The method of K harmonics and the shell model

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The aim of the review is to compare the method of K harmonics (hyperspherical harmonics) and the translationally invariant shell model. Only nuclei with $A > 4$ will be considered. For them, as will be seen, many results of the method of K harmonics are very similar to the corresponding results of shell calculations. This similarity is due to the fact that the wave functions of the method of K harmonics and the translationally invariant shell model have much in common. The structure of the wave functions of the two methods is considered, their connection with one another is analyzed, and then the results of calculations of various nuclear properties obtained by these methods are compared.

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INTRODUCTION

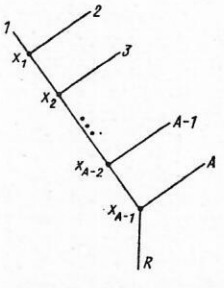
In recent years, a new interesting method—the method of K harmonics proposed by Simonov[1]—has been developed in the theory of light nuclei. Originally, it was applied mainly to nuclei of three and four nucleons and was very effective.[2] However, the method can be readily generalized to an arbitrary number of nucleons.[3] In the literature, there are now descriptions of a number of calculations in the main approximation of the method of K harmonics for a number of light nuclei with $A > 4$ made under the direction of Blaz'.[4] For the same nuclei, the shell model, the Hartree–Fock method, and other methods have also been successfully used. It is therefore of interest to compare the results obtained in the framework of the new method with the results of the older traditional approaches. This would make it possible to clarify the advantages and disadvantages of the various methods, the extent to which they are equivalent, and to understand which additional degrees of freedom are taken into account in the various particular methods. The main aim of the present review is to compare the method of K harmonics and the shell model, or, more precisely, the translationally invariant shell model, whose wave functions correctly describe the center-of-mass motion of the nucleus. However, we shall also briefly touch on other methods in the theory of light nuclei, for example, the method of generator coordinates. We shall be concerned only with nuclei with $A > 4$, for which, as we shall see, many results of the method of K harmonics are very similar to the corresponding results of calculations in the shell model. This is not fortuitous but due to the fact that the structure of the wave functions in the two methods has much in common. Below, we shall briefly recall the structure of the wave functions in the two methods, we shall analyze the connection between the two, and we shall then compare the results of calculations of certain properties of nuclei obtained by these methods.

1. STRUCTURE OF THE WAVE FUNCTIONS OF A TRANSLATIONALLY INVARIANT OSCILLATOR BASIS

Coordinates and Jacobi "Trees". The wave function Ψ of a nucleus consisting of A nucleons must be translationally invariant, and its arguments are therefore usually taken to be the Jacobi coordinates $\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_{A-1}$. Each of these coordinates $\bar{X}_i$ is the distance between the centers of mass of two groups of nucleons, one of which contains p_i nucleons and the other q_i nucleons ($p_i, q_i \geqslant 1$). Such a coordinate $\bar{X}_i$ can be put in correspondence with reduced mass $\mu_i = p_i q_i m/(p_i + q_i)$, where m is the nucleon mass. It is convenient to introduce "normalized" Jacobi coordinates $\bar{x}_i = \bar{X}_i \sqrt{\mu_i/m}$. We separate the standard set of Jacobi coordinates

$$\left.\begin{aligned}
&\bar{x}_1 = (1/\sqrt{2})(\bar{r}_1 - \bar{r}_2);\\
&\bar{x}_2 = \sqrt{2/3}\,[(1/2)(\bar{r}_1 + \bar{r}_2) - \bar{r}_3];\\
&\bar{x}_3 = \sqrt{3/4}\,[(1/3)(\bar{r}_1 + \bar{r}_2 + \bar{r}_3) - \bar{r}_4];\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\bar{x}_{A-1} = \sqrt{\frac{A-1}{A}}\left[\frac{1}{A-1}\sum_{i=1}^{A-1}\bar{r}_i - \bar{r}_A\right].
\end{aligned}\right\} \qquad (1)$$

Here, $\bar{r}_i$ are the coordinates of nucleon i; each coordinate $\bar{x}_i$ of the standard set determines the distance between nucleon $i+1$ and the center of mass of the group of nucleons with the numbers $1, 2, \ldots, i$. We shall say that other sets of Jacobi coordinates are *nonstandard*. In particular, if a permutation of the nucleon coordinates $\bar{r}_i$ is made in the set (1), a nonstandard set of Jacobi coordinates is obtained. The choice of the Jacobi coordinates can be illustrated graphically in the form of a Jacobi "tree":



(I)

The free ends of this graph correspond to the coordinates $\bar{r}_i$ of the nucleons; each internal line denotes the coordinates of the centers of mass of the group of p nucleons whose numbers are designated at the free ends that "grow" from this internal line. The mass pm is associated with the internal line and the mass m with the free end. To each node there corresponds a definite Jacobi coordinate composed of the coordinates

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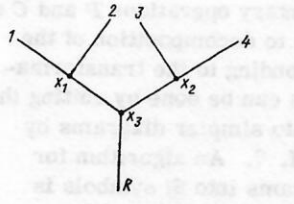
of the centers of mass of the two groups of p and q nucleons corresponding to the lines going upward out of this node. If a line goes to the left, then the coordinate corresponding to it is taken with plus sign; if to the right, with minus sign. The difference of the coordinates of the two centers of mass must be multiplied by $[pq/(p+q)]^{1/2}$. To the standard set of Jacobi coordinates (1) there corresponds the Jacobi tree given above. The "nonstandard Jacobi tree" shown in the graph



(II)

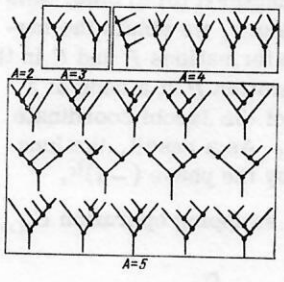
describes the following set of Jacobi coordinates:

$$\left.\begin{aligned}\bar{x}_1&=(1/\sqrt{2})\,(\bar{r}_1-\bar{r}_2);\\ \bar{x}_2&=(1/\sqrt{2}(\,(\bar{r}_3-\bar{r}_4);\\ \bar{x}_3&=(1/2)\,(\bar{r}_1+\bar{r}_2-\bar{r}_3-\bar{r}_4).\end{aligned}\right\} \tag{2}$$

General Properties of "Treelike" Graphs. In what follows, we shall encounter other treelike graphs, and we shall therefore consider some general properties of such graphs. First of all, we establish what is the total number of trees with A free ends. It is clear that if there is one tree of a certain type, for example, the one shown in (I), then one can obtain $A!$ trees of the same structure by permuting the numbers of the free ends in all possible ways. Therefore, we must consider how many different trees there are if one discounts permutations of the numbers of the free ends. Treelike graphs for which the number of the free ends have been omitted will be called *skeleton* graphs. We denote the number of skeleton graphs for A free ends by $K(A)$; then for this number we can write down the recursion relation[5]

$$K(A)=\sum_{A_1+A_2=A} K(A_1)\,K(A_2). \tag{3}$$

The numerical values of $K(A)$ for $A=2$–6 are given in Table I, and the corresponding skeleton graphs for $A=2$–5 have the form



(III)

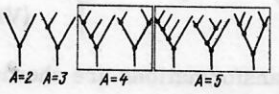
In Ref. 6, Kuznetsov and Smorodinskiĭ obtain the general formula $K(A)=(2A-2)!!/[(A-1)!A!]$. It can be seen from the graph given above that skeleton graphs can be combined into pairs related by mirror reflection with respect to the trunk of the tree. If we separate the graphs that cannot be obtained from one another by mirror reflection, we arrive at essentially different trees. We denote their number by $M(A)$, and for it we again obtain a recursion relation of the type (3):

TABLE I. Number of skeleton graphs as a function of the number A of nucleons.

A	2	3	4	5	6
$K(A)$	1	2	5	14	49
$M(A)$	1	1	2	3	6
Total number	2	12	120	1 680	35 280

$$M(A)=\sum_{\substack{A_1\geq A_2\\ A_1+A_2=A}} M(A_1)\,M(A_2). \tag{4}$$

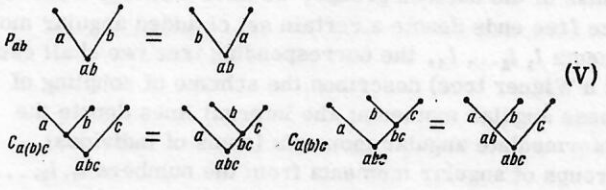
The values of $M(A)$ for $A\leq 6$ are given in Table I, and the essentially different trees for $A\leq 5$ are illustrated by the graph



(IV)

Note that $M(A)$, $K(A)$, and the total number of trees, which is equal to $K(A)A!=[(2A-2)!!]/[(A-1)!]$, increase very rapidly with increasing A.

Operations on Trees. Let us consider transitions from one tree to another. The set of all transformations from one tree to another is called the *groupoid of rearrangements* (*recouplings*).[5] This set differs from a group in that a product is not defined for all its elements and multiplication is not associative for all products. Details about the structure of the groupoids for different A and their representations are given in Ref. 5. Here, we only mention that the number of elements of the groupoid also increases extremely rapidly with increasing A. However, any arbitrarily complicated transformation of trees can be reduced to a succussion of a certain number of elementary transformations. These elementary operations on trees are the following (we use the terminology of Ref. 6):

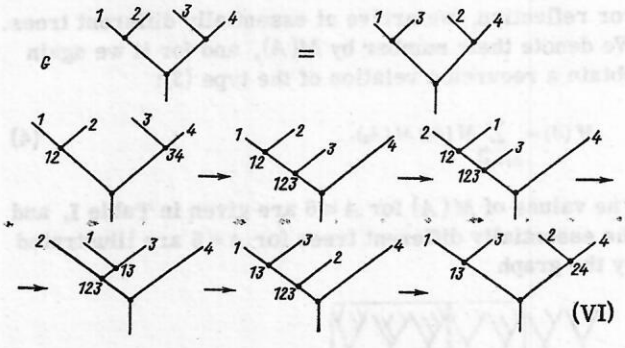


(V)

1) the reflection P_{ab} in the fork ab of the tree from which branches a and b grow; the structure of the crown whose base is branch a and the structure of the crown above branch b do not change, the crowns being merely moved into a new position; 2) the swapping (transplating) $C_{a(b)c}$ of one branch b at the fork abc from the left to the right or vice versa. Here, as in the case of reflection, the structures of the crowns above the branches a, b, and c are not changed. In what follows, we shall consider only those cases when the matrices of the transformation $C_{a(b)c}$ are unitary and real, i.e.,

the two swappings shown in the graph will be characterized by the same transformation coefficients, and we shall therefore denote such mutually invertible transitions by the single symbol $C_{a(b)c}$.

Thus, any rearrangement of trees can be reduced to a sequence of the elementary operations P and C, in each stage only one branch being taken from one side of the fork to the other. As an example, we represent the transition between two trees with four free ends as follows:



(VI)

Here, all the intermediate transformations are shown. The resulting transformation G has the form

$$G = C_{13\,(2)\,4} P_{13,2} C_{2\,(1)\,3} P_{12} C_{12\,(3)\,4}. \tag{5}$$

This representation of G in terms of the operations C and P is not unique; for example, one can write

$$G = C_{1\,(3)\,24} P_{24,3} C_{2\,(4)\,3} P_{34} C_{1\,(2)\,34} \text{ etc.} \tag{6}$$

However, the expressions (5) and (6) are minimal in the sense that G cannot be reduced to a smaller number of operations P and C.

Wigner Tree and Decomposition of Tree Rearrangement Operations into Elementary Operations. We have considered operations on trees of a general kind, without considering what particular meaning attaches to the free ends and other lines of these trees. If the free ends denote the coordinates of individual nucleons and the internal lines the coordinates of the centers of mass of the nucleon groups, we have a Jacobi tree. If the free ends denote a certain set of added angular momenta $l_1\ l_2 \ldots\ l_A$, the corresponding tree (we shall call it a Wigner tree) describes the scheme of coupling of these angular momenta; the internal lines denote the intermediate angular momenta (sums of individual groups of angular momenta from the numbers $l_1, l_2, \ldots, l_A$); and the trunk of the tree is the total angular momentum $\overline{L} = \overline{l}_1 + \overline{l}_2 + \ldots + \overline{l}_A$.

If at any fork the *left-hand* angular momentum is l_i, and the *right-hand* is l_j, it is assumed that they are added by means of the Clebsch–Gordan coefficients $(l_i m_i l_j m_j / l_i j m_{ij}$. The transition from one Wigner tree to another means a certain change in the scheme of coupling of the angular momenta implemented by means of a definite $3nj$ symbol. The diagram of the corresponding $3nj$ symbol can be obtained from the graphs of the two trees between which the transition is made by combining the corresponding free branches with equal numbers and combining the two trunks into one common line. The elementary operations on Wigner trees are the following: a) $P j_1 j_2$ denotes the simple change of phase $(-1)^{j_1+j_2-j_{12}}$; b) the coefficient of the transition $C j_1 (j_2) j_3$ reduces to the Racah coefficient:

$$C j_1 (j_2) j_3 = U (j_1 j_2 j_3; j_{12} j_{23}).$$

Therefore, the decomposition of a complex transformation G over trees into elementary operations P and C of the type (5)–(6) is equivalent to decomposition of the complex $3nj$ symbol corresponding to the transformation G into $6j$ symbols. This can be done by cutting the diagram of the $3nj$ symbol into simpler diagrams by the methods described in Ref. 7. An algorithm for cutting a $3nj$ symbol in diagrams into $6j$ symbols is developed in Ref. 8. The expression (5) for Wigner trees means that the $9j$ symbol is reduced to a sum of products of three Racah coefficients:

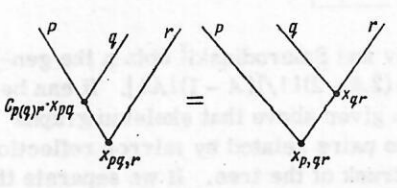
$$[(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j_{1234} \end{Bmatrix}$$
$$= \sum_{j_{123}} U (j_{12} j_3 j_{1234} j_4; j_{123} j_{34}) (-1)^{j_{13}+j_2-j_{12}}$$
$$\times U (j_2 j_1 j_{123} j_3; j_{12} j_{13}) (-1)^{j_{13}+j_{12}-j_{123}} U (j_{13} j_2 j_{1234} j_4; j_{123} j_{24}). \tag{7}$$

Transitions from One Jacobi Tree to Another in Harmonic Oscillator Basis. In the translationally invariant shell model, we shall be dealing with products of oscillator functions of Jacobi coordinates: $\Psi = \prod_{i=1}^{A-1} \Psi_{n_i l_i m_i} (\overline{x}_i)$, where n, l, m are the number of quanta, the orbital angular momentum, and its projection for the corresponding oscillator state. It is helpful to add the angular momenta $l_1 \ldots l_{A-1}$ in some manner into the total angular momentum L; we denote the wave function obtained as a result by

$$| l_1 l_2 \ldots l_{A-1} : LMD_W D_J \rangle, \tag{8}$$

where $\mathcal{D}_J$ denotes the Jacobi tree that defines the structure of the employed Jacobi coordinates $\overline{x}_1 \overline{x}_2 \ldots \overline{x}_{A-1}$; $\mathcal{D}_W$ is the Wigner tree which specifies the scheme of adding the angular momenta of the motion with respect to the individual Jacobi coordinates $l_1 l_2 \ldots l_{A-1}$ into the total angular momentum L. In what follows, we shall need to transform the functions (8) to other sets of Jacobi coordinates. Therefore, we obtain the matrices of the elementary transformations P and C in the basis (8). The reflection operation P at a node of a Jacobi tree changes the sign of the Jacobi coordinate $\overline{x}_i$ corresponding to this node. As a result, the function (8) is simply multiplied by the phase $(-1)^{l_i}$,

Let us consider the branch swapping operation $C_{p(q)r}$:



(VII)

Here, the free ends are the coordinates R_p, R_q, R_r of

the centers of mass of the three nucleon groups with masses p, q, and r, respectively. The nodes correspond to the two pairs of Jacobi coordinates

$$\left.\begin{aligned} x_{pq} &= \sqrt{\frac{pq}{p+q}}(R_p - R_q);\quad x_{pq,\,r} \\ &= \sqrt{\frac{r}{(p+q+r)(p+q)}}[pR_p + qR_q - (p+q)R_r]; \\ x_{qr} &= \sqrt{\frac{qr}{q+r}}(R_q - R_r);\quad x_{p,\,qr} \\ &= \sqrt{\frac{y}{(p+q+r)(q+r)}}[(q+r)R_p - qR_q + rR_r]. \end{aligned}\right\} \tag{9}$$

These two sets of Jacobi coordinates are related by the following orthogonal transformation:

$$\left.\begin{aligned} x_{p,\,qr} &= \sqrt{\frac{pr}{(p+q)(q+r)}}\,x_{pq,\,r} + \sqrt{\frac{q(p+q+r)}{(p+q)(q+r)}}\,x_{pq}; \\ x_{qr} &= \sqrt{\frac{q(p+q+r)}{(p+q)(q+r)}}\,x_{pq,\,r} - \sqrt{\frac{pr}{(p+q)(q+r)}}\,x_{pq}, \end{aligned}\right\} \tag{10}$$

which can be regarded as a kinematic rotation through the angle

$$\varphi = \tan^{-1}\sqrt{pr/[q(p+q+r)]}.$$

We separate the part of the function (8) related to the Jacobi coordinates x_{pq}, $x_{pq,r}$. Then in accordance with Refs. 10 and 11,

$$\begin{aligned} &|n_1 l_1(x_{pq,\,r}),\ n_2 l_2(x_{pq}) : \Lambda M\rangle \\ &= \sum_{n_1' l_1' n_2' l_2'} \langle n_1 l_1 n_2 l_2 : \Lambda \,|\, pr/q(p+q+r) \,|\, n_1' l_1' n_2' l_2' : \Lambda\rangle \\ &\quad \times |n_1' l_1'(x_{p,\,qr})\ n_2' l_2'(x_{qr}) : \Lambda M\rangle, \end{aligned} \tag{11}$$

where $\langle n_1 l_1 n_2 l_2 : \Lambda \,|\, \mu_1/\mu_2 \,|\, n_1' l_1' n_2' l_2' : \Lambda\rangle$ are generalized Talmi–Moshinsky coefficients.[12] Thus,

$$\begin{aligned} &\langle n_1 l_1(x_{pq})\ n_2 l_2(x_{pq,\,r}) : \Lambda M \,|\, C_{pq,\,r} \,|\, n_1' l_1'(x_{pq})\ n_2' l_2'(x_{pq,\,r}) : \Lambda\rangle \\ &\quad = \langle n_2 l_2 n_1 l_1 : \Lambda \,|\, pr/q(p+q+r) \,|\, n_2' l_2' n_1' l_1' : \Lambda\rangle. \end{aligned} \tag{12}$$

Note that if the oscillator states $|n_1 l_1 m_1\rangle$, $|n_2 l_2 m_2\rangle$ in (11) are not simply added vectorially but a combination is formed from them with definite symmetry $(\lambda\mu)$ under $SU(3)$, i.e.

$$|n_1 n_2 : (\lambda\mu)\, LM\rangle = \sum_{l_1 l_2} \langle l_1^{(n_1 0)} l_2^{(n_2 0)} \,|\, L^{(\lambda\mu)}\rangle \,|\, n_1 l_1 n_2 l_2 : LM\rangle, \tag{13}$$

where $\langle l_1^{n_1 0} l_2^{n_2 0} \,|\, L^{(\lambda\mu)}$ are the Clebsch–Gordan coefficients for $SU(3)$ in the reduction $SU(3) \supset SO(3)$ (Ref. 13), then the matrix element of the swapping $C_{p(q)r}$ can be expressed even more simply[14,15]:

$$\langle n_1 n_2 : (\lambda\mu)\, LM \,|\, C_{p(q)r} \,|\, n_1' n_2' : (\lambda\mu)\, LM\rangle = d^{\lambda/2}_{mm'}(\varphi), \tag{14}$$

where on the right-hand side we have an ordinary Wigner D function[16] [functions of a symmetric top with $m = (n_1 - n_2)/2$, $m' = (n_1' - n_2')/2$], and φ is the angle of kinematic rotation. Using Eqs. (11) and (12), we can express the matrix of any transition from one Jacobi tree to another in the oscillator basis (8) as a sum of products of several generalized Talmi–Moshinsky coefficients. It must be borne in mind that before the matrix element of any operation C is calculated the angular momenta must be regrouped in such a way that the angular momenta corresponding to Jacobi coordinates involved in the transformation C are added as in (12). Therefore, the general expression will contain, besides Talmi–Moshinsky coefficients and phase factors, the requisite number of $3nj$ symbols as well.

Thus, we have seen how the oscillator functions (8) transform under rearrangement of Jacobi trees. Such operations must be performed to construct a wave function with given permutational symmetry. However, this question will be discussed below, and we now turn to describing the structure of the wave functions in the method of K harmonics.

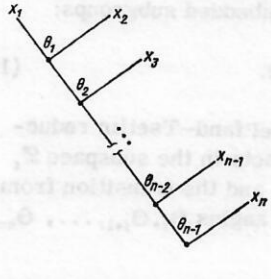
2. STRUCTURE AND PROPERTIES OF HYPERSPHERICAL HARMONICS

Hyperspherical Coordinates and Trees. The distinguishing feature of the method of K harmonics is that spherical coordinates are introduced into the $3(A-1)$-dimensional space of Jacobi coordinates: the hyperradius ρ and hyperspherical angles ($3A-4$ of them); the wave function Ψ of the nucleus is expanded with respect to standard functions of these angles—hyperspherical harmonics. Since for what follows it is important to know the properties of these harmonics, we recall the structure of these functions, which was investigated in detail in Refs. 17 and 18.

