One-boson exchange potentials and their use in low energy nuclear physics

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Some aspects of the modern meson theory of the *NN* interaction and its use to describe nuclear matter and finite nuclei are considered. The one-boson exchange potential (OBEP) is introduced, and considered in application to nucleon-nucleon scattering, the properties of the deuteron, and quasinuclear resonances. Its application to nuclear matter and also to finite nuclei in the framework of relativistic Hartree and Hartree-Fock theories is described.

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INTRODUCTION

The attempt to describe the interaction between nucleons and the properties of nuclei in the framework of a unified approach on the basis of meson exchange is an old but attractive idea. In recent years, interest in this problem has grown in connection with the construction of models of the nucleon–nucleon interaction (one-boson exchange potentials, abbreviated OBEP), which take into account the exchange of mesons and resonances with different space-time transformation properties. This interest is fully understandable, since the use of meson potentials puts the theory of nuclear structure on a new footing, eliminating phenomenology in the choice of the nucleon–nucleon interaction and relegating it to the description of elementary particles and their interactions. This establishes an intimate connection between the theory of nuclear structure and elementary particles, so that in each of them one can obtain new results by using regularities established in the adjacent field.

The properties of OBE potentials have been reviewed several times. Babikov[1] has surveyed the field up to 1970, with brevity and clarity, including both the meson theory of nuclear forces and the main directions in its development as well as the possibilities for applying the meson theory of nucleon–nucleon interactions to heavy nuclei. We should also mention Erkelenz's review,[2] which brings the field up to 1974. Some other literature will be considered below.

In this review we discuss the results so far achieved in the implementation of the above program, and we have taken our aim to be that of combining in one article a discussion of the theory of nuclear matter and the theory of finite nuclei on the unified basis of the meson theory of nuclear interactions.

1. ONE-BOSON EXCHANGE POTENTIALS

We consider a system of two nucleons interacting with the meson field. For the moment, we shall not particularize the meson field (which may be a scalar, vector, or pseudoscalar field).

Suppose the state of the system is described by the wave function

$$\Psi(\mathbf{r}_1, \xi_1; \mathbf{r}_2, \xi_2; Q; t), \tag{1}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of the first and the second nucleon; ξ_1 and ξ_2 are their spin and isospin variables; Q is the variable of the meson field.

We write down the Schrödinger equation for the wave function (1):

$$\begin{aligned}&[H_D(\mathbf{r}_1, \xi_1)+H_D(\mathbf{r}_2, \xi_2)+H_{\text{int}}(\mathbf{r}_1, \xi_1)\\&+H_{\text{int}}(\mathbf{r}_2, \xi_2)+H_{\text{mes}}]\,\Psi(\mathbf{r}_1, \xi_1; \mathbf{r}_2, \xi_2; Q, t)\\&\quad=i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}_1, \xi_1; \mathbf{r}_2, \xi_2; Q; t),\end{aligned} \tag{2}$$

where

$$H_D(\mathbf{r}_i, \xi_i)=\boldsymbol{\alpha}_i\mathbf{p}_i+\beta_i Mc^2 \tag{3}$$

are the Dirac operators for the first and second nucleon (for $i=1$ and 2, respectively); $H_{\text{int}}(\mathbf{r}_i, \xi_i)$ is the operator of the interaction of nucleon i with the meson field. The form of $H_{\text{int}}(\mathbf{r}, \xi)$ depends on the species of meson with which the interaction takes place. If the nucleons interact with scalar mesons, the interaction operator is

$$H^S_{\text{int}}(\mathbf{r}_i, \xi_i)=-\sqrt{2\pi}\,g_S\beta_i Q(\mathbf{r}_i), \tag{4}$$

and if they interact with vector mesons,

$$\begin{aligned}H^V_{\text{int}}(\mathbf{r}_i, \xi_i)=-\sqrt{2\pi}\Big\{&g_V\Big[\sum_{k=1}^{3}\alpha_{ik}Q_k(\mathbf{r}_i)+iQ_0(\mathbf{r}_i)\Big]\\&+\frac{m}{4M^2}f_V\sigma^i_{\mu\nu}Q_{\mu\nu}(\mathbf{r}_i)\Big\},\end{aligned} \tag{5}$$

where m is the meson mass and M the nucleon mass.

The operator of the interaction of the nucleon with pseudoscalar meson field has the form[1)]

$$H^P_{\text{int}}(\mathbf{r}_i, \xi_i)=-\sqrt{2\pi}\,g_P i\beta_i\gamma_{5,i}Q(\mathbf{r}_i), \tag{6}$$

where $Q(\mathbf{r}_i)$ is the amplitude of the scalar or pseudo-

[1)]The interaction of a nucleon with a pseudoscalar meson field can also be considered in the form $-(f/m)\beta\gamma_5\gamma_\mu\partial_\mu Q(\mathbf{r})$. At the present time, it is not clear which of these two forms of the interaction is actually realized, though the expression (6) is more widely used.

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scalar field at the position of nucleon i; $Q_k(\mathbf{r}_i)$ and $Q_0(\mathbf{r}_i)$ are the three spatial and the one time component of the vector field;

$$\sigma_{\mu\nu}=\frac{1}{2i}[\gamma_\mu\gamma_\nu];\quad Q_{\mu\nu}(\mathbf{r})=\partial_\mu Q_\nu-\partial_\nu Q_\mu.$$

The constants g characterize the strength of the interaction in the case of the vector field (5) the additional constant f_V is also needed.

Besides the operators given above, there may be other forms of the operator of the interaction between the nucleons and mesons field, but Eqs. (4)–(6) give the simplest forms of the interaction. The operator H_{mes} in (2) describes the free meson field. The matrices α, β, γ_5, γ_μ are the traditional matrices of relativistic theory.

Using the method of Fock functionals, one can show that the problem of the interaction of two nucleons through a meson field in accordance with Eqs. (2)–(6) can be reduced in the framework of the one-boson approximation (see, for example, Ref. 4) to the following problem:

$$[H_D(\mathbf{r}_1,\xi_1)+H_D(\mathbf{r}_2,\xi_1)+V^D]\,\Phi_0(\mathbf{r}_1,\xi_1;\mathbf{r}_2,\xi_2;t)=i\hbar\frac{\partial}{\partial t}\Phi_0(\mathbf{r}_1,\xi_1;\mathbf{r}_2,\xi_2;t),\tag{7}$$

where $\Phi_0(\mathbf{r}_1,\xi_1;\mathbf{r}_2,\xi_2;t)$ is the wave function of the system of two nucleons; V^D is the one-boson exchange potential (OBEP).

If the interaction between the nucleons is realized through scalar mesons, then (see Ref. 4)

$$V_S^D=V^S(r)\,\beta_1\beta_2.\tag{8}$$

For vector mesons,

$$V_V^D=V^V(r)\,(\alpha_1\cdot\alpha_2-1).\tag{9}$$

Finally, for pseudoscalar mesons

$$V_P^D=-V^P(r)\,\beta_1\gamma_{5,1}\beta_2\gamma_{5,2}.\tag{10}$$

The potentials (8)–(10) have been written down for the case when the mesons exchanged between the nucleons are isoscalar, i.e., have zero isospin.

The interaction potentials resulting from the exchange of isovector mesons have the form

$$V_{S\tau}^D=V^{S\tau}(r)\,\beta_1\beta_2\,(\tau_1\cdot\tau_2);\tag{11}$$

$$V_{V\tau}^D=V^{V\tau}(r)\,(\alpha_1\cdot\alpha_2-1)\,\tau_1\cdot\tau_2;\tag{12}$$

$$V_{P\tau}^D=-V^{P\tau}(r)\,\beta_1\gamma_{5,1}\beta_2\gamma_{5,2}\tau_1\cdot\tau_2.\tag{13}$$

The interaction V^D in Eq. (7) contains a sum over interactions for all the mesons taken into account in the description of the interaction.

Note that the derivation of the OBE potentials entails a cutoff procedure and the introduction of the so-called interaction form factors $F(\mathbf{k}^2)$, where $\mathbf{k}$ is the momentum of the exchange meson. This quantity is introduced in order to make the potential obtained from meson theory finite at zero distances between the nucleons. In this way, one avoids the difficulties associated with divergences of the potential at the origin. The introduction of $F(\mathbf{k}^2)$ corresponds to "smearing" point nucleons over a finite region of space and is thus a phenomenological method for taking into account the finite sizes of the nucleons.

The meson–nucleon form factors are introduced rather arbitrarily: For example, a form factor in the form $[\Lambda^2/(\mathbf{k}^2+\Lambda^2)]^{1/2}$ (where Λ is the cutoff parameter) was used in Refs. 4 and 5 for elastic scattering and in Ref. 6 for inelastic pp scattering. A form factor in the form $\Lambda^2/(\mathbf{k}^2+\Lambda^2)$, corresponding to the so-called dipole regularization, was used, for example, in Ref. 7. The form factor problem has been discussed in Refs. 1, 2, 4, and 8. Note that in approximate relativistic calculations (to accuracy V^2/c^2) which use OBE potentials (as nucleon–nucleon interactions) the introduction of form factors eliminates the $1/r^3$ singularities of the tensor and spin–orbit forces.

Choice of the factor in the form $F(\mathbf{k}^2)=\Lambda^2/(\Lambda^2+\mathbf{k}^2)$ leads to a function $V(r)$ of the form

$$V(r)=-g^2[\Lambda^2/(\Lambda^2-m^2)]^2\{\exp(-mr)/r-[\exp(-\Lambda r)/r]\,[1+(\Lambda^2-m^2)\,r/2\Lambda]\}.\tag{14}$$

Note that in the present review we use functions $V(r)$ that differ in sign from the functions $J(r)$ used in Refs. 4, 7, 9, and 10; the form of the function $V(r)$ corresponding to different types of regularization is given, for example, in Ref. 10,

The procedure for deriving the OBE potential (8)–(13) in the coordinate representation includes two approximations: the adiabatic approximation $\mathbf{q}^2\ll M^2$, where $\mathbf{q}$ is the relative momentum of the nucleons in the initial state and M is its mass, and neglect of retardation effects (see, for example, Refs. 2 and 11).

Since in what follows we shall be interested in laboratory system energies not exceeding 450 MeV, the Dirac equation (7) for the two nucleons 1 and 2 can be reduced to an equation of the Breit–Pauli type taking into account relativistic corrections to order v^2/c^2.

In the present review, we shall not dwell on these transformations (they are presented, for example, in Ref. 4) but give only the final result. In the approximation $v^2/c^2\ll 1$, Eq. (7) takes the form[4]

$$[p_1^2/2M+p_2^2/2M+\sum_{\text{mesons}}V_{\text{tot}}(\mathbf{r}_1,\xi_1;\mathbf{r}_2,\xi_2)]\,\Psi(\mathbf{r}_1,\xi_1;\mathbf{r}_1,\xi_2;t)=i\hbar\,\partial\Psi(\mathbf{r}_1,\xi_1;\mathbf{r}_2,\xi_2;t)/\partial t,\tag{15}$$

where the interaction V_{tot} between the two nucleons of mass M has the form

$$V_{\text{tot}}=V_C(r)+V_\sigma(r)\,(\sigma_1\cdot\sigma_2)+V_{LS}(r)\,\mathbf{l}\cdot\mathbf{S}+V_T(r)\,S_{12}+V_\Delta(r)\,\nabla^2+V_\nabla(r)\,(\mathbf{r}\cdot\nabla),\tag{16}$$

where $\mathbf{r}=\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$ is the radius vector joining the nucleons; $\mathbf{p}=\mathbf{p}_{12}=(\mathbf{p}_1-\mathbf{p}_2)/2$ is the relative momentum of the nucleons; $\mathbf{l}=\mathbf{r}\times\mathbf{p}$ is the angular momentum of the relative motion; $\mathbf{S}=(\sigma_1+\sigma_2)/2$ is the operator of the to-

TABLE I. Mesons and resonances with $J \leqslant 1$ (I is the isospin).

I \ J^P	0^-	0^+	1^-
0	η (549 MeV)	ε (800—1000 MeV), S^* (1000 MeV)	ω (784 MeV) φ (1019 MeV)
1	π (140 MeV)	π_N (950—1020 MeV)	ρ (765 MeV)

tal spin of the two nucleons; $S_{12} = [(3/r^2)(\sigma_1 \cdot \mathbf{r})(\sigma_2 \cdot \mathbf{r}) - (\sigma_1 \cdot \sigma_2)]$ is the tensor operator; the form of the functions $V_C(r)$, $V_\sigma(r)$, $V_{LS}(r)$, $V_T(r)$, $V_\Delta(r)$, $V_\nabla(r)$ is determined by the spacetime properties of the exchange mesons. We give the contribution of the various mesons to each component of (16). The scalar mesons:

$$V_C(r) = V^S(r) + a^2 \langle \nabla^2 V^S(r) \rangle / 4; \tag{17}$$

$$V_\sigma(r) = V_T(r) = 0; \; V_{LS}(r) = -\frac{1}{2} a^2 \frac{1}{r} \frac{dV^S}{dr}; \tag{18}$$

$$V_\Delta(r) = a^2 V^S(r); \tag{19}$$

$$V_\nabla(r) = a^2 \frac{1}{r} \frac{d}{dr} V^S(r). \tag{20}$$

The vector mesons:

$$\left.\begin{aligned} V_C(r) &= -V^V(r) - \tfrac{1}{2} a^2 \tfrac{f}{g} \langle \nabla^2 V^V(r) \rangle; \\ V_\sigma(r) &= -\tfrac{1}{6} a^2 \left(1 + \tfrac{f}{g}\right)^2 \langle \nabla^2 V^V(r) \rangle; \\ V_{LS}(r) &= -\tfrac{3}{2} a^2 \left(1 + \tfrac{4}{3} \tfrac{f}{g}\right) \tfrac{1}{r} \tfrac{dV^V(r)}{dr}; \\ V_T(r) &= \tfrac{1}{2} a^2 \left(1 + \tfrac{f}{g}\right)^2 r \tfrac{d}{dr} \left(\tfrac{1}{r} \tfrac{dV^V(r)}{dr}\right); \\ V_\Delta(r) &= a^2 V^V(r); \\ V_\nabla(r) &= a^2 \tfrac{1}{r} \tfrac{dV^V(r)}{dr}. \end{aligned}\right\} \tag{21}$$

The pseudoscalar mesons:

$$\left.\begin{aligned} V_C(r) &= V_{LS}(r) = V_\Delta(r) = V_\nabla(r) = 0; \\ V_\sigma(r) &= -\tfrac{1}{12} a^2 \langle \nabla^2 V^P(r) \rangle; \\ V_T(r) &= -\tfrac{1}{12} a^2 r^2 \tfrac{1}{r} \tfrac{d}{dr} \left(\tfrac{1}{r} \tfrac{dV^P(r)}{dr}\right), \end{aligned}\right\} \tag{22}$$

where $a^2 = \hbar^2/Mc^2$; the angular brackets indicate that the action of the operator ∇^2 is not extended to functions outside the angular brackets; f and g are, respectively, the constants of the tensor and vector coupling of the vector meson to the nucleons [see (5)]. Note that the individual terms in the expression (16) are not Hermitian, although this operator taken as a whole is. This follows from the fact that for scalar mesons the operator (16) with allowance for (17)–(20) can be reduced to the form

$$V^S(r) - \tfrac{1}{2} a^2 [p^2 V^S(r) + V^S(r)\, p^2]$$
$$- \tfrac{1}{4} a^2 \langle \nabla^2 V^S(r) \rangle - \tfrac{1}{2} a^2 \tfrac{1}{r} \tfrac{dV^S(r)}{dr} \mathbf{l} \cdot \mathbf{S},$$

all of whose components are Hermitian.

