Microscopic theory of nuclear quadrupole excitations

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The method of boson representations of fermion operators is used to obtain a closed expression for the collective quadrupole Hamiltonian of a nucleus. Some solutions of the Schrödinger equation with this collective Hamiltonian are investigated and it is shown that, in principle, the Hamiltonian describes spherical nuclei with their vibrational multiplets and deformed nuclei with rotational bands based on the vibrational states. The properties of the collective quadrupole states of a number of spherical and transition nuclei are analyzed.

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1. BOSON REPRESENTATIONS OF FERMION OPERATORS

The experimental data on low-lying nuclear states indicate that the division of nuclei into spherical and deformed nuclei with characteristic corresponding collective excitations is frequently arbitrary. However, one can always identify a sequence of collective states of a given nucleus that differ only through the values of certain quantum numbers and are coupled by large matrix elements of the quadrupole moment operator.[1] If one is interested in only their energies, quadrupole moments, and $E2$ transitions, these collective quadrupole excitations can be described by five collective coordinates $\alpha_{2\mu}(\mu = 0, \pm 1, \pm 2)$. The problem is that the microscopic Hamiltonian of a nucleus used to describe low-lying states depends on a large number of dynamical variables, among which the collective quadrupole variables are in no way distinguished. One can attempt to distinguish them in the microscopic Hamiltonian in a number of ways.[2–4] Here, we consider only methods in which fermion operators are represented in terms of boson operators, and we consider only even–even nuclei, in which states with an even number of quasiparticles are realized. We shall therefore be concerned with only the algebraic properties of the binary fermion operators: $a_\alpha^+ a_\beta^+$, $a_\alpha^+ a_\beta$, $a_\beta a_\alpha$, which in the given configuration space form the Lie algebra $R_{2\Omega}$. For our purposes, it is more convenient to go over to the operators

$$\left.\begin{aligned} \hat{q}^n_{JM} &= \frac{1}{2}\sum_{a,b} q^{nJ}_{ab}\left[A^+_{JM}(ab) + (-1)^{J-M} A_{J-M}(ab)\right];\\ \hat{p}^n_{JM} &= -\frac{i}{2}\sum_{a,b} p^{nJ}_{ab}\left[A_{JM}(ab) - (-1)^{J-M} A^+_{J-M}(a,b)\right], \end{aligned}\right\} \quad (1)$$

where

$$A^+_{JM} = \sum_{m_a m_b} C^{JM}_{j_a m_a j_b m_b} a^+_{am_a} a^+_{bm_b}.$$

The tensor operators $\hat{q}^n_{JM}$ and $\hat{p}^n_{JM}$ are characterized by the angular momentum J, its third component M, and the additional index n. The amplitudes q^{nJ}_{ab} and p^{nJ}_{ab} satisfy the following orthogonality relations:

$$\sum_{a,b} q^{nJ}_{ab} p^{n'J}_{ab} = \delta_{nn'}, \quad \sum_n q^{nJ}_{ab} p^{nJ}_{cd} = \frac{1}{2}\left[\delta_{ac}\delta_{bd} - (-1)^{j_a+j_b+J}\delta_{ad}\delta_{bc}\right]. \quad (2)$$

Using these relations, we can calculate the commutator

$$[\hat{q}^n_{JM}, \hat{p}^{n'}_{J'M'}] = i\delta_{JJ'}\delta_{MM'}\delta_{nn'} - i\sum_{a,b,c,L,K} q^{nJ}_{ac} p^{n'J'}_{bc} \begin{Bmatrix} j_a & j_b & L\\ J' & J & j_c \end{Bmatrix} \times C^{LK}_{JMJ'-M'}(-1)^{J'-M'}\left[1 + (-1)^{j_a+j_b+L}\right] B_{LK}(a,b), \quad (3)$$

where

$$B_{LK}(ab) = \sum_{m_a m_b} (-1)^{j_b - m_b} C^{LK}_{j_a m_a j_b - m_b} a^+_{am_a} a_{bm_b}.$$

If we ignore the second term in (3), as is done in the *RPA*, the operators $\hat{q}^n_{JM}$ and $\hat{p}^n_{JM}$ satisfy the commutation relations for coordinates and momenta.

The operators (1) are only part of the Lie algebra mentioned above. To close the algebra, we must add to the generalized coordinates $\hat{q}^n_{JM}$ and momenta $\hat{p}^n_{JM}$ all their linearly independent commutators, i.e., $[\hat{p}^n_{JM}, p^{n'}_{J'M'}]$, $[\hat{q}^n_{JM}, \hat{q}^{n'}_{J'M'}]$, and $[\hat{q}^n_{JM}, \hat{p}^{n'}_{J'M'}]$. Using the orthogonality relations (2), we can readily show[2] that the set of operators that consists of the generalized coordinates and momenta and their linearly independent commutators are completely equivalent to the set of binary operators $a^+_\alpha a^+_\beta$, $a^+_\alpha a_\beta$, $a_\beta a_\alpha$. Therefore, one can express the Hamiltonian of the nucleus and all the single-particle operators in terms of $\hat{q}^n_{JM}$, $\hat{p}^n_{JM}$, and their commutators.

To illustrate our method, we assume that there is a single collective variable, which is weakly coupled to the other degrees of freedom. For example, suppose this is a variable describing β vibrations of nuclei. In this case, we can separate out from the set (2) the generalized coordinate $\hat{q}_0$ and the momentum $\hat{p}_0$. Our algebra contains the operators $\hat{q}_0$, $\hat{p}_0$, and $[\hat{q}_0, \hat{p}_0]$. The double commutators have the form

$$[[\hat{q}_0, \hat{p}_0], \hat{q}_0] = -L\hat{p}_0 - \sum_{n\neq 0} L_n \hat{p}_n; \quad [[\hat{q}_0, \hat{p}_0], \hat{p}_0] = K\hat{q}_0 + \sum_{n\neq 0} K_n \hat{q}_n;$$
$$L = 4\sum_{a,b,c,d} q^0_{ab} p^0_{cb} q^0_{ad} q^0_{cd}; \quad K = 4\sum_{a,b,c,d} q^0_{ab} p^0_{cb} p^0_{ad} p^0_{cd}.$$

Thus, strictly speaking, the algebra of the operators $\hat{q}_0$, $\hat{p}_0$, $[\hat{q}_0, \hat{p}_0]$ is not closed. But note, first, that the coefficients L_n and K_n, in contrast to L and K, do not contain coherent parts, and are therefore small. Second, if we are interested in only those matrix elements of the double commutators that couple collective states, the contribution of the noncollective operators $\hat{q}_n$ and $\hat{p}_n$ can be ignored. In this approximation, the operators

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$\hat{q}_0, \hat{p}_0, [\hat{q}_0, \hat{p}_0]$ form a closed algebra, which is identical with the algebra of $SU(2)$.

We now consider the case of two collective variables, for example, β and γ vibrations. We introduce the two collective coordinates $\hat{q}_0$ and $\hat{q}_2$, the two momenta $\hat{p}_0$ and $\hat{p}_2$, and also the commutators $[\hat{q}_0, \hat{q}_2]$, $[\hat{p}_0, \hat{p}_2]$, $[\hat{q}_2, \hat{p}_2]$, $[\hat{q}_0, \hat{p}_0]$, $[\hat{q}_0, \hat{p}_2]$, and $[\hat{q}_2, \hat{p}_0]$. Among these operators, eight are linearly independent, and these form the algebra of $SU(3)$.

We now consider all five quadrupole degrees of freedom, including β and γ vibrations as well as rotations. To this end, we study the generalized coordinates and momenta with angular momentum $J = 2$:

$$\hat{q}^{\text{coll}}_{2\mu} = \frac{1}{2}\sum_{a,\,b} q^{\text{coll}}_{ab}\,[A^+_{2\mu}(ab) + (-1)^{\mu} A_{2-\mu}(ab)];$$

$$\hat{p}^{\text{coll}}_{2\mu} = -\frac{i}{2}\sum_{a,\,b} \hat{p}^{\text{coll}}_{ab}\,[A_{2\mu}(ab) - (-1)^{\mu} A^+_{2-\mu}(ab)].$$

In the same approximation as we discussed above, the operators $q^{\text{coll}}_{2\mu}$, $p^{\text{coll}}_{2\mu}$ and their commutators form a closed algebra, which is identical with the algebra of $SU(6)$

$$\left.\begin{aligned}
&[\hat{q}_\mu, \hat{p}_{\mu'}] = (-1)^{\mu+\mu'}\,[\hat{q}_{-\mu'}, \hat{p}_{-\mu}];\\
&[\hat{p}_\mu, \hat{p}_{\mu'}] = (-1)^{\mu+\mu'}\,[\hat{q}_{-\mu}, \hat{q}_{-\mu'}];\\
&[[\hat{q}_\mu, \hat{p}_{\mu'}], \hat{q}_{\mu''}] = 2\delta_{\mu\mu'}(-1)^{\mu''}\hat{p}_{-\mu''} - \delta_{-\mu\mu'}(-1)^{\mu''}\hat{p}_{\mu'}\\
&\qquad - \delta_{\mu'\mu''}(-1)^{\mu}\hat{p}_{-\mu};\\
&[[\hat{q}_\mu, \hat{p}_{\mu'}], \hat{p}_{\mu''}] = 2\delta_{\mu\mu'}(-1)^{\mu''}\hat{q}_{-\mu''} - \delta_{\mu'\mu''}(-1)^{\mu'}\hat{q}_{\mu}\\
&\qquad + \delta_{\mu\mu''}(-1)^{\mu'}\hat{q}_{-\mu'};\\
&[[\hat{q}_\mu, \hat{q}_{\mu'}], \hat{q}_{\mu''}] = \delta_{-\mu'\mu''}(-1)^{\mu'}\hat{q}_{\mu} - \delta_{-\mu\mu''}(-1)^{\mu}\hat{q}_{\mu'};\\
&[[\hat{q}_\mu, \hat{q}_{\mu'}], \hat{p}_{\mu''}] = \delta_{\mu'\mu''}(-1)^{\mu'}\hat{p}_{-\mu} - \delta_{\mu\mu''}(-1)^{\mu}\hat{p}_{-\mu'};\\
&[[\hat{q}_\mu, \hat{p}_{\mu'}], [\hat{q}_{\mu''}, \hat{p}_{\mu'''}]] = \delta_{\mu'\mu''}(-1)^{\mu+\mu'+\mu'''}[\hat{q}_{-\mu'''}, \hat{q}_{-\mu'}]\\
&\quad + \delta_{\mu\mu'}(-1)^{\mu'}[\hat{q}_{\mu''}, \hat{q}_{-\mu'}] + \delta_{-\mu'\mu''}(-1)^{\mu''}[\hat{q}_{\mu''}, \hat{q}_{\mu'}]\\
&\qquad + \delta_{\mu'\mu''}[\hat{q}_{-\mu'''}, \hat{q}_{\mu}];\\
&[[\hat{q}_\mu, \hat{p}_{\mu'}], [\hat{q}_{\mu''}, \hat{q}_{\mu'''}]] = \delta_{-\mu\mu''}(-1)^{\mu}[\hat{q}_{\mu''}, \hat{p}_{\mu'}]\\
&\quad - \delta_{-\mu\mu''}(-1)^{\mu}[\hat{q}_{\mu'''}, \hat{p}_{\mu'}] + \delta_{\mu'\mu''}(-1)^{\mu'''}[\hat{q}_{\mu}, \hat{p}_{-\mu'''}]\\
&\qquad - \delta_{\mu'\mu''}(-1)^{\mu''}[\hat{q}_{\mu}, \hat{p}_{-\mu''}];\\
&[[\hat{q}_\mu, \hat{q}_{\mu'}][\hat{q}_{\mu''}, \hat{q}_{\mu'''}]] = \delta_{-\mu'\mu''}(-1)^{\mu'}[\hat{q}_{\mu}, \hat{q}_{\mu'''}]\\
&\quad + \delta_{-\mu\mu'''}(-1)^{\mu}[\hat{q}_{\mu}, \hat{q}_{\mu''}] - \delta_{-\mu\mu''}(-1)^{\mu}[\hat{q}_{\mu'}, \hat{q}_{\mu'''}]\\
&\qquad - \delta_{-\mu'\mu''}(-1)^{\mu'}[\hat{q}_{\mu}, \hat{q}_{\mu'''}],
\end{aligned}\right\} \quad (4)$$

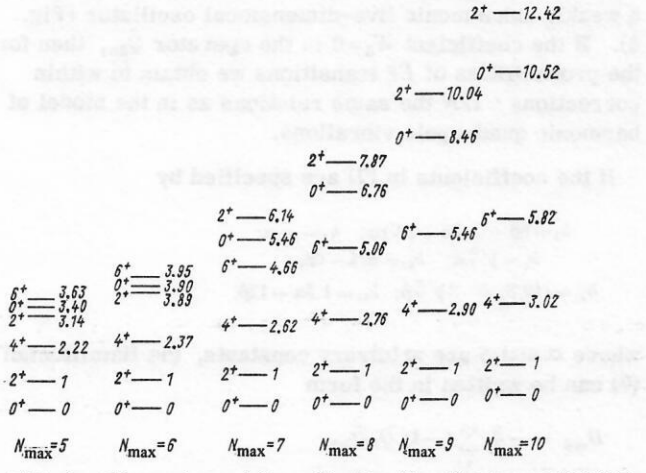


FIG. 1. Eigenvalues of the collective Hamiltonian containing terms to fourth order in powers of the boson operators as functions of the dimension of the space of collective states: $h_1 = 1.0$, $h_2 = -3.6$, $h_3 = 0.9$, $h_{4L} = 0$.

where $\hat{q}_\mu = \hat{q}^{\text{coll}}/\sqrt{L}$ and $\hat{p}_\mu = \hat{p}^{\text{coll}}_\mu/\sqrt{K}$.