Suppose give Cartesian coordinates in an n-dimensional space $x_1, x_2, \ldots, x_n$. The hyperspherical angles $\Theta_1, \Theta_2, \ldots, \Theta_{n-1}$ can be chosen in such a way that the Cartesian coordinates are related to the hyperspherical by

$$\left.\begin{aligned} x_1 &= \rho \sin\Theta_{n-1} \ldots \sin\Theta_2 \sin\Theta_1, \quad 0 \leqslant \rho \leqslant \infty; \\ x_2 &= \rho \sin\Theta_{n-1} \ldots \sin\Theta_2 \cos\Theta_1, \quad 0 \leqslant \Theta_1 < 2\pi; \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_{n-1} &= \rho \sin\Theta_{n-1} \cos\Theta_{n-2}; \\ x_n &= \rho \cos\Theta_{n-1}; \\ \rho^2 &= \sum_{i=1}^{n} x_i^2. \end{aligned}\right\} \tag{15}$$

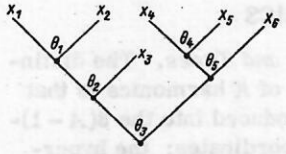
A set of hyperspherical coordinates satisfying the relations (15) can be represented graphically in the form of the canonical hyperspherical tree:



(VIII)

Other choices of the hyperspherical angles can be made. Each can be represented by its own hyperspherical tree. In the general case, the free branches of the hyperspherical tree correspond to the Cartesian coordinates of the point in the n-dimensional space. The nodes correspond to the hyperspherical angles Θ_i. The internal lines can be interpreted as the hyper-radius in the subspace spanned by the axes x_i whose numbers appear at the free ends "growing" out of this internal line.

In order to write down the expression for the Cartesian coordinate x_i in terms of the hyperspherical angles, it is necessary to follow the path from the lowest node of the hyperspherical tree to the branch x_i and if, as one moves from a certain node Θ_k, it is necessary to go to the left then the factor $\sin\Theta_k$ is added to the expression relating x_i and ρ; but if one must go to the right, the factor $\cos\Theta_k$ is added. If two branches come out of Θ_k upward, then the angle Θ_k varies in the range $0 \leqslant \Theta_k < 2\pi$; if two internal lines come out of it, then $0 \leqslant \Theta_k \leqslant \pi/2$; if one of the lines is internal and the other free, $0 \leqslant \Theta_k \leqslant \pi$. Thus, the tree



(IX)

stands for a set of hyperspherical angles $\Theta_1 \ldots \Theta_5$ in which the Cartesian coordinates are related to the hyperspherical by

$$\left.\begin{aligned} x_1 &= \rho\sin\Theta_3\sin\Theta_2\sin\Theta_1;\\ x_2 &= \rho\sin\Theta_3\sin\Theta_2\cos\Theta_1;\\ x_3 &= \rho\sin\Theta_3\cos\Theta_2;\\ x_4 &= \rho\cos\Theta_3\sin\Theta_5\sin\Theta_4;\\ x_5 &= \rho\cos\Theta_3\sin\Theta_5\cos\Theta_4;\\ x_6 &= \rho\cos\Theta_3\cos\Theta_5, \end{aligned}\right\} \tag{16}$$

$$0 \leqslant \Theta_1;\ \Theta_4 \leqslant 2\pi;\ 0 \leqslant \Theta_2;\ \Theta_5 \leqslant \pi;\ 0 \leqslant \Theta_3 \leqslant \pi/2.$$

We now turn to the canonical set of hyperspherical angles defined by the relations (15). The set of all rotations in the space $\mathcal{L}_n$ with coordinates $x_1, x_2, \ldots, x_n$ forms the group of n-dimensional rotations $O(n)$. If we restrict ourselves to rotations in the subspace $\mathcal{L}_{n-1}$ with coordinates $x_1, \ldots, x_{n-1}$, we go over to the subgroup $O(n-1)$. The transition from $\mathcal{L}_n$ to $\mathcal{L}_{n-1}$ entails making the angle Θ_{n-1} in (15) equal to zero. The vector parallel to the n-th axis has coordinate $\Theta_{n-1} = 0$. It is invariant under the group $O(n-1)$. In this connection, the canonical hyperspherical tree (VIII) can be given a definite group-theoretical meaning: This tree corresponds to reduction of the group $O(n)$ in accordance with the following chain of successively embedded subgroups:

$$O(n) \supset O(n-1) \supset \ldots O(3) \supset O(2), \tag{17}$$

which is called the canonical Gel'fand–Tsetlin reduction. In (17), each group $O(i)$ acts in the subspace $\mathcal{L}_i$ with coordinates $x_1, x_2, \ldots, x_i$ and the transition from $\mathcal{L}_n$ to $\mathcal{L}_i$ is made by setting the angles $\Theta_i, \Theta_{i+1}, \ldots, \Theta_{n-1}$ equal to zero.

Similarly, a certain chain of subgroups can be put into correspondence with a hyperspherical tree of general form. To select this chain, it is only necessary to bear in mind that the vanishing of the angle Θ_k corresponding to the node at which two internal lines converge from which grow m and n free $\mathcal{L}$ branches, respectively, corresponds to a transition from the space $\mathcal{L}_{m+n}$ of $m \oplus n$ dimensions to the space that is the direct sum of subspaces $\mathcal{L}_m \oplus \mathcal{L}_n$. Therefore, the vanishing of the angle Θ_k leads to the reduction $O_{m+n} \supset O_m \times O_n$. This remark makes it clear that the tree in (V) corresponds to the reduction

$$\begin{array}{ccc} O(6) \supset O(3) & \times & O(3) \\ \cup & & \cup \\ O(2) & & O(2) \end{array}, \tag{18}$$

where the first group $O(3)$ acts in the subspace with coordinates x_1, x_2, x_3, and the second in the subspace with coordinates x_4, x_5, x_6. The angle Θ_3 connects the radius vectors ρ_1 and ρ_2 in these subspaces to the hyper-radius ρ:

$$\rho_1 = \rho\sin\Theta_3,\ \rho_2 = \rho\cos\Theta_3,\quad 0 \leqslant \Theta_3 \leqslant \pi/2; \tag{19}$$

the remaining angles Θ_1, Θ_2, Θ_3, Θ_4 are ordinary spherical angles in the given three-dimensional subspaces.

Volume Element and the Laplacian (Canonical Tree). If one uses the canonical set of hyperspherical coordinates (15), the volume element dV in the n-dimensional space has the form

$$dV = dx_1\,dx_2 \ldots dx_n = \rho^{n-1}\,d\rho\,d\Omega, \tag{20}$$

where the element of solid angle $d\Omega$ is equal to

$$d\Omega = \sin^{n-2}\Theta_{n-1}\sin^{n-3}\Theta_{n-2} \ldots \sin\Theta_2\,d\Theta_{n-1}\,d\Theta_{n-2} \ldots d\Theta_1. \tag{21}$$

The total solid angle can be readily calculated by integrating (21) over all angles Θ_i within the limits given in (15). This yields

$$\Omega = \int d\Omega = 2\pi^{n/2}/\Gamma(n/2). \tag{22}$$

The Laplacian is defined by

$$\Delta_n = \sum_n \frac{\partial^2}{\partial x_n^2} = \frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\Delta_{\Omega_n}, \tag{23}$$

where the angular part of the n-dimensional Laplacian satisfies the recursion relation

$$\Delta_{\Omega_n} = \frac{1}{\sin^{n-2}\Theta_{n-1}}\frac{\partial}{\partial\Theta_n}\left(\sin^{n-2}\Theta_{n-1}\frac{\partial}{\partial\Theta_{n-1}}\right) + \frac{1}{\sin^2\theta_{n-1}}\Delta_{\Omega_{n-1}}. \tag{24}$$

Hyperspherical Harmonics (Canonical Tree). Hyperspherical harmonics are eigenfunctions of the angular part Δ_{Ω_n} of the Laplacian that are regular on the n-dimensional sphere. The eigenvalues of the Laplacian are equal to $-l_n(l_n+n-2)$, where l_n are integral nonnegative numbers, and they are highly degenerate. To distinguish degenerate hyperspherical harmonics with the same n-dimensional "hyper angular momentum" l_n from one another, one must require that these functions be eigenfunctions of all the Laplacians Δ_{Ω_i} $(i = 2, \ldots, n-1)$ in the spaces $\mathcal{L}_i$ of lower dimension with coordinates $x_1, x_2, \ldots, x_i$. Then the hyperspherical harmonics will be characterized by not only the hyper angular momentum l_n but also by the hyper angular momenta $l_2, \ldots, l_{n-1}$ in the spaces of lower dimensions. We denote such a hyperspherical harmonic by the symbol $Y_{LM}(\Theta_1, \ldots, \Theta_{n-1})$, where $L = l_n$, and M replaces the set of hyper angular momenta $l_{n-1} \ldots l_2$. We then obtained

$$\left.\begin{aligned}&\Delta_{\Omega_n} Y_{LM}(\Theta_q) = -l_n(l_n+n-2)\,Y_{LM}(\Theta_q);\\&\Delta_{\Omega_i} Y_{LM}(\Theta_q) = -l_i(l_i+i-2)\,Y_{LM}(\Theta_q);\\&-i\frac{\partial}{\partial\Theta_2} Y_{LM}(\Theta_q) = l_2 Y_{LM}(\Theta_q).\end{aligned}\right\} \tag{25}$$

The set of hyper angular momenta M is sufficient for complete and unambiguous labeling of degenerate hyperspherical harmonics. The following values are allowed:

$$l_n \geqslant l_{n-1} \geqslant l_{n-2} \geqslant \ldots \geqslant l_3 \geqslant |l_2|; \tag{26}$$

l_2 may be negative and takes integral values satisfying

$$-l_3 \leqslant l_2 \leqslant l_3. \tag{27}$$

The corresponding hyperspherical function has the analytic form

$$Y_{LM}(\Theta_q) = NC^{(n-2)/2+l_{n-1}}_{l_n-l_{n-1}}(\cos\Theta_{n-1})\sin^{l_{n-1}}\Theta_{n-1} \times C^{(n-3)/2+l_{n-2}}_{l_{n-1}-l_{n-2}}(\cos\Theta_{n-2})\sin^{l_{n-2}}\Theta_{n-2}-\ldots C^{1/2+l_2}_{l_3-|l_2|}(\cos\Theta_2)\sin^{|l_2|}\Theta_2 \begin{cases}\exp(il_2\Theta_1), & l_2>0;\\ (-1)^{l_2}\exp(il_2\Theta_1), & l_2<0,\end{cases}$$

where

$$N^2 = \frac{1}{2\pi}\prod_{j=0}^{n-3}\frac{2^{2l_{n-j-1}+n-j-4}\,(l_{n-j}-l_{n-j-1})!\,(n-j+2l_{n-j}-2)\,\Gamma^2\left(\frac{n-j-2}{2}+l_{n-j-1}\right)}{\Gamma(l_{n-j}+l_{n-j-1}+n-j-2)\,\pi}, \tag{28}$$

$$C^p_m(t) = \frac{2^m\Gamma(p+m)}{m!\,\Gamma(p)}\sum_{k=0}^{[m/2]}(-1)^k\frac{m!\,(p+m-1-k)!}{(m-2k)!\,2^{2k}k!\,(p+m-1)!}t^{m-2k}. \tag{29}$$

In Eq. (29) for the Gegenbauer polynomials $C^p_m(t)$ the symbol $[m/2]$ denotes the integral part of the number $m/2$. In the case $n=3$, Eq. (28) goes over rigorously into the well known expression for the ordinary spherical functions

$$Y_{lm}(\Theta,\varphi) = \sqrt{\frac{(l-m)!\,(2l+1)}{(l+m)!\,4\pi}} \times\sum_{s=0}(-1)^s\frac{(2l-2s-1)!!\,(\cos\Theta)^{l-m-2s}}{2^s s!\,(l-m-2s)!}\sin^m\Theta\exp(im\varphi),\quad m\geqslant 0. \tag{30}$$

The harmonics (28) are orthonormal:

$$\int d\Omega Y_{LM}(\Theta_q)\,Y^*_{L'M'}(\Theta_q) = \delta_{LL'}\delta_{MM'};\quad \delta_{MM'}\equiv\delta_{l_{n-1}l'_{n-1}}\ldots\delta_{l_2l'_2}. \tag{31}$$

It follows from the relations (25) that the hyper angular momentum l_n characterizes the irreducible representation $\mathcal{T}_{l_n}$ $O(n)$ with respect to which the hyperspherical function $Y_{LM}(\Theta_q)$ transforms. This representation belongs to the so-called irreducible representations of class I with respect to the subgroup $O(n-1)$.

The hyper angular momenta l_i $(i=2,\ldots,n-1)$ characterize irreducible representations of the subgroups $O(i)$ in the chain (17) to which the corresponding hyperspherical harmonics belong. The hyperspherical harmonics Y_{LM} form the canonical Gel'fand–Tsetlin basis for the irreducible representation $\mathcal{T}^{l(n)}$ of class I of $O(n)$ (Ref. 19). According to Ref. 15, it is designated by the signature $(l_n O,\ldots,O)$ (there must be altogether $[n/2]$ numbers in the parentheses). Therefore, the matrix elements of operators of infinitesimally small rotations must be found from the functions (28) in accordance with the Gel'fand–Tsetlin formulas.[19,20] However, we shall not require them in what follows and we shall not calculate them. We mention only certain properties of hyperspherical harmonics similar to those of ordinary spherical harmonics.

Some Properties of Hyperspherical Harmonics. In (28), we set $\Theta_{n-1}=0$. Then $Y_{LM}(\Theta_{n-1}=0)$ is nonvanishing only for $l_{n-1}=0$ and does not depend on the remaining hyperspherical angles. At the same time, in accordance with (26), we have $l_{n-2}=l_{n-3}=\ldots=l_2=0$ and $Y_{LM}(\Theta_{n-1}=0)$ does not depend on the angles Θ_{n-2}, Θ_{n-3}, $\ldots$, Θ_2, Θ_1, which can also be assumed equal to zero. As a result, we obtain the relation

$$Y_{LM}(0,0) = \delta_{M0}\sqrt{\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(n+l_n-2)\,(2l_n+n-2)}{2\pi^{n/2}l_n!\,\Gamma(n-1)}};\qquad \delta_{M0}\equiv\delta_{l_{n-1}0}\delta_{l_{n-2}0}\ldots\delta_{l_20}, \tag{32}$$

which is the analog of the well known property of three-dimensional spherical functions expressed by

$$Y_{lm}(0,0) = \delta_{m0}\sqrt{(2l+1)/4\pi}. \tag{33}$$

The vanishing of l_{n-1}, $l_{n-2},\ldots,l_2$ for $\Theta_{n-1}=0$ is perfectly natural since, going over to $\Theta_{n-1}=0$, we obtain a function that depends only on the Cartesian variable x_n and is invariant with respect to the subgroups $O(n-1)$, $O(n-2),\ldots,O(2)$ of the chain (17), i.e., corresponds to zero hyper angular momenta in the spaces $\mathcal{X}_i$ with coordinates $x_1, x_2,\ldots,x_i$ $(i=1,2,\ldots,n-1)$.

It is also easy to prove the relation

$$Y_{L_1M_1}(\Theta)\,Y_{L_2M_2}(\Theta) = \sum_{L=|L_1-L_2|}^{L_1+L_2}(L_1M_1L_2M_2|LM)\,(L_10L_20|L0)\,Y_{LM}(\Theta) \times\sqrt{\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(n+L_1-2)\,\Gamma(n+L_2-2)\,(2L_1+n-2)\,(2L_2+n-2)\,L!}{2\pi^{n/2}L_1!\,L_2!\,\Gamma(n-1)\,(2L+n-2)\,\Gamma(n+L-2)}}, \tag{34}$$

which enables one to decompose the product of two hyperspherical harmonics of the same arguments with respect to hyperspherical harmonics of the same arguments. In (34), $(L_1M_1L_2|LM)$ are Clebsch–Gordan coefficients for the groups O_n.[1)] Note that the direct product $\mathcal{D}^{L_1}\times\mathcal{D}^{L_2}$ of irreducible representations of O_n contains not only the irreducible representations $\mathcal{D}^L$ of class I but also more general representations. However, the presence of the coefficient $(L_10L_20|L0)$ ensures that the sum on the right-hand side of (34) in fact contains only the irreducible representations $\mathcal{D}^L$ of class I[2)] and, in addition, as in the analogous rela-

[1)] The formulas for calculating these coefficients were obtained in Ref. 21.

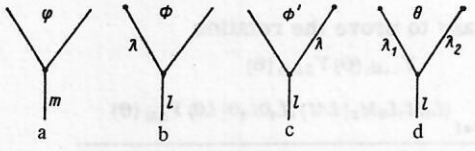
[2)] We recall that an irreducible representation of a group G is said to be a representation of the class I with respect to the subgroup H if the space of the representation D has a vector invariant under H. It can be seen from the meaning of the coefficient $(L_10L_20/L0)$ that the irreducible representation $\mathcal{D}^L$ must be such that it contains a vector with $M=0$, i.e., a vector invariant with respect to $O(n-1)$. Therefore, the given Clebsch–Gordan coefficient is nonzero only for representations $\mathcal{T}^L$ of class I with respect to the subgroup $O(n-1)$.

tion for ordinary spherical functions, we must have

$$L_1+L_2+L=\text{ even number}, \tag{35}$$

which can be readily understood by comparing the parities of the two sides of (34). The parity of the hyperspherical harmonic Y_{LM} is $(-1)^L$. This last fact becomes obvious if, instead of hyperspherical harmonics, one considers the harmonic polynomials $J_{LM}(x_1x_2\ldots x_n)=\rho^L Y_{LM}(\Theta_q)$, It is easy to see that $J_{LM}(x_1\ldots x_n)$ are homogeneous polynomials of degree L which are solutions of Laplace's equation.

Hyperspherical Harmonics (Noncanonical Trees). We now formulate rules for writing down hyperspherical harmonics for an arbitrary hyperspherical tree. In order to specify hyperspherical harmonics graphically, it is necessary to associate each internal branch of the hyperspherical tree with a certain hyper angular momentum l_i, which labels an irreducible representation of the subgroup of rotations in the space of variables x_i, $x_j,\ldots,$ x_k corresponding to the free branches growing out of branch l_i. A general n-dimensional hyperspherical function is a product of $n-1$ factors, each of which depends on one of the hyperspherical angles. Since a node of the hyperspherical tree corresponds to each hyperspherical angle, it is sufficient to say which expressions must be associated with each type of node. In a hyperspherical tree general form, one can find four types of node or fork, which are represented in the following graphs:



(X)

Let us consider each type separately.[17,22]

1. *Node with two free ends* (a). An angle $0\leqslant\varphi<2\pi$ is associated with such a node. This fork makes a contribution $d\Omega_a=d\varphi$ to the expression for the element $d\Omega$ of the total solid angle. The dependence of the hyperspherical harmonic on this angle has the form

$$J^a_m(\varphi)=(1/\sqrt{2\pi})\exp(im\varphi),\quad m=0,\ \pm1,\ \pm2,\ \ldots, \tag{36}$$

where m specifies the irreducible representation of $SO(2)$ that acts on the space of variables x_i, x_j corresponding to the two free ends coming out of the node.

We may mention here, for reasons which will become clear later, that we shall sometimes find it more convenient to use, not the complex quantities (36), but the real expressions

$$(1/\sqrt{\pi})\cos m\varphi,\ (1/\sqrt{\pi})\sin m\varphi,\quad m=0,\ 1,\ 2,\ \ldots \tag{36a}$$

A diagram of the form (*Xa*) is associated with these expressions, but the plus sign is placed next to the node for cosine dependence and the minus sign for sine dependence.

