Optical potential for ultracold neutrons

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The theory of the passage of ultracold neutrons through matter is reviewed. The methods of the theory of multiple scattering are used to obtain an expression for an optical potential describing the interaction of neutrons of this energy range with matter. The influence of inhomogeneity of the scattering medium on the damping and dispersion law of a coherent neutron wave is considered.

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INTRODUCTION

Investigations of neutrons of very low energies (ultracold neutrons), some unusual properties of which were first pointed out by Zel'dovich,[1] were begun by Shapiro and his collaborators.[2] Currently, such investigations are made at various centers and the number of problems being studied is very considerable: from measurement of the electric dipole moment of the neutron to the study of the structure of matter in the condensed state.[3–5] The fundamentals of the theory of the interaction of neutrons in this energy range with matter[1)] were laid by Shapiro.[2] To describe the propagation of ultracold neutrons in matter, one uses the potential

$$U_0 = U_{R0} + iU_{I0}, \quad (1)$$

where

$$U_{R0} = (2\pi\hbar^2/m)\,\rho\,\mathrm{Re}\,b, \quad (2)$$

b is the length of coherent scattering of neutrons on an infinitely heavy nucleus; m is the neutron mass; and ρ is the number of scattering nuclei in 1 cm^3 of the irradiated material. For simplicity, we shall restrict the discussion to interaction of neutrons with nonmagnetic materials.

A formula analogous to (2) is used to calculate the imaginary part of the potential

$$U_{I0} = (2\pi\hbar^2/m)\,\rho\,\mathrm{Im}\,b. \quad (3)$$

In accordance with the optical theorem,

$$\mathrm{Im}\,b = -k\,[\sigma_{\mathrm{abs}} + \sigma_{\mathrm{inel}}]/4\pi, \quad (4)$$

where **k** is the wave vector of the incident neutrons; σ_{abs} is the absorption cross section; σ_{inel} is the cross section of inelastic scattering resulting in the neutron's leaving the low-energy region. It is expedient to call the sum $\sigma_{\mathrm{abs}} + \sigma_{\mathrm{inel}}$ the neutron removal cross section (by analogy with the term used in reactor physics). Many authors have attempted to justify the semiphenomenological expressions (1)–(4) (see, for example, Refs. 7 and 8). An important question in need of careful examination is the possibility of using the optical theorem in the form (4) in this case. The point is that in this relation the scattering amplitude f ($f = -b$) is taken on the "mass" shell, and therefore the optical potential can be determined only in a restricted range of the variables. Second, Eq. (4) in the case of zero absorption of neutrons relates the imaginary part of the scattering amplitude f on one nucleus to the cross section of inelastic scattering on a collection of nuclei. This relation is to a certain extent justified if the inelastic scattering is treated in the incoherent approximation, i. e., when, like Imb, σ_{inel} describes scattering on one nucleus (see below). Interference effects cannot be taken into account in such a scheme. A derivation free of this inconsistency of an expression for an optical potential describing the propagation of a coherent neutron wave through matter was given in Refs. 9 and 10. The point of departure of this calculation is the well known formulas of the theory of multiple scattering.[11] We regard this potential of the optical model (in the same way as is done in the theory of nuclear reactions[11–13]) as an effective potential in the Schrödinger equation, which describes a coherent neutron wave that is equivalent to a coherent field in the original many-channel problem. The imaginary part of this potential determines the attenuation of the coherent wave in the ingoing channel. Note that for such an interpretation the applicability of the expressions for the optical potential is not restricted to the region of very low neutron energies. In addition, in such an approach there is no need to use the optical theorem to find ImU_0; instead, one can calculate this quantity directly. Nevertheless, we shall consider in detail the calculation of ImU_0 by means of the standard procedure based on the optical theorem, and we shall estimate the corrections due to the departure from the "mass" shell.

The calculation made in the second approximation of the theory of multiple scattering (see below) shows that for a coherently scattering system of nuclei the cross section of inelastic scattering in (4) contains a contribution of interferences inelastic scattering with absorption of phonons (a model of an harmonic crystal is studied as scatterer). According to the estimates obtained by Placzek and Van Hove,[14] the correction due to the interference of the neutron waves scattered by different nuclei is, in the region of low energies, 10–20% of the value of σ_{inel} calculated in the incoherent approximation.

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1)Frank[6] and Steyerl[4] have proposed the following more detailed classification of very low energy neutrons: very cold neutrons (10^{-7} eV $< E_n < 10^{-4}$ eV), and ultracold neutrons ($E_n < 10^{-7}$ eV). The justification for this division of the range of very low energies is the physically different behavior of the neutrons of these groups in matter: The behavior of very cold neutrons is characterized by a wave process in matter, whereas the wave function of ultracold neutrons is rapidly damped in the medium (the damping constant is $\varkappa_0 \approx 10^5$–10^6 cm^{-1}). In the present review, we shall for brevity refer to all neutrons with energy near $E_n \approx 10^{-7}$ eV and above as ultracold, giving additional explanations whenever this term could lead to confusion.

The correction of second order in Reb to the real part of the optical potential is in order to magnitude $(b/0.1a)U_{R_0} = (10^{-3}-10^{-4})U_{R_0}$, where U_{R_0} is defined by (2); a is the mean distance between the scattering nuclei.

In the general case, the optical potential is nonlocal and depends on the neutron energy. Such a result is familiar in optics and nuclear physics. If one considers the propagation of neutrons for $E_n \lesssim 10^{-4}$ eV, then the energy dependence and the nonlocality of the optical potential both have very little influence on the propagation of the neutron wave in matter.

Inhomogeneity of the medium changes the excitation spectrum of the scattering system and introduces corrections to the cross section of inelastic scattering. In the majority of cases of practical interest, these corrections are small. Exchange of a phonon between very cold neutrons propagating in a medium leads to a very slight departure from ideal behavior of the neutron gas. In the case of ultracold neutrons in a bottle, such an interaction between the neutrons is even weaker.

In this review, we shall also study the propagation of neutrons through inhomogeneous media.[9,10] After the analysis of weak single scattering on an inhomogeneous target, for which the Born approximation applies, we consider the corrections to these results due to strong scattering on one inhomogeneity and multiple scattering in a large volume of scattering matter. To describe multiple scattering in an inhomogeneous medium, we again use a variant of the "optical" model. In the framework of this model, the coherent neutron field satisfies a Schrödinger equation with an effective potential that depends regularly on the spatial coordinates; this is the analog of the optical potential introduced above to take into account inelastic processes.[9,10,15] Elastic incoherent scattering on static inhomogeneities leads to an attenuation of the coherent wave analogous to the removal of neutrons from the ingoing channel as a result of inelastic scattering. Inhomogeneity of the medium may be due to either dynamical fluctuations of the density (which are, for example, strong near critical points) or have a static nature. An example of an inhomogeneity of this last type is an inhomogeneity in alloys, ideal polycrystals, and in real crystals with structure defects.

The effective optical potential for an unbounded scattering medium enables one to find the refractive index for a neutron wave and then consider the interaction of neutrons with a medium that occupies a bounded volume. For ultracold neutrons ($E_n \lesssim 10^{-7}$ eV), which have a small penetration depth into a bottle wall, one must carefully examine the influence of the structure of the surface bounding the scattering medium on the behavior of a neutron wave. We discuss the influence of roughness of the bottle walls on the coefficient of absorption of ultracold neutrons[16,17] and the results of calculations of the angular distribution of neutrons reflected from a nonideal surface.[17,18] We also discuss the absorption of ultracold neutrons in a surface film contaminating a bottle wall,[19] and we estimate the mean square of the fluctuations of the neutron field in an inhomogeneous medium and the influence of roughness of the walls of a neutron guide tube on a neutron wave propagating along the tube.

1. DERIVATION OF GENERAL FORMULAS OF THE OPTICAL MODEL FOR A HOMOGENEOUS MEDIUM

In the theory of multiple scattering (see, for example, Ref. 11) it is shown that the expression for the optical potential U_0 describing elastic scattering of particles on a system of N scatterers in the state $|i\rangle$ can be written in the form of the following expansion:[2)]

$$U_0 = \sum_{\alpha=1}^{N} \langle, i|\hat{t}_\alpha|i,\rangle + \sum_{\alpha\neq\beta=1}^{N} \left\langle, i\left|\hat{t}_\alpha \frac{1-\Lambda_i}{d}\hat{t}_\beta\right|i,\right\rangle + \dots \qquad (5)$$

Here, the angular brackets $\langle, i|\cdots|i\rangle$ denote the matrix element between the wave functions $|i\rangle$ of the scattering medium. For brevity, we shall not particularize the representation in which the potential U_0 is expressed with respect to the variables of the neutron. In Eq. (5), $1/d$ is the Green's function of the Schrödinger equation with the interaction between the neutron and the scattering nuclei switched off, i.e.,

$$d = E_{0i} + i\eta - \hat{K}; \qquad (6)$$

$$\hat{K} = \hat{K}_n + \hat{H}, \qquad (7)$$

where $\hat{K}_n = -\hbar^2\nabla_n^2/2m$ is the operator of the neutron kinetic energy; $\hat{H} = \Sigma_{\alpha=1}^N \hat{K}_\alpha + U$ is the Hamiltonian of the scattering system; $\hat{K}_\alpha = -\hbar^2\nabla_\alpha^2/2M_\alpha$ is the operator of the kinetic energy of the αth scattering nucleus; U is the operator of the potential energy of the interaction of the nuclei of matter with one another; η is a small positive number; E_{0i} is the total energy of the system consisting of the neutron and the scatterer in state $|i\rangle$.

Under real conditions, the quantum-mechanical state of the scatterer is not defined since it is in thermodynamic equilibrium at temperature T. Therefore, in order to obtain the final result, the neutron wave function ψ_c satisfying the Schrödinger equation with the optical potential U_0 (5) must be averaged over the statistical distribution of the states $|i\rangle$. We shall assume approximately that this function is equal to the solution of the wave equation with optical potential $\langle U_0\rangle_T$ averaged over the equilibrium distribution of the states of the scattering system. In doing this, we ignore the equilibrium fluctuations of the optical potential. In order to take into account such fluctuations, it is necessary to average, not the optical potential U_0, but the scattering operator $\hat{T}_c$ corresponding to it. The error introduced by our approximation is $\sim\langle(U_0 - \langle U_0\rangle_T)^2\rangle_T$, where the symbol $\langle\cdots\rangle_T$ denotes averaging over the equilibrium distribution of the scatterer states. Since the dependence of the optical potential on the scatterer state is primarily determined by the density ρ of the scattering nuclei (ρ is the number of nuclei in 1 cm^3 of scattering matter), we have $\langle(U_0 - \langle U_0\rangle)^2\rangle_T \sim \langle(\rho - \langle\rho\rangle)^2\rangle_T$. These fluctuations give rise to scattering analogous to the well known molecular scattering of light in optics.