Thus, β and γ vibrations can be described in our approach by the Lie algebra of $SU(3)$, while the inclusion of rotational degrees of freedom leads us to the algebra of $SU(6)$. This extension of the algebra is analogous to the situation in elementary-particle physics. The internal degrees of freedom (in our case, β and γ vibrations) are associated with $SU(3)$. The combination of this group with $SU(2)$, which describes the ordinary spin, leads to $SU(6)$.

For the operators $\hat{q}_\mu$ and $\hat{p}_\mu$ and their commutators the following representation holds in terms of the operators of quadrupole phonons $b^+_{2\mu}$ and $b_{2\mu}$:

$$\left.\begin{aligned}
&b^+_{2\mu},\ b_{2\mu}:\\
&\hat{q}_\mu = \sqrt{N - \sum_\nu b^+_{2\nu}b_{2\nu}}\,(-1)^{\mu} b_{2-\mu} + b^+_{2\mu}\sqrt{N - \sum_\nu b^+_{2\nu}b_{2\nu}};\\
&\hat{p}_\mu = i\left[(-1)^{\mu} b^+_{2-\mu}\sqrt{N - \sum_\nu b^+_{2\nu}b_{2\nu}} - \sqrt{N - \sum_\nu b^+_{2\nu}b_{2\nu}}\,b_{2\mu}\right];\\
&i\,[\hat{q}_\mu, \hat{p}_{\mu'}] = b^+_{2\mu}b_{2\mu'} + (-1)^{\mu+\mu'} b^+_{2-\mu'}b_{2-\mu}\\
&\qquad - \delta_{\mu\mu'}(N - \sum_\nu b^+_{2\nu}b_{2\nu});\\
&[\hat{q}_\mu, \hat{q}_{\mu'}] = (-1)^{\mu'} b^+_{2\mu}b_{2-\mu} - (-1)^{\mu} b^+_{2\mu'}b_{2-\mu'}.
\end{aligned}\right\} \quad (5)$$

This is a boson representation of the same type as the one proposed for the algebra of $SU(2)$ by Holstein and Primakoff.[5] Here, N is a positive integer that restricts the maximal number of boson states in the wave functions. From the point of view of group theory, the representation (5) realizes a completely symmetric representation of $SU(6)$, and N is an eigenvalue of the Casimir operator for this group.

As can be seen from (5), in the approach presented above, the fermion operators are represented by infinite series if they are expanded in powers of the operators $b^+_{2\mu}$ and $b_{2\mu}$. In (5), these series have been summed. In Eqs. (5), in each order in $b^+_{2\mu}$ and $b_{2\mu}$, we have taken into account not all the existing terms,[6] but only those whose coefficients are coherent sums over the single-particle quantum numbers. The following question arises: Is it necessary to sum the infinite series in powers of the phonon operators if by no means all terms are taken into account in each order? The answer to this question is as follows. If all terms are taken into account in the expansion of the fermion operators in powers of the collective phonons, the procedure becomes so complicated that in practice one must restrict the treatment to a finite number of terms.[2] As a result, one obtains a collective Hamiltonian whose degree in the operators $b^+_{2\mu}$ and $b_{2\mu}$ is finite. Usually, we restrict ourselves to terms to fourth order. In calculations based on such Hamiltonians with the properties of collective states, convergence is achieved (if nuclei with appreciable anharmonic effects are considered) at unphysically large values of the phonon numbers. As an example, Fig. 1 gives the results of calculation of the spectra of low-lying collective states in ^{56}Fe. The calculation was made with a collective Hamiltonian containing terms to fourth order in powers of $b^+_{2\mu}$ and $b_{2\mu}$. The parameters of the collective Hamiltonian are chosen in such a way as to describe approximately the ^{56}Fe spectrum when the Hamiltonian is diagonalized in the space

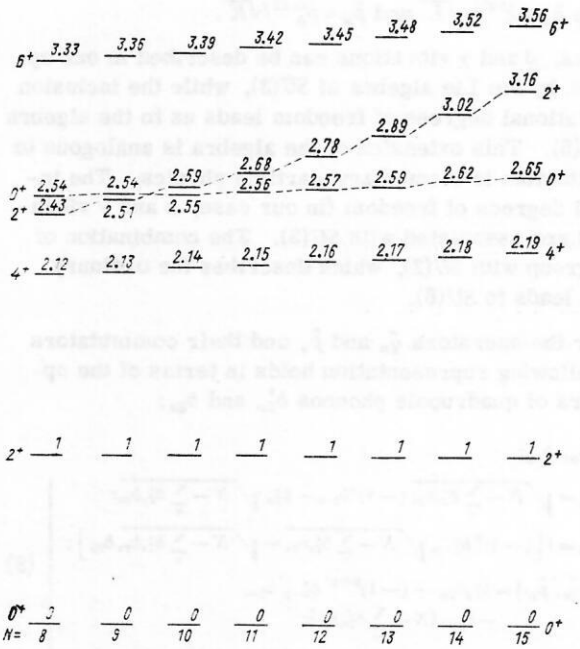


FIG. 2. The same as in Fig. 1, but with $h_1 = 1.0$, $h_2 = 0.59$, $h_3 = 0.12$.

of phonon states with maximal number of phonons $N = 5$. (In ^{56}Fe, there are two pairs of nucleons above the ^{56}Ni core and eight pairs of nucleons above the ^{40}Ca core.) The space is then extended to $N = 6, 7, 8, \ldots, 12$. As can be seen from Fig. 1, convergence is not achieved even for $N = 12$. Analogous results for ^{108}Pd are given in Fig. 2.

Since the treatment is restricted to the collective operators $\hat{q}_\mu$ and $\hat{p}_\mu$, to construct a collective Hamiltonian in terms of the operators of the quadrupole phonons $b^+_{2\mu}$ and $b_{2\mu}$ it is first of all necessary to express the fermion Hamiltonian approximately in terms of the operators $\hat{q}_\mu$ and $\hat{p}_\mu$. This can be done by means of the following prescription:

$$\left.\begin{aligned} &a^+a \to g\,[\hat{q},\ \hat{p}];\\ &a^+a^+a^+a^+ \to g_1(\hat{q}-i\hat{p})^2;\\ &a^+a^+a^+a \to g_2(\hat{q}-i\hat{p})\,[\hat{q},\ \hat{p}];\\ &a^+a^+aa \to g_3\,[\hat{q},\ \hat{p}]\,[\hat{q},\ \hat{p}]+g_4\,[\hat{q},\ \hat{q}]\,[\hat{q},\ \hat{q}]+g_5(\hat{q}-i\hat{p})(\hat{q}+i\hat{p}). \end{aligned}\right\} \quad (6)$$

To find the coefficients g_i, it is necessary to calculate the matrix elements of both sides of Eq. (6) between the collective states and a small number of quasiparticles (or phonons). After this, to construct the collective Hamiltonian, one can use the relations (5). We then obtain the collective Hamiltonian

$$\begin{aligned} \hat{H}_{\text{coll}} &= h_0 - h_1\hat{N} + h_2\sum_\nu (-1)^\nu \\ &\times \left(b^+_{2\nu}b^+_{2-\nu}\sqrt{(N-\hat{N})(N-1-\hat{N})} + \text{h.c.}\right) \\ &+ h_3\sum_\nu (-1)^\nu \left(b^+_{2\nu}[b^+_2b_2]_{2-\nu}\sqrt{N-\hat{N}} + \text{h.c.}\right) \\ &+ \sum_{\substack{L=0,\,2,\,4\\ M}} h_{4L}\,[b^+_2b^+_2]_{LM}\,[b_2b_2]_{LM}, \end{aligned} \quad (7)$$

where $\hat{N} = \sum_\mu b^+_{2\mu}b_{2\mu}$. The square brackets denote vector coupling. The constants h_0, h_1, h_2, h_3, h_{4L} are expressed in terms of the single-particle energies and the matrix elements of the interaction. The exact expressions for them are given in Ref. 2. One can obtain a similar expression for the operator of the quadrupole moment $\hat{Q}_{2\mu}$:

$$\hat{Q}_{2\mu} = W_1\left(b^+_{2\mu}\sqrt{N-\hat{N}} + \sqrt{N-\hat{N}}\,(-1)^\mu b_{2-\mu}\right) + W_2\,[b^+_2b_2]_{2\mu}. \quad (8)$$

2. CONSTRUCTION OF COLLECTIVE HAMILTONIAN AND ANALYSIS OF SOME SOLUTIONS OF IT

The Hamiltonian (7) and $\hat{Q}_{2\mu}$ are constructed from the operators

$$b^+_{2\mu}\sqrt{N-\hat{N}},\quad \sqrt{N-\hat{N}}\,b_{2\mu},\quad b^+_{2\mu}b_{2\mu} \quad (9)$$

and are combinations of linear and quadratic terms in these operators. In their turn, the 35 operators (9) are a boson realization of the algebra of $SU(6)$ for the completely symmetric representation characterized by the quantum number N. Thus, the Hamiltonian (7) is diagonalized in the space of states that realize the representation of the $SU(6)$ algegra with quantum number N.

Let us consider a number of special solutions[7] of the Schrödinger equation with the Hamiltonian (7). If $h_2 = h_3 = 0$ in (7), then the Hamiltonian commutes with $\hat{N}$, and and its eigenfunctions are characterized by a definite number of quadrupole phonons. The spectrum of eigenvalues of the Hamiltonian has the form

$$E(I, n, v) = a_1n + a_2n^2 + a_3v(v+3) + a_4I(I+1),$$

where I is the total angular momentum; n is the number of phonons and v is the seniority;

$$\begin{aligned} a_1 &= h_1 - 0.6h_{40} - 4\sqrt{5}\,h_{42}/35 - 12h_{44}/35;\\ a_2 &= 0.2h_{40} + 2\sqrt{5}\,h_{42}/35 + 6h_{44}/35;\\ a_3 &= -0.2h_{40} + 2\sqrt{5}\,h_{42}/35 + h_{44}/35;\\ a_4 &= h_{44}/21 - h_{42}/7\sqrt{5}. \end{aligned}$$

To classify states with more than five phonons, an additional quantum number is required. If a_2, a_3, and a_4 are small compared with a_1, we obtain the spectrum of a weakly anharmonic five-dimensional oscillator (Fig. 3). If the coefficient $W_2 = 0$ in the operator $\hat{Q}_{2\mu}$, then for the probabilities of $E2$ transitions we obtain to within corrections $\sim 1/N$ the same relations as in the model of harmonic quadrupole vibrations.

If the coefficients in (7) are specified by

$$\begin{aligned} h_1 &= 6\beta + (4.25 + 2N)\alpha;\quad h_2 = -\alpha;\\ h_3 &= \sqrt{7}\,\alpha;\quad h_{40} = \alpha/4 - 6\beta;\\ h_{42} &= (19/8)\alpha - 3\sqrt{5}\,\beta;\quad h_{44} = 1.5\alpha + 12\beta, \end{aligned}$$

where α and β are arbitrary constants, the Hamiltonian (7) can be written in the form

$$\begin{aligned} H_{\text{coll}} &= -\frac{\alpha}{8}\sum_\nu (-1)^\nu \hat{Q}_{2\nu}\hat{Q}_{2-\nu} \\ &+ \beta\sum_\nu (-1)^\nu I_\nu I_{-\nu}. \end{aligned} \quad (10)$$

In this expression

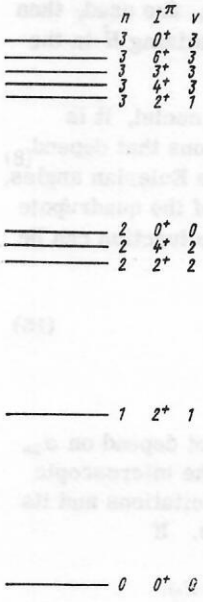


FIG. 3. Spectrum of collective states of a weakly anharmonic five-dimensional oscillator.

$$\left.\begin{aligned}\tilde{Q}_{2\nu}&=-\sqrt{8}\left(b^{+}_{2\nu}\sqrt{N-\hat{N}}\right.\\&\left.+\sqrt{N-\hat{N}}\,(-1)^{\nu}b_{2-\nu}\right)\\&-\sqrt{14}\,[b^{+}_{2}b_{2}]_{2\nu};\\I_{\nu}&=\sqrt{10}\,[b^{+}_{2}b_{2}]_{1\nu}.\end{aligned}\right\}\tag{11}$$

One can show that the eight operators (11) form the algebra of $SU(3)$, and the Hamiltonian (10) is a linear combination of a Casimir operator of $SU(3)$ and the square of the angular momentum operator $\sum_{\nu}(-1)^{\nu}I_{\nu}I_{-\nu}$. The spectrum of eigenvalues of the Hamiltonian (10) has the form

$$\begin{aligned}&E(I,\lambda,\mu)\\&=-\alpha(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)/2-(\beta+3\alpha/8)\,I(I+1).\end{aligned}\tag{12}$$

Here, λ and μ are positive integers that characterize the irreducible representations of $SU(3)$. It must be borne in mind that not all irreducible representations of $SU(3)$ are realized as eigenfunctions of the Hamiltonian (10), but only those that belong to completely symmetric representations of $SU(6)$ characterized by given N. Such representations of $SU(3)$ can be found by means of the method described in Ref. 8. The representations (λ, μ) belonging to symmetric $SU(6)$ representations for $N=15$ are given in Table I. It can be seen, for example, that representations with odd λ or μ are absent. This is the main difference between the $SU(3)$ scheme introduced by Elliot to describe rotational excitations of nuclei, and the solutions of the Schrödinger equation with Hamiltonian (7) that we considered above. In Elliot's scheme, all the representations (λ, μ) of $SU(3)$ are realized, although it is known experimentally that a number of rotational bands of this scheme do not occur in even-even nuclei.