2. *Node with one free end* (b and c). To it, there corresponds the angle $0\leqslant\Phi\leqslant\pi$ $(0\leqslant\Phi'\leqslant\pi)$. To the element of total solid angle, this angle makes the contribution $d\Omega_b=\sin^{p-2}\Phi\ d\Phi$, and in the hyperspherical harmonic the dependence on this angle has a form analoggous to (28):

$$J^b_{l\lambda}(\Phi)=N^b_{l\lambda}C^{(p-2)/2+\lambda}_{l-\lambda}\cos\Phi\sin^\lambda\Phi, \tag{37}$$

where

$$N^b_{l\lambda}=\left[\frac{2^{2\lambda+p-4}(l-\lambda)!\,(2l+p-2)\,\Gamma^2(\lambda+(p-2)/2)}{\pi\Gamma(l+\lambda+p-2)}\right]^{1/2}.$$

Here, p is the number of free branches connected to the node under consideration. The index l specifies the irreducible representation of $SO(p)$ which acts in the space of variables corresponding to the free lines connected to the given node. The index $\lambda=l,\ l-1,\ l-2,\ 1,$ or 0 labels the irreducible representation of $SO(p-1)$ acting on the space of variables corresponding to the free lines connected to the internal line λ. Using the connection between the Gegenbauer and Jacobi polynomials,[23] we can rewrite (37) in the form[22]

$$\left.\begin{aligned}&J^b_{l\lambda}(\Phi)=\overline{N}^b_{l\lambda}\sin^\lambda\Theta\,P^{\lambda+(p-3)/2,\ \lambda+(p-3)/2}_{l-\lambda}\cos\Theta;\\&\overline{N}^b_{l\lambda}=\left[\frac{\Gamma(l+\lambda+p-2)\,(2l+p-2)\,(l-\lambda)!}{\pi2^{2\lambda+p-2}\Gamma(l+(p-1)/2)\Gamma(l+(p-1)/2)}\right]^{1/2}.\end{aligned}\right\} \tag{38}$$

The transition from the fork of graph b to graph c is obviously made by the trivial substitution $\Phi'=\pi/2-\Phi$. However, for completeness, we give the resulting expression:

$$J^b_{l\lambda}(\Phi')=N^b_{l\lambda}\cos^\lambda\Phi'(-1)^{l-\lambda}C^{\lambda+(p-2)/2}_{l-\lambda}(\sin\Phi'). \tag{39}$$

In (38) and below, we use the standard definition of the Jacobi polynomials[17,23]:

$$\begin{aligned}P^{\alpha\beta}_k(Z)&=\frac{(-1)^k}{2^k k!}(1-Z)^{-\alpha}(1+Z)^{-\beta}\frac{d^k}{dZ^k}\left[(1-Z)^{\alpha+k}(1+Z)^{\beta+k}\right]\\&=\frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}F\left(-k,\ k+\alpha+\beta+1,\ \alpha+1;\ \frac{1-Z}{2}\right)\\&=\frac{(-1)^k\Gamma(k+\beta+1)}{k!\,\Gamma(\beta+1)}F\left(-k+\alpha+\beta+1,\ \beta+1,\ \frac{1+Z}{2}\right).\end{aligned} \tag{40}$$

3. *Node without free branches* (d). To it corresponds the angle $\Theta\,(0\leqslant\Theta\leqslant\pi/2)$. This angle makes a contribution to the element of total solid angle equal to $d\Omega_\Gamma=\sin^{p-1}\Theta\cos^{q-1}\Theta d\Theta$, where p is the number of free branches connected to the line λ_1 and q is the number connected to λ_2. The corresponding part of the hyperspherical harmonic has the form

$$J^\Gamma_{l\lambda_1\lambda_2}(\Theta)=N^\Gamma_{l\lambda_1\lambda_2}\sin^{\lambda_1}\Theta\cos^{\lambda_2}\Theta P^{\lambda_1+\frac{p}{2}-1,\ \lambda_2+\frac{q}{2}-1}_{\frac{1}{2}(l-\lambda_1-\lambda_2)}(\cos2\Theta), \tag{41}$$

where

$$N^\Gamma_{l\lambda_1\lambda_2}=\left[\frac{(2l+p+q-2)\,\Gamma\left(\frac{\lambda_1+\lambda_2+l+p+q-2}{2}\right)\left(\frac{l-\lambda_1-\lambda_2}{2}\right)!}{\Gamma\left(\frac{l+\lambda_1-\lambda_2+p}{2}\right)\Gamma\left(\frac{l+\lambda_2-\lambda_1+q}{2}\right)}\right].$$

The allowed values of $\lambda_1,\ \lambda_2=0,\ 1,\ 2,\ldots$ for fixed l are determined by the relation

$$l-\lambda_1-\lambda_2=\text{ even numbers}. \tag{42}$$

As an illustration, let us give the explicit expression

for the hyperspherical harmonic corresponding to the tree represented by the graph (IX):

$$Y_{ll_1m_1l_2m_2}(\Theta_1\Theta_2\ldots\Theta_5) = \sqrt{\frac{(2l+4)\left(\frac{l-l_1-l_2}{2}\right)!\,\Gamma\left(\frac{l_1+l_2+l+4}{2}\right)!}{\Gamma\left(\frac{l+l_1-l_2+3}{2}\right)\Gamma\left(\frac{l+l_2-l_1+3}{2}\right)!}} \times (\sin\Theta_3)^{l_1}(\cos\Theta_3)^{l_2}P^{l_1+1/2,l_2+1/2}_{(l-l_1-l_2)/2}(\cos 2\Theta_3)\,Y_{l_1m_1}(\Theta_2\Theta_1) \times Y_{l_2m_2}(\Theta_5\Theta_4), \quad (43)$$

where $Y_{l_im_i}(\Theta_j\Theta_k)$ are ordinary three-dimensional spherical harmonics. We call a function of the form (43), which is made up of a product of ordinary spherical functions, a polyspherical function. In what follows, we shall have to deal with spaces of dimension $3A-3$, and it will therefore be convenient to use hyperspherical trees in which all the free branches are combined into triplets (these will be three Cartesian components of the corresponding Jacobi coordinates). The corresponding hyperspherical harmonic will be a polyspherical function because it is made up like (27) from ordinary spherical harmonics corresponding to each Jacobi coordinate. We shall say that trees of such structure are polyspherical. Thus, on the transition from one set of Jacobi coordinates to another, we must deal with a transformation of one polyspherical tree into another. However, we shall not restrict ourselves to this case but consider in general form the question of transformation of hyperspherical harmonics for an arbitrary hyperspherical tree on the transition to any other tree.

"Timber" Coefficients. Suppose there is an n-dimensional hyperspherical harmonic

$$J_{k_1k_2\ldots k_{n-2}}(\Theta_1\ldots\Theta_{n-1})\mathscr{Z}_\Gamma, \quad (44)$$

that corresponds to some hyperspherical tree $\mathscr{Z}^\Gamma$ and in which k is the total hyper angular momentum and $k_1k_2\ldots k_{n-2}$ are the hyper angular momenta in the subspaces of the n-dimensional space with basis $x_1, x_2, \ldots, x_n$. We consider the transition G from the tree $\mathscr{Z}^\Gamma$ to a different hyperspherical tree $\mathscr{Z}^{\Gamma'}$. Then the hyperspherical functions for the two trees are related to one another as follows:

$$J_{kk_1'k_2'\ldots k_{n-2}'}(\Theta_i)\mathscr{D}_{\Gamma'} = \sum_{k_1k_2\ldots k_{n-2}} \langle k_1k_2\ldots k_{n-2}:k\mathscr{D}_\Gamma \,|\, k_1'k_2'\ldots k_{n-2}':k\mathscr{D}^{\Gamma'}\rangle \times J_{kk_1k_2\ldots k_{n-2}}(\Theta_1)\mathscr{Z}_\Gamma. \quad (45)$$

The coefficients of the transition from one set of harmonics to the other:

$$\langle k_1k_2\ldots k_{n-2}:k\mathscr{D}_\Gamma \,|\, k_1'k_2'\ldots k_{n-2}':k\mathscr{D}^{\Gamma'}\rangle \quad (46)$$

will be called "timber" coefficients. The operation of this transition G reduces to a product of the elementary transformations P and C. Therefore, the matrix of the transition (46) will also be a product of the matrices of the individual transitions C and P.

Let us consider the action of the reflection P on the function (44). If the reflection takes place at a node at which two free branches converge (see (Xa)), the operation P corresponds to the substitution $\varphi \to \pi/2-\varphi$:

$$PJ^a_m(\varphi) = J^{a'}_m(\varphi) = \exp(im\pi/2)\,J^{(a)}_{-m}(\varphi). \quad (47)$$

A node of the type (Xb) goes over under the reflection P into a node of type (Xc) and vice versa:

$$PJ^b_{l\lambda}(\Phi) = J^c_{l\lambda}(\Phi) = (-1)^{l-\lambda}J^b_{l\lambda}(\Phi); \quad (48)$$

$$PJ^c_{l\lambda}(\Phi) = J^b_{l\lambda}(\Phi) = (-1)^{l-\lambda}J^c_{l\lambda}(\Phi). \quad (49)$$

Finally, for the case (Xd)

$$PJ^d_{l\lambda_1\lambda_2}(\Theta) = J^d_{l\lambda_2\lambda_1}(\Theta) = (-1)^{l-\lambda_1-\lambda_2}J^d_{l\lambda_1\lambda_2}(\Theta). \quad (50)$$

Thus, the coefficients of transformation of the hyperspherical harmonics under tree rearrangements of the type of reflections P can be found trivially. Using the explicit form of the hyperspherical harmonics, we can also obtain the coefficients for the operations of branch swapping $C_{a(b)c}$. This was done in Ref. 22 for all variants of swappings. However, subsequent analysis showed[24,25] that these coefficients are identical to the $6j$ symbols for the noncompact group $SU(1,1)$. Therefore, in this review we shall consider only this later elegant treatment of the timber coefficients (see Sec. 4).

For exposition of this approach, it is necessary to know the connection between oscillator functions in Jacobi coordinates and the same functions in multidimensional spherical coordinates.

3. OSCILLATOR FUNCTIONS IN JACOBI COORDINATES AND IN HYPERSPHERICAL COORDINATES

Solution of the Schrödinger Equation for an n-Dimensional Oscillator in Hyperspherical Coordinates. The Hamiltonian of the translationally invariant shell model for A particles in Jacobi coordinates has the form

$$H = \sum_{i=1}^{A-1}\left[-\frac{\hbar^2}{2m}\Delta_{x_i}+\frac{m}{2}\omega^2x_i^2\right], \quad (51)$$

and its eigenfunctions are products of three-dimensional oscillator functions of the individual Jacobi coordinates x_i. In the $(3A-3)$-dimensional space of Jacobi coordinates, one can introduce hyperspherical coordinates: the hyper-radius

$$\rho^2 = \sum_{i=1}^{A-1}x_i^2 = \sum_{i=1}^{A}(r_i-R)^2$$

(r_i are the coordinates of individual nucleons; R is the center of mass of the nucleus) and the hyperspherical angles Θ_j $(j=1,\ldots,3A-4)$. Then the Hamiltonian (51) is rewritten as follows:

$$H = -\frac{\hbar^2}{2m}\left[\frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\left(\rho^{n-1}\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\Delta_{\Theta_i}\right]+\frac{1}{2}m\omega^2\rho^2, \quad (51a)$$

where $n=3A-3$, and Δ_{Θ_i} is the angular part of the Laplacian. The energy eigenvalues are given by

$$E_N = (2\varkappa+K+n/2)\,\hbar\omega \quad (52)$$

($2\varkappa + K = N$ is the total number of quanta), and the eigenfunctions have the form

$$\Psi_{NK\nu} = R_{\varkappa K}(\rho)\, Y_{K\nu}(\Theta_i)_{\mathscr{D}_\Gamma}, \tag{53}$$

where $Y_{K\nu}(\Theta_i)$ is some hyperspherical harmonic with total hyper angular momentum K corresponding to the chosen hyperspherical tree $\mathscr{D}_\Gamma$; ν is the set of the remaining hyper angular momenta that characterize the state:

$$R_{\varkappa K}(\rho) = \sqrt{\frac{2\varkappa!}{\Gamma(\varkappa + K + n/2)}}\, e^{-(1/2)\rho^2} \rho^K L_\varkappa^{K+(1/2)(n-2)}(\rho^2). \tag{54}$$

Here, ρ is given in units of $r_0^2 = \hbar/m\omega$ and the associated Laguerre polynomials are defined in the usual manner[24]:

$$L_n^\alpha(x) = \sum_{k=0}^{n} (-1)^k \frac{(n+\alpha)!}{(n-k)!\,(\alpha+k)!\,k!} x^k. \tag{55}$$

It is well known[10] that a set of oscillator functions with fixed number N of quanta forms the basis of a symmetric irreducible representation with Young tableau $[N]$ of the group $U(n) = U(3A-3)$ whose generators are the operators

$$A_{i\alpha,\, j\beta} = a_{i\alpha}^+ a_{j\beta}, \quad i,\ j = 1,\ 2 \ldots A-1, \quad \alpha, \beta = x,\ y,\ z, \tag{56}$$

where $a_{i\alpha}^+$ and $a_{i\alpha}$ are the operators of creation and annihilation of quanta with respect to the α-th Cartesian component of the Jacobi coordinate x_i. It is clear that wave functions of the form (53) with fixed NK and all possible ν belong to the irreducible representation of $SO(n) = SO(3A-3)$ with hyper angular momentum K.

Therefore, the set of functions (53) corresponds to the reduction $SU(3A-3) \supset SO(3A-3)$. We shall be interested in the following question: How are the functions (53) and the oscillator functions in the Jacobi coordinates (8) related? To answer this question, we solve an auxiliary problem. We split the space with basis $x_1, x_2, \ldots, x_n$ into two subspaces of dimension n_1 and n_2 ($n_1 + n_2 = n$) with bases $x_1 x_2 \ldots x_{n_1}$ and $x_{n_1+1} \ldots x_n$, repectively. We introduce hyperspherical coordinates $\rho_1 \Theta_i^1$ and $\rho_2 \Theta_j^2$ in these subspaces and construct oscillator functions $\Psi_{N_1 K_1 \nu_1}$, $\Psi_{N_2 K_2 \nu_2}$ of the type (53) in each of these subspaces. We express the functions (53) of ρ and Θ in terms of functions of $\rho_1 \Theta_i^1$ and $\rho_2 \Theta_j^1$:

$$\Psi_{NK\nu} = \sum_{N_1 N_2} \langle N_1 K_1 N_2 K_2 | NK \rangle\, \Psi_{N_1 K_1 \nu_1} \Psi_{N_2 K_2 \nu_2}. \tag{57}$$

Here, it is assumed that the spherical functions $Y_{K\nu}$ on the left-hand side is chosen in accordance with the reduction $SO(n) \supset SO(n_1) \times SO(n-n_1)$, i.e., it coincides with the function (43). Therefore, the coefficients of the transition (57) do not depend on the quantum numbers ν, ν_1, ν_2 and

$$\sum_{N_1 + N_2 = N} \langle N_1 K_1 N_2 K_2 | NK \rangle\, R_{N_1 K_1}(\rho_1)\, R_{N_2 K_2}(\rho_2) = R_{NK}(\rho)\, N_{KK_1K_2}^2 \sin^{K_1}\Theta \cos^{K_2}\Theta P_{(K-K_1-K_2)/2}^{(K_1+n_1/2-1),\,(K_2+n_2/2-1)}(\cos 2\Theta), \tag{58}$$

where

$$\rho_2 = \rho\cos\Theta, \quad \rho_1 = \rho\sin\Theta.$$

It can be shown that the expansion coefficients in (57) can be related to the Clebsch–Gordan coefficients for the group $Sp(2, R)$ or its locally isomorphic group $SU(1, 1)$ (Refs. 25 and 26). As a result, the "timber" coefficients are proportional to the $3nj$ symbols for this group, which will contain moments that are multiples of 1/4. This connection between corresponding quantities for the two different groups $SO(n)$ and $Sp(2, R)$ is based on the fact that these two groups are complementary in the sense of the definition proposed by Moshinsky and Quesne.[27]

Complementary Groups. Suppose there is a direct product $H_1 \times H_2$ of two groups H_1 and H_2 and we are given a linear space $\mathscr{L}$ in which a representation (in general, reducible) of $H_1 \times H_2$ is realized. We shall say that H_1 and H_2 are complementary groups within the space $\mathscr{L}$ if there exists a one-to-one correspondence between all irreducible representations of the groups H_1 and H_2 contained in D, i.e., in the direct sum

$$D = \sum_i D^{j_i \lambda_i} \tag{59}$$

every irreducible representation D^{j_i} of H_1 is encountered in combination with only one definite irreducible representation D^{λ_i} of H_2 (and vice versa) and every such combination D^{j_i}, D^{λ_i} is encountered in the decomposition of D not more than once.

It is known that the noninvariance group of a system of p particles in a q-dimensional oscillator is the group $Sp(2pq; R)$ of canonical transformations whose generators are the operators $a_{i\alpha}^+ a_{j\beta}^+$, $(1/2)(a_{i\alpha}^+ a_{j\beta} + a_{j\beta} a_{i\alpha}^+)$, $a_{i\alpha} a_{j\beta}$ ($i = 1, 2, \ldots, p$; $\alpha = 1, 2, \ldots, q$). The set of all states with even number of quanta $N = 0, 2, \ldots, \infty$ forms a basis of an infinite-dimensional representation with lowest weight $[1/4\ 1/4 \ldots 1/4] \equiv [1/4^{pq}]$ of this noncompact group. All odd states of the oscillator occur in a different representation with weight $[3/4\ 1/4 \ldots 1/4] \equiv [3/4\ 1/4^{pq-1}]$. Moshinsky and Quesne[27] showed that within the spaces of these representations $Sp(2q, R)$ and $O(p)$ are complementary groups. In what follows, we shall be interested in the case $Sp(2n, R) \to Sp(2, R) \times O(n)$. For this case, the case I representation D^K of $O(n)$ with global angular momentum K combines with a definite representation D^J of the group $Sp(2, R)$, which is locally isomorphic to the group $SU(1, 1)$. If K is odd, then the "angular momentum" J in the notation adopted for $SU(1, 1)$ is related as follows to the global angular momentum K:

$$J = (1/2)(K + n/2) - 1. \tag{60}$$

This means that the oscillator function for a state with number N of quanta and global angular momentum K having the form (52) belongs to an irreducible representation of $Sp(2, R)$ with angular momentum $J = (1/2) \times (K-2) + n/4$ and is characterized by its projection $M = (1/2)N + n/4$. Thus, in this problem we encounter two-valued unitary irreducible representations of $SU(1, 1)$ from the positive discrete series.[17] Note that

with every function of an orthogonal scheme and, therefore, with the corresponding hyperspherical harmonic, one can naturally associate an angular momentum which is a multiple of [1/4]. Ultimately, this has the consequence that the "timber" coefficients are proportional to nj symbols with quarter-integral angular momenta.

"Timber" Coefficients and j Symbols for the Sp(2, R). Let us consider an n-dimensional harmonic oscillator in a space with basis $x_1, x_2, \ldots, x_n$. Its Hamiltonian is equal to the sum of the oscillator Hamiltonians for each of these variables, and the wave function is constructed from products of wave functions of n one-dimensional oscillators. We recall that in accordance with Ref. 27 all eigenfunctions of a one dimensional oscillator of positive parity belong to a single irreducible representation of the group Sp(2, R) of canonical transformations with angular momentum $j = -3/4$, and the set of functions of negative parity form the basis of another irreducible representation of this group with angular momentum $j = -1/4$. The group Sp(2, R) is locally isomorphic to $SU(1,1)$ and is specified by the generators

$$J_+ = a^+a^+, \; J_- = -aa; \; J_0 = (1/2)(a^+a + aa^+), \tag{61}$$

which satisfy the usual commutation relations for angular momentum operators:

$$[J_0, J_\pm] = \pm J_\pm; \; [J_+, J_-] = 2J_0, \tag{62}$$

but have different properties under Hermitian conjugation:

$$(J_0)^+ = J_0; \quad (J_\pm)^+ = -J_\mp. \tag{63}$$

We shall be interested in the positive discrete series of unitary irreducible representations of this group. These representations are specified by the positive number j [the eigenvalue of the operator $J^2 = J_+J_- + J_0^2 - J_0$ is $j(j+1)$]. The basis vectors Ψ_{jm} of the representation D^j are labeled by the projections m of j, and $m = j+1, j+2, \ldots$. The lowest vector with minimal projection satisfies the condition

$$J_-\Psi_{jj+1} = 0; \tag{64}$$

$$J^2\Psi_{jj+1} = j(j+1)\Psi_{jj+1}. \tag{65}$$

Since the minimal value of the projection m for even states of a one-dimensional oscillator is 1/4, we have $j = -3/4$; for odd states, $m_{\min} = 3/4$ and $j = -1/4$. We may mention that, having in mind mirror symmetry $j \to -j-1$ (Ref. 7), we may assume that even states are also characterized by angular momentum $j = -1/4'$. We have added the prime to distinguish it from the inequivalent representation having odd states as basis.