Let us return to the discussion of Eq. (5). We denote by $\hat{t}_\alpha$ the t matrix of scattering of a neutron on a nucleus

[2)]One can have other forms of the optical potential. However, the differences from (5) appear only in the terms of higher order, whose treatment goes beyond the accuracy of the present calculation.

bound in the matter. This operator satisfies the equation[11]

$$\hat{t}_\alpha = V_\alpha + V_\alpha [(1 - \Lambda_i)/d]\, \hat{t}_\alpha, \tag{8}$$

where $1 - \Lambda_i$ is the projection operator that eliminates state $|i\rangle$ from the complete set of intermediate states of the scatterer; V_α is the operator of the potential energy of the interaction of the neutron with the αth nucleus. We emphasize that $\hat{t}_\alpha$ is a many-particle operator (see the definition of d above). Such a definition of the operator $\hat{t}_\alpha$ is convenient for analyzing the influence of inelastic processes: in the ingoing channel we have a state corresponding to elastic scattering on a system of N scatterers. Indeed, for $\Lambda_i = 1$ we have $\hat{t}_\alpha = V_\alpha$ and $U_0 = \Sigma_{\alpha=1}^{N} \hat{t}_\alpha = \Sigma_{\alpha=1}^{N} V_\alpha = \hat{V}$. If by the wave function of the optical model we understand a coherent wave, then incoherent elastic scattering will also remove particles from the ingoing channel. To take into account this effect, we can again use the expansion (5), but in Eq. (8) for $\hat{t}_\alpha$ we must omit the projection operator Λ_i.

In the approximation of the Fermi pseudopotential[20–23] we have in the coordinate representation

$$\langle \mathbf{r}_n | V_\alpha | \mathbf{r}'_n \rangle = (2\pi\hbar^2/m)\, b_\alpha \delta(\mathbf{r}_n - \mathbf{R}_\alpha)\, \delta(\mathbf{r}'_n - \mathbf{R}_\alpha); \tag{9}$$

where $\mathbf{R}_\alpha$ and $\mathbf{r}_n$ are the radius vectors of the αth nucleus and the neutron, respectively. When we find $\mathrm{Re}\,\hat{t}_\alpha$ by means fo Eq. (8), we can restrict ourselves to a single iteration, i.e., assume approximately[3)]

$$\mathrm{Re}\,\hat{t}_\alpha \approx \mathrm{Re}\,V_\alpha \sim \mathrm{Re}\,b_\alpha. \tag{10}$$

The correction to this result is in order of magnitude $r_{\mathrm{nuc}}/0.1a \lesssim 10^{-3}$ (r_{nuc} is the radius of the nucleus; a is the mean distance between the nuclei of the scattering matter (see, for example, Ref. 23).

The quantity $\mathrm{Im}\,\hat{t}_\alpha$ contains a contribution from $\mathrm{Im}\,V_\alpha$ proportional to $\mathrm{Im}\,b_\alpha$ and therefore to the cross section of capture of a neutron by the αth nucleus. This cross section does not depend on the chemical binding of the absorbing nucleus. In order to take into account the contribution of inelastic scattering to $\mathrm{Im}\,\hat{t}_\alpha$, we make two iterations of Eq. (8) and obtain as a result

$$\hat{t}_\alpha \approx V_\alpha + V_\alpha [(1 - \Lambda_i)/d]\, V_\alpha \tag{11}$$

and accordingly

$$\mathrm{Im}\,\hat{t}_\alpha \approx \mathrm{Im}\,V_\alpha + \mathrm{Im}\,(V_\alpha [(1 - \Lambda_i)/d]\, V_\alpha). \tag{12}$$

Let us consider the second term in (12), which is largely due to inelastic scattering of neutrons. In the approximation of the pseudopotential (9),

$$\langle, i | \mathrm{Im}\,(\hat{t}_\alpha - V_\alpha) | i, \rangle$$
$$= (2\pi\hbar^2 b/m)^2 \mathrm{Im}\,\langle, i | \delta(\mathbf{r}_n - \mathbf{R}_\alpha) [(1 - \Lambda_i)/d]\, \delta(\mathbf{r}_n - \mathbf{R}_\alpha) | i, \rangle. \tag{13}$$

In (13), we omit the imaginary part of b since its allowance goes beyond the accuracy of our calculation.

3)Application of perturbation theory with the pseudopotential (9) as perturbation must, strictly speaking, lead already in the second order to singular expressions. However, the averaging over the equilibrium distribution of the scatterer states effectively spreads out the interaction region and removes the singularity.

In the momentum representation with respect to the neutron coordinates, substituting the explicit expression for the propagator $1/d$ and introducing a complete set of intermediate states of the system, we obtain for the matrix elements $\hat{t}_\alpha - V_\alpha$ the expression:

$$\langle \mathbf{q}' | \hat{t}_\alpha - V_\alpha | \mathbf{q} \rangle = \left(\frac{2\pi\hbar^2}{m} b \right)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_{\gamma \neq i} \langle i | \exp[-\mathrm{i}\mathbf{R}_\alpha (\mathbf{q}' - \mathbf{k})] | \gamma \rangle$$
$$\times \left[E_i + \varepsilon_i + \mathrm{i}\eta - E_\gamma - \frac{\hbar^2 k^2}{2m} \right]^{-1} \langle \gamma | \exp[\mathrm{i}\mathbf{R}_\alpha (\mathbf{q} - \mathbf{k})] | i \rangle. \tag{14}$$

Here, E_i and E_γ are the scattering energy of the system in the initial and intermediate states; $\mathbf{k}_i$ and $\varepsilon_i = \hbar^2 k_i^2/2m$ are the wave vector and energy of the incident neutron. Taking into account the well known relation $1/(x + \mathrm{i}\eta) = \mathscr{P}/x - \mathrm{i}\pi\delta(x)$ and separating out explicitly the term corresponding to the intermediate state $|i\rangle$, we have

$$\mathrm{Im}\,\langle \mathbf{q}' | \hat{t}_\alpha - V_\alpha | \mathbf{q} \rangle = -\pi \left(\frac{2\pi\hbar^2}{m} b \right)^2$$
$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} \Big\{ \sum_{\gamma} \langle i | \exp[-\mathrm{i}\mathbf{R}_\alpha (\mathbf{q}' - \mathbf{k})] | \gamma \rangle\, \delta\left[E_i + \varepsilon_i - E_\gamma - \frac{\hbar^2 k^2}{2m} \right]$$
$$\times \langle \gamma | \exp[\mathrm{i}\mathbf{R}_\alpha (\mathbf{q} - \mathbf{k})] | i \rangle - \delta\left[\frac{\hbar^2 k^2}{2m} - \varepsilon_i \right] \langle i | \exp[-\mathrm{i}\mathbf{R}_\alpha (\mathbf{q}' - \mathbf{k})] | i \rangle$$
$$\times \langle i | \exp[\mathrm{i}\mathbf{R}_\alpha (\mathbf{q} - \mathbf{k})] | i \rangle \Big\}. \tag{15}$$

One can perform the summation over the intermediate states of the scatterer. As a result, $\mathrm{Im}\langle \mathbf{q}' | \hat{t}_\alpha - V_\alpha | \mathbf{q} \rangle$ can be expressed in terms of the time correlation coefficient[20,21]:

$$\chi^{(i)}_{\alpha\beta}(t; \mathbf{q}', \mathbf{q}, \mathbf{k}) = \langle i | \exp[-\mathrm{i}\hat{\mathbf{R}}_\alpha(t)(\mathbf{q}' - \mathbf{k})] \exp[\mathrm{i}\hat{\mathbf{R}}_\beta(0)(\mathbf{q} - \mathbf{k})] | i \rangle, \tag{16}$$

namely

$$\mathrm{Im}\,\langle \mathbf{q}' | \hat{t}_\alpha - V_\alpha | \mathbf{q} \rangle = -\pi \left(\frac{2\pi\hbar^2}{m} b \right)^2$$
$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} \Big\{ \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \exp(\mathrm{i}\omega t)\, \chi^{(i)}_{\alpha\alpha}(t; \mathbf{q}', \mathbf{q}, \mathbf{k})$$
$$- \frac{1}{\hbar} \delta(\omega) \langle i | \exp[-\mathrm{i}\hat{\mathbf{R}}_\alpha(0)(\mathbf{q}' - \mathbf{k})] | i \rangle \langle i | \exp[\mathrm{i}\hat{\mathbf{R}}_\alpha(0)(\mathbf{q} - \mathbf{k})] | i \rangle \Big\}. \tag{17}$$

Here

$$\hat{\mathbf{R}}_j(t) = \exp[\mathrm{i}\hat{H}t/\hbar]\, \hat{\mathbf{R}}_j(0) \exp[-\mathrm{i}\hat{H}t/\hbar]$$

is the coordinate operator of the jth scattering nucleus written down in the Heisenberg representation, and

$$\omega = \hbar k_i^2/2m - \hbar k^2/2m.$$

We average (17) over the equilibrium distribution of the states of the scattering system and obtain

$$\langle \mathrm{Im}\,\langle \mathbf{q}' | \hat{t}_\alpha - V_\alpha | \mathbf{q} \rangle \rangle_T = -\pi \left(\frac{2\pi\hbar^2}{m} b \right)^2$$
$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} \Big\{ \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dt \exp(\mathrm{i}\omega t)\, \chi^{T}_{\alpha\alpha}(t; \mathbf{q}', \mathbf{q}, \mathbf{k})$$
$$- \frac{1}{\hbar} \delta(\omega) \langle \langle i | \exp[-\mathrm{i}\hat{\mathbf{R}}_\alpha(0)(\mathbf{q}' - \mathbf{k})] | i \rangle$$
$$\times \langle i | \exp[\mathrm{i}\hat{\mathbf{R}}_\alpha(0)(\mathbf{q} - \mathbf{k})] | i \rangle \rangle_T, \tag{18}$$

where

$$\chi^{T}_{\alpha\beta}(t; \mathbf{q}', \mathbf{q}, \mathbf{k})$$
$$= \langle \langle i | \exp[-\mathrm{i}\hat{\mathbf{R}}_\alpha(t)(\mathbf{q}' - \mathbf{k})] \exp[\mathrm{i}\hat{\mathbf{R}}_\beta(0)(\mathbf{q} - \mathbf{k})] | i \rangle \rangle_T. \tag{19}$$

Remembering that (see, for example, Refs. 20 and 21)

$$\lim_{t \to \infty} \chi^{T}_{\alpha\beta}(t; \mathbf{q}', \mathbf{q}, \mathbf{k}) = \langle \langle i | \exp[-\mathrm{i}\hat{\mathbf{R}}_\alpha(0)(\mathbf{q}' - \mathbf{k})] | i \rangle \rangle_T$$
$$\times \langle \langle i | \exp[\mathrm{i}\mathbf{R}_\beta(0)(\mathbf{q} - \mathbf{k})] | i \rangle \rangle_T, \tag{20}$$

we write the expression (18) in the more compact form[4)]

$$\langle \mathrm{Im}\,\langle \mathbf{q}'\,|\,\hat{t}_\alpha - V_\alpha\,|\,\mathbf{q}\rangle\rangle_T = -\pi\left(\frac{2\pi\hbar^2}{m}b\right)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\left\{\frac{1}{2\pi\hbar}\right.$$
$$\left.\times\int_{-\infty}^{\infty}dt\exp(i\omega t)\,[\chi^T_{\alpha\alpha}(t;\,\mathbf{q}',\mathbf{q},\mathbf{k})-\lim_{t\to\infty}\chi^T_{\alpha\alpha}(t;\,\mathbf{q}',\mathbf{q},\mathbf{k})]\right\}. \quad (21)$$