As can be seen from (12), the spectrum of eigenvalues of the Hamiltonian (10) is a collection of rotational bands with equal moments of inertia based on different internal states. Each band is characterized by the quantum numbers (λ, μ, K). Here, K is an additional

TABLE I. $SU(3)$ representations belonging to symmetric representations of $SU(6)$ for $N=15$.

n_b^*	0	1	2	3	4
$(\lambda,\mu)^n$	$(30,0)^1$	$(26,2)^1$	$(24,0)^1$ $(22,4)^1$	$(18,6)^1$ $(20,2)^1$	$(14,8)^1$ $(16,4)^1$ $(18,0)^1$

n_b^*	5	6	7
$(\lambda,\mu)^n$	$(10,10)^1$ $(12,6)^2$ $(14,2)^2$	$(8,8)^1$ $(10,4)^2$ $(12,0)^2$	$(6,6)^1$ $(8,2)^2$ $(4,4)^1$ $(6,0)^2$ $(2,2)^1$ $(0,0)^1$

*n_b is the number of internal phonons.

quantum number equal to the minimal spin of the band that takes the values μ, $\mu-2, \ldots$ (Ref. 8). For $N \geq 10$, the band with the lowest energy has the quantum numbers $(\lambda, \mu, K) = (2N, 0, 0)$. The next two bands are degenerate and have the quantum numbers $(\lambda, \mu, K) = (2N-4, 2, 0)$ and $(2N-4, 2, 2)$. Then follows a group of four bands with quantum numbers $(\lambda, \mu, K) = (2N-8, 4, 0)$, $(2N-8, 4, 2)$, $(2N-8, 4, 4)$ and $(2N-6, 0, 0)$ (Fig. 4). If N is a sufficiently large number, then the ratio of the energies of the bases of the second and the first group of bands is approximately equal to 2, and the spectrum in Fig. 4 can be interpreted in the framework of a traditional picture of collective excitations of strongly deformed nuclei. The band with lowest energy is based on the ground state. The band $(\lambda = 2N-4, \mu = 2, K = 0)$ is based on the β vibrational state, and the band $(\lambda = 2N-4, \mu = 2, K = 2)$ on the γ vibrational state. The energies of the quanta of both vibrations are approximately equal and $\omega_\beta = \omega_\gamma = 6\alpha N$. A second group of bands is based on two-phonon states. These are either two β phonons $(K = 0)$ or two γ phonons $(K = 0, 4)$ or one β phonon and one γ phonon $(K = 2)$. In the general case, one can show that if $N \gg n_b$, where n_b is the total number of β and γ phonons, the spectrum of eigenvalues of the Hamiltonian (10) coincides with the spectrum

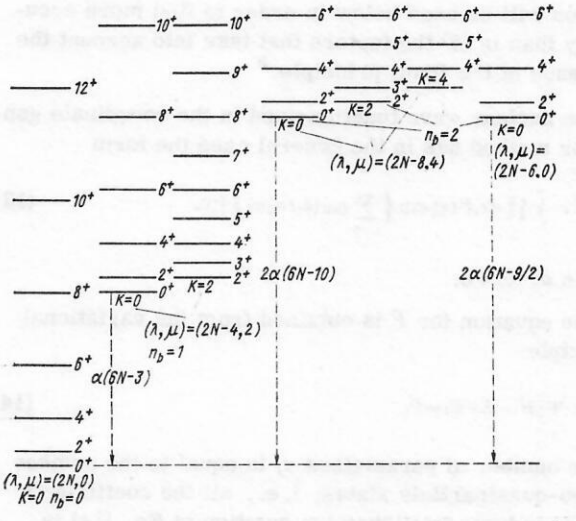


FIG. 4. Classification of collective states by means of $SU(3)$ quantum numbers.

$$E(n_\beta, n_\gamma, I) = 6\alpha N(n_\beta + n_\gamma) + (\beta + 3\alpha/8) I(I+1),$$

where n_β and n_γ are the number of β and γ phonons, respectively.

If it is assumed that the operator of the electric quadrupole moment is proportional to $\tilde{Q}_{2\mu}$, the probability of $E2$ transitions between bands with different (λ, μ) is zero. The probabilities of transitions between bands with the same (λ, μ) but different K remain undetermined, since these bands are degenerate. For transitions within one band, the following expression is obtained[8]:

$$B(E2; I \to I') = 4(4N^2 + 6N + 3) \times (C^{I'0}_{I020})^2 \left(1 - \frac{1}{2}\frac{1}{4N^2+6N+3}[I(I+1) + I'(I'+1)]\right),$$

where $C^{I'0}_{I020}$ is a Clebsch–Gordan coefficient. For $I^2 \ll 4N^2$, this expression gives Alga's rules for the probabilities of E_2 transitions.

Thus, we have shown that the Hamiltonian (7) contains solutions corresponding to the model of harmonic quadrupole vibrations and the model of an axisymmetric rotator executing small vibrations about its equilibrium shape.

3. ALLOWANCE FOR THE PAULI PRINCIPLE IN THE CONSTRUCTION OF BOSON REPRESENTATIONS OF FERMION OPERATORS

The Hamiltonian (7) obtained above differs from the Hamiltonian found by the method of boson expansion of fermion operators (Ref. 2) by the presence of the factor $\sqrt{N - \hat{N}}$, which takes into account approximately the influence of the Pauli principle. That this factor depends only on the number of phonons is due to the approximation. Below, we shall develop a method that enables one to take into account the dependence of this factor on not only the number of phonons but also the seniority and the spin.

In Ref. 2c, a connection was established between the method of the coordinate generator and the method of boson representation of fermion operators. This connection will be used below in order to find more accurately than in (5) the factors that take into account the influence of the Pauli principle.[9]

The nuclear wave function used in the coordinate generator method has in the general case the form

$$\Psi = \int \prod_i dx_i F(x) \exp\left\{\sum_{\alpha,\beta} f_{\alpha\beta}(x_i)\, a^+_\alpha a^+_{\bar{\beta}}\right\} |0\rangle. \qquad (13)$$

where $a_\alpha |0\rangle = 0$.

The equation for F is obtained from the variational principle

$$\delta \langle \Psi | \hat{H} - E | \Psi \rangle = 0. \qquad (14)$$

If the number of parameters x_i is equal to the number of two-quasiparticle states, i.e., all the coefficients $f_{\alpha\beta}$ are independent, then the solution of Eq. (14) is equivalent to exact solution of the Schrödinger equation. If the number of parameters x_i is appreciably less, and only the variables β and γ, for example, are used, then the solution (14) is equivalent to diagonalizing $\hat{H}$ in the space of internal collective states.

To describe spherical and transition nuclei, it is necessary to use generator wave functions that depend on five parameters, β, γ, and the three Eulerian angles, or, equivalently, the five components of the quadrupole deformation tensor $\alpha_{2\mu}$. Then the wave function can be written in the form

$$\Psi = \int d\alpha_{-2} \ldots d\alpha_2 F(\alpha) \exp\left\{\sum_\mu \alpha^*_\mu A^+_{\bar{2}\mu}\right\} |0\rangle, \qquad (15)$$

where $A^+_{2\mu} = \sum_{a,b} \Phi_{ab} C^{2\mu}_{j_a m_a j_b m_b} a^+_{am_a} a^+_{bm_b}$.

The fact that the amplitudes Φ_{ab} do not depend on $\alpha_{2\mu}$ means we have assumed constancy of the microscopic structure of the collective branch of excitations and its independence of the vibration amplitude. If

$$\Phi_{ab} = (1 + \delta_{ab})\sqrt{5}\, \langle a | r^2 Y_2 | b \rangle (u_a v_b + u_b v_a)(E_a + E_b),$$

and the parameters $\alpha_2\mu$ are fixed such that $\alpha_{2\mu\neq 0}$, $\alpha_0 = \delta$, then $\exp\{\sum_\mu \alpha^*_{2\mu} A^+_{2\mu}\}$ is equal to the Nilsson wave function for small values of the deformation parameter δ. If the Tamm–Dancoff method is used to determine the structure of the collective variable, then the amplitudes are

$$\Phi_{ab} = (1 + \delta_{ab})\sqrt{5}\, \langle a | r^2 Y_2 | b \rangle (u_a v_b + u_b v_a)(E_a + E_b - \omega),$$

where ω is the frequency of the collective vibrations in the Tamm–Dancoff approximation.

The expression of the nuclear wave function in the form (15) means that we shall diagonalize the Hamiltonian, not in the complete space of fermion states, but in the space constructed by means of only the operators $A^+_{2\mu}$. Since this space acquires importance in the method developed below, we shall study it in more detail. But let us first consider the properties of the space of states constructed by means of the operators $b^+_{2\mu}$ of quadrupole bosons:

$$[b_{2\mu}, b^+_{\bar{2}\mu'}] = \delta_{\mu\mu'}, \quad [b_{2\mu}, b_{2\mu'}] = [b^+_{\bar{2}\mu}, b^+_{\bar{2}\mu'}] = 0.$$

Each boson state is uniquely determined by five quantum numbers, for which the eigenvalues of the following operators are usually employed in nuclear physics:

$$\left.\begin{aligned}
&\hat{N}_B = \sum_\mu b^+_{\bar{2}\mu} b_{2\mu};\ \hat{I}^2_B = 10 \sum_\mu (-1)^\mu [b^+_{\bar{2}} b_2]_{1\mu} [b^+_{\bar{2}} b_2]_{1-\mu};\\
&(\hat{I}_B)_Z = \sqrt{10}\, [b^+_{\bar{2}} b_2]_{10};\\
&\hat{T}^2_B = 2\Big|\sum_\mu (-1)^\mu [b^+_{\bar{2}} b_2]_{3\mu} [b^+_{\bar{2}} b_2]_{3-\mu}\\
&\quad - \sum_\mu (-1)^\mu [b^+_{\bar{2}} b_2]_{1\mu} [b^+_{\bar{2}} b_2]_{1-\mu}\Big|;\\
&\hat{\Omega}_B = \sum_\mu (-1)^\mu \{[[b^+_{\bar{2}} b_2]_3 [b^+_{\bar{2}} b_2]_1]_{2\mu} [[b^+_{\bar{2}} b_2]_3 [b^+_{\bar{2}} b_2]_1]_{2-\mu}\}.
\end{aligned}\right| \qquad (16)$$

The physical meaning of the operators $\hat{N}_B$, $\hat{I}^2_B$, and $(\hat{I}_B)_Z$ is well known. The eigenvalues of the operator $\hat{T}^2_B$ can be written in the form $v(v+3)$, where v is a positive integer, called the seniority. The operator T^2_B commutes with the Bohr–Mottelson Hamiltonian if the potential energy and the mass coefficients do not depend on γ. The operator $\hat{\Omega}_B$ is needed for complete classification of the boson states. Thus, the states constructed

by means of the operator $b^+_{2\mu}$ can be written as follows:

$$|Nv\Omega IM)_B. \tag{17}$$

In the basis (17), the matrix elements $(Nv\Omega IM|\, b^+_{2\mu}\,|N-1, v'\Omega'I'M')$ can be found analytically.[8]

We now consider the space of states constructed by means of the quadrupole fermion operators $A^+_{2\mu}$: $|0\rangle$, $A^+_{2\mu}|0\rangle$, $A^+_{2\mu_1}A^+_{2\mu_2}|0\rangle$.... Since all the five operators $A^+_{2\mu}$ commute with one another and all five are independent, like the phonon operators, it is in principle possible to define uniquely the operators $\hat N_{\rm ph}$, $\hat T^2_{\rm ph}$, $\hat I^2_{\rm ph}$, $(\hat I_{\rm ph})_Z$, and $\hat\Omega_{\rm ph}$ by the requirement that they act on the fermion states $|\mu_1 \dots \mu_n\rangle \equiv A^+_{2\mu_1}\dots A^+_{2\mu_n}|0\rangle$ in the same way as the operators (16) on the boson states $b^+_{2\mu_1}\dots b^+_{2\mu_n}|0\rangle$. Thus, the set of fermion states $|\mu_1\dots\mu_n\rangle$ is equivalent to the set $[Nv\Omega IM\rangle$. The correspondence between the fermion and boson state vectors has been established. However, there is an important difference between the fermion and boson bases. The number of boson states is not bounded, whereas the number of fermion states is if the space of single-particle states is bounded. This follows from the Pauli principle. With each fermion state $|Nv\Omega IM\rangle$ one can associate a boson state, but the converse is not true. The boson states with which one can associate fermion states form the physical subspace in the complete boson space.

We now turn to the construction of the boson images of the fermion operators. Equation (14) can be rewritten as follows:

$$\int \prod_\mu d\alpha'_{2\mu}\,\langle\alpha|\hat H - E|\alpha'\rangle F(\alpha') = 0, \tag{18}$$

where $|\alpha\rangle = \exp\{\sum_\mu \alpha^*_{2\mu}A^+_{2\mu}\}|0\rangle$.