For vector mesons, instead of (16) and (21),

$$-V^V(r) - \tfrac{1}{2} a^2 \left(1 + \tfrac{f}{g}\right) \langle \nabla^2 V^V(r) \rangle$$
$$- \tfrac{1}{2} a^2 [p^2 V^V(r) + V^V(r)\, p^2] - \tfrac{1}{6} a^2 \left(1 + \tfrac{f}{g}\right)^2$$
$$\times \langle \nabla^2 V^V(r) \rangle\, \sigma_1 \cdot \sigma_2 - \tfrac{1}{12} a^2 \left(1 + \tfrac{f}{g}\right)^2 \left(\tfrac{d^2 V^V}{dr^2} - \tfrac{1}{r} \tfrac{dV^V}{dr}\right) S_{12}$$
$$- \tfrac{3}{2} a^2 \left(1 + \tfrac{4}{3} \tfrac{f}{g}\right) \tfrac{1}{r} \tfrac{dV^V}{dr} \mathbf{l} \cdot \mathbf{S}. \tag{23}$$

Note that the potentials (8)–(10) in the static limit ($v/c \to 0$) reduce to very simple interactions:

$$V_S = V^S(r), \; V_V = -V^V(r); \; V_P = 0, \tag{24}$$

and the operator (16) can be represented as a sum of the velocity-independent static part (24) and the relativistic corrections (of order v^2/c^2); moreover, the structure of the operator (16) is such that for every type of meson for given g, f, m, and Λ all the relativistic corrections (spin–orbit and tensor forces included) are uniquely determined by the static limit of the corresponding OBE potential and do not require additional adjustable parameters when this operator is used to describe nucleon–nucleon scattering or if used in a many-body problem.

As the meson coordinate Q, in (1) we considered pseudoscalar, vector, and scalar fields corresponding to bosons with spin J equal to 0 and 1. Mesons have now also been discovered with $J \geqslant 2$ (axial-vector, tensor), although their masses are so large that the exchange of such mesons is unimportant for low-energy nucleon–nucleon interactions. Therefore, as a rule, one considers the so-called PVS models of the NN potential, which include pseudoscalar ($J^P = 0^-$), vector ($J^P = 1^-$), and scalar ($J^P = 0^+$) mesons and resonances.

So far, thirteen mesons and resonances with $J \leqslant 1$ have been discovered.[12] However, not all of them have uniquely determined quantum numbers. The basic set of mesons and resonances with $J \leqslant 1$ observed experimentally is given in Table I.

From the basic set, the OBEP models include exchange of π, η, ρ, ω, and Φ mesons (though some authors do not consider the exchange of Φ mesons but take into account the contribution of this meson indirectly through the effective ωN coupling constant).

All the OBEP models include exchange of scalar mesons, which is introduced to explain the attraction in the NN potential at intermediate distances. However, a number of authors[13–15] have given a more physical justification of this attraction, attributing it to two-pion exchange. In view of this circumstance, which is underlined by the fact that to achieve best fitting of the NN data it is, as a rule, necessary to introduce a scalar–isoscalar meson with mass around 500 MeV (and not 800–1000 MeV), the introduction of scalar isoscalar mesons in the OBEP models must be regarded as an artificial procedure that approximates two-pion forces; three-pion exchange is simulated in some models by a scalar isovector meson. We note, however, that in Ref. 9 NN scattering was fitted by means of the scalar mesons ε (782.8 MeV) and δ (963 MeV) with masses close to those of experimentally established scalar mesons; the most important achievement of Ref. 9 (see also Ref. 16) is the introduction of a large width of the ε meson, which makes it possible to dispense with the fictitious scalar meson with small mass.

TABLE II. Parameters of OBE potentials.

Meson	Type of potential	π	η	ρ	ω	σ_1	π_p, π_N, δ	σ_c	σ_0	ε	η_p, η_0^+
Coupling constant g^2; f/g dimensionless	a (Ref. 4)	14.7	—	—	25	—	—	—	—	14.7	—
	b (Ref. 7, model II)	14.37	—	0.81; 4.70	7.83	—	4.47	1.97	—	—	3.98
	c (Ref. 7, model III)	14.61	—	0.65; 5.06	9.68	1.01	—	1.52	7.32	—	—
	d (Ref. 9)	14.26	2.53	0.583; 5.18	10.00	—	1.39	—	—	13.9	—
Meson mass m, MeV	a (Ref. 4)	138.7	—	—	783	—	—	—	—	600	—
	b (Ref. 7, model II)	138.7	—	763	782.8	—	1016	416.1	—	—	1070
	c (Ref. 7, model III)	138.7	—	763	782.8	763	—	416.1	782.8	—	—
	d (Ref. 9)	138.7	548.7	763	782.8	—	963	—	—	782.8	—
Cutoff parameter Λ, MeV	a (Ref. 4)	600	—	—	1880	—	—	—	—	1880	—
	b (Ref. 7, model II)	2293	—	1200	1200	—	1200	1200	—	—	1200
	c (Ref. 7, model III)	1299	—	1299	1299	1299	—	1299	1299	—	—
	d (Ref. 9)	1879.1	1879.1	1879.1	1879.1	—	1879.1	—	—	416.1	—

The existence of repulsion at short internucleon distances, the so-called soft core, can now be regarded as an established fact. In the light of the new relativistic treatment of the NN interaction,[17,18] one-pion exchange also contributes to this repulsion.

Finally, the π mesons determine the long-range part of the NN potential; this component of the NN interaction (OPEP) has been very well studied and was included earlier as one of the components of the phenomenological models of the NN potential.

All the PVS models contain a combination of scalar and vector potentials which is such as to give partial compensation of the strong attractive static term produced by the scalar mesons and the strong repulsive static term produced by the vector mesons (the idea of this type of model was first put forward by Green[19]), so that the resulting (attractive) static potential is relatively weak. This combination also leads to a pronounced enhancement of the role played by relativistic effects in the nucleon–nucleon interaction (the relativistic corrections of order v^2/c^2 associated with the exchange of scalar and vector mesons are determined by the very strong static parts of the corresponding OBE potentials and, in contrast to the static parts, have the same signs); this manifestation of the relativistic effects may be characterized as "maximal relativization".

There are relativistic effects of a different category, which may be characterized by the name minimal relativization; these are due to the relativistic relation between the energy and momentum, the condition of Lorentz covariance, and so forth. Finally, the relativistic effects include allowance for retardation (the finite time of propagation of the exchange meson), which is evidently very important when these potentials are used in a Hartree–Fock scheme to solve the many-body problem; in particular, at the present time it is assumed that it is precisely the neglect of retardation in the potential resulting from the one-pion exchange in the form (13) that leads to the collapse of the nucleus when the relativistic self-consistency procedure is carried out (see below). Nevertheless, all models of OBE potentials in the coordinate representation so far proposed[5,7,20,21] have ignored retardation and, moreover, it has been assumed that it is impossible in principle to take into account the effects of retardation in the coordinate representation. However, it was shown in Ref. 22 that retardation may be taken into account in the coordinate representation, though the use of the momentum representation for this purpose has certain advantages[23] (in Ref. 23, an OBEP is constructed with allowance for retardation in the momentum representation). A number of OBE potentials have now been constructed in the momentum and the coordinate representation (in this connection, we also mention Refs. 24–26); the various PVS models of the OBE potential differ not only in the choice of the representation (coordinate or momentum) but also in the masses and coupling constants of various exchange mesons, the regularization procedure, and the ability to reproduce the experimental data and the properties of nuclear matter.

The one-boson exchange potentials contain fewer adjustable parameters (5–10) than the phenomenological potentials, and these have a clear physical meaning (the adjustable parameters are: the cutoff parameters, the coupling constants of the scalar, ρ, ω, and φ mesons, and the masses of the scalar mesons), and they describe a large body of experimental data; a list of the mesons and the parameters of some of the modern OBEP models is given in Table II (see Refs. 4, 7, and 9). In Table II we give as an example the parameters of the OBEP models developed by Green *et al*; the parameters of the other models can be found in Ref. 2. As a rule, the adjustable parameters in all the models are determined by the description of NN scattering over a wide energy range (0–450 MeV) and the properties of the deuteron (binding energy, quadrupole moment, D-state admixture, etc.[2]); the quality of the description[23] is not inferior to the description obtained with one of the best phenomenological potentials—Reid's potential with soft core[27] (and this potential is not defined for $J \geq 3$). Here, we shall not dwell in detail on the description of the properties of the NN system in one-boson exchange models (for a complete review see Ref. 2). Note that a convenient tool for solving this problem is the quasipotential approach developed by Logunov and Tavkhelidze,[28] which makes possible a unified treatment of the two-nucleon problem at low and high energies. This approach is characterized by

the fact that its equations are written in a form that makes it possible to include the relativistic kinematic effects in the two-nucleon equation and the NN potentials, i.e., to formulate the problem from the very start with complete allowance for relativity. We should also mention the variant of quasipotential theory of scattering developed by Kadyshevskiĭ *et al.*[29,30]; the basic ideas of the quasipotential approach are reviewed in Ref. 31. The fruitfulness of using the quasipotential approach in the problem of internucleon forces was noted in Ref. 32 (see also Ref. 33). Partovi and Lomon[14] used such an approach[28,34] to calculate the NN potential with allowance for two-pion exchange. In Ref. 35 (see also Ref. 36) the Kadyshevskiĭ–Matveev equation[29,30] was used in the framework of the one-boson exchange scheme to describe nucleon–nucleon scattering (see also Refs. 24 and 37).

Note that the OBE potentials admit a direct transition to the $N\bar{N}$ (nucleon–antinucleon) channel, the connection between the NN and the $N\bar{N}$ potential being determined by the G parity (the G parity of those components of the potential that are associated with the exchange of several mesons is defined as the product of the G parities of the individual mesons). It can be shown that a potential associated with the exchange of vector isoscalar mesons changes sign on the transition to the $N\bar{N}$ channel, while a potential associated with the exchange of scalar–isoscalar mesons does not, which leads to a strong increase in the interaction potential energy in the $N\bar{N}$ system, from which we can expect the spectrum of bound $N\bar{N}$ states to be much richer than the spectrum in the NN system. The spectrum of states in the $N\bar{N}$ system with small mass defect was studied in Refs. 38 and 39 with the OBE potentials of Bryan and Phillips.[40] The calculations revealed the presence in this system of a large number of resonances and bound states with nonvanishing orbital angular momentum, the resonances being characterized by comparatively large values of the ratio $\Gamma_{N\bar{N}}/\Gamma$ ($\Gamma_{N\bar{N}}$ is the width of decay of the state through the elastic channel and Γ is the annihilation width). In Ref. 41, an attempt was made (for different variants of the OBE potential) to obtain as bound $N\bar{N}$ states the mesons that transmit the interaction in the NN channel (the Fermi–Yang model). These questions are discussed in detail in the review Ref. 42.

In many models of OBE potentials, the coupling constants of the vector mesons (this applies basically to the ω meson) are found to be larger than their experimental values. This defect is not present in the model proposed in Ref. 43 (see also Ref. 44), in which the actual values of all the meson–nucleon coupling constants are obtained. The aim set in Refs. 43 and 44 is to construct an OBEP that simultaneously describes the low energy nucleon–nucleon (NN) and hyperon-nucleon scattering. In Ref. 43, such a model is constructed in conformity with $SU(3)$ symmetry, which makes possible a transition to the YN channel. In this model, the ε and ρ mesons are regarded as mesons that have a width; the potential associated with the exchange of this type of meson can be represented as a sum of the potentials of the two effective narrow mesons with different masses, the meson with smaller mass (≈ 510 MeV) in this two-pole approximation corresponding to the traditional fictitious σ meson. The interaction at very short distances ($r \lesssim 0.5$ F) is described in Ref. 43 phenomenologically by means of hard repulsive cores whose radii are adjustable parameters. In Ref. 43, the phase shifts are described very well, though the difficulties remain with regard to obtaining quantitative agreement in the fitting of the $^3D_2$ phase shifts (which are inherent in all meson potentials); in Ref. 43, a good description is also obtained for the scattering lengths, the effective radii, and the deuteron parameters (the results on the fitting of YN scattering have not yet been published).

In Refs. 45 and 46, the OBE potentials are used to describe a three-nucleon system and, in particular, to calculate the triton binding energies (E_T); in Ref. 47, the contribution made to the triton binding energy by three-particle forces based on meson theory (see below) was investigated.

2. USE OF OBE POTENTIALS TO CALCULATE THE POTENTIALS OF NUCLEAR MATTER

Besides problems relating to the scattering of free nucleons, the deuteron, and the triton, calculations of the properties of infinite nuclear matter give considerable information (in addition to that given by the two-nucleon problem) on many-nucleon forces since one may say that there exists a theory of nuclear matter in which, starting from the interaction of free nucleons, one can approximately calculate the density and binding energy per nucleon of the nuclear matter, making comparatively small errors which can be estimated. This comparatively favorable situation in the theory of nuclear matter arose after it had been established that the Brueckner–Goldstone expansions converge with respect to the number of independent hole states[2)] that participate in the process.[48] In other words, it was found that to determine the properties of nuclear matter it is sufficient to consider only the diagrams of first order in the effective interaction between the nucleons that is calculated in theory. This follows from the fact that processes of higher order contain not less than three independent hole states and, therefore, make a comparatively small contribution (processes of second order in the effective interaction are absent in the Brueckner–Goldstone theory).

Note that the criterion of convergence with respect to the number of independent hole states coincides with the criterion of convergence with respect to the powers of the density of the nuclear matter. The point is that n independent holes can be formed as a result of the interaction of not less than n particles, which is possible if particles approach one another simultaneously to a sufficiently short distance. As measure of this

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[2)]By independent hole states we understand states whose momentum may be taken arbitrarily within the Fermi sphere without violating the law of conservation of momentum as a whole for the process.