The function $\chi^T_{\alpha\beta}$ for $\mathbf{q}'=\mathbf{q}=\mathbf{k}_i$ and $\mathbf{k}=\mathbf{k}_f$ defines the double differential cross section of inelastic scattering of neutrons on a system of N isotopically identical spinless nuclei[20,21]:

$$\left[\frac{d^2\sigma}{d\Omega\,d\varepsilon_f}\right]^{\mathrm{inel}} = \frac{b^2}{2\pi\hbar}\frac{k_f}{k_i}\sum_{\alpha,\,\beta=1}^{N}\int_{-\infty}^{\infty}dt\exp(i\omega t)$$
$$\times[\chi^T_{\alpha\beta}(t;\,\mathbf{k}_i,\mathbf{k}_i,\mathbf{k}_f)-\lim_{t\to\infty}\chi^T_{\alpha\beta}(t;\,\mathbf{k}_i,\mathbf{k}_i,\mathbf{k}_f)]. \quad (22)$$

Here $\varepsilon_f=\hbar^2k_f^2/2m$ and $\mathbf{k}_f$ are the energy and wave vector of the scattered nucleon, respectively. In the incoherent approximation ($\alpha=\beta$), the total cross section of inelastic scattering per nucleus has the form

$$[\sigma^{\mathrm{inel}}_{\mathrm{tot}}(\varepsilon_i)]^{\mathrm{incoh}} = \frac{\hbar^2}{mk_i}b^2$$
$$\times\int d\mathbf{k}_f\left\{\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dt\exp(i\omega t)\,[\chi^T_{\alpha\alpha}(t;\,\mathbf{k}_i,\mathbf{k}_i,\mathbf{k}_f)\right.$$
$$\left.-\lim_{t\to\infty}\chi^T_{\alpha\alpha}(t;\,\mathbf{k}_i,\mathbf{k}_i,\mathbf{k}_f)]\right\}. \quad (23)$$

Setting $\mathbf{q}=\mathbf{q}'=\mathbf{k}_i$ and $\mathbf{k}=\mathbf{k}_f$ in (21), we can readily show that

$$\langle \mathrm{Im}\,\langle \mathbf{q}'\,|\,\hat{t}_\alpha - V_\alpha\,|\,\mathbf{q}\rangle\rangle_T = -\frac{2\pi\hbar^2}{m}\left[\frac{k_i}{4\pi}(\sigma^{\mathrm{inel}}_{\mathrm{tot}}(\varepsilon_i))^{\mathrm{incoh}}\right]. \quad (24)$$

Thus, the imaginary part of the matrix element $\langle \mathbf{q}'\,|\,\hat{t}_\alpha - V_\alpha\,|\mathbf{q}\rangle$ calculated on the "mass" shell, i.e., for $\mathbf{q}'=\mathbf{q}=\mathbf{k}_i$, is uniquely determined by the total cross section of inelastic scattering calculated in the incoherent approximation ("optical theorem"). Anticipating slightly, we note that the transition to a spatially homogeneous scattering medium imposes a restriction on the range of variation of the wave vectors in the matrix elements: $\mathbf{q}=\mathbf{q}'$. If we restrict the treatment to operators that are local in the coordinate representation with respect to the neutron variables, we can set $\mathbf{q}'=\mathbf{q}=0$ and in the region of very low neutron energies ($\mathbf{k}_i=0$) the condition of calculation of the matrix element on the mass shell is automatically satisfied. This means that if these requirements are fulfilled the optical theorem in the form (24) holds. Bearing in mind that

$$\langle \mathbf{q}'\,|\,\mathrm{Im}\,V_\alpha\,|\,\mathbf{q}\rangle = -\frac{2\pi\hbar^2}{m}\frac{k_i}{4\pi}\sigma_{\mathrm{abs}} \quad (\mathbf{q}'=\mathbf{q}=\mathbf{k}_i), \quad (25)$$

we write this relation in the different form

$$\mathrm{Im}\,\langle\langle \mathbf{q}'\,|\,\hat{t}_\alpha\,|\,\mathbf{q}\rangle\rangle_T = -\frac{2\pi\hbar^2}{m}\frac{k_i}{4\pi}\{[\sigma^{\mathrm{inel}}_{\mathrm{tot}}(\varepsilon_i)]^{\mathrm{incoh}}+\sigma_{\mathrm{abs}}(\varepsilon_i)\} \quad (\mathbf{q}=\mathbf{q}'=\mathbf{k}_i). \quad (26)$$

In the first approximation of the theory of multiple scattering,

$$U_{01}=\sum_{\alpha=1}^{N}\langle i,\,|\,\hat{t}_\alpha\,|\,i,\rangle. \quad (27)$$

Taking into account (9), (10), and (26), we write down in the coordinate representation

$$\langle \mathbf{r}'_n\,|\,U_{01}\,|\,\mathbf{r}_n\rangle = \frac{2\pi\hbar^2}{m}\sum_{\alpha=1}^{N}\tilde{b}_\alpha\langle i\,|\,\delta(\mathbf{r}_n-\mathbf{R}_\alpha)\,|\,i\rangle\,\delta(\mathbf{r}_n-\mathbf{r}'_n). \quad (28)$$

[4)]In the case of macroscopic systems, when there are no distinguished degrees of freedom, $\langle\langle i\,|\hat{A}|\,i\rangle\langle i\,|\hat{B}\,|i\rangle\rangle_T \approx \langle\langle\hat{A}\rangle\rangle_T\langle\langle\hat{B}\rangle\rangle_T$.

Here

$$\tilde{b}_\alpha = \mathrm{Re}\,b_\alpha+\frac{m}{2\pi\hbar^2}\,i\,\mathrm{Im}\,\langle \mathbf{k}_i\,|\,\hat{t}_\alpha\,|\,\mathbf{k}_i\rangle = \mathrm{Re}\,b_\alpha - i\frac{k_i}{4\pi}(\sigma^{\mathrm{incoh}}_{\mathrm{inel}}+\sigma_{\mathrm{abs}}). \quad (29)$$

Thus, the calculation of $\mathrm{Im}U_0$ in the first approximation of the theory of multiple scattering takes into account the contribution of inelastic scattering in the incoherent approximation. Accordingly, it follows from (28) in the momentum representation that

$$\langle \mathbf{q}'\,|\,U_{01}\,|\,\mathbf{q}\rangle = \iint d\mathbf{r}_n\,d\mathbf{r}'_n\exp[-i\mathbf{q}'\mathbf{r}'_n]\exp(i\mathbf{q}\mathbf{r}_n)\langle \mathbf{r}'_n\,|\,U_{01}\,|\,\mathbf{r}_n\rangle$$
$$= \frac{2\pi\hbar^2}{m}\sum_{\alpha=1}^{N}\tilde{b}_\alpha\langle i\,|\exp[i(\mathbf{q}-\mathbf{q}')\mathbf{R}_\alpha]\,|\,i\rangle. \quad (30)$$

For a spatially homogeneous medium, the expressions (28) and (30) must be averaged over the configurations of the scattering nuclei. Then

$$\overline{\langle \mathbf{r}'_n\,|\,U_{01}\,|\,\mathbf{r}_n\rangle} = (2\pi\hbar^2/m)\,\tilde{b}\,(N/V)\,\delta(\mathbf{r}'_n-\mathbf{r}_n) \quad (31)$$

and

$$\overline{\langle \mathbf{q}'\,|\,U_{01}\,|\,\mathbf{q}\rangle} = (2\pi\hbar^2/m)\,\tilde{b}\,(N/V)\,(2\pi)^3\,\delta(\mathbf{q}-\mathbf{q}'). \quad (32)$$

Here, V is the volume of the scattering system, $\tilde{b}=\Sigma^N_{\alpha=1}\tilde{b}_\alpha/N$. The bar denotes averaging, i.e.,

$$\overline{f(\mathbf{R}_\alpha)} = \int_V f(\mathbf{R}_\alpha)\,d\mathbf{R}_\alpha/V. \quad (33)$$

The averaging over the configurations of the scattering nuclei, i.e., over the equilibrium positions, is an independent operation only in the case of a crystal. In the case of a gaseous target or a liquid scatterer, the calculation of the matrix element $\langle i\,|\cdots|\,i\rangle$ already leads to expressions that depends only on the difference of the spatial coordinates. Note that the averaging which we have performed of the optical potential introduces an error into the neutron wave function. By the wave function of the optical model we shall understand a coherent neutron wave, i.e., a wave function averaged over the configurations of the scattering nuclei (see, for example, Ref. 24). This use of the averaged optical potential obviously means that we neglect its fluctuations due to the inhomogeneity of the scattering medium and, in particular, its microscopic structure.

We now turn to the calculation of the correction of second order to the optical potential. We consider the second term in (5):

$$U_{02} = \sum_{\alpha\neq\beta=1}^{N}\left\langle,\,i\left|\,\hat{t}_\alpha\frac{1-\Lambda_i}{d}\,\hat{t}_\beta\,\right|i,\right\rangle. \quad (34)$$

If the spins of the scattering nuclei are uncorrelated, then, taking into account Eqs. (9), (10), and (29), we write

$$U_{02} = \left(\frac{2\pi\hbar^2}{m}\right)^2\sum_{\alpha\neq\beta=1}^{N}\tilde{b}_\alpha\tilde{b}_\beta\left\langle,\,i\left|\,\delta(\mathbf{r}_n-\mathbf{R}_\alpha)\frac{1-\Lambda_i}{d}\delta(\mathbf{r}_n-\mathbf{R}_\beta)\right|i,\right\rangle. \quad (35)$$

We shall assume for simplicity that all the scattering nuclei are the same, i.e., $\tilde{b}_\alpha=\tilde{b}_\beta=\tilde{b}$. In addition, in the expression for U_{02} we ignore the imaginary part of $\tilde{b}$. It is readily seen that allowance for $\mathrm{Im}\tilde{b}$ goes beyond the accuracy of our calculation.

