Symmetry groups in quantum field theory

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Symmetry and supersymmetry groups in quantum field theory are reviewed. These include spacetime symmetry groups (Lorentz, Poincaré, Weyl, conformal) and internal symmetry groups with scalar and spinor parameters (supersymmetry groups). The main attention is devoted to the existence of the various symmetry groups. It is shown that the requirement of relativistic invariance leads to strong restrictions on the structure of the possible symmetry groups.

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INTRODUCTION

It is well known that symmetry groups are very important in quantum theory. Invariance of a theory under a group is reflected in the existence of certain conserved quantities, selection rules, restrictions on the form of the Lagrangian and scattering amplitudes, etc. Symmetry principles give a method for classifying elementary particles (unitary symmetry) and are used to construct diverse field models (models with spontaneous symmetry breaking).

In connection with the wide use of group methods in quantum field theory, it is of interest to establish what symmetry groups quite generally are possible; in other words, what groups can be regarded as symmetry groups. The discussion of these possibilities is the main aim of the present review. We shall see that invariance of quantum field theory under the Poincaré group imposes strong restrictions on the structure of the possible symmetry groups.

In the review, we shall consider spacetime symmetry groups and internal symmetry. Among the spacetime groups, we devote particular attention to the conformal group. The Lorentz and Poincaré groups are considered very briefly. We shall discuss the structure of possible Lie groups of transformations in Minkowski space. We shall show that the possible finite groups of spacetime symmetries compatible with relativistic invariance are exhausted by the Lorentz, Poincaré, Weyl, and conformal groups. A large part of the review is devoted to supersymmetry groups—groups of internal symmetry with spinor parameters. Supersymmetry groups have been widely discussed recently in connection with a number of possibilities they offer. In the review, we consider the possible form of the algebra of supersymmetry groups and realization of symmetry groups as groups of transformations of superspaces.

1. LORENTZ AND POINCARÉ GROUPS

Since the Lorentz and Poincaré groups are so well known, we shall consider them only briefly. We denote the coordinates of Minkowski space by x_μ, where $\mu = 0$, 1, 2, 3.

The Lorentz group is the group of continuous, homogeneous coordinate transformations

$$x_\mu \to x'_\mu = \Lambda_{\mu\nu} x_\nu, \tag{1}$$

that leave the Lorentz square $x^2 = x_\mu x_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$ invariant. The Lorentz group, which is the group of rotations of four-dimensional pseudo-Euclidean space, has six continuous parameters and, accordingly, six generators $Y_{\mu\nu}$ ($Y_{\mu\nu} = -Y_{\mu\nu}$). The generators $Y_{\mu\nu}$ satisfy the well-known commutation relations

$$[Y_{\mu\nu}, Y_{\rho\Sigma}] = i(g_{\mu\Sigma} Y_{\nu\rho} + g_{\nu\rho} Y_{\mu\Sigma} - g_{\mu\rho} Y_{\nu\Sigma} - g_{\nu\Sigma} Y_{\mu\rho}), \tag{2}$$

where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The Casimir operators of the Lorentz group have the form

$$C_1 = Y_{\mu\nu} Y_{\mu\nu}; \quad C_2 = \varepsilon_{\mu\nu\rho\Sigma} Y_{\mu\nu} Y_{\rho\Sigma}. \tag{3}$$

Thus, the irreducible representations of the Lorentz group are classified by the values of two numbers. We shall be interested in irreducible finite-row representations. We denote these representations by $D(j, k)$; j and k take the values 0, 1/2, 1, 3/2, ... The dimension of the representation $D(j, k)$ is $(2j+1)(2k+1)$ (Refs. 1 and 2).

All Lorentz covariant fields that transform in accordance with particular representations of the Lorentz group can be divided into two classes: tensor fields, with $j+k$ integral, and spinor fields, with $j+k$ half-integral. This division of the fields arises because the fields of the first kind transform in accordance with single-valued representations, while those of the second transform in accordance with two-valued representations of the Lorentz group. In addition, for the quantization of the former fields, it is necessary to consider the commutator, and for the latter, the anticommutator (the theorem on the connection between spin and statistics[3]). All possible types of fields can be expressed in a unified manner using spinor notation. For this, one introduces two types of indices, with and without dots, which each take two values. An entity that transforms in accordance with the irreducible representation $D(j, k)$ of the Lorentz group has the form of a spin tensor $T_{A_1 \ldots A_{2j} \dot{B}_1 \ldots \dot{B}_{2k}}$ that is completely symmetric in each type of index ($A_i = 1, 2$; $B_i = \dot{1}, \dot{2}$). The antisymmetric tensors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$ ($\varepsilon_{AB} = -\varepsilon_{BA}$ and $\varepsilon_{\dot{A}\dot{B}} = -\varepsilon_{\dot{B}\dot{A}}$) play the role of an invariant metric in the space of the spin tensors. The spin tensor formalism is convenient for considering a number of questions, and is used below.

The Poincaré group contains, besides the Lorentz transformations (1), the displacements

$$x_\mu \to x'_\mu = x_\mu + a_\mu \tag{4}$$

and leaves invariant the cone $(x-y)^2=0$. The Poincaré group is a ten-parameter group, and its generators $Y_{\mu\nu}$, P_ρ satisfy the commutation relations [in addition to the relations (2)]

$$\left.\begin{array}{l}[Y_{\mu\nu}, P_\rho]=i(g_{\mu\rho}P_\nu-g_{\nu\rho}P_\mu);\\ [P_\mu, P_\nu]=0.\end{array}\right\} \qquad (5)$$

The Poincaré group has two Casimir operators:

$$C_1=P^2;\quad C_2=Y_{\mu\nu}Y_{\mu\nu}P^2/2-Y_{\mu\sigma}Y_{\nu\sigma}P_\mu P_\nu. \qquad (6)$$

In irreducible representations with $P^2=M^2>0$, $C_2=M^2s(s+1)$, where s is the spin ($s=0$, 1/2, 1, 3/2, ...). Thus, for $M^2>0$ the irreducible representations of the Poincaré group are classified by the mass M and the spin s. In representations with $M=0$, the second Casimir operator is the helicity λ ($\lambda=0$, $\pm 1/2$, ± 1, $\pm 3/2$, ...). Note that in representations with $M^2\geqslant 0$ there is an additional invariant $\operatorname{sgn} p_0=p_0/|p_0|$.

By virtue of relativistic invariance, vectors of the state space of quantum field theory transform in accordance with unitary representations of the Poincaré group. Accordingly, the quantum fields that describe elementary particles transform in accordance with irreducible unitary representations of the Poincaré group and, therefore are classified by their mass M and spin s (for $M=0$ by the helicity).

However, in many cases (for example, in the construction of Lagrangians) it is convenient to use fields that transform in accordance with reducible representations of the Poincaré group, namely, fields $\psi_M^{(j,k)}$ with mass M transforming in accordance with the representations $D(j,k)$ of the Lorentz group. Since the represenation $D(j,k)$ of the Lorentz group is the direct sum $[D(j,k)=\sum_{M=|j-k|}^{j+k}\otimes D_{SO(3)}(m)]$ of the group of three-rotations, the field $\psi_M^{(j,k)}$ for $M^2>0$ transforms in accordance with the direct sum of irreducible representations of the Poincaré group with the spin spectrum $j+k$, $j+k-1$, ..., $|j-k|$. The best known examples of fields of this type are the vector field V_μ ($D(1/2, 1/2)$) and symmetric tensor field $h_{\mu\nu}$ ($D(1,1)$). The presence of different spin values of the fields of the type $\psi_{M>0}^{(j,k)}$ leads to a number of problems associated with the elimination of redundant components.

In the massless case, the correspondence between the pair (j,k) and the helicity is established by Weinberg's theorem (Ref. 4): Under the assumption of positivity of the metric, the field $\psi_{M=0}^{(j,k)}$ describes a massless particle with helicity $\lambda=k-j$ and, conversely, a massless particle with helicity λ can be described only by the field $\psi_{M=0}^{(j,k)}$ with $k-j=\lambda$. In what follows, we shall be concerned with irreducible and reducible ($\psi_M^{(j,k)}$) quantum fields.

In conclusion, we give the commutation relations of a Poincaré covariant quantum field with the generators of the Poincaré group, i.e., the infinitesimal expression of the transformation properties of the field $\psi_M^{(j,k)}$:

$$\left.\begin{array}{l}[Y_{\mu\nu}, \psi_M^{(j,k)}(x)]=i\left(x_\mu\dfrac{\partial}{\partial x^\nu}-x_\nu\dfrac{\partial}{\partial x^\mu}+\Sigma_{\mu\nu}\right)\psi_M^{(j,k)}(x);\\ [P_\mu, \psi_M^{(j,k)}(x)]=i\dfrac{\partial}{\partial x^\mu}\psi_M^{(j,k)}(x).\end{array}\right\} \qquad (7)$$

Thus, the requirement of relativistic invariance of the theory severely restricts the possible types of fields, and also their properties. However, there still remains a great freedom, and this is where the other symmetries, related in some way or other with dynamics, come in. The coordinates of Minkowski space and the Lorentz covariant fields ($\psi_M^{(j,k)}$) are the elements of the space on which these groups act.

2. CONFORMAL GROUP

The Lorentz and Poincaré groups are exact symmetry groups of quantum field theory, while the majority of the remaining symmetry groups are approximate and become exact only under certain limiting conditions. We shall be interested here in the basic possibility of the existence of a particular symmetry group, and we shall therefore discuss the limiting conditions very briefly.

We begin with the spacetime groups. Since quantum field theory must be relativistically invariant, any spacetime symmetry group of quantum field theory (approximate or not) must contain the Poincaré group as a subgroup. Let us consider the possible groups of transformations in Minkowski space that are extensions of the Poincaré group.

The possible structure of such groups is suggested by the properties of the equations that describe massless fields. Namely, the equations for massless fields are invariant under transformations that form the so-called Weyl group and the conformal group. The conformal group contains[5, 5a]:

1) the Poincaré subgroup

$$x_\mu\to x'_\mu=\Lambda_{\mu\nu}x_\nu+a_\mu;$$

2) the special conformal transformations

$$x_\mu\to x'_\mu=(x_\mu-c_\mu x^2)/(1-2cx+c^2x^2);$$

3) the dilation

$$x_\mu\to x'_\mu=\lambda x_\mu,$$

where c_μ and λ are parameters of the corresponding transformations. The Poincaré group and the dilation form the Weyl group, which, in its turn, is a subgroup of the conformal group. We shall consider the conformal group, stating the results for the Weyl group at the same time.

The transformations of the 15-parameter conformal group leave invariant the light cone $(x-y)^2=0$ and the angles between world lines. For transformations of the Poincaré group and the dilation this is obvious, and for a special conformal transformation this property follows from the formula for the transformation of $(x-y)^2$:

$$(x-y)^2\to(x'-y')^2=(x-y)^2/[(1-2cx+c^2x^2)\times(1-2cy+c^2y^2)].$$

It follows from this that a special conformal transformation can change the sign of $(x-y)^2$ and x_0-y_0. The difficulties that then arise with causality in Minkowski space can be eliminated by the transition to the infinitely-sheeted covering space.[5a-8]

Above, we have considered continuous conformal transformations. The most important discrete conformal transformation is the inversion R with respect to the unit hypersphere:

$$x_\mu \xrightarrow{R} x'_\mu = Rx_\mu = x_\mu/x^2. \tag{8}$$

Using the R transformation, the special conformal transformation K can be represented in the compact form

$$x_\mu \xrightarrow{k} x'_\mu = Kx_\mu = RPRx_\mu,$$

where P is the shift $x_\mu \to x'_\mu = x_\mu + c_\mu$. Because of the complicated form of the nonlinear special conformal transformation it is frequently much more convenient to work with the R transformation.

Let us consider the algebra of the conformal group. The generators of Lorentz transformations $Y_{\mu\nu}$, shifts P_ρ, special conformal transformations K_μ, and the dilation D satisfy the following commutation relations[5]:

$$\left.\begin{aligned}
&[Y_{\mu\nu}, Y_{\rho\Sigma}] = \mathrm{i}\,(g_{\mu\Sigma}Y_{\nu\rho} + g_{\nu\rho}Y_{\mu\Sigma} - g_{\mu\rho}Y_{\nu\Sigma} - g_{\nu\Sigma}Y_{\mu\rho});\\
&[Y_{\mu\nu}, P_\rho] = \mathrm{i}\,(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu);\\
&[Y_{\mu\nu}, K_\rho] = \mathrm{i}\,(g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu);\\
&[P_\mu, P_\nu] = 0; \quad [K_\mu, K_\nu] = 0;\\
&[P_\mu, K_\nu] = 2\mathrm{i}\,(g_{\mu\nu}D + Y_{\mu\nu});\\
&[Y_{\mu\nu}, D] = 0; \quad [P_\mu, D] = \mathrm{i}P_\mu; \quad [K_\mu, D] = -\mathrm{i}K_\mu.
\end{aligned}\right\} \tag{9}$$

Note that the conformal algebra contains two subalgebras isomorphic to the Poincaré algebra: the subalgebra with basis elements $Y_{\mu\nu}$ and P_ρ and the subalgebra $Y_{\mu\nu}$ and K_ρ. If $RY_{\mu\nu}R = Y_{\mu\nu}$, $RP_\mu R = K_\mu$, these subalgebras are related by the R transformation.

The conformal group has three Casimir operators

$$\left.\begin{aligned}
&C_2 = Y_{\mu\nu}Y_{\mu\nu}/2 + K_\mu P_\mu - 4\mathrm{i}D - D^2;\\
&C_3 = (W_\mu K_\mu + K_\mu W_\mu)/4 - \varepsilon_{\mu\nu\rho\Sigma}DY_{\mu\nu}Y_{\rho\Sigma}/8;\\
&C_4 = \{K_\mu K_\mu P_\nu P_\nu - 4K_\mu Y_{\mu\nu}Y_{\nu\rho}P_\rho - 4K_\mu Y_{\mu\nu}P_\nu(D+6\mathrm{i})\\
&+3\,(Y_{\mu\nu}Y_{\mu\nu})^2/4 + (\varepsilon_{\mu\nu\rho\Sigma}Y_{\mu\nu}Y_{\rho\Sigma})^2/16\\
&+Y_{\mu\nu}Y_{\mu\nu}(D^2 + 8\mathrm{i}D - c_2 - 22) - D^4 - 16\mathrm{i}D^3 + 80D^2\\
&+128\mathrm{i}D + 36c_2 - 16\mathrm{i}c_2 D - 2c_2 D^2\}/4,
\end{aligned}\right\} \tag{10}$$

where $W_\mu = \varepsilon_{\mu\nu\rho\Sigma}P_\nu Y_{\rho\Sigma}/2$. Thus, irreducible representations of the conformal group are specified by three numbers.