We now take the products of oscillator functions depending on the variables x_1 and x_2 and form the vector composition of the corresponding angular momenta j_1 and j_2 equal to $-3/4$ or $-1/4$, respectively:

$$|j_1j_2 : j_{12}m_{12}\rangle = \sum_{m_1+m_2=m_{12}} (j_1m_1j_2m_2 \,|\, j_{12}m_{12}) \,|\, j_1m_1(x_1)\rangle \,|\, j_2m_2(x_2)\rangle. \tag{66}$$

We recall that the rule of vector addition for the angular momenta considered gives the following allowed values for the total angular momentum:

$$j_{12} = j_1 + j_2 + 1, \quad j_1 + j_2 + 2 \ldots \tag{67}$$

The function (66) describes the state of a two-dimensional oscillator with $2m_{12} - 1$ quanta. In addition, in accordance with the fact that Sp(2, R) and $O(2)$ are complementary, it belongs to a definite representation D^Λ of $O(2)$. The quantum number Λ specifies the modulus of the rotational angular momentum in the plane $x_1x_2\ldots$. Therefore, the function (66) can be put in correspondence with the wave function of two-dimensional oscillator in cylindrical coordinates:

$$\rho^2 = x_1^2 + x_2^2; \quad \tan\varphi = x_1/x_2;$$

$$|j_1j_2 : j_{12}m_{12}\rangle \rightleftarrows |n_\perp\Lambda\rangle_\pm = \frac{1}{\sqrt{\pi}} R_{n_\perp\Lambda}(\rho) \begin{cases} \cos\Lambda\varphi, \\ \sin\Lambda\varphi. \end{cases} \tag{68}$$

Here, $n_\perp = 2m_{12} - 1$; $\Lambda = 2j_{12} + 1$; $j_1 = -3/4$ and $j_2 = -3/4$ correspond to even values of Λ and an angular function of the type $\cos\Lambda\varphi$ (subscript +); $j_1 = -3/4$ and $j_2 = -1/4$ correspond to odd Λ and an angular function of the type $\sin\Lambda\varphi$; for $j_1 = -1/4$ and $j_2 = -3/4$ we have odd Λ and a dependence of the type $\cos\Lambda\varphi$ on the angle φ; finally, the case $j_1 = -1/4$, $j_2 = -1/4$, leads to even Λ and an angular function of the type $\sin\Lambda\varphi$. By virtue of this correspondence, we can regard (66) as a decomposition of the wave function of a two-dimensional oscillator in cylindrical coordinates in terms of functions in Cartesian coordinates:

$$|n_1(x_1)\rangle\,|n_2(x_2)\rangle = |n_\perp\Lambda\rangle_\pm = \sum_{n_1+n_2=n_\perp} \langle n_\perp\Lambda \,|\, n_1n_2\rangle \,|\, n_1(x_1)\rangle\,|\, n_2(x_2)\rangle. \tag{69}$$

If we use for the radial oscillator function the standard expression

$$R_{n_\perp\lambda}(\rho) = \sqrt{2(n_\perp/2 + \Lambda/2)!\,(n_\perp/2 - \Lambda/2)!} \times \sum_{b=|\Lambda|,\,|\Lambda|+2,\,\ldots,\,n_\perp} (-1)^{\frac{1}{2}(b-|\Lambda|)} \exp(-\rho^2/2) \times \frac{\rho^b}{(b/2 - \Lambda/2)!\,(b/2 + \Lambda/2)!\,(n_\perp/2 - b/2)!},$$

then for the matrix of the transition from cylindrical to Cartesian coordinates we obtain the expression

$$\langle n_\perp\Lambda \,|\, n_1n_2\rangle = \langle j_1m_1j_2m_2 \,|\, jm\rangle \begin{cases} (-1)^{n_\perp/2}, & \text{if } n_\perp \text{ is even;} \\ (-1)^{n_\perp/2 - 1/2}, & \text{if } n_\perp \text{ is odd.} \end{cases} \tag{70}$$

The connection between the parameters on the two sides of this equation was given above.

Adding the angular momentum j_{12} to the angular momentum $j_3 = -3/4$ or $-1/4$ corresponding to the coordinate x_3, we obtain the wave function $|j_1j_2(j_{12}), j_3 : j_{123}\, m_{123}\rangle$ which corresponds to solution of the Schrödinger equation for a three-dimensional oscillator in spherical coordinates: $|nl\Lambda\rangle_\pm = R_{nl}(r)\, Y^\pm_{l\Lambda}(\Theta, \varphi)$. The number of quanta is $n = 2(m_{123} - 3/4)$, $l = 2j_{123} + 1/2$. Note that here and in what follows we use real spherical harmonics:

$$Y^{\pm}_{l\Lambda}(\Theta,\varphi)=\begin{Bmatrix}(1/\sqrt{2})\,[Y_{l\Lambda}(\Theta,\varphi)(-1)^m+Y_{l-\Lambda}(\Theta,\varphi)]\\(1/\sqrt{2})\,[Y_{l\Lambda}(\Theta,\varphi)(-1)^m-Y_{l-\Lambda}(\Theta,\varphi)]\end{Bmatrix}$$
$$=\frac{1}{\sqrt{\pi}}\Theta_{l\Lambda}(\Theta)\begin{cases}\cos\Lambda\varphi;\\ \sin\Lambda\varphi,\end{cases}\qquad(71)$$

where $Y_{l_1\pm\mu}$ are the ordinary spherical harmonics.

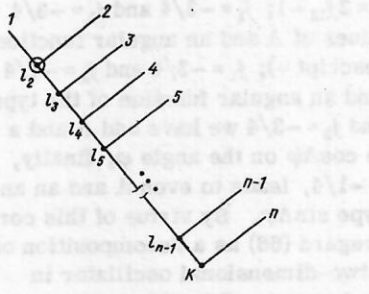
Adding to j_{123} successively $j_4, j_5, \ldots, j_n$, we obtain a function of the form

$$|j_1j_2(j_{12}),\ j_3(j_{123}),\ j_4\ldots j_{n-1}(j_{12}\ldots j_{n-1});\ j_n:JM\rangle,$$

which corresponds to solution of the oscillator Hamiltonian in n-dimensional spherical coordinates:

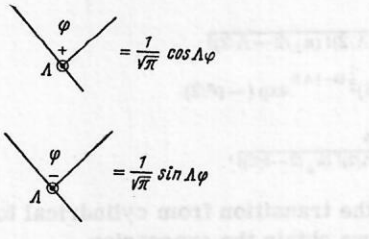
$$|j_1j_2(j_{12}),\ j_3\ldots j_n:JM\rangle=\Psi_{NK\nu}=R_{NK}(\rho)\,Y_{K\nu}(\Theta_1\Theta_2\ldots\Theta_{n-1}),\qquad(72)$$

where $\rho^2=x_1^2+x_2^2+\ldots+x_n^2$; $Y_{K\nu}$ is the hyperspherical harmonic corresponding to the tree[25,26] shown in the following graph.



(XI)

ν is the set of hyper angular momenta $l_1, l_2, \ldots, l_{n-1}$ in spaces of dimension $2, 3, \ldots, n-1$. These hyper angular momenta are indicated next to the nodes of the tree, and K is the global angular momentum in the n-dimensional space. Note that, in contrast to Refs. 22 and 26, we use real spherical harmonics. Therefore, at the tree fork with two free ends shown in the graph



(XII)

it is necessary to specify the sign + or −.

The tree in graph (XI) can be regarded from two points of view: On the one hand, it characterizes a selection of hyperspherical angles; on the other hand, it specifies the scheme for adding angular momenta of the type −3/4, −1/4. If one does not use sequential addition but some other scheme, it will correspond to a different choice of hyperspherical angles. It is clear from this that the scalar product of two n-dimensional oscillator functions $\Psi_{NK\nu}$ and $\Psi_{NK\nu'}$ with fixed N and K for two variants ν and ν' of the choice of the hyperspherical coordinates coincides with a certain j symbol for the group Sp(2,R), this corresponding to the transition from the scheme ν for coupling the angular momenta $j_1j_2\ldots j_n$ to the coupling scheme ν':

$$\langle\Psi_{NK\nu}|\Psi_{NK\nu'}\rangle\equiv\langle Y_{K\nu}|Y_{K\nu'}\rangle=\langle(j_1j_2\ldots j_n)j_n)_\nu\|(j_1j_2\ldots j_n)_{\nu'}\rangle.\qquad(73)$$

Thus, we have shown that the "timber" coefficient $\langle Y_{K\nu}|Y_{K\nu'}\rangle$ characterizing the unitary transformation of the hyperspherical harmonics on the transition from one tree to another is equal to the j symbol for the group Sp(2,R) containing angular momenta that are multiples of 1/4.

One can show that these j symbols are analytic continuations of the ordinary j symbols known in the quantum theory of angular momentum. In particular, the Clebsch–Gordan coefficients can be calculated in accordance with the formula

$$\langle j_1m_1j_2m_2|JM\rangle$$
$$=(-1)^{M-J-1}\sqrt{\frac{(2J+1)\,\Gamma(j_1+m_1+1)\,\Gamma(j_2+m_2+1)\,\Gamma(J-j_1-j_2)}{\Gamma(J+M+1)\,\Gamma(M-J)\,\Gamma(m_1-j_1)}}$$
$$\times\sqrt{\frac{\Gamma(J+j_2-j_1-1)\,\Gamma(m_2-j_2)}{\Gamma(J+j_1-j_2+1)\,\Gamma(J+j_1+j_2+2)}}$$
$$\times\sum_{r=M-J-1}\frac{(-1)^r\,\Gamma(r+1)\,\Gamma(M+J-r)}{\Gamma(r+J-M+2)\,\Gamma(r+j_1-m_1+2)\,\Gamma(j_2-j_1+M-r)\,\Gamma(M-r-j_1-j_2-1)}.\qquad(74)$$

If the resulting expression is compared with the expression for the Clebsch–Gordan coefficients of $SU(2)$ given in Ref. 28, it can be seen that (74) is obtained from the latter by the replacement of each j by $-j-1$ and m by $-m$, i.e.,

$$\langle j_1m_1j_2m_2|JM\rangle_{SU(2)}\to\langle j_1m_1j_2m_2|JM\rangle_{\mathrm{Sp}(2,R)},\qquad(75)$$

if $j_1\to-j_1-1;\ j_2\to-j_2-1;\ J\to-J-1;$

$m_1\to-m_1;\ m_2\to-m_2;\ M\to-M.$

The transition from the Clebsch–Gordan coefficients of $SU(2)$ to the corresponding quantities for Sp(2,R) is similar to the operation of reflection in the theory of ordinary angular momentum (Ref. 24).[3)]

The Clebsch–Gordan coefficients of the group Sp(2,R) satisfy the following symmetry relation:

$$\langle j_1m_1j_2m_2|JM\rangle=(-1)^{j_1+j_2-J-1}\langle j_2m_2j_1m_1|JM\rangle.\qquad(76)$$

The numerical values of the Racah coefficients can be found by means of the formula[24]

$$U(j_1j_2Jj_3;\ J_{12}J_{23})=(-1)^{J_{12}-j_2-J+J_{23}}$$
$$\times\sqrt{\Gamma(J_{12}-j_2+j_1+1)\,\Gamma(J_{12}+j_1+j_2+2)\,\Gamma(J_{23}+j_2-j_3+1)\times\to}$$
$$\to\times\,\Gamma(J_{23}+j_2+j_3+2)$$
$$\times\sqrt{\frac{(2J_{12}+1)(2J_{23}+1)\,\Gamma(J-J_{23}-j_1)\,\Gamma(j_1+J-J_{23}+1)\,\Gamma(j_2+J_{12}-j_1+1)}{\Gamma(J+J_{12}-j_3+1)\,\Gamma(J+J_{12}+j_2+2)\,\Gamma(J+J_{23}-j_1+1)\,\Gamma(J+J_{23}-j_1+2)}}\to$$
$$\to\times\,\Gamma(J_{12}-j_1-j_2)$$

[3)] Note that although the expressions for the Clebsch–Gordan coefficients of $SU(2)$ and Sp(2,R) are similar, it must be borne in mind that these coefficients contain completely different angular momenta j_1, j_2, J_1, since these triplets of angular momenta in the case of the groups $SU(2)$ and Sp(2,R) satisfy absolutely different rules of vector composition [see (65)]. But one can still say that the Clebsch–Gordan coefficients for Sp(2,R) containing only integral and half-integral angular momenta have values numerically equal to certain coefficients of $SU(2)$. Explicit relations of this kind are obtained in Ref. 29.

$$\times\sqrt{\Gamma(j_3+J-J_{12}+1)\,\Gamma(J-J_{12}-j_3)\,\Gamma(J_{23}-j_2-j_3)\,\Gamma(j_3-j_2+J_{23}+1)}$$
$$\times\frac{\sum_{r=0}(-1)^r\,\Gamma(2J-r+1)}{\Gamma(r+1)\,\Gamma(j_1+J-J_{23}+1-r)\,\Gamma(j_3+J-J_{12}+1-r)\,\Gamma(J-j_3-J_{12}-r)}\to,$$
$$\to\times\Gamma(J_{23}-J+j_2+2+J_{12}+r)\,\Gamma(J-J_{23}-j_1-r)\,\Gamma(J_{23}-j_2-J+J_{12}+1+r) \qquad (77)$$

which can again be obtained from the general expression for the Racah coefficient of $SU(2)$ if all j are replaced by $-j-1$ in Eq. (77).

4. TRANSFORMATION OF OSCILLATOR FUNCTIONS IN HYPERSPHERICAL COORDINATES ON THE TRANSITION FROM ONE SET OF JACOBI COORDINATES TO ANOTHER

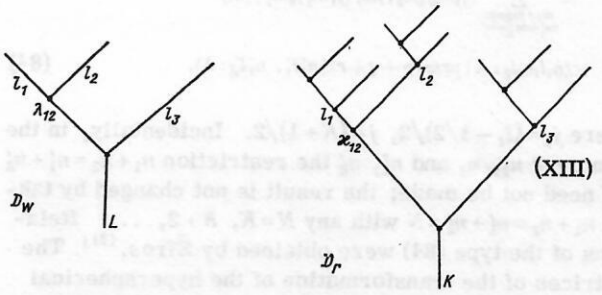
Above, we have considered the transformation of oscillator functions on the transition from one set of Jacobi coordinates to another. An analogous problem also arises in the method of K harmonics. To solve it, we must operate with oscillator functions in multidimensional spherical coordinates. We shall use polyspherical coordinates chosen in such a way that the total hyperspherical function $Y^{\mathscr{D}_\Gamma,\ \varkappa_i}_{Kl_1,\ l_2\dots l_{A-1}}$ is constructed from spherical functions with angular momenta l_1, $l_2,\dots,\ l_{A-1}$ for the individual Jacobi coordinates x_1, $x_2,\dots,\ x_{A-1}$ in accordance with the rules formulated in Sec. 2. The symbol $\mathscr{D}_\Gamma$ specifies the shape of the hyperspherical tree that defines the construction of the hyperspherical harmonics; $\varkappa_i$ are the intermediate hyper angular momenta of the given tree. It is expedient to add the angular momenta $l_1,\ l_2,\dots,\ l_{A-1}$ in some way to form the total angular momenta L. In this connection, we consider the Wigner tree $\mathscr{D}_W$, which specifies the scheme for adding the angular momenta and the intermediate angular momenta λ_i. We denote the hyperspherical function obtained at the end by $Y^{\mathscr{D}_\Gamma\varkappa_i,\mathscr{D}_W\lambda_i}_{KLM,\ l_1l_2\dots l_{A-1}}$. Then the oscillator function with N quanta in these hyperspherical coordinates has the form

$$|l_1l_2\dots l_{A-1},\ \mathscr{D}_\Gamma\varkappa_i,\ \mathscr{D}_W\lambda_i : NKLM\rangle = R_{NK}(\rho)\,Y^{\mathscr{D}_\Gamma\varkappa_i\mathscr{D}_W\lambda_i}_{KLM,\ l_1l_2\dots l_{A-1}}. \qquad (78)$$

In the preceding section, we have noted that the construction of the oscillator function in multidimensional spherical coordinates from a given hyperspherical tree $\mathscr{D}_\Gamma$ can be regarded as addition of $A-1$ angular momenta of the form $j_i=(l_i-1/2)/2$ by means of the Clebsch–Gordan coefficients for $Sp(2,R)$ or $SU(1,1)$ to the total angular momentum $[K+(3A-7)/2]/2$ with intermediate angular momenta j_i whose values are determined by the intermediate hyper angular momenta $\varkappa_i$. For brevity, we shall call this operation simply addition of the angular momenta $l_1,\ l_2,\dots,\ l_{A-1}$ to the total angular momentum K in the sense of the group $Sp(2,R)$ with intermediate angular momenta $\varkappa_i$. Further, to be specific we shall assume that the scheme for adding the angular momenta l_i in the sense of $Sp(2,R)$ is the same as for the ordinary rotation group, i.e., the trees $\mathscr{D}_\Gamma$ and $\mathscr{D}_W$ have the same structure. We denote a function (78) of this type by

$$|l_1l_2\dots l_{A-1};\ {}^{\lambda_i}_{\varkappa_i} : NKLM\rangle. \qquad (79)$$

Such a function can be readily expressed in terms of the wave functions of a three-dimensional oscillator for each of the Jacobi coordinates. For example, if $A=4$ and the trees $\mathscr{D}_W$ and $\mathscr{D}_\Gamma$ shown in the following graph are chosen,



(XIII)

then the function (79) has the form

$$|l_1l_2l_3;\ {}^{l_{12}}_{\varkappa_{12}} : NKLM\rangle = \sum_{\substack{M_{12}N_{12}\\ m_1m_2n_1n_2m_3n_3}} \langle n_1l_1n_2l_2\,|\,N_{12}\varkappa_{12}\rangle\,\langle l_1m_1l_2m_2\,|\,l_{12}m_{12}\rangle \times\langle N_{12}\varkappa_{12}n_3l_3\,|\,NK\rangle\,\langle l_{12}m_{12}l_3m_3\,|\,LM\rangle\,|\,n_1l_1m_1(x_1)\rangle \times|\,n_2l_2m_2(x_2)\rangle\,|\,n_3l_3m_3(x_3)\rangle. \qquad (80)$$

The functions for more complicated systems are constructed similarly: There is a parallelism between addition of the angular momenta $l_1,\ l_2,\dots,\ l_{A-1}$ with respect to the rotation group and with respect to $Sp(2,R)$.

We now discuss the transition from certain Jacobi coordinates $x_1,\ x_2,\dots,\ x_{A-1}$ to others $x_1',\dots,\ x_{A-1}'$ in the functions (79). The operation of reflection P of a coordinate x_i is performed trivially:

$$P=(-1)^{l_i}. \qquad (81)$$

Let us consider the elementary kinematic rotation (10) through the angle φ; it relates the two Jacobi coordinates shown in (VII):

$$|l_1l_2;\ NKLM(x_{pq},\ x_{pq,\,r})\rangle = \sum_{l_1'l_2'}\langle l_1l_2\,|\,l_1'l_2'\rangle^{K\Lambda}_{\varphi}\,|\,l_1'l_2' : NKLM(x_{qr},\ x_{p,\,qr})\rangle. \qquad (82)$$

The coefficients of the transformation (82) were introduced in Ref. 30, and are called Raynal–Revai coefficients. Since the radius vector

$$\rho=[x_{pq}^2+x_{pq,\,r}^2]^{1/2}$$

does not change under the transformation (10), the transformation (82) essentially affects only the angular parts of the oscillator functions—the hyperspherical harmonics:

$$Y_{l_1l_2K\Lambda M}(\bar{x}_{pq},\ \bar{x}_{pq,\,r}) = \sum_{l_1'l_2'}\langle l_1l_2\,|\,l_1'l_2'\rangle^{K\Lambda}_{\varphi}\,Y_{l_1'l_2'K\Lambda M}(\bar{x}_{qr},\ \bar{x}_{p,\,qr}). \qquad (83)$$

Using the relations (82) and (11), we can readily find the connection between the Raynal–Revai coefficients and the Talmi–Moshinsky coefficients (12):

$$\langle l_1 l_2 | l_1' l_2' \rangle_\varphi^{K\Lambda}$$

$$= \sum_{\substack{n_1+n_2=K \\ n_1'+n_2'=K}} \langle n_1 l_1 n_2 l_2 | K\Lambda \rangle \langle n_1' l_1' n_2' l_2' | K\Lambda \rangle$$

$$\times \langle n_1 l_1 n_2 l_2 : \Lambda | pr/q\,(p+q+r) | n_1' l_1',\ n_2' l_2' : \Lambda \rangle$$

$$= \sum_{\substack{m_1+m_2=m \\ m_1'+m_2'=m'}} (j_1 m_1 j_2 m_2 | jm)\,(j_1' m_1' j_2' m_2' | jm)$$

$$\times \langle n_1 l_1 n_2 l_2 : \Lambda | pr/q\,(p+q+r) | n_1' l_1',\ n_2' l_2' : \Lambda \rangle, \qquad (84)$$

where $j_i = (l_i - 1/2)/2$, $j = (K+1)/2$. Incidentally, in the sum over n_1, n_2 and n_1', n_2' the restriction $n_1 + n_2 = n_1' + n_2' = K$ need not be made; the result is not changed by taking $n_1 + n_2 = n_1' + n_2' = N$ with any $N = K,\ K+2, \ldots$. Relations of the type (84) were obtained by Efros.[31] The matrices of the transformation of the hyperspherical functions on the transition from one set of Jacobi coordinates to another are in the general case sums of products of certain Raynal–Revai coefficients. These matrices can also be related to the Talmi coefficients[32] or simple Raynal–Revai coefficients.

It now becomes clear how we must obtain the matrices for transforming the hyperspherical harmonics on the transition from one set of Jacobi coordinates to another if we know the expression for the matrix of the transformation of oscillator functions depending on the same Jacobi coordinates. It is as follows: In the expression known it is necessary to replace all the Talmi coefficients by the corresponding Raynal–Revai coefficients and every Racah coefficient or higher symbol for angular momenta $l_1 l_2 \ldots$ must be doubled with the corresponding nj symbol for Sp(2R) and to each phase $(-1)^{-L_i j + l_i + l_j}$ it is necessary to add the phase $(-1)^{-J_{ij}+j_i+j_j+1}$. Below, we shall use this rule to obtain the coefficients of fractional parentage in the method of K harmonics.