In the momentum representation with respect to the neutron coordinates,

$$\langle \mathbf{q}'\,|\,U_{02}\,|\,\mathbf{q}\rangle = \left(\frac{2\pi\hbar^2}{m}b\right)^2\sum_{\alpha\neq\beta=1}^{N}\int\frac{d\mathbf{k}}{(2\pi)^3}\sum_{\gamma\neq i}\langle i\,|\exp[-i\mathbf{R}_\alpha(\mathbf{q}'-\mathbf{k})]\,|\,\gamma\rangle$$
$$\times\left[E_i+\varepsilon_i+i\eta-E_\gamma-\frac{\hbar^2k^2}{2m}\right]^{-1}\langle\gamma\,|\exp[i\mathbf{R}_\beta(\mathbf{q}-\mathbf{k})]\,|\,i\rangle. \quad (36)$$

Using the above calculations, we can obtain from (36)

$$\langle \mathrm{Im}\,\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T = -\pi\left(\frac{2\pi\hbar^2}{m}b\right)^2 \sum_{\alpha\neq\beta=1}^{N}\int\frac{d\mathbf{k}}{(2\pi)^3}$$

$$\times\left\{\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt\exp(i\omega t)\,[\chi^T_{\alpha\beta}(t;\,\mathbf{q}',\,\mathbf{q},\,\mathbf{k}) - \lim_{t\to\infty}\chi^T_{\alpha\beta}(t;\,\mathbf{q}',\,\mathbf{q},\,\mathbf{k})]\right\}. \quad (37)$$

The function $\chi^T_{\alpha\beta}$ is defined by (19). Setting $\mathbf{q}'=\mathbf{q}=\mathbf{k}_i$ in (37) and lifting the restriction on the summation over α and β (in this way we include in the treatment the part of $\mathrm{Im}U_{01}$ due to inelastic incoherent scattering), we have

$$\langle \mathrm{Im}\,\langle \mathbf{k}_i \,|\, U^{\mathrm{coh}}_{02} \,|\, \mathbf{k}_i\rangle\rangle_T = -\pi\left(\frac{2\pi\hbar^2}{m}b\right)^2$$

$$\times\sum_{\alpha,\,\beta=1}^{N}\int\frac{d\mathbf{k}_f}{(2\pi)^3}\left\{\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt\exp(i\omega t)\,[\chi^T_{\alpha\beta}(t;\,\mathbf{k}_i,\,\mathbf{k}_i,\,\mathbf{k}_f)\right.$$

$$\left.-\lim_{t\to\infty}\chi^T_{\alpha\beta}(t;\,\mathbf{k}_i,\,\mathbf{k}_i,\,\mathbf{k}_f)]\right\} = -\frac{2\pi\hbar^2}{m}N\left[\frac{k_i}{4\pi}\frac{\sigma^{\mathrm{inel}}_{\mathrm{tot}}(\varepsilon_i)}{N}\right]. \quad (38)$$

Here

$$\sigma^{\mathrm{inel}}_{\mathrm{tot}}(\varepsilon_i) = \frac{\hbar^2}{mk_i}b^2\int d\mathbf{k}_f\sum_{\alpha,\,\beta=1}^{N}\left\{\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt\exp(i\omega t)\right.$$

$$\left.\times[\chi^T_{\alpha\beta}(t;\,\mathbf{k}_i,\,\mathbf{k}_i,\,\mathbf{k}_f) - \lim_{t\to\infty}\chi^T_{\alpha\beta}(t;\,\mathbf{k}_i,\,\mathbf{k}_i,\,\mathbf{k}_f)]\right\}. \quad (39)$$

Equations (38) and (39) are a generalization of the optical theorem (24) to the case of a coherently scattering system of N nuclei. Note that (38) and (39) also hold in the case of a local optical potential ($\mathbf{q}'=\mathbf{q}=0$) in a spatially homogeneous medium for neutrons of very low energies ($\mathbf{k}_i=0$) when the condition of calculating the matrix element on the mass shell is fulfilled: $\mathbf{q}'=\mathbf{q}=\mathbf{k}_i$.

We consider the case of practical interest of an harmonic crystal.[5)] In the framework of this model, using the standard procedure,[20,21] we can represent the expression for $\langle \mathrm{Im}\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T$ in the form

$$\langle \mathrm{Im}\,\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T = -\frac{\pi}{\hbar}\left(\frac{2\pi\hbar^2}{m}b\right)^2\sum_{\alpha\neq\beta=1}^{N}\int\frac{d\mathbf{k}}{(2\pi)^3}\exp[-i\mathbf{R}_{\alpha0}(\mathbf{q}'-\mathbf{k})]$$

$$\times\exp[i\mathbf{R}_{\beta0}(\mathbf{q}-\mathbf{k})]\exp[-W(\mathbf{q}'-\mathbf{k})]\exp[-W(\mathbf{q}-\mathbf{k})]$$

$$\times\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\exp(i\omega t)\,[K_{\alpha\beta}(\mathbf{q}'-\mathbf{k},\,\mathbf{q}-\mathbf{k};\,t)-1]. \quad (40)$$

Here

$$W(\mathbf{q}-\mathbf{k}) = \frac{1}{2}\sum_j(2\langle n_j\rangle_T+1)\,|(\mathbf{q}-\mathbf{k})\,\mathbf{S}^j_\alpha|^2; \quad (41)$$

$$K_{\alpha\beta}(\mathbf{q}'-\mathbf{k},\,\mathbf{q}-\mathbf{k};\,t)$$

$$=\exp\{\sum_j[(\langle n_j\rangle+1)\,((\mathbf{q}'-\mathbf{k})\,\mathbf{S}^j_\alpha)\,((\mathbf{q}-\mathbf{k})\,\mathbf{S}^{j*}_\beta)\exp(-i\omega_j t)$$

$$+\langle n_j\rangle_T\,((\mathbf{q}'-\mathbf{k})\,\mathbf{S}^{j*}_\alpha)\,((\mathbf{q}-\mathbf{k})\,\mathbf{S}^j_\beta)\exp(i\omega_j t)]\}, \quad (42)$$

$$\mathbf{S}^j_\alpha = i\sqrt{\frac{\hbar}{2MN\omega_j}}\,\mathbf{e}_{j\alpha}\exp(i\mathbf{f}_j\mathbf{R}_{\alpha0}). \quad (43)$$

Here $\mathbf{R}_{\alpha0}$ is the radius vector to the equilibrium position of the αth nucleus; j is the index of the normal vibration of the crystal; ω_j, $\mathbf{f}_j$, and $\mathbf{e}_j$ are, respectively, the frequency, wave vector, and unit polarization vector of vibration j; $\langle n_j\rangle_T = \{\exp([\hbar\omega_j/(k_BT)]-1)\}^{-1}$ is the mean number of phonons of vibration j at temperature T; k_B is Boltzmann's constant; and M is the mass of a scattering nucleus (it is assumed that all the nuclei are identical).

Expansion of the exponential function in (42) in a power series leads to a representation of $K_{\alpha\beta}$ in the form of a series each of whose terms describes a scattering process with the excitation or absorption of a definite number of phonons. We restrict ourselves to analyzing the contribution of single-phonon processes. In this case,

$$\langle \mathrm{Im}\,\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T = -\frac{\pi}{\hbar}\left(\frac{2\pi\hbar^2}{m}b\right)^2$$

$$\times\sum_{\alpha\neq\beta=1}^{N}\int\frac{d\mathbf{k}}{(2\pi)^3}\exp[-i\mathbf{R}_{\alpha0}(\mathbf{q}'-\mathbf{k})]\exp[i\mathbf{R}_{\beta0}(\mathbf{q}-\mathbf{k})]$$

$$\times\sum_j\{((\mathbf{q}'-\mathbf{k})\,\mathbf{S}^j_\alpha)\,((\mathbf{q}-\mathbf{k})\,\mathbf{S}^{j*}_\beta)\,[\langle n_j\rangle_T+1]\,\delta(\omega-\omega_j)$$

$$+((\mathbf{q}'-\mathbf{k})\,\mathbf{S}^{j*}_\alpha)\,((\mathbf{q}-\mathbf{k})\,\mathbf{S}^j_\beta)\,\langle n_j\rangle_T\,\delta(\omega+\omega_j)\}$$

$$\times\exp[-W(\mathbf{q}'-\mathbf{k})-W(\mathbf{q}-\mathbf{k})].$$

In the case of a spatially homogeneous medium, it is not difficult to average this formula over the distribution $\mathbf{R}_{\beta0}$. Taking into account the relation

$$\frac{1}{NV_{\mathrm{uc}}}\int d\mathbf{R}_{\beta0}\exp[i\mathbf{R}_{\beta0}(\mathbf{q}-\mathbf{q}')] = [(2\pi)^3/(NV_{\mathrm{uc}})]\,\delta(\mathbf{q}-\mathbf{q}'),$$

where V_{uc} is the volume of the unit cell of the crystal, and using the explicit form of the amplitudes of the normal vibrations $\mathbf{S}^j_\alpha$ (43), we have

$$\langle \mathrm{Im}\,\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T = -\pi\frac{(2\pi)^3\,\delta(\mathbf{q}-\mathbf{q}')}{NV_{\mathrm{uc}}}\frac{b^2}{2NM}\left(\frac{2\pi\hbar^2}{m}\right)^2$$

$$\times\sum_{\alpha\neq\beta=1}^{N}\int\frac{d\mathbf{k}}{(2\pi)^3}\exp[i(\mathbf{R}_{\alpha0}-\mathbf{R}_{\beta0})(\mathbf{k}-\mathbf{q})]$$

$$\times\sum_j\frac{|(\mathbf{q}-\mathbf{k})\,\mathbf{e}_j|^2}{\omega_j}\langle n_j\rangle_T\exp[-i\mathbf{f}_j(\mathbf{R}_{\alpha0}-\mathbf{R}_{\beta0})]$$

$$\times\exp[-2W(\mathbf{q}-\mathbf{k})]\,\delta\left[\frac{\hbar k_i^2}{2m}-\frac{\hbar k^2}{2m}+\omega_j\right]. \quad (44)$$

In (44), we have omitted the term that describes processes with the emission of a phonon. In the neutron energy range considered, this term is small. We know that[20,21]

$$\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\exp[i\varkappa(\mathbf{R}_{\alpha0}-\mathbf{R}_{\beta0})] = \frac{(2\pi)^3}{V_{\mathrm{uc}}}N\sum_\tau\delta(\varkappa-2\pi\tau). \quad (45)$$

Here, τ is a vector of the reciprocal lattice. Taking into account this relation, we obtain

$$\langle \mathrm{Im}\,\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T = -\pi\frac{(2\pi)^3\,\delta(\mathbf{q}-\mathbf{q}')}{NV_{\mathrm{uc}}}$$

$$\times\frac{b^2}{2M}\left(\frac{2\pi\hbar^2}{m}\right)^2\int\frac{d\mathbf{k}}{(2\pi)^3}\sum_j\frac{|(\mathbf{q}-\mathbf{k})\,\mathbf{e}_j|^2}{\omega_j}\langle n_j\rangle_T$$

$$\times\delta\left[\frac{\hbar k_i^2}{2m}-\frac{\hbar k^2}{2m}+\omega_j\right]\exp[-2W(\mathbf{q}-\mathbf{k})]$$

$$\times\frac{(2\pi)^3}{V_{\mathrm{uc}}}\left[\sum_\tau\delta[\mathbf{k}-\mathbf{q}-\mathbf{f}_j-2\pi\tau]-\frac{V_{\mathrm{uc}}}{(2\pi)^3}\right]. \quad (46)$$

Note that the summation over all normal vibrations of the lattice can be replaced by the operation

$$\sum_{j=1}^{N}\cdots = \sum_{s=1}^{3}\frac{V_{\mathrm{uc}}N}{(2\pi)^3}\int d\mathbf{f}\ldots.$$