The classification of the irreducible representations of the conformal group is a fairly complicated problem. Of particular assistance here is the local isomorphism between the conformal group and the group $SO(2, 4)$.[5] One can see that this isomorphism exists by going over from $Y_{\mu\nu}$, P_ρ, K_Σ, D to the combinations $L_{\mu\nu} = Y_{\mu\nu}$; $L_{5\mu} = (P_\mu + K_\mu)/2$; $L_{6\mu} = (P_\mu - K_\mu)/2$; $L_{56} = D$. From the relations (9) we find that the operators L_{ab} $(a, b = 0, 1, 2, 3, 5, 6)$ satisfy the commutation relations of the algebra $SO(2, 4)$: $[L_{ab}, L_{cd}] = \mathrm{i}(g_{ad}L_{bc} + g_{bc}L_{ad} - g_{bc}L_{bd} - g_{bd}L_{ac})$, where $g_{ab} = \mathrm{diag}(1, -1, -1, -1, -1, +1)$, which proves the local isomorphism of the groups.

The local isomorphism between the conformal group and $SO(2, 4)$ enables one, first, to construct a 6-model of Minkowski space (the Dirac model[9]), in which the nonlinear conformal transformations become linear homogeneous transformations and, second, and this is more important, enables one to use for the classification of the irreducible representations of the conformal group the methods developed for the pseudo-orthogonal and pseudounitarity groups [$SO(2, 4)$ is locally isomorphic to $SU(2, 2)$].

The irreducible representations of the conformal group were classified in Refs. 10–14. Physically, the most interesting are the representations in which $p^2 \geq 0$. Since $p^2 \to \lambda^{-2}p^2$ under the dilation, all representations with $p^2 \geq 0$ divide into two classes: massless representations ($p^2 = 0$) and representations with a continuous mass spectrum (p^2 is any positive number).

The irreducible unitary massless representations are the simplest unitary representations of the conformal group. In them, the Casimir operators can be expressed in terms of the single number q (Ref. 12):

$$C_2 = 3\,(q^2 - 1); \quad C_3 = \pm q\,(q^2 - 1); \quad C_4 = -3\,(q^2 - 1)^2/4, \tag{11}$$

where q takes the values 0, 1/2, 1, 3/2, ... In the massless representations, the sign of the energy $\varepsilon_0 = p_0/|p_0|$ is an invariant. Therefore, for each value of q there exist four inequivalent representations [combinations of the sign of p_0 and the sign of C_3 in (11)].

The unitary irreducible massless representations of the conformal group coincide with the irreducible unitary massless representations of the Poincaré group with helicity $\lambda = \pm q$ (Ref. 15). It is for this reason that the equations describing massless particles are conformally invariant.

Let us now consider irreducible representations with $p^2 > 0$. In these representations, the sign of p_0 is also an invariant, and the Casimir operators[12] are

$$\left.\begin{aligned}
&C_2 = \Lambda_m(\Lambda_m + 4) + 2Y_m(Y_m + 1) + 2K_m(K_m + 1);\\
&C_3 = -(\Lambda_m + 2)(Y_m - K_m)(Y_m + K_m + 1);\\
&C_4 = (\Lambda_m + 2)^4/4 - (\Lambda_m + 2)^2\\
&-(\Lambda_m + 2)[Y_m(Y_m + 1) + K_m(K_m + 1)]\\
&+4Y_m(Y_m + 1)K_m(K_m + 1).
\end{aligned}\right\} \tag{12}$$

In the unitary irreducible representations Y_m, $K_m = 0, 1/2, 1, 3/2, \ldots$, and Λ_m can take real values bounded by the inequalities (Ref. 16) $\Lambda_m \leq -(Y_m + K_m + 2)$ for $Y_m K_m \neq 0$ (the so called nondegenerate representations) and $\Lambda_m < -(1 + Y_m + K_m)$ for $Y_m K_m = 0$ (degenerate representations). Note that the representations with $\Lambda_m = l/n$ (l and n are integers) are n-valued, and representations with irrational Λ_m are infinite-valued. Thus, the universal covering of the conformal group is infinitely sheeted.[16,5a-8]

We now turn to quantum fields that transform in accordance with irreducible unitary representations of the conformal group. The general procedure for constructing arbitrary conformally covariant fields has been set forth in Refs. 5 and 17. We consider the class of conformal fields defined by

$$[Y_{\mu\nu}, \psi^{(j_1, j_2)d}(x)] = i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + \Sigma_{\mu\nu}\right)\psi^{(j_1, j_2)d}(x);$$
$$[P_\mu, \psi^{(j_1, j_2)d}(x)] = i\frac{\partial}{\partial x^\mu}\psi^{(j_1, j_2)d}(x);$$
$$[K_\mu, \psi^{(j_1, j_2)d}(x)] = i\left[(x^2 g_{\mu\nu} - 2x_\mu x_\nu)\frac{\partial}{\partial x^\nu} - 2x_\mu d + 2ix_\nu\Sigma_{\mu\nu}\right]\psi^{(j_1, j_2)d}(x);$$
$$[D, \psi^{(j_1, j_2)d}(x)] = -i\left(d + x_\nu\frac{\partial}{\partial x^\nu}\right)\psi^{(j_1, j_2)d}(x). \quad (13)$$

The superscripts (j_1, j_2) mean that the field $\psi^{(j_1, j_2)d}(x)$ transforms in accordance with the representation $D(j_1, j_2)$ of the Lorentz group; $\Sigma_{\mu\nu}$ is the corresponding spin matrix. The real number d is the so called scale dimension of the conformal field $\psi^{(j_1, j_2)d}$.

Equations (13) mean that the transformation properties of the conformal field are given by the three numbers j_1, j_2, and d. Whereas the meaning of the numbers j_1 and j_2 is obvious, the scale dimension d is a new quantum number characterizing conformally covariant fields.

What values can be taken by the scale dimension d and j_1 and j_2 for fields that are of physical interest, i.e., for fields that transform in accordance with unitary irreducible representations of the conformal group with $p^2 \geq 0$? To answer this question, it is necessary to consider the correspondence between the triplets of numbers j_1, j_2, d, and Y_m, K_m, Λ_m (and q for $p^2 = 0$).

We calculate the values of the Casimir operators for single-particle states of the field $\psi^{(j_1, j_2)d}$, (x), i.e., $C_2\psi^{(j_1, j_2)d}(x)|0\rangle$; $C_3\psi^{(j_1, j_2)d}|0\rangle$; $C_4\psi^{(j_1, j_2)d}(x)|0\rangle$, where $|0\rangle$ is the invariant vacuum. Using Eqs. (10) and (13) and the translational invariance of C_2, C_3, C_4, we find[18]

$$\left.\begin{aligned} &C_2 = (d-2)^2 - 4 + \Sigma_{\mu\nu}\Sigma_{\mu\nu}/2;\\ &C_3 = i(d-2)\,\varepsilon_{\mu\nu\rho\Sigma}\Sigma_{\mu\nu}\Sigma_{\rho\Sigma}/8;\\ &C_4 = (d-2)^4/4 - (d-2)^2 + (\Sigma_{\mu\nu}\Sigma_{\mu\nu})^2/16\\ &\quad + (\varepsilon_{\mu\nu\rho\Sigma}\Sigma_{\mu\nu}\Sigma_{\rho\Sigma})^2\cdot 64 - (d-2)^2\,\Sigma_{\mu\nu}\Sigma_{\mu\nu}/4, \end{aligned}\right\} \quad (14)$$

where on the left- and right-hand sides of the equations application to the vectors $\psi^{(j_1, j_2)d}(x)|0\rangle$ is understood. Since for the spin matrix

$$\Sigma_{\mu\nu}\Sigma_{\mu\nu} = 4j_1(j_1+1) + 4j_2(j_2+1);$$
$$\varepsilon_{\mu\nu\rho\Sigma}\Sigma_{\mu\nu}\Sigma_{\rho\Sigma} = 8i\,[j_1(j_1+1) - j_2(j_2+1)],$$

we finally obtain[16,19]

$$\left.\begin{aligned} &C_2 = (d-2)^2 - 4 + 2j_1(j_1+1) + 2j_2(j_2+1);\\ &C_3 = -(d-2)\,[j_1(j_1+1) - j_2(j_2+1)];\\ &C_4 = (d-2)^4/4 - (d-2)^2 - (d-2)^2\,[j_1(j_1+1) + j_2(j_2+1)]\\ &\quad + 4j_1(j+1)\,j_2(j_2+1). \end{aligned}\right\} \quad (15)$$

Let us consider first representations with continuous spectrum p^2. Comparing (12) and (15), we find $j_1 = Y_m$, $j_2 = K_m$, $d = -\Lambda_m$. As a result, in the unitarity irreducible representations of the conformal group with continuous mass spectrum we have:

1) nondegenerate representations

$$j_1,\ j_2 = 1/2,\ 1,\ 3/2,\ \ldots;\ d = 2 + j_1 + j_2 + \rho,\ \rho \geq 0;$$

2) degenerate representations

$$\left.\begin{aligned} &j_1 = 0,\ j_2 = j = 0,\ 1/2,\ 1,\ \ldots\\ \text{or}\quad &j_2 = 0,\ j_1 = j = 0,\ 1/2,\ 1,\ \ldots \end{aligned}\right\} d = 1 + j + \rho',\ \rho' > 0.$$

For the massless representations, it follows from comparison of (11) and (15) that there are the following possibilities:

$$\left.\begin{aligned} &j_1 = 0,\ j_2 = j = 0,\ 1/2,\ 1,\ \ldots\ (\lambda = p = j)\\ \text{or}\quad &j_2 = 0,\ j_1 = j = 0,\ 1/2,\ 1,\ \ldots\ (\lambda = -p = -j) \end{aligned}\right\} d = 1 + j.$$

Thus, only those massless fields that transform in accordance with the representations $D(j, 0)$ and $D(0, j)$ of the Lorentz group are conformal covariants [of course under the assumption of the transformation laws (13)]. This result can also be obtained by analyzing the equations for massless fields.[20] The dimension $d = 1 + j$ of the free massless fields is called the canonical dimension and is equal to the physical dimension of the fields.

Scale dimensions of fields that differ from the canonical dimensions and satisfy, in theories with positive metric, the inequalities $d(j) > 1 + j$ are called anomalous dimensions. As can be seen from the foregoing, fields with anomalous dimensions transform in accordance with degenerate and nondegenerate infinite-valued representations of the conformal group with continuous p^2 spectrum $(0 < p^2 < \infty)$. Conformally covariant conserved tensors transforming in accordance with the representations $D(n/2, n/2)$ of the Lorentz group, of the type of the current J_μ and the energy–momentum tensor with dimension $d = 2 + n$, transform in accordance with nondegenerate representations.

Note that the irreducible unitary representations of the conformal group listed above are also irreducible unitary representations of the Weyl group. The conformal group also has infinite-component nondecomposable representations, which have been considered in Refs. 21 and 22.

We now formulate briefly the main properties of quantum conformal fields. Free conformal fields can be constructed by a method analogous to that in the construction of free Poincaré covariant fields as proposed by Weinberg.[23,4] The free massless conformal fields simply coincide with the corresponding free massless Poincaré covariant fields. The free conformal fields $\psi^{(j_1, j_2)d}(x)$ with continuous p^2 spectrum are generalized free fields, and they can be represented in the form[16,18]

$$\psi^{(j_1, j_2)d}(x) = \sum_{s=|j_1-j_2|}^{j_1+j_2} \int_0^\infty dm^2\,(m^2)^{(d-2)/2}\,\psi^{(j_1, j_2)}_{m,s}(x),$$

where $\psi^{(j_1, j_2)d}_{m,s}(x)$ is the free relativistic field with mass m and spin s (Ref. 24). The explicit form of the conformal fields $\psi^{(j, j_2)d}(x)$ is given in Refs. 16, 18, and 25.

Interacting conformal fields exhibit a number of interesting features. The main difference from ordinary relativistically invariant theory is the following. In the theory of interacting relativistic fields $\psi_{m,s}(x)$, the vectors $\psi_{m,s}(x)|0\rangle$ necessarily (by virtue of the Lehmann representation) contain, besides the mass m, a certain spectrum of masses, i. e., they transform in accordance with a reducible representation of the Poincaré group. But in the theory of interacting conformal fields $\psi^{(j_1, j_2)d}$ the vectors $\psi^{(j_1, j_2)d}(x)|0\rangle$ form the space of an irreducible representation of the conformal group (with dimension d).[16,19] It is important that this is a purely kinematic result. Thus, in a conformally invariant theory the properties of single-particle states are completely fixed and do not depend on the particular dynamics.

With this circumstance is associated the fact that the requirement of conformal invariance completely fixes the form of the two- and three-point Green's functions (except for arbitrary constants).[26,27] This property of conformally invariant theory enables one to formulate a bootstrap program for calculating anomalous dimensions.[27–30] A discussion of bootstrap equations and also other applications of conformal invariance in quantum field theory would go outside the scope of this review (see, for example, Refs. 31 and 32). We merely mention here that the requirement of conformal invariance of a theory fixes a considerable part of the dynamics.

In real theories, conformal invariance is approximate because the physical particle masses are finite, and it becomes exact only in the limit when one can ignore the characteristic masses.

