag{12}$$

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_0(z)$ (see Fig. 1f)

$$\left.\begin{aligned}
&A^+(s,t,u)=\frac{64f^4m^3}{g_A^2}\sqrt{\frac{t}{t-4m^2}}\ln\frac{\sqrt{t/(t-4m^2)}+1}{\sqrt{t/(t-4m^2)}-1};\\
&A^-(s,t,u)=\frac{16f^4m^3}{g_A^2}\frac{s-u}{\sqrt{t(t-4m^2)}}\ln\frac{\sqrt{t/(t-4m^2)}+1}{\sqrt{t/(t-4m^2)}-1};\\
&B^-(s,t,u)=\frac{16f^4m^2}{g_A^2}\sqrt{\frac{t-4m^2}{t}}\ln\frac{\sqrt{t/(t-4m^2)}+1}{\sqrt{t/(t-4m^2)}-1};\\
&B^+(s,t,u)=0.
\end{aligned}\right\}\quad(13)$$

In obtaining these expressions, we have approximated certain Feynman integrals at an average point:

$$\int_0^1 dx\frac{(x-1)^2}{\sqrt{tx^2-4m^2t(1-x)^2}}\ln\left|\frac{x+\sqrt{x^2-4m^2(1-x)^2/t}}{x-\sqrt{x^2-4m^2(1-x)^2/t}}\right| =\frac{1}{2}\frac{1}{\sqrt{t(t-4m^2)}}\ln\frac{\sqrt{t/(t-4m^2)}+1}{\sqrt{t/(t-4m^2)}-1}.$$

Such an approximation correctly reflects the analytic properties of the scattering amplitude. At the same time, numerical calculations of the integral made in the interval $0\leqslant\omega<2$ indicate that such an approximation deviates from the exact value by less than 30% for large ω;

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_3(z)$ (see Fig. 1e)

$$\left.\begin{aligned}
&A^+(s,t,u)=32f^4m^3\left(1-\frac{1}{g_A^2}\right)\\
&\times\left[L\left(\frac{s}{m^2}\right)-L(1)+\frac{s-m^2}{s}\ln\frac{m^2}{m^2-s}+(s\leftrightarrow u)\right];\\
&A^-(s,t,u)=B^-(s,t,u)=0;\\
&B^+(s,t,u)=32f^4m^2\left(1-\frac{1}{g_A^2}\right)\left[\frac{2m^2}{m^2-s}\left[L\left(\frac{s}{m^2}\right)-L(1)\right]\right.\\
&\left.-\frac{s+m^2}{s}\ln\frac{m^2}{m^2-s}-(s\leftrightarrow u)\right];
\end{aligned}\right\}\quad(14)$$

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_3(z)$ (see Fig. 1g)

$$\left.\begin{aligned}
&A^+(s,t,u)=B^+(s,t,u)=0;\\
&A^-(s,t,u)=8f^4m^3\left(1-\frac{1}{g_A^2}\right)\left[\frac{s-u}{\sqrt{t(4m^2-t)}}\tan^{-1}\sqrt{\frac{t}{4m^2-t}}\right];\\
&B^-(s,t,u)=8f^4m^2\left(1-\frac{1}{g_A^2}\right)\frac{t-6m^2}{\sqrt{t(4m^2-t)}}\tan^{-1}\sqrt{\frac{t}{4m^2-t}};
\end{aligned}\right\}\quad(15)$$

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_3(z)$ (see Fig. 1d)

$$\left.\begin{aligned}
&A^\pm(s,t,u)\\
&=16f^4m^3\left(1-\frac{1}{g_A^2}\right)\left[\left(\frac{s-m^2}{s}\right)^2\ln\frac{m^2}{m^2-s}+\frac{s-m^2}{m^2}\pm(s\leftrightarrow u)\right];\\
&B^\pm(s,t,u)=m^{-1}A^\mp(s,t,u).
\end{aligned}\right\}\quad(16)$$

The contributions of the diagrams in Figs. 1a–1c can be written in the form

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_1(z)\,\mathscr{L}_1(t)$ (see Fig. 1a)

$$\left.\begin{aligned}
&A^\pm(s,t,u)=\binom{1}{-1/3}48f^4m^3\left\{\frac{m^2}{s-m^2}\left[L\left(\frac{s}{m^2}\right)-L(1)\right]\right.\\
&\left.+\frac{2\theta}{\sin 2\theta}\left[1+h(\theta)+\ln\frac{m^2}{m^2-s}\right]\pm(s\leftrightarrow u)\right\};\\
&B^\pm(s,t,u)=\binom{-1}{1/3}48f^4m^2\left\{\frac{m^2(s+m^2)}{(s-m^2)^2}\left[L\left(\frac{s}{m^2}\right)-L(1)\right]\right.\\
&+\frac{2\theta}{\sin 2\theta}\frac{2m^2}{s-m^2}\left[h(\theta)+\ln\frac{m^2}{m^2-s}\right]\mp(s\leftrightarrow u),
\end{aligned}\right\}\quad(17)$$

where $\sin^2\theta=t/4m^2$; $h(\theta)=\theta^{-1}\int_0^\theta\zeta\,\mathrm{ctg}\,\zeta\,d\zeta$;

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_1(z)\,\mathscr{L}_1(t)$ (see Fig. 1b)

$$\left.\begin{aligned}
&A^\pm(s,t,u)=32f^4m^3\left\{\frac{-m^2}{m^2-s}\left[L\left(\frac{s}{m^2}\right)-L(1)\right]\right.\\
&\left.+\frac{m^2}{s}\ln\frac{m^2}{m^2-s}\pm(s\leftrightarrow u)\right\};\\
&B^\pm(s,t,u)=-32f^4m^2\left\{\frac{2m^4}{(s-m^2)^2}\left[L\left(\frac{s}{m^2}\right)-L(1)\right.\right.\\
&\left.+\frac{s-m^2}{s}\ln\frac{m^2}{m^2-s}\right]+\frac{2m^2}{s-m^2}+\frac{m^2}{s}\ln\frac{m^2}{m^2-s}\mp(s\leftrightarrow u)\Big\};
\end{aligned}\right\}\quad(18)$$

$\mathscr{L}_1(x)\,\mathscr{L}_1(y)\,\mathscr{L}_1(z)\,\mathscr{L}_1(t)$ (see Fig. 1c)

$$\left.\begin{aligned}
&A^\pm(s,t,u)=24f^4m^3\left[\frac{m^2-s}{s}+\frac{m^2(s-m^2)}{s^2}\ln\frac{m^2}{m^2-s}\pm(s\leftrightarrow u)\right];\\
&B^\pm(s,t,u)=m^{-1}A^\mp(s,t,u).
\end{aligned}\right\}\quad(19)$$

Here, we shall use the natural system of units ($\hbar=c=m_\pi=1$) and the Goldberger–Treiman relation $f^2=g_A^2/16\pi F_\pi^2$ ($f^2\approx 0.8$).