The replacement of the summation over the vectors of the reciprocal lattice by integration suppresses interference effects. Indeed, assuming $\Sigma_\tau\cdots = V_{\mathrm{uc}}\int d\tau\cdots$, we have $\langle \mathrm{Im}\langle \mathbf{q}' \,|\, U_{02} \,|\, \mathbf{q}\rangle\rangle_T = 0$. This result is a con-

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5) To be specific, we write down all formulas for the contribution to the imaginary part of the optical potential of interference scattering, $\mathrm{Im}U_{02}$. Obviously, if we set $\alpha=\beta$ in these expressions we must arrive at the result of the calculation in the first approximation. Lifting the restriction on the summation over α and β, we obtain $\mathrm{Im}(U_{01}+U_{02})$.

sequence of our choice of the expansion (5), when the contribution of inelastic scattering in the incoherent approximation was included in the optical potential U_{01}. Omitting the second term in the square brackets in (46), we arrive at an expression for the imaginary part of the optical potential $\langle\langle \mathbf{q}' | U_{02}^{\mathrm{coh}} | \mathbf{q}\rangle\rangle_T$, calculated with allowance for the coherence of the inelastic scattering of the neutrons. The presence of the δ functions of the wave vectors enables us to integrate over **k**. Bearing in mind that in the considered neutron-energy range we can set $\mathbf{k}_i = 0$, we obtain

$$\langle \mathrm{Im}\,\langle \mathbf{q}' | U_{02}^{\mathrm{coh}} | \mathbf{q}\rangle\rangle_T = -\pi \frac{(2\pi)^3 \delta(\mathbf{q}-\mathbf{q}')}{NV_{\mathrm{uc}}} \frac{Nb^2}{2M} \left(\frac{2\pi\hbar^2}{m}\right)^2$$
$$\times \sum_{s=1}^{3} \sum_{\tau} \int \frac{d\mathbf{f}}{(2\pi)^3} \frac{\exp[-2W(\mathbf{f}+2\pi\tau)]}{\omega_s(f)} \delta\left[\omega_s(f) - \frac{\hbar(\mathbf{f}+2\pi\tau+\mathbf{q})^2}{2m}\right]$$
$$\times \langle n(\omega_s)\rangle_T |(\mathbf{f}+2\pi\tau)\mathbf{e}_s|^2. \qquad (47)$$

Neglect of the nonlocality of $\mathrm{Im}\,U_{02}^{\mathrm{coh}}$ in the coordinate representation corresponds to the approximation $\mathbf{q} = 0$ in (47). In this approximation, we have

$$\langle \mathrm{Im}\,\langle \mathbf{q}' | U_{02}^{\mathrm{coh}} | \mathbf{q}\rangle\rangle_{\mathrm{loc}.T} = -\frac{(2\pi)^3 \delta(\mathbf{q}-\mathbf{q}')}{NV_{\mathrm{uc}}}$$
$$\times Nb^2 \frac{2\pi\hbar^2}{m} \frac{m}{M} \frac{1}{2\pi} \sum_{s=1}^{3} \sum_{\tau} \int d\mathbf{f} \exp[-2W(\mathbf{f}+2\pi\tau)] \langle n(\omega_s)\rangle_T$$
$$\times \frac{|(\mathbf{f}+2\pi\tau)\mathbf{e}_s|^2}{(\mathbf{f}+2\pi\tau)^2} \delta\left[\frac{2m\omega_s(\mathbf{f})}{\hbar} - (\mathbf{f}+2\pi\tau)^2\right]. \qquad (48)$$

Noting that the cross section of coherent inelastic scattering with the absorption of a phonon is given by the expression[20, 21, 25]

$$\sigma_{\mathrm{tot}}^{\mathrm{coh}} = \frac{4\pi b^2}{k_i} \frac{1}{2\pi} \frac{m}{M} \sum_{s=1}^{3} \sum_{\tau} \int d\mathbf{q} \exp[-2W(\mathbf{q}+2\pi\tau)]$$
$$\times \langle n(\omega_s)\rangle_T \frac{(\mathbf{e}_s(\mathbf{q}+2\pi\tau))^2}{(\mathbf{q}+2\pi\tau)^2} \delta\left[(\mathbf{q}+2\pi\tau)^2 - \frac{2m\omega_s(\mathbf{q})}{\hbar}\right], \qquad (49)$$

we can readily obtain the following result (see also (38)):

$$\langle \mathrm{Im}\,\langle \mathbf{q}' | U_{02}^{\mathrm{coh}} | \mathbf{q}\rangle\rangle_{\mathrm{loc}.T} = -\frac{(2\pi)^3 \delta(\mathbf{q}-\mathbf{q}')}{NV_{\mathrm{uc}}} N \frac{2\pi\hbar^2}{m} \frac{k_i \sigma_{\mathrm{tot}}^{\mathrm{coh}}}{4\pi}; \mathbf{k}\to 0. \qquad (50)$$

The cross section σ^{coh} is for a single nucleus. In the coordinate representation,

$$\langle \mathrm{Im}\,\langle \mathbf{r}' | U_{02}^{\mathrm{coh}} | \mathbf{r}\rangle\rangle_{\mathrm{loc}.T} = -\delta(\mathbf{r}-\mathbf{r}') \frac{2\pi\hbar^2}{m} \langle\rho\rangle \frac{k_i \sigma_{\mathrm{tot}}^{\mathrm{coh}}}{4\pi}. \qquad (51)$$

Equations (50) and (51) are the justification for using the optical theorem to calculate the imaginary part of the optical potential. We write (51) for $\sigma_{\mathrm{abs}} = 0$ in the form

$$\langle \underset{k_i\to 0}{\mathrm{Im}}\,\langle \mathbf{r}' | U_{02}^{\mathrm{coh}} | \mathbf{r}\rangle\rangle_{\mathrm{loc}.T} = -\frac{2\pi\hbar^2}{m} \langle\rho\rangle \frac{k_i \sigma_{\mathrm{inel}}^{\mathrm{incoh}}}{4\pi} \left[1 + \frac{\sigma^{\mathrm{interf}}}{\sigma^{\mathrm{incoh}}}\right] \delta(\mathbf{r}-\mathbf{r}'). \qquad (52)$$

As was noted in the introduction, the interference correction to the total cross section of inelastic scattering at very low energies does not exceed 10–20% of the cross section calculated in the incoherent approximation.[14]

Equations (50)–(52) were obtained under the assumption that the imaginary part of the optical potential is a local operator in the coordinates representation. Generally speaking, $\mathrm{Im}\,U_{02}^{\mathrm{coh}}$ can be calculated without recourse to this assumption but directly by means of Eq. (47). However, this method of calculation entails laborious summations over lattice vectors, which determine the interference path, depend essentially on the structure of the crystal, and vary irregularly from substance to substance. Therefore, taking into account the relative smallness of the contribution of interference scattering to $\mathrm{Im}\,U_{02}^{\mathrm{coh}}$, we shall restrict ourselves below to the incoherent approximation when we consider the influence of nonlocality of the optical potential on the propagation of a coherent neutron wave through matter.

Calculations analogous to those made above show that the contribution to the imaginary part of the optical potential [due to the second term in Eq. (12)] has the form

$$\langle \mathrm{Im}\,\langle \mathbf{q}' | U_{01}^{\mathrm{inel}} | \mathbf{q}\rangle\rangle_T = -\pi \frac{(2\pi)^3 \delta(\mathbf{q}-\mathbf{q}')}{NV_{\mathrm{uc}}} \frac{b^2}{2M} \left(\frac{2\pi\hbar^2}{m}\right)^2$$
$$\times \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_j \frac{|(\mathbf{q}-\mathbf{k})\mathbf{e}_j|^2}{\omega_j} \langle n_j\rangle_T \delta\left[\frac{\hbar k_i^2}{2m} - \frac{\hbar k^2}{2m} + \omega_j\right]. \qquad (53)$$

We recall that here b is the real part of the coherent scattering length.

In this formula, we have for simplicity omitted the factor $\exp[-2W]$, which is valid in the case of small momentum transfers. In the coordinate representation, we have accordingly

$$\langle \mathrm{Im}\,\langle \mathbf{r}' | U_{01}^{\mathrm{inel}} | \mathbf{r}\rangle\rangle_T = \frac{1}{(2\pi)^6} \int d\mathbf{q} \int d\mathbf{q}' \exp[i\mathbf{q}'\mathbf{r}'] \exp(-i\mathbf{q}\mathbf{r})$$
$$\times \langle \mathrm{Im}\,\langle \mathbf{q}' | U_{01}^{\mathrm{inel}} | \mathbf{q}\rangle\rangle_T. \qquad (54)$$

Let us find this quantity in the Debye approximation for a crystal of cubic symmetry, when

$$\sum_j (\varkappa \mathbf{e}_j)^2 f(\omega_j) = \frac{\varkappa^2}{3} \int g(\Omega)\, d\Omega f(\Omega) \qquad (55)$$

and the spectrum of frequencies of the normal vibrations is

$$g(\Omega) = \begin{cases} \dfrac{9N\Omega^2}{\Omega_{\max}^3} & \Omega \leqslant \Omega_{\max}; \\ 0 & \Omega > \Omega_{\max}. \end{cases} \qquad (56)$$

Integrating, we obtain

$$\langle \mathrm{Im}\,\langle \mathbf{r}' | U_{01}^{\mathrm{inel}} | \mathbf{r}\rangle\rangle_T = -\frac{\pi}{V_{\mathrm{uc}}} \frac{b^2}{2M} \left(\frac{2\pi\hbar^2}{m}\right)^2$$
$$\times \frac{3}{\Omega_{\max}^3} \int \frac{d\varkappa}{(2\pi)^3} \varkappa^2 \exp[i\varkappa(\mathbf{r}-\mathbf{r}')]$$
$$\times \frac{1}{2\pi^2} \int_0^{k_{\max}} k\, dk \frac{\sin k|\mathbf{r}'-\mathbf{r}|}{|\mathbf{r}'-\mathbf{r}|} \left(\frac{\hbar k^2}{2m} - \frac{\hbar k_i^2}{2m}\right)$$
$$\times \left[\exp\left[\frac{\hbar^2}{2mk_B T}(k^2 - k_i^2)\right] - 1\right]^{-1};$$
$$k_{\max} = \left(\frac{2m}{\hbar}\Omega_{\max} + k_i^2\right)^{1/2}. \qquad (57)$$

In the region of relatively low temperatures, when $\hbar^2 k_{\max}^2/(2mk_B T) \geqslant 1$, the integration over k can be extended to infinity. The integrals which are then obtained have been tabulated. As a result, we have

$$\langle \mathrm{Im}\,\langle \mathbf{r}' | U_{01}^{\mathrm{inel}} | \mathbf{r}\rangle\rangle_T = \frac{\pi}{V_{\mathrm{uc}}} \frac{b^2}{2M} \left(\frac{2\pi\hbar^2}{m}\right)^2 \frac{1}{2\pi^2} \delta^2(\mathbf{r}'-\mathbf{r}) J(\mathbf{r}'-\mathbf{r}), \qquad (58)$$

where

$$J(\mathbf{r}-\mathbf{r}') = \frac{3\sqrt{2\pi}}{4} \frac{\hbar}{m\Omega_{\max}^3} \left[\frac{mk_B T}{\hbar^2}\right]^{7/2} (\mathbf{r}'-\mathbf{r})^2$$
$$\times \sum_{n=1}^{\infty} \frac{1}{n^{7/2}} \exp\left[-\frac{(\mathbf{r}-\mathbf{r}')^2}{2\hbar^2 n} mk_B T\right] - \frac{9\sqrt{2\pi}}{4} \frac{\hbar}{m\Omega_{\max}^3} \left[\frac{mk_B T}{\hbar^2}\right]^{5/2}$$
$$\times \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \exp\left[-\frac{(\mathbf{r}-\mathbf{r}')^2}{2\hbar^2 n} mk_B T\right]. \qquad (59)$$

In the calculation of (59) we have set $k_i = 0$. The error due to this approximation is $\sim\hbar^2 k_i^2/(2mk_BT) = \varepsilon_i/(k_BT) \ll 1$.