3. STRUCTURE OF TRANSFORMATION GROUPS IN MINKOWSKI SPACE

In the preceding section, we have seen that invariance under the conformal group imposes much stronger restrictions on quantum theory than invariance under just the Poincaré group, which is a subgroup of the conformal group. It would be attractive to find a group of transformations in Minkowski space that, in its turn, would contain the conformal group as a subgroup and fix even more of the dynamics. Independently of this, the following question is of interest: What spacetime symmetry groups higher than the conformal exist? (We emphasize that we have in mind finite continuous groups of transformations.) The answer to this question is negative. Namely, there do not exist finite Lie groups of transformation in Minkowski space that contain the conformal group as a subgroup; thus, there do not exist finite Lie groups of spacetime symmetries that are higher than the conformal group.[33]

Let us prove this. It is well known that the generators of an arbitrary Lie group of transformations of the coordinates of Minkowski space have the form

$$L_i(x) = \mathrm{i} f_{i\mu}(x)\frac{\partial}{\partial x^\mu} \quad (\mu = 0, 1, 2, 3), \tag{16}$$

where the functions $f_{i\mu}(x)$ are determined by the law of transformation of the coordinates; the subscript i takes the values $1, 2, \ldots, N$ (N is the number of parameters of the group). In particular, for the conformal group

$$\left.\begin{aligned} &Y_{\mu\nu} = \mathrm{i}\,(x_\mu \partial/\partial x^\nu - x_\nu \partial/\partial x^\mu); \quad P_\mu = \mathrm{i}\partial/\partial x^\mu;\\ &K_\mu = \mathrm{i}\,(x^2 g_{\mu\nu} - 2x_\mu x_\nu)\,\partial/\partial x^\nu; \quad D = \mathrm{i} x_\rho \partial/\partial x^\rho. \end{aligned}\right\} \tag{17}$$

We show that in Minkowski space it is impossible to realize an algegra of a finite Lie group that has generators of the form (16) and contains the conformal group as a subgroup.

Suppose otherwise. For clarity, we shall consider a special case. Suppose that in Minkowski space there exists a group of transformations that is locally isomorphic to $SO(p,q)$ ($p+q>6$, $p\geqslant 2$, $q\geqslant 4$). The generators $\mathscr{L}_{AB}$ ($A, B = 0, 1, 2, 3, 5, 6, 7, \ldots, p+q$) of $SO(p,q)$ satisfy the standard commutation relations

$$[\mathscr{L}_{AB}, L_{CD}] = \mathrm{i}\,(g_{AD}L_{BC} + g_{BC}L_{AD} - g_{AC}L_{BD} - g_{BD}L_{AC}), \tag{18}$$

where $g_{AB} = \mathrm{diag}(1, -1, -1, -1, -1, 1, \ldots,)$. The algebra $SO(p,q)$ contains the subalgebra $SO(2,4)$ (isomorphic to the conformal algebra) with generators $L_{ab} = \mathscr{L}_{ab}$ ($a, b = 0, 1, 2, 3, 5, 6$). The correspondence between the generators L_{ab} and the generators of the conformal group has been given above.

By virtue of the relations (18), the generators $\mathscr{L}_{AB}$ always include at least one generator Λ (for example, L_{67}) that commutes simultaneously with $L_{\mu\nu}$ and $L_{5\mu}$, which form the algebra $SO(1,4)$. Let us find the form of this generator. From the condition $[\Lambda, L_\mu] = [\Lambda, Y_{\mu\nu}] = 0$ it follows that

$$\Lambda(x) = \mathrm{i}\Phi(x^2)x_\nu(\partial/\partial x^\nu), \tag{19}$$

where $\Phi(x^2)$ is an arbitrary scalar function of x^2. Further, from the condition $[\Lambda, L_{5\mu}] = [\Lambda, P_\mu + K_\mu]/2 = 0$, taking into account (17) and (19), we obtain the equation

$$(1-x^2)\,g_{\mu\rho}\Phi(x^2) + 2x_\mu x_\rho\,[(1-x^2)\,d\Phi(x^2)/dx^2 + \Phi(x^2)] = 0, \quad \mu, \rho = 0, 1, 2, 3. \tag{20}$$

The only nonsingular solution of the system of equations (20) is, as we readily see, $\Phi = 0$.

Thus, $\Lambda(x) = 0$, i. e., in Minkowski space there does not exist an operator of the form (16) that commutes simultaneously with $L_{\mu\nu} = Y_{\mu\nu}$ and $L_{5\mu} = (P_\mu + K_\mu)/2$. Therefore, the group $SO(p,q)$ with ($p+q>6$, $p\geqslant 2$, $q\geqslant 4$) cannot be realized as a group of transformations of Minkowski space. Similarly, one can show that the group $SU(p,q)$ with ($p+q>4$, $p\geqslant 2$, $q\geqslant 4$) cannot be realized as a group of transformations of Minkowski space. For this it is necessary to note that the algebra $SO(2,4)$ is isomorphic to the algebra $SU(2,2)$, and the algebra $SO(1,4)$, which is formed by the generators $Y_{\mu\nu}$ and $(p_\mu + K_\mu)/2$, is isomorphic to the unitary symplectic group $U_{sp}(2,2)$.

The proof given above can be generalized to an arbitrary finite Lie group containing the conformal subgroup. For any finite Lie group is locally isomorphic to some linear group.[34] But for linear groups one can show that there always exists at least one generator that does not belong to the subalgebra $SO(2,4)$ [$\sim SU(2,2)$] and commutes with the generators L_μ and $L_{5\mu}$ of the subgroup $SO(1,4)$ [$\sim U_{sp}(2,2)$], where $SO(1,4)$ $SO(2,4)$. But, as we have seen, such a generator cannot be realized in Minkowski space, and therefore such a group cannot be realized as a group of transformations of Minkowski space.

Thus, in Minkowski space there do not exist finite Lie groups of transformations of coordinates containing the conformal group as a subgroup. Thus, spacetime symmetries higher than the conformal symmetry do not exist. We emphasize that an important role in the derivation of this assertion is played by the requirement of invariance under Lorentz transformations.

Besides the conformal group, there exists one further group in Minkowski space that contains the Poincaré group as subgroup. This is the group of all linear transformations of the coordinates (the affine group):

$$x_\mu \to x'_\mu = a_{\mu\nu}x_\nu + b_\mu. \tag{21}$$

The 20-parameter affine group is a semidirect product of the group $L(4, E)$ and the group of displacements. The algebra of the affine group consists of the generators $Y_{\mu\nu}$ of Lorentz transformations, the generators $R_{\mu\nu}$ ($R_{\mu\nu} = R_{\nu\mu}$, $R_{\mu\mu} = 0$) of properly-linear [Russian: Sobstvenno-lineĭnye] transformations, the dilation generator D, and the generators P_ρ of displacements. The generators $Y_{\mu\nu}$, P_ρ, and D satisfy the commutation relations of the Weyl algebra (see Sec. 2). The remaining commutation relations have the form

$$\left.\begin{array}{l}[R_{\mu\nu}, R_{\rho\Sigma}] = \mathrm{i}\,(g_{\mu\Sigma}Y_{\nu\rho} + g_{\nu\rho}Y_{\mu\Sigma} + g_{\mu\rho}Y_{\nu\Sigma} + g_{\nu\Sigma}Y_{\mu\rho});\\ [R_{\mu\nu}, D] = 0;\\ [Y_{\mu\nu}, R_{\rho\Sigma}] = \mathrm{i}\,(g_{\mu\rho}R_{\nu\Sigma} + g_{\mu\Sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\Sigma} - g_{\nu\Sigma}R_{\mu\rho});\\ [R_{\mu\nu}, P_\rho] = \mathrm{i}\,(g_{\mu\rho}P_\nu + g_{\nu\rho}P_\mu - g_{\mu\nu}P_\rho/2).\end{array}\right\} \tag{22}$$

Note that the Lorentz transformations and the properly-linear transformations form the special linear group $SL(4, E)$, which is locally isomorphic to $SO(3, 3)$.

The affine and the conformal group, which act on the same space (Minkowski space), have the same subgroups—the Weyl group. But here their similarity apparently stops. The conformal group contains, besides the Weyl group, the nonlinear special conformal transformations, and the affine group contains the linear properly-linear transformations. This difference leads to an important difference when one is considering these groups as symmetry groups. Whereas the conformal group has representations with fixed mass ($M^2 = P^2 = 0$), the affine group, by virtue of the relation $[R_{\mu\nu} P^2] = 4i(P_\mu P_\nu - g_{\mu\nu}P^2/4)$, has representations only with continuous mass spectrum. Whereas conformal symmetry, which is approximate for real particles, becomes exact in the limit, i.e., one can ignore the characteristic masses, affine symmetry (symmetry with respect to properly-linear transformations) remains approximate for all relationships between the characteristic momenta and masses. Any finite Lie group of transformations in Minkowski space containing the affine group as a subgroup has the same property.

Thus, summarizing the results of the above, we see that the number of possible exact finite Lie groups of symmetries in Minkowski space is limited and exhausted by the Poincaré, Weyl, and conformal groups[1)] (in Ref. 35, this fact is proved in a different way). It follows from this in particular that the number of globally conserved quantities having a spacetime origin is finite.

All the above finite continuous transformation groups in Minkowski space are subgroups of the infinite group of general coordinate transformations

$$x_\mu \to x'_\mu = f_\mu(x_0, x_1, x_2, x_3),$$

where $f_\mu(x)$ are arbitrary functions of the coordinates. The physical meaning of the group of general coordinate transformations appears in the general theory of relativity, which is based on this group.

It is well known that there is a profound difference between finite and infinite symmetry groups. However, as we shall see below, the infinite group of general coordinate transformations reduces in a certain sense to its two finite subgroups—the affine and the conformal.

The group of general coordinate transformations has infinitely many parameters. The parameters of this group can be regarded as coefficients of the expansions of the functions $f_\mu(x)$ in infinite series in powers of the coordinates. The algebra of the group of general coordinate transformations contains infinitely many generators of the form

$$L_\mu^{n;\,n_0,\,n_1,\,n_2,\,n_3} = \mathrm{i}x_0^{n_0}x_1^{n_1}x_2^{n_2}x_3^{n_3}\frac{\partial}{\partial x^\mu}\,(n = n_0 + n_1 + n_2 + n_3). \tag{23}$$

The algebra of the generators $L_\mu^{n;\,n_0,n_1,n_2,n_3}$ contains two closed subalgebras: the algebra of the conformal group with the generators $Y_{\mu\nu}$, P_ρ, K_Σ, D (17) and the algebra of the affine group with the generators $Y_{\mu\nu}$, $R_{\mu\nu}$, P_ρ, D, where the generator $R_{\mu\nu}$ of properly-linear transformations has the form

$$R_{\mu\nu} = \mathrm{i}\left(x_\mu\frac{\partial}{\partial x^\nu} + x_\nu\frac{\partial}{\partial x^\mu} - \frac{1}{2}g_{\mu\nu}\left(x_\rho\frac{\partial}{\partial x^\rho}\right)\right). \tag{24}$$

The algebra of the affine and conformal groups have common subalgebras—the Weyl algebra; the remaining generators differ from one another. As a result, the affine and the conformal algebra do not commute with one another.

Let us consider the minimal algebra containing the affine and the conformal algebra as subalgebras. Such an algebra must contain, besides the generators $Y_{\mu\nu}$, $R_{\mu\nu}$, P_ρ, K_Σ, and D, all possible commutators of these generators with one another, and all possible repeated commutators. It can be shown that the required algebra is infinite and coincides with the algebra of the group of general coordinate transformations. In other words, any generator $L_\mu^{n;\,n_0,n_1,n_2,n_3}$ of the group of general coordinate transformations can be represented as a linear combination of repeated commutators of the generators of the affine and the conformal group.[36]

To prove this theorem, it is necessary to show that the repeated commutators of the generators and their linear combinations exhaust all generators of the form (23). We consider successively generators $L_\mu^{n;\,\cdots}$ with $n = 0, 1, 2, \ldots$. The generators of the displacements $P_\mu = i(\partial/\partial x^\mu)$ give all the generators $L_\mu^{n;\,\cdots}$ with $n = 0$. Further, the generators of the Lorentz transformations

[1)]Analogous results also hold for spacetime of arbitrary dimension (⩾2).

$Y_{\mu\nu} = i(x_\mu(\partial/\partial x^\nu) - x_\nu(\partial/\partial x^\mu))$, the properly-linear transformations $R_{\mu\nu} = i(x_\mu(\partial/\partial x^\nu) + x_\nu(\partial/\partial x^\mu) - g_{\mu\nu}/2[x_\rho(\partial/\partial x^\rho)])$, and the dilation generator $D = ix_\rho(\partial/\partial x^\rho)$ exhaust all the L_μ with $n=1$. The generators of the special conformal transformations $K_\mu = i(x^2 g_{\mu\nu} - 2x_\mu x_\nu)(\partial/\partial x^\nu)$ belong to $L_\mu^{2;\cdots}$. We show that all generators $L_\mu^{n;\cdots}$ with $n=2$ are contained in the closing algebra. Consider the commutator of $R_{\mu\mu}$ with K_ν $(\mu \neq \nu)$:

$$[R_{\mu\mu}, K_\nu] = -2x_\mu^2(\partial/\partial x^\nu).$$

We have obtained the generator $ix_\mu^2(\partial/\partial x^\nu)$ $(\mu \neq \nu)$. Further

$$\left[R_{\nu\mu}, ix_\mu^2 \frac{\partial}{\partial x^\nu}\right] = [2x_\mu x_\nu \frac{\partial}{\partial x^\nu} - x_\mu^2 \frac{\partial}{\partial x^\mu}]. \quad (25)$$

Comparing (25) with K_μ, we see that the closing algebra contains the generators $ix_\mu x_\nu(\partial/\partial x^\nu)$ $(\mu \neq \nu)$ and $ix_\mu^2(\partial/\partial x^\mu)$. Finally, the generator $ix_\mu x_\nu(\partial/\partial x^\Sigma)(\mu \neq \nu \neq \Sigma)$ arises in the commutator

$$\left[R_{\mu\nu}, ix_\rho x_\nu \frac{\partial}{\partial x^\nu}\right] = x_\mu x_\rho \frac{\partial}{\partial x^\nu}.$$

Thus, all generators $L_\mu^{n;\cdots}$ with $n=2$ have been exhausted.