5. RECURSIVE METHOD FOR ANTISYMMETRIZING K HARMONICS

Fractional-Parentage Coefficients in the Method of K Harmonics. In the preceding sections, we have considered how we can construct hyperspherical harmonics with given orbital angular momentum L and hyper angular momentum K for a particular set of Jacobi coordinates. We must now find out how to combine these harmonics with spin–isospin wave functions into a complete antisymmetric wave function. For this, it is first of all necessary to construct hyperspherical harmonics $Y_{KLM[f](r)}$ that have definite permutational symmetry, i.e., transforms in accordance with the irreducible representation $\mathscr{D}^{[f]}$ with the Young tableau $[f]$ of the group of permutations S_A of A particles in accordance with the row (r) of this representation ((r) is a Yamanouchi symbol; see, for example, Ref. 33). If this is done, then the total antisymmetric function Ψ is constructed from $Y_{KLM[f](r)}$ and the spin–isospin functions $\chi_{ST[\tilde{f}](\tilde{r})}$ with the conjugate Young tableau and Yamanouchi symbol in accordance with the well-known formula

$$\Psi = \frac{1}{\sqrt{n_f}} \sum_r Y_{KLM[f](r)} \chi_{ST[\tilde{f}](\tilde{r})}, \qquad (85)$$

where n_f is the dimension of the irreducible representation $\mathscr{D}^{[f]}$ of S_A. In their turn, harmonics with definite permutational symmetry can be constructed from harmonics of the type $Y^{\mathscr{D}_\Gamma \times_i \mathscr{D}_W \lambda_i}_{l_1 l_2 \ldots l_{A-1} KLM}$ by means of the Young operators

$$C^{[f]}_{(r)(r')} = \frac{n_f}{A!} \sum_P \langle [f](r) | P | [f](r) \rangle\, P, \qquad (86)$$

where the sum is over all permutations P of S_A; $\langle [f](r) | P | [f](r) \rangle$ is the matrix element of P in the standard orthogonal irreducible Young–Yamanouchi representation.[33]

The Young operator has the property that, applied to an arbitrary function Φ, it transforms it into a function with the permutational symmetry $[f](r)$:

$$\Psi_{[f](r)} = \frac{1}{N} C^{[f]}_{(r)(r')} \Phi, \qquad (87)$$

with a normalization coefficient given by

$$N^2 = \langle \Phi | C^{[f]}_{(r')(r')} | \Phi \rangle. \qquad (88)$$

Rewriting this formula for hyperspherical harmonics, we obtain

$$Y_{KLM[f](r)\varepsilon} = \frac{1}{N} C^{[f]}_{(r)(r')} Y^{\mathscr{D}_\Gamma \times_i \mathscr{D}_W \lambda_i}_{l_1 l_2 \ldots l_{A-1} KLM}. \qquad (89)$$

Here, the index ε distinguishes different harmonics with identical sets of quantum numbers $KLM[f](r)$. The concrete form of the harmonic (89) depends on the choice of the auxiliary Yamanouchi symbol (r') and the quantum numbers $l_1,\ l_2,\ \ldots,\ l_{A-1} \mathscr{D}_{\Gamma \times_i} \mathscr{D}_W \lambda_i$ of the "unrenormalized" harmonic on the right-hand side of (89). If the harmonics (89) are decomposed with respect to the harmonics $Y^{\mathscr{D}_\Gamma \times_i \mathscr{D}_W \lambda_i}_{l_1 l_2 \cdots l_{A-1} KLM}$, we find that

$$\left.\begin{aligned} Y_{KLM[f](r)\varepsilon} = \frac{1}{N} \sum_{\substack{l_1' l_2' \ldots l_{A-1}' \varkappa_i' \lambda_i' \\ l_1' l_2' \ldots l_{A-1}' \varkappa_i' \lambda_i'}} & \langle l_1' l_2' \ldots l_{A-1}' \varkappa_i' \lambda_i' : KL | C^{[f]}_{(r')(r)} | ; \\ | l_1 \ldots l_{A-1}^{\varkappa_i} \lambda_i : KL \rangle & Y^{\mathscr{D}_{l \varkappa_i'} \mathscr{D}_W \lambda_i'}_{l_1' l_2' \ldots l_{A-1}' KLM}. \end{aligned}\right\} \qquad (90)$$

The expansion coefficients in (90) reduce to the matrix elements of permutations of the particles, i.e., to the matrix elements of operations of transition from one set of Jacobi coordinates to another. The calculation of these quantities was considered in the previous section. Therefore, Eq. (90) in principle solves the problem of constructing hyperspherical harmonics with definite symmetry. However, from the practical point of view these calculations are rather cumbersome for large values of A and K. It is more convenient to have a recursive procedure to construct harmonics with given permutational symmetry. The problem is formulated as follows: Suppose we know hyperspherical harmonics with permutational symmetry $Y_{K'L'M'[f'](r')\varepsilon'}$ for $A-1$ particles and they must be combined with spherical harmonics $Y_{l_{A-1} m_{A-1}}$ for the degree of freedom

$$x_{A-1} = \sqrt{(A-1)/A}\left(r_A - \frac{1}{A-1}\sum_{i=1}^{A-1} r_i\right)$$

to make a total hyperspherical harmonic with fixed permutational symmetry.

We now form products of the oscillator function $|N'K'L'M'[f'](r')\varepsilon'\rangle$ for $A-1$ particles and the oscillator function for the last Jacobi coordinate x_{A-1} and apply a Young operator to it. As a result, we obtain a certain oscillator function of A particles with given permutational symmetry:

$$|NKLM[f](r)\varepsilon\rangle = \frac{1}{N} C^{[f]}_{(r)(r')} \sum_{N'n} \langle N'K'nl\,|\,NK\rangle\,|\,N'K'L'\varepsilon'[f'](r'),\ nl:LM\rangle. \qquad (91)$$

To make the total function correspond to hyper angular momentum K, we have added the angular momenta with respect to the group $Sp(2R)$.[4)]

We decompose the function (91) with respect to products of the functions of $A-1$ particles and the function of the motion of the last particle with respect to the center of mass of the group of $A-1$ particles:

$$|ANKLM[f](r)\varepsilon\rangle = \sum_{N''K''n'l'\varepsilon''L''} \langle AK[f]L\varepsilon\,|\,A-1K''[f'']L''\varepsilon'',\ l'\rangle \times \langle N''K''n'l'\,|\,NK\rangle\,|\,N''K''L'',\ \varepsilon''[f'](r'),\ n'l':LM\rangle. \qquad (92)$$

The expansion coefficients on the right-hand side of (92) are called *fractional-parentage coefficients* for separating one particle in the orthogonal sum.[33] If they are known, then the problem of constructing symmetrized functions for A particles from symmetrized functions of $A-1$ particles can be completely solved. To calculate these coefficients, we can use the formulas obtained in the translationally invariant shell model, modifying them slightly in accordance with the method set forth in the preceding section. Namely, Eqs. (7.23)–(7.25) of Ref. 10, in which the fractional-parentage coefficients $\langle A|A-1\rangle$ are expressed in terms of $\langle A-1|A-2\rangle$, are rewritten as follows:

$$\langle AK[f]L\varepsilon\,|\,A-1K_1[f]L_1\varepsilon_1 l_1\rangle = \frac{1}{n_{f_1}}\sqrt{\frac{n_f n_{f'}}{A}}\frac{1}{Q}\Big\{\delta_{[f_1][f']}\delta_{L_1L'}\delta_{K_1K'}\delta_{l_1l_2} + (A-1)\sum_{\substack{K_1'[f_1']L_1'\varepsilon_1'\\ l_2'l_1'L_0K_0}} \frac{n_{f_1'}}{n_{f'}}\langle[f_1'][f_1]\,|\,P_{A-1\,A}\,|\,[f_1'][f']\rangle^{[f]} \times (-1)^{l_1'+l_1}(-1)^{j_1'-j_1+j_2-j_2'} U(L_1'l_2'Ll_1;\,L_1L_0)\,U(L_1'l_1'Ll_2;\,L'L_0) \times \langle A-1K_1[f_1]L_1\varepsilon_1\,|\,A-2K_1'[f_1']L_1'\varepsilon_1';\,l_2'\rangle \times \langle A-1K'[f']L'\varepsilon'\,|\,A-2K_1'[f_1']L_1'\varepsilon_1',\,l_1'\rangle\,U[J_1'j_2'Jj_1;\,J_1J_0] \times U[J_1'j_1'Jj_2;\,J'J_0]\,\langle l_1l_2'\,|\,l_2l_1'\rangle_\varphi^{L_0K_0}\Big\}; \qquad (93)$$

$$Q^2 = 1 + (A-1)\sum_{\substack{K_1'[f_1']L_1'\varepsilon_1'\\ l_2'l_1'L_0K_0}} \frac{n_{f_1'}}{n_{f'}}\langle[f_1'][f']\,|\,P_{A-1\,A}\,|\,[f_1'][f']\rangle^{[f]} \times \langle A-1K'[f']L'\varepsilon'\,|\,A-2K'[f_1']L_1'\varepsilon_1',\,l_2'\rangle \times \langle A-1K'[f']L'\varepsilon'\,|\,A-2K_1'[f_1']L_1'\varepsilon_1',\,l_1'\rangle \times (-1)^{l_2+l_1'}(-1)^{j_1'-j_2'}\,U(L_1'l_2'Ll_2;\,L'L_0)\,U(L_1'l_1'Ll_2;\,L'L_0) \times U[J_1'j_2'Jj_2;\,J'J_0]\,U[J_1'j_1'Jj_2;\,J'J_0]\,\langle l_2l_2'\,|\,l_2l_1'\rangle_\varphi^{L_0K_0}, \qquad (94)$$

$$\varphi = \tan^{-1}\sqrt{\frac{1}{(A-2)A}};\quad j_i = \frac{1}{2}\left(l_i - \frac{1}{2}\right);\quad J = \frac{1}{2}\left(K + \frac{3A-7}{2}\right);$$
$$J_1' = \frac{1}{2}\left(K_1' + \frac{3A-13}{2}\right);\quad J_1 = \frac{1}{2}\left(K_1 + \frac{3A-10}{2}\right);$$
$$J' = \frac{1}{2}\left(K' + \frac{3A-10}{2}\right);\quad J_0 = \frac{1}{2}(K_0+1); \qquad (95)$$

[4)] It should be noted that functions (91) with the same quantum numbers NKLM[f] obtained by symmetrizing states with different values of [f'], (r'), ε' are not necessarily orthogonal to each other and they are not even all linearly independent. Therefore, to find a complete set of states one must construct from the functions (91) the requisite number of linearly independent and orthonormalized states. This rearrangement of the functions (91) leads to a corresponding change in the coefficients of fractional parentage (92)–(96).

n_f is the dimension of the irreducible representation $[f]$ of S_A; $\langle[f_1'][f_1]|P_{A-1,A}|[f_1'][f']\rangle^{[f]}$ is the matrix element of the standard irreducible representation of the same group.

Note that $\langle A=2|\,(A-1)\equiv 1\rangle \equiv 1$. Therefore, using the recursion relations (93)–(95), we can successively calculate all the necessary fractional-parentage coefficients. They symbol $U[\ldots]$ is used to denote the Racah coefficients for Sp(2, R); the symbol $U(\ldots)$ denotes the ordinary Racah coefficients. If only the angular parts of the functions are separated in (92), then, using (58), we can rewrite the decomposition of the hyperspherical harmonics for A particles with permutational symmetry with respect to the same harmonics for $A-1$ particles:

$$Y_{KLM[f](r)\varepsilon} = \sum_{\substack{K''L''l'\varepsilon''\\ M''m'}} \langle AK[f]L\varepsilon\,|\,A-1K''[f']L''\varepsilon'',\ l'\rangle\,(L''M''l'm'\,|\,LM) \times Y_{K''L''M''[f'](r')\varepsilon''}Y_{l'm'}N^r_{KK''l'}\sin^{K''}\Theta\cos^{l'}\Theta \times P^{K''+(3A-6)/2-1;\ l'+1/2}_{K-K''-l'}(\cos 2\Theta); \qquad (96)$$

the fractional-parentage coefficients for the method of K harmonics were introduced in Refs. 34 and 35.

Coefficients of fractional parentage for separating two or more particles can be introduced similarly. In particular, the two-particle coefficients are defined as follows:

$$Y_{KLM[f](r)\varepsilon} = \sum_{\substack{[f_0]L_2\varepsilon_2K_2\\ L'K'\Lambda L_0M'M_0\mu M_2}} \langle[f](r)\,|\,[f_2](r_2),\ [f_0]\rangle \times \langle AK[f]L\varepsilon\,|\,A-2K_1[f_2]L_2\varepsilon_2,\ \Lambda\{L'K'\};\ L_0\rangle\,(L'M'L_0M_0\,|\,LM) \times (L_2M_2\Lambda\mu\,|\,L'M')\,Y_{K_2L_2M_2[f_2](r_2)\varepsilon_2}Y_{\Lambda\mu}(\mathbf{R}_{A-2}-\mathbf{R}_2) \times Y_{L_0M_0}(\mathbf{r}_{A-1}-\mathbf{r}_A)\,N_{KK'L_0}N_{K'K_2\Lambda}\sin^{K_2'}\Theta_1\cos^{L_0}\Theta_1 \times \sin^{K_2}\Theta_2\cos^{\Lambda}\Theta_2 P^{K_2'+(3A-9)/2-1;\ \Lambda+1/2}_{K'-K_2-\Lambda}(\cos 2\Theta_2) \times P^{K'+(3A-6)/2-1;\ L_0+1/2}_{K-K'-L_0}(\cos 2\Theta_1). \qquad (97)$$

The following scheme is proposed for adding the angular momenta in accordance with the rotation group and the group Sp(2, R):

$$\left.\begin{aligned} &\mathbf{L}_2+\boldsymbol{\Lambda}=\mathbf{L}';\quad \mathbf{L}'+\mathbf{L}_0=\mathbf{L};\quad \mathbf{J}_2+\mathbf{J}_\Lambda=\mathbf{J}';\quad \mathbf{J}'+\mathbf{J}_0=\mathbf{J};\\ &J_\Lambda=\frac{1}{2}(\Lambda-1/2);\quad J_0=\frac{1}{2}\left(L_0-\frac{1}{2}\right);\quad J=\frac{1}{2}\left(K+\frac{3A-7}{2}\right);\\ &J'=\frac{1}{2}\left(K'+\frac{3A-10}{2}\right);\quad J_2=\frac{1}{2}\left(K_2+\frac{3A-13}{2}\right). \end{aligned}\right\} \qquad (98)$$

Here, L_0 is the angular momentum of the relative motion of the pair of nucleons $A-1$ and A [of the degree of freedom $x'_{A-1}=(r_A-r_{A-1})/\sqrt{2}$], and Λ is the angular momentum of this pair with respect to the remaining $A-2$ particles (the degree of freedom $x'_{A-2}=[(1/2)(\bar r_{A-1}+\bar r_A)-(1/A-2)\sum_{i=1}^{A-2}r_i]\times\sqrt{2(A-2)/A}$). The two-particle coefficients of fractional parentage can be expressed in terms of the one-particle coefficients as follows[5)]:

$$\langle AK[f]L\varepsilon\,|\,A-2K_2[f_2]L_2\varepsilon_2,\ \Lambda\{L'K'\};\ L_0\rangle = \langle[f](r)\,|\,[f_2](r_2),\ [f_0](r)\rangle^{-1} \times \sum_{K_1\varepsilon_1L_1l_1l_2\lambda\varkappa}\langle AK[f]L\varepsilon\,|\,A-1K_1[f_1]L_1\varepsilon_1,\ l_1\rangle \times \langle A-1K_1[f_1]L_1\varepsilon_1\,|\,A-2K_2[f_2]L_2\varepsilon_2,\ l_2\rangle\,(-1)^{\lambda-l_1-l_2} \times (-1)^{J_\varkappa-j_1-j_2-1}\,U(L_2l_2Ll_1;\,L_1\lambda)\,U[J_2j_2Jj_1;\,J_1J_\varkappa] \times \langle l_1l_2\,|\,\Lambda L_0\rangle^{\varkappa\lambda}_\varphi\,U(L_2\Lambda LL_0;\,L'\lambda)\,U[J_2J_\Lambda JJ_0;\,J'J_\varkappa]. \qquad (99)$$

[5)] $\varphi=\tan^{-1}\sqrt{A-2/A}$, the tableau $[f_1]$, and the corresponding Yamanouchi symbols (r_2) and (r) are chosen in an arbitrary but fixed manner (see Eq. (2.43) in the book Ref. 10)).

The fractional-parentage decompositions (96) and (97) are convenient for calculating the matrix elements of operators in the method K harmonics.

6. CALCULATION OF MATRIX ELEMENTS OF OPERATORS IN THE METHOD OF K HARMONICS

Let us consider the calculation of matrix elements of the Hamiltonian in the method of K harmonics. We denote the antisymmetrized combinations of hyperspherical harmonics with spin–isospin functions $\chi^{[\tilde{f}](\tilde{r})}_{SM_S TM_T}$ by means of the symbol

$$|AK[f]\varepsilon LSTM_L M_S M_T\rangle = \frac{1}{\sqrt{n_f}} \sum_r Y_{KLM[f](r)} \varepsilon \chi^{[\tilde{f}](\tilde{r})}_{SM_S TM_T}. \quad (100)$$

Here, $[\tilde{f}](\tilde{r})$ are the Young tableau and Yamanouchi symbol conjugate to $[f](r)$; n_f is the dimension of the irreducible representation $D^{[f]}$ of the symmetric group S_A.

We introduce the total two-particle coefficients of fractional parentage for the K harmonics (100):

$$\begin{aligned}
&|AK[f]\varepsilon LSTM_L M_S M_T\rangle \\
&= \sum_{\substack{K_2[f_2]\varepsilon_2 S_2 T_2 \Lambda \\ K'L'L_0 S_0 T_0 \\ M'M_0 \\ M_{S_0} M_{T_0}}} \langle AK[f]\varepsilon LST \,|\, A-2\,[f_2]\,\varepsilon_2 L_2 S_2 T_2,\ \Lambda(L'K');\ L_0 S_0 T_0\rangle \\
&\times (L'M'L_0 M_0 \,|\, LM)\,(S_2 M_{S_2} S_0 M_{S_0} \,|\, SM_S)\,(T_2 M_{T_2} T_0 M_{T_0} \,|\, TM_T) \\
&\times N_{KK'\lambda} \sin^{K'}\Theta_1 \cos^{L_0}\Theta_1 P^{K'+[(3A-6)/2]-1}_{K-K'-L_0}(\cos 2\Theta_1) \\
&\times |A-2\,[f_2]\,\varepsilon_2 S_2 T_2 L_2 K_2,\ \Lambda(L'K')\,M'M_{S_2}M_{T_2}\rangle\,|L_0 M_0 S_0 M_{S_0} T_0 M_{T_0}\rangle.
\end{aligned} \quad (101)$$

Here, the last factor depends on the angular variables of the relative motion of particles $A-1$ and A, and the penultimate factor includes the angular variables of the relative motion of the $A-2$ particles, and also the coordinates of the radius vector joining the centers of mass of the $A-2$ particles and the separated pair of nucleons; $r_{A-1,A} = \rho\cos\Theta_1$.

The total coefficient of fractional parentage is constructed from the orbital and spin–isospin coefficients of fractional parentage:

$$\begin{aligned}
&\langle AK[f]\varepsilon LST \,|\, A-2\ K_2[f_2]\,\varepsilon_2 L_2 S_2 T_2,\ \Lambda(L'K');\ L_0 S_0 T_0\rangle \\
&= \sqrt{n_{f'}/n_f}\,\langle AK[f]\varepsilon L \,|\, A-2\ K[f_2]\,\varepsilon_2 L_2,\ \Lambda(L'K');\ L_0\rangle \\
&\times \langle (st)^A\,[\tilde{f}]\,ST \,|\, (st)^{A-2}\,[f']\,S'T',\ (st)^2\,S_0 T_0\rangle.
\end{aligned} \quad (102)$$

By virtue of the antisymmetry of the functions (100), the matrix element of the potential energy of the interaction between the nucleons:

$$V = \sum_{i<j}^{A} V(r_{ij}) \text{ and } V(r_{ij}) = f(r_{ij})\,W_{\sigma\tau} \quad (103)$$

can be written in the form

$$\begin{aligned}
&\langle AK[f]\varepsilon LSTM_L M_S M_T \,|\, \hat{V} \,|\, A\bar{K}\,[\bar{f}]\,\bar{\varepsilon}\bar{L}\bar{S}\bar{T} M_L M_S M_T\rangle \\
&= \frac{A(A-1)}{2}\,\langle AK[f]\varepsilon LST \,|\, V(r_{A-1,A}) \,|\, A\bar{K}\,[\bar{f}]\,\bar{\varepsilon}\bar{L}\bar{S}\bar{T}\rangle \\
&= \frac{A(A-1)}{2} \sum_{\substack{K_2[f_2]\varepsilon_2 L_2 \\ S_2 T_2 L_0 S_0 T_0 \\ \Lambda L'K'}} \langle AK[f]\varepsilon LST \,|\, A-2\ K_2[f_2] \\
&\times \varepsilon_2 L_2 S_2 T_2,\ \Lambda(L'K');\ L_0 S_0 T_0\rangle \\
&\times \langle A\bar{K}\,[\bar{f}]\,\bar{\varepsilon}\bar{L}\bar{S}\bar{T} \,|\, A-2\ K_2[f_2]\,\varepsilon_2 L_2 S_2 T_2,\ \Lambda(L'K');\ L_0 S_0 T_0\rangle \\
&\times \langle S_0 T_0 \,|\, W_{\sigma\tau} \,|\, S_0 T_0\rangle\, R^{K\bar{K}}_{K',L_0}(\rho).
\end{aligned} \quad (104)$$

Here

$$\begin{aligned}
R^{K\bar{K}}_{K'L_0}(\rho) &= \int d\Theta_1 (\sin\Theta_1)^{3A-7} (\cos\Theta_1)^2 N_{KK'L_0} N_{\bar{K}K'L_0} f(\rho\cos\Theta_1) \\
&\times (\sin\Theta_1)^{2K'} (\cos\Theta_1)^{2L_0} P^{K'+(3A-6)/2-1,\ L_0+1/2}_{K-K'-L_0}(\cos 2\Theta_1) \\
&\times P^{K'+(3A-6)/2-1,\ L_0+1/2}_{\bar{K}-K'-L_0}(\cos 2\Theta_1);
\end{aligned} \quad (105)$$

$\langle S_0 T_0 \,|\, W_{\sigma\tau} \,|\, S_0 T_0\rangle$ is the spin–isospin part of the matrix element of the interaction. For simplicity, we have taken only central forces. Equation (104) can be readily generalized to the case of tensor and vector forces, and also forces that depend on velocity.