The diagram $\mathfrak{L}_2(x)\mathfrak{L}_2(y)$ (see Fig. 1h) can also be renormalized in the usual manner despite its linear divergence. Therefore, calculating it in the usual way, we obtain

$$\left.\begin{aligned}
&A^+(s,t,u)=8f^4m^3\left[\frac{m^2-s}{s}+\frac{(s-m^2)(3s+m^2)}{s^2}+(s\leftrightarrow u)\right];\\
&A^-(s,t,u)=B^-(s,t,u)=0;\\
&B^+(s,t,u)=8f^4m^2\left[\frac{m^2-s}{s}+\frac{s^2-m^4}{s^2}\ln\frac{m^2}{m^2-s}-(s\leftrightarrow u)\right].
\end{aligned}\right\}\quad(20)$$

We shall calculate the remaining diagrams using the super-superpropagator method of regularization.

4. USE OF THE SUPERPROPAGATOR METHOD TO CALCULATE DIVERGENT DIAGRAMS

Since renormalization of the πN scattering amplitude by the superpropagator method has not hitherto been considered by anyone, it would be worthwhile demonstrating the method of calculation for at least the very simple example of the diagram $\mathfrak{L}_0(x)\mathfrak{L}_2(y)$ (see Fig. 1i). In the calculations, we shall follow Volkov's method, which has certain advantages over the methods of other authors.[1] We shall proceed from the standard definition of the S matrix in terms of the interaction Lagrangian. Pairing with respect to the internal fields in the S matrix by means of the Hori operator (for details see Ref. 24), we obtain for the matrix element

$$\begin{aligned}
\langle f|S^{(2)}|i\rangle=&\frac{m}{2}\langle f|\int d^4x_1\,d^4x_2\bar{\psi}_2\psi_2\\
&\times[\partial_\mu\pi_1\partial^\mu\pi_1 I(x_1-x_2)+\pi_1\partial^\mu\pi_1 I_{2\mu}(x_1-x_2)\\
&+\pi_1\pi_1 I_3(x_1-x_2)]|i\rangle,
\end{aligned}\quad(21)$$

where

$$\left.\begin{aligned}
&I_1(x_1-x_2)=\sum_{n=0}^\infty d(n)\,[-\mathrm{i}\Delta^c(x_1-x_2)]^{2n+2},\\
&I_{2\mu}(x_1-x_2)=2\sum_{n=0}^\infty d(n)\,[-\mathrm{i}\Delta^c_\mu(x_1-x_2)]\,[-\mathrm{i}\Delta^c(x_1-x_2)]^{2n+1};\\
&I_3(x_1-x_2)=\sum_{n=0}^\infty d(n)\,[-\mathrm{i}\Delta^c_\mu(x_1-x_2)]^2\,[-\mathrm{i}\Delta^c(x_1-x_2)]^{2n};\\
&d(n)=\frac{2^{2n+4}(2n+3)\,g_A^{2n+2}}{3\,(2n+4)!\,F_\pi^{4n+4}}.
\end{aligned}\right\}\quad(22)$$

The finding of the expression (21) is simplified considerably if the tabulated differentiation formulas given in Ref. 24 are used in the integration.

Carrying out the necessary commutations of the nucleon and meson operator-valued fields with the operators of creation and annihilation of these fields and going over to the new variables $x_1 - x_2 = u$, $x_1 + x_2 = v$, we can integrate with respect to the variable v, obtaining

$$\langle j | S^{(2)} | i \rangle = \frac{m}{2} \frac{m\delta_{\alpha\gamma}\delta_{\beta\delta}}{(2\pi)^2 \sqrt{4q_1^0 q_2^0 p_1^0 p_2^0}} \times \bar{u}(p_2) \int d^4u \exp(iQu) [2q_1q_2 I_1(u) + iQ_\mu I_2^\mu(u) + 2I_3(u)] u(p_1) \delta^4(P_i - P_f); \quad Q = q_2 - q_1. \tag{23}$$

In order to integrate in (23) with respect to u, it is necessary to go over to Euclidean space. Going over in (23) to spherical coordinates in Euclidean space and expressing the results in terms of invariant amplitudes in accordance with (8), and then integrating with respect to the angles, we obtain (using the explicit expressions for the causal functions and their derivatives[24])

$$\left.\begin{aligned} &A^+(q_2 - q_1) = \frac{m}{2i} \int_0^\infty dr \sum_{n=0}^\infty \frac{d(n)}{(2\pi)^{4n+2}} \\ &\times \left[2q_1q_2 r^{-4n+2} - 4Q_\mu r^{-4n-4} \frac{\partial}{\partial Q_\mu} - 4r^{-4n-4}\right] \frac{J_1 \sqrt{-Q^2}\, r}{\sqrt{-Q^2}}; \\ &B^\pm(s, t, u) = A^-(s, t, u) = 0, \end{aligned}\right\} \tag{24}$$

where $J_1(x)$ is the cylinder function of the first kind.