Commuting the generators of the form $L_\mu^{2;\cdots}$ with one another, we arrive at the generators $L_\mu^{3;\cdots}$, for example,

$$\left[ix_\mu^2 \frac{\partial}{\partial x^\nu}, ix_\nu^2 \frac{\partial}{\partial x^\nu}\right] = -2x_\mu^2 x_\nu \frac{\partial}{\partial x^\nu}.$$

Successive commutation of the repeated commutators leads to generators $L_\mu^{n;\cdots}$ with ever higher n. Therefore, the assertion of the theorem becomes natural. A rigorous proof can be given by the method of mathematical induction.[36]

Thus, the algebra of the group of general coordinate transformations is the closure of the algebra of the affine and the conformal group. Thus, any theory that is simultaneously invariant under the affine and the conformal group is also invariant under the group of general coordinate transformations. Since the transformation properties of any entity with respect to the infinite generally covariant group are determined by its transformation properties with respect to the much simpler finite-dimensional affine and conformal groups, one can construct Einstein's theory of gravitation on the basis of joint nonlinear realizations of these groups.[37]

We note the intimate connection between this theorem and the result contained above, at the start of the section, about the nonexistence of a finite group of transformation of Minkowski space containing the conformal subalgebra. The point is that the addition to the conformal algebra of any generator of the form (23) different from $Y_{\mu\nu}$, P_ρ, K_ρ, D necessarily leads to the infinite algebra of the group of general coordinate transformations. It follows from this that the structure of the infinite-dimensional group of general coordinate transformations differs essentially from the structure of Lie groups [for example, $SO(p,q)$, $SU(p,q)$] in the formal limit of infinitely many parameters $(p, q \to \infty)$.

4. INTERNAL SYMMETRY GROUPS

A considerable enlargement of the class of possible symmetry groups arises when one considers groups of transformations that do not affect the coordinates of Minkowski space—groups of internal symmetries. The simplest example of a group of internal symmetry is the group of gauge transformations:

$$\varphi(x) \to \varphi'_i(x) = \exp(i\alpha)\varphi(x), \quad (26)$$

where $\varphi(x)$ is a complex field and α an arbitrary constant. The group of gauge transformations is the single-parameter unitary group $U(1)$ and the field $\varphi(x)$ in (26) transforms in accordance with a unitary (single-row) representation of this group. More complicated internal symmetry groups are $SU(2)$ and $SU(3)$. In these cases, the field transforms in accordance with an n-row irreducible representation of these groups:

$$\varphi_i(x) \to \varphi'_i(x) = U_{ik}\varphi_k(x),$$

where the subscript i takes the values $1, \ldots, n$; U_{ik} is an n-row unitary matrix.

In the general case, any continuous group can be an internal symmetry group. The quantum fields then transform in accordance with some representation of the group G:

$$\varphi_\alpha(x) \xrightarrow{G} \varphi'_\alpha(x) = S_{\alpha\beta}\varphi_\beta(x).$$

The main restriction imposed on internal symmetry groups in the framework of the usual axioms of quantum field theory[3] is positivity of the metric of the state space. As a result, one can consider only unitary representations. In conjunction with the requirement that the number of fields (particles) in multiplets be finite, we conclude that internal symmetry groups must be compact.[2)] Thus, the class of symmetry groups is augmented with compact Lie groups.

There is an important difference between spacetime symmetry groups and internal symmetry groups: Whereas the latter must be compact, all the former that we have considered are noncompact, although in both cases we are concerned with fields with finitely many components. This difference is due to the fact that in the case of the spacetime groups the coordinates of Minkowski space, on which the field depends, are additional indices that label the vectors of the representation space, and this makes it possible to realize infinite-dimensional unitary representations by fields with finitely many components.

We have seen in the preceding section that the dimension (number of parameters) of the spacetime symmetry groups is bounded. For internal symmetry groups, no such bounds arise. The point is that the internal symmetry groups [for example, $SU(n)$] are groups of transformations of abstract spaces (n-dimensional complex space in the case of $SU(n)$). The coordinates of these spaces are in no way related to the coordinates of Minkowski space, and the number of these coordinates is not restricted by any physical arguments. As

2)Only compact groups have nontrivial finite-dimensional unitary representations (see, for example, Fronsdal's paper in Ref. 38).

a result, arbitrary (compact) transformation groups are allowed.[3)] But the structure of Minkowski space is completely fixed, and this leads to the above restrictions. We emphasize that the abstract spaces (and their coordinates) considered above are completely fictitious. Only the coordinates of Minkowski space and quantum fields that transform in accordance with certain representations of the various groups are observable (physical).

We shall not here consider the various groups of internal symmetry (the most popular among them are the unitary groups $SU(3)$, $SU(4)$, and $SU(6)$), nor their applications, in particular in hadron physics, since detailed reviews and monographs on these questions exist.[38–40] We merely point out that the majority of internal symmetry groups are broken.

The internal symmetry groups lead to a classification of the elementary particles in accordance with internal quantum numbers of the type of charge, isospin, strangeness, unitary spin. On the other hand, the very important characteristic mass, for example, of particles remains completely outside the scope of internal symmetry groups. It would therefore be good if one could find a symmetry group that simultaneously classifies particles with respect to internal quantum numbers and masses. Since the concept of mass of particles in the framework of quantum field theory is related to the invariant p^2 of the Poincaré group, such a group must be a nontrivial union of the internal symmetry groups and the Poincaré group. Such a group would be simultaneously a nontrivial relativization of internal symmetry groups. The trivial solution—the direct product of the internal symmetry groups and the Poincaré group—gives equal masses of particles in the multiplets of the internal symmetry groups. The solution of this problem is given by O'Raifeartaigh's theorem.[41]

Let G be the algebra of an arbitrary finite Lie group that contains the Poincaré algebra as a subalgebra and a set of generators of, for example, an internal symmetry group. Then if the operator $p^2 = p_\mu p_\mu$ of the square of the momentum and any power of it are self-adjoint operators in the Hilbert space H of a representation of G and if the spectrum of the operator p^2 in H contains a discrete point m^2, then the eigenspace H_m corresponding to the eigenvalue m^2 is closed and invariant with respect to the operators that represent the algebra G in H. We shall not prove this theorem, but turn directly to its consequences.

It follows from O'Raifeartaigh's theorem that, under the assumption that the required Lie group is finite, the metric is positive (the state space is a Hilbert space), the eigenvalues of p^2 corresponding to the particle masses are discrete, and p^2 is an observable (p^2 is self-adjoint), the mass spectrum within multiplets of internal symmetry groups either consists of a single point or is continuous. Thus, no finite Lie symmetry group can explain the observed mass spectrum of the particles.

Further, if one requires that the mass takes only a finite number of values in multiplets of the internal symmetry groups, the required group reduces to the direct product of the Poincaré group and the internal symmetry group. Various dynamical assumptions[42] lead to the same result.

We emphasize that in the proof of O'Raifeartaigh's theorem the requirement of relativistic invariance plays a very important role. It is the structure of the Poincaré algebra that leads to the theorem.

Thus, the attempt to combine internal symmetry groups with spacetime symmetry groups does not lead to a new type of group. The internal symmetry groups and the spacetime groups act completely independently: The spaces on which these groups act are not coupled—a transformation in one of them does not induce transformation in the other; the quantum fields transform in accordance with a direct product of the representations of the internal symmetry groups and the spacetime groups. It is important to note that we restrict ourselves to finite Lie groups. If we consider infinite groups or groups that are not Lie groups, then new possibilities arise.

One of the best known classes of infinite Lie groups is the class of local internal symmetry groups. In the internal symmetry groups considered above, the group parameters are the same at all points of Minkowski space. But in local groups, which realize the idea of a short interaction range in field theory, the group parameters are functions of the coordinates (see, for example, Refs. 43 and 44). In the same way, one can consider local spacetime groups.[43,44]

In local groups, there arises a definite connection between Minkowski space and the space on which the considered group acts. This is described by the concept of a fiber bundle, which leads to a beautiful geometrical interpretation of local groups (see, for example, Ref. 44). An extensive literature (see, for example, Refs. 43–50), to which we refer the reader, has been devoted to local gauge groups and theories invariant under these groups.

5. GROUPS OF TRANSFORMATIONS WITH SPINOR GENERATORS

All the transformation groups considered in the previous sections have been Lie groups and the corresponding algebras are Lie algebras, i.e., the binary operation in the algebras is the commutator of two operators. This is due to the fact that the parameters of Lie groups are commuting numbers. With respect to the Lorentz group, the parameters of Lie groups have a tensor nature, i.e., they may be scalars, vectors, tensors, etc. As a result, the multiplets of symmetry groups contain either fields with one and the same spin value or fields with the same statistics, i.e., fields with integral or half-integral spin values.

It is readily noted that we have not used the possibility in which the parameters transform in accordance with

[3)]However, if one fixes the number of fields in the theory, restrictions on the possible symmetry groups then arise. For example, in the case of three fields, the maximal symmetry group is $U(3)$.

spinor representations of the Lorentz group. However, in this case there is an important difference between the Lie groups and algebras. The point is that the generators of transformations with spinor parameters are spinors with respect to the Lorentz group.

Therefore, to preserve the correct connection between spin and statistics, the corresponding algebra must contain the anticommutators of the spinor generators with one another, and this means that we must go beyond the framework of Lie algebras.

It is obvious that transformations with spinor parameters cannot be transformations of coordinates of Minkowski space. These transformations act on the "space" of quantum fields and in this respect are similar to the transformations of the internal symmetry groups. Further, since $\delta\varphi \sim \varepsilon\psi$ in the case of an infinitesimal transformation with spinor parameter ε, the fields φ and ψ are described by different statistics. Thus, irreducible representations of groups that include transformations with spinor parameters contain fields with integral and half-integral spins. Thus, groups of transformations with spinor parameters (generators) lead to new possibilities.

We consider a group containing the Poincaré subgroup and a transformation with parameter that is the simplest Lorentz spinor. The algebra of such a group consists of the generators of Lorentz transformations, displacements, and the spinor generator Q, i.e., the algebra is the extension of the Poincaré algebra by the spinor generator Q.

Let us find the form of the simplest possible spinor extensions of the Poincaré algebra; for this, it is convenient to use the spinor notation considered in Sec. 1. In this notation, the generators of Lorentz transformations are expressed in the form of two symmetric spin tensors Y_{AB} and $\overline{Y}_{\dot{A}\dot{B}}$, which transform, respectively, in accordance with the representations $D(1, 0)$ and $D(0, 1)$ of the Lorentz group; the generator of displacements is expressed in the form $P_{A\dot{B}}$ [the representation $D(1/2, 0)$], and the spinor generators in the form of a pair of two-component spinors, the spinor Q_A, which transforms in accordance with $D(1/2, 1/2)$, and the adjoint spinor $\overline{Q}_{\dot{B}}$ [the representation $D(0, 1/2)$]. We choose the basis in the representation space in such a way that $\overline{Y}_{\dot{A}\dot{B}} = Y^+_{AB}$ and $\overline{Q}_{\dot{A}} = (Q_A)^+$, where the symbol + denotes Hermitian conjugation. The correspondence between the tensor and spinor forms of expression is given by the following formulas:

for the generators of Lorentz transformations

$$\left.\begin{aligned} &Y_{\mu\nu} = \{(\sigma_\mu)_{A\dot{C}}(\sigma_\nu)_{B\dot{C}} Y_{AB} + (\sigma_\mu)_{A\dot{C}}(\sigma_\nu)_{A\dot{D}} \overline{Y}_{\dot{C}\dot{D}}\}/4;\\ &Y_{AB} = (\sigma_\mu)_{A\dot{E}}(\sigma_\nu)_{B\dot{E}} Y_{\mu\nu}/2;\\ &\overline{Y}_{\dot{C}\dot{D}} = (\sigma_\mu)_{N\dot{C}}(\sigma_\nu)_{N\dot{D}} Y_{\mu\nu}/2; \end{aligned}\right\} \tag{27}$$

for the generators of displacements

$$P_\mu = (\sigma_\mu)_{A\dot{B}} P_{A\dot{B}}/2; \quad P_{A\dot{B}} = (\sigma_\mu)_{A\dot{B}} P_\mu, \tag{28}$$

where $\sigma_\mu = (\sigma_0, \sigma_i)$ are Pauli matrices. The usual Einstein summation is performed over repeated vector indices and over repeated spinor indices (here and below) the spinor summation $\xi_A\eta_A = \xi_A\varepsilon_{AB}\eta_B \equiv \xi_1\eta_2 - \xi_2\eta_1$.

Thus, we consider the structure of the algebra $S^{(1/2)}$, which contains the generators Y_{AB}, $\overline{Y}_{\dot{A}\dot{B}}$, $P_{A\dot{B}}$, Q_A, and $\overline{Q}_{\dot{B}}$. The generators Y_{AB}, $\overline{Y}_{\dot{C}\dot{D}}$, and $P_{A\dot{B}}$ satisfy the ordinary commutation relations of the Poincaré algebra written in spinor form. The commutation relations of the spinor generators Q_A and $\overline{Q}_{\dot{B}}$ with the generators of Lorentz transformations are uniquely fixed by the transformation properties of Q and $\overline{Q}$ with respect to the Lorentz group and have the form

$$\left.\begin{aligned} &[Y_{AB}, Q_C] = i(\varepsilon_{AC}Q_B + \varepsilon_{BC}Q_A);\\ &[\overline{Y}_{\dot{A}\dot{B}}, \overline{Q}_{\dot{C}}] = i(\varepsilon_{\dot{A}\dot{C}}\overline{Q}_{\dot{B}} + \varepsilon_{\dot{B}\dot{C}}\overline{Q}_{\dot{A}});\\ &[Y_{AB}, \overline{Q}_{\dot{C}}] = 0; \; [\overline{Y}_{\dot{A}\dot{B}}, Q_C] = 0. \end{aligned}\right\} \tag{29}$$

Thus, it remains to find the commutation relations of the generator P of displacements with Q and $\overline{Q}$ and of the spinor generators with one another. To preserve the correct connection between spin and statistics, it is necessary to consider, as already noted, anticommutators of spinor generators.