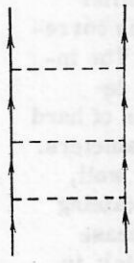

FIG. 1. Ladder diagram.

distance one should take, not the range of nuclear forces, but rather the radius of the repulsive core (which is soft in the case of OBE potentials), since it is essentially the large core interaction that leads to the strong correlation effects in nuclear matter. But the probability for n particles to be in a region measured by a length of the order of the core radius is proportional to ρ^n, where ρ is the density of the nuclear matter.

Let us discuss briefly the nature of the effective interaction. The physical meaning of its introduction into theory consists of allowing for the so-called ladder diagrams corresponding to scattering of a pair of particles above the Fermi limit,[3)] these interacting through the original two-nucleon potential. Figure 1 shows a ladder diagram, which may be a composite part of different diagrams. Sequences of diagrams of two types containing ladder diagrams are shown in Figs. 2 and 3. Diagrams that are a sum of sequences of the diagrams shown in Figs. 2 and 3, respectively, are shown in Fig. 4. These two diagrams are the only diagrams of first order in the effective interaction. It is easy to see that they are none other than the Hartree–Fock direct and exchange diagrams in which the effective nucleon–nucleon interaction is taken as interaction.

We write the energy of the nuclear matter in the form

$$E=\sum_{m<k_F}\left\{\langle m|\hat{T}|m\rangle+\frac{1}{2}\sum_{n<k_F}[\langle mn|G|mn\rangle-\langle mn|G|nm\rangle]\right\}. \qquad (25)$$

where m and n are states of the nucleon (its momentum and spin and isospin projections), the summations are restricted to states within the Fermi surface, and the Fermi momentum k_F is related to the density ρ of the nuclear matter by $\rho = 2k_F^3/3\pi^2$. In (25), G is the effective nucleon–nucleon interaction, its matrix elements satisfying the Bethe–Goldstone equation

$$\langle \mathbf{q}'|G(\mathbf{P})|\mathbf{q}\rangle=\langle \mathbf{q}'|V|\mathbf{q}\rangle - \int \frac{Q(\mathbf{k},\mathbf{P})\langle \mathbf{q}'|V|\mathbf{k}\rangle\langle \mathbf{k}|G(\mathbf{P})|\mathbf{q}\rangle}{E(\mathbf{P}+\mathbf{k})+E(\mathbf{P}-\mathbf{k})-W(\mathbf{q},\mathbf{P})}d^3k. \qquad (26)$$

[3)] By scattering in nuclear matter we shall understand any change in the state of the particles in which there is no phase shift when the particles move apart to an infinite distance between them. Because of the Pauli exclusion principle, a phase shift resulting from an interaction is impossible in nuclear matter.[48]

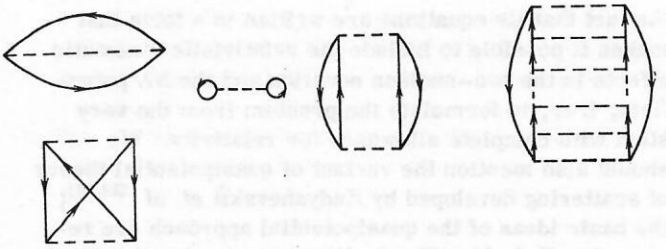

FIG. 2. Two-hole nonexchange diagrams. A continuous line stands for particles if the arrow points upward and holes if it points downward; the dashed line represents interaction through the original two-nucleon forces.

Here, V is the original internucleon potential; $\mathbf{q}$ is the relative momentum of the interacting nucleons; $\mathbf{P}=(\mathbf{k}_1+\mathbf{k}_2)/2$ is their mean momentum; $E(\mathbf{P}\pm\mathbf{k})$ are the energies of the two nucleons of a considered pair with momenta outside the Fermi sphere:

$$E(\mathbf{k}_m)=k_m^2/2M; \qquad (27)$$

$Q(\mathbf{k},\mathbf{P})$, the Pauli operator, is equal to 1 if both the momenta $\mathbf{k}+\mathbf{P}$ and $\mathbf{k}-\mathbf{P}$ lie outside the Fermi sphere and 0 otherwise; $W(\mathbf{q},\mathbf{P})$ is the so-called starting energy. For the processes shown in Figs. 2 and 3,

$$W(\mathbf{q},\mathbf{P})=\tilde{E}(\mathbf{P}+\mathbf{q})+\tilde{E}(\mathbf{P}-\mathbf{q}), \qquad (28)$$

where $\tilde{E}(\mathbf{P}\pm\mathbf{q})$ are the Hartree–Fock energies of the nucleons below the Fermi limit:

$$\tilde{E}(\mathbf{k}_m)=\frac{k_m^2}{2M}+\sum_{n<k_F}[\langle mn|G(\mathbf{P})|mn\rangle-\langle mn|G(\mathbf{P})|nm\rangle]. \qquad (29)$$

As can be seen from (26), the effective interaction depends on the mean momentum $\mathbf{P}$ of the pair of nucleons and, therefore, is not Galileo invariant, in contrast to the original nucleon interaction in the nonrelativistic limit ($v/c \to 0$). This is explained by the presence of the medium, which distinguishes the system at rest with respect to the medium; in this system, the total momentum of all the particles of the medium is zero. The dependence on the total momentum of the

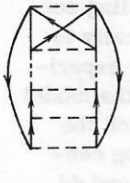

FIG. 3. Two-hole exchange diagrams. The notation is the same as in Fig. 2.

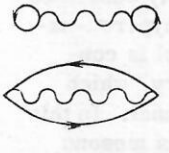

FIG. 4. Direct and exchange diagrams of first order in the effective interaction. The wavy line is the interaction through the effective nucleon–nucleon forces, which include the corresponding ladder processes.

particles has the consequence that the effective interaction also depends on the angle between the relative momentum and the vector **P**. This means there is violation of isotropy in the space of the relative momentum, resulting in nonconservation of the relative angular momentum of a pair of nucleons interacting with effective forces. However, the dependence of G on **P** is comparatively weak, so that this violation of symmetry is slight and is usually ignored (see below).

Note that the relations (26), (28), and (29) form a system of coupled equations—in accordance with (29), the single-particle energies $\tilde{E}(\mathbf{k}_m)$ depend on the effective interaction G, which, in its turn, depends on the single-particle energies through the starting energy (28). Therefore, to find the effective interaction, it is necessary to solve a self-consistency problem. Note that in this case there is no need to find the single-particle wave functions since they can only, because of the symmetry of the problem (translational invariance), be plane waves. Nevertheless, a self-consistency problem arises because the interaction is not taken to be the fixed vacuum internucleon interaction but rather an effective interaction, which itself depends on the state of the nucleons.

The Bethe–Goldstone equation is usually solved approximately under the two following simplifying assumptions:

1) the Pauli operator $Q(\mathbf{k}, \mathbf{P})$ is averaged over the angle between the vectors **k** and **P**:

$$\bar{Q}(\mathbf{k}, \mathbf{P}) = \frac{1}{2}\int_0^{\pi} Q(\mathbf{k}, \mathbf{P}) \sin\theta \, d\theta,$$

where θ is the angle between **k** and **P**. Bearing in mind that only angles θ for which

$$|\cos\theta| < (P^2 + k^2 - k_F^2)/(2kP)$$

contribute to the integral, we obtain

$$\bar{Q}(\mathbf{k}, \mathbf{P}) = \begin{cases} 0, & \text{if} \quad P^2 + k^2 \leqslant k_F^2; \\ 1, & \text{if} \quad k - P \geqslant k_F; \\ (P^2 + k^2 - k_F^2)/2kP & \text{otherwise,} \end{cases} \tag{30}$$

where we have also taken into account the fact that the mean momentum satisfies $P \leqslant k_F$ because of the conservation of the momentum of the nucleon pair in the homogeneous nuclear matter;

2) the Hartree–Fock energies are taken in the form

$$\tilde{E}(\mathbf{k}_m) = k_m^2/2M^* + A, \tag{31}$$

where M^* and A are adjustable parameters and M^* is an effective mass. The values of M^* and A are determined by the self-consistency condition: In accordance with the initial approximation for M^* and A the value of G is calculated, and then, using (29), the values of $\tilde{E}(\mathbf{k}_m)$ $(k_m < k_F)$ and the new values of M^* and A that give the best approximation of $\tilde{E}(\mathbf{k}_m)$ are found; such iterations are repeated until the process converges.

Both approximations lead to only slight errors in the calculation of the energy of nuclear matter: In the case of the first approximation, this is pointed out in Ref. 2; the second approximation is discussed in Ref. 49.

Under the adopted approximations, the Bethe–Goldstone equation (26) can be solved in the partial-wave representation. In this representation, one can write the matrices of the interaction V and the effective interaction G in the form (see, for example, Ref. 50)

$$\langle \mathbf{q}' | G(\mathbf{P}) | \mathbf{q} \rangle = \frac{2}{\pi} \sum_{\alpha l l' M} i^{l-l'} G^{\alpha}_{ll'}(P, q', q)\, \mathcal{Y}^{JM}_{lS}(\hat{q}')\, \mathcal{Y}^{JM+}_{l'S}(\hat{q}); \tag{32}$$

$$\langle \mathbf{q}' | V | \mathbf{q} \rangle = \frac{2}{\pi} \sum_{\alpha l l' M} i^{l-l'} V^{\alpha}_{ll'}(q', q)\, \mathcal{Y}^{JM}_{lS}(\hat{q}')\, \mathcal{Y}^{JM+}_{l'S}(\hat{q}), \tag{33}$$

where α denotes the set of quantum numbers: J, S, T, are the angular momentum, the spin angular momentum, and the isospin of the pair of nucleons in the given partial wave; $\mathcal{Y}^{JN}_{lS}(\hat{q})$ are spherical spinors:

$$\mathcal{Y}^{JM}_{lS}(\hat{q}) = \sum_{m_l m_S} \langle l S m_l m_S | JM \rangle Y_{lm_l}(\hat{q}) | Sm_S \rangle. \tag{34}$$

Substituting the expressions (32) and (33) into Eqs. (26), we obtain the following integral equation for the matrices of the effective interaction in the partial-wave representation:

$$G^{\alpha}_{ll'}(P, q', q) = V^{\alpha}_{ll'}(q', q) - \frac{2}{\pi} \sum_{l''} \int_0^{\infty} \frac{V^{\alpha}_{ll''}(q', k)\, \bar{Q}(k, P)\, G^{\alpha}_{l''l'}(P, k, q)}{P^2/2M + k^2/2M - W(q, P)}\, k^2\, dk, \tag{35}$$

where we have remembered that, in accordance with (28) and in the approximation (31), the starting energy $W(q, P)$ does not depend on the angle between the vectors **k** and **P**.

Let us consider the results of numerical calculations based on Eq. (35), using the OBE potentials given in Ref. 49. In Ref. 49, Eq. (35) was simplified somewhat more by replacing in it the mean momentum **P**, which can vary in the interval $[0, k_F]$, by a certain average value P_{av}. This does not strongly distort the solution of the equation since the relative momentum **k** of the pair of nucleons appreciably exceeds on the average the limiting momentum k_F, and therefore, P as well. This last is due to the fact that the short-range repulsive core knocks nucleon pairs into states high above the Fermi sphere (the mean value of the relative momentum is $\approx 1/$(core radius), which corresponds to an energy $\gtrsim 300$ MeV, which is appreciably higher than the Fermi energy ≈ 50 MeV).

With allowance for this remark, Eq. (35) can be solved by matrix inversion, which entails replacing the integral by a sum over a finite number of momenta k; this transforms the integral equation into a system of linear algebraic equations for $G^{\alpha}_{ll'}(P, q', q)$, where q' belongs to the set of momentum values over which the summation is made.

The empirical value of the density of nuclear matter is taken equal to the nucleon density in the center of heavy nuclei. The binding energy per nucleon is taken from the Weizsäcker formula extrapolated to $A = 2N = 2Z \to \infty$ with neglect of the Coulomb interaction between the protons.

TABLE III. Binding energy of nuclear matter as function of the Fermi momentum k_F.

k_F, F^{-1}	1.20	1.36	1.50	1.55	1.60	1.70	1.80
Potential energy, MeV	−27.63	−34.51	−40.33	−42.28	−44.15	−47.57	−50.38
Kinetic energy, MeV	17.92	23.01	27.99	29.89	31.85	35.95	40.31
Binding energy, MeV	−9.72	−11.50	−12.33	−12.40	−12.30	−11.61	−10.07

Results of calculation of the binding energy of nuclear matter per nucleon in accordance with the above scheme for the OBE potential in the momentum representation introduced in Ref. 23 are given in Table III. These calculations were made in Ref. 49. Saturation (minimal energy) corresponds to Fermi momentum $k_F \approx 1.55$ F^{-1} and binding energy per nucleon 12.4 MeV. The empirical values of these quantitites are 1.4 F^{-1} and 15.7 MeV, respectively. Although the insufficient binding energy can be increased by taking into account three-, four-, ... particle diagrams, the too large value of the density is only worsened when n-particle diagrams ($n \geq 3$) are taken into account. This is due to the fact that when the density is increased the relative importance of many-particle correlations increases since their contribution contains a higher power of the density than the contribution from the main two-particle diagrams.

The potential used in the Bethe–Goldstone equation in the calculations just described has the following form in the momentum representation:

$$V(\mathbf{p}, \mathbf{q}) = \sqrt{M/E_p}\, V_{OBE}(\mathbf{p}, \mathbf{q}) \sqrt{M/E_q}, \tag{36}$$

where $V_{\text{OBE}}(\mathbf{p},\mathbf{q})$ is the one-boson exchange potential; the factors $(M/E_{p,q})^{1/2}$ are the "minimal relativization" correction. Let us discuss the meaning of this correction, following Ref. 51. The scattering amplitude $\tilde{R}(\mathbf{k},\mathbf{k}')$ of free relativistic nucleons satisfies the equation[28,34]

$$\tilde{R}(\mathbf{k}, \mathbf{k}') = V_{OBE}(\mathbf{k}, \mathbf{k}') - \frac{2M^2}{(2\pi)^3} \int \frac{V_{OBE}(\mathbf{k}, \mathbf{q})\, \tilde{R}(\mathbf{q}, \mathbf{k}')}{(q^2 - k'^2 + i\varepsilon)(M^2 + q^2)^{1/2}} d^3q. \tag{37}$$

The nonrelativistic Lippmann–Schwinger equation for the scattering amplitude has the form

$$R(\mathbf{k}, \mathbf{k}') = V_{OBE}(\mathbf{k}, \mathbf{k}') - \frac{2M}{(2\pi)^3} \int \frac{V_{OBE}(\mathbf{k}, \mathbf{q})\, R(\mathbf{q}, \mathbf{k}')}{q^2 - k'^2 + i\varepsilon} d^3q. \tag{38}$$

The amplitude $R(\mathbf{k},\mathbf{k}')$ obtained from Eq. (38) does not satisfy the relativistic unitarity condition, which entails violation of the general relativistic relation between the forward scattering amplitude and the total scattering cross section. However, if the potential (36) is used in Eq. (38) instead of V_{OBE}, and one takes the relativistic amplitude to be

$$\tilde{R}(\mathbf{p}, \mathbf{q}) = \sqrt{M/E_p}\, R(\mathbf{p}, \mathbf{q}) \sqrt{M/E_q},$$

then Eq. (37) for $\tilde{R}(\mathbf{p},\mathbf{q})$ will be satisfied and relativistic unitarity established. Therefore, we may assume that if we use the nonrelativistic equation then we must use in it not the OBE potential itself but a potential of the form (36). This has just been demonstrated for the Lippmann–Schwinger equation describing two free particles. This conclusion can also be extrapolated to the Bethe–Goldstone equation (26), which describes a pair of particles in the nuclear medium.