In order to integrate with respect to r in (24), it is necessary to use Volkov's method of analytic regularization. For this, we write the sums in (24) in the form of a Mellin–Barnes integral (Sommerfeld–Watson integral) and deform the contour of integration in the complex z plane in such a way that $-1 < \mathrm{Re}\, z < 0$. In the complete region of integration we obtain (it is here necessary to introduce a partial regularization, as is usually done[1])

$$A^+(Q^2) = -\frac{m}{4i} \int_{\alpha+i\infty}^{\alpha-i\infty} dz \operatorname{ctg} \pi z\, d(z) (2\pi)^{-4z-2} \times \left[2q_1q_2 \frac{\Gamma(-2z)}{\Gamma^2(2z+2)} \frac{(\sqrt{-Q^2})^{4z}}{2^{4z+2}} - 4Q_\mu \frac{\Gamma(-1-2z)}{\Gamma(2z+3)} \frac{\partial}{\partial Q_\mu} \frac{(\sqrt{-Q^2})^{4z+2}}{2^{4z+2}} - \frac{\Gamma(-1-2z)}{\Gamma(2z+3)} \frac{(\sqrt{-Q^2})^{4z+2}}{2^{4z+4}}\right], \tag{25}$$

where $0 < \alpha < 1$ and $\Gamma(z)$ is the gamma function. The expression (25) corresponds to summation of a chain of diagrams (see Fig. 1i) with two, four, etc., internal lines. In order to obtain the contribution corresponding to one loop, it is necessary to make a subtraction only at the point $z = 0$. Thus, we finally obtain

$$A^+(s, t, u) = -\frac{m}{6} \frac{(2\pi)^2 g_A^2}{(4\pi F_\pi)^4} \left[2 \ln \frac{-t g_A^2}{8\pi^2 F_\pi^2} + 6C - \frac{11}{2}\right] t, \tag{26}$$

where C is the Euler–Mascheroni constant.

Similarly, we can obtain the contribution of $\mathfrak{L}_0(x)\mathfrak{L}_3(y)$ (see Fig. 1i):

$$B^-(s, t, u) = \frac{1 - g_A^2}{3} \frac{4f^4}{g_A^2} \left[2 \ln \frac{-t}{8\pi^2 F_\pi^2} + g_A \ln \frac{g_A + 1}{g_A - 1} + \ln(g_A^2 - 1) + 6C - \frac{17}{2}\right]. \tag{27}$$

The calculation of the diagram $\mathfrak{L}_2(x)\mathfrak{L}_3(y)$ (see Fig. 1h) corresponding to the contribution from the s and u channels is somewhat complicated by the presence of the massive nucleon propagators. However, in this case too the integrals with respect to r can be performed and the results of integration can be expressed in terms of the hypergeometric function.[24] The final contribution to the invariant amplitudes of πN scattering from the diagram of Fig. 1h with the $\mathfrak{L}_2(x)\mathfrak{L}_3(y)$ vertices has the form

$$\left.\begin{aligned} &A^-(s, t, u) = 8f^4 m^3 \left(1 - \frac{1}{g_A^2}\right) \\ &\times \left[\left(2\widetilde{C} + \frac{1}{4}\right) \frac{s - m^2}{m^2} + \frac{s^2 - m^4}{sm^2} + \frac{(s - m^2)^2}{s^2} \ln \frac{m^2}{m^2 - s} - (s \leftrightarrow u)\right]; \\ &B^-(s, t, u) = 8f^4 m^2 \left(1 - \frac{1}{g_A^2}\right) \\ &\times \left[\frac{m^2 - s}{s} - \frac{(s - m^2)^2}{s^2} \ln \frac{m^2}{m^2 - s} + (s \leftrightarrow u)\right], \end{aligned}\right\} \tag{28}$$

where

$$\widetilde{C} = \{6C - 7/3 + g_A \ln[(g_A + 1)/(g_A - 1)] + \ln[(m/4\pi F_\pi)^4 g_A^2 (g_A^2 - 1)]\}/4.$$

The contributions from the diagram of Fig. 1h with the $\mathfrak{L}_3(x)\mathfrak{L}_3(y)$ vertices are even smaller because of the factor $(g_A^2 - 1)^2/16$. In what follows, we shall not consider them.

5. CALCULATION OF THE PARTIAL-WAVE AMPLITUDES

To obtain the partial-wave amplitudes, we shall use the well tested method of separating the partial-wave amplitudes by combining the invariant amplitudes for the forward and backward scattering angles. The procedure for finding the expressions for the partial-wave amplitudes is set forth in detail in Ref. 5, and also in Refs. 23 and 24. In what follows, for convenience of controlling the crossing symmetry properties, we restore the pion mass, setting $\omega^2 = \nu + 1$ (ω is the pion energy and $\nu = \mathbf{q}^2$ is the square of the cms pion momentum).

In obtaining the partial-wave amplitudes, it is convenient to use an expansion in the amplitude in powers of ω/m and restrict oneself to the first terms of the expansion. Then the invariant amplitudes expressed in terms of the pion energy have the following crossing symmetry properties: $A^+(\omega)$ and $B^-(\omega)$ forward are symmetric under the substitution $\bar{\omega} \to -\bar{\omega}$ ($\bar{\omega} = 2\omega + 2\nu/m$), $\omega \to -\omega$; $A^-(\omega)$ and $B^+(\omega)$ are antisymmetric under this substitution. These amplitudes have the same properties for backward scattering under the substitution, but $\omega \to -\omega$, which enables us to obtain the partial s and p waves, which also satisfy simple crossing symmetry properties under the substitution $\omega \to -\omega$ (see the Appendix). In order to obtain the expansions of the invariant amplitudes in powers of ω/m, we use the expansion for the Euler dilogarithm given in Ref. 29:

$$\frac{m^2}{m^2 - s}\left[L\left(\frac{s}{m^2}\right) - L(1)\right] = 1 + \frac{1}{4}\left(1 - \frac{s}{m^2}\right) + \frac{1}{9}\left(1 - \frac{s}{m^2}\right)^2 + \ldots + \left[1 + \frac{1}{2}\left(1 - \frac{s}{m^2}\right) + \frac{1}{3}\left(1 - \frac{s}{m^2}\right)^2 + \ldots\right] \ln \frac{m^2}{m^2 - s}. \tag{29}$$

To simplify the expressions, we set $\omega^3 = \omega\nu$, $\omega^2 = \nu$,

$\tilde{\omega}=2\omega$. The explicit form of the total contribution of the diagrams of the higher orders is given in Ref. 36. Using the amplitudes (9)–(20) and (26)–(28) we can obtain expressions for the real parts of the partial s- and p-wave amplitudes, using the relations[10]

$$s^{\pm}=[\varphi^{\pm}(+1)+\varphi^{\pm}(-1)]/2; \quad (30)$$
$$h_{33}(\omega)=[\Delta\varphi^{+}-\Delta\varphi^{-}-2(\Delta\beta^{+}-\Delta\beta^{-})]/6\nu; \quad (31)$$
$$h_{13}(\omega)=[\Delta\varphi^{+}+2\Delta\varphi^{-}-2(\Delta\beta^{+}+2\Delta\beta^{-})]/6\nu; \quad (32)$$
$$h_{31}(\omega)=[\Delta\psi^{+}+2\Delta\varphi^{-}+4(\Delta\beta^{+}+2\Delta\beta^{-})]/6\nu; \quad (33)$$
$$h_{11}(\omega)=[\Delta\varphi^{+}+2\Delta\varphi^{-}+4(\Delta\beta^{+}+2\Delta\beta^{-})]/6\nu, \quad (34)$$