The possible form of these commutation relations can be written down on the basis of Lorentz invariance. The use of the spinor form of expression makes it possible to do this almost automatically, and this is the main advantage of spinor notation. For the required commutation relations,

$$\left.\begin{aligned} &[P_{A\dot{B}}, Q_C] = a_1\varepsilon_{AC}\overline{Q}_{\dot{B}};\\ &[P_{A\dot{B}}, \overline{Q}_{\dot{C}}] = a_2\varepsilon_{\dot{B}\dot{C}}Q_A \end{aligned}\right\} \tag{30}$$

and

$$\left.\begin{aligned} &Q_A\overline{Q}_{\dot{B}} + \overline{Q}_{\dot{B}}Q_A = bP_{A\dot{B}};\\ &Q_AQ_B + Q_BQ_A = c_1Y_{AB};\\ &\overline{Q}_{\dot{A}}\overline{Q}_{\dot{B}} + \overline{Q}_{\dot{B}}\overline{Q}_{\dot{A}} = c_2\overline{Y}_{\dot{A}\dot{B}}, \end{aligned}\right\} \tag{31}$$

where a_1, a_2, b, c_1, c_2 are arbitrary constants.

Further restrictions on the form of the algebras (30) and (31), i.e., on the possible values of the constants a_1, a_2, b, c_1, c_2, are given by the generalized Jacobi identities that the generators Y, $\overline{Y}$, P, Q, and $\overline{Q}$ must satisfy. The generalized Jacobi identities are a modification of the ordinary Jacobi identities to the case of algebra containing anticommutators, and they have the form[50]

$$\begin{aligned} &(-1)^{g(x_1)g(x_3)}[x_1, [x_2, x_3]] + (-1)^{g(x_2)g(x_1)}[x_2, [x_3, x_1]]\\ &\qquad + (-1)^{g(x_3)g(x_2)}[x_3, [x_1, x_2]] = 0, \end{aligned} \tag{32}$$

where x_1, x_2, x_3 are any three generators of the algebra $[x_1, x_2] = x_1x_2 - (-1)^{g(x_1)g(x_2)}x_2x_1$; $g(x) = 0$ $(g(Y, P) = 0)$ for tensor generators x; $g(x) = 1$ for spinor generators x $(g(Q, \overline{Q}) = 1)$.

We consider the identities (32) for all possible triplets of the generators Y, $\overline{Y}$, P, Q, and $\overline{Q}$. The generators Y, $\overline{Y}$, and P of the Poincaré group naturally satisfy these identities. Restrictions on a_1, a_2, b, c_1, c_2 can arise only in the identities (32) for triplets containing spinor generators. It is easy to show that the identities (32) for the triplets $(Y, Y, Q(\overline{Q}))$, $(Y, P, Q(\overline{Q}))$, $(Y, (\overline{Y}), Q(\overline{Q}), Q(\overline{Q}))$, and $(Q(\overline{Q}), Q(\overline{Q}), Q(\overline{Q}))$ are satisfied identically and do not lead to restrictions.[4] From the generalized Jacobi identities for p, P, and $Q\,(\overline{Q})$, we find $a_1a_2=0$. Further, from the identities (32) for the triplets (P, Q, Q) and $(P, \overline{Q}, \overline{Q})$ we obtain, respectively, $c_1=a_1b$ and $c_2=a_2b$. Finally, for the generators P, Q, and $\overline{Q}$ the identities (32) give $a_2c_1=0$ and $a_1c_2=0$. These relations exhaust the restrictions on the constants a_1, a_2, b, c_1, c_2 that follow from group considerations.

Thus, there exist two types of $S^{(1/2)}$ algebra:

$$\left.\begin{array}{l}\text{I. } [P_{A\dot{B}}, Q_C]=a\varepsilon_{AC}\overline{Q}_{\dot{B}};\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}]=0;\\ \{Q_A, \overline{Q}_{\dot{B}}\}=bP_{A\dot{B}};\ \{Q_A, Q_B\}=abY_{AB};\ \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\}=0.\end{array}\right\} \quad (33)$$

$$\left.\begin{array}{l}\text{II. } [P_{A\dot{B}}, Q_C]=0;\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}]=a\varepsilon_{\dot{B}\dot{C}}Q_A;\\ \{Q_A, \overline{Q}_{\dot{B}}\}=bP_{A\dot{B}};\ \{Q_A, Q_B\}=0;\ \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\}=ab\overline{Y}_{\dot{A}\dot{B}},\end{array}\right\} \quad (34)$$

where a and b are arbitrary numbers.

We emphasize that the commutation relations (33) and (34) in conjunction with the relations (29) and the commutation relations for Y_{AB}, $\overline{Y}_{\mathring{A}\mathring{B}}$, and $P_{A\mathring{B}}$ give the form of all possible extensions of the Poincaré algebra by the spinor generators Q_A and $\overline{Q}_{\mathring{B}}$. Algebras of the type (33) and (34) are called supersymmetry algebras and were considered for the first time in Refs. 51–55.

Supersymmetry algebras are not Lie algebras since they contain anticommutators as well as commutators. However, they can be regarded as a generalization of Lie algebras in which the groups have as parameters, not ordinary commuting numbers, but elements of a Grassman algebra.[50,56] The parameters of transformations with tensor generators are commuting quantities (even elements of the Grassman algebra) and the parameters of transformations with spinor generators are completely anticommuting (odd elements of the Grassman algebra). The general theory of algebras of this type has been considered in Refs. 50 and 57.

An interesting feature of supersymmetry algebras is that the generators of shifts (momentum) and Lorentz transformations (angular momenta) are bilinear combinations of spinor generators.

Not all algebras of the form (33) and (34) have the same physical interest: First, the $S^{(1/2)}_{\mathrm{I,II}}$ algebras for $a\neq 0$ are not invariant under Hermitian conjugation ($S^{(1/2)}_{\mathrm{I}} \to S^{(1/2)}_{\mathrm{I,II}}$); this is because $Q_A \to \overline{Q}_{\mathring{B}}$, $Y_{AB}\to \overline{Y}_{\mathring{A}\mathring{B}}$, $\overline{Y}_{\mathring{A}\mathring{B}} \to Y_{AB}$. Thus, the algebras $S^{(1/2)}_{\mathrm{I,II}}$ for $a\neq 0$ do not have Hermitian representations, and therefore the corresponding groups do not have unitary representations. Second, since $[p^2, Q_A]=aP_{A\mathring{B}}\overline{Q}_{\mathring{B}}/2$, for the algebra $S^{(1/2)}_{\mathrm{I,II}}$ and $[p^2, \overline{Q}_{\mathring{B}}]=aP_{C\mathring{B}}Q_C/2$ for the algebra $S^{(1/2)}_{\mathrm{I,II}}$, the spectrum of the operator p^2 for $a\neq 0$ is continuous in irreducible representations of these algebras. Thus, only algebra $S^{(1/2)}$ for $a=0$, which has Hermitian representations and in which the operator p^2 is an invariant, can be regarded as a symmetry algebra of theories describing the interaction of particles.

Field-theory models that are invariant with respect to a group with the algebra $S^{(1/2)}$ ($a=0$) have been studied in Refs. 51, 53, 58–62. The most interesting features of these models are the existence of a large number of cancellations of divergences and the appearance of spinor gauge fields[58–62] (see also the reviews Refs. 63–66).

6. GENERAL STRUCTURE OF SPINOR EXTENSIONS OF THE POINCARÉ ALGEBRA

In Sec. 5, we found the general form of extensions of the Poincaré algebra with spinor generators transforming in accordance with the two-row representations of the Lorentz group [$D(1/2, 0)$ and $D(0, 1/2)$].

We consider here possible nontrivial extensions of the Poincaré algebra by means of arbitrary spinor generators, i.e., S algebras consisting of the generators of Lorentz transformations $\overline{Y}_{AB}$, $Y_{\mathring{A}\mathring{B}}$, displacements $P_{A\mathring{B}}$, and spinor generators $Q^{(j,k)}_{A_1\cdots A_{2j}\mathring{B}_1\cdots\mathring{B}_{2k}}$ transforming in accordance with the irreducible representations $D(j,k)$ of the Lorentz group ($j+k$ is half-integral). We shall say that the extension of the Poincaré algebra is nontrivial if at least one of the commutators $[P,Q]$ or anticommutators $\{Q,Q\}$ is nonzero.

The generators Y_{AB}, $\overline{Y}_{\mathring{A}\mathring{B}}$, and $P_{A\mathring{B}}$ form the Poincaré algebra with the standard commutation relations. The commutation relations of the spinor generator $Q^{(j,k)}$ with Y and $\overline{Y}$ are fixed by the transformation properties

$$[Y_{AC}, Q^{(j,k)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2k}}]=\mathrm{i}\underset{A_1,\ldots,A_{2j}}{\mathrm{Sym}}\ \varepsilon_{AA_1}Q^{(j,k)}_{CA_2\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2k}};$$

$$[\overline{Y}_{\dot{B}\dot{C}}, Q^{(j,k)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2k}}]=\mathrm{i}\underset{\dot{B}_1,\ldots,\dot{B}_{2k}}{\mathrm{Sym}}\ \varepsilon_{\dot{B}\dot{B}_1}Q^{(j,k)}_{A_1\ldots A_{2j}\dot{C}\dot{B}_2\ldots\dot{B}_{2k}}.$$

where Sym denotes symmetrization with respect to the corresponding indices.

To find the form of the commutators $[P,Q]$ and the anticommutators $\{Q,Q\}$ we use, as in the previous section, the transformation properties of the generators, Y, $\overline{Y}$, P, and Q and the generalized Jacobi identities (32).

We consider the S algebra containing the single spinor generator $Q^{(j,k)}$. From the formulas for the decomposition of tensor products of representations of the Lorentz group, namely $D(1/2,\ 1/2)\otimes D(j,k)=\sum_{+,-}\otimes D(j\pm 1/2,\ k\pm 1/2)$ and $D(j,k)\otimes D(j,k)=D(2j, 2k)\otimes\cdots\otimes D(0,0)$, it follows that the commutator $[P, Q^{(j,k)}]$ is zero and the anticommutator $\{Q,Q\}$ can be proportional only to the generator Y. However, it follows from the identities (32) for $P, Q, \overline{Q}$ that the corresponding coefficient of proportionality is zero. Thus, there do not exist nontrivial S algebras containing a single spinor generator.

The S algebra containing the two spinor generators $Q^{(j,k)}$ and $\tilde{Q}^{(\tilde{j},\tilde{k})}$ has been studied. It follows from the decompositions $D(1/2,\ 1/2)\otimes D(j,k)=\sum_{+,-}\otimes D(j\pm 1/2,\ k\pm 1/2)$ and $D(j,k)\otimes D(\tilde{j},\tilde{k})=D(j+\tilde{j}, k+\tilde{k})\otimes\cdots\otimes D(|j-\tilde{j}|,\ |k-\tilde{k}|)$ that there are two possibilities. The first is

[4]The generalized Jacobi identities containing the generators of Lorentz transformations do not lead to restrictions for an obvious reason—all transformations associated with Lorentz invariance have already been taken into account in Eqs. (29)–(31).

$j-\tilde{j}=0$, $|k-\tilde{k}|=1$ (or $|j-\tilde{j}|=1$, $k-\tilde{k}=0$). Then $[P,Q]=0$, $[P,\tilde{Q}]=0$ and the anticommutators $\{Q,Q\}$, $\{\tilde{Q},\tilde{Q}\}$, $\{Q,\tilde{Q}\}$ are proportional to the generators $Y(\bar{Y})$. However, by virtue of the generalized Jacobi identities for P, $Q(\tilde{Q})$, $Q(\tilde{Q})$, the corresponding coefficients of proportionality are zero, and this possibility therefore corresponds to a trivial S algebra. The second possibility is $|j-\tilde{j}|=1/2$, $|k-\tilde{k}|=1/2$. In this case, the following commutation relations are possible:

$$[P,Q]\approx\tilde{Q},\ [P,\tilde{Q}]\sim Q,\ \{Q,\tilde{Q}\}\sim P,$$
$$\{Q,Q\}\sim Y(\bar{Y}),\ \{\tilde{Q},\tilde{Q}\}\sim Y(\bar{Y}).$$

All S algebras of this type can be divided into two classes: In the first, we put the $S_{\rm I}^{(j,k)}$ algebra containing the spinor generators $Q^{(j,k)}$ and $\tilde{Q}^{(j-1/2,k+1/2)}$ (or, equivalently, $Q^{(j,k)}$ and $\tilde{Q}^{(j+1/2,k-1/2)}$); in the second, the $S_{\rm II}^{(j,k)}$ algebra containing the spinor generators $Q^{(j,k)}$ and $\tilde{Q}^{(j+1/2,k+1/2)}$. These two classes exhaust the nontrivial S algebras containing two spinor generators. From the properties of the $S_{\rm I}$ and $S_{\rm II}$ algebras we note that in the general case they are not invariant with respect to spatial reflection.

For physical applications, the greatest interest attaches to Hermitian representations of algebras (unitary representations of the groups). Since the representation $D(j,k)$ of the Lorentz group goes over into the representation $D(k,j)$ under Hermitian conjugation, an S algebra can have Hermitian representations only if it contains both the generator $Q^{(j,k)}$ and $\bar{Q}^{(k,j)}$. It is readily seen that this requirement is satisfied only by the algebra $S_{\rm I}^{(j,k)}$ with $j-k=1/2$ (the case $j-k=-1/2$ is equivalent to it and corresponds to the pair $Q^{(j,k)}$ and $\bar{Q}^{(j+1/2,k-1/2)}$).