If we use the fact that for $N = N_{\min}$ the wave function of the translationally invariant shell model is, on the one hand, related in a very simple way to the K harmonics ($K = N_{\min}$):

$$|AN_{\min}\varepsilon[f]\,LST\rangle = R_{N_{\min}K=N_{\min}}(\rho)\,|AK[f]\varepsilon LST\rangle; \quad (106)$$

$$R_{KK} = \sqrt{\frac{2}{\Gamma[K+(3A-3)/2]}}\,\rho^K \exp(-\rho^2/2r_0^2)\,(r_0)^{-(1/2)(2K+3A-3)}, \quad (107)$$

and, on the other, can be related simply to a shell wave function, we can considerably simplify the calculation of the matrix elements (106). For example, for nuclei of the p shell, the shell function $|s^4 p^{A-4}[f]\,LST\rangle$ can be expressed in terms of the function of the translationally invariant shell model as follows:

$$|s^4 p^{A-4}[f]\,LST\rangle = \psi_{00}(R_A)\,|AN_{\min}[f]\varepsilon LST\rangle, \quad (108)$$

where $\Psi_{00}(R_A)$ is the wave function of the zero-point vibrations of the center of mass. It is obvious that

$$\begin{aligned}
&\langle s^4 p^{A-4}[f]\,\varepsilon LST \,|\, V \,|\, s^4 p^{A-4}[\bar{f}]\,\bar{\varepsilon}\bar{S}\bar{T}\rangle \\
&= \langle AN_{\min}[f']\,\varepsilon LST \,|\, V \,|\, AN_{\min}[\bar{f}]\,\bar{\varepsilon}\bar{L}\bar{S}\bar{T}\rangle.
\end{aligned} \quad (109)$$

A matrix element between shell wave functions can be reduced by the ususal methods[10,36] to two-particle matrix elements of the type $\langle nl \,|\, V(r_{A-1,A}) n'l'\rangle$ and ultimately to Talmi integrals of the form

$$\begin{aligned}
I_K &= \int \exp(r^2/r_0^2)\, r^{K+2} V(r)\, dr / \langle AN_{\min}[f']\,\varepsilon LST \,|\, V \,|\, AN_{\min}[\bar{f}]\,\bar{\varepsilon}\bar{L}\bar{S}\bar{T}\rangle \\
&= \sum_k I_k Q_k.
\end{aligned} \quad (110)$$

If on the left-hand side of this equation we integrate, not with respect to all variables, but only with respect to the hyperspherical angles, then, taking into account (107), we obtain

$$\langle AK[f]\varepsilon LST \,|\, V \,|\, AK[f]\varepsilon LST\rangle = \sum_k I_k(\rho)\, Q_k, \quad (111)$$

where

$$I_k(\rho) = N_k \int_0^1 (1-Z)^{K-k+1/2}\, Z^{k+3/2} V(\rho\sqrt{Z})\, dZ; \quad (112)$$

$$N_k = \Gamma(K+3/2,\ A-3/2)/\Gamma(K-k+3/2)\,\Gamma(k+3/2);$$

$$\rho_{A-2} = \rho\sin\Theta;\quad r_{A-1,A} = \rho\cos\Theta;\quad Z = \cos^2\Theta;\quad 1-Z = \sin^2\Theta.$$

Thus, to calculate the matrix elements of the potential energy with respect to K harmonics with the lowest attainable hyper angular momentum $K_{\min}$ for given A we can use shell results, in which it is necessary to replace the ordinary Talmi integrals I_K (110) by the functions $I_K(\rho)$ (112). This result was obtained in Ref. 35.

In the same way, one can obtain the matrix elements of not only the potential energy but also the other operators. The result (111) can also be found from Eq. (104) by writing out explicitly the Jacobi polynomial in (105) in terms of trigonometric functions of Θ.

Above, we have considered the calculation of the matrix elements of the potential energy of nucleons. Let us say a few words about the role they play in the method of K harmonics. In this method, the wave function of nucleus A is sought in the form of an expansion with respect to K harmonics $|AK[f]\varepsilon LST\rangle$:

$$\Psi = \sum_{K \geqslant K_{\min}} \varphi_K^{K[f]\varepsilon LST}(\rho)\,|AK[f]\varepsilon LST\rangle, \qquad (113)$$

where

$$\int \varphi_K^2(\rho)\,\rho^{3A-4}\,d\rho = 1.$$

The Hamiltonian of the nucleus has the form

$$H = -\frac{\hbar^2}{2m}\frac{1}{\rho^{3A-4}}\frac{\partial}{\partial\rho}\left(\rho^{3A-4}\frac{\partial}{\partial\rho}\right) - \frac{\hbar^2}{2m\rho^2}\Delta_\Theta + V. \qquad (114)$$

If we substitute the function (113) into the Schrödinger equation and multiply the result from the left by $\langle AK[f]\times\varepsilon LST|$; we obtain a system of equations of the form

$$\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\rho^2} + \frac{\hbar^2 L_K(L_K+1)}{2m\rho^2} - E\right\}\chi_K^{K\varepsilon[f]LST}(\rho) + \sum_{\overline{K}\varepsilon\overline{LST}} W_{K\varepsilon[f]LST}^{\overline{K}\varepsilon[\overline{f}]\overline{LST}}(\rho)\,\chi_{\overline{K}[\overline{f}]\varepsilon\overline{LST}} = 0. \qquad (115)$$

Here, $L_K = K + (3A-6)/2$, $\chi_K = \rho^{(3A-4)/2}\varphi_K(\rho)$; the number of equations in this system and the number of terms in the sum (115) is equal to the number of terms included in the expansion (113).

The potentials $W_K^{\overline{K}}(\rho)$ are, as it happens, the very matrix elements that we discussed above:

$$W_{K\varepsilon[f]LST}^{\overline{K}\varepsilon[\overline{f}]\overline{LST}}(\rho) = \langle AK[f]\varepsilon LST\,|\,V\,|\,A\overline{K}[\overline{f}]\varepsilon\overline{LST}\rangle. \qquad (116)$$

Note the extremely strong "centrifugal potential" $L_K(L_K+1)/\rho^2$ which occurs in these equations and is a consequence of the Pauli principle since the large values of L_K are due to the fact that the values $K_{\min}$ compatible with the Pauli principle are already large.

We have shown above that the methods developed in the shell model can be fairly readily transferred and generalized to the method of K harmonics. The shell approach is especially effective for approximate calculations of the potentials $W_K^{\overline{K}}(\rho)$ in accordance with the Baz'–Zhukov–Surdov method.[4] This is an asymptotic method and can give accurate results only for fairly large mass numbers A. Its essence is as follows. Using Eqs. (106), (107), (109), and (116), we arrive at the result

$$\langle s^4p^{A-4}[f]\varepsilon LST\,|\,V\,|\,s^4p^{A-4}[\overline{f}]\overline{\varepsilon LST}\rangle \equiv V_{[f]\varepsilon LST}^{[\overline{f}]\varepsilon\overline{LST}}(r_0) = \text{const}\int d\rho\rho^{3A-4+2K}\exp(-\rho^2/r_0^2)\,W_{K[f]\varepsilon LST}^{K[\overline{f}]\varepsilon\overline{LST}}(\rho). \qquad (117)$$

We represent the integral on the right-hand side of this equation in the form

$$\int d\rho W_{K[f]\varepsilon LST}^{K[\overline{f}]\varepsilon\overline{LST}}(\rho)\exp(3A-4+2K)\left(\ln\rho/r_0 - \frac{\rho^2}{(3A-4+2K)\,r_0^2}\right)$$

and calculate it by Laplace's method.[37] Restricting ourselves to the lowest term in powers of $(3A-4+2K)^{-1}$, we find that

$$V_{[f]\varepsilon LST}^{[\overline{f}]\varepsilon\overline{LST}}(r_0) = W_{K[f]\varepsilon LST}^{K[\overline{f}]\varepsilon\overline{LST}}\left(r_0\sqrt{K+\frac{3A-4}{2}}\right). \qquad (118)$$

Thus, to obtain the approximate values of the potential W at the point ρ, it is sufficient to calculate the shell matrix element (117) with the following value of the shell parameter:

$$r_0 = \frac{\rho}{\sqrt{K+(3A-4)/2}}.$$

7. COMPARISON OF THE RESULTS OF CALCULATIONS IN THE TRANSLATIONALLY INVARIANT SHELL MODEL AND BY THE METHOD OF K HARMONICS FOR NUCLEI OF THE $1p$ SHELL

In the overall study of nuclear structure, the method of K harmonics has been mainly used to describe systems of three or four nucleons; for heavier nuclei, only magic nuclei have been considered. True, a calculation has been made[38] for the nucleus ^{12}C, but the computational scheme used in it also corresponds to a nucleus with closed shells in jj coupling. In Ref. 39, a broader investigation was made of the method of K harmonics in a minimal approximation in the case of nonmagic nuclei of the p shell and with the use of a single nucleon–nucleon potential.

In the same paper, a first attempt was made to study the spectra of the lowest excited states of nonmagic nuclei in the method of K harmonics. The computational method was an approximate variant of the Baz'–Zhukov–Surkov method described in the previous section. All the calculations of Ref. 39 were made in the minimal approximation when only a single term with $K = K_{\min}$ is taken into account in the expansion (113).[6)] In this approximation, the system (115) reduces to the equation

$$\left\{\frac{d^2}{d\rho^2} - \frac{L_K(L_K+1)}{\rho^2} - \frac{2m}{\hbar^2}[E + W_\gamma^\nu(\rho)]\right\}\chi_\nu(\rho) = 0. \qquad (119)$$

This equation with the potential found from (118) was solved numerically on a computer; the binding energies, wave functions, and other physical quantities such as the ground and excited states of $1p$-shell nuclei of positive and negative parity for various forms of the nucleon–nucleon potential were calculated.[41]

Results of Calculations for Ground States of $1p$-Shell Nuclei. Figure 1 shows the results of calculations of the binding energy of p-shell nuclei in the

[6)] In the literature, there are calculations which are more exact, taking into account states with global angular momentum $K_{\min}$ and $K_{\min}+2$ (Ref. 40), but such calculations are fairly complicated and have been made so far only for the single nucleus ^{16}O.

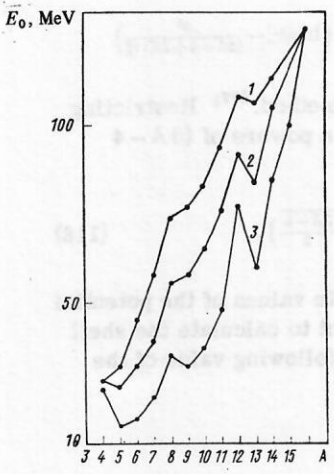


FIG. 1. Binding energies of p-shell nuclei: 1) shell-model calculations; 2) experimental values; 3) calculations in lowest approximation of the method of K harmonics.

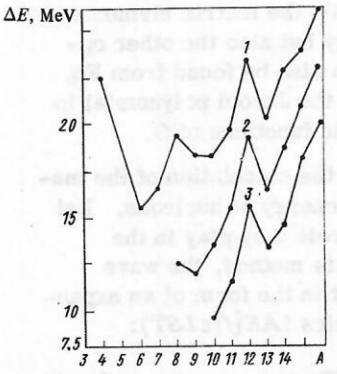


FIG. 2. Distance ΔE between monopole excitations in p-shell nuclei calculated by the method of K harmonics: 1) interval between ground state and first excited state, $E_{1K}-E_{0K}$; 2) difference of energies between first and second excited states, $E_{2K}-E_{1K}$; 3) difference of energies between second and third excited states, $E_{3K}-E_{2K}$.

method of K harmonics and compares them with the results of the shell calculations of Ref. 42 and also the experimental[43] binding energies of these nuclei.

As can be seen in Fig. 1, the binding energy of these nuclei for the chosen potential (variant No. 1 of the Volkov potential) are appreciably lower than the experimental values , except for ^{16}O. This result is not unexpected, since the calculation was made in the approximation $K=K_{min}$, and allowance for the following harmonics raises the binding energy. In addition, the absolute value of binding energy depends strongly on the choice of the nucleon–nucleon potential, as was shown in Ref. 44; this can also be seen Table II. Therefore, the discrepancy between theory and experiment may be due not only to the inadequacy of the K_{min} approximation but also to poor choice of the nucleon–nucleon potential. The shape of the calculated binding-energy curve as a function of the mass number reproduces the shape of the experimental curve, so that the anomalous increase in the binding energy for nuclei having an α-particle structure is well reproduced in the K-harmonic calculation, as in the shell-model calculations. In Fig. 2, we show the change in the distances between neighboring levels of excitation with respect to the collective variable ρ with increasing A (for functions with one, two, and three nodes, respectively). It can be seen that, with increasing A, the intervals between the excitations exhibit the same periodicity in fours as the binding energy. It can also be seen in Fig. 2 that the levels excited with respect to the collective variable ρ and which are equidistant in the oscillator model are significantly bunched in the K harmonic approach.

Table III gives the results of calculations of the rms radii of p-shell nuclei derived from the functions to which the binding energies in Fig. 1 correspond. It can be seen that the theoretical rms radii are higher than the experimental. This may be attributed to the variant of the Volkov potential investigated. A deeper potential gives larger binding energies and decreases the width of the well, so that the rms radius also decreases.

Description of Excited States of Nuclei in the Method of K Harmonics. The description of excited states of nuclei imposes greater requirements on the accuracy of the calculation. We shall discuss some results of spectroscopic calculations by the method of K harmonics for light nuclei. Figure 3 shows the spectra of the lowest excited states of the nuclei ^{6}Li and ^{7}Li in the K harmonics approach for two types of central potential, those of Volkov[41] and of Eikemeier and Hakenbroich,[45] and it also compares them with the experimental spectra of these nuclei. Since the calculation was made with central forces and no allowance

TABLE II. Dependence of binding energy of p-shell nuclei on the choice of the nucleon–nucleon potential.

Nucleus	E_{be}^{theor}, MeV	Variant of potential	E_{be}^{exp}, MeV
^{4}He	23.46	1 (Ref. 5)	
	25.79	6 (Ref. 5)	28.3
	26.05	3 (Ref. 5)	
^{6}Li	16.98	1 (Ref. 5)	
	20.86	6 (Ref. 5)	31.99
	13.03	1 (Ref. 6)	
^{7}Li	26.1	1 (Ref. 5)	
	13.675	1 (Ref. 6)	39.25
^{16}O	142.4	1 (Ref. 5)	
	158.4	2 (Ref. 5)	127.62
	151.4	3 (Ref. 5)	

TABLE III. Comparison of calculated rms radii of ground states of p-shell nuclei and the experimental values.

A	$\overline{(r^2)}^{1/2}_{theor}$, F	$\overline{(r^2)}^{1/2}_{exp}$, F	A	$\overline{(r^2)}^{1/2}_{theor}$, F	$\overline{(r^2)}^{1/2}_{exp}$, F
4	1.81	1.708	11	2.87	2.25
5	2.31	—	12	2.94	2.496
6	2.74	2.535	13	2.92	—
7	2.65	—	14	2.85	2.48
8	2.71	—	15	2.80	—
9	2.84	2.26	16	2.75	2.741
10	2.72	—			

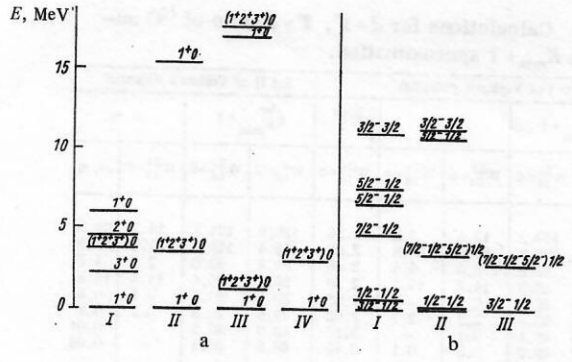


FIG. 3. Spectra of lowest levels of nuclei for ^{6}Li (a) and ^{7}Li (b): I) experimental spectrum; II) results of calculation with variant 1 of the Volkov forces; III) the same for Volkov's variant 6 (a); IV) the same for Eikemeier and Hakenbroich forces.

for spin-orbit decoupling, in the nucleus ^{6}Li one must compare the relative position of the levels LST-010 and LST-210 with the experimental value of the center of gravity of the lowest levels $J^\pi = 1^+$, 2^+, 3^+, $T = 0$. As can be seen in Fig. 3, the relative position of the levels for these three potentials varies from 1 to 3.5 MeV, whereas the experimental value of the center of gravity of the levels is 3.74 MeV. However, in all cases the order in which the levels follow one another is preserved. One also finds a first excited state with respect to the collective variable with $J^\pi = 1^+$, $T = 0$, this lying in the region 15–17 MeV for several variants of the potential. Experimentally, this nucleus has a level at 15.6 MeV with $J^\pi = 1^+$. These results suggest that the relative position in the spectra of the levels is less sensitive to the choice of the parameters of the potential than the absolute value of the binding energy. Similar conclusions can be drawn from the calculations of the spectra of the lowest excited states in ^{7}Li. The order in which the levels follow and their relative position is reproduced sensibly for different variants of the nucleon–nucleon potential.

States of Anomalous Parity of the Nucleus ^{16}O in the Method of K Harmonics. Excited states of anomalous parity, which have parity opposite to the ground state's, cannot be obtained in the approximation $K = K_{min}$. In this case, the expansion of the wave function in the method of K harmonics begins with the approximation $K = K_{min} + 1$.

In Ref. 44, Zhukov *et al.* calculated the ground state of ^{16}O and also excited dipole states with $J^\pi = 1^-$, $T = 1$. They also estimated the probabilities of dipole transitions from the ground state to the $J^\pi = 1^-$, $T = 1$ excited states. To describe dipole states in the nucleus ^{16}O in the method of K harmonics, it is necessary to take into account at least two components in the expansion (113), one of which corresphonds to transition of a nucleon from the $1p$ to the $2s$ shell, and the other to a $1p \to 1d$ transition. The corresponding angular and radial functions and the potentials are labeled by the indices $2s$ and $1d$.

Thus, the wave function was written in the form

$$\Psi(\bar{\xi}) = [\chi_{2s}(\rho)\, Y_{K=1,3,2s} + \chi_{1d}(\rho)\, Y_{K=1,3,2d}] \frac{1}{\rho^{(3A-4)/2}}. \qquad (120)$$

Figure 4 shows the matrix elements $W_{1d}^{1s}(\rho)$ and $W_{1d}^{2s}(\rho)$ that occur in this system for the first set of Volkov potentials. [41]

To estimate the probabilities of dipole transitions and the density of the nucleon distribution in the nucleus, it is necessary to know the matrix elements of the density operator and the dipole-transition operator.

The matrix element of the dipole-transition operator from the ground state $J^\pi = 0^+$, $T = 0$ of ^{16}O to the excited state $J^\pi = 1^-$, $T = 1$ takes the following form after all the intermediate integrations have been performed:

$$\langle \Psi_{exc} | \hat{d} | \Psi_{gnd} \rangle = \frac{3 \cdot 69!!}{2^{36} \cdot 35!} e \left(\int_0^\infty \chi_0(\rho)\, \chi_{2s}(\rho)\, \rho\, d\rho \right.$$
$$\left. + \sqrt{5} \int_0^\infty \chi_0(\rho)\, \chi_{1d}(\rho)\, \rho\, d\rho, \right. \qquad (121)$$

where e is the proton charge; $\chi_0(\rho)$ is the radial function of the ground state. The density of nucleons $n(\bar{r})$ in the state Ψ of nucleus A is found after calculation of the expectation value of the operator

$$\hat{n}(r) = \frac{1}{A} \sum_{i=1}^{A} \delta(\bar{r} - \bar{r}_i),$$

i.e.,

$$n(r) = \langle \Psi | \hat{n}(r) | \Psi \rangle. \qquad (122)$$

Here, the density $n(r)$ is normalized to one particle in the volume. The mean square radius is

$$\bar{r}^2 = \int_0^\infty r^4 n(r)\, dr. \qquad (123)$$

The results of the calculations are shown in Figs. 5 and 6 and in Table IV. The density of the nucleon distribution in ^{16}O for $J = 1^-$, $T = 1$ states is shown in Figs. 7a, 7b, and 7c.

The investigation enables one to draw a number of conclusions.

1. The absolute values of the binding energies of levels in both the K_{min} and the $K_{min} + 1$ approximation changes by 10–17 MeV for different variants of the potential, so that the change at the same time of the excitation energy ΔE_{min}, $K_{min} + 1$ is slight—by about 3

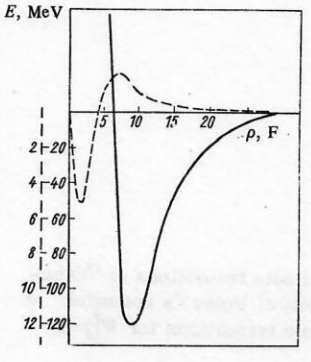


FIG. 4. Matrix elements $W_{1d}^{1d}(\rho) + L(L+1)/\rho^2$ (continuous curve) and $W_{1d}^{2s}(\rho)$ (dashed curve) for first set of Volkov's potential.