where

$$\Delta\varphi^{\pm}=\varphi^{\pm}(1)-\varphi^{\pm}(-1); \quad \Delta\beta^{\pm}=[\beta^{\pm}(+1)+\beta^{\pm}(-1)]/2;$$

and

$$\left.\begin{aligned}\varphi^{\pm}&=\frac{1}{4\pi}\left[A^{\pm}(\omega,\cos\theta)+\frac{s-u}{4m}B^{\pm}(\omega,\cos\theta)\right],\\ \beta^{\pm}&=\frac{1}{8\pi m}B^{\pm}(\omega,\cos\theta).\end{aligned}\right\} \quad (35)$$

The partial s waves in the isotopic state ± and also the p-wave amplitudes h_{ij} are related to the scattering phases as follows:

$$s^{\pm}=\frac{1}{3}\left[S_1+\binom{2}{-1}S_3\right]; \quad \Delta S_j=[\exp(i\delta_j)\sin\delta_j]/q; \qquad h_{ij}=[\exp(i\delta_{ij}\sin\delta_{ij}]/q^3 \quad (36)$$

($W^2=s$, $E=\sqrt{m^2+\nu}$ is the cms nucleon energy).

Let us first consider the partial s^- wave. Substituting into (30) the contributions of the tree diagrams (9), we obtain in the second order

$$\mathrm{Re}\, s_{\mathrm{II}}^{-}\approx 2f^2\omega/g_A^2. \quad (37)$$

Summing the contributions of the fourth-order diagrams with allowance for (30), we obtain

$$\mathrm{Re}\, s_{\mathrm{IV}}^{-}\approx\frac{8f^4}{\pi}\left[\left(2\tilde{C}-\frac{4\tilde{C}+3}{2g_A^2}\right)m^2\omega\right.$$
$$\left.+\left(\frac{(g_A^2-1)\tilde{C}}{g_A^2}+\frac{1{,}17}{g_A^2}\right)\omega\nu+\left(-1+\frac{5}{3g_A^2}+\frac{1}{3g_A^4}\right)\omega\nu\ln\frac{m}{2\omega}\right]. \quad (38)$$

It would seem that the presence of the terms linear in ω in (38) violates the well known low energy theorems. On the other hand, on the basis of analysis of the dispersion relations it was established that there are no terms of order $m^2\omega$ and higher in the s waves.[3,4] However, these terms of the type $m^2\omega$ do not occur in the s^- wave if

$$4g_A^2\tilde{C}-4\tilde{C}-3=0. \quad (39)$$

Using now the expressions for the real parts of the partial-wave amplitudes s_l in the second and fourth pertrubation order in the main coupling constant f^2 of the type

$$s_l=f^2 s_l^{\mathrm{II}}+f^4 s_l^{\mathrm{IV}}+\dots \quad (40)$$

we can, following Refs. 13 and 17, recover the amplitude in the (1, 1) Padé approximation in accordance with the expression

$$s_l=[f^2 s_l^{\mathrm{II}}]^2/(f^2 s_l^{\mathrm{II}}-f^4 s_l^{\mathrm{IV}}). \quad (41)$$

Then from the expressions (37) and (38) with allowance for (39) and using the (1, 1) Padé approximation (41), we obtain an expression for the partial-wave amplitude:

$$\mathrm{Re}\, s^{-}\approx\frac{2f^2}{g_A^2}\frac{\omega}{1+(4f^2\nu/\pi)\,[1.83g_A^2-1.92+(g_A^2-1.67+1/3g_A^2)\ln(m/2\omega)]}. \quad (42)$$

Similarly, for the s wave in the isotopic state + after accurate allowance for terms of the type m^{-1} in the first nonvanishing approximation, we obtain

$$s_{\mathrm{II}}^{+}\approx 2f^2\nu/m. \quad (43)$$

In the second nonvanishing approximation, using the expressions (9) and (30), we find

$$\mathrm{Re}\, s_{\mathrm{IV}}^{+}\approx\frac{8f^4}{\pi}\left\{\left(-\frac{11}{4}+\frac{79}{16g_A^2}\right)m\nu\right.$$
$$\left.+\left(2.86+\frac{0.56}{g_A^2}\right)\frac{\nu^2}{m}+\left[-\frac{2m\nu}{3g_A^2}+\left(1.5-\frac{9.33}{g_A^2}\right)\frac{\nu^2}{m}\right]\ln\frac{m}{2\omega}\right\}. \quad (44)$$

The condition of absence of terms of order $m\nu$ in the s^+ wave has the form

$$132g_A^2-205=0, \quad (45)$$

if one uses here the numerical approximation $\nu\ln(m/2\omega)\approx\nu(1-4\omega^2/m^2)$, which is completely adequate in the bounded range $\omega=1$–3 of values in which we are interested. With allowance for the condition (45) and using the (1, 1) Padé approximation (41), we obtain from (43) and (44) the final expression for the s^+ partial-wave amplitude:

$$\mathrm{Re}\, s^{+}\approx-\frac{2f^2}{m}\frac{\nu}{1+(4f^2\nu/\pi)\,[2.86+3.23/g_A^2+(1.5-9.33/g_A^2)\ln(m/2\omega)]}. \quad (46)$$

To make (42) and (46) more perspicuous, we use the circumstance that in this range of low energies ω and for values $g_A^2\approx 1.5$–2 of the axial constant one can with good accuracy approximate the square brackets in (42) and (46) by the expressions $2(g_A^2-1)$ and $5-6g_A^{-2}$, respectively. Then (42) and (46) take the simple form

$$\mathrm{Re}\, s^{-}\approx\frac{2f^2}{g_A^2}\frac{\omega}{1+8f^2\nu\,(g_A^2-1)/\pi}; \quad (42')$$
$$\mathrm{Re}\, s^{+}\approx-\frac{2f^2}{m}\frac{\nu}{1+(f^2\nu/\pi)\,[(20g_A^2-24)/g_A^2]}. \quad (46')$$

It is worth emphasizing that the s waves obtained in this way contain no arbitrariness since they are determined by the well known constants f^2 and g_A and they also have the correct crossing symmetry properties under the substitution $\omega\to-\omega$ (see the Appendix).