Note that besides taking into account relativity by changing the potential in accordance with (36), there is a further way which consists of replacing the nonrelativistic kinetic energy operator (27) in the denominator of Eq. (26) by the relativistic form

$$E(\mathbf{k}_m) = (M^2 + k_m^2)^{1/2} - M.$$

This way of taking into account relativity was proposed by Lee and Tabakin.[50]

In Ref. 51, the influence of the correction for minimal relativization in accordance with (36) was clarified by means of numerical calculations, and it was found that the binding energy when this correction is ignored changes by 0.5 MeV, i.e., not very strongly. The reason for this weak influence of minimal relativization need not, however, be due to its unimportance but rather to the fact that the parameters of the potential used in Ref. 51 with correction for minimal relativization were adjusted without it independently from the data for the NN system and, therefore in one case the correction for minimal relativization was effectively taken into account in the choice of the parameters of the potential.

The following qualitative argument indicates that relativistic corrections and, in particular, those for minimal relativization are important: the nucleon–nucleon correlations due to the core lead to states with relative momentum corresponding to energies 300 MeV, at which relativistic effects appear perfectly naturally.

Calogero and Levi[10] put forward the following argument for taking into account relativity when relativistic equations are used, not in their original form, but in reduced form. They calculated the binding energy of nuclear matter in the nonrelativistic single-particle approximation for an OBE potential in the coordinate representation with the parameters chosen by Miller and Green[52] in such a way as to obtain the correct binding energy of doubly magic nuclei. Miller and Green also worked in the single-particle approximation, but they used the relativistic Dirac equation. The binding energy obtained by Calogero and Levi was very high (−38.9 MeV per nucleon). This difference between the binding energies can hardly be attributed to the finiteness of the nuclei or exchange effects ignored by Miller and Green. The difference was more probably due to insufficiently complete allowance for the relativistic nature of the nucleon motion in the calculations of Calogero and Levi.

However, the strength of this conclusion is much

TABLE IV. Binding energy of nuclear matter for different OBE potentials obtained by various authors.

OBE potential	Holinde and Machleidt (Ref. 11)	Schierholz (Ref. 26)	Gersten, Thompson, and Green (Ref. 24)	Ueda, Nack and Green (Ref. 53)
Binding energy, MeV	−11.83	−17.84	−18.82	−19.23
Saturation binding energy, MeV	−11.9	−21.4	−22.2	−23.3
Saturation Fermi momentum, F^{-1}	1.48	1.88	1.83	1.87

weakened by the fact that in both cases the calculations were made in the single-particle approximation.

Calculations of the properties of nuclear matter with a number of different OBE potentials were made by Holinde and Machleidt.[11] In Table IV, we give some of the results they obtained for the binding energy of nuclear matter. The calculations for the binding energy were made with $k_F = 1.5\ F^{-1}$.

It may be concluded from the above data that the Holinde–Machleidt potential gives the best results if one bears in mind that the corrections to the Brueckner–Bethe–Goldstone theory associated with three-and-more-particle diagrams increase the binding energy and the density (see Ref. 3), and both these quantities in any case exceed the experimental values for all potentials except the first.

Note that in recent years, as a consequence of the ever greater improvement in the calculations of the binding energy of nuclear matter, interest has arisen in meson theory in the problem of three-particle forces due to the exchange of two pions (interactions due to the exchange of two heavier mesons are negligibly small). The three-particle interaction may take place either by direct exchange or by scattering of a π meson on a third nucleon through an intermediate state N^* [through the well known resonance $\Delta(1236)$]. This problem is discussed in the reviews Refs. 3 and 32. The corresponding literature quoted in them should be augumented by the three later papers Refs. 54–56. According to Ref. 57, the three-particle forces make a contribution to the nuclear-matter binding energy equal to 1.7 MeV of attraction energy per nucleon, but this cannot be regarded as final[54–56]; the sign of this contribution also speaks in favor of the potential of Ref. 49. Note that the OBE potential introduced in Ref. 11 gives the best quantitative description of NN scattering and the deuteron properties.

In Ref. 10, calculations were made of nuclear-matter properties in the Fermi gas approximation. This means that the wave function of the nucleons was taken in the form of a Slater determinant. It is clear that in the Fermi-gas approximation one must obtain a lower binding energy since the correlations taken into account in Brueckner–Bethe–Goldstone theory are largely due to the damping of the two-particle wave function when nucleons approach to distances of the order of the core radius and less, whereas in the Fermi-gas model such damping in the region of the core is absent, which leads to an additional positive contribution to the energy of nuclear matter. Indeed, in Ref. 10 unbound nuclear matter was obtained for all OBE potentials with parameters obtained from data for two free nucleons. This is an indication that the Fermi-gas model is unsatisfactory when applied to nuclear matter.

To conclude this section, we note that in Ref. 36 an extension was proposed to the standard Brueckner theory with OBE forces by the inclusion of meson degrees of freedom, i.e., in which the Hilbert space contains states of nucleons and mesons. In this approach, the basis is a field Hamiltonian of the type of Chew and Low, which is considered in the framework of noncovariant perturbation theory (ignoring antinucleon states). At first, the two-nucleon problem was considered, the solution of this leading to the Kadyshevskiĭ–Matveev equation,[29,30] and then a theory of nuclear matter was developed that goes over in the adiabatic approximation (see, for example, Ref. 36) into the traditional Brueckner theory. Numerical calculations of nuclear matter in the framework of this formalism are to be published.[4)]

3. RELATIVISTIC SELF-CONSISTENT NUCLEAR THEORY

We shall use the relativistic potentials introduced in Refs. 4, 7, and 9 and in Refs. 23 and 25. We shall present two approaches to the many-body problem using OBE potentials.

We begin our discussion of the first approach by discussing Refs. 52 and 58–60. In Ref. 52, to construct a meson theory of nuclear structure, use is made of a relativistic self-consistent theory based on OBE potentials in the form (8)–(13) (see, for example, Ref. 4).

Consider a system of A nucleons interacting with one another by exchanging scalar, vector, and pseudoscalar mesons (both isoscalar and isovector). The Hamiltonian of this system has the form

$$H=\Big[\sum_{i=1}^{A}(c\boldsymbol{\alpha}_i\cdot\mathbf{p}_i+\beta_i Mc^2)+\frac{1}{2}\sum_{i,k=1}^{A}\Big\{-\beta_i\beta_k\Big(\sum_S V_{ik}^S+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k\sum_{S\tau}V_{ik}^{S\tau}\Big)+(1-\boldsymbol{\alpha}_i\cdot\boldsymbol{\alpha}_k)\Big(\sum_V V_{ik}^V+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k\sum_{V\tau}V_{ik}^{V\tau}\Big)+\beta_i\gamma_{5,i}\beta_k\gamma_{5,k}\Big(\sum_P V_{ik}^P+\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_k\sum_{P\tau}V_{ik}^{P\tau}\Big)\Big\}\Big], \tag{39}$$

where $\sum_S$, $\sum_V$, and $\sum_P$ are summations over the isoscalar, scalar, vector, and pseudovector mesons, and $\sum_{S\tau}$, $\sum_{V\tau}$, and $\sum_{P\tau}$ are the summations over the isovector mesons with the same spacetime transformation properties. In Ref. 52, the "dipole" regularization of the functions V_{ik} is used.

To obtain the single-particle self-consistent relativistic equations from (39), a variational procedure was used in Ref. 52, this leading to a local Dirac equation of the most general form:

[4)] These results have been published in: K. Kotthoff, R. Machleidt, and D. Schutte, Nucl. Phys. A **264**, 484 (1976).

TABLE V. Calculations of total binding energy and rms radii of doubly magic nuclei.

Nucleus	Binding energy/A, MeV						Rms radius, F					
	Ref. 61	Ref.62	Ref.63	Ref.64	Ref.52	Exp.	Ref. 63	Ref. 61	Ref.62	Ref.64	Ref.52	Exp.
^{16}O	6.76	8.10	6.75	7.73	7.35	7.98	2.71	2.59	2.84	2.76	2.60	2.73
^{40}Ca	4.99	8.53	7.49	8.32	8.25	8.55	3.41	3.19	3.67	3.45	3.49	3.49
^{48}Ca	4.00	8.70	7.48	7.87	8.55	8.67	3.45	3.25	3.75	3.52	3.49	3.48
^{90}Zr	—	8.59	7.85	8.00	8.87	8.71	4.18	—	4.58	4.23	4.30	4.30
^{208}Pb	2.49	7.90	7.53	7.31	8.05	7.87	5.37	4.87	5.94	5.44	5.54	5.50

$$\{c\alpha\cdot\mathbf{p}+\beta[Mc^2+U_S(\mathbf{r})+\gamma^\mu U_V^\mu(\mathbf{r}) + \gamma^5\gamma^\mu U_A^\mu(\mathbf{r})+\gamma^5 U_P(\mathbf{r})+\gamma^\mu\gamma^\nu U_T^{\mu\nu}(\mathbf{r})]\}\Phi = E'\Phi,\quad E'=E+Mc^2, \tag{40}$$

where $U_S(\mathbf{r})$, $U_V^\mu(\mathbf{r})$, $U_A^\mu(\mathbf{r})$, $U_P(\mathbf{r})$, $U_T^{\mu\nu}(\mathbf{r})$ are, respectively, the scalar, vector, pseudovector, pseudoscalar, and tensor relativistic one-particle potentials. Although all these components are local, they depend on the state (their form is also determined by the functions $V(ik)$). In Refs. 52 and 59, Eq. (40) is applied to nuclei with filled shells. In this case, the single-nucleon states are

$$\Phi(r)=\begin{pmatrix} if_\varkappa(r)\,\mathcal{Y}_{lS}^{jm} \\ g_\varkappa(r)\,\frac{(\sigma\cdot\mathbf{r})}{r}\,\mathcal{Y}_{lS}^{jm}\end{pmatrix}, \tag{41}$$

where $f_\varkappa(r)$ and $g_\varkappa(r)$ are the large and the small component of the radial wave function, respectively; $\mathcal{Y}_{lS}^{jm}$ is a spherical spinor; $\varkappa=\pm(j+1/2)$ for $j=(l\mp1/2)$ are eigenstates of the operators of the angular momentum J and the parity P, and therefore the following commutation relations must be satisfied:

$$[U(\mathbf{r}),\ \mathbf{J}]=[U(\mathbf{r}),\ P]=0, \tag{42}$$

where $U(\mathbf{r})$ is the total potential of Eq. (40), and

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\frac{\hbar}{2}\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix},\quad P=\beta P_0$$

(P_0 is the nonrelativistic parity operator).

The requirement (42) considerably simplifies the Dirac equation (40), reducing it to the form

$$\{c\alpha\cdot\mathbf{p}+\beta[Mc^2+U(r)]\}\Phi=E'\Phi \tag{43}$$

with the potential

$$U(r)=[U_S(r)+\gamma^0U_V^0(r)-\gamma^rU_V^r(r)-\gamma^0\gamma^rU_T^r(r)], \tag{44}$$

in which none of the components depends on the angles; pseudoscalar and pseudovector potentials are absent; only the radial components of the two three-vectors $U_V^j(r)$ and $U_T^{0j}(r)$ ($j=1,\ 2,\ 3$), denoted by $U_V^r(r)$ and $U_T^r(r)$, are nonzero. The operators $U_V^r(r)$ and $U_T^r(r)$ appear because of allowance for Fock exchange effects.

In the numerical calculations in Ref. 52, the exchange effects were ignored; then for nuclei with closed shells, only the scalar interaction and the zeroth component of the vector interaction contribute to the direct potential (we emphasize that in this case π mesons do not contribute to the one-particle self-consistent relativistic potential).

In Ref. 52, calculations were made of the total binding energies and the rms radii of doubly magic nuclei. The results of the calculations made in Ref. 52 are compared in Table V with the results of Hartree–Fock phenomenological calculations by Davies and McCarthy,[61] Köhler,[62] Negele,[63] and Németh and Vautherin.[64]

As can be seen in Table V, the best rms radii were obtained by Miller and Green, whereas the calculations of Köhler give the best total binding energy. In Ref. 52, Miller and Green also calculated the energy eigenvalues of various single-particle states of doubly magic nuclei, and these agree reasonably with the experimental separation energies even for deep states. This is one further advantage of the relativistic approach compared with nonrelativistic phenomenological calculations, which require the inclusion of forces that depend on the density or phenomenological three-particle forces.

The success of the relativistic Hartree theory in reproducing the saturation properties and the single-particle levels of nuclei with closed shells[58] is largely due to the fact that the mathematical expectation of the relativistic kinetic energy operator $\hat{T}=c\alpha\cdot\mathbf{p}+(\beta-1)Mc^2$ differs appreciably from its nonrelativistic analog. Indeed, calculation of the mathematical expectation of the operator $\hat{T}$ with relativistic wave functions (41) leads to the expression[58]

$$\langle\Phi_k|\hat{T}|\Phi_k\rangle=\int_0^\infty[E_k-U_S(r)-U_V^0(r)]\,f_\varkappa^2(r)\,r^2\,dr + \int_0^\infty[U_S(r)-U_V^0(r)+E_k]\,g_\varkappa^2(r)\,r^2\,dr.$$

The second term gives an important relativistic effect, this being due to the mutual enhancement of the very deep potentials $U_S(r)$ and $U_V^0(r)$ in this term (whereas in the first term their combination is weakened to a considerable degree). The contribution of the second term is between −5 and −10 MeV. It is this renormalization of the kinetic energy that is responsible for the success of the relativistic approach mentioned above.[52,59]

In Ref. 52, Miller and Green used a four-parameter model of OBE potentials including the mesons given in Table VI.