Let us find the commutation relations for the algebra $S^{(j)}\equiv S_{\rm I}^{(j,j-1/2)}$ for $j>1/2$. (The case $j=1/2$ was considered in the previous section.) The requirement of Lorentz invariance fixes the form of these commutation relations to within arbitrary constants:

$$[P_{A\dot{B}},Q^{(j,j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}]=a_1\operatorname*{Sym}_{A_1,\ldots,A_{2j}}\varepsilon_{AA_1}\bar{Q}^{(j-1/2,j)}_{A_2\ldots A_{2j}\dot{B}\dot{B}_1\ldots\dot{B}_{2j-1}};$$

$$[P_{A\dot{B}},\bar{Q}^{(j-1/2,j)}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}]=a_2\operatorname*{Sym}_{\dot{B}_1,\ldots,\dot{B}_{2j}}\varepsilon_{\dot{B}\dot{B}_1}Q^{(j,j-1/2)}_{AA_1\ldots A_{2j-1}\dot{B}_2\ldots\dot{B}_{2j}};$$

$$\{Q^{(j,j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}},\bar{Q}^{(j-1/2,j)}_{C_1\ldots C_{2j-1}\dot{D}_1\ldots\dot{D}_{2j}}\}$$
$$=b\operatorname*{Sym}_{A,\dot{B},C,\dot{D}}\varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j-1},C_{2j-1}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-1}\dot{D}_{2j-1}}P_{A_{2j}\dot{D}_{2j}};$$

$$\{Q^{(j,j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}},Q^{(j,j-1/2)}_{C_1\ldots C_{2j}\dot{D}_1\ldots\dot{D}_{2j-1}}\}$$
$$=c\operatorname*{Sym}_{A,\dot{B},C,\dot{D}}\varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j}C_{2j}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-2}\dot{D}_{2j-2}}\bar{Y}_{\dot{B}_{2j-1}\dot{D}_{2j-1}}$$
$$+d_1\operatorname*{Sym}_{A,\dot{B},C,\dot{D}}\varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j-1}C_{2j-1}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-1}\dot{D}_{2j-1}}Y_{A_{2j}C_{2j}};$$

$$\{\bar{Q}^{(j-1/2,j)}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}},\bar{Q}^{(j-1/2,j)}_{C_1\ldots C_{2j-1}\dot{D}_1\ldots\dot{D}_{2j}}\}$$
$$=c_2\operatorname*{Sym}_{A,\dot{B},C,\dot{D}}\varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j-1}C_{2j-1}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-1}\dot{D}_{2j-1}}\bar{Y}_{\dot{B}_{2j}\dot{D}_{2j}}$$
$$+d_2\operatorname*{Sym}_{A,\dot{B},C,\dot{D}}\varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j-2}C_{2j-2}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j}\dot{D}_{2j}}Y_{A_{2j-1}C_{2j-1}}.$$

The generalized Jacobi identities (32) yield restrictions on the possible values of a_1, a_2, b, c_1, d_1, c_2, d_2, namely ($j>1/2$): $a_1a_2=0$, $a_1b=0$, $a_2b=0$, $c_1=d_1=c_2=d_2=0$. Thus, one can have the three following types of algebra $S^{(j)}$ for $j>1/2$ (for $j=1/2$ see (33) and (34) (Ref. 67)):

$$\left.\begin{array}{l}\text{I. }[P_{A\dot{B}},Q^{(j,j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}]\\ \qquad=a\operatorname*{Sym}_{A_1,\ldots,A_{2j}}\varepsilon_{AA_1}\bar{Q}^{(j-1,2,j)}_{A_2\ldots A_{2j}\dot{B}\dot{B}_1\ldots\dot{B}_{2j-1}};\\ [P,\bar{Q}]=0;\ \{Q,\bar{Q}\}=0;\ \{QQ\}=0;\ \{\bar{Q},\bar{Q}\}=0.\end{array}\right\}\quad(35)$$

$$\left.\begin{array}{l}\text{II. }[P_{A\dot{B}},\bar{Q}^{(j-1/2,j)}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}]\\ \qquad=a\operatorname*{Sym}_{\dot{B}_1,\ldots,\dot{B}_{2j}}\varepsilon_{\dot{B}\dot{B}_1}Q^{(j,j-1/2)}_{AA_1\ldots A_{2j-1}\dot{B}_2\ldots\dot{B}_{2j}};\\ [P,Q]=0;\ \{Q,\bar{Q}\}=0;\ \{Q,Q\}=0;\ \{\bar{Q},\bar{Q}\}=0.\end{array}\right\}\quad(36)$$

$$\left.\begin{array}{l}\text{III. }[P,Q]=0;\ [P,\bar{Q}]=0;\ \{Q,Q\}=0;\ \{\bar{Q},\bar{Q}\}=0;\\ \{Q^{(j,j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-2}},\bar{Q}^{(j-1/2,j)}_{C_1\ldots C_{2j-2}\dot{D}_1\ldots\dot{D}_{2j}}\}\\ =b\operatorname*{Sym}_{A,\dot{B},C,\dot{D}}\varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j-1}C_{2j-1}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-2}\dot{D}_{2j-1}}P_{A_{2j}\dot{D}_{2j}}.\end{array}\right\}\quad(37)$$

Note that whereas $P_{A\dot{B}}$ and $Y_{AB}\times(\bar{Y}_{\dot{A}\dot{B}})$ are bilinear combinations of spinor generators for $j=1/2$ ($a\neq0$), in the case $j>1/2$ only the translation generators $P_{A\dot{B}}$ have the form of bilinear combinations of spinor generators (the algebra $S_{\rm III}$). This difference is due to the appearance when $j>1/2$ in the anticommutators $\{Q,Q\}$ and $\{\bar{Q},\bar{Q}\}$ of terms with coefficients d_1 and d_2, which leads by virtue of the identity (32) for $Q(\bar{Q})$, $Q(\bar{Q})$, and $Q(\bar{Q})$ to the vanishing of C_1, C_2 and d_1,d_2.

From the general properties of the algebras $S^{(j)}$, we note that $S_{\rm I}^{(j)}$ and $S_{\rm II}^{(j)}$ are not invariant with respect to spatial reflection. Indeed, under spatial reflection $S_{\rm I}^{(j)}\leftrightarrow S_{\rm II}^{(j)}$. Thus, in theories that are invariant with respect to groups with the algebras $S_{\rm I,II}^{(j)}$, parities are not conserved. It is more important that $S_{\rm I}^{(j)}$ and $S_{\rm II}^{(j)}$ are not invariant with respect to Hermitian conjugation (under which $S_{\rm I}^{(j)}\leftrightarrow S_{\rm II}^{(j)}$), and, therefore, they do not have Hermitian representations (the corresponding groups do not have unitary representations). In addition, for $S_{\rm I}^{(j)}$ and $S_{\rm II}^{(j)}$ the operator p^2 is not an invariant and it has a continuous spectrum of values. Thus, only the algebras $S_{\rm III}^{(j)}$ ($S^{(1/2)}$ ($a=0$)) are of interest as symmetry algebras of theories describing the interaction of particles.

Similarly, one can consider S algebras containing more than two spinor generators, and also algebras containing spinor generators that transform in accordance with irreducible representations of the Lorentz group (for example, the representations $(D(1/2,1/2)^n\otimes D(1/2,0))$. Note that algebras containing an odd number of spinor generators do not have Hermitian representations.

Finally we draw attention to an error in Ref. 35. Considering extensions of the Poincaré algebra by the spinor generators Q and $\bar{Q}$, Haag, Lopuszański, and Sohnius concluded that the algebra $S^{(1/2)}$ ($a=0$) is the only possible spinor extension of the Poincaré algebra. The source of this error is the incorrect assertion in Sec. 3 of Ref. 35 that the anticommutator $(Q^{(j,j-1/2)}_{A_1\cdots A_{2j}\dot{B}_1\cdots\dot{B}_{2j-1}}\bar{Q}^{(j-1/2,j)}_{C_1\cdots C_{2j-1}\dot{D}_1\cdots\dot{D}_{2j}}+\bar{Q}^{(j-1/2,j)}_{C_2\cdots C_{2j-1}\dot{D}_1\cdots\dot{D}_{2j}}\times Q^{(j,j-1/2)}_{A_1\cdots A_{2j}\dot{B}_1\cdots\dot{B}_{2j-1}}$ is completely symmetric with respect to the indices of every type.

7. SPINOR EXTENSIONS OF THE WEYL ALGEBRA AND THE CONFORMAL ALGEBRA

The method of finding spinor extensions used in Sec. 6 can be applied to all spacetime groups.

Weyl group. We consider separately extensions of the Weyl algebra with spinor generators that transform: 1) in accordance with the representations $D(1/2, 0)$ and $D(0, 1/2)$ of the Lorentz group, and 2) in accordance with arbitrary spinor representations.

1. The generators Y_{AB}, $\overline{Y}_{\dot{A}\dot{B}}$, $P_{A\dot{B}}$, and D of the Weyl algebra satisfy the standard commutation relations. The commutation relations of the spinor generators Q_A and $\overline{Q}_{\dot{B}}$ with the generators Y_{AB}, $\overline{Y}_{\dot{A}\dot{B}}$, $P_{A\dot{B}}$ are given by (29) and (30), and the anticommutators of the spinor generators with one another by (31). Finally, the commutation relations of the dilation generator D and the spinor generators have the form

$$[D, Q_A] = f_1 Q_A;\ [D, \overline{Q}_{\dot{A}}] = f_2 \overline{Q}_{\dot{A}}, \tag{38}$$

where f_1 and f_2 are arbitrary constants.

The generalized Jacobi identities give a number of restrictions on the constants a_1, a_2, b, c_1, c_2, f_1, f_2. From the identities (32) for the generators Y, $\overline{Y}$, P, Q and $\overline{Q}$ we naturally obtain (33) and (34). In order to obtain restrictions on f_1 and f_2, we must, taking into account (33) and (34), consider the identities (32) for triplets of generators containing D and $Q(\overline{Q})$. It is readily verified that the identities (32) for the triplets of generators $[D, Y(\overline{Y}), Q(\overline{Q})]$ and $[D, D, Q(\overline{Q})]$ do not give any restrictions on f_1 and f_2. For the triplets $(D, P, \overline{Q})$ and $(D, \overline{Q}, \overline{Q})$ in the case (33) the generalized Jacobi identities are satisfied identically, while in the cases (34) they give, respectively, $a(f_1 - f_2 - i) = 0$ and $f_2 ab = 0$. Further, for the triplets (D, P, Q) and (D, Q, Q), the identities (32) in the case (33) give $a(f_2 - f_1 - i) = 0$ and $f_1 ab = 0$, while in the cases (34) no restrictions arise. Finally, the generalized Jacobi identities for $D, Q, \overline{Q}$ give $b(f_1 + f_2 - i) = 0$ in both cases.

As a result, we obtain the six following types of extensions of the Weyl algebra by the generators Q_A and $\overline{Q}_{\dot{B}}$ [we shall not write out the commutation relations for Y, $\overline{Y}$, P, D and the relations (32), which are the same in all cases]:

1) $[P_{A\dot{B}}, Q_C] = a\varepsilon_{AC}\overline{Q}_{\dot{B}};\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}] = 0;$
$\{Q_A, \overline{Q}_{\dot{B}}\} = bP_{A\dot{B}};\ \{Q_A, Q_B\} = abY_{AB};\ \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = 0;$
$[D, Q_A] = 0;\ [D, \overline{Q}_{\dot{B}}] = i\overline{Q}_{\dot{B}}.$

2) $[P_{A\dot{B}}, Q_C] = 0;\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}] = a\varepsilon_{\dot{B}\dot{C}}Q_A;$
$\{Q_A, \overline{Q}_{\dot{B}}\} = bP_{A\dot{B}};\ \{Q_A, Q_B\} = 0;\ \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = ab\overline{Y}_{\dot{A}\dot{B}};$
$[D, Q_A] = iQ_A;\ [D, \overline{Q}_{\dot{B}}] = 0.$

3) $[P_{A\dot{B}}, Q_C] = a\varepsilon_{AC}\overline{Q}_{\dot{B}};\ [P, \overline{Q}] = 0;$
$\{Q, \overline{Q}\} = 0;\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0;$
$[D, Q_A] = ifQ_A;\ [D, \overline{Q}_{\dot{A}}] = i(1+f)\overline{Q}_{\dot{A}}.$

4) $[P, Q] = 0;\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}] = a\varepsilon_{\dot{B}\dot{C}}Q_A;$
$\{Q, \overline{Q}\} = 0;\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0;$
$[D, Q_A] = i(1+f)Q_A;\ [D, \overline{Q}_{\dot{B}}] = if\overline{Q}_{\dot{B}}.$

5) $[P, Q] = 0;\ [P, \overline{Q}] = 0;$
$\{Q_A, \overline{Q}_{\dot{B}}\} = bP_{A\dot{B}};\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0;$
$[D, Q_A] = ifQ_A;\ [D, \overline{Q}_{\dot{A}}] = i(1-f)\overline{Q}_{\dot{A}}.$

6) $[P, Q] = 0;\ [P, \overline{Q}] = 0;$
$\{Q, \overline{Q}\} = 0;\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0;$
$[D, Q_A] = ifQ_A;\ [D, \overline{Q}_{\dot{A}}] = id\overline{Q}_{\dot{A}},$

where a, b, f, and d are arbitrary numbers. Note that only algebras of the fifth type for $f = 1/2$ and the sixth type for $f = d$ are invariant under Hermitian conjugation and therefore have Hermitian representations.

2. One can consider similarly extensions of the Weyl algebra by means of arbitrary spinor generators. In this case, there are four types of algebras:

1) the commutation relations (35) and

$$[D, Q^{(j, j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}] = ifQ^{(j, j-1/2)}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}};$$
$$[D, \overline{Q}^{(j-1/2, j)}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}] = i(1+f)\overline{Q}^{(j-1/2, j)}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}.$$

2) the relations (36) and

$$[D, Q^{(j, j-1/2)}_{\ldots}] = i(1+f)Q^{(j, j-1/2)}_{\ldots};$$
$$[D, \overline{Q}^{(j-1/2, j)}_{\ldots}] = if\overline{Q}^{(j-1/2, j)}_{\ldots}.$$

3) the commutation relations (37) and

$$[D, Q^{(j, j-1/2)}_{\ldots}] = ifQ^{(j, j-1/2)}_{\ldots};$$
$$[D, \overline{Q}^{(j-1/2, j)}_{\ldots}] = i(1-f)\overline{Q}^{(j-1/2, j)}_{\ldots}.$$

4) $[P, Q^{(j, j-1/2)}] = 0;\ [P, \overline{Q}^{(j-1,2, j)}] = 0;$
$\{Q, \overline{Q}\} = 0;\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0;$
$[D, Q] = ifQ;\ [D, \overline{Q}] = id\overline{Q},$

where f and d are arbitrary constants. Algebras of the type 3–6 for $j = 1/2$ are special cases of the algebras 1–4 for arbitrary j. As in the case of $j = 1/2$, only algebras of the third ($f = 1/2$) and fourth ($f = d$) type have Hermitian representations.