For the p waves in the first nonvanishing approximation,

$$h_{33}\approx\frac{4}{3}\frac{f^2}{\omega}; \quad h_{13}=h_{31}\approx-\frac{2}{3}\frac{f^2}{\omega}; \quad h_{11}\approx-\frac{8}{3}\frac{f^2}{\omega}. \quad (47)$$

Taking into account the contributions of the fourth-order diagrams (26)–(28) in accordance with (31)–(34)

and applying the (1, 1) Padé approximation (41) to the resulting expressions, we obtain, having performed a very simple unitarization, the following expressions for the p-wave amplitudes of πN scattering:

$$\mathrm{Re}\, h_{33}^{-1} \approx \frac{3\omega}{4f^2}\left\{1-\frac{2f^2\omega}{\pi}\left[\left(\frac{108g_A^2-77}{48g_A^2}-\frac{2}{3g_A^2}\ln\frac{m}{2\omega}\right)m - \left(2-\frac{3.75}{g_A^2}\right)\omega-\frac{\omega}{3}\left(2+\frac{g_A^2-1}{g_A^4}\right)\ln\frac{m}{2\omega}\right]\right\}; \quad (48)$$

$$\mathrm{Re}\, h_{13}^{-1} \approx -\frac{3\omega}{2f^2}\left\{1+\frac{4f^2\omega}{\pi}\left[\left(\frac{108g_A^2-77}{48g_A^2}-\frac{2}{3g_A^2}\ln\frac{m}{2\omega}\right)m + \omega\left(4.79-\frac{7.5}{g_A^2}+\frac{2}{3}\left(1+\frac{g_A^2-1}{g_A^4}\right)\ln\frac{m}{2\omega}\right)\right]\right\}, \quad (49)$$

$$\mathrm{Re}\, h_{31}^{-1} \approx -\frac{3\omega}{2f^2}\left\{1+\frac{4f^2\omega}{\pi}\left[\left(\frac{108g_A^2-77}{48g_A^2}-\frac{2}{3g_A^2}\ln\frac{m}{2\omega}\right)m - \omega\left(2.82-\frac{3.75}{g_A^2}-\frac{1}{3}\left(1-\frac{g_A^2-1}{g_A^4}\right)\ln\frac{m}{2\omega}\right)\right]\right\}; \quad (50)$$

$$\mathrm{Re}\, h_{11}^{-1} \approx \left\{-\frac{8}{3}\frac{f^2}{\omega}-\frac{16}{9}\frac{\gamma_{33}}{\omega+\omega_{33}}+\frac{16}{9}\frac{\gamma_{33}}{\omega+\omega_{33}} \times\left[1-\frac{9f^2(\omega+\omega_{33})}{8\pi}\left[\left(\frac{108g_A^2-77}{48g_A^2}-\frac{2}{3g_A^2}\ln\frac{m}{2\omega}\right)m + \omega\left(4-\frac{7.5}{g_A^2}+\frac{2}{3}\left(2+\frac{g_A^2-1}{g_A^4}\right)\ln\frac{m}{2\omega}\right)\right]\right]^{-1}\right\}^{-1}. \quad (51)$$

The derivation of the expression for the wave h_{11} requires a special explanation. Use of the expression from (47) as the first term of the Padé approximation for the h_{11} wave leads to a negative phase shift δ_{11}, which is in crass disagreement with experiment. It is well known[30] that the 11 phase shift becomes positive at $\omega = 1.7$ and passes through resonance in the region of $\omega = \omega_{11} \approx 3.7$. The second-order term $h_{11} = -8f^2/3\omega$ derives from nucleon exchange in the s channel $(-9f^2/3\omega)$ and in the u channel $(f^2/3\omega)$. Only this second, numerically small term can be regarded as the potential ensuring the necessary attraction for the formation of the positive phase shift and the resonance. Clearly, the term $-3f^2/\omega$ cannot be regarded as the original one for constructing the Padé approximation. It follows from dispersion relations[31] that the main contribution to the attraction in this wave is due to the exchange (33)—the resonance in the u channel—and has the form $(16/9)\gamma_{33}(\omega+\omega_{33})^{-1}$, where $\gamma_{33} = 4f^2/3$. On the basis of these arguments, we have separated out in the 11 wave the term $(16/9) \times \gamma_{33}(\omega+\omega_{33})^{-1}$ and regarded it as the basic one for constructing the Padé approximation for this wave. The p-wave amplitudes also approximately satisfy the crossing symmetry conditions (see the Appendix).

6. COMPARISON WITH EXPERIMENT

Investigation of the transcendental equation (39) shows that one of the roots is in the range $1.3 < g_A < 1.35$, which is completely acceptable from the physical point of view. Thus, the superpropagator method does indeed lead to amplitudes for which the low energy theorems are satisfied at close to the experimental values of g_A.

It is known that the s^- wave is due principally to exchange of a ρ meson in the t channel. The expression for the s^- wave calculated by means of a linear ρ-exchange Lagrangian of the Yukawa type has the form

$$\mathrm{Re}\, s^- \approx \frac{g_{\rho\pi\pi}g_{\rho NN}}{4\pi}\frac{2\omega}{m_\rho^2+4\nu}, \quad (52)$$

if at the same time one takes into account the universality hypothesis $g_{\rho\pi\pi} = 2g_{\rho NN}$.

The expression (42′) for Res^- has the same form as (52) with the important difference that the parameters of the ρ meson are here not introduced from without but are obtained dynamically. Indeed, comparing the numerators and denominators in (42′) and (52), we obtain the relations

$$\left.\begin{aligned} 2g_{\rho\pi\pi}g_{\rho NN}/4\pi &\approx \pi/[g_A^2(g_A^2-1)]; \\ m_\rho^2 &\approx \pi/[2f^2(g_A^2-1)]. \end{aligned}\right\} \quad (53)$$

If the second of these relations is substituted in the first and the hypothesis of universality of the interaction of the ρ meson with the π-meson and nucleon fields is invoked, and also the Goldberger–Treiman relation, we obtain

$$g_{\rho\pi\pi}^2 = m_\rho^2/2F_\pi^2,$$

which is the well known Kawarabayashi–Suzuki–Fayyazuddin–Riazuddin relation.