The equality of the parameters of the ω and ε mesons serves for complete compensation of the static parts of the vector and scalar potentials. In Ref. 52, attraction between the nucleons is ensured by a scalar σ meson with mass equal to twice the π meson mass. The vector–isovector ρ meson is responsible for producing the symmetry energy, although terms corresponding to

TABLE VI. Mesons of the OBE potential of Miller and Green[52] and its parameters

Meson	IJ^P	Mass, MeV	g^2
ω	01^-	782.8	25.0
ε	00^+	782.8	25.0
σ	00^+	277.4	0.675
ρ	11^-	763.0	2.5
		$\Lambda = 2M$	

derivative coupling [see (5)] were not considered in Ref. 52; the dipole regularization is used and the adjustable parameters are the coupling constants of the ω, σ, and ρ mesons and the regularization parameter Λ, which is equal to twice the nucleon mass.

Note that the OBE potential model used in Ref. 52 to calculate nuclear properties did not use nucleon–nucleon scattering for fitting purposes, although it has all the main properties of "realistic" OBE potentials. The difference between the effective interaction[52] and the free OBE potentials[4,7,9] appears in the slight difference of the adjustable parameters and is partly due to the difference between the schemes for describing *NN* scattering and nuclear structure: The parameters of the realistic OBE potentials[4,7,9] were obtained by fitting *NN* scattering and the deuteron properties in the framework of a Schrödinger equation with allowance for relativistic corrections of order v^2/c^2 (the Breit–Pauli equation),[4] whereas the parameters of the effective OBE potentials were obtained in Ref. 52 by solving relativistic Hartree Dirac equations with OBE potentials expressed in the form (8)–(13). The remaining difference can be ascribed to correlations. Correlation effects are very important for nonrelativistic local *NN* potentials (with hard cores responsible for the greater part of the correlation effects of nonrelativistic models).

Before we turn to a discussion of the role of correlations in the considered relativistic self-consistent theory, let us point out that the realization in Ref. 52 of a relativistic formalism with one-boson potential expressed in the form (8)–(13) leads to the absence of a complicated dependence of the effective potentials (present in the effective interactions of the nonrelativistic series[62]) on the relative two-particle states and the local density. This circumstance is due to the fact that the Dirac γ matrices labeling the species of exchange meson take onto themselves the dependence on the states.

The relativistic effects of the OBEP models make it unnecessary to introduce a hard core. The possibility of exact fitting of the *S*-wave phase shifts without the introduction of hard cores in the models of one-boson exchange is a direct consequence of the presence of velocity-dependent components. Further, the weakening of correlation effects when the relativistic self-consistent theory is used is due, as is asserted in Ref. 58, to one of the manifestations of the "maximal relativization" effects—one we have already considered above: renormalization of the mathematical expectation of the single-particle relativistic kinetic-energy operator. The fact that the correlation effects when OBEP models are used are less important than in nonrelativistic phenomenological calculations was established in calculations of inifinite nuclear matter in Refs. 65 and 66, in which a strong reduction of the "wound integral" (see, for example, Ref. 48) was obtained for one-boson exchange models.

However, correlation effects are also important in the approach under consideration; this was shown in Sec. 2. Correlation effects in the nucleus were taken into account in Ref. 52 indirectly by changing the parameters of the OBE potential from their values in the free OBE potentials, and also by the particular choice of the OBE potential model itself, which, as we have already mentioned, retains all the main properties of realistic OBE models.

In Ref. 59, a relativistic self-consistent formalism taking into account the exchange (Fock) interaction was developed. In this formalism, one starts with the relativistic Hartree–Fock equations for the set of occupied single-nucleon states $\{\Phi_i\}$ in the nucleus:

$$(c\boldsymbol{\alpha}\cdot\mathbf{p}+\beta Mc^2)\,\Phi_i(\mathbf{r}_1)+\sum_{j=1}^{A}\int \Phi_j^\dagger(\mathbf{r}_2)\,V(|\mathbf{r}_1-\mathbf{r}_2|)\,\Phi_j \times(\mathbf{r}_2)\,d^3r_2\Phi_i(\mathbf{r}_1)-\sum_{j=1}^{A}\int \Phi_j^\dagger(\mathbf{r}_2)\,V(|\mathbf{r}_1-\mathbf{r}_2|)\,\Phi_i \times(\mathbf{r}_2)\,d^3r_2\Phi_j(\mathbf{r}_1)=E_i^i\Phi_i(\mathbf{r}_1),$$

these containing the direct potential

$$U(\mathbf{r}_1)=\sum_{j=1}^{A}\int \Phi_j^\dagger(\mathbf{r}_2)\,V(|\mathbf{r}_1-\mathbf{r}_2|)\,\Phi_j(\mathbf{r}_2)\,d^3r_2 \tag{45}$$

and the exchange potential with nonlocal kernel

$$K(\mathbf{r}_1,\mathbf{r}_2)=-\sum_{j=1}^{A}\Phi_j^\dagger(\mathbf{r}_2)\,V(|\mathbf{r}_1-\mathbf{r}_2|)\,\Phi_j(\mathbf{r}_1). \tag{46}$$

The nonlocal exchange potential can be reduced to a local but velocity-dependent one (also multiplying from the left by $\Phi_i^\dagger(\mathbf{r}_1)\,\Phi_i(\mathbf{r}_1)$):

$$U_{\text{ex}}^i(\mathbf{r}_1)=-\sum_{j=1}^{A}\frac{\int \Phi_j^\dagger(\mathbf{r}_2)\,V(|\mathbf{r}_1-\mathbf{r}_2|)\,\Phi_i(\mathbf{r}_2)\,d^3r_2\,\{\Phi_j(\mathbf{r}_1)\,\Phi_i^\dagger(\mathbf{r}_1)\}}{\Phi_i^\dagger(\mathbf{r}_1)\,\Phi_i(\mathbf{r}_1)}. \tag{47}$$

The most interesting feature of the relativistic exchange potential is that the factor $\{\Phi_j(\mathbf{r}_1)\,\Phi_i^\dagger(\mathbf{r}_1)\}$ is a Dirac matrix. To transform this quantity, we use the general expansion

$$M=\sum_A \mathrm{Tr}\,(M\gamma_A)\,\gamma^A/4$$

and we obtain

$$\{\Phi_j(\mathbf{r}_1)\,\Phi_i^\dagger(\mathbf{r}_1)\}=\sum_A[\Phi_i^\dagger(\mathbf{r}_1)\,\gamma_A\Phi_j(\mathbf{r}_1)]\,\gamma^A/4,$$

where γ^A are sixteen linearly independent matrices formed from the products of the Dirac matrices.

TABLE VII. Parameters of OBE potentials

Meson	I	J^P	Mass, MeV	g^2	f/g	Λ, MeV	Λ_V, MeV	Literature
π	1	0^-	138.5	13	—	1950	—	Ref. 23
η	0	0^-	548.5	6	—	1950	—	
σ_0	0	0^+	500	4.63	—	1950	—	
δ	1	0^+	960	4.74	—	1950	—	
ρ	1	1^-	763	1.5	3.5	1950	1250	
ω	0	1^-	782.8	14	0	1950	1250	
Φ	0	1^-	1020	7	0	1950	1250	
π	1	0^-	138.5	14	—	2500	—	Ref. 25
η	0	0^-	548.5	2	—	2500	—	
σ_0	0	0^+	500	5.04	—	2000	—	
δ	1	0^+	960	6.12	—	2500	—	
ρ	1	1^-	763	0.7	4.5	1300	2500	
ω	0	1^-	782.8	9.8	0	1300	2500	
Φ	0	1^-	1020	4.9	0	1300	2500	

Note. For scalar and pseudoscalar mesons the form factors have the form $\Lambda^4/(\Lambda^2-\Delta^2)^2$; for vector mesons, the form $[\Lambda^4/(\Lambda^2-\Delta^2)^2] \times (\Lambda_V^2-m^2)/(\Lambda_V^2-\Delta^2)$, where $\Delta^2=(\Delta E)^2-(\Delta q)^2$, ΔE and Δq are the energy transfer and the momentum transfer.

It can be seen from Eq. (47) that if the two-particle interaction $V(|\mathbf{r}_1-\mathbf{r}_2|)$ has a special transformation law (for example, scalar), then even in this case the one-particle exchange potential will contain components that transform as all possible Dirac tensors: scalar, vector, pseudoscalar, pseudovector, and tensor. Thus, we arrive at a Dirac equation of the type (40), which reduces for nuclei with closed shells to equations (43) and (44). The expression for the individual components of the potentials (44) in terms of the wave functions (41) and OBE potentials are given in Ref. 59. In this paper, the calculations were made using the same vector–scalar model of the *NN* interaction as in Ref. 52. Comparison of the Hartree–Fock calculations of Ref. 59 with the Hartree calculations of Ref. 52 shows that allowance for the exchange interaction increases the binding energy per nucleon in light nuclei by 1.5 MeV and reduces the rms radius by 0.03 F (in Ref. 67 it was pointed out that an error was made in Ref. 59, but that this does not change this result significantly). It is interesting that the exchange contribution to the binding energy in Ref. 59 corresponds to attraction, whereas in atomic Hartree–Fock calculations the exchange associated with electron–electron interaction has the opposite sign with respect to the direct energy for this interaction. This relation also holds in Ref. 59, although the appreciable compensation of the two-particle matrix elements of the scalar and vector interactions is the reason why the exchange energy in Ref. 59 corresponds to attraction: The direct energy in Ref. 59 is obtained as a result of compensation of scalar (attractive) and vector (repulsive) matrix elements, the scalar being somewhat larger than the vector matrix elements; the exchange energy is associated with the same compensation, but in this case the vector matrix elements are somewhat larger.

The advantage of the approach of Refs. 52 and 59 resides in its elegance derived from the relativistic form of expression of the self-consistent equations on which this approach is based; however, it too has its difficulties.

The first problem is the correct inclusion of one-pion exchange. The pseudoscalar interaction in the form (6) does not contribute to the direct (Hartree) potentials for nuclei with closed shells. It is, however, shown in Ref. 59 that the contribution of one-pion exchange to the (Fock) exchange interaction calculated in the first order of perturbation theory may be fairly large.

Unfortunately, the attempts to include one-pion interaction in the coordinate representation in the form (13) in the relativistic self-consistent Hartree–Fock scheme were unsuccessful: Inclusion of the subsequent Hartree–Fock iterations led to collapse of the nucleus. This circumstance may be attributed to neglect of retardation effects in the coordinate form of representation of the one-pion exchange potential (13). Thus, in the approach proposed in Ref. 59 there is the problem of finding the form of the potential of one-pion interaction suitable for use in the relativistic Hartree–Fock theory.

A second difficulty is that the formalism developed in Ref. 59 is not adapted for using derivative meson–nucleon couplings. This shortcoming means that it is impossible to describe realistically the contribution of the ρ meson in the framework of the theory developed in Refs. 52 and 59 (Pauli tensor coupling of ρ mesons and nucleons was ruled out in Refs. 52 and 59). The investigations of Refs. 68–70 are free of these two difficulties.

In Ref. 69, calculations are made by the Hartree–Fock method for nuclei with closed shells. Two OBE potentials in the momentum representation introduced in Refs. 23 and 25 are used as the original interaction. The parameters of both potentials are given in Table VII. The use of the momentum representation in Ref. 69 made it possible to take into account retardation effects. In addition, it is no longer necessary to extrapolate the coordinate dependence of the potential obtained on the energy shell (at small momentum transfers) to the region off this shell, which is important for many-nucleon problems. The advantages of the momentum representation of OBE potentials over their coordinate representation are set forth in Ref. 2.

Note that, in contrast to Miller and Green's calculations in Refs. 52, 58, and 59, the Hartree–Fock calculations of Ref. 69 were based not on the relativistic single-particle equation (Dirac equation) but on the Schrödinger equation, which however, used, not directly an OBE potential, but a potential in the form (36). This procedure—use of a nonrelativistic equation and a modified potential—is equivalent in a certain sense to considering the relativistic equation with the original potential (this question was discussed in more detail in Sec. 2).

In Ref. 69, the local-density approximation investigated in Refs. 71 and 72 was used. Let us explain the essence of this approximation. We write the potential energy of the nucleus in the form

$$\langle V\rangle=\frac{1}{2}\sum_{\alpha,\beta}\langle\alpha\beta|G(\varepsilon_\alpha+\varepsilon_\beta)|\alpha\beta\rangle, \qquad (48)$$

where the sum is over the filled states; $G(\varepsilon_\alpha+\varepsilon_\beta)$ is

the effective interaction calculated by Brueckner's method, i.e., as in infinite nuclear matter (see Sec. 2). The starting energy $\varepsilon_\alpha+\varepsilon_\beta$ is made up of Hartree–Fock energies.

Clearly, the expression (48) is approximate, being based on the assumption that in the calculation of the effective interaction the nucleus in each of its small volumes can be regarded with sufficient accuracy as a piece of infinite nuclear matter with density equal to some average of the actual density of the nucleons at the points at which the interacting particles are (this may be, in particular, the geometric mean density, as was assumed in Refs. 71 and 72). We write Eq. (48) in the form

$$\langle V\rangle=\int d^3r\tilde{D}[\rho(r)]+\Big\{\frac{1}{2}\sum_{\alpha\beta}\langle\alpha\beta|G(\varepsilon_\alpha+\varepsilon_\beta)|\alpha\beta\rangle - \int d^3rD[\rho(r),\omega(r)]\Big\} + \Big\{\int d^3rD[\rho(r),\omega(r)]-\int d^3r\tilde{D}[\rho(r)]\Big\}. \quad (49)$$

Here, $D[\rho(r),\omega(r)]$ is the potential-energy density of nuclear matter with density $\rho(r)$ and starting energy $\omega(r)$ common to all two-particle states; $\tilde{D}[\rho(r)]$ is the same quantity calculated for the self-consistent starting energies $\tilde{\omega}(r)$. In (49), $\omega(r)$ can be chosen arbitrarily. Using this, we take $\omega(r)$ to be a function that approximates the self-consistent starting energies $\varepsilon_\alpha+\varepsilon_\beta$ of the finite nucleus. The introduction of starting energies that depend on the coordinates but not on two-particle states agrees with the basic assumption about the possibility of regarding nuclei as nuclear matter with local values $\rho(r)$ and $\omega(r)$ of the parameters at every point.

Proceeding from the definitions of $\tilde{\omega}(r)$ and $\omega(r)$, it is natural to assume that

$$\omega(r)-\tilde{\omega}(r)=2[U^F(r)-U^M(r)], \quad (50)$$

where $U^F(r)$ and $U^M(r)$ are the Hartree–Fock fields of the finite nucleus and infinite nuclear matter, respectively.