These algebras exhaust the nontrivial extensions of the Weyl algebra by a pair of arbitrary spinor generators that transform in accordance with conjugate representations of the Lorentz group.

Conformal group. The algebra of the conformal group consists of the generators Y_{AB}, $\overline{Y}_{\dot{A}\dot{B}}$, $P_{A\dot{B}}$, D and the generator of the special conformal transformation $K_{A\dot{B}}$; these satisfy the commutation relations (10). Let us consider extension of the algebra (10) by means of the spinor generators Q_A and $\overline{Q}_{\dot{B}}$. The general form of the commutation relations of the spinor generators with Y_{AB}, $\overline{Y}_{\dot{A}\dot{B}}$, $P_{A\dot{B}}$, and D follows from Lorentz invariance and is given by (29), (30), and (38). From the same considerations there follows the form of the commutators of $K_{A\dot{B}}$ with Q and $\overline{Q}$:

$$[K_{A\dot{B}}, Q_C] = g_1\varepsilon_{AC}\overline{Q}_{\dot{B}};\ [K_{A\dot{B}}, \overline{Q}_{\dot{C}}] = g_2\varepsilon_{\dot{B}\dot{C}}Q_A, \tag{39}$$

and the anticommutators of the spinor generators with one another:

$$\{Q_A, Q_B\} = c_1Y_{AB};\ \{\overline{Q}_{\dot{A}}, \overline{Q}_{\dot{B}}\} = c_2\overline{Y}_{\dot{A}\dot{B}};$$
$$\{Q_A, \overline{Q}_{\dot{B}}\} = bP_{A\dot{B}} + \rho K_{A\dot{B}}, \tag{40}$$

where $g_1, g_2, c_1, c_2, b, \rho$ are arbitrary constants.

As in the previous sections, we now use the generalized Jacobi identities. We find the possible values of the constants b and ρ. Taking into account the relations $[D, P_{A\dot{B}}] = iP_{A\dot{B}}$ and $[D, K_{A\dot{B}}] = -iK_{A\dot{B}}$, we obtain from the identities (32) for D, Q, $\overline{Q}$ the system of equations $b(f_1 + f_2 - i) = 0$ and $\rho(f_1 + f_2 + i) = 0$. Thus, there are three possibilities: 1) $b \neq 0$, $\rho = 0$, $f_1 + f_2 = i$; 2) $b = 0$, $\rho \neq 0$, $f_1 + f_2 = -i$; and 3) $b = \rho = 0$ (f_1 and f_2 are arbitrary numbers).

Further, from the identities (32) for $P_{A\dot{B}}$, $K_{A\dot{B}}$, and Q_A we find $f_1 + f_2 = 0$; $g_1 a_2 - 4 + 8if_1 = 0$; $a_1 g_2 - 4 - 8if_1 = 0$. It is readily seen that these equations are compatible with only the possibility 3). Finally, since the generators y, $\bar{y}$, and K form the Poincaré algebra, the extension of the algebra y, $\bar{y}$, K by means of the generators Q and $\overline{Q}$ has the same form as the extensions of the algebra y, $\bar{y}$, P. In particular, $g_1 g_2 = 0$.

As a result, we find that there are two possibilities:

1. $[P_{A\dot{B}}, Q_C] = a\varepsilon_{AC}\overline{Q}_{\dot{B}}$; $[P, \overline{Q}] = 0$;

$$[K, Q] = 0;\ [K_{A\dot{B}}, \overline{Q}_{\dot{C}}] = 8\varepsilon_{\dot{B}\dot{C}}Q_A/8;$$
$$[D, Q_A] = -iQ_A/2;\ [D, \overline{Q}_{\dot{B}}] = i\overline{Q}_{\dot{B}}/2;$$
$$\{Q, \overline{Q}\} = 0;\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0.$$

2. $[P, Q] = 0$; $[P_{A\dot{B}}, \overline{Q}_{\dot{C}}] = a\varepsilon_{\dot{B}\dot{C}}Q_A$;

$$[K_{A\dot{B}}, Q_C] = 8\varepsilon_{AC}\overline{Q}_{\dot{B}}/a;\ [K, \overline{Q}] = 0;$$
$$[D, Q_A] = iQ_A/2;\ [D, \overline{Q}_{\dot{B}}] = -i\overline{Q}_{\dot{B}}/2;$$
$$\{Q, \overline{Q}\} = 0;\ \{Q, Q\} = 0;\ \{\overline{Q}, \overline{Q}\} = 0.$$

There are also two possibilities, completely analogous to those given here, when the conformal algebra is extended by two arbitrary spinor generators, namely, $Q^{(j, j-1/2)}$ and $\overline{Q}^{(j-1/2, j)}$. However, it is readily seen that these algebras are not invariant under Hermitian conjugations and therefore do not have Hermitian representations.

The extensions of the conformal algebra by two pairs of spinor generators $Q_1^{(j, j-1/2)}$, $\overline{Q}_1^{(j-1/2, j)}$ and $Q_2^{(j, j-1/2)}$, $\overline{Q}_2^{(j-1/2, j)}$ have Hermitian representations. The general form of these extensions of the conformal algebra can be readily obtained by using the results of the previous sections. In the simplest case $j = 1/2$ we have[35]

$$[P_{A\dot{B}}, Q_C^{(1)}] = 0;\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}^{(1)}] = 0;$$
$$[P_{A\dot{B}}, Q_C^{(2)}] = 2i\varepsilon_{AC}\overline{Q}_{\dot{B}}^{(1)};\ [P_{A\dot{B}}, \overline{Q}_{\dot{C}}^{(2)}] = 2i\varepsilon_{\dot{B}\dot{C}}Q_A^{(1)};$$
$$[K_{A\dot{B}}, Q_C^{(1)}] = 2i\varepsilon_{AC}\overline{Q}_{\dot{B}}^{(2)};\ [K_{A\dot{B}}, \overline{Q}_{\dot{C}}^{(1)}] = 2i\varepsilon_{\dot{B}\dot{C}}Q_A^{(1)};$$
$$[K_{A\dot{B}}, Q_C^{(2)}] = 0;\ [K_{A\dot{B}}, \overline{Q}_{\dot{C}}^{(2)}] = 0;$$
$$[D, Q_A^{(1)}] = -iQ_A^{(1)}/2;\ [D, \overline{Q}_{\dot{A}}^{(1)}] = -i\overline{Q}_{\dot{A}}^{(1)}/2;$$
$$[D, Q_A^{(2)}] = iQ_A^{(2)}/2;\ [D, \overline{Q}_{\dot{A}}^{(2)}] = i\overline{Q}_{\dot{A}}^{(2)}/2;$$
$$\{Q_A^{(1)}, \overline{Q}_{\dot{B}}^{(1)}\} = P_{A\dot{B}};\ \{Q_A^{(1)}, Q_B^{(1)}\} = 0;\ \{\overline{Q}_{\dot{A}}^{(1)}, Q_{\dot{B}}^{(1)}\} = 0;$$
$$\{Q_A^{(2)}, \overline{Q}_{\dot{B}}^{(2)}\} = K_{A\dot{B}};\ \{Q_A^{(2)}, Q_B^{(2)}\} = 0;\ \{\overline{Q}_{\dot{A}}^{(2)}, \overline{Q}_{\dot{B}}^{(2)}\} = 0.$$
$$\{Q_A^{(1)}, Q_B^{(2)}\} = \varepsilon_{AB}D - Y_{AB};\ \{\overline{Q}_{\dot{A}}^{(1)}, \overline{Q}_{\dot{B}}^{(2)}\} = \varepsilon_{\dot{A}\dot{B}}D - Y_{\dot{A}\dot{B}};\ \{Q_A^{(1)}, \overline{Q}_{\dot{B}}^{(2)}\} = 0$$

together with the relations (32) and the commutation relations of the generators of the conformal group.

In order to obtain this and other spinor extensions of the conformal algebra it is convenient to use conformal spinor analysis, which enables one to take into account the requirements of conformal invariance almost automatically.

Since the Poincaré, Weyl, and conformal groups exhaust the possible spacetime symmetry groups (Sec. 3), in Secs. 5–7 we have considered all possible supersymmetry algebras.

8. UNION OF SUPERSYMMETRY AND INTERNAL SYMMETRY GROUPS

The method used many times above can be employed to study as well symmetry groups that contain not only spacetime transformation and supertransformations (transformations with spinor generators) but also transformations of internal symmetry groups. The corresponding algebras for the case of the Poincaré and conformal groups and spinor generators that transform in accordance with the representations $D(1/2, 0)$ and $D(0, 1/2)$ of the Lorentz group have been considered in Ref. 35. In the case of Poincaré group and arbitrary spinor generators, the required algebra has the form

$$[P_{A\dot{B}}, Q^{(j, j-1/2)\alpha}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}] = 0;\ [P_{A\dot{B}}, \overline{Q}^{(j-1/2, j)\alpha}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}] = 0;$$
$$[Q^{(j, j-1/2)\alpha}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}, B_i] = -\sum_\beta \overline{S}_i^{\alpha\beta} Q^{(j, j-1/2)\beta}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}};$$
$$[\overline{Q}^{(j-1/2, j)\alpha}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}, B_i] = \sum_\beta S_i^{\alpha\beta} \overline{Q}^{(j-1/2, j)\beta}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}};$$
$$\{Q^{(j, j-1/2)\alpha}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}, \overline{Q}^{(j-1/2, j)\beta}_{C_1\ldots C_{2j-1}\dot{D}_1\ldots\dot{D}_{2j}}\}$$
$$= \delta^{\alpha\beta} \underset{A, \dot{B}, C, \dot{D}}{\mathrm{Sym}}\ \varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j-1}C_{2j-1}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-1}\dot{D}_{2j-1}}P_{A_{2j}\dot{D}_{2j}};$$
$$\{Q^{(j, j-1/2)\alpha}_{A_1\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}, Q^{(j, j-1/2)\beta}_{C_1\ldots C_{2j}\dot{D}_1\ldots\dot{D}_{2j-1}}\}$$
$$= Z^{\alpha\beta} \underset{A, \dot{B}, C, \dot{D}}{\mathrm{Sym}}\ \varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j}C_{2j}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j-1}\dot{D}_{2j-1}};$$
$$\{\overline{Q}^{(j-1/2, j)\alpha}_{A_1\ldots A_{2j-1}\dot{B}_1\ldots\dot{B}_{2j}}, \overline{Q}^{(j-1/2, j)\beta}_{C_1\ldots C_{2j-1}\dot{D}_1\ldots\dot{D}_{2j}}\}$$
$$= Z^{\alpha\beta} \underset{A, \dot{B}, C, \dot{D}}{\mathrm{Sym}}\ \varepsilon_{A_1C_1}\ldots\varepsilon_{A_{2j}C_{2j-1}}\varepsilon_{\dot{B}_1\dot{D}_1}\ldots\varepsilon_{\dot{B}_{2j}\dot{D}_{2j}};$$
$$[P_{A\dot{B}}, B_i] = 0;\ [Y_{AB}, B_i] = 0;\ [\overline{Y}_{\dot{A}\dot{B}}, B_i] = 0;$$
$$[B_i, B_k] = \sum_l C_{ikl}B_l,$$

where B_i are the generators of the internal symmetry group; the spinor generators Q and $\overline{Q}$ transform in accordance with an arbitrary (n-row) representation of the internal symmetry group, i.e., the index α takes the values $\alpha = 1, \ldots, n$; $S_i^{\alpha\beta}$ are the matrices of the n-row Hermitian representation of the internal symmetry group; C_{ikl} are the structure constants of this group; $Z^{\alpha\beta}$ commutes with all the generators of this group.

9. REALIZATION OF SUPERSYMMETRY GROUPS AS GROUPS OF TRANSFORMATIONS OF SUPERSPACES

The commutation relations considered above for the generators of supersymmetry groups are completely sufficient for the construction of irreducible representations and their classification, the construction of quantum fields that are covariant with respect to the supersymmetry groups, etc. As always, the study of a symmetry group is simplified and all expressions become much more apparent if the symmetry group can be represented as a group of transformations of a space. In the case of supersymmetry groups, only the spacetime subgroups (Poincaré, Weyl, and conformal) are groups of transformations (the coordinates of Minkowski space), while the supertransformations are transformations of quantum fields that do not affect the coordinates. Thus it is necessary to find an extension of Minkowski space that would enable one to represent supertransformations also as transformations of certain coordinates.

The role of such an extension of Minkowski space is played by *superspace*, which is a space some of whose coordinates are the coordinates of Minkowski space while the remaining coordinates are completely anticommuting spinors. We emphasize that only some of the coordinates of superspace, the coordinates of Minkowski space, have a physical meaning; the remaining coordinates are fictitious and none of the physical quantities depend on them.

To be specific, we consider the algebra $S^{1/2}$ ($a=0$), the extension of the Poincaré algebra by the spinor generators Q_A and $\overline{Q}_{\dot B}$ ($b=2$):

$$\left.\begin{aligned}&[P_{A\dot B}, Q_C]=0;\;\; [P_{A\dot B}, \overline{Q}_{\dot C}]=0;\\ &\{Q_A, \overline{Q}_{\dot B}\}=2P_{A\dot B};\; \{Q_A, Q_B\}=0;\; \{\overline{Q}_{\dot A}, \overline{Q}_{\dot B}\}=0\end{aligned}\right\} \quad (41)$$

together with the relations (32) and the commutation relations for the generators of the Poincaré group.