In the Schrödinger equation $\mathrm{Im}\langle \mathbf{r}' \mid U_{01}^{\mathrm{inel}} \mid \mathbf{r}\rangle\rangle_T$ enters as the kernel of the integral term, i.e., the equation for a coherent neutron wave contains the integral $\mathrm{i}\int d\mathbf{r}\langle \mathrm{Im}\langle \mathbf{r}' \mid U_{01}^{\mathrm{inel}} \mid \mathbf{r}\rangle\rangle_T \langle \psi_c \langle \mathbf{r}\rangle\rangle$, which corresponds to a nonlocal potential. Substitution into this integral of the expressions (58) and (59) and integration lead to the result

$$\mathrm{i}\int d\mathbf{r}\,\langle \mathrm{Im}\langle \mathbf{r}' \mid U_{01}^{\mathrm{inel}} \mid \mathbf{r}\rangle\rangle_T \langle \psi_c(\mathbf{r})\rangle = -\mathrm{i}\frac{\pi}{V_{\mathrm{uc}}}\frac{b^2}{2M}\left(\frac{2\pi\hbar^2}{m}\right)^2 \frac{3\sqrt{2\pi}}{4}\frac{\hbar}{m\Omega_{\max}^3}\frac{1}{2\pi^2}\left[\frac{mk_BT}{\hbar^2}\right]^{5/2}$$
$$\times\left[\frac{15mk_BT}{\hbar^2}\sum_{n=1}^{\infty}\frac{1}{n^{7/2}}\langle\psi_c(\mathbf{r}')\rangle - 3\sum_{n=1}^{\infty}\frac{1}{n^{5/2}}\nabla_{\mathbf{r}'}^2\langle\psi_c(\mathbf{r}')\rangle\right]. \quad (60)$$

The second term in (60) is due to the nonlocality of the optical potential. It is of order of a fraction $\hbar^2 k_i^2/(2mk_BT) \ll 1$ of the first term. In the region of neutron energies $\varepsilon > U_R$, this correction has the same sign as the first term, while it is of the opposite sign when $\varepsilon < U_R$. Retaining only the first term in (60), we write

$$\mathrm{i}\int d\mathbf{r}\,\langle \mathrm{Im}\langle \mathbf{r}' \mid U_{01}^{\mathrm{inel}} \mid \mathbf{r}\rangle\rangle_T \langle \psi_c(\mathbf{r})\rangle$$
$$= -\mathrm{i}\langle\psi_c(\mathbf{r})\rangle\frac{45}{4}\pi\frac{m}{M}\frac{b^2}{V_{\mathrm{uc}}}\left(\frac{k_BT}{\hbar\Omega_{\max}}\right)^3\hbar\sqrt{\frac{2\pi k_BT}{m}}\,\zeta(7/2), \quad (61)$$

where $\zeta(7/2)$ is the Riemann zeta function.

Remembering that the cross section for scattering of a neutron with the absorption of one phonon calculated in the incoherent approximation has the form[25]

$$\sigma_{\mathrm{inel}}^{\mathrm{incoh}} = \frac{45}{2}\pi\zeta(7/2)\frac{m}{M}\frac{b^2}{v_i}\sqrt{\frac{2\pi k_BT}{m}}\left(\frac{k_BT}{\hbar\Omega_{\max}}\right)^3, \quad (62)$$

where v_i is the velocity of the incident neutrons, we obtain from (61)

$$\frac{1}{\langle\psi_c(\mathbf{r})\rangle}\int d\mathbf{r}'\,\langle \mathrm{Im}\langle \mathbf{r}' \mid U_{01}^{\mathrm{inel}} \mid \mathbf{r}\rangle\rangle_T \langle \psi_c(\mathbf{r}')\rangle = -\frac{\sigma_{\mathrm{inel}}^{\mathrm{incoh}}}{2V_{\mathrm{uc}}}\hbar v_i. \quad (63)$$

This result completes the investigation of the region of applicability of (4). Note that the spatial scale of the nonlocality of the imaginary part of the optical potential due to interference scattering is determined by the phonon mean free path Λ. At sufficiently low temperatures, Λ may be comparable with or even exceed the wavelength of ultracold neutrons.

As model of a scatterer we have above considered an ideal harmonic crystal. Let us now consider briefly how a nonideal nature of the medium affects the excitation spectrum of the scattering system, and through it the inelastic scattering of neutrons. The effect of weak anharmonicity of the vibrations of the scattering nuclei in a lattice can be taken into account approximately by replacing the δ function of the energy (or of $\omega = E/\hbar$) in (44) and the equations derived from it by certain "smoothed" functions with effective width inversely proportional to the phonon lifetime. Obviously, this leads to only small corrections to the cross section of scattering with the absorption of a phonon and to the imaginary part of the optical potential: The imaginary part of the optical potential is determined by an integral over the frequency spectrum of the vibrations and the fine details of the frequency-distribution function in this case are not important.[6)] The presence in the crystal of microscopic defects in cases of practical interest also has no significant influence on the estimate of the cross section of inelastic scattering and $\mathrm{Im}\langle U_0\rangle$. For example, the presence in a crystal lattice of dislocations reduces the number of degrees of freedom associated with the vibrations of the atoms. Accordingly, the effective Debye temperature of the crystal is reduced by[26]

$$\Delta\Theta_D \approx -\rho_D a^2 \Theta_D, \quad (64)$$

where ρ_D is the density of the dislocations and a is the lattice constant. For $\rho_D \approx 10^9 - 10^{10}\ \mathrm{cm}^{-2}$, $a \approx 10^{-8}$ cm,

$$|\Delta\Theta_D|/\Theta_D \approx 10^{-7} - 10^{-6}. \quad (65)$$

Since[25] when $k_B\theta_D \gg k_BT \gg \varepsilon_n$ the cross section for scattering with the absorption of a phonon is

$$\sigma_{(-)}^{(1)} \sim 4\pi(\mathrm{Re}\,b)^2 (k_BT/\varepsilon_n)^{1/2}\frac{m}{M}(T/\Theta_D)^3, \quad (66)$$

the reduction of the effective Debye temperature increases the cross section of inelastic scattering and the imaginary part of the optical potential by

$$\Delta\,\mathrm{Im}\langle U_0\rangle/\mathrm{Im}\langle U_0\rangle \propto \Delta\sigma_{(-)}^{(1)}/\sigma_{(-)}^{(1)} \leqslant 3\cdot 10^{-6}. \quad (67)$$

A greater change in the inelastic scattering cross section is due directly to the contribution of the vibrations of the dislocations. Indeed, the dislocation degrees of freedom increase the phase volume for the low-frequency vibrations of the crystal. If it is assumed that the energy of a quantum of a dislocation vibration $\hbar\omega_0$ is $10^{-6} - 10^{-4}$ eV (Ref. 26), then the correction due to the nonideal nature of the crystal to the cross section of inelastic scattering for $T = 300\,°\mathrm{K}$, $\Theta_D = 1000\,°\mathrm{K}$ ($k_B\Theta_D \gg k_BT \gg \hbar\omega_0 \gg \varepsilon_n$) is

$$\Delta\sigma_{(-)}^{(1)}/\sigma_{(-)}^{(1)} = \rho_D a^2/3\Gamma(7/2)\,[k_BT/(\hbar\omega_0)]^{1/2}(\theta_D/T)^3 \leqslant 10^{-3}; \quad (68)$$
$$\rho_D = 10^{10}\ \mathrm{cm}^{-2}.$$

With increasing temperature, this correction decreases. It is small because of the relatively low density of dislocations. Dynamical effects of this type of departure from an ideal crystal lattice can be estimated in this manner.

Near the surface of the crystal, the density of dislocations may be higher since the surface is a sink for different kinds of lattice defects. The surface of the crystal is itself a two-dimensional defect that distorts the frequency spectrum of the vibrations of the atoms in the surface layer of the lattice. The weakening of the coupling of these atoms can be taken into account in a very crude approximation by introducing a lower effective Debye temperature Θ_D' satisfying $\Theta_D \gg T \gg \Theta_D'$. This slackening of the lattice rigidity and increase in the amplitude of the atomic vibrations disappears over $\lesssim 10$ atomic layers.[27] A simple calculation shows that an increase of σ^{inel} by 10 times leads to the following restriction on Θ_D':

[6)]Oscillations of the matter density in an anharmonic crystal are due not only to the motion of phonons but are also associated with fluctuations in the temperature of the scattering volume; these lead to the appearance of neutrons that are scattered quasielastically. In the case of a solid reflector, the ratio of the intensities of the quasielastically and inelastically scattered neutrons is small and of order of magnitude $(c_p - c_v)/c_v$.

$$(T/\Theta_D')^{1/2} \gg (2.25\Gamma(1/2)/48)\, 10^4\, (T/\Theta_D)^3. \tag{69}$$

Both for $\Theta_D > \Theta_D' \gg T$ and for $\Theta_D \gg T \gg \Theta_D'$ one can arrive at unreal values of the amplitudes of the atomic vibrations near the surface.[27] One can consider a different model to estimate the softening of the vibrations of the atom in the surface layer by assuming that this "softening" affects only those vibrations that are polarized along the normal to the surface. In the framework of this model, the surface layer can be regarded as part of an anisotropic laminated crystal with two effective Debye temperatures $\Theta_D^{\parallel} \ll \Theta_D^{\perp} = \Theta_D$. It is well known (see, for example, Ref. 28) that for $\Theta_D^{\perp} \gg T \gg \Theta_D^{\parallel}$ the largest phase volume corresponds to vibrations polarized in the plane parallel to the surface, and the so-called bending vibrations. The distribution functions of these groups of vibrations have the form

$$g_{\perp}(\omega) = (4/3)(\omega/\omega_{\perp}^2); \tag{70}$$

$$\omega_{\perp} = k\Theta_D^{\perp}/\hbar.$$
$$g_{\parallel}(\omega) = 1/3\omega_{\perp}. \tag{71}$$

It is not difficult to verify that vibrations of the first type make the following contribution to the cross section of inelastic scattering:

$$\sigma_{(-)}^{\perp} = 0.1\sigma_{(-)}^{(D)}(8/45)\,[\hbar\omega_{\perp}/(k_B T)] \quad (k_B T \ll \hbar\omega_{\perp}). \tag{72}$$

The contribution of the bending vibrations is

$$\sigma_{(-)}^{\parallel} = 0.1\sigma_{(-)}^{(D)}(4/135)\quad [\hbar\omega_{\perp}/(k_B T)]^2.^{7)} \tag{73}$$

Here, $\sigma_{(-)}^{D}$ is the total cross section of inelastic scattering on volume vibrations of the crystal in the Debye approximation:

$$\sigma_{(-)}^{D} = \sigma_s(m/M)\sqrt{k_B T/\varepsilon_H}\; 3\Gamma(7/2)\,(T/\Theta_D)^3;$$

the factor 0.1 is the relative fraction of the number of degrees of freedom associated with the atoms in the surface layer; $\sigma_s = 4\pi\,\mathrm{Re}\,b^2$.