We now consider the s^+ wave. The condition that there be no terms of order $m\nu$ in (45) is satisfied at values $g_A \approx 1.25$, which is very close to the values of g_A taken from Eq. (39). It is well known that the s^+ wave can be described by σ-meson exchange, and, as was pointed by Shirkov and Serebryakov,[10] an important role is here played by the allowance for short-wavelength repulsion. The expression for Res^+ calculated by means of a linear σ-exchange Lagrangian of the Yukawa type with allowance for short-wavelength repulsion, has the form[10]

$$\mathrm{Re}\, s^+ \approx -\frac{g_{\sigma\pi\pi}g_{\sigma NN}}{4\pi m_\sigma^2}\frac{4\nu}{m_\sigma^2+4\nu}, \quad (54)$$

which again has the same form as the expression (46′) for Res^+. Comparing (46′) and (54), we obtain

$$g_{\sigma\pi\pi}g_{\sigma NN}/4\pi \approx t_\sigma^2 f^2/2m, \quad (55)$$

which is a kind of analog of the Kawarabayashi–Suzuki–Fayyazuddin–Riazuddin relation. For the square of the mass of the σ meson we also obtain a fairly simple approximate relation:

$$m_\sigma^2 \approx \pi g_A^2/[f^2(5g_A^2-6)]. \quad (56)$$

In a number of cases, it is important to know the constants $g_{\sigma\pi\pi}$ and $g_{\sigma NN}$ separately. From (55) we can obtain an estimate for $g_{\sigma\pi\pi}$ if, for example, we start with the relation $g_{\pi NN} = g_{\sigma NN}$, which follows from the Gell-Mann–Lévy σ model. For $g_{\sigma\pi\pi}$ we then find

$$g_{\sigma\pi\pi}^2/4\pi = t_\sigma^4 f^2/16m^4, \quad (57)$$

which for values $t_\sigma = 20$ to 34 gives $g_{\sigma\pi\pi} = 4\pi(0.4$ to $3.3)$. It is interesting to note that if we set $t_\sigma = 50$ then Eq. (57) yields the relation $g_{\sigma\pi\pi} = g_{\sigma NN} = g_{\pi NN} \approx 60\pi$. But if we

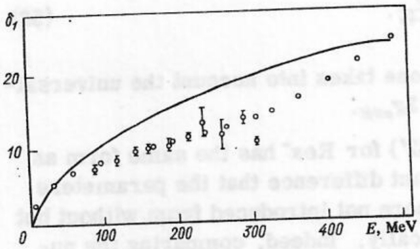

FIG. 2. Energy dependence of the phase shift δ_1. The continuous curve represents the theoretical data obtained from the expressions (42) and (46) for $g_A=1.4$; the open circles (Ref. 30) and the open circles with bars (Ref. 33) are the experimental points.

assume that a specific analog of Sakurai's universality hypothesis holds,[7] i.e., $g_{\sigma\pi\pi}=g_{\sigma NN}$, it then follows from (55) that

$$g^2_{\sigma\pi\pi}/4\pi=t_\sigma^2 f^2/2m. \tag{58}$$

For $t_\sigma\approx 30$, Eq. (58) also yields values of $g_{\sigma NN}$ for which a good description is obtained for the low energy nucleon–nucleon and pion–nucleon interactions calculated on the basis of dispersion relations.[32] The energy dependence of the s-wave phase shifts δ_1 and δ_3 calculated in accordance with (42) and (46) and with allowance for the relation (30) is shown in Figs. 2 and 3.

It can be seen that the agreement with the experimental data is better at values $g_A=1.25$–1.4, which is in agreement with Eqs. (39) and (45). At these values of g_A, the scattering length a^- varies in the range $a^-=0.107$ to 0.08, the square of the mass of the ρ meson in the range $t_\rho\approx 21$ to 35, and $g^2_{\rho\pi\pi}/4\pi=2.6$ for $g_A=1.3$ (recall that we use the system $\hbar=c=m_\pi=1$). The scattering length $a^+=0$, the square of the mass of the σ meson is $t_\sigma=(20–34)m_\pi^2$, and $g_{\sigma\pi\pi}g_{\sigma NN}=(2.5–6.5)4\pi$.

If we compare with the experimental data not the s-wave phase shifts δ_1 and δ_3 themselves but the isotopic amplitudes $s^\pm$, we see that the agreement between the s^- amplitude and the experimental data is very good, whereas the s^+ amplitude agrees with these data somewhat less well, which is reflected in Figs. 2 and 3. Numerically, this is due to the fact that the residue at

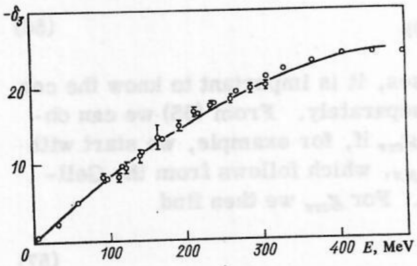

FIG. 3. Energy dependence of the phase shift δ_3. The continuous curve represents the theoretical data obtained from the expressions (42) and (46) for $g_A=1.3$; the open circles (Ref. 30) and the open circles with bars (Ref. 33) are the experimental points.

the σ pole obtained from the chiral Lagrangian is underestimated in accordance with (55). However, we should still like to emphasize the very reasonable agreement between the calculated arbitrary parameters.

Note in this connection that the s waves in the $\pm$ isotopic state are the most suitable for theoretical analysis since the partial waves are due to the exchange of σ and ρ mesons in the t channel. It is this circumstance which indicates that the Padé approximation must be applied to these amplitudes. The singularities in these amplitudes obtained by means of the Padé approximation have a clear physical meaning. In contrast, the singularities of the amplitudes with total isotopic spin $I=1/2$ and $3/2$ obtained by means of the Padé procedure (as it is usually done; see, for example, Refs. 16 and 18) do not have such a clear physical meaning, and they are merely a certain average with certain weights of the real physical singularities.

It is interesting to note that whereas the use of a linear σ-exchange Lagrangian requires the additional introduction of a short-wavelength repulsion, the nonlinear chiral Lagrangian in the single-loop approximation reproduces the correct result automatically. The condition of absence of terms of order $m^2\omega$ and $m\nu$ in the s^- and s^+ waves is satisfied automatically at slightly different but still very close values of g_A. There is nothing remarkable about this difference, since in the derivation of these equations we have used a whole series of approximations, for example, the approximate separation of the partial-wave amplitudes, incorrect allowance for the meson mass, various numerical approximations, etc.