If it were possible to express $\langle\alpha|H|\alpha'\rangle$ in the form $\hat h(\alpha_{2\mu}, \partial/\partial\alpha_{2\mu})\langle\alpha|\alpha'\rangle$, we should obtain instead of (18)

$$\{\hat h(\alpha_{2\mu}, \partial/\partial\alpha_{2\mu}) - E\}\int\prod_{\mu'} d\alpha'_{2\mu'}F(\alpha')\langle\alpha|\alpha'\rangle = 0.$$

Thus, we obtain a Schrödinger equation in which $h(\alpha_{2\mu}, \partial/\partial\alpha_{2\mu})$ is the collective Hamiltonian and $\Pi_\mu d\alpha'_\mu F(\alpha')\times\langle\alpha|\alpha'\rangle$ the collective wave function. Before we construct $h(\alpha_{2\mu}, \partial/\partial\alpha_{2\mu})$, it is necessary to calculate the matrix elements

$$\langle\alpha|A_{2\mu}|\alpha'\rangle = \langle 0|\exp\{\sum_\nu \alpha_{2\nu}A_{2\nu}\}A_{2\mu}|\alpha'\rangle = \partial/\partial\alpha_{2\mu}\langle\alpha|\alpha'\rangle;$$

$$\langle\alpha|A^+_{2\mu}|\alpha'\rangle = \langle 0|\exp\Big\{\sum_\nu \alpha_{2\nu}A_{2\nu}\Big\}A^+_{2\mu}|\alpha'\rangle$$

$$= \alpha_{2\mu}\langle\alpha|\alpha'\rangle + \frac12\sum_{\nu,\nu'}\alpha_{2\nu}\alpha_{2\nu'}\langle\alpha|[A_{2\nu'}[A_{2\nu}, A^+_{2\mu}]]|\alpha'\rangle$$

$$= \Big(\alpha_{2\mu} - \Big(\frac1L + \frac{1}{K+L}\Big)\alpha_{2\mu}\sum_\nu \alpha_{2\nu}\frac{\partial}{\partial\alpha_{2\nu}} - \frac{1}{2L(K+1)}$$

$$\times\sum_\nu(-1)^\nu\alpha_{2\nu}\alpha_{2-\nu}(-1)^\mu\frac{\partial}{\partial\alpha_{2-\mu}} + \frac1R[\alpha_{2\mu}, \hat I^2] + \dots\Big)\langle\alpha|\alpha'\rangle,$$

where the coefficients K, L, and R are expressed in terms of the amplitudes Φ_{ab}. The following boson representations for the operators $A_{2\mu}$ and $A^+_{2\mu}$ are obtained:

$$A_{2\mu}\to\partial/\partial\alpha_{2\mu}; \tag{19}$$

$$A^+_{2\mu}\to\alpha_{2\mu} - \Big(\frac1L + \frac{1}{K+L}\Big)\alpha_{2\mu}\sum_\nu\alpha_{2\nu}\frac{\partial}{\partial\alpha_{2\nu}}$$

$$-\frac{1}{2L(K+1)}\sum_{\nu,\mu}(-1)^{\nu+\mu}\alpha_\nu\alpha_{-\nu}\frac{\partial}{\partial\alpha_{-\mu}} + \frac1R[\alpha_{2\mu}, \hat I^2] + \dots \tag{20}$$

In what follows, we shall use the substitution

$$\alpha_{2\mu}\to b^+_{2\mu},\ \partial/\partial\alpha_{2\mu}\to b_{2\mu}, \tag{21}$$

which preserves the commutation relations. We denote the result of substituting (21) into the right-hand side of (20) by $B^+_{2\mu}$. The boson representations (19) and (20) are representations of the Dyson type. They do not preserve the properties of Hermitian conjugation of the operators $A_{2\mu}$ and $A^+_{2\mu}$ since $b_{2\mu}\neq B_{2\mu}$. But, as will be shown below, the Hermitian conjugate boson operators $A_{2\mu}$ and $A^+_{2\mu}$ can be readily obtained.

Since

$$[A^+_{2\mu}, A^+_{2\mu'}] = 0, \tag{22}$$

we require

$$[B^+_{2\mu}, B^+_{2\mu'}] = 0. \tag{23}$$

In (20), we retain in addition to the first four terms only those needed to satisfy the condition $[A^+_{2\mu}, A^+_{2\mu'}] = 0$.

If $K\to\infty$ and $R\to\infty$ the simplest solution for $B^+_{2\mu}$ satisfying the condition (23) is $B^+_{2\mu} = b^+_{2\mu}(1-\hat N/L)$. If only $R\to\infty$, then the condition (23) can be satisfied by means of a finite number of terms in (20):

$$B^+_{2\mu} = b^+_{2\mu} - \Big(\frac1L + \frac{1}{K+1}\Big)b^+_{2\mu}\hat N - \frac{1}{2L(K+1)}\sum_\nu(-1)^{\nu+\mu}b^+_{2-\nu}b_{2-\mu}$$

$$+\frac{1}{L(K+1)}b^+_{2\mu}\hat N^2 - \frac{1}{2L(K+1)}\sum_\nu(-1)^\nu b^+_{2\nu}b^+_{2-\nu}\hat N(-1)^\mu b_{2-\mu}$$

$$\equiv B^+_{2\mu}(L, K).$$

If all three coefficients K, L, and R are finite, the condition (23) cannot be satisfied if (20) is a finite series. However, it is easy to see that this condition can be satisfied by writing $B^+_{2\mu}$ in the form

$$B^+_{2\mu} = F^{-1}(\hat I^2/R)B^+_{2\mu}(K, L)F(\hat I^2/R)F^{-1}(6/R),$$

where F is an arbitrary function. The factor $F^{-1}(6/R)$ is substituted in order to leave the coefficient of $b^+_{2\mu}$ equal to unity.

The operator $B^+_{2\mu}(K, L)$ can be represented in the form $B^+_{2\mu}(K, L) = \Phi^{-1}b^+_{2\mu}\Phi$, where Φ is an Hermitian operator diagonal in the basis $|Nv\Omega IM\rangle$. Its matrix elements depend only on N and v:

$$\left.\begin{aligned}&\Phi(N, v)/\Phi(N+1, v+1) = [1-(N+v)/2K](1-N/L);\\ &\Phi(N, v)/\Phi(N-1, v-1) = [1-(N+v-3)/2K](1-N/L).\end{aligned}\right\} \tag{24}$$

Thus, we have obtained the following boson representation for the operators $A_{2\mu}$ and $A^+_{2\mu}$:

$$\left.\begin{aligned}&A_{2\mu}\to b_{2\mu};\\ &A^+_{2\mu}\to F^{-1}\Phi^{-1}b^+_{2\mu}\Phi F.\end{aligned}\right\} \tag{25}$$

As we noted above (25) does not preserve the proper-

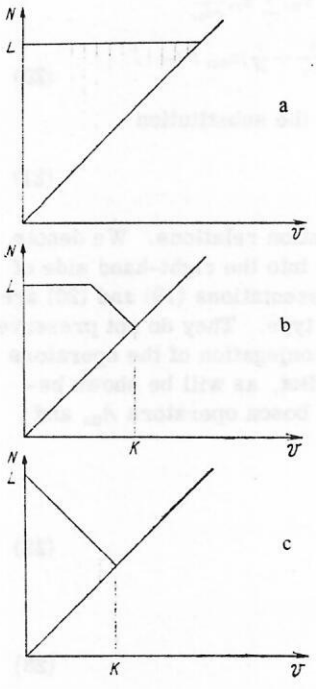


FIG. 5. Boundaries of physical regions of the space of selective states: a) $L \leq K$; b) $K < L < 2K$; c) $2K \leq L$.

ties of Hermitian conjugation of the operators $A_{2\mu}$ and $A^+_{2\mu}$. The requirement of Hermitian conjugation can be satisfied by means of the boson representation

$$\left.\begin{aligned} A_{2\mu} &\to F^{1/2}\Phi^{1/2}b_{2\mu} \\ &\times \Phi^{-1/2}F^{-1/2}; \\ A^+_{2\mu} &\to F^{-1/2}\Phi^{-1/2}b^+_{2\mu} \\ &\times \Phi^{1/2}F^{1/2}. \end{aligned}\right\} \tag{26}$$

Analytic expressions for the matrix elements of the operators (26) are given in Appendix 1. It can be seen from them that the boundaries of the physical region in the complete boson space are determined by means of the constants K and L. In Fig. 5, the boundaries of the physical region are shown for the cases $L \ll K$(a), $K < L < 2K$(b), and $L \geq 2K$(c).

To construct the collective Hamiltonian, one can use the method formulated at the end of Sec. 1 of this paper, except now it is necessary to find approximate expressions for the fermion operators, not in terms of $\hat{q}_{2\mu}$ and $\hat{p}_{2\mu}$, but of $A^+_{2\mu}$ and $A_{2\mu}$. As an example of the use of the boson representation (26), we give the eigenvalue spectrum of the Hamiltonian

$$\hat{H} = c_1 \sum_{\mu} [A_{2\mu}, A^+_{2\mu}] + c_2 \sum_{\mu} A^+_{2\mu} A_{2\mu}. \tag{27}$$

If K, L, $R \to \infty$, then $E = 5c_1 + c_2 N$. If only $R \to \infty$, then

$$E = c_1\left\{5 - N\frac{6L-6K-7}{L(K+1)} + N^2\frac{6}{L(K+1)} + \frac{N(N+3)-v(v+3)}{2L(K+1)}\right\}$$
$$+ c_2\left\{N\frac{LK-4L-5K-5}{L(K+1)} - N^2\frac{K+L-3}{L(K+1)} - N\frac{N(N+3)-v(v+3)}{2L(K+1)}\right\}. \tag{28}$$

The Hamiltonian (27) is an approximate expression in terms of $A^+_{2\mu}$ and $A_{2\mu}$ of the Hamiltonian used in the Tamm–Dancoff approximation. It can be seen from (28) that the deviations from equal spacing are due to the influence of the Pauli principle, especially in the lighter nuclei (for example, $A \approx 70$), where $L \approx 5$–8.

Thus, the method developed in this section has made it possible to find comparatively easily the dependence of the factors that determine the influence of the Pauli principle, i.e., $\Phi(N, v)$ and $F(I)$, on not only the phonon number N but also the seniority v and spin I.

We shall show below how one can calculate the matrix elements of the operators $A_{2\mu}$ and $A^+_{2\mu}$ and construct a collective Hamiltonian without introducing explicitly the phonon operators $b^+_{2\mu}$ and $b_{2\mu}$ or using the approximate expression for the microscopic Hamiltonian in terms of $A^+_{2\mu}$ and $A_{2\mu}$ as an intermediate stage.

4. COLLECTIVE HAMILTONIAN WITHOUT COLLECTIVE VARIABLES

We shall use here the same basis of collective fermion states as in Sec. 3:

$$|Nv\Omega IM\rangle = N^{-1/2}_{Nv\Omega I}[A^+_2 \ldots A^+_2]_{Nv\Omega IM}|0\rangle, \tag{29}$$

where $N_{Nv\Omega I}$ is a normalization factor that must be calculated. We illustrate the method of calculating the matrix elements of the microscopic Hamiltonian between the states (29) by calculating the matrix elements of H_{40}, the part of the total Hamiltonian containing terms of the form $a^+a^+a^+a^+$:

$$\begin{aligned} &\langle N+2, v'\Omega' IM|H_{40}|Nv\Omega IM\rangle \\ &= N^{-1/2}_{Nv\Omega I}\langle N+2, v'\Omega' IM|[A^+_2 \ldots A^+_2]_{Nv\Omega IM}H_{40}|0\rangle \\ &= N^{-1/2}_{Nv\Omega I}\sum_k \langle N+2, v'\Omega' IM|[A^+_2 \ldots A^+_2]_{Nv\Omega IM}|k\rangle\langle k|H_{40}|0\rangle. \end{aligned} \tag{30}$$

Here, $|k\rangle$ is a complete set of four-quasiparticle states with zero angular momentum, these also including the collective state $(1/\sqrt{2})N^{-1/2}_{2000}[A^+_2A^+_2]_0|0\rangle$. The approximation will be to retain in the sum over k only the collective state that gives the largest contribution on account of the coherent effect:

$$\langle N+2, v'\Omega' IM|H_{40}|Nv\Omega IM\rangle \approx N^{-1/2}_{Nv\Omega I}\frac{1}{\sqrt{2}}N^{-1}_{2000}\sqrt{\frac{(N+2)(N+1)}{2}}$$
$$\times (Nv\Omega I;\ 0,2|\}N+2,\ v'\Omega' I) \times N^{-1/2}_{N+2,\ v'\Omega' I}\langle 0|[A_2A_2]_{00}H_{40}|0\rangle. \tag{31}$$

Here, $(Nv\Omega I;\ 0.2|\}N+2,\ v'\Omega I$ is a fractional-percentage coefficient.

Thus, the problem of calculating the matrix element of H_{40} reduces to calculating the normalization factors $N_{Nv\Omega I}$ and the matrix element of H_{40} between states with few quasiparticles, $\langle 0[A_2A_2]_{00}H_{40}|0\rangle$, which can be readily done. The matrix elements of the remaining terms of the microscopic Hamiltonian are calculated similarly. A certain care must be used to calculate the matrix elements of $H_{31} \sim a^+a^+a^+a$, which do not contain coherent parts. The expression (31) is such that only the simplest matrix element, $\langle 20\,000|H_{40}|0\rangle$, can be calculated exactly; all the remainder must be calculated approximately. The approximate expression for the matrix element $\langle 3102\mu|H_{40}|1102\mu\rangle$, found from (31) can be compared with the one calculated exactly for ^{126}Ba. The discrepancy is a fraction of a percent. [2h] The exact and

TABLE II. Comparison of exact and approximate values of the matrix element $M \equiv \langle 3112\,\mu | H_{40} | 1112\mu\rangle$ for Se isotopes.

Isotopes	M_{exact}	M_{approx}	Δ_M, %	Isotopes	M_{exact}	M_{approx}	Δ_M, %
^{72}Se	−3.277	−3.556	8.8	^{76}Se	−3.082	−3.181	3.2
^{74}Se	−3.387	−3.444	1.7	^{78}Se	−2.631	−2.868	9.0

approximate matrix elements for lighter nuclei are compared in Table II.