The effective interaction G satisfied the Bethe–Goldstone equation (26), which in operator form may be written as

$$G=V-V\frac{Q}{e}G, \quad (51)$$

where V is the original two-particle interaction; $e=E-\omega$ is the difference between the kinetic-energy operator of the pair of nucleons above the Fermi limit and the operator of the starting energy. Proceeding from (51) and making algebraic transformations with the operators, we readily obtain

$$\frac{dG}{d\omega}=-G\frac{Q}{e}\frac{Q}{e}G. \quad (52)$$

We calculate the matrix element of the operator $dG/d\omega$ between plane waves:

$$\left\langle kk'\left|\frac{dG}{d\omega}\right|kh'\right\rangle=\langle\chi_{kk'}|\chi_{kk'}\rangle, \quad (53)$$

where $\chi_{kk'}=|kk'\rangle-\psi_{kk'}$ is the defect function, i.e., the difference between the plane wave $|kk'\rangle$ and the distorted wave function $\psi_{kk'}$ taking into account the two-particle correlations:

$$G|kk'\rangle=V\psi_{kk'}. \quad (54)$$

The relation (53) follows from simultaneous use of Eqs. (51) and (54).

Let us consider the term in the second square brackets in Eq. (49). Using (50) and (53), we obtain

$$D[\rho(r),\omega(r)]-\tilde{D}[\rho(r)]=\frac{1}{2\Omega}\sum_{kk'}\langle kk'|G[\omega(r)] -G[\tilde{\omega}(r)]|kk'\rangle=-[\omega(r)-\tilde{\omega}(r)]\frac{1}{2\Omega}\sum_{kk'}\langle\chi_{kk'}|\chi_{kk'}\rangle. = -[U^F(r)-U^M(r)]\varkappa(r)\rho(r), \quad (55)$$

where

$$\varkappa(r)=\frac{1}{\Omega\rho(r)}\sum_{kk'}\langle\chi_{kk'}|\chi_{kk'}\rangle$$

is the so-called wound integral. In accordance with Ref. 71, the equivalent local potential for a finite system is defined as follows:

$$U^F(r)=\sum_\alpha\varphi_\alpha^*(r)\int d^3r'U(\mathbf{r},\mathbf{r}')\varphi_\alpha(\mathbf{r}')\Big/\sum_\alpha\varphi_\alpha^*(r)\varphi_\alpha(\mathbf{r}), \quad (56)$$

and for the nonlocal potential $U(\mathbf{r},\mathbf{r}')$ the following relation holds by definition:

$$\langle\alpha|U|\alpha\rangle=\sum_\beta\langle\alpha\beta|G(\varepsilon_\alpha+\varepsilon_\beta)|\alpha\beta\rangle. \quad (57)$$

From (56) and (57), we obtain

$$\int d^3r\rho(r)U^F(r)=\sum_{\alpha\beta}\langle\alpha\beta|G(\varepsilon_\alpha+\varepsilon_\beta)|\alpha\beta\rangle, \quad (58)$$

$$\int d^3r\rho(r)\varkappa[\rho(r)]U^F(r)=\sum_{\alpha\beta}\langle\alpha\beta|\varkappa(r_1)G(\varepsilon_\alpha+\varepsilon_\beta)|\alpha\beta\rangle. \quad (59)$$

Taking into account the relation

$$\rho(r)U^M(r)=2D[\rho(r),\omega(r)],$$

we write Eq. (49) in the form

$$\langle V\rangle=\int d^3r\tilde{D}[\rho(r)]+\frac{1}{2}\sum_{\alpha\beta}\langle\alpha\beta|[1 -2\varkappa(r_1)]G(\varepsilon_\alpha+\varepsilon_\beta)|\alpha\beta\rangle-\int d^3r[1 -2\varkappa(r)]D[\rho(r),\omega(r)]=\int d^3r\tilde{D}[\rho(r)]+(RST), \quad (60)$$

where the term (RST) gives the correction to the potential energy of the nucleus due to its finite size and is called the residual surface term. We write down (RST) approximately, replacing in it the effective interaction with self-consistent starting energies $\varepsilon_\alpha+\varepsilon_\beta$ of the finite system by the effective interaction with local starting energy $\omega(r)$ introduced above:

$$(RST) \approx \frac{1}{2}\int d^3r_1\, d^3r_2 \{\rho(r_1)\rho(r_2) V[r, \rho(r_1)\rho(r_2)] - \rho^2(r_1) V[r, \rho^2(r_1)]\} + \frac{1}{2}\int d^3r_1\, d^3r_2 \{|\rho(\mathbf{r}_1, \mathbf{r}_2)|^2 V[r, \rho(r_1)\rho(r_2)] - |\rho_M(\mathbf{r}_1, \mathbf{r}_2)|^2 V[r, \rho^2(r_1)]\},$$

where $\rho(\mathbf{r}_1, \mathbf{r}_2) = \sum_\alpha \varphi_\alpha^*(\mathbf{r}_1)\varphi_\alpha(\mathbf{r}_2)$ is the mixed (exchange) density; $\rho_M(\mathbf{r}_1, \mathbf{r}_2)$ is the mixed density for infinite nuclear matter:

$$\rho_M(\mathbf{r}_1, \mathbf{r}_2) = \frac{2}{\pi^2}\frac{\sin[k_F(r_1)r] - k_F(r_1) r\cos[k_F(r_1)r]}{r^3}; \quad (61)$$

and

$$r = |\mathbf{r}_1 - \mathbf{r}_2|, \quad k_F(r) = [(3/2)\pi^2\rho(r)]^{1/3}.$$

Further

$$V[r, \rho(r_1)\rho(r_2), \omega(r_1)] = [1 - 2\varkappa(r_1)] G[r, k_F(r_1) k_F(r_2), \omega(r_1)]. \quad (62)$$

It is easy to see that only the long-range part of the effective interaction contributes to (RST); substitution of δ-function forces as G makes (RST) vanish. On the other hand, Sprung and Banerjee[73] have shown that a slight change in the starting energy is equivalent to introducing an additional interaction with very short range. Taking this into account, we can replace the starting energy $\omega(r)$ of the finite system in the expression (60) by the starting energy $\tilde{\omega}(r)$ of infinite nuclear matter, which leads to the relation

$$\langle V\rangle = \frac{1}{2}\int d^3r_1\, d^3r_2 \{\rho(r_1)\rho(r_2) W + |\rho(\mathbf{r}_1, \mathbf{r}_2)|^2 W\}, \quad (63)$$

where

$$\left.\begin{aligned} &W = \Gamma[r_1, \rho(r_1)]\,\delta(\mathbf{r}_1 - \mathbf{r}_2)/2 + \tilde{V}[r, \rho(r_1)\rho(r_2)];\\ &\Gamma[r_1, \rho(r_1)] = 2\tilde{D}[\rho(r_1)]/\rho^2(r_1) - \int d^3r_2 \tilde{V}[r, \rho^2(r_1)]\\ &- \int d^3r_2 [|\rho_M(\mathbf{r}_1, \mathbf{r}_2)|^2/\rho^2(r_1)]\,\tilde{V}[r, \rho^2(r_1)], \end{aligned}\right\} \quad (64)$$

where the interaction $\tilde{V}$ is determined by

$$\tilde{V}[r, \rho(r_1)\rho(r_2)] = [1 - 2\varkappa(r_1)]\,\tilde{G}[r, k_F(r_1) k_F(r_2)], \quad (65)$$

and $\tilde{G}[r, k_F(r_1) k_F(r_2)]$ is the effective interaction for infinite nuclear matter calculated for self-consistent starting energies.

In Ref. 69, the Hartree–Fock calculations were based on the interaction represented by Eq. (64). The potential energy density of the nuclear matter as a function of the density can be written in the form[72]

$$\left.\begin{aligned} &D_{\substack{nn\\pp}} = a\hat{\rho}_{\substack{n\\p}}^{5/3} + b\hat{\rho}_{\substack{n\\p}}^{2} + c\hat{\rho}_{\substack{n\\p}}^{7/3} + d\hat{\rho}_{\substack{n\\p}}^{8/3};\\ &D_{np} = \hat{\rho}_n\hat{\rho}_p(\alpha\hat{\rho}^{-1/3} + \beta + \gamma\hat{\rho}^{1/3} + \delta\hat{\rho}^{2/3});\\ &\hat{\rho} = \rho/\rho_{NM}, \quad \hat{\rho}_{\substack{n\\p}} = \rho_{\substack{n\\p}}\frac{1}{\rho_{NM}/2}, \end{aligned}\right\} \quad (66)$$

where ρ_{NM} is the density of nuclear matter as saturation.

The effective interaction $V[r, \rho(r_1)\rho(r_2)]$ is found by interpolating the effective interaction obtained in calculations of nuclear matter in accordance with the formula

TABLE VIII. Binding energy per nucleon and rms radius of charge distribution calculated by Hartree–Fock method without fitting of $\tilde{D}[\rho(r)]$.

Nucleus	Parameter	OBEP I (Ref. 23)	OBEP II (Ref. 25)	Reid	Experiment
^{16}O	E/A, MeV	−4.78	−4.34	−3.77	−7.98
	r_c, F	2.86	2.85	3.03	2.73
^{40}Ca	E/A, MeV	−5.13	−4.67	−3.82	−8.55
	r_c, F	3.47	3.51	3.76	3.49
^{48}Ca	E/A, MeV	−5.23	−4.76	−3.80	−8.67
	r_c, F	3.49	3.54	3.79	3.48
^{90}Zr	E/A, MeV	−5.31	−4.77	−3.63	−8.71
	r_c, F	4.13	4.22	4.54	4.23
^{208}Pb	E/A, MeV	−4.46	−3.89	−2.67	−7.87
	r_c, F	5.19	5.36	5.68	5.50

$$V^{ST}[r, \rho(r_1)\rho(r_2)] = V_1^{ST}(r) - [(k_F^3(r_1) + k_F^3(r_2))/2]^{1/3} V_2^{ST}(r), \quad (67)$$

where S and T are the total spin and isospin of the interacting nucleons. The functions $V_{1,2}^{ST}(r)$ are found by fitting the effective interaction in the form (67) written down for the case of nuclear matter:

$$V_{NM}^{ST}(r, k_F) = V_1^{ST}(r) + k_F V_2^{ST}(r), \quad (68)$$

for two values of the Fermi momentum $k_F = 1.0$ F^{-1} and 1.4 F^{-1}, which correspond to the most probable nucleon densities in the nucleus. The radial dependence $V_{1,2}^{ST}(r)$ is taken in the form

$$V_i^{ST}(r) = \sum_{m=1}^{7} V_{mi}^{ST}\frac{\exp(-m\mu r)}{\mu r} \quad (69)$$

with $\mu = 0.7$ F^{-1}. The parameters V_{mi}^{ST} were chosen in such a way as to give the best approximation of the radial dependence of the effective interaction obtained from calculations of nuclear matter. As was noted above, only the long-range part of $V(r, \rho(r))$ contributes to the effective interaction (64), and therefore the effective interaction is cut off with respect to the radius: $V_i^{ST}(r) = 0$ if $r < 0.5$ F.

In Ref. 69, two series of calculations were made in accordance with the Hartree–Fock method: Without additional fitting of the potential energy density $\tilde{D}[\rho]$ of nuclear matter and with fitting of it to the "empirical" values (binding energy and density of nuclear matter).

Tables VIII and IX give the results of calculations by the Hartree–Fock method of the binding energy per nucleon and the rms radii of the charge distribution in the nucleus without additional fitting of $\tilde{D}[\rho(r)]$ (see Table VIII) and with such fitting (see Table IX). The calculations were made for two OBE potentials (see Table VII) and also for Reid's phenomenological potential with soft core (see, for example, Ref. 27).

As can be seen from Tables VIII and IX, the calculations based on the OBE potential without additional fitting give better results than the calculations with

TABLE IX. Binding energy per nucleon and rms radius of charge distribution calculated by Hartree–Fock method with fitting of $\tilde{D}[\rho(r)]$.

Nucleus	Parameter	OBEP I (Ref. 23)	OBEP II (Ref. 25)	Reid	Experiment
^{16}O	E/A, MeV	−7.98	−7.98	−7.96	−7.98
	r_c, F	2.87	2.74	2.78	2.73
^{40}Ca	E/A, MeV	−8.47	−8.50	−8.43	−8.55
	r_c, F	3.40	3.41	3.50	3.49
^{48}Ca	E/A, MeV	−8.56	−8.61	−8.49	−8.67
	r_c, F	3.45	3.46	3.57	3.48
^{90}Zr	E/A, MeV	−8.70	−8.73	−8.58	−8.71
	r_c, F	4.13	4.14	4.30	4.23
^{208}Pb	E/A, MeV	−7.86	−7.87	−7.75	−7.87
	r_c, F	5.20	5.29	5.51	5.50

Reid's phenomenological potential. The OBE potentials lead to a greater attraction, which is expressed in the binding energy and the radius of the charge distribution. Although the OBE potentials also give insufficient binding of the nuclei, there is hope of improving the situation in this regard because allowance for higher orders eliminates the discrepancy with the experimental values. Note that calculations of the contribution of the three-particle correlations for ^{16}O using the Reid potential[74] do not give sufficient binding energy.

In calculations with additional fitting of equally good results $\tilde{D}[\rho,(r)]$, for all three potentials are obtained. However, these calculations have less physical interest since the advantage of OBE potentials derived from their connection to the meson parameters is largely reduced to nothing. It is probable that the procedure of additional fitting is also not particularly meaningful for phenomenological potentials such as Reid's. Indeed, the justification for fitting the potential energy density $\tilde{D}[\rho(r)]$ is the assumption that such fitting may take into account effectively the contribution from many-nucleon correlations. Nevertheless, direct allowance for such a contribution does not give agreement with the empirical potential energy for Reid's potential.[74]

In calculations of both types and for all potentials, the radius of the charge distribution is too small. At the same time, allowance for the higher orders in Brueckner theory leads, as one would expect, to even smaller radii. A way out of this situation may be to take into account in the meson-exchange potentials the nucleon isobar $\Delta(1236)$ and to make correct allowance for two-pion exchange (see, for example, Ref. 75).

Calculations of the ^{16}O ground state were made in the spirit of Brueckner–Hartree–Fock theory in Ref. 68; the agreement between experiment and calculation in Ref. 68 is the same as in Ref. 69. In Ref. 70, use is made of an extension of the Brueckner–Hartree–Fock theory (see, for example, Ref. 36) by the inclusion of mesonic degrees of freedom; the results of calculations made by this method for ^{16}O reveal considerable improvement in the agreement with experiment.