Note one further important circumstance: Not only in the first but also in the second nonvanishing order of perturbation theory there is a very strong mutual compensation of the terms of order m^0 and m^{-1}, i.e., the terms that give the physical results. Moreover, it is basically the diagrams of fourth and third orders that are compensated (of the types in Figs. 1a–1g), and the main contribution is made by the loop diagrams of second order (of the types in Figs. 1h and 1i). It is therefore not remarkable that the results of this paper are very close to those of Ref. 24, in which allowance was made for the contributions of only the loop diagrams. In other words, the situation is such as if the series expansion by the perturbation method in the nonlinear chiral Lagrangian were made with respect to a constant much smaller than $g^2_{\pi NN}$.

We now come to consider p waves. The energy dependence of the p-wave phase shifts is shown in Figs. 4–6. As can be seen from these figures, the calculated curves are in satisfactory agreement with the experimental data right up to energies ~400 MeV. One could hardly require better agreement with the experimental phase shifts in the region of pion kinetic energies above 400 MeV since we have entirely ignored inelastic effects. This applies in the first place to the strongly inelastic (1, 1) phase shift. The investigation of the phase shift δ_{11} by means of dispersion relations indicates that satisfactory agreement with the experiment can be obtained with allowance for the additional inelastic channel. With regard to the

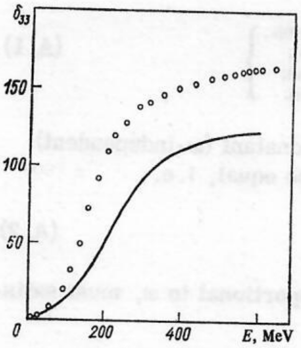

FIG. 4. Energy dependence of the phase shift δ_{33}. The continuous curve represents the theoretical data obtained from (48) for $g_A = 1.4$; the open circles are the experimental points taken from Ref. 30.

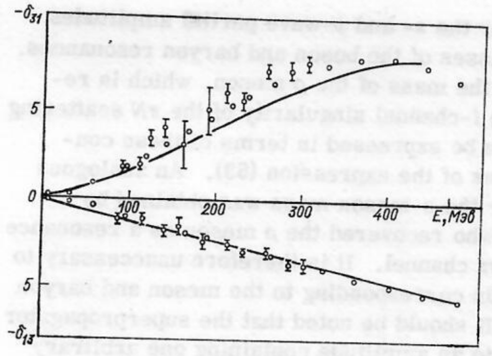

FIG. 6. Energy dependence of the phase shifts δ_{13} and δ_{31}. The continuous curves represent the theoretical data obtained from the expressions (49) and (50) for $g_A = 1.4$ and $g_A = 1.3$, respectively; the experimental points are taken from Refs. 30 and 33.

phase δ_{33}, all models lead to one and the same result—an underestimated value of this phase shift in the region beyond the resonance (cf. Ref. 17). The low energy theory is evidently capable of describing the δ_{33} phase shift only in the pre-resonance region (it is sufficient to recall the Chew–Low effective range theory.) The results obtained correctly reflect the most important and, at the same time, complicated features of the p-wave amplitudes: the presence of resonances in the (1, 1) and (3, 3) waves, transition of the phase shift δ_{11} through zero in the neighborhood of $\omega \approx 1.5$, and the absence of low energy resonances in the small p_{13} and p_{31} amplitudes. In addition, closed expressions are also obtained for the position of the p-wave resonances in terms of the fundamental constants, namely

$$\omega_{33} \approx \frac{\pi g_A^2}{4.4 f^2 m (g_A^2 - 1)}; \quad \omega_{11} \approx \frac{4\pi g_A^2}{9.9 f^2 m (g_A^2 - 1)} - \omega_{33}. \tag{59}$$

From (59) there follow the numerical values $\omega_{33} \approx 2.6$ and $\omega_{11} \approx 3$ for the positions of the resonances, while their experimental values are, respectively, $\omega_{33} \approx 2$ and $\omega_{11} \approx 3.7$ (Refs. 30 and 33).

From the expressions (48)–(51) taken at the threshold we obtain the following expressions for the scattering "lengths":

$$a_{33} \approx 0.18; \quad a_{13} \approx -0.026; \quad a_{31} \approx -0.032; \quad a_{11} \approx -0.07, \tag{60}$$

whereas their experimental values are[34]

$$\left.\begin{array}{ll} a_{33} = (204.1 \pm 4.5)\,10^{-3}; & a_{13} = (-26.6 \pm 6.3)\,10^{-3}; \\ a_{31} = (-42.9 \pm 7.1)\,10^{-3}; & a_{11} = (-84.5 \pm 10.2)\,10^{-3}. \end{array}\right\} \tag{61}$$

As follows from (60) and (61), the p-wave scattering lengths agree with the experiment to within the limits of the errors, except for a_{33}. The somewhat too small length of the (3, 3) wave leads to an underestimated value of the (3, 3) phase shift, as can be seen in Fig. 4. We should like to point out that a 30% deviation of this phase shift from the experimental data is within the range of our approximations. It can be seen from the expressions (48) and (51) that the energy dependence of the (3, 3) and (1, 1) phase shifts is determined to a considerable extent by the small terms of order ω in the square brackets, whose accuracy is below the accuracy in the determination of the principal terms $\sim m$. The positive coefficient of ω in (48) and the smaller (in absolute magnitude) coefficient of ω in (51) can lead to a very good description of these waves. At the same time, the small p waves are determined primarily by the terms of order m and are not very sensitive to the coefficients of ω. Thus, quantitative description of the p waves seems to require the inclusion of two additional parameters. This could be taken as a phenomenological indication of the role of the following approximations in the perturbation method.

From the expressions (9)–(20) and (26)–(28) for the invariant amplitudes $A^{\pm}(s, t, u)$ and $B^{\pm}(s, t, u)$ and their derivatives with respect to the angle in the (1, 1) Padé approximation (41), one can also obtain analytic expressions for the partial d waves. Such calculations are made in Ref. 35. The results obtained are in satisfactory agreement with the experimental data in the energy range we consider for all the d phase shifts except d_{13}.