Equations of the type (31) can be improved by calculating not one, but several of the simplest matrix elements exactly. We now turn to the calculation of the normalization factors $N_{Nv\Omega I}$. This can be done in the same approximation as in the calculation of the matrix elements of the microscopic Hamiltonian:

$$\begin{aligned} N_{Nv\Omega I} &\equiv \langle Nv\Omega IM | Nv\Omega IM\rangle \\ &= \sum_{v'\Omega' I'} (N-1, v'\Omega' I'; 2 | \} Nv\Omega I)^2 N_{N-1, v', \Omega', I'} \\ &\quad + \frac{\sqrt{N-1}}{2} \Sigma (N-1, v'\Omega' I'; 2 | \} Nv\Omega I) \\ &\quad \times (N-2, v''\Omega'' I''; J, 2 | \} Nv\Omega I)\, C^{IM}_{I'M'2\mu} C^{IM}_{I''M''Jm} \\ &\quad \times C^{JM}_{2v2v'} \langle 0 | [A_2 \dots A_2]_{N-1, v'\Omega' I' M'} \\ &\quad \times [A_2^+ \dots A_2^+]_{N-2, v''\Omega'' I'' M''} | k\rangle \langle k | [[A_{2\mu'}, A^+_{2v}], A^+_{2v'}] | 0\rangle. \end{aligned}$$

Here, $|k\rangle$ is a complete set of two-quasiparticle states, among which we shall in the future retain only the collective state $A^+_{2\mu}|0\rangle$, which separates out the coherent effects. As a result, we obtain

$$\begin{aligned} N_{Nv\Omega I} &= \sum_{v'\Omega' I'} (N-1, v'\Omega' I'; 2 | \} Nv\Omega I)^2 \\ &\times N_{N-1, v'\Omega' I'} + (N-1) \sum (N-1, v'\Omega' I'; 2 | \} Nv\Omega I) \\ &\times (N-2, v''\Omega'' I''; J, 2 | \} Nv\Omega I)\, (N-2, v''\Omega'' I''; 2 | \} N-1, \\ &v'\Omega' I')\, [22(J) I'' I | 2.2 I'' (I') \times I] C_J N_{N-1, v'\Omega' I'}, \end{aligned} \tag{32}$$

where $[22(J)I''I|\,2.2I''(I')I]$ is a Racah coefficient; $c_J \equiv 1 - \frac{1}{2} < 0 | [A_2 A_2]_{JM} [A_2^+ A_2^+]_{JM} | 0\rangle$. The recursion relation (32) can be used to calculate the normalization factors.[2h] These calculations have shown that as N and v increase the coefficients $N_{Nv\Omega I}$ gradually decrease and, above certain values of N and v which depend on the amplitudes Φ_{ab}, become zero, so that in actual calculations the space of collective fermion states in which the microscopic Hamiltonian is diagonalized is finite. Clearly, this is a result of the approximation made in the derivation of (32), if, of course, no restriction is made on the space of single-particle states in the calculation of the amplitudes Φ_{ab}. The approximation made in deriving (32) becomes unsatisfactory when $N_{Nv\Omega I} \ll 1$.

To get an idea of the behavior of $N_{Nv\Omega I}$ at large N and v, we analyze the following model. Suppose the space of collective states is constructed by means of a single operator: $A^+ = \sum_{s>} \psi_s a^+_s a^+_{-s}$, $\sum_{s>} \psi_s^2 = 1$. Then the space of collective states will include only the following vectors: $|n\rangle \equiv N_n^{-1/2} A^{+n} |0\rangle$; $N_n = \langle 0 | A^n A^{+n} | 0\rangle$. We calculate the normalization factor N_n. For this, we consider

$$\begin{aligned} \langle 0 | \exp [A \exp (it)] \exp (A^+) | 0\rangle &= \exp \Big[\sum_s \ln (1 + \exp (it)\, \psi_s^2) \Big] \\ &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \exp (int)\, N_n. \end{aligned}$$

It follows that the normalization factor in which we are interested is given by

$$N_n = \frac{(n!)^2}{2\pi} \int_{-\pi}^{\pi} \exp (-int) \exp \Big[\sum_s \ln (1 + \exp (it)\, \psi_s^2) \Big].$$

To obtain an analytic expression for N_n, it is convenient to go over from summation over s to integration:

$$\psi_s^2 \to \psi^2(x) = \frac{2a}{\pi} \frac{1}{a^2 + x^2}; \quad \int_0^{\infty} \psi^2(x)\, dx = 1;$$

$$\sum_s \to \int_0^{\infty} dx; \quad \sum_s \ln (1 + \exp (it)\, \psi_s^2) \to \int_0^{\infty} dx \ln (1 + \exp (it)\, \psi^2(x))$$

$$= \pi a \left(\sqrt{1 + \frac{2}{\pi a} \exp (it)} - 1 \right)$$

and

$$N_n = \frac{(n!)^2}{2\pi} \exp (-\pi a) \int_{-\pi}^{\pi} dt \exp (-int) \times \exp \left(\pi a \sqrt{1 + \frac{2}{\pi a} \exp (it)} - 1 \right).$$

The final expression has the form

$$N_n = n! \sum_{k=0}^{n-1} \frac{(n+k-1)!}{k!\,(n-k-1)!\,(-2\pi a)^k}.$$

For example,

$$N_2 = 2!\,(1 - 1/\pi a); \quad N_3 = 3!\,(1 - 3/\pi a + 3/\pi^2 a^2);$$
$$N_4 = 4!\,(1 - 6/\pi a + 15/\pi^2 a^2 - 15/\pi^3 a^3).$$

The results of calculations for $\pi a = 6$ are the following: $N_1 = 1$, $N_2 = 0.833$, $N_3 = 0.583$, $N_4 = 0.347$, $N_5 = 0.178$, $N_6 = 0.080$, $N_7 = 0.032$, $N_8 = 0.011$. They agree fairly well with calculations in accordance with the recursion relation

$$N_n = N_{n-1} [1 + (n-1)/\Omega] \tag{33}$$

for $\Omega = 5$: $N_1 = 1$, $N_2 = 0.833$, $N_3 = 0.595$, $N_4 = 0.372$, $N_5 = 0.206$, $N_6 = 0.103$, $N_7 = 0.047$, $N_8 = 0.020$.

Thus, for small n one can certainly use the expressions obtained above for the normalization factors, and for large n expressions of the type (33).

5. TRANSITION FROM SPHERICAL TO DEFORMED NUCLEI

We consider here the question of the types of transitions from spherical to deformed nuclei that can be described by means of the Hamiltonian (7). To simplify the treatment, we set $h_{4L} = 0$, and investigate the spectrum of collective excitations as a function of h_2 and h_3. Setting $h_3 = 0$ and increasing $|h_2|$ (h_2 is here negative), we obtain the spectrum of collective excitations shown in Fig. 6. It is identical with the spectrum of collective excitations in the Jean–Wilets model. With increasing $|h_2|$, the ratio $E(4_1^+)/E(2_1^+)$ increases. This means that $|h_2|$ plays the role of the parameter β_{eff} and the latter increases with the former.

Setting $h_2 = 0$, and increasing h_3, we obtain the spectrum of collective excitations shown in Fig. 7, from

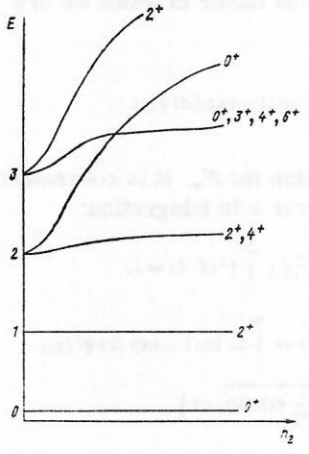


FIG. 6. Spectrum of excited collective states of the Hamiltonian (7) for $h_3 = h_{4L} = 0$.

which it can be seen that the 0_2^+ state sinks with increasing h_3, so that in the transition region it is near the 2_1^+ state. With a further increase of h_3, this state becomes the ground state, and we obtain the excitation spectrum characteristic of deformed nuclei. But among the excited states there appears a spherical 0^+ state. A situation of this kind is realized in the isotopes ^{96,98,100,102}Mo. It can be seen from this example that h_3 plays the same role in the Hamiltonian (7) as the parameter γ_{eff} in the phenomenological collective model of the nucleus.

If h_2 and h_3 increase simultaneously, there is a smooth transition from the typical vibrational spectrum of collective excitations to a typical rotational spectrum. This transition is described in Ref. 10.

6. E2 TRANSITIONS BETWEEN LEVELS OF A TWO-PHONON TRIPLET

Interesting experimental data on the $E2$ transition $4_1^+ \to 2_2^+$ were obtained in Ref. 11, in which the lower bound $B(E2;\ 4_1^+ \to 2_1^+)/B(E2;\ 4_1^+ \to 2_2^+) > 80$ was found for ^{108}Pd. This result characterizes a strong prohibition on $E2$ transitions between the levels 2_2^+ and 4_1^+ of the two-phonon triplet. It is very hard to explain it and simultaneously the large quadrupole moment of the 2_1^+ and the small value of $B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$, if one stays within the framework of a model that takes into account only mixing of components with few phonons. This can be seen from the following arguments.[12]

1. The data on the inelastic scattering of protons on spherical nuclei and the quadrupole moments $Q(2_2^+)$ of 2_1^+ states can be explained[13] by assuming that the 2_1^+ and 2_2^+ states are a mixture of a one- and a two-phonon component. However, under this assumption the theoretical $B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$ values are several times larger than the experimental. Large probabilities of $E2$ transitions between the triplet levels 0_2^+, 2_2^+, 4_1^+ are also obtained.

2. Allowance in the Hamiltonian for both types of third-order anharmonic terms:

$$\{\sum_\mu (b_2^+ b_2^+)_{2\mu} b_{2\mu} + \text{h.c.}\} \text{ and}$$
$$\{\sum_\mu (b_2^+ b_2^+)_{2\mu} (-1)^\mu b_{2-\mu}^+ + \text{h.c.}\}$$

enables one to explain[14] the small value of $B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$ and the large quadrupole moment of the 2_1^+ state. But then the branching ratio $B(E2;\ 0_2^+ \to 2_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$ is found to be 2–3 times larger than the experimental. The predictions for the probabilities of $E2$ transitions between the terms 0_2^+, 2_2^+, 4_1^+ of the triplet are too large.

3. In Ref. 15, it was shown to be possible to explain simultaneously and correctly the data on $Q(2_1^+)$ and $B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$ without losing the agreement with experiments for the other quantities; it was shown that, besides the collective part of the quadrupole moment operator $Q_{2\mu}$, it is also necessary to take into account its single-particle part, i.e., the part that does not change the number of quasiparticles. Both terms $Q_{2\mu}$ make a coherent contribution if one calculates matrix elements between states whose energy difference is less than the energy $E(2_1^+)$ of the state 2_1^+, which is the case when $Q(2_1^+)$ is calculated. The contributions of these terms to $Q_{2\mu}$ have opposite signs and compensate each other in the calculation of the matrix elements between states whose energy difference is greater than $E(2_1^+)$, as in the calculation of $B(E2;\ 2_2^+ \to 0_1^+)$. Unfortunately, in Refs. 15 results are not given for the $E2$ transitions between the levels of the triplet 0_2^+, 2_2^+, 4_1^+. The ratio $B(E2;\ 4_1^+ \to 2_1^+)/B(E2;\ 4_1^+ \to 2_2^+)$ is calculated in a model that allows for mixing of one-, two-, and three-phonon states and the one-particle and collective parts of $Q_{2\mu}$ are taken into account. The branching ratio was found to be of the order 25–30, which is much lower than the experimental value.

4. If one takes into account not only the anharmonic terms of third order but also those of fourth order,[16] satisfactory agreement with experiment can be achieved for many quantities. But $B(E2;\ 4_1^+ \to 2_1^+)/B(E2;\ 4_1^+ \to 2_2^+)$ is found to be of the order of 10, which is much smaller than the experimental value.

5. For ^{114}Cd, a calculation has been made[17] in the framework of a model that takes into account not only collective excitations but also states of a noncollective nature; it proved possible to reconcile the values for $Q(2_1^+)$ and $B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$ with the experimental values and obtain for the ratio $B(E2;\ 4_1^+ \to 2_1^+)/B(E2;\ 4_1^+ \to 2_2^+)$ the value 85, which corresponds to the strong prohibition of the transition between the 4_1^+ and 2_2^+ levels. But in this model one can consider only nuclei in which the protons or neutrons have only two particles

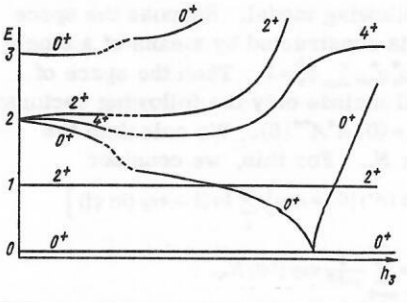


FIG. 7. The same as in Fig. 6, but for $h_2 = h_{4L} = 0$.

TABLE III. Energies of collective states, probabilities of $E2$ transitions, and quadrupole moment of the 2_1^+ state in ^{106}Pd.

Parameter	Theory	Experiment
$E(2_2^+)/E(2_1^+)$	2.18	2.15
$E(0_2^+)/E(2_1^+)$	2.41	2.43
$E(4_1^+)/E(2_1^+)$	2.22	2.415
$B(E2;\ 4_1^+ \to 2_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$	1.52	1.8 ± 0.2 1.3 ± 0.3
$B(E2;\ 2_2^+ \to 2_1^+)\ B(E2;\ 2_1^+ \to 0_1^+)$	1.09	1.6 ± 0.4
$B(E2;\ 0_2^+ \to 2_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$	0.68	0.9 ± 0.2
$B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$	0.008	0.022
$B(E2;\ 4_1^+ \to 2_1^+)/B(E2;\ 4_1^+ \to 2_1^+)$	121	>80
$B(E2;\ 2_2^+ \to 0_1^+)/B(E2;\ 2_1^+ \to 0_1^+)$	$6\cdot10^{-4}$	$<6\cdot10^{-3}$
$\langle 2_1^+ \Vert Q_2 \Vert 2_1^+\rangle/\langle 2_1^+ \Vert Q_2 \Vert 0_1^+\rangle$	-0.68 1	-0.9 ± 0.1; -0.55 ± 0.08
$B(E2;\ 0_2^+ \to 2_1^+)/B(E2;\ 0_2^+ \to 2_2^+)$		

or two holes above the closed shell. We cannot consider the nucleus ^{108}Pd in this model.