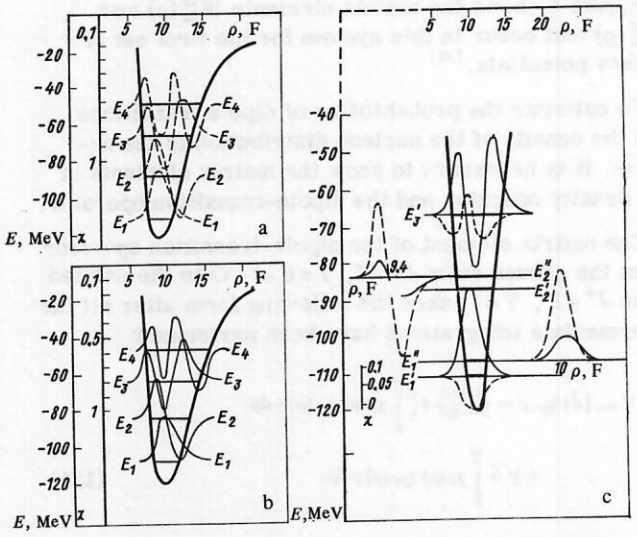


FIG. 5. Binding energies and wave functions of the first few levels for the first set of Volkov's potential. The dashed curves are the wave function of the ^{16}O ground state in the K_{min} approximation; a) and b) are the positions of the levels and the wave functions for the potentials $W^{2s}_{2s}(\rho)$ and $W^{1d}_{1d}(\rho)$ for $W^{2s}_{1d}(\rho) = 0$, respectively; c) the same with allowance for $2s-1d$ state mixing.

MeV. At the same time, the interval between the $1d-2s$ states remains constant and is about 3 MeV.

Allowance for the Coulomb energy raises the absolute position of the binding energies of the levels by 16–17 MeV in both the K_{min} and the $K_{min}+1$ approximation; at the same time, the relative energies ΔEK_{min}, $K_{min}+1$ and ΔE_{1d-s} do not change. Therefore, all the results of the calculation are given without allowance for Coulomb corrections.

2. Allowance for mixing of $1d-2s$ states leads to additional splitting of the levels by ~1 MeV. This effect is virtually independent of the chosen potential and changes little with increasing excitation energy. As can be seen in Fig. 4, the matrix element for $1d-2s$ state coupling is small.

3. As can be seen from the table, the rms radius of the nucleus in excited states changes by 10% as the excitation energy increases.

4. In Ref. 46, Komar *et al.* found from measurements of the cross sections of total absorption of γ rays by the nucleus ^{16}O that about half the integrated

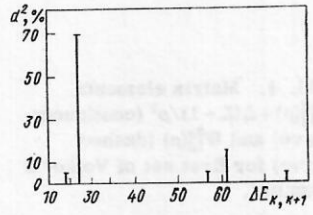


FIG. 6. Strengths of $J=1^-$, $T=1$ dipole transitions of ^{16}O nucleus in percentages for the first set of Volkov's potential: a) and b) are the strengths of the dipole transitions for $W^{2s}_{1d}=0$ and $W^{2s}_{1d}=10$, respectively.

TABLE IV. Calculations for $J=1^-$, $T=1$ state of ^{16}O nucleus in the $K_{min}+1$ approximation.

	Set I of Volkov's potential					Set II of Volkov's potential			
	$E^{be}_{K_{min}+1}$		d^2, %		$(\overline{r^2})^{1/2}$, F	$E^{be}_{K_{min}+1}$		d^2, %	
	$W^{2s}_{1d}=0$	$W^{2s}_{1d}\neq 0$	$W^{2s}_{1d}=0$	$W^{2s}_{1d}\neq 0$	$W^{2s}_{1d}=0$	$W^{2s}_{1d}=0$	$W^{2s}_{1d}\neq 0$	$W^{2s}_{1d}=0$	d^2, %
$2s$	109.5	110.3	13.4	9.2	2.26	120.9	121.5	14.2	21.5
$1d$	107.2	106.7	66.3	67.8	2.28	118.4	118.1	71.2	57.9
$2s$	86.6	87.2	3.0	4.5	2.56	99.3	99.6	2.3	4.7
$1d$	83.5	83.3	14.2	15.7	2.70	96.9	96.1	11.4	14.5
$2s$	66.4	66.8	0.5	—	2.87	80.1	80.5	0.1	0.4
$1d$	63.7	63.5	2.2	2.7	2.92	77.6	77.3	0.7	0.8
$2s$	48.9	49.1	0.1	—	3.15	63.0	63.3	—	0.01
$1d$	46.6	46.2	0.5	0.1	3.15	60.8	60.4	—	0.06

Note. E^{be} is the total binding energy; d^2 is the square of the dipole transition operator matrix element; $\overline{r^2}$ is the mean square radius for four states with different number of nodes with respect to the collective variable ρ. All results are given for two variants of the Volkov potential both with and without allowance for $1d-2s$ state mixing.

cross section determined by the dipole sum rule occurs in the region of energies of the giant dipole resonance (more than half of the integrated cross section of photoabsorption is at the region of higher γ-ray energies. An attempt to explain this phenomenon was made in Ref. 47, in which it was shown that, if one adds to the giant dipole resonance three-particle–three-hole $(3p-3h)$ states, than about 18% of the total dipole sum is in the region of energies above the giant resonance. This effect is to a large extent taken into account in the method of K harmonics; in the K_{min} approximation, allowance is made for an infinite number of $|npnh\rangle$ monopole shell

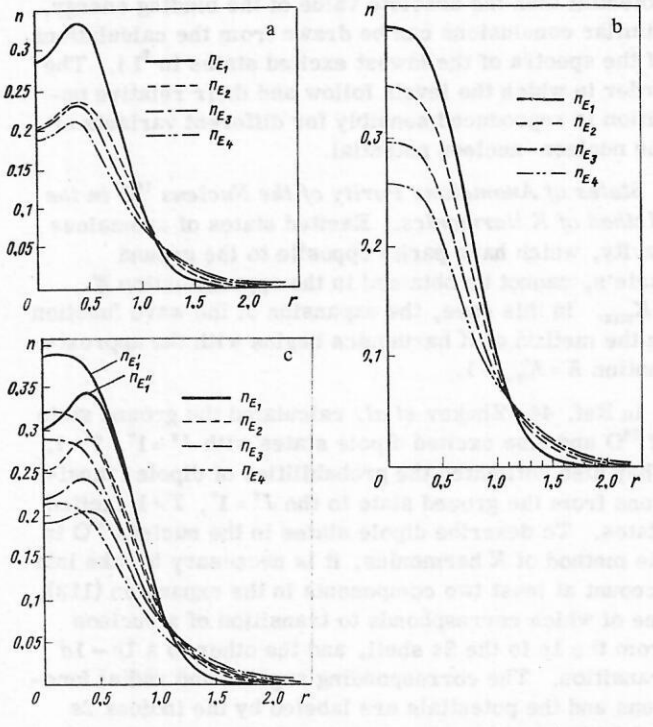


FIG. 7. Density of nucleon distribution in ^{16}O nucleus for the first four $J=1^-$, $T=1$ states with the first set of Volkov's potentials: a) when particle in $1d$ shell is excited; b) when particles in $2s$ shell are excited; c) with allowance for $1d-2s$ state mixing.

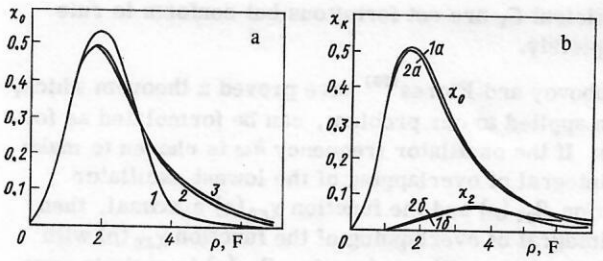


FIG. 8. Results of calculation of longitudinal (a) and transverse (b) form factors of inelastic electron scattering on ^{16}O nucleus with excitation of $J=1^-$, $T=1$ dipole states in the method of K harmonics (continuous curves) and in the shell model (dashed curves) (sic).

configurations. As a result, 20 % of the strength of the dipole transitions corresponds to a state having one node with respect to the variable ρ and 5% to states with two nodes with respect to this variable. In the language of particle–hole excitations, this corresponds to the fact that admixture of both $|3p-3h\rangle$ and $|5p-5h\rangle$ states is important in the dipole resonance. It is be hoped that this result is not changed qualitatively when the nucleon–nucleon potentials are chosen in such a way as to give the correct binding energies of the ground state and excited states of ^{16}O and also when following terms in the K-harmonic series are added.

In Ref. 48, Goncharova *et al.* calculated the form factors of inelastic scattering of electrons on ^{16}O with excitation of $J=1^-$, $T=1$ dipole states. The results are shown in Figs. 8a and 8b, in which the continuous curves are the form factors calculated by the method of K harmonics, and the dashed curves are the form factors obtained in the shell model [translator's note: Figs. 8a and 8b contain no dashed curves]. It can be seen from Figs. 8a and 8b that the inelastic electron form factors found by the method of K harmonics do not have the deep diffraction minimum characteristic of shell model calculations. The minimum is filled by the contribution of the monopole excitations taken into account in the method of K harmonics. The results of the method of K harmonics and the shell model are strongly different in the region of large momentum transfers.

Comparison of Results of the Method of K Harmonics with the Results of the Translationally Invariant Shell Model for p-Shell Nuclei. The results of Refs. 39 enable us to make a quantitative comparison of the method of K harmonics in the lowest approximation with the translationally invariant shell model. To this end, the values of the binding energy E_{be} for all p-shell nuclei were calculated by means of the wave functions of the translationally invariant shell model for a number of quanta $N=K_{min}$. In addition, the wave functions of the method of K harmonics were expanded with respect to functions of the translationally invariant shell model. These calculations were made in accordance with (53)–(55), from which it can be seen that the expansion of the functions of the method of K harmonics with respect to functions of the translationally invariant shell model reduces to expansion of $\chi_{K\nu}(\rho)$ with respect to the radial wave functions of a $(3A-3)$-dimensional harmonic oscillator:

$$\chi^{\nu}_{K}(\rho)=\sum_{\varkappa} C^{\nu}_{\varkappa K} R_{\varkappa K}(\rho), \tag{124}$$

and the expansion coefficients are given by

$$C^{\nu}_{\varkappa K}(\rho)=\int \chi^{\nu}_{K}(\rho)\, R_{\varkappa K}(\rho)\, \rho^{3A-4}\, d\rho. \tag{125}$$

The results of calculations by the method of K harmonics in the lowest approximation and in the translationally invariant shell model are compared in Table V, which gives the values of the binding energy calculated by the method of K harmonics and in the translationally invariant shell model, the oscillator frequency $\hbar\omega$ at which the maximum of the lowest oscillator function $R_{0K}(\rho)$ coincides with the maximum, and also the coefficients of the expansions of the bound-state functions of the method of K harmonics with respect to functions of the translationally invariant shell model with the above frequency:

$$\chi_{K\nu}(\rho)=C_0 R_{0K}(\rho) + C_1 R_{1K}(\rho)+C_2 R_{2K}(\rho)+\ldots \tag{126}$$

Since only some of the expansion coefficients are given, it is not necessary for them to satisfy the orthonormality relations, in particular for states excited with respect to ρ. Examination of Table V shows that the binding energies for all p-shell nuclei obtained for the Volkov potential (variant 1) in the lowest approximation of the method of K harmonics and in the translationally invariant shell model with allowance for only the lowest configuration differ from one another by not more than 0.5 MeV. This close correspondence be-

TABLE V. Results of calculations by the method of K harmonics and the translationally invariant shell model (TISM).

A	ν	$E_{K\nu}$	E_{TISM}	C_0	C_1	C_2	$\hbar\omega$	K_0	K_1
^{5}He	0	−15.55	−14.01	0.980	0.093	0.173	16.84	—	—
^{6}Li	0	−16.98	−15.43	0.984	−0.061	0.158	13.87	24.77	—
	1	−1.56	11.7	0.123	0.863	−0.361			
^{7}Li	0	−26.11	−25.31	0.988	−0.071	0.133	16.17	26.33	—
	1	−9.64	−1.89	0.171	0.671	−0.375			
^{8}Be	0	−40.09	−39.45	0.984	−0.071	0.142	16.44	38.45	16.16
	1	−20.67	−15.22	0.119	0.830	−0.352			
	2	−8.08	8.31	−0.034	0.497	0.257			
^{9}Be	0	−39.16	−38.42	0.983	−0.083	0.146	15.76	37.38	16.23
	1	−20.8	−15.71	0.123	0.818	−0.337			
	2	−8.7	6.86	−0.052	0.469	0.264			
^{10}B	0	−43.99	−43.31	0.989	−0.055	0.146	15.6	38.77	21.33
	1	−25.8	−21.05	0.083	0.853	−0.307			
	2	−12.3	0.78	−0.075	0.463	0.404			
^{11}B	0	−57.8	−57.34	0.988	−0.064	0.140	16.44	44.22	27.02
	1	−38.1	−33.45	0.119	0.897	−0.300			
	2	−22.7	−9.98	−0.066	0.486	0.456			
^{12}C	0	−88.6	−87.71	0.987	−0.085	0.132	16.01	66.68	44.38
	1	−65.3	−63.91	0.143	0.900	−0.298			
	2	−46.0	−40.61	−0.073	0.531	0.453			
^{13}C	0	−70.63	−70.13	0.986	−0.092	0.115	16.58	48.57	32.34
	1	−50.35	−46.08	0.109	0.910	−0.295			
	2	−33.8	−22.43	−0.103	0.519	0.470			
^{14}C	0	−96.5	−95.75	0.993	−0.082	0.061	18.2	56.6	38.07
	1	−73.7	−68.76	0.187	0.913	0.294			
	2	−55.0	−42.21	−0.016	0.486	0.504			
^{15}N	0	−115.6	−115.13	0.994	−0.042	0.093	18.5	61.05	48.94
	1	−91.7	−87.61	0.105	0.907	−0.268			
	2	−70.3	−60.49	−0.077	0.398	0.621			
^{16}O	0	−142.4	−141.8	0.995	−0.034	0.112	19.39	88.22	55.5
	1	−113.4	−110.5	0.098	0.902	−0.264			
	2	−90.4	−85.74	−0.050	0.370	0.653			

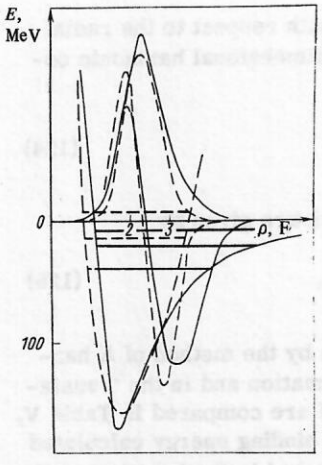


FIG. 9. Results of calculation of effective potential $W(\rho)$ with inclusion of "centrifugal" term $L_K(L_K+1)/\rho^2$ in the method of K harmonics for the ^{8}Be nucleus, the wave functions of the ground state and first excited state in this potential, and also the lowest oscillator functions R_{0K} and R_{1K} and the oscillator potential (also including the "centrifugal" term and with the choice of the frequency $\hbar\omega$ described in the text). The origin for the oscillator potential (dashed curve) is chosen to make the energies of the ground states in the two wells equal.

tween the lowest states of the shell model and the method of K harmonics holds for all nuclei of the $1p$ shell.[7] The reason for this similarity is evident in Fig. 9, which shows the effective potential $W(\rho)$ (with inclusion of the "centrifugal" term $L_k(L_k+1)/\rho^2$) of the method of K harmonics for the nucleus ^{8}Bd, the wave functions of the ground state and the first excited state in this potential, and also the lowest oscillator function R_{0K} and R_{1K} and the oscillator potential (also with inclusion of the "centrifugal" term and the choice of the frequency $\hbar\omega$ described above). The origin of the calculation for the oscillator potential, which is shown by the dashed curve, is chosen in such a way as to make the ground-state energies in the two wells equal. It can be seen from Fig. 9 that the oscillator potential is close to the K-harmonic potential near the bottom of the well, is somewhat narrower near the first nodeless solution, and differs strongly in the region of states with respect to the collective variable ρ. It is therefore natural that the ground-state energies in the two potentials should be similar. It can also be seen that the wave functions of the ground states differ little except in the asymptotic region. But in the two methods there are great difference between the energies and the wave functions of ρ-excited states. These conclusions are valid not only for the nucleus ^{8}Be, which is chosen as a typical example, but also for the remaining $1p$-shell nuclei (see Table V). In Table V, it is striking that the integral of overlapping of the K-harmonic function of the ground state, $\chi_{K0}(\rho)$, with the lowest oscillator function is 98%. At the same time, the contribution to $\chi_{K0}(\rho)$ of the two-quantum oscillator excitation is exceptionally small. The very small values of the coefficient C_1 are not fortuitous but conform to rule completely.

Dubovoy and Flores[50] have proved a theorem which, when applied to our problem, can be formulated as follows: If the oscillator frequency $\hbar\omega$ is chosen to make the integral of overlapping of the lowest oscillator function $R_{0K}(\rho)$ and the function $\chi_{K0}(\rho)$ maximal, then the integral of overlapping of the function $\chi_{K0}(\rho)$ with the following oscillator function $R_{1K}(\rho)$ is strictly zero. Note that this theorem was proved in Ref. 50 only for a three-dimensional oscillator, but it can be readily generalized to the radial functions of an oscillator of any dimension.[51]

As we have already noted, in choosing $\hbar\omega$ we did not achieve maximal value of the integral of overlapping of the lowest function of the translationally invariant shell model and the wave function of the method of K harmonics, but assured only coincidence of the maxima of these functions. However, if such a condition is satisfied, the overlap integral of the functions is fairly near the masimum, and the coefficients C_1 are very small, although nonvanishing. This circumstance casts light on the origin of the close agreement between the results of the translationally invariant shell model and the method of K harmonics for the ground state. Since the function R_{1K} cannot be mixed with the wave function R_{0K}, the shell function in the method of K harmonics is improved by admixture of states with oscillator energy $4\hbar\omega$ and higher energies. Because of the large difference between the energies of these states and the ground state the admixture is small. With regard to the ρ-excited states, their structure is much more complicated and their wave functions do not reduce to any one shell function. It is therefore to be expected that the method of K harmonics will give a much better description of the properties of giant monopole resonances than the shell model in the simplest single-configuration approximation.

Method of K Harmonics and Monopole Excitations of Light Nuclei. Monopole excitations correspond to vibrations of the density of nuclear matter and can contribute to different effects. It was shown above that these states have a strong influence on the probabilities of dipole transitions. The new interest in these "breathing modes" arose recently in connection with the discovery in inelastic scattering of electrons and hadrons on a number of nuclei[52] of giant resonances of collective type that can apparently be identified with $E0$ and $E2$ transitions. In Refs. 49 and 53 it was shown that the results of a phenomenological hydrodynamic model for the 0^+ breathing modes arise naturally from the microscopic method of K harmonics with $K=K_{\min}$. The numerical calculations showed that the hydrodynamic model of monopole vibrations is a consequence of the microscopic method of K harmonics in the limit of large values of the mass number. The connection between the rigidity parameters of the hydrodynamic model and the method of hyperspherical functions was traced in these papers. It was shown that the two models give similar results in the region $A>16$ and that strong discrepancies are observed in

[7] This fact was first established for the example of the nucleus ^{16}O in Ref. 49.

TABLE VI. Results of calculations in the method of K harmonics of the compressibility parameter K_0 for the ground state and the first excited state (K_1) of p-shell nuclei.

Nucleus	^{6}Li	^{7}Li	^{8}Be	^{9}Be	^{10}B	^{11}B	^{12}C	^{13}C	^{14}C	^{15}N	^{16}O
E_{be}, MeV	16.98	26.11	40.09	39.16	44	57.8	88.6	70.63	96.5	115.6	142.4
r_0, F	2.12	2.05	2.1	2.19	2.25	2.22	2.27	2.26	2.17	2.15	2.13
ΔE_{10}, MeV	15.42	16.47	19.42	18.36	18.2	19.7	23.3	20.28	22.8	23.9	29
ΔE_{21}, MeV	—	—	12.59	12.1	13.5	15.4	19.3	16.55	18.7	21.4	23
K_0, MeV	24.77	26.33	38.45	37.38	38.77	44.22	66.68	48.57	56.6	61.05	88.22
K_1, MeV	—	—	16.16	16.23	21.33	27.02	44.38	32.34	38.07	48.94	55.5
	—	—	0.43	0.43	0.55	0.61	0.69	0.66	0.67	0.8	0.63

the region of nuclei $A<16$. It became clear that the behavior of the rigidity parameter K of the nucleus as a function of the number of nucleons (in the region of light nuclei), the excitation energy of the nucleus, and of the quantum numbers of the levels ($JLST$) warrants special study. In Ref. 54, a calculation was made of the nuclear compressibility for all p-shell nuclei in the framework of a single nucleon–nucleon potential. States of both K_{min} and $K_{min}+1$ type were studied. The compressibility parameter K was estimated in accordance with the expression of the hydrodynamic model, which for the method of K harmonics can be written in the form[53]

$$K=(m/\hbar^2)\,r_0^2\,(\Delta E)^2;\quad r_0=\rho/\sqrt{A}, \tag{127}$$

where r_0 is the ground-state radius of the nucleus; ΔE is the difference between the excitation energies of the neighboring monopole states. Note that Eq. (127) uses the radius r_0 corresponding to the ground state of the nucleus in the excitation band with given quantum numbers irrespective of the excited state considered. Naturally, when r_0 is chosen in this way, the estimate for K is better for the lower excited states. The corresponding values of the compressibility parameter for high excited states can pretend only to a qualitative description of this parameter as a function of the excitation energy. The results of the calculation are given in Table VI, which shows the values of the compressibility parameter for the ground state, K_0, and also the compressibility parameter for the first excited state, K_1. It can be seen from this table that the compressibility parameter increases with increasing number A within the p shell by about a factor 3.5. With increasing excitation energy for a given nucleus, the compressibility parameter decreases in a number of cases very considerably, and, moreover, differently for different A. In α-cluster nuclei, this parameter exhibits an anomalous increase compared with the values for neighboring nuclei. The dependence of the compressibility parameter on the excitation energy, the quantum numbers of the states, and the potentials is demonstrated for the example of ^{16}O, in Table VII, which shows that the rigidity depends on the quantum numbers of the states even when the energies of these states are almost equal. One observes a certain dependence of the nuclear compressibility on the choice of the parameters of the nucleon–nucleon interaction, but the difference between the results for different variants of the interaction decreases with increasing excitation energy.