In the following section, we shall present an approach to the investigation of various questions of nuclear theory using OBE potentials based on approximate inclusion (to order v^2/c^2) of relativistic effects in the nucleon–nucleon interaction; such an approach loses the elegance inherent in relativistic equations but has greater perspicuity and enables one to take into account derivative meson–nucleon coupling on an equal footing with direct coupling (see Eq. (5)).

The main subject to be discussed in the following section is the origin of the spin-orbit interaction in nuclei.

4. OBE POTENTIALS AND THE SPIN-ORBIT INTERACTION IN NUCLEI

A spin-orbit interaction in nuclei was first introduced in the framework of the shell model,[76,77] but there is still no clarity about the origin of the spin-orbit interaction in nuclei, although its magnitude clearly indicates that relativistic effects play an important role in the occurrence of such forces in nuclei.

At first, it was natural to attempt to explain the occurrence of these forces in nuclei by analogy with the situation in an atom, in which the greater part of the observed spin-orbit coupling is explained as the relativistic correction of the motion of the electron in the field of the nucleus (Thomas coupling). Furry[78] suggested that the different signs of the doublet splittings in atomic and nuclear spectra should be explained by the different type of interaction of the particles: The electrostatic potential in which the electron moves in an atom transforms as the zeroth (time) component of the four-vector of the electromagnetic field. Furry pointed out that if the nuclear average field is regarded as a universal scalar, then in this case one obtains a Thomas operator of spin-orbit coupling that differs only in sign from the analogous operator of the atomic problem. However, these early attempts to ascribe the spin-orbit interaction in nuclei to relativity were abandoned after calculations showed that Thomas coupling gives a doublet splitting about 30 times less than the experimentally observed splitting.

In Ref. 79, a phenomenological model with compensating contributions of two relativistic fields—scalar and vector—leading to a large spin-orbit coupling was considered by Duerr.

In Refs. 80 and 81 it was shown that in nuclei one can obtain strong spin-orbit coupling of relativistic origin if two-particle spin-orbit forces generated by relativistic OBE potentials are used in the framework of Hartree and Hartree–Fock theory.

The relations (16) and (18), (21) determine the Galileo-invariant component of the spin-orbit forces of an OBE potential, this component depending only on the relative momentum $\mathbf{p}=(\mathbf{p}_1-\mathbf{p}_2)/2$. The total operator of the two-particle spin-orbit forces, to which only scalar and vector mesons, both isoscalar and iso-

vector, contribute, has a more complicated structure. The final two-particle operator of the spin-orbit interaction can be represented as a sum of two operators[80,81]:

$$V_{ik}^{so}=-\frac{\hbar}{8M^2c^2}\mathbf{r}_{ik}\times\mathbf{p}_{ik}(\sigma_i+\sigma_k)\frac{1}{r_{ik}}\frac{d}{dr_{ik}}\Big\{\Big(\sum_S V_{ik}^S+3\sum_V V_{ik}^V\Big)$$
$$+\tau_i\cdot\tau_k\Big[\sum_{S\tau}V_{ik}^{S\tau}+\sum_{V\tau}\Big(3+4\frac{f}{g}\Big)V_{ik}^{V\tau}\Big]\Big\}; \quad (70a)$$

$$\delta V_{ik}^{so}=-\frac{\hbar}{8M^2c^2}\mathbf{r}_{ik}\times(\mathbf{p}_i+\mathbf{p}_k)\cdot(\sigma_i-\sigma_k)\frac{1}{r_{ik}}$$
$$\times\frac{d}{dr_{ik}}\Big[\Big(\sum_S V_{ik}^S-\sum_V V_{ik}^V\Big)+\tau_i\cdot\tau_k\Big(\sum_{S\tau}V_{ik}^{S\tau}-\sum_{V\tau}V_{ik}^{V\tau}\Big)\Big], \quad (70b)$$

that is, a sum of the Galileo-invariant V_{ik}^{so}, determined by the relative momentum $(\mathbf{p}_i-\mathbf{p}_k)$, and the Galileo-noninvariant δV_{ik}^{so}, which depends on the total momentum $(\mathbf{p}_i+\mathbf{p}_k)$ of the two nucleons.

The Galileo-noninvariant corrections of relativistic origin were first considered in Refs. 82 and 83. In a system consisting of two nucleons in free space, Galileo-noninvariant forces are absent by virtue of the relativity principle, whereas in a many-particle system Galileo noninvariant forces are present since every pair of particles moves in the field of the other particles, i.e., the relativity principle no longer applies separately to each pair of particles.

Note that the π mesons do not contribute to the two-particle operator of the spin-orbit interaction, or at least not in the approximation of one-boson exchange.

On the basis of the operators (70), a one-particle spin-orbit operator was determined for doubly magic plus one nucleon nuclei, the mathematical expectation of the operators (70) being calculated with nonsymmetrized wave function of the core of the nucleus. The calculation was made in the short-range approximation, i.e., the action of the operator (70) was restricted to P states, the procedure in this respect being equivalent to the one used to calculate the one-particle spin-orbit potential from phenomenological two-particle spin-orbit forces.[84] As a result, the following one-particle spin-orbit operator is obtained:

$$V_{so}(i)=\frac{\pi}{3}\frac{\hbar^2}{M^2c^2}\frac{1}{r}\frac{d\rho}{dr}\int_0^\infty\frac{d}{dx}\Big[\Big(\sum_S V_{ik}^S(x)$$
$$+\sum_V V_{ik}^V(x)\Big)-\tau_3^i\frac{N-Z}{A}\Big\{\sum_{S\tau}V_{ik}^{S\tau}(x)$$
$$+\sum_{V\tau}\Big(1+2\frac{f}{g}\Big)V_{ik}^{V\tau}(x)\Big\}\Big]x^3\,dx\,\mathbf{l}\cdot\sigma, \quad (71)$$

where $A-1=N+Z$; τ_3^i is the projection of the isospin of nucleon i, $\tau_3^i/2=\pm1/2$ (the positive sign corresponds to the proton); $\rho(r)$ is the matter density in the nucleus.

The average nuclear field is calculated in Ref. 80 using only the static part of the OBE potential. The calculation is made without allowance for the correct symmetry of the wave function in the short-range approximation; in this case, the average field is obtained in the form of a local velocity-independent potential with profile that repeats the profile of the density distribution (see also Ref. 85):

$$V_{av}(\mathbf{r})=4\pi\rho(\mathbf{r})\int_0^\infty\Big\{\Big(\sum_S V_{ik}^S(x)-\sum_V V_{ik}^V(x)\Big)$$
$$-\tau_3^i\frac{N-Z}{A}\Big(\sum_{S\tau}V_{ik}^{S\tau}(x)-\sum_{V\tau}V_{ik}^{V\tau}(x)\Big)\Big\}x^2\,dx. \quad (72)$$

Note the following important circumstance: The contributions of the vector and scalar mesons to the spin-orbit potential (71) have the same signs, while their contributions to the average field (72) have opposite signs. One then obtains a situation in which similar contributions compensate each other to a considerable degree in the average field but enhance one another in the spin-orbit potential. It is the circumstance that leads to the strong enhancement of spin-orbit coupling in nuclei. Note also that the operator δV_{ik}^{so} in (70b) plays a less important role than the Galileo-invariant operator V_{ik}^{so} in (70a), since δV_{ik}^{so} is determined by the difference between the contribution of the scalar and vector mesons. Therefore, the contribution of δV_{ik}^{so} to the spin-orbit splittings is approximately of the same order of magnitude as the one-particle Thomas coupling in the old relativistic approach, i.e., is an order of magnitude smaller than the V_{ik}^{so} contribution.

Thus, in Ref. 80 a one-particle spin-orbit potential (71) of Thomas form is obtained with radial dependence that, as in the phenomenological approach,[84] is determined by the density gradient but, in contrast to the phenomenological approach, additional spin-orbit forces requiring special adjustable parameters were not introduced into the *NN* interaction in Ref. 80 (in (71), the functions V_{ik} differ by no more than the sign from the static limits of the OBE potentials, so that all the quantities in (71) are determined from data that do not depend on the spin-orbit interaction).

In Ref. 80, the potential (71) was used to calculate the spin-orbit splittings ΔE_{nl} of a number of single-particle states $|nl=j\mp1/2\rangle$ of the nuclei ^{41}Ca and ^{209}Pb. The results of the calculations of the spin-orbit splittings due to the operator (71) and the depths of the average field (72) are given in Table X.

In Ref. 81, an investigation is made of the influence of the Pauli principle on the spin-orbit splitting and the average nuclear field, the treatment taking into account a wave function with the correct symmetry. The spin-orbit splittings are expressed in terms of the contributions of the spin-orbit operator to the energies of the single-particle states as follows:

$$\Delta E_{nl}^{so}=\varepsilon_i^{so}(j=l-1/2)-\varepsilon_i^{so}(j=l+1/2), \quad (73a)$$

where in the Hartree–Fock approximation

$$\varepsilon_i^{so}=\sum_{k\le A}(\langle ik\,|\,V_{ik}^{so}\,|\,ik\rangle-\langle ik\,|\,V_{ik}^{so}\,|\,ki\rangle). \quad (73b)$$

Remembering that the operator (70a) acts only in triplet states and in the adopted short-range approximation only in triplet P states, the exchange matrix elements can be reduced to direct matrix elements.[81] In this case, for nuclei with one nucleon above closed shells we obtain a single-particle spin-orbit operator of the following form[81]:

TABLE X. Spin–orbit splittings ΔE^{so}_{nl} and depth of average nuclear field potential V^0_{av} for neutrons obtained with allowance for the Pauli principle (upper row) and without it (lower row) for different variants of two-particle forces (see Table II).

Splitting	Ref. 4	Model II (Ref. 7)	Model III (Ref. 7)	Ref. 9	Experiment
ΔE^{so}_{1f} (^{41}Ca), MeV	13.8 9.32	7.65 4.25	9.45 5.56	10.1 5.98	6.50
ΔE^{so}_{2p} (^{41}Ca), MeV	3.30 2.22	1.82 1.01	2.25 1.32	2.37 1.41	2.00
ΔE^{so}_{2g} (^{209}Pb), MeV	3.53 2.22	2.62 1.16	2.88 1.43	2.94 1.52	2.47
ΔE^{so}_{1i} (^{209}Pb), MeV	11.2 7.08	8.33 3.70	9.22 4.59	9.37 4.82	4.57
ΔE^{so}_{3d} (^{209}Pb), MeV	2.03 1.28	1.49 0.677	1.66 0.830	1.69 0.874	0.98
V^0_{av} (^{41}Ca), MeV	−34.0 −45.3	+6.79 −40.8	−56.9 −87.1	−70.9 −103	−46.1
V^0_{av} (^{209}Pb), MeV	−31.7 −45.3	−4.53 −51.2	−56.3 −89.4	−68.3 −105	−53.4

$$V^{HF}_{so}(i)=\frac{\pi}{4}\frac{h^2}{M^2c^2}\frac{1}{r}\frac{d\rho}{dr}\int_0^\infty\frac{d}{dx}\Big\{\Big(\sum_S V^S_{ik}(x)+3\sum_V V^V_{ik}(x)\Big)+\Big[\sum_{S\tau}V^{S\tau}_{ik}(x)+\sum_{V\tau}\Big(3+4\frac{f}{g}\Big)V^{V\tau}_{ik}(x)\Big]\Big\}x^3\,dx\Big(1-\tau^i_3\frac{N-Z}{3A}\Big)\mathbf{l}\cdot\boldsymbol{\sigma}, \tag{74}$$

where it is also assumed that the neutron density $\rho_n(r)$ and the proton density $\rho_p(r)$ are related to the total density by the relations

$$\rho_n(r)=(N/A)\,\rho(r);\quad \rho_p(r)=(Z/A)\,\rho(r). \tag{75}$$

In the same approximations, the average field is local and has the form[81]

$$V^{HF}_{av}(\mathbf{r})=3\pi\rho(\mathbf{r})\int_0^\infty\Big[\Big(\sum_S V^S_{ik}(x)-\sum_V V^V_{ik}(x)\Big)-\Big(\sum_{S\tau}V^{S\tau}_{ik}(x)-\sum_{V\tau}V^{V\tau}_{ik}(x)\Big)\Big]x^2\,dx. \tag{76}$$

The results of the calculations of Ref. 81 are given in Table X together with the results obtained without allowance for the Pauli principle.

As can be seen from Table X, allowance for the correct symmetry of the wave function improves the agreement between theory and experiment (the bad results for V^0_{av} obtained for model II of Ref. 7 are due to the too large contribution of the scalar–isovector and vector–isovector mesons in this model). Such a circumstance demonstrates that correct description of the properties of nuclear structure by means of a particular model of the NN interaction may be an important factor in selecting it among other models that give an equal description of the properties of the two-nucleon sytem.

Before we turn to discussing possible sources of discrepancy between theory and experiment and ways of improving the theory, let us also consider Ref. 86, in which it is shown how one could verify the manifestation of this idea in exotic objects such as antiproton atoms, which, however, are being intensively studied at the present time. In particular, it is noted in Refs. 87 and 88 that in recent years x-ray transitions in $\bar{p}$ atoms have been observed. These measurements are interesting in that they give direct and as yet the only information about the $\bar{p}$–nucleus interaction at low energies and, in particular, the interaction of the $\bar{p}n$ system at low energies.[89] From the theoretical point of view, the potential of the interaction of $\bar{p}$ and the nucleus must be determined from the interaction of the nucleons in the nucleus and their interaction with the antiproton. If these elementary interactions were exactly determined at the present time, one could construct the interaction potential of the $\bar{p}$–nucleus system. In Ref. 86, one of the present authors considered the procedure for constructing such a potential (or rather, its real and spin-orbit parts), using the OBE potentials discussed above and assuming at the same time that these potentials can be used for the transition to the $N\bar{N}$ channel[42]; the behavior of the potential on the transition to the $N\bar{N}$ channel is determined by the quantum numbers of the exchange mesons.

If one is speaking about the real part of the potential of the interaction between the antiproton and a nucleus, it must be borne in mind that in this case the components (72) corresponding to exchange of vector–isoscalar and scalar–isovector mesons change their sign, while the components corresponding to the exchange of scalar–isoscalar and vector–isovector mesons do not change their sign. This means that the potential of the interaction between the antiproton and a nucleus (see the calculations made in Ref. 86) is much deeper than the potential (72).