6. For completeness, we note that in the Davydov–Chaban phenomenological model[18] with values of the parameters μ and γ typical of spherical nuclei ($\mu = 0.6$, $\gamma = 20° - 25°$) the branching ratio $B(E2;\ 4_1^+ - 2_1^+)/B(E2;\ 4_1^+ - 2_2^+)$ takes a value approximately equal to 30, which is much smaller than the one found for ^{108}Pd.

It remains to assume that allowance for mixing of only one-, two-, and three-phonon states is inadequate in the case of ^{108}Pd. A method without this restriction is developed in Ref. 2g. Using this method, we calculate the energy, the probability of $E2$ transitions involving the 2_1^+ state and the triplet levels 0_2^+, 2_2^+, 4_1^+, and also the quadrupole moment of the 2_1^+ state. The calculations show that: a) the nucleus ^{108}Pd cannot be treated in the framework of a model that takes into account the mixing of only one-, two-, and three-phonon states, since admixtures of states with a large number of phonons are important; b) the potential energy of the quadrupole vibrations, which one can separate out from the total collective Hamiltonian (as is shown in Ref. 24), has at $\beta = 0$ a minimum, and not a maximum. Therefore, it is in principle impossible to treat ^{108}Pd in the framework of RPA. The minimum of the potential energy occurs at $\beta \neq 0$. Its depth is much less than the energy of the zero-point vibrations, and therefore the properties of ^{108}Pd are very different from those of deformed nuclei.

The results of the calculation are given in Table III. All the calculated quantities agree satisfactorily with the experimental quantities.[19] Note that allowance for the contribution of the single-particle part of $Q_{2\mu}$ is important for the calculation of the quadrupole moment of the 2_1^+ state. The ratio $B(E2;\ 4_1^+ - 2_1^+)/B(E2;\ 4_1^+ - 2_2^+)$ is also sensitive to the contribution of this part of $Q_{2\mu}$. A small change of the parameters, without significantly worsening the agreement with experiment for the remaining quantities, can increase $B(E2;\ 4_1^+ - 2_1^+)/B(E2;\ 4_1^+ - 2_2^+)$ from 10^2 to 10^3.

Experimentally, the ratio $B(E2;\ 0_2^+ - 2_1^+)/B(E2;\ 0_2^+ - 2_2^+)$ as yet remains unknown. In calculations, it has been found to be appreciably smaller than $B(E2;\ 4_1^+ - 2_1^+)/B(E2;\ 4_1^+ - 2_2^+)$, which indicates a weak forbiddenness for the transition $0_2^+ - 2_2^+$. To test the theory, it is very important to obtain experimental information about this ratio. An analogous result for the ratio $B(E2;\ 0_2^+ - 2_1^+)/B(E2;\ 0_1^+ - 2_1^+)$ was obtained in Ref. 20 for ^{78}Se and in Ref. 16 for a number of spherical nuclei, but on the basis of a collective Hamiltonian with a form different from (7).

The above results can be understood by considering the structure of the wave functions of the ^{108}Pd collective states found in these calculations (Fig. 8). The graphs are very similar for the 0_1^+, 2_1^+, and 4_1^+ states. They are merely shifted relative to one another along the abscissa by $\Delta N = 1$. Since the principal part of the quadrupole moment operator changes the number of phonons by unity, this explains why the $E2$ transitions $4_1^+ - 2_1^+$ and $2_1^+ - 0_1^+$ are distinguished. The structure of the 2_2^+ state is the same as that of the 4_1^+ state. In both cases, the two- and four-phonon components are the principal ones, making up 85% of the state norm. Since the principal part of the quadrupole moment operator changes the number of phonons by unity, the transition $4_1^+ - 2_2^+$ is forbidden.

We now consider the 0_2^+ state. In it, none of the components make a contribution of more than 30% to the norm. The contributions of the no-phonon, two-, three-, and four-phonon components are very noticeable. In this, the 0_2^+ state differs strongly from the other two-phonon states.

7. COLLECTIVE STATES OF TRANSITION ISOTOPES OF Sm AND Gd (Ref. 21)

The properties of the collective states of the isotopes of many elements vary smoothly with varying number of nucleons from typical vibrational to typical rotational states. One could explain, qualitatively at least, the main properties of transition nuclei by interpolating between the vibrational and the rotational limits. How-

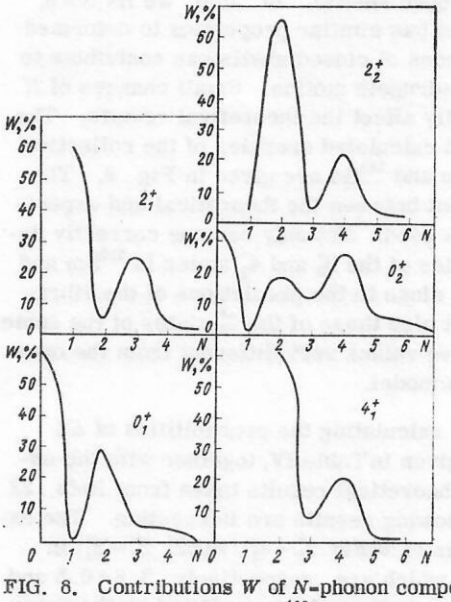


FIG. 8. Contributions W of N-phonon components to the 0_1^+, 2_1^+, 4_1^+, 0_2^+, 2_2^+ wave functions of ^{108}Pd.

TABLE IV. Ratio of reduced probabilities of $E2$ transitions in ^{150,152}Sm and ^{152}Gd (Rot are the predictions of the rotational model; Vib, of the vibrational model).

$\frac{B(E2;\ I\to I')}{B(E2;\ J\to J')}$	^{152}Gd		^{150}Sm				^{152}Sm			Rot	Vib
	Experiment	Theory	Experiment	Theory	Ref. [29]	Ref. [28]	Experiment	Theory	Ref. [29]		
$\frac{2_2^+\to 2_1^+}{2_2^+\to 0_1^+}$	52±5	48	12±1	42	0.7	0.5	6.2±0.9	3.7	9.1	1.43	∞
$\frac{2_2^+\to 4_1^+}{2_2^+\to 2_1^+}$	2.0±0.3	2.2	3.8±0.5	3.2	45	66	2.7±0.9	6.3	4.9	1.8	0
$\frac{2_2^+\to 0_2^+}{2_2^+\to 2_1^+}$	107±11	141	28±4	197	48	65	—	—	—	—	0
$\frac{2_2^+\to 0_2^+}{2_2^+\to 0_1^+}$	2.05	2.96	2.3	4.7	69	127	—	—	—	—	—
$\frac{4_2^+\to 2_1^+}{4_2^+\to 4_1^+}$	—	—	0.002	0.001	0.28		0.10±0.03	0.22	0.08	1.1	0
$\frac{4_2^+\to 2_2}{4_2^+\to 4_1^+}$	6.8±1.1	8.9	5.9±0.8	17.9	75.4		41±18	103	41	—	1.1
$\frac{4_2^+\to 2_1^+}{4_2^+\to 2_2^+}$	—	—	$4\cdot10^{-4}$	$5\cdot10^{-4}$	$4\cdot10^{-3}$		0.002	0.004	0.0002	—	0
$\frac{2_3^+\to 0_1^+}{2_3^+\to 2_1^+}$	0.14±0.01	0.36	0.23±0.03	0.33	0.08	0.06	0.39±0.03	0.10	0.43	0.78	—
$\frac{2_3^+\to 4_1^+}{2_3^+\to 2_1^+}$	—	—	0.6±0.2	0.01	0.27	0.01	0.09±0.01	0.02	0.05	0.05	∞
$\frac{\langle 2_1^+\|Q_2\|2_1^+\rangle}{\langle 2_1^+\|Q_2\|0_1^+\rangle}$	—	—	−1.4±0.2	−1.5	—	—	—	—	—	—	0

ever, a number of facts do not fit into this scheme. One such example is provided by the properties of the low-lying states in the isotopes ^{150}Sm and ^{152}Gd. In these isotopes the energies of the 0_2^+ and 4_1^+ states are close to the predictions of the vibrational model, whereas the ratio $E(2_1^+)/E(2_1^+) = 2.7-3.0$ strongly contradicts this model. In ^{152}Gd, the ratio $B(E2;\ 2_2^+\to 4_1^+)/B(E2;\ 2_2^+\to 2_1^+) = 2.0$ agrees well with the prediction 1.8 of the rotational model. But the ratio $B(E2^+;\ 2_2^+\to 2_1^+)/B(E2;\ 2_2^+\to 0_1^+) = 52$ is typical of spherical nuclei.

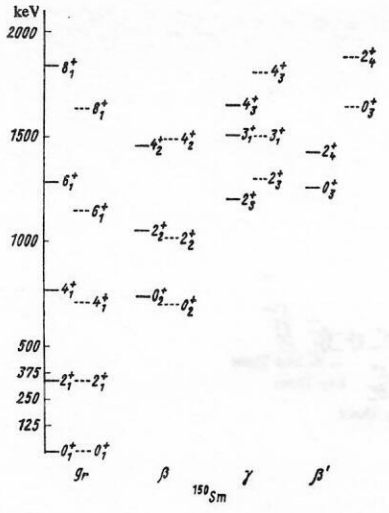
Let us analyze these experimental data on the basis of the collective Hamiltonian (7). To simplify the analysis, we assume that $h_{4I} = 0$. We fix the remaining parameters in such a way as to describe the experimental situation in ^{150,152}Sm and ^{152}Gd. Since ^{150}Sm and ^{152}Gd have similar properties taking N the same for these nuclei and equal to (7) (somewhat less than the number of nucleon pairs in open shells). In ^{152}Sm, we fix $N = 9$, since this nucleus has similar properties to deformed nuclei, and nucleons of closed shells can contribute to the collective quadrupole motion. Small changes of N do not significantly affect the theoretical results. The experimental and calculated energies of the collective states of ^{150,152}Sm and ^{152}Gd are given in Fig. 9. The general agreement between the theoretical and experimental results is good. Not only can one correctly describe the energies of the 0_2^+ and 4_1^+ states in ^{150}Sm and ^{152}Gd, which are close to the predictions of the vibrational model, but also those of the 2_2^+ states of the same nuclei, which have values very different from the ones predicted by this model.

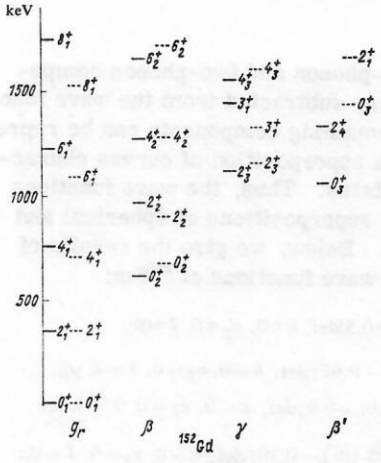
The results of calculating the probabilities of $E2$ transitions are given in Table IV, together with the experimental and theoretical results taken from Refs. 22 and 23. The following results are interesting. The experimental values of $B(E2;\ 2_2^+\to 4_1^+)/B(E2;\ 2_2^+\to 2_1^+)$ in ^{150}Sm and ^{152}Gd, which are, respectively, 3.8 ± 0.5 and 2.0 ± 0.3 and close to the values predicted by the rotational model, agree well with our calculations. At the same time, we have succeeded in explaining the large experimental values of the ratio $B(E2;\ 2_2^+\to 2_1^+)/B(E2;\ 2_2^+\to 0_1^+)$ in ^{150}Sm and ^{152}Gd which are, respectively, equal to 12 ± 1 and 52 ± 5. For this ratio, the rotational model predicts the value 1.43.

Thus, it is possible to explain the fact that in one and the same transition nucleus the values of certain quantities are close to the ones obtained from the rotational model, while the values of others are typical of spherical nuclei. As follows from our calculations, the reason for this is as follows. The wave functions of the ground states of ^{150}Sm and ^{152}Gd consist to 75–90% of the no-phonon component. At the same time, the wave functions of the excited states have a more complicated structure. In them, none of the components makes a contribution greater than 50%. And, as a rule, three or four components make an appreciable contribution to the wave functions. One can therefore say that the ground states of ^{150}Sm and ^{152}Gd are spherical, whereas the structure of the excited states corresponds to an intermediate position between spherical and deformed. The ratio $B(E2;\ 2_2^+\to 4_1^+)/B(E2;\ 2_2^+\to 2_1^+)$ depends only on the structure of the excited states and is therefore close to the value predicted by the rotational model. The ratio $B(E2;\ 2_2^+\to 2_1^+)/B(E2;\ 2_2^+\to 0_1^+)$ depends on the structure of the ground state, and its value is typical of spherical nuclei.

It is well known that in strongly deformed nuclei the spectrum of collective states consists of rotational bands based on states with different internal structures. The $E2$ transitions between levels of one band are much stronger than those between levels of different rotational bands. The reason for this is well known and is due to the fact that in strongly deformed nuclei the rotational motion is decoupled from the internal (at least for small spins).