A further question has direct bearing on the subject of this review: the connection between the method of K harmonics and the method of generator coordinates. It is interesting in connection with the fact that, as shown in Ref. 55, generator coordinates can be used to formulate an approach that, on the one hand, is completely equivalent to the method of K harmonics and, on the other, is convenient in that calculations are made in it entirely by means of the standard computational mathematical formalism of the shell model: At no stage in the calculation it is necessary to use multidimensional spherical coordinates.

8. CONNECTION BETWEEN THE METHOD OF K HARMONICS AND THE METHOD OF GENERATOR COORDINATES

Generator Formulation of the Method K Harmonics. We write the function (113) of the method of K harmonics in generator form, using as generator coordinate the parameter α (or, which is the same thing, the oscillator radius r_0):

$$\Psi_{K_0}(\mathbf{r}_1 \ldots \mathbf{r}_A, \sigma_i\tau_i)=\int_0^\infty d\alpha f_{K\nu}(\alpha)\,|\,AKK\nu\rangle_\alpha. \tag{128}$$

This equation means that we have expanded the wave function of the method of K harmonics in an integral with respect to oscillator shell functions with different radii or frequencies. What is the meaning of the expression (128) and when is it possible? Using the explicit form of the wave function $\Psi_{K\nu}$ in the method of K harmonics (113), we obtain an expression for the radial part of the wave function:

$$\varphi_{K\nu}(\rho)=\int f_{K\nu}(\alpha)\exp(-\alpha\rho^2)\,\rho^K\,d\alpha, \tag{129}$$

i.e., the transition to the generator form of the wave function simple means representation of the function $\varphi_{K\nu}\rho^{-K}$ in the form of a Laplace integral. If we consider ground states and if the function $\varphi_{K\nu}(\rho)$ decreases at infinity, then such a representation is possible. It

TABLE VII. Dependence of the results of calculating the compressibility parameter in the method of K harmonics on the excitation energy, the quantum numbers of the state, and the choice of the potential for the example of the nucleus ^{16}O.

Parameter	$J^\pi=0^+$, potential 1	$J^\pi=1^-$, potential 1	$J^\pi=1^-$, potential 2
E_{be}, MeV	142.4	110.3	121.5
r_0, F	2.13	2.31	2.26
ΔE_{10}, MeV	29	23.1	21.9
ΔE_{21}, MeV	23	20.4	19.1
ΔE_{32}, MeV	20	17.7	17.2
K_0, MeV	88.22	65.83	56.64
K_1, MeV	55.5	51.34	43.08
K_2, MeV	41.96	38.65	34.9

is clear from this that the generator procedure is equivalent to the method of K harmonics (at least, for bound states) and the essence of the procedure consists of replacing the unknown radial functions of the method of K harmonics by their Laplacian originals $f_{K\nu}(\alpha)$.

For these new unknown functions, one can write down an equation of motion that follows from the Schrödinger equation for the function (113). This will be an ordinary system of coupled integral equations of Hill–Wheeler type of the method of generator coordinates[5]:

$$\sum_{K\nu}\int d\alpha f_{K\nu}(\alpha)\,[H_{K'\nu',\,K\nu}(\alpha',\alpha)-\delta_{K\nu,\,K'\nu'}I_{K\nu}(\alpha,\alpha')\,E]=0, \quad (130)$$

where

$$H_{K'\nu',\,K\nu}(\alpha',\alpha)=\langle AN=K'K'\nu'\,|\,H\,|\,AN=KK\nu\rangle_{\alpha'\alpha} \quad (131)$$

is the matrix element of the Hamiltonian H between shell wave functions with different oscillator radii $r_0'=\sqrt{2/\alpha'}$ (on the left-hand side of the matrix element) and $r_0=\sqrt{2/\alpha}$ (on the right-hand side of the element). The overlap integral

$$I_{K\nu}(\alpha,\alpha')=\langle AN'=KK\nu\,|\,AN=KK\nu\rangle_{\alpha'\alpha} \quad (132)$$

is different from zero and unity since the shell functions with different r_0 are not orthogonal to one another. Only when $\alpha'=\alpha$ does $I_{K\nu}(\alpha,\alpha)=1$. Thus, the expressions (128) and (129) can be regarded as an expansion of the radial wave function of the method of K harmonics with respect to a system of nonorthogonal wave functions of Gaussian type or as expansion of a complete harmonic with fixed values of K and ν with respect to shell functions with the same $K\nu$ and different radii r_0.

Instead of the system of differential equations for $\varphi_{K\nu}(\rho)$, we obtain the system of integral equations (130). It is striking that these equations do not contain multidimensional spherical coordinates at all but only the matrix elements (131) and (132) between shell wave functions. It is true that the price for this convenience is the use of the nonorthogonal system of functions. It was shown in Ref. 55 that this does not introduce serious difficulties; working formulas were derived for calculation of the integrals $I_{K\nu}(\alpha,\alpha')$ and $H_{K'\nu',\,K\nu}(\alpha,\alpha')$ for p-shell nuclei in the $K_{\min}$ approximation. Note also that the wave functions (128) and (130) coincide literally with the corresponding expressions of Ref. 56, in which monopole vibrations in ^{16}O were considered. In these investigations, the nuclear wave function was approximated by a single function (128), corresponding to the lowest shell configuration s^4p^{12}, i.e., this calculation corresponded to the $K_{\min}$ approximation of the method of K harmonics.

Interesting approximate methods of solving equations of the type (130) for the case when the overlap integral can be represented to a good approximation in the form of the Gaussian function $I(\alpha,\alpha')\sim\exp[E(\alpha-\alpha')^2]$ have been developed in the literature.[56,57] In Ref. 56 a somewhat different choice was made of the monopole generator coordinate and it was shown that in this case the Gaussian approximation of the overlap integral is valid to a high accuracy. Thus there are good grounds for investigating the methods of solution proposed in Refs. 56 and 57. Moreover, the examination of this question is of great interest since, as is shown in Ref. 57, Gaussian approximation of the overlap integral leads naturally to the random-phase approximation. Therefore, the possibility is opened up of following the connection between the large number of varied methods in modern nuclear theory.

However, we shall not consider here this interesting possibility, but we shall instead use the standard procedure of solving integral equations in which the integral with respect to the variable α is replaced by a finite sum over discrete values of α_i (or r_{0i}):

$$\Psi_{K\nu}(r_1\ldots r_A,\sigma_j\tau_j)=\sum_{i=1}^{n}C_{K\nu}(i)\,\Phi_{K\nu}(\alpha_i). \quad (133)$$

Then the values of the coefficients $C_{K\nu}(i)$ at these points can be found from the system of algebriac equations

$$\sum_{i=1}^{n}C_{K\nu}(i)\,[H_{K'\nu',\,K\nu}(\alpha_j,\alpha_i)-I_{K\nu}(\alpha_j,\alpha_i)\,E]=0, \quad (134)$$

where $j=1,\,2,\ldots,\,n$. For simplicity, we have restricted ourselves in (134) to inclusion of only one value of $K\nu$ in (128).

For generalization to the case of several sets of $K\nu$, we must obviously introduce for each set of quantum numbers a corresponding n, this being the number of the terms taken into account in the quadrature formula (133) for these values of $K\nu$, and write down a system of equations for the coefficients $C_{K\nu}(i)$, which will also contain $K\nu$-nondiagonal matrix elements of the Hamiltonian H. The ground-state energy $E_0^{(n)}$ is determined by the lowest root of the secular equation

$$\det_{i,\,j=1,\,2,\,\ldots,\,n}|H_{K\nu,\,K\nu}(\alpha_j,\alpha_i)-EI_{K\nu}(\alpha_j,\alpha_i)|=0. \quad (135)$$

It is known from the theory of Fredholm integral equations that with increasing n the lowest root of this equation converges uniformly to the exact eigenvalue of Eq. (130) and the approximate wave function (133) converges to the true eigenfunction of the ground state. The only problem is the rate of convergence. This depends strongly on the values of the parameters α_i.

To determine the optimal values of the parameters α_i, one can use a variational principle and determine the parameters $\alpha_1\ldots\alpha_n$ by minimizing the eigenvalue of the determinant (135).

Thus, the computational procedure is as follows:

1. The shell wave functions $|ANK_\nu\rangle_\alpha$ with oscillator parameter α are constructed.

2. The matrix elements $H_{K'\nu',\,K\nu}(\alpha',\alpha)$ and $I_{K\nu}(\alpha',\alpha)$ are calculated from these functions with different oscillator parameters α and α'.

3. The numbers n in Eq. (133) are specified for each of K_ν sets taken into account, and the matrix of the secular equation (135) is calculated.

TABLE VIII. Results of calculation of the binding energy of ^{3}H, the wave functions, and the rms radii of the ground state of this nucleus.

Number of functions		r_{Ki}	C_{KU}	E_{be}	$\overline{\langle r^2\rangle}$	Number of functions		r_{Ki}	C_{KU}	E_{be}	$\overline{\langle r^2\rangle}$
$K=0$	$K=2$					$K=0$	$K=2$				
1	—	1.36	1	−5.96	1.85	1	1	1.36	—	−6.85	—
		1.17	0.59					1.78			
2	—			−7.35	2.35			1.16	0.61		
		2.11	0.49			2	1	2.08	0.46	−8.12	2.60
		1.12	0.51					1.75	0.15		
3	—	1.78	0.47	−7.38	2.53			1.16	0.62		
		2.86	0.18			2	2	2.08	0.46	−8.19	2.70
4	—	—	—	−7.39	—			1.48	0.10		
								2.62	0.10		

4. The lowest solution of this secular equation is found and it is minimized with respect to the parameter α_i.

5. Once the optimal parameters α_i have been found and the roots of the secular equation determined, one can find the coefficients $C_{K\nu}(i)$, and with them the approximation wave functions (133).

The convergence with respect to n of the expansion (133) has been studied in detail in quantum chemistry problems for the expansion of Slater or Hartree–Fock atomic orbitals with respect to gaussoids.[58,59] Experience with the calculations shows that it is sufficient to take into account between three and six gaussoids in order to achieve good accuracy for different quantities. It is reasonable to think that in nuclear problems the convergence will be no worse than in atomic calculations. With regard to the convergence with respect to the quantum numbers K and ν, the situation is the same here as in the ordinary method of K harmonics.

6. The calculations are repeated with increasing number of included terms, etc., until a stable value of the energy of the ground state (or excited state in which we are interested) has been reached and there is no effective change of the wave function, which can then be used to calculate different properties of nuclei.

Quantitative Verification of the Generator Version of the Method of K Harmonics. For practical verification of the efficacy of the above approach, the calculation of Simonov and Badalyan[2] for the ^{3}H nucleus with nucleon–nucleon forces in the form of a rectangular well was repeated. The calculations were made with allowance in the wave function for only terms with $K=0$ and 2, which, according to Ref. 2, make the main contribution to the solution. The parameters of the potential were taken to be the same as in Refs. 62: $V_s=-13.639$ MeV, $R_s=2.583$ F, $V_T=-35.303$ MeV, $R_T=2.043$ F. Interaction in odd states was ignored. The wave functions of the translationally invariant shell model were used; namely, the value $K=0$ was taken to correspond to a function of the translationally invariant shell model of the form $|A=3, N=0$ (Ref. 3), $L=0, S=1/2, T=1/2\rangle$, and for $K=2$ the function $|A=3, N=2$ (Ref. 21), $L=0, S=1/2, T=1/2\rangle$ was taken; the coefficients of fractional parentage for them are given in Refs. 10, 33, and 60.

The nonorthogonality of the wave functions introduces only minimal changes into the technique of calculations in the translationally invariant shell model. For example, the matrix elements of central forces $V=\sum_{i<j}V_{ij}$ are calculated in accordance with the formula

$$\begin{aligned}&\langle A=3\ N'[f']LST\,|\sum V_{ij}|\,A=3\ N[f]LST\rangle_{\alpha'\alpha}\\&=3\sum_{\nu\nu'\Lambda n n' l S_0T_0}\langle A=3\ N'[f']LST\,|\,A=2\ \nu'\Lambda S_0T_0,\ n'l\rangle\\&\times\langle A=3\ N[f]LST\,|\,A=2\ \nu\Lambda S_0T_0,\ nl\rangle\, I^l_{n',n}(\alpha'\alpha)\\&\times V^{\Lambda S_0T_0}_{\nu'\nu}(\alpha',\alpha)\,B_{S_0T_0},\end{aligned}\tag{136}$$

where

$$I^l_{n'n}(\alpha',\alpha)=\int R_{n'l}(r/r_0')\,R_{nl}(r/r_0)\,r^2\,dr\tag{137}$$

is the overlap integral of the wave functions of a three-dimensional oscillator with different radii r_0' and r_0;

$$V^{\Lambda S_0T_0}_{\nu',\nu}=\int R_{\nu'\Lambda}(r_{12})\,V_{12}R_{\nu\Lambda}(r_{12})\,r^2\,dr\tag{138}$$

is the two-particle matrix element of the interaction; $B_{S_0T_0}$ are the ordinary spin–isospin factors. The overlap integral $I(\alpha',\alpha)$ is calculated in accordance with (136) (it must be borne in mind that it is diagonal with respect to K), in which it is necessary to replace the factor $V^{\Lambda S_0T_0}_{\nu'\nu}(\alpha'\alpha)$ by $I^{\Lambda}_{\nu'\nu}(\alpha',\alpha)$.

In order to gauge the convergence of the expansion with respect to gaussoids, calculations were made for several values of n_0 (the number of included functions with $K=0$) and n_2 (the number of gaussoids in the expansion of the component with $K=2$). The results of calculation of the binding energy of ^{3}H, the functions, and the mean square radii of the ground state of this nucleus are given in Table VIII. In the third column of this table, instead of the parameters α_i^K of the gaussoids found by minimization, we have given the oscillator radii $r_{Ki}=\sqrt{2/\alpha_i^K}$., which are equivalent to them.

If we recall that exact calculation by the method of K harmonics with allowance for only the harmonic with $K=0$ gives the binding energy $E_{be}=-7.375$ to -7.5 MeV and $E_{be}=-8.094$ to -8.125 MeV with allowance for the harmonic with $K=2$, the convergence of the expansion with respect to the gaussoids is fairly good. A value of the binding energy equal to that of the exact calculation to the second figure after the decimal point is obtained already when one takes into account two or three gaussoids for each value of K. One can clearly follow the rapid saturation of the binding energy when n_0 and n_2 are increased. Good accuracy is also achieved for the mean square radius of the nucleus (the last result should be compared with the result 2.11 F $=\langle r_0^2\rangle$ in Ref. 2, and not with the charge radius of the nucleus ^{3}H, which also includes the radius of the charge distribution in the proton).

It can be seen from Figs. 10a and 10b that the transition from $n_0=1$ to $n_0=2$ considerably changes the wave function $\chi_{K\nu}(\rho)=\rho\varphi_{K\nu}(\rho)$ with $K=0$. A further increase in n_0 hardly changes the shape of the wave function but improves its asymptotic behavior. Our function $\chi_{00}(\rho)$ agrees well with the result of the exact calculation in Ref. 1.

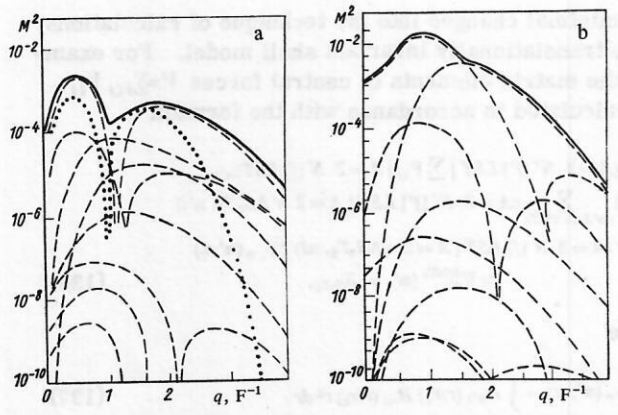


FIG. 10. Wave functions of the nucleus ^{3}H in the method of generator coordinates: a) the function $\psi_{K=0}(\rho)$. The curves are obtained with allowance for one, two, and three terms, respectively, in the expansion (39). The state with $K=2$ is not taken into account; b) the functions $\chi_{K=0}(\rho)$ and $\chi_{K=2}(\rho)$. Curves 1a) and 1b) obtained with allowance for two terms with $K=0$ and two terms with $K=2$.

Less accurate is the function $\chi_{K=2}(\rho)$, whose maximum is displaced somewhat to the right of the function in Ref. 1. Transition from $n_2=1$ to $n_2=2$ improves the function $\chi_{K=2}(\rho)$, but still does not lead to complete agreement with the exact result. However, this is natural since $\chi_{K=2}(\rho)$ is a small correction to the ^{3}H wave function and, therefore, the relative error with which it is calculated must be lower than for the function χ_0 in the same approximation. However, these defects of the function χ_2 have little influence, as we shall see, on the values of E_{be} and $\langle r^2\rangle$, which are virtually equal to the exact values.

Thus, the results show that the convergence of the expansion with respect to the gaussoids (136) of the radial functions of the method of K harmonics is good. These functions can be approximated with sufficiently high accuracy by three to four gaussoids for each value of K and ν. If it is recalled that to obtain the same binding energy from expansion of the ^{3}H functions with respect to a set of orthogonal oscillator functions it was necessary to take into account six to eight components[61] instead of our four, we see that our method takes, as regards the number of terms in the expansion that play a main role in the nuclear wave function, an intermediate position between the method of K harmonics and an expansion with respect to an orthogonal oscillator basis. The convenience of the approach derives from from the fact that, using it, one can completely eliminate from the calculation multidimensional spherical coordinates and, at the same time, there is no need to construct a large number of wave functions of an orthogonal set, the number and complexity of which rapidly increase with increasing number N of quanta. The expansion (139) contains functions of absolutely identical structure, differing only in the oscillator radius. The nonorthogonality of these functions does not cause any difficulties. Such an approach is particularly convenient for making calculations in the approximation that takes into account only harmonics with $K=K_{min}$. Recently, the method of expanding the wave function with respect to gaussoids was successfully applied to three- and four-body problems.[63] The good correspondence of the results obtained by the method of K harmonics and the method of generator coordinates was confirmed for the example of the nuclei ^{16}O and ^{40}Ca in Ref. 64.

Thus, we have considered above a number of questions relating to the method of K harmonics as compared with the shell model and other methods of nuclear theory. This comparison makes it possible to establish the place of this method among the other microscopic approaches in nuclear theory. It is clear that the method of K harmonics is in principle more flexible and more accurate than the simple shell model since the construction of the shell function is rigid and essentially contains only one variable parameter—the oscillator energy $\hbar\omega$. In the method of K harmonics, one of the degrees of freedom—the hyper-radius ρ— is "unfrozen" and the wave function for this degree of freedom is found from the Schrödinger equation, i.e., it corresponds to an absolute minimum of the energy with respect to this degree of freedom, which makes is possible to obtain better results than in the shell model. True, as estimates show, this gain in the energy for the ground state is quantitatively small and the results are similar to those of the shell model. However, for excited states the situation in different and in the region of giant monopole resonances the quality of the method of K harmonics must be appreciably better than the shell model. When speaking of the advantages of the method of K harmonics, we must also mention the application of the method in the region of the continuum, where it has proved very valuable in, for example, study of many-particle resonances,[65] in the formulation of asymptotic boundary conditions for nuclear reactions with production of three or more particles,[66] and also in the development of an interpolation approach to the theory of nuclear reactions with the participation of clusters.[67] As a whole, the tendency to "unfreeze" individual nuclear degrees of freedom exhibited by the transition from the shell model to the method of K harmonics is opportune and progressive. It has been further developed in the method of generalized hyperspherical functions.[68] In this last, three collective degrees of freedom (ρ, β, and γ vibrations) are freed. A full exposition and analysis of this method can be found in the reviews Ref. 69.

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Note Added in Proof. As an addition to Sec. 6, we may mention one further interesting possibility of obtaining exact expressions for the potentials $W^{K'\nu}_{K\nu}(\rho)$; this is described in Refs. 70 and 71. By means of an inverse Laplace transformation, the calculation of these integrals is reduced to finding integrals of the Talmi type

$$\int_0^{\rho} r^{2p+2}(\rho^2-r^2)^{(K+K'+1)/2-p}\, V(\sqrt{2}r)\, dr$$

of the two-particle nucleon–nucleon interaction $V(\sqrt{2}r)$.

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