In the derivation of the single-particle operator of the spin-orbit interaction for an antiproton atom on the basis of (70a) and (70b), it is also necessary to take into account the change in the signs of the vector–scalar and scalar–isovector components in (70a) and (70b), which means that in this case the Galileo-noninvariant part of the spin-orbit forces (70b) will also contribute to the single-particle spin-orbit potential. However, the final operator of the spin-orbit coupling in an antiproton is approximately 30 times weaker than the analogous operator in a nucleus; this circumstance should be manifested in the x-ray spectra of antiproton atoms and may serve as a confirmation of the validity of the idea of a scalar–vector compensation of the static components of the OBE potentials (and concomitant enhancement of the scalar and vector contributions in the relativistic corrections) characteristic of all one-boson exchange potentials.

Hitherto, we have considered only the contribution of the two-particle spin-orbit forces to the one-particle operator of the spin-orbit coupling in nuclei. However, it is known[90–92] that the one-particle spin-orbit operator in nuclei is completely determined by the contribution of two-particle spin-orbit forces only in the case of spin-saturated nuclei, i.e., nuclei for which the two levels of the doublet $j=l\mp1/2$ are either both filled or

TABLE XI. Contribution of the different components of the OBE potential to the spin–orbit splittings for the nucleus ^{209}Pb.

Parameter	ΔE^{so}_{1i}, MeV		ΔE^{so}_{2g}, MeV		ΔE^{so}_{3d}, MeV	
	Model II	Model III	Model II	Model III	Model II	Model III
Finite range of static OBEP components	−3.02	−2.54	−1.23	−1.02	−0.548	−0.456
$-\frac{1}{2M^2}[p^2V+Vp^2]$	−0.546	−0.571	−0.226	−0.232	−0.100	−0.103
$-\frac{1}{4M^2}\langle\nabla^2V\rangle$	1.18	1.17	0.501	0.493	0.22	0.217
$-\frac{1}{12M^2}\langle\nabla^2V\rangle\,\sigma_1\cdot\sigma_2$	1.42	1.42	0.568	0.576	0.253	0.258
Tensor forces	−3.55	−3.62	−1.42	−1.44	−0.633	−0.646
Spin-orbit forces	8.33	9.22	2.62	2.88	1.49	1.66
Total splitting	3.81	5.11	0.813	1.26	0.682	0.93
Experiment	4.57		2.47		0.98	

both empty. For the final elucidation of the question of the origin of the spin-orbit coupling in nuclei in the framework of OBE potentials, we have considered the contributions of all components of the OBE potential (16) to the spin-orbit splitting of single-particle states in spherical and deformed nuclei.

Let us consider first the example of doubly magic plus one nucleon nuclei. In this case, we use the technique developed in Ref. 91. The nucleon–nucleon interaction potentials used in Ref. 92 arise when the matrix elements of finite-range potentials are expanded in Taylor series. If the range of the potential is short compared with k_F, then in this expansion we can restrict ourselves to terms quadratic in the relative momenta (of the initial and final state). Applying such an expansion to the one-boson exchange potential (note that in this case it is all the more justified by the fact the OBE potential (16) is itself determined to within operators quadratic in the relative momentum of the nucleons), we reduce the operator (16) to the form

$$t_0\{1+x_0P_\sigma\}\,\delta(\mathbf{r}_1-\mathbf{r}_2)-\frac{1}{8}\{t_1+P_\sigma x_1\}\,[(\nabla_1'-\nabla_2')^2\,\delta(\mathbf{r}_1-\mathbf{r}_2)+\delta(\mathbf{r}_1-\mathbf{r}_2)(\nabla_1-\nabla_2)^2]+\frac{1}{4}\{t_2+P_\sigma x_2\}\,(\nabla_1'-\nabla_1')\,\delta(\mathbf{r}_1-\mathbf{r}_2)(\nabla_1-\nabla_2)+(-i)\,\frac{1}{2}V^{so}\,\frac{\delta(\mathbf{r}_1-\mathbf{r}_2)}{r^2}(\mathbf{r}_1-\mathbf{r}_2)\times(\nabla_1-\nabla_2)\cdot(\sigma_1+\sigma_2)+[V_0^T+V_\tau^T(\tau_1\cdot\tau_2)]\,\frac{\delta(\mathbf{r}_1-\mathbf{r}_2)}{r^2}S_{12}, \tag{77}$$

where P_σ is the operator of spin exchange; the operator ∇' acts only from the left and the operator ∇ from the right; t_0, x_0; t_1, x_1; t_2, x_2; V^{so}; V_0^T; V_τ^T are constants determined by the OBEP parameters. The terms with x_1 and x_2 in (77) are due either to the presence of isovector mesons or the presence of the operators $\sigma_1\cdot\sigma_2$ in the OBEP components, or to both of these simultaneously.

Once the interaction has been represented in the form (77), we can use directly the technique developed in Refs. 92 and 93; in particular, one can show that all the OBEP components (77) except the spin-orbit forces lead to a one-particle spin-orbit potential of the following form for doubly magic plus one nucleon nuclei[94]:

$$U_q^{so}(r)=\frac{1}{8}\left[(t_1-t_2)\,\frac{1}{r}J_q(r)-(x_1+x_2)\,\frac{1}{r}J(r)\right]1\cdot\sigma, \tag{78}$$

where the index q distinguishes the protons and neutrons; $J(r)$ is the spin density determined by means of the relation[92]

$$J(r)=\frac{1}{4\pi r^3}\sum_\alpha(2j_\alpha+1)\,[j_\alpha(j_\alpha+1)-l_\alpha(l_\alpha+1)-3/4]\,R_\alpha^2(r), \tag{79}$$

where $R_\alpha(r)$ is the radial wave function; α is a complete set of quantum numbers characterizing the spherically symmetric one-particle state; j_α is the total angular momentum; l_α is the orbital angular momentum of this state.

The contribution of the various OBEP components to the spin-orbit splitting of single-particle states was determined for the example of the nucleus ^{209}Pb, which has two spin-nonsaturated shells (the proton $1h_{11/2}$ and the neutron $1i_{13/2}$). Calculations were made with wave functions of the model discussed in Refs. 80 and 81. The results of calculations of the splittings of three states of the ^{209}Pb nucleus are given in Table XI for the two PVS models of Ueda and Green[7] (model II and III; see Table II). It can be seen from Table XI that the individual OBEP components make an important contribution in the nucleus ^{209}Pb to the final spin-orbit splitting, this being comparable in some cases with the contribution of the two-particle spin-orbit forces. However, the two-particle spin-orbit forces are nevertheless a distinguished OBEP component (and of all the nucleon–nucleon potentials) since in the case of spin-saturated nuclei the final spin-orbit splitting is completely determined by the two-particle spin-orbit forces (this last assertion is valid to the extent that the radial wave functions of the spin-orbit doublet $j=l\mp1/2$ can be assumed to be coincident). It can be seen from Table XI that the final spin-orbit splitings produced by all the OBEP components agree reasonably with experiment.

We note the following possible sources of discrepancy between this theory and experiment: 1) restriction to the first step in the self-consistency procedure; 2) the use of the short-range approximation to obtain the operator (74) from the two-particle spin-orbit forces; 3) the neglect of residual interactions; 4) the approximate nature of the single-particle wave functions that were used; 5) the restriction of the scheme to one-boson exchange.

For deformed nuclei (see, for example, Ref. 95) it is convenient to use the following expansion [equivalent to (77)] of the OBEP potentials in powers of the relative momentum:

$$[V_0+V_\tau(\tau_1\cdot\tau_2)]\,\delta(\mathbf{r})+\left\{[V_0^\nabla+V_\tau^\nabla(\tau_1\cdot\tau_2)+V_\sigma^\nabla(\sigma_1\cdot\sigma_2)+V_{\sigma\tau}^\nabla(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2)]\,\frac{\delta(\mathbf{r})}{r^2}\mathbf{r}\cdot\nabla+\text{H.c.}\right\}+(-i)\,V^{so}\,\frac{\delta(\mathbf{r})}{r^2}\mathbf{r}\times\nabla(\sigma_1+\sigma_2)+[V_0^T+V_\tau^T(\tau_1\cdot\tau_2)]\,\frac{\delta(\mathbf{r})}{r^2}S_{12}. \tag{80}$$

We consider axisymmetric nuclei, assuming that the even–even core is invariant under the operation of time reversal. To describe the nucleons, we shall use the asymptotic Nilsson representation.

The spin-orbit splitting of the state of an odd nucleus with quantum numbers n, $\tilde{n}$, Λ, τ ($|n\tilde{n}\Lambda\tau\rangle \equiv |Nn\Lambda\tau\rangle$, where τ is the isospin variable and Λ is the projection of the orbital angular momentum onto the quantization axis) from the operator (80), whose individual components we denote by $V^i(\mathbf{r}, \mathbf{p}; \sigma_1, \sigma_2; \tau_1, \tau_2)$ with ($i = so, T, \nabla$), can be represented in the form

$$\Delta E^i_{so}(n\tilde{n}\Lambda\tau) = E^i(n\tilde{n}\Lambda\tau\sigma = -1) - E^i(n\tilde{n}\Lambda\tau\sigma = +1),$$

where E^i is the contribution from the potential V^i to the single-particle energy of the nucleon and with allowance for antisymmetrization of the wave functions:

$$E^i = \sum_2 [\langle \Phi_1(1)\Phi_2(2)|V^i|\Phi_1(1)\Phi_2(2)\rangle - \langle \Phi_1(1)\Phi_2(2)|V^i|\Phi_2(1)\Phi_1(2)\rangle], \quad (81)$$

where $\Phi_1(k) \equiv \langle \mathbf{r}_k, \sigma_k, \tau_k | n\tilde{n}\Lambda\tau\sigma\rangle$ is the wave function of the nucleon; $\Phi_2(k) \equiv \langle \mathbf{r}_k, \sigma_k, \tau_k | n'\tilde{n}'\Lambda'\tau'\sigma'\rangle$ is the wave function of the nucleon core ($k = 1, 2$). The sum $\sum_2$ is over all nucleons of the core. After separation of the center-of-mass variable and the relative variable, the problem of calculating the two-particle matrix elements (81) reduces to the calculation of the "single-particle" matrix elements between functions of the relative variables. Leaving aside the technique for calculating the matrix elements, we give the results for the spin-orbit splittings for all components of the OBE potential (80) (all the components in (80) except the first two determined by the constants V_0 and V_τ contribute to the splitting):

$$\Delta E^{so}_{so}(n\tilde{n}\Lambda\tau) = -\frac{\tilde{V}^{so}}{3\pi}\Lambda \sum_{\substack{n'\tilde{n}'\Lambda'\sigma'\tau' \\ (\Omega'>0)}} (1+\delta_{\tau\tau'}) \times \int_{-\infty}^{\infty} dl\,(J_n(l)J_{n'}(l))^2 \int_0^{\infty} dx\, \varphi_{\tilde{n}|\Lambda|}\varphi_{\tilde{n}'|\Lambda'|}(\varphi'_{\tilde{n}'|\Lambda|}\varphi_{\tilde{n}'|\Lambda'|} - \varphi_{\tilde{n}|\Lambda|}\varphi'_{\tilde{n}'|\Lambda'|}); \quad (82a)$$

$$\Delta E^T_{so}(n\tilde{n}\Lambda\tau) = -\frac{2}{15\pi}\Lambda \sum_{\substack{n',\tilde{n}'\Lambda'\sigma'\tau' \\ (\Omega'>0)}} [(\tilde{V}^T_0 - \tilde{V}^T_\tau)\delta_{\tau\tau'} + 2\tilde{V}^T_\tau]\sigma'\Lambda' \int_{-\infty}^{\infty} dl\,(J_n(l)J_{n'}(l))^2 \int_0^{\infty} dx \frac{(\varphi_{\tilde{n}|\Lambda|}\varphi_{\tilde{n}'|\Lambda'|})^2}{x}; \quad (82b)$$

$$\Delta E^{\nabla}_{so}(n\tilde{n}\Lambda\tau) = \frac{1}{3\pi}\Lambda \sum_{\substack{n'\tilde{n}'\Lambda'\sigma'\tau' \\ (\Omega'>0)}} [(\tilde{V}^{\nabla}_0 - \tilde{V}^{\nabla}_\sigma - \tilde{V}^{\nabla}_\tau + \tilde{V}^{\nabla}_{\sigma\tau})\delta_{\tau\tau'} + 2(\tilde{V}^{\nabla}_\tau - \tilde{V}^{\nabla}_{\sigma\tau})]\sigma'\Lambda' \times \int_{-\infty}^{\infty} dl\,(J_n(l)J_{n'}(l))^2 \int_0^{\infty} dx \frac{(\varphi_{\tilde{n}|\Lambda|}\varphi_{\tilde{n}'|\Lambda'|})^2}{x}. \quad (82c)$$

Here

$$\left.\begin{aligned} J_n(l) &= \left(\frac{1}{\pi^{1/2}2^n n!}\right)^{1/2} \exp(-l^2/2) H_n(l); \\ \varphi_{\tilde{n}|\Lambda|}(x) &= \left[\frac{2\tilde{n}!}{(\tilde{n}+|\Lambda|)!}\right]^{1/2} \exp(-x/2)\, x^{|\Lambda|/2} L^{|\Lambda|}_{\tilde{n}}(x); \\ \tilde{V}^i &= \left(\frac{M\omega_0^\circ}{\hbar}\right)^{3/2} \frac{M\omega_\perp}{\hbar} V^i; \end{aligned}\right\} \quad (83)$$

$H_n(l)$ is an Hermite polynomial; $L^{|\Lambda|}_{\tilde{n}}(x)$ is a Laguerre polynomial.

Results of calculations of the spin-orbit splitting of the neutron level $|Nn\Lambda\sigma\rangle = |202\pm\rangle$ in the nuclei ^{25}Mg and ^{27}Si [in accordance with (82a)–(82c)] can be found in Ref. 95.

It follows from (82a)–(82b) that for spin-saturated nuclei (among the occupied states, every space-isospin state is encountered twice—for the two spin states) only the two-particle spin-orbit forces contribute to the spin-orbit splitting (as for spherical nuclei).

CONCLUSIONS

The approach to the problem of nucleon–nucleon interaction presented in the present review has been developed intensively during the last ten years. It is now clear what are its advantages and shortcomings. It describes a large body of experimental data and establishes connections between different directions of nuclear and elementary-particle physics. The further development of this approach will require allowance for the role of many-meson exchange, retardation effects, and three-particle forces; detailed knowledge of the meson–nucleon form factors and the contributions of nucleon isobars is also important for the construction of the meson theory of nuclear forces.

The use of meson potentials in many-particle problems leads to a description of nuclear matter and finite nuclei that is at least not inferior to the one obtained with purely phenomenological models. The development of this aspect of relativistic theory presupposes the inclusion of many-particle forces and correlations in the theory of nuclear matter and an advance beyond the framework of the single-particle description in the theory of finite nuclei.

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