In the transitional nuclei, the rotational motion is not





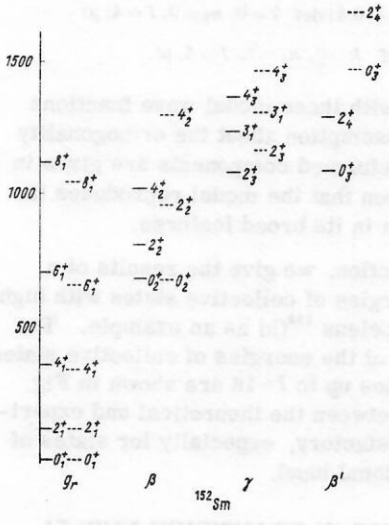


FIG. 9. Spectrum of low-lying collective states of ^{150,152}Sm and ^{152}Gd. The solid lines are the experimental positions; the dashed lines, the theoretical positions.

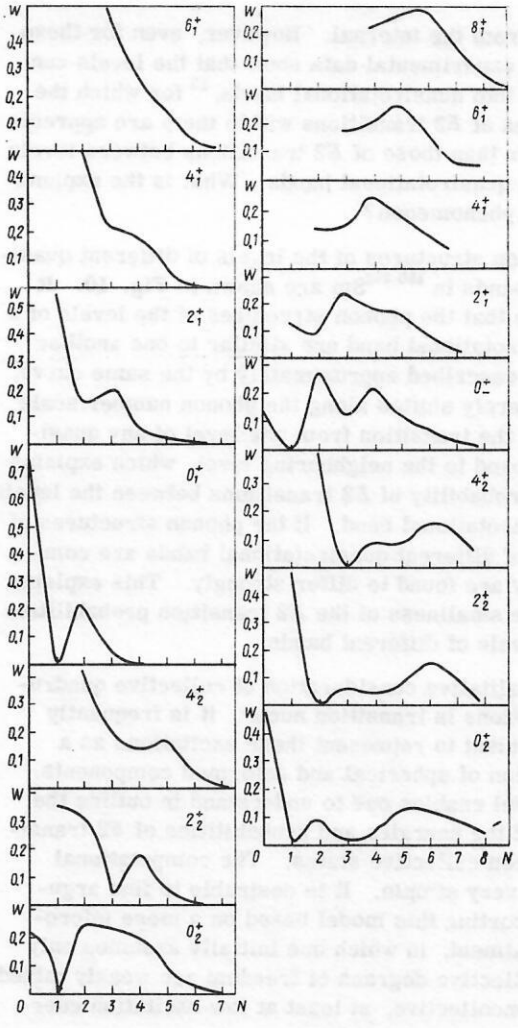


FIG. 10. Contributions W of N-phonon components to the wave functions of states belonging to the ground and β-quasirotational bands of ^{150}Sm (a) and ^{152}Sm (b).

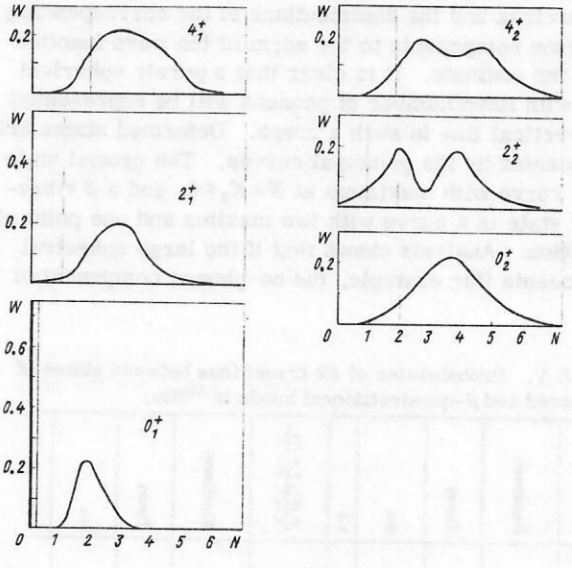


FIG. 11. Separation of spherical components (vertical lines) and deformed components (smooth curves) in the wave functions of collective states of ^{150}Sm.

decoupled from the internal. However, even for these nuclei, the experimental data show that the levels can be grouped into quasirotational bands,[10] for which the probabilities of $E2$ transitions within them are appreciably greater than those of $E2$ transitions between levels of different quasirotational bands. What is the explanation of this phenomenon?

The phonon structures of the levels of different quasirotational bands in ^{150,152}Sm are shown in Fig. 10. It can be seen that the phonon structures of the levels of a given quasirotational band are similar to one another and can be described approximately by the same curve, which is merely shifted along the phonon number scale by unity on the transition from one level of any quasirotational band to the neighboring level, which explains the large probability of $E2$ transitions between the levels of the quasirotational band. If the phonon structures of the levels of different quasirotational bands are compared, they are found to differ strongly. This explains the relative smallness of the $E2$ transition probabilities between levels of different bands.

In the qualitative consideration of collective quadrupole excitations in transition nuclei, it is frequently very convenient to represent these excitations as a superposition of spherical and deformed components. Such a model enables one to understand in outline the behavior of the energies and probabilities of $E2$ transitions between collective states. The computational scheme is very simple. It is desirable to find arguments supporting this model based on a more microscopic treatment, in which one initially assumes only that the collective degrees of freedom are weakly mixed with the noncollective, at least at low excitation energies.

Such calculations were made in Ref. 21 for the transition isotopes Sm and Gd. Information on the structure of the wave functions is given in Ref. 21 in the form of graphs, in which the phonon number N is plotted along the abscissa and the contributions of the corresponding N-phonon components to the norm of the wave function along the ordinate. It is clear that a purely spherical state with fixed number of phonons will be represented by a vertical line in such a graph. Deformed states are represented by the principal curves. The ground state is the curve with maximum at $N=N_0\neq 0$, and a β vibrational state is a curve with two maxima and one point of inflection. Analysis shows that if the large spherical components (for example, the no-phonon component of the 0_1^+ state and the no-phonon and two-phonon components of the 0_2^+ state) are subtracted from the wave functions (Fig. 11), the remaining components can be represented in the form of a superposition of curves characteristic of deformed states. Thus, the wave functions can be represented as superpositions of spherical and deformed components. Below, we give the results of this treatment for the wave functions of ^{150}Sm:

$$|0_1^+\rangle = 0.86|0\rangle + 0.5|\text{def},\ k=0,\ n_\beta=0,\ I=0\rangle;$$

$$|2_1^+,\ \mu\rangle = 0.74 b_{2\mu}^+|0\rangle - 0.67|\text{def},\ k=0,\ n_\beta=0,\ I=2,\ \mu\rangle;$$

$$|4_1^+,\ \mu\rangle = 0.6\frac{[b_2^+b_2^+]_{4\mu}}{\sqrt{2}}|0\rangle - 0.8|\text{def},\ k=0,\ n_\beta=0,\ I=4,\ \mu\rangle;$$

$$|0_2^+\rangle = 0.46\left(|0\rangle - \frac{[b_2^+b_2^+]_0}{\sqrt{2}}|0\rangle\right) - 0.76|\text{def},\ k=0,\ n_\beta=0,\ I=0\rangle;$$

$$|2_2^+\mu\rangle = 0.55 b_{2\mu}^+|0\rangle - 0.30\frac{[b_2^+b_2^+]_{2\mu}}{\sqrt{2}}|0\rangle - 0.62|\text{def},\ k=0,\ n_\beta=0,\ I=2,\ \mu\rangle + 0.45|\text{def},\ k=0,\ n_\beta=1,\ I=2,\ \mu\rangle;$$

$$|4_2^+\mu\rangle = 0.55\frac{[b_2^+b_2^+]_{4\mu}}{\sqrt{2}} - 0.4|\text{def},\ k=0,\ n_\beta=0,\ I=4,\ \mu\rangle + 0.75|\text{def},\ k=0,\ n_\beta=1,\ I=4,\ \mu\rangle.$$

The results obtained with these model wave functions under a very crude assumption about the orthogonality of the spherical and deformed components are given in Table V. It can be seen that the model reproduces the experimental situation in its broad features.

To conclude this section, we give the results of a calculation of the energies of collective states with high spins. We took the nucleus ^{154}Gd as an example. The results of calculation of the energies of collective states of ^{154}Gd with spin values up to $I=18$ are shown in Fig. 12. The agreement between the theoretical and experimental results is satisfactory, especially for states of the ground quasirotational band.

TABLE V. Probabilities of $E2$ transitions between states of the ground and β-quasirotational bands in ^{150}Sm.

$\frac{B(E2;\ I\to I')}{B(E2;\ J\to J')}$	Experiment	Theory	Rot	Vib	$\frac{B(E2;\ I\to I')}{B(E2;\ J\to J')}$	Experiment	Theory	Vib	Rot
$\frac{2_2^+\to 4_1^+}{2_2^+\to 2_1^+}$	3.8±0.5	1.8	1.8	0	$\frac{2_2^+\to 2_1^+}{2_2^+\to 0_1^+}$	12±1	13	1.43	∞
$\frac{4_2^+\to 4_1^+}{4_2^+\to 2_1^+}$	400	100	0.91	∞	$\frac{2_2^+\to 0_2^+}{2_2^+\to 0_1^+}$	28±4	64	$\frac{\beta_0^2}{\langle(\Delta\beta)^2\rangle}$	$\frac{0}{0}$

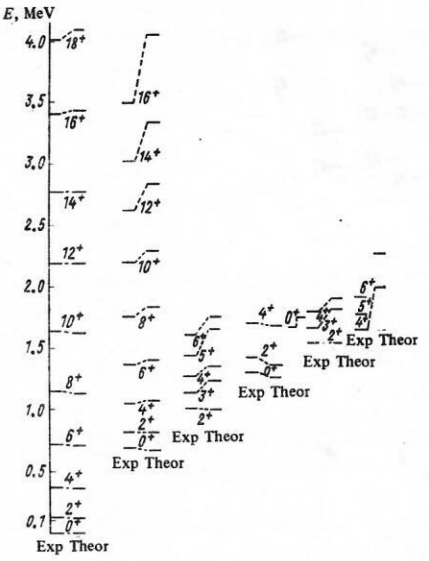


FIG. 12. Spectrum of collective excitations of ^{154}Gd.

8. SPHERICAL STATES IN TRANSITION NUCLEI

In Ref. 24, the spectra of collective excitations of the isotopes ^{190,192,194}Pt were investigated in (α, xn) reactions. Of the results obtained in Ref. 24, the most interesting are the following:

1) at spin $I=12$ in ^{190,192}Pt and $I=10$ in ^{194}Pt the smooth growth with increasing I of the difference between the energies $E(I+2)-E(I)$ of the levels of the ground quasirotational band breaks down;

2) the $E2$ transitions $12^+ \to 10^+$ in ^{190,192}Pt and the $E2$ transition $10^+ \to 8^+$ in ^{194}Pt are hindered compared with the purely collective transitions. In the case of ^{194}Pt the effect is appreciable;

3) in ^{190,192}Pt several levels that have $I^\pi = 10^+$ and are nearly degenerate with respect to the energy are found near the 12^+ state.

Let us analyze this situation on the basis of the collective model of the nucleus. We consider the collective quadrupole Hamiltonian containing terms to fourth order in powers of the phonon operators. We require the seniority v to be a good quantum number. This means that the collective Hamiltonian must not contain anharmonic terms of third order:

$$\begin{aligned} H_{\text{coll}} &= c_1 \textstyle\sum_\mu b^+_{2\mu} b_{2\mu} + c_2 \sum_\mu (-1)^\mu (b^+_{2\mu} b^+_{2-\mu} \\ &+ b_{2-\mu} b_{2\mu}) + c_{41} \left(\sum_\mu (-1)^\mu b^+_{2\mu} b^+_{2-\mu} \sum_\nu (-1)^\nu b^+_{2\nu} b^+_{2-\nu}\right. \\ &\left. + \text{h. c.}\right) + c_{43} \left(\sum_\mu (-1)^\mu b^+_{2\mu} b^+_{2-\mu} \sum_\nu b^+_{2\nu} b_{2\nu} + \text{h. c.}\right) \\ &+ \textstyle\sum_{L=0,\,2,\,4} c_{4L} \left([b^+_2 b^+_2]_L\, [b_2 b_2]_L\right)_{00}. \end{aligned} \tag{34}$$

Using a linear canonical transformation of the operators $b^+_{2\mu}$ and $b_{2\mu}$, we can transform the Hamiltonian in such a way that in terms of the new phonon operators the term $\sum_\mu (-1)^\mu b^+_{2\mu} b^+_{2-\mu} \sum_\nu (-1)^\nu b^+_{2\nu} b^+_{2-\nu} + \text{h. c.}$ is absent. Therefore, without loss of generality, we can set $c_{41} = 0$.

The eigenstates of the Hamiltonian (34) are characterized by the following quantum numbers: I and M, the angular momentum and its projection; v, the seniority; Ω, an additional quantum number. The spectrum is degenerate with respect to Ω, and the multiplicity of degeneracy is[8]

$$k_\Omega = \left[\tfrac{1}{6}\left(2v - I - 3\,\frac{1-(-1)^I}{2}\right)\right] - \left[\tfrac{1}{3}(v-I+2)\right] + 1 - \delta_{I,\,2v-1},$$

where $[A]$ is the integral part of the number A.