Thus, these various mechanisms of removal of ultracold neutrons by the inelastic interaction with the atoms of the crystal lattice do not explain the anomalously short lifetime of ultracold neutrons in bottles.

There are two corrections to the real part of the optical potential (2). First, the correction to the pseudopotential approximation, which in order of magnitude is $b/0.1a$ (b is the neutron scattering length and a is the mean distance between the scattering nuclei in the matter). This correction can in particular be obtained by estimating the real part of the second term in (11). Second, there is a correction due to interference inelastic scattering. It is described by the real part of the second term in Eq. (5). As is readily verified, if one does not restrict oneself to calculating the imaginary part, then (15)–(21), (37), and (40) can also be used to analyze the real part of the optical potential if in these equations $(-1/2\hbar)\int_{-\infty}^{\infty} dt \exp(i\omega t)\cdots$ is replaced by $(-i/\hbar)\int_0^{\infty} dt \exp[(i\omega t - \eta)t]\cdots$. We give the result of these estimates.[10] In the incoherent approximation, i.e., when allowance is made for only the contribution of the second term in (11),

$$\Delta\,\mathrm{Re}\,\langle\langle \mathbf{r}' | U_{02}^{\mathrm{incoh}} | \mathbf{r}\rangle\rangle_T = \frac{2\pi\hbar^2}{m} b \frac{1}{V_{uc}}\,\delta(\mathbf{r}-\mathbf{r}')$$
$$\times b\left[\frac{2\pi}{3}\frac{\hbar}{MN}\int \frac{g(\Omega)\,d\Omega}{\Omega}(2n_\Omega+1)\right]^{-1/2}. \tag{74}$$

Thus, the correction to the optical potential U_{R_0} (2) in the approximation of the Fermi pseudopotential is a fraction of U_{R_0} equal to

$$b\left[\frac{2\pi}{3}\frac{\hbar}{MN}\int \frac{g(\Omega)\,d\Omega}{\Omega}(2n_\Omega+1)\right]^{-1/2} \sim b\left[2\pi\frac{\hbar}{M\langle\Omega\rangle}\right]^{-1/2}$$
$$\sim \frac{b}{0.1a} = 10^{-3} - 10^{-4}.$$

Allowance for the correction for inelastic interference scattering does not change the magnitude of this estimate.

Above, we have considered the influence of the phonon field on the propagation of a neutron in matter in the calculation of the self-energy part of the Green's function of the neutron field. In principle, there may be exchange of a phonon between two neutrons propagating in the matter. This process leads to a weak attraction between them.[10] Calculation of the amplitude of elastic scattering of a neutron on a neutron due to the exchange of a virtual phonon leads to the following estimate for the potential energy of the interaction between the neutrons in the matter:

$$\mathrm{Re}\int d\mathbf{r} U_{\mathrm{eff}}(\mathbf{r}) \exp(i\varkappa\mathbf{r}) = -\frac{3}{2}\frac{\varkappa^2}{M}\frac{1}{V_{uc}}$$
$$\times\left(\frac{2\pi\hbar^2}{m}b\right)\exp[-2W(\varkappa)]\frac{2}{\Omega_{\max}^2}, \tag{75}$$

where $\hbar\varkappa$ is the change in the momentum of one of the neutrons due to scattering. The ratio of this quantity to the energy of a neutron confined in a bottle is

$$x = \mathrm{Re}\int d\mathbf{r} U_{\mathrm{eff}}(\mathbf{r}) \exp(i\varkappa\mathbf{r}) \Big/ \int d\mathbf{r}\,\langle U_{R0}\rangle \exp(i\varkappa\mathbf{r})$$
$$= -3\frac{m}{M}\frac{(\hbar\varkappa)^2/m\,[(2\pi\hbar^2)/m]\,b\rho}{(\hbar\Omega_{\max})^2} \approx -3\,(m/M)\,(\langle U_R\rangle/\hbar\Omega_{\max})^2 \ll 1. \tag{76}$$

Thus, the intensity of this interaction between the neutrons is extremely small. In the case of ultracold neutrons in a bottle, this interaction is even weaker. Note that the interaction range when allowance is made for interference scattering is determined by the phonon mean free path, which under favorable conditions may be 10^{-5}–10^{-4} cm (incoherent scattering leads to interaction with an effective range 10^{-9} cm).

Our treatment refers to the case of an unbounded medium. The expressions obtained for the optical potential enable us to calculate the refractive index of the scattering medium:

$$n = k_I/k_0 = \sqrt{1 - \langle U_0\rangle/\varepsilon_n}, \tag{77}$$

where $\mathbf{k}_0 = m\mathbf{v}/\hbar$ is the neutron wave vector in vacuum.[6,29] In the case of a homogeneous medium, we know that

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7) The expressions (68), (72), and (73) were obtained in the approximation of single-phonon scattering. Allowance for the contribution of many-phonon processes for $T \gg \Theta_D$, $\hbar\omega_0/k_B$ in the model of an ideal gas does not essentially change the estimates. The enhancement in the anomaly in the removal of ultracold neutrons with increasing penetration depth $\varkappa_0$ as $E_n \to U_0$ also indicates the ineffectiveness of the "softening" of the vibrations of the atoms near the surface of the reflector.

tion of a coherent neutron wave. Above, we have already briefly discussed how inhomogeneity of the medium changes the nature of the excitations in the scatterer and thus affects the inelastically scattered wave. In addition, fluctuations in the density of the scattering matter directly interact with the neutron field and give rise to an elastically scattered neutron wave. An obvious example illustrating this assertion is the boundary of the scattering medium. First, the presence of the boundary changes the vibration spectrum of the crystal and, second, it leads to the appearance of an elastically reflected neutron wave. Let us now consider processes of this last type.

We begin the exposition by studying weak single scattering of a neutron wave by an inhomogeneous material which occupies the volume V. To calculate the scattered neutron field $\psi_s(\mathbf{r})$, we restrict ourselves to the Born approximation of perturbation theory. We can write

$$\psi_s(\mathbf{r}) \approx \int_V G_0(\mathbf{r}|\mathbf{r}')\, \delta U(\mathbf{r}')\, \psi_0(\mathbf{r}')\, d\mathbf{r}'. \tag{89}$$

Here, δU is the perturbation due to the static inhomogeneities of the medium. The volume V within which δU is nonzero may be in vacuum or surrounded by homogeneous matter.

The Green's function of the wave equation without perturbation, $G_0(\mathbf{r}|\mathbf{r}_0)$, has the form[8)]

$$G_0(\mathbf{r}|\mathbf{r}_0) = -\frac{m}{2\pi\hbar^2}\frac{\exp[ik|\mathbf{r}-\mathbf{r}_0|]}{|\mathbf{r}-\mathbf{r}_0|}. \tag{90}$$

The wave number k of a neutron in a homogeneous medium is determined by the energy ε_n of the neutron and its potential energy in the unperturbed medium $\langle U\rangle$:

$$k = \frac{1}{\hbar}\sqrt{2m(\varepsilon_n - \langle U\rangle)}. \tag{91}$$

In this section, we restrict ourselves to analyzing the scattering of very cold neutrons. We discuss the propagation of ultracold neutrons below. We shall assume that the incident wave can be represented in the form

$$\psi_0(\mathbf{r}) = \tilde{\psi}_0(\mathbf{r}) \exp[ik|\mathbf{R}-\mathbf{r}|], \tag{92}$$

where $\mathbf{R}$ is the point at which the neutron source is placed. The complex amplitude $\tilde{\psi}_0(\mathbf{r})$ varies appreciably over distances $L = V^{1/3} \gg 2\pi/k$.

We consider the correlation function of the scattered wave $K_{\psi\psi}(\mathbf{r}'|\mathbf{r}) = \langle \psi_s^*(\mathbf{r}')\psi_s(\mathbf{r})\rangle$. The angular brackets denote averaging with respect to the distribution law of the inhomogeneities in the medium. The function $K_{\psi\psi}$ is an important characteristic of the neutron field. For example, $K_{\psi\psi}(\mathbf{r}_1|\mathbf{r}_1)$ is the mean density of scattered neutrons, $\langle|\psi_s(\mathbf{r}_1)|^2\rangle$, at the point $\mathbf{r}_1$. Taking into account the well known quantum-mechanical expression for the current density:

$$\mathbf{j}(\mathbf{r}) = \frac{i\hbar}{2m}[\psi(\mathbf{r})\nabla_{\mathbf{r}}\psi^*(\mathbf{r}) - \psi^*(\mathbf{r})\nabla_{\mathbf{r}}\psi(\mathbf{r})], \tag{93}$$

we can show that

$$\langle \mathbf{j}(\mathbf{r}_1)\rangle = \frac{i\hbar}{2m}(\nabla_{\mathbf{r}_1} - \nabla_{\mathbf{r}_2}) K_{\psi\psi}(\mathbf{r}_1|\mathbf{r}_2)|_{\mathbf{r}_1=\mathbf{r}_2}. \tag{94}$$

Taking into account the expressions (89)–(92), we can express the function $K_{\psi\psi}(\mathbf{r}_1|\mathbf{r}_2)$ in terms of the correlation function of the perturbation δU by means of the relation

$$K_{\psi\psi}(\mathbf{r}_1|\mathbf{r}_2) = (m/2\pi\hbar^2)^2 \int_V\int_V d\mathbf{r}'\, d\mathbf{r}''\, \frac{\exp[-ik|\mathbf{r}_1-\mathbf{r}'|]}{|\mathbf{r}_1-\mathbf{r}'|} \times \frac{\exp[ik|\mathbf{r}_2-\mathbf{r}''|]}{|\mathbf{r}_2-\mathbf{r}''|} K_{UU}(\mathbf{r}'|\mathbf{r}'') \times \tilde{\psi}_0^*(\mathbf{r}')\tilde{\psi}_0(\mathbf{r}'')\exp[-ik|\mathbf{R}-\mathbf{r}'|]\exp[ik|\mathbf{R}-\mathbf{r}''|]. \tag{95}$$

Here,

$$K_{UU}(\mathbf{r}_1|\mathbf{r}_2) = \langle \delta U(\mathbf{r}_1)\,\delta U(\mathbf{r}_2)\rangle \tag{96}$$

is the correlation function of the perturbation δU.