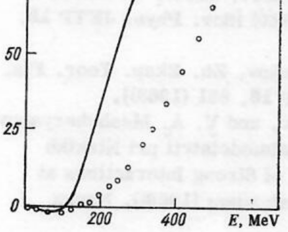

FIG. 5. Energy dependence of the phase shift δ_{11}. The continuous curve represents the theoretical data obtained from (51) for $g_A = 1.27$; the open circles are the experimental points taken from Ref. 30.

CONCLUSIONS

The results considered in this paper indicate that, using nonlinear chiral Lagrangians, one can obtain a dynamical description of low energy πN scattering.[36] The essence of the dynamical description of the interaction of elementary particles based on the use of such an interaction Lagrangian is that all the physical amplitudes are determined by a small number of basic fundamental parameters, in our case by the masses of the nucleon and the π meson (m, m_π) and the coupling constants f^2 and g_A. No arbitrary parameters whatsoever are introduced. It is possible to obtain parameter-free expressions for the s- and p-wave partial amplitudes and for the masses of the boson and baryon resonances. For example, the mass of the ρ meson, which is recovered as the t-channel singularity of the πN scattering amplitude, can be expressed in terms of these constants by means of the expression (53). An analogous expression for the ρ-meson mass was obtained by Lehmann,[28] who recovered the ρ meson as a resonance in the direct $\pi\pi$ channel. It is therefore unnecessary to introduce fields corresponding to the meson and baryon resonances. It should be noted that the superpropagator method leads to an amplitude containing one arbitrary parameter η. To eliminate this arbitrariness, it is necessary to invoke additional physical arguments. The usually employed conditions lead to the choice $\eta=0$ (see, for example, Refs. 1 and 37). Above, we have followed this choice.

With regard to the prospects for applying nonlinear Lagrangians and these methods to describe other important processes of strong interactions such as nucleon–nucleon scattering and the electromagnetic nucleon form factors, it must be pointed out that direct use of the simplest nonlinear chiral invariant Lagrangians of the type (4) does not lead to a satisfactory description of these processes. For example, the Lagrangian (4) leads to zero nucleon–nucleon scattering lengths, which clearly contradicts the experimental data. This shortcoming can be eliminated in several ways. One of them is to leave the mass shell. This way, which leads to the matrix variant of the Padé approximation, was proposed by Bessis *et al.*[38,39] Another way, proposed in Ref. 40, is to employ more complicated chiral invariant terms (of the four-fermion type) in the interaction Lagrangian.

It follows from the results that the nonlinear $SU(2)\times SU(2)$ chiral invariant Lagrangian enables one, in conjunction with the superpropagator method of regularization, the generalized method of summation of the renormalized perturbation series, and the Padé approximation, to obtain a dynamical description of not only the $\pi\pi$ interaction and the electromagnetic properties of the π and K mesons[19,20] but also of the very important πN scattering process.[36]

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APPENDIX

The s waves (37), (38) and (43), (44) obtained in the first and second order of the perturbation method satisfy the simple crossing symmetry conditions (Ref. 5) $s^{\pm}(\omega)=\pm s^{\pm}(-\omega)$. It is interesting that the p waves obtained also satisfy the approximation of slightly more complicated crossing symmetry conditions. Namely, as is shown in Refs. 4 and 5, the p waves satisfy the condition

$$\left.\begin{aligned} &h_{31}(\omega)-h_{11}(\omega)=-[h_{31}(-\omega)-h_{11}(-\omega)];\\ &2h_{31}(\omega)+h_{11}(\omega)=2h_{31}(-\omega)+h_{11}(-\omega);\\ &h_{33}(\omega)-h_{13}(\omega)=-h_{33}(-\omega)+h_{13}(-\omega);\\ &2h_{33}(\omega)+h_{13}(\omega)=2h_{33}(-\omega)+h_{13}(-\omega).\end{aligned}\right\} \tag{A.1}$$

It follows from this that the constant (ω-independent) terms in the h_{ij} waves must be equal, i.e.,

$$c_{31}=c_{13}=c_{33}=c_{11}. \tag{A.2}$$

The terms a_{ij}, which are proportional to ω, must satisfy the conditions

$$2a_{31}=-a_{11};\ 2a_{33}=-a_{13}. \tag{A.3}$$

As follows from the relations (36)–(38), the following equation will be approximately satisfied[5] (with neglect of $\Delta\beta^{\pm}$):

$$h_{11}(\omega)+\binom{2}{-1}h_{31}(\omega)=h_{13}(\omega)+\binom{2}{-1}h_{33}(\omega), \tag{A.4}$$

from which it follows that

$$2a_{31}=-a_{13}=2a_{33}=-a_{11}. \tag{A.5}$$

In the fourth order of the perturbation method, on the basis of (10)–(20), (26)–(28), and (31)–(34),

$$\operatorname{Re} h_{33}\approx\frac{8}{3}\frac{f^4}{\pi}\left[\frac{108g_A^2-109}{48g_A^2}m+\omega\left(-2+3.75g_A^{-2}-\frac{\widetilde{G}}{3}\ln\frac{m}{2\omega}\right)\right];$$

$$\operatorname{Re} h_{13}\approx\frac{8}{3}\frac{f^4}{\pi}\left[\frac{108g_A^2-109}{48g_A^2}m+\omega\left(4.79-7.5g_A^{-2}+\frac{2}{3}\widetilde{G}\ln\frac{m}{2\omega}\right)\right];$$

$$\operatorname{Re} h_{31}(\omega)=\frac{8}{3}\frac{f^4}{\pi}\left[\frac{108g_A^2-109}{48g_A^2}m+\omega\left(-2{,}82+3.75g_A^{-2}-\frac{\widetilde{G}}{3}\ln\frac{m}{2\omega}\right)\right];$$

$$\operatorname{Re} h_{11}(\omega)=\frac{8}{3}\frac{f^4}{\pi}\left[\frac{108g_A^2-109}{48g_A^2}m+\omega\left(4-7.5g_A^{-2}+\frac{2}{3}\widetilde{G}\ln\frac{m}{2\omega}\right)\right];$$

where $\tilde{G}=(g_A^2-1)/g_A^4$.

As one would expect, the conditions (A.2) for the constant terms are satisfied exactly. The conditions for the coefficients a_{ij} of ω are not satisfied exactly for all terms. This is due to the approximate fulfillment of the condition (A.4). The same applies to the fulfillment of the conditions for the p waves in the (1, 1) order of the perturbation method (47). Nevertheless, we see that the crossing symmetry conditions are approximately satisfied. If a Padé inversion is made, the crossing symmetry property is lost.

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