In the general case, the eigenstates of the Hamiltonian (34) are a superposition of states with different phonon numbers n. But for definite values of the coefficients c_2 and c_{43}, the eigenstates include some with fixed phonon number n_0:

$$|n = n_0,\ v = n_0,\ \Omega,\ IM\rangle. \tag{35}$$

We now show that the states (35) are eigenstates of (34). In the Hamiltonian (34), the phonon number is changed by the following terms:

$$\begin{aligned} &c_2 \left(\textstyle\sum_\mu (-1)^\mu b^+_{2\mu} b^+_{2-\mu} + \sum_\mu (-1)^\mu b_{2-\mu} b_{2\mu}\right) \\ &+ c_{43} \left(\textstyle\sum_\mu (-1)^\mu b^+_{2\mu} b^+_{2-\mu} \sum_\nu b^+_{2\nu} b_{2\nu} + \text{h. c.}\right). \end{aligned} \tag{36}$$

Since n_0 is the minimal number of phonons for a state with $v = n_0$, the result of applying the operator (36) to the state vector (35) can be written as follows:

$$(c_2 + c_{43} n_0) \textstyle\sum_\mu (-1)^\mu b^+_{2\mu} b^+_{2-\mu} |n = n_0,\ v = n_0,\ \Omega,\ IM\rangle.$$

If $n_0 = -c_2/c_{43}$, then (35) is an eigenvector of the Hamiltonian with eigenvalue

$$\begin{aligned} E_{n_0,\,I} = &\left(c_1 + \frac{2}{7\sqrt{5}} c_{42} - \frac{3}{7} c_{44}\right) n_0 + \left(\frac{4}{7\sqrt{5}} c_{42} + \frac{1}{7} c_{44}\right) n_0^2 \\ &+ \left(\frac{1}{21} c_{44} - \frac{1}{7\sqrt{5}} c_{42}\right) I(I+1). \end{aligned} \tag{37}$$

The ratio c_2/c_{43} need not be an integer, and if it is not, the state $|n_0, v = n_0, \Omega, IM\rangle$ acquires admixtures of components with other phonon numbers. But they are small because of the factor $(n_0 + c_2/c_{43}) < 1$.

In nuclei in which the potential energy of quadrupole vibrations has a maximum at $\beta = 0$ and minimum at $\beta \neq 0$, the coefficient c_2 is negative, and c_{43} positive. If in these nuclei the potential energy does not depend on γ, then states of the type (35) can be realized in them. These are purely spherical states with fixed phonon number. There is a complete set of such states with different values of I. The deeper the minimum at $\beta \neq 0$, the larger is the value of n_0 at which these states appear. Thus, there is an entire band of spherical states, in contrast to the majority of other eigenstates of (34), whose wave functions are concentrated in the region of the minimum at $\beta \neq 0$. As can be seen from (37), the energies of the states of the spherical band depend on I in the same way as the energies of the states of a deformed rotational band. If the coefficient $(c_{44}/21 - c_{42}/7\sqrt{5})$ is sufficiently small, the spherical band intersects the ground-state quasirotational band, and backbending will be observed. If the value of the seniority for the spherical band is appreciably larger than for the ground-state band at the point of intersection, the seniority selection rule for $E2$ transitions will result in a state with a greater lifetime, i.e., an isomeric state. In addition, for sufficiently large n_0 and I, the states of the spherical band are degenerate with multiplicity k_Ω.

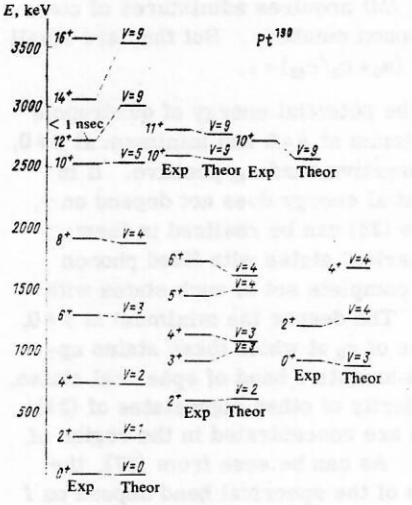
To calculate the energies of collective quadrupole excitations in the isotopes ^{190,192,194}Pt, we use the Hamiltonians (7). As was pointed in Ref. 24, the maximal phonon number N can be taken to be approximately equal to the number of nucleon pairs in open shells. For ^{190}Pt, we obtain $N = 9$. In what follows, we shall use this value in calculations for ^{192,194}Pt. A test showed that small changes do not essentially alter the results of the calculations. The numerical values of the coefficients h_1, h_2, h_3, h_{4I} were fixed in such a way as to minimize the deviations of the theoretical values of the energies from the experimental values for the first six low-lying states 2^+_1, 2^+_2, 4^+_1, 4^+_2, 3^+_1, 0^+_2 (Table VI). It can be seen from Table VI that the coefficient h_3 is appreciably smaller than h_2, and its influence can therefore be ignored. If, in addition, in (7) we ignore $1/N = 1/9$ in the

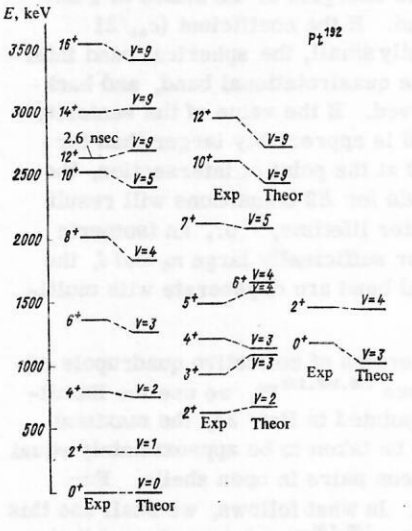
TABLE VI. Allowed values of the spin and degeneracy of states of the spherical band with $N = v = 9$.

I^π	0^+	3^+	4^+	6^+	7^+	8^+	9^+	10^+	11^+	12^+	13^+	14^+	15^+	16^+	18^+
k_Ω	1	1	1	2	1	1	2	2	1	2	1	1	1	1	1

square root compared with the unity, Eq. (7) can be rewritten as

$$H_{\text{coll.}} = h_0 + h_1 \sum_\mu b_{2\mu}^+ b_{2\mu} + h_2 \Big\{ \sum_\mu (-1)^\mu b_{2\mu}^+ b_{2-\mu}^+ \times \Big(1 - \frac{\sum_\nu b_{2\nu}^+ b_{2\nu}}{N}\Big) + \text{h. c.} \Big\} + \sum_I h_{4I} [[b_2^+ b_2^+]_I [b_2 b_2]_I]_0. \quad (38)$$





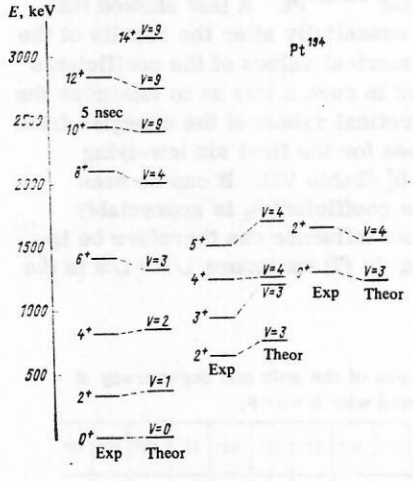


FIG. 13. Theoretical and experimental energies of the collective states of ^{190,192,194}Pt.

TABLE VII. Values of the coefficients h_1, h_2, h_3, h_{4L} (keV) and

$$\chi = \sqrt{\sum_{i=1}^{6} \frac{(E_{\text{exp}}^i - E_{\text{theor}}^i)^2}{(E_{\text{exp}}^i)^2}}$$

in the isotopes ^{190,192,194}Pt.

Nucleus	190Pt	192Pt	194Pt
h_1	14.3	−73.3	123.0
h_2	−1505.9	−1828.4	−1682.7
h_3	−23.3	−4.7	−1.56
h_{40}	57.7	97.7	−146.9
h_{42}	−121.6	−105.2	−122.1
h_{44}	39.1	−10.3	−22.3
χ	0.11	0.26	0.32

This expression is equivalent to the Hamiltonian (43), and the spherical band with $n_0 = 9$ is an eigenstate of the Hamiltonian (34) with energy

$$E_I = 9\left(h_1 + \frac{2}{7\sqrt{5}} h_{42} - \frac{3}{7} h_{44}\right) + 81\left(\frac{4}{7\sqrt{5}} h_{42} + \frac{1}{7} h_{44}\right) + \left(\frac{1}{21} h_{44} - \frac{1}{7\sqrt{5}} h_{42}\right) I(I+1).$$

The multiplicities of degeneracy of the states of the spherical band are given in Table VI.

To find out whether the spherical band intersects the ground-state band for the isotopes ^{190,192,194}Pt, we found the eigenvalues of the Hamiltonian (7) for states with large spin. The values of the coefficients were taken from Table VII. The results of the calculation are shown in Fig. 13, in which we give only those states of

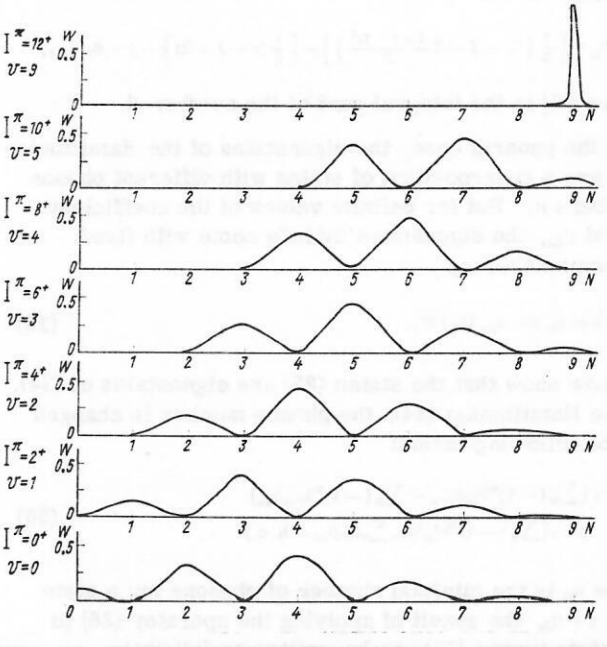


FIG. 14. Phonon structure of wave functions of collective states of the ground quasirotational band. N is the number of phonons and W the relative contribution to the norm of the wave function.

the spherical band that belong to the yrast line. Fig. 14 shows the wave functions of the states belonging to the yrast line in the isotope ^{190}Pt. It can be seen that up to spin $I^\pi = 10^+$ the yrast line is based on states of the ground quasirotational band, but that from the spin $I^\pi = 12^+$ a spherical band begins. Because of the large difference in the seniority values between the states $I^\pi = 10^+$ and $I^\pi = 12^+$, the $E2$ transition $12^+ \to 10^+$ is hindered. However, if N_0 is not an integer, the spherical state acquires admixtures of states with smaller seniority value and the factor prohibiting the $E2$ transition is reduced. In ^{190}Pt, the experimental factor is 2–3. It is interesting that ^{190}Pt also exhibits two almost degenerate 10^+ states. The energies of these states are practically equal to the energies of the corresponding states of the spherical band. Experimentally, an 11^+ state of the spherical band has also been discovered.

Similar results are obtained for the isotopes ^{192,194}Pt. The theory explains the appearance of additional states with $I^\pi = 10^+, 12^+$ in ^{192}Pt and the abrupt change in the variation of the energy difference $[E(I+2) - E(I)]$ with increasing I in the ground-state quasirotational band.

From the result for states with small values of the angular momentum, it follows that in these nuclei the correspondence rule proposed in Ref. 9 for describing quasirotational bands in transition nuclei breaks down. The state which forms the base of the β vibrational band has seniority $v = 3$, and not $v = 0$ as predicted in Ref. 10. The large value of the seniority of the states of the β vibrational band leads to a strong reduction of the probabilities of $E2$ transitions from this band to the ground-state band. This effect is observed experimentally. The ratio $B(E2; 2_3^+ \to 0_1^+)/B(E2; 2_3^+ \to 0_2^+)$ in the isotopes ^{190,192}Pt is $2 \cdot 10^{-4}$, whereas in transition nuclei like ^{150}Sm, ^{152}Gd, and ^{98}Mo it may be 0.4–0.5.

APPENDIX I

$$\langle I+2, U+1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I+2, U+1, N+1 \mid b_{2\mu}^+ \mid IUN\rangle \times \sqrt{(1-N/L)\,[1-(N+U)/2k]\,(1-2I/R)};$$

$$\langle I+1, U+1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I+1, U+1, N+1 \mid b_\mu^+ \mid IUN\rangle \times \sqrt{(1-N/L)\,[1-(N+U)/2k]\,[1-(2I-1)/2R]};$$

$$\langle I, U+1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I, U+1, N+1 \mid b_\mu^+ \mid IUN\rangle \sqrt{(1-N/L)\,[1-(N+U)/2k]};$$

$$\langle I-1, U+1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I-1, U+1, N+1 \mid b_\mu^+ \mid IUN\rangle \sqrt{(1-N/L)\,[1-(N+U)/2k]\,[1-(2I-3)/2R]};$$

$$\langle I-2, U+1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I-2, U+1, N+1 \mid b_\mu^+ \mid IUN\rangle \sqrt{(1-N/L)\,[1-(N+U)/2k]\,[1-(2I-4)/R]};$$

$$\langle I+2, U-1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I+2, U-1, N+1 \mid b_\mu^+ \mid IUN\rangle \times \sqrt{(1-N/L)\,[1-(N-U-3)/2k]\,(1-2I/R)};$$

$$\langle I, U-1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I, U-1, N+1 \mid b_\mu^+ \mid IUN\rangle \sqrt{(1-N/L)\,[1-(N-U-3)/2k]};$$

$$\langle I-1, U-1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I-1, U-1, N+1 \mid b_\mu^+ \mid IUN\rangle \times \sqrt{(1-N/L)\,[1-(N-U-3)/2k]\,[1-(2I-3)/R]};$$

$$\langle I-2, U-1, N+1 \mid A_\mu^+ \mid IUN\rangle = \langle I-2, U-1, N+1 \mid b_\mu^+ \mid IUN\rangle \times \sqrt{(1-N/L)\,[1-(N-U-3)/2k]\,[1-(2I-4)/R]}.$$

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