If the scattering medium is spatially homogeneous (homogeneous on the average) and the scattering volume is sufficiently large: $L = V^{1/3} \gg l_0$ (l_0 is the correlation length of the perturbation δU, i.e., the distance over which K_{UU} vanishes), then

$$K_{UU}(\mathbf{r}_1|\mathbf{r}_2) \equiv K_{UU}(\mathbf{r}_1 - \mathbf{r}_2). \tag{97}$$

If the distances from the scattering volume to the source R and to the point of observation r are large, so that $kl_0^2/r,\ kl_0^2/R \ll 1$, i.e., when the conditions for Fraunhofer diffraction are realized, the expression (95) can be written approximately in the form

$$K_{\psi\psi}(\mathbf{r}_1|\mathbf{r}_2) = \left(\frac{m}{2\pi\hbar^2}\right)^2 \int_V d\mathbf{r}' \frac{|\tilde{\psi}_0(\mathbf{r}')|^2}{|\mathbf{r}_1-\mathbf{r}'||\mathbf{r}_2-\mathbf{r}'|}\exp[-ik|\mathbf{r}_1-\mathbf{r}'|] \times \exp[ik|\mathbf{r}_2-\mathbf{r}'|]\int d\boldsymbol{\rho}\, K_{UU}(\boldsymbol{\rho})\exp(i\boldsymbol{\varkappa}\boldsymbol{\rho}), \tag{98}$$

where $\boldsymbol{\varkappa} = \mathbf{k}_f(\mathbf{r}) - \mathbf{k}_i(\mathbf{r}')$ is the change in the wave vector resulting from the scattering. If the volume is sufficiently large, the integral of K_{UU} in (98) is the spectral density of $\Phi_U(\mathbf{q})$, and

$$K_{UU}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{q}\exp(-i\mathbf{q}\mathbf{r})\,\Phi_U(\mathbf{q}). \tag{99}$$

Thus

$$K_{\psi\psi}(\mathbf{r}_1|\mathbf{r}_2) = \left(\frac{m}{2\pi\hbar^2}\right)^2 \int_V d\mathbf{r}' \frac{|\tilde{\psi}_0(\mathbf{r}')|^2}{|\mathbf{r}_1-\mathbf{r}'||\mathbf{r}_2-\mathbf{r}'|} \times \exp[-ik|\mathbf{r}_1-\mathbf{r}'|]\exp[ik|\mathbf{r}_2-\mathbf{r}'|]\,\Phi_U(\boldsymbol{\varkappa}). \tag{100}$$

Here

$$\Phi_U(\boldsymbol{\varkappa}) \approx \int_V K_{UU}(\boldsymbol{\rho})\exp(i\boldsymbol{\varkappa}\boldsymbol{\rho})\,d\boldsymbol{\rho}. \tag{101}$$

Let us calculate the scattered flux from the infinitesimal volume dV'. Taking into account (100), we differentiate in the expression (94). For $k|\mathbf{r}_1 - \mathbf{r}'|$, $k|\mathbf{r}_2 - \mathbf{r}'| \gg 1$ we can differentiate only the exponentials. As a result, we obtain

$$\langle d\mathbf{j}\rangle = \left(\frac{m}{2\pi\hbar^2}\right)^2 dV' \frac{\hbar\mathbf{k}_f(\mathbf{r}')}{m}\frac{|\tilde{\psi}_0(\mathbf{r}')|^2}{|\mathbf{r}_1-\mathbf{r}'|^2}\Phi(\boldsymbol{\varkappa}). \tag{102}$$

The density of the incident flux is

$$j_0 = [\hbar k_i(\mathbf{r}')/m]\,|\tilde{\psi}_0(\mathbf{r}')|^2. \tag{103}$$

From (102) and (103) we obtain directly the following expression for the mean differential scattering cross section:

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle = |\mathbf{r}-\mathbf{r}'|^2(\mathbf{k}_f\langle d\mathbf{j}\rangle)/j_0|k_f = [m/(2\pi\hbar^2)]^2\, dV'\Phi_U(\boldsymbol{\varkappa}). \tag{104}$$

To simplify the expressions, we introduce the new quantities

[8)]The use in (87) of more complicated expressions for the Green's function $G_0(\mathbf{r}|\mathbf{r}')$ and the wave function $\Psi_0(\mathbf{r})$ corresponding to motion in potentials that vary smoothly in space leads to the well known results of the theory of scattering in the distorted-wave Born approximation.[11,15]

$$\delta\gamma(\mathbf{r}) = [m/(2\pi\hbar^2)]\,\delta U(\mathbf{r}) \tag{105}$$

and

$$\Phi_\gamma(\varkappa) = [m/(2\pi\hbar^2)]^2\,\Phi_U(\varkappa). \tag{106}$$

Then

$$\langle d\sigma/d\Omega\rangle = dV'\Phi_\gamma(\varkappa). \tag{107}$$

If the scattering volume V is small compared with r^3 and R^3, then the angular distribution of the neutrons scattered by the complete volume is given by

$$\langle d\sigma/d\Omega\rangle = V\Phi_\gamma(\varkappa) = V\int d\boldsymbol{\rho}\exp(i\varkappa\boldsymbol{\rho})\,K_{\gamma\gamma}(\boldsymbol{\rho}). \tag{108}$$

Recall that although $V \gg l_0^3$ the value of $V^{1/3}$ is small compared with the mean free path against scattering on inhomogeneities of the medium. Accordingly, the cross section per scattering nucleus is

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_1 = \frac{1}{N_{\text{nuc}}}\left\langle\frac{d\sigma}{d\Omega}\right\rangle = \frac{V}{N_{\text{nuc}}}\Phi_\gamma(\varkappa). \tag{109}$$

For $\varkappa l_0 \ll 1$ in the case of an isotropic medium, when $K_{\gamma\gamma}(\rho) \equiv K_{\gamma\gamma}(|\rho|)$, we have

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_1 \approx \frac{V}{N_{\text{nuc}}}\int d\rho K_{\gamma\gamma}(\rho)\left[1-\frac{\varkappa^2}{6}\frac{\int d\rho K_{\gamma\gamma}(\rho)\,\rho^2}{\int d\rho K_{\gamma\gamma}(\rho)}\right]. \tag{110}$$

The expressions (107)—(110) can be obtained by directly averaging the well known quantum-mechanical formula for the scattering cross section in the Born approximation.[4] It follows from (107)—(110) that the intensity of the neutron wave scattered through angle Θ is determined by the spectral component of the inhomogeneity with spatial scale

$$l(\theta) = 2\pi/\varkappa = \pi/(k\sin\theta/2) = \lambda/(2\sin\theta/2). \tag{111}$$

Because the scattering volume is finite, neighboring spectral components (in an interval $\sim\lambda/V^{1/3}$ (Ref. 32)) participate in the scattering through a given angle.

The correlation function $K_{\gamma\gamma}$ (or its first moments $\int K_{\gamma\gamma}(\rho)\rho^n d\boldsymbol{\rho}$) can be recovered from the experimentally determined angular distribution of scattered neutrons by Fourier analysis. We estimate the cross section of fluctuation scattering for a simple model of localized identical scatterers (we shall call each of these scatterers an impurity). In this case

$$\delta\gamma(\mathbf{r}) = \sum_{j=1}^{N}\delta\gamma_0(\mathbf{r}-\mathbf{R}_j), \tag{112}$$

where $\mathbf{R}_j$ is the point at which impurity j is situated; $\delta\gamma_0$ describes the action of one scatterer. Accordingly, for the correlation function $K_{\gamma\gamma}(\mathbf{r}-\mathbf{r}')$ we have

$$K_{\gamma\gamma}(\mathbf{r}-\mathbf{r}') = \sum_{j,\,j'=1}^{N}\langle\delta\gamma_0(\mathbf{r}-\mathbf{R}_j)\,\delta\gamma_0^*(\mathbf{r}'-\mathbf{R}_{j'})\rangle$$
$$= \sum_{j,\,j'=1}^{N}\frac{1}{(2\pi)^6}\int\int d\mathbf{q}\,d\mathbf{q}'\,\langle\exp[-i\mathbf{q}(\mathbf{r}-\mathbf{R}_j)]\exp[i\mathbf{q}'(\mathbf{r}'-\mathbf{R}_{j'})]\rangle$$
$$\times\,\delta\tilde{\gamma}_0(\mathbf{q})\,\delta\tilde{\gamma}_0^*(\mathbf{q}'). \tag{113}$$

Here

$$\delta\tilde{\gamma}_0(\mathbf{q}) = \int d\mathbf{r}\exp(i\mathbf{q}\mathbf{r})\,\delta\gamma_0(\mathbf{r}).$$

If the positions of the scatterers are uncorrelated, the terms with $j \neq j'$ vanish, and the expression for $K_{\gamma\gamma}$ takes the form

$$K_{\gamma\gamma}(\mathbf{r}-\mathbf{r}') = \frac{N}{V}\frac{1}{(2\pi)^3}\int d\mathbf{q}\exp[-i\mathbf{q}(\mathbf{r}-\mathbf{r}')]\,|\delta\tilde{\gamma}_0(\mathbf{q})|^2. \tag{114}$$

Accordingly, for the spectral density of such an inhomogeneous medium we have

$$\Phi_{\gamma\gamma}(\varkappa) = \frac{N}{V}|\delta\tilde{\gamma}_0(\varkappa)|^2. \tag{115}$$

We substitute (115) into (109). Then for the differential scattering cross section per nucleus we obtain the expression

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_1 = \frac{N}{N_{\text{nuc}}}|\delta\tilde{\gamma}_0(\varkappa)|^2. \tag{116}$$

The individual scattering centers, each of which contain on the average N_{nuc}/N nuclei, scatter incoherently in our model. If the scatterer is a homogeneous sphere of radius R_0, so that

$$\delta\gamma_0(\mathbf{r}) = \begin{cases}\Gamma^{(0)} & r \leqslant R_0;\\ 0 & r > R_0,\end{cases} \tag{117}$$

then

$$\delta\tilde{\gamma}_0(\mathbf{q}) = (4\pi\Gamma^{(0)}/q^3)[\sin qR_0 - qR_0\cos qR_0]; \tag{118}$$
$$\approx (4\pi R_0^3/3)\,\Gamma^{(0)}, \qquad qR_0 \ll 1; \tag{119}$$
$$\approx -(4\pi\Gamma^{(0)}/q^2)\,R_0\cos qR_0, \qquad qR_0 \gg 1 \tag{120}$$

and

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_1 = \left(\frac{N}{N_{\text{nuc}}}\right)|\Gamma^{(0)}|^2(4\pi R_0^3/3)^2, \qquad qR_0 \ll 1;$$
$$= \left(\frac{N}{N_{\text{nuc}}}\right)|\Gamma^{(0)}|^2\left(\frac{2\pi}{q^4}\right)4\pi R_0^2, \quad qR_0 \gg 1. \tag{121}$$

Suppose the perturbation is due to