The corresponding superspace is an eight-dimensional space with coordinates $x_{A\dot B}$, and θ_A, $\overline{\theta}_{\dot B}$, where $x_{A\dot B} = (\sigma_\mu)_{A\dot B}x_\mu$ are the coordinates of Minkowski space and θ and $\overline{\theta}$ are constant, completely anticommuting two-component spinors.[68,69] The algebra (41) is the algebra of the group of transformations of superspace, and it contains

1) the Lorentz transformations

$$x_\mu \to x'_\mu = \Lambda_{\mu\nu}x_\nu;\;\; \theta_A \to \theta'_A = S^{(1/2,\,0)}_{AB}\theta_B,\;\; \overline{\theta}_{\dot A} \to \overline{\theta}'_{\dot A} = S^{0,\,1/2}_{\dot A\dot B}\overline{\theta}_{\dot B},$$

where $S^{(1/2,0)}$ and $S^{(0,1/2)}$ are two-row representations of the Lorentz group;

2) the displacements

$$x_\mu \to x'_\mu = x_\mu + a_\mu;\;\; \theta_A \to \theta'_A = \theta_A;\;\; \overline{\theta}_{\dot A} \to \overline{\theta}'_{\dot A} = \overline{\theta}_{\dot A};$$

3) the supertransformations

$$\begin{aligned}&\theta_A \to \theta'_A = \theta_A + \xi_A;\;\; \overline{\theta}_{\dot A} \to \overline{\theta}'_{\dot A} = \overline{\theta}_{\dot A} + \overline{\xi}_{\dot A};\\ &x_{A\dot B} \to x'_{A\dot B} = x_{A\dot B} - i\xi_A\overline{\theta}_{\dot B} + i\overline{\theta}_A\overline{\xi}_{\dot B}.\end{aligned}$$

The corresponding generators have the form

$$\left.\begin{aligned}&Y_{\mu\nu} = i\left(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}\right) + \frac{i}{4}\theta_A[\sigma_\mu, \sigma_\nu]_{AB}\frac{\partial}{\partial\theta^B}\\ &\quad - \frac{i}{4}\overline{\theta}_{\dot A}[\sigma_\mu, \sigma_\nu]^+_{\dot A\dot B}\frac{\partial}{\partial\overline{\theta}^{\dot B}};\\ &P_\mu = i\frac{\partial}{\partial x^\mu};\;\; Q_A = \frac{\partial}{\partial\theta^A} - i\overline{\theta}_{\dot B}\partial_{A\dot B};\;\; \overline{Q}_{\dot A} = -\frac{\partial}{\partial\overline{\theta}^{\dot A}} - i\theta_B\partial_{B\dot A},\end{aligned}\right\} \quad (42)$$

where $\partial_{A\dot B} = (\sigma_\mu)_{A\dot B}(\partial/\partial x^\mu)$. It is not difficult to see that the generators $Y_{\mu\nu}$, P_μ, Q_A, $\overline{Q}_{\dot B}$, defined by (42) do indeed satisfy the commutation relations (41).

We emphasize the important difference between supertransformations and transformations of the internal symmetry groups. Whereas the latter do not involve the spacetime coordinates, the supertransformations are simultaneous transformations of the unphysical spinor coordinates θ and $\overline{\theta}$ and the coordinates of Minkowski space.

Note that transformations of this supersymmetry group leave the differential form $(dx_\mu + i\overline{\theta}\sigma_\mu d\overline{\theta} - i\theta\sigma_\mu d\overline{\theta})^2$ invariant. The formalism of superspace enables one to express compactly quantities that are covariant with respect to the supersymmetry group. For this, we introduce the concept of the superfield $\psi(x, \theta, \overline{\theta})$, which is an operator-valued function of the coordinates x_μ, θ, and $\overline{\theta}$ (Refs. 68–70). The law of transformation of the superfield with respect to the supersymmetry group is given by analogy with the ordinary fields:

$$U\psi(x, \theta, \overline{\theta})U^{-1} = \psi'(x, \theta, \overline{\theta}) = \psi(x', \theta', \overline{\theta}'). \quad (43)$$

The superfield $\psi(x, \theta, \overline{\theta})$ can also have Lorentz (spinor or tensor) indices, and then a corresponding matrix arises on the right-hand side of (43) as a result of Lorentz transformations. Since the supertransformations do not affect the Lorentz indices, for a superfield with any indices we have under supertransformations

$$\begin{aligned}U(\xi)\psi\ldots(x, \theta, \overline{\theta})U^{-1}(\xi) = \psi\ldots(x_{AB} - i\xi_A\overline{\theta}_{\dot B} + i\theta_A\overline{\xi}_{\dot B},\\ \theta_A + \xi_A,\; \overline{\theta}_{\dot A} + \overline{\xi}_{\dot A}).\end{aligned} \quad (44)$$

Under infinitesimal supertransformations

$$\begin{aligned}\delta\psi\ldots(x, \theta, \overline{\theta}) = \xi_A\left(\frac{\partial}{\partial\theta^A} - i\overline{\theta}_{\dot B}\partial_{A\dot B}\right)\psi\ldots(x, \theta, \overline{\theta})\\ + \overline{\xi}_{\dot A}\left(\frac{\partial}{\partial\overline{\theta}^{\dot A}} + i\theta_B\partial_{B\dot A}\right)\psi\ldots(x, \theta, \overline{\theta}).\end{aligned} \quad (45)$$

It is easy to see that the derivatives $\partial/\partial\theta$ and $\partial/\partial\overline{\theta}$ are noncovariant with respect to supertransformations. It follows from (45) that the covariant derivatives have the form

$$D_A = \frac{\partial}{\partial\theta^A} + i\overline{\theta}_{\dot B}\partial_{A\dot B};\;\; \overline{D}_{\dot B} = -\frac{\partial}{\partial\overline{\theta}^{\dot B}} - i\theta_A\partial_{A\dot B}.$$

The commutation relations for D_A and $\overline{D}_{\dot B}$ coincide with the commutation relations for Q_A and $\overline{Q}_{\dot B}$ to within a common sign. The concept of covariant derivative is very helpful when one is considering irreducible superfields, the construction of supersymmetric Lagrangians, etc.[68–72]

A superfield is equivalent to a certain multiplet of ordinary fields. Let us show this for the example of a superfield $\psi(x, \theta)$ that does not depend on the coordinate $\overline{\theta}$. Expanding $\psi(x, \theta)$ in a power series in θ and remembering that the square of each component of the spinor θ vanishes since $\theta_A\theta_B + \theta_B\theta_A = 0$, we obtain for the scalar superfield

$$\psi(x, \theta) = \psi^{(1)}(x) + \theta_A\psi^{(2)}_A(x) + \theta_A\varepsilon_{AB}\theta_B\psi^{(3)}(x). \quad (46)$$

Thus, the scalar superfield is equivalent to a multiplet consisting of two ordinary scalar fields [$\psi^{(1)}(x)$ and $\psi^{(3)}(x)$] and one ordinary spinor field [$\psi^{(2)}_A(x)$]. For the superfield $\psi^{(j,0)}_{A_1\cdots A_{2j}}(x,\theta)$ transforming in accordance with the representation $D(j,0)$ of the Lorentz group, we find

$$\psi^{(j,0)}_{A_1\ldots A_{2j}}(x,\theta)=\psi^{(1)}_{A_1\ldots A_{2j}}(x)+\theta_{A_1}\psi^{(2)}_{A_2\ldots A_{2j}}(x)+\theta_A\psi^{(3)}_{AA_1\ldots A_{2j}}(x)+\theta_B\varepsilon_{BC}\theta_C\psi^{(4)}_{A_1\ldots A_{2j}}(x). \quad (47)$$

Therefore, the superfield $\psi^{(j,0)}_{A_1\cdots A_{2j}}(x,\theta)$ is equivalent to the multiplet of ordinary fields consisting of the fields $\psi^{(1)}$ and $\psi^{(4)}$, which transform in accordance with the representations $D(j,0)$ of the Lorentz group, and the fields $\psi^{(2)}$ and $\psi^{(3)}$, which transform in accordance with the representations $D(j-1/2,0)$ and $D(j+1/2,0)$, respectively.

Similarly, the superfield $\psi^{(0,j)}(x,\bar{\theta})$, which transforms in accordance with the representation $D(0,j)$ of the Lorentz group, is a multiplet of ordinary fields that transform in accordance with the representations $D(0,j-1/2)$ $D(0,j)$, and $D(0,j+1/2)$. One can show that the superfields $\psi^{(j,0)}(x,\theta)$ and $\psi^{(0,j)}(x,\bar{\theta})$ form a complete set of irreducible superfields. The irreducible superfields are classified in accordance with the values of $Y=j$, the superspin, which can take the values 0, 1/2, 1, ... (Refs. 73 and 64). The ordinary spin is not an invariant characteristic of the superfield and it takes for the superfield with superspin Y, as can be seen from (47), the values

$$Y-1/2,\ Y,\ Y+1/2.$$

The formalism of superspace and superfields can also be constructed for extensions of the Poincaré algebra by arbitrary spinor generators (the algebra $S^{(j)}_{\mathrm{III}}$).[74] In this case, superspace is an $4+4j(2j+1)$-dimensional space with coordinates $x_{A\dot{B}}$ and $\theta_{A_1\cdots A_{2j}\dot{B}_1\cdots\dot{B}_{2j-1}}$, $\bar{\theta}_{C_1\cdots C_{2j-1}\dot{D}_1\cdots D_{2j}}$, where θ and $\bar{\theta}$ are completely anticommuting spinors. The algebra $S^{(j)}_{\mathrm{III}}$ can be realized as the algebra of the group of superspace transformations containing the Poincaré subgroup and the supertransformations

$$\theta\to\theta'=\theta+\xi;\quad \bar{\theta}\to\bar{\theta}'=\bar{\theta}+\bar{\xi};$$
$$x_{A\dot{B}}\to x'_{A\dot{B}}=x_{A\dot{B}}-i\xi_{AA_2\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}\bar{\theta}_{A_2\ldots A_{2j}\dot{B}\dot{B}_1\ldots\dot{B}_{2j-1}}+i\theta_{AA_2\ldots A_{2j}\dot{B}_1\ldots\dot{B}_{2j-1}}\bar{\xi}_{A_2\ldots A_{2j}\dot{B}\dot{B}_1\ldots\dot{B}_{2j-1}}.$$

As in the case $j=1/2$, one can introduce superfields and show that they are equivalent to multiplets of ordinary fields. Also as in the case $j=1/2$, the superfields $\psi^{(Y,0)}(x,\theta)$ and $\psi^{(0,Y)}(x,\bar{\theta})$ are irreducible. Expanding the superfields $\psi^{(Y,0)}(x,\theta)$ and $\psi^{(0,Y)}(x,\theta)$ into polynomials of degree $2j(2j+1)$ in powers of θ and $\bar{\theta}$, respectively, we see that these superfields are equivalent to a set of ordinary fields with spin spectrum[74]

$$Y-j(4j-1),\ Y-j(4j-1)+1/2,\ \ldots,\ Y-1/2,\ Y,\ Y+1/2,\ \ldots,\ Y+j(4j-1)-1/2,\ Y+j(4j-1).$$

Finally, for the groups considered in Sec. 8, the corresponding spinor coordinates have, besides the Lorentz indices, indices α corresponding to an internal symmetry group. The irreducible superfields with superspin Y are equivalent to a set of ordinary fields with different internal quantum numbers and spin spectrum:

$$n(Y-j(4j-1)),\ n(Y-j(4j-1))+1/2,\ \ldots,\ Y-1/2,\ Y,\ Y+1/2,\ \ldots,\ n(Y+j(4j-1))-1/2,\ n(Y+j(4j-1)),$$

where n is the dimension of the representation of the internal symmetry group with respect to which the spinor generators Q and $\bar{Q}$ transform. Thus, spinor extensions of the Weyl and conformal groups can also be represented in the form of groups of transformations of corresponding superspaces.

10. EXTENSIONS OF THE POINCARÉ ALGEBRA BY MEANS OF TENSOR GENERATORS

The transformations of ordinary internal symmetry groups are transformations with scalar (with respect to the Lorentz group) parameters. Supertransformations can be regarded as internal symmetry transformations with spinor parameters. In the framework of such an approach, it is natural to consider also internal symmetry transformations with vector and tensor parameters, i.e., extensions of the Poincaré algebra by generators that are tensors of various ranks.

We begin by considering extensions of the Poincaré algebra by means of the vector generator $Y_{A\dot{B}}=(\sigma_\mu)_{A\dot{B}}Y_\mu$. By Lorentz invariance,

$$[P_{A\dot{B}},\ V_{C\dot{D}}]=a(\varepsilon_{AC}\bar{Y}_{\dot{B}\dot{D}}+\varepsilon_{\dot{B}\dot{D}}Y_{AC});$$
$$[V_{A\dot{B}},\ V_{C\dot{D}}]=b(\varepsilon_{AC}\bar{Y}_{\dot{B}\dot{D}}+\varepsilon_{\dot{B}\dot{D}}Y_{AC}),$$

where a and b are arbitrary constants. Using the Jacobi identities for P, P, and V, we find $a=0$, and from the Jacobi identities for P, V, and V we obtain $b=0$.

Thus, the extension of the Poincaré algebra by the vector generator is trivial. It is readily seen that extensions of the Poincaré algebra by arbitrary tensor generators are also trivial. However, if one considers extensions of the Poincaré algebra by vector and scalar generators and by tensor $T_{\mu\nu}$ $(T_{\mu\nu}=T_{\nu\mu},\ T_{\mu\mu}=0)$ and scalar generators, nontrivial algebras are obtained: in the first case an algebra isomorphic to the conformal algebra and in the second an algebra isomorphic to the affine algebra. Thus, the properties of these extensions are intimately related to the structure of the conformal and affine groups. In particular, the multiplets have infinitely many components. These algebras exhaust the possible extensions of the Poincaré algebra by tensor generators.[35]

CONCLUSIONS

Thus, there exist only a limited number of different types of symmetry. In conclusion, I should like to emphasize the fact that the main reason for the existence of the restrictions is the relativistic invariance of quantum field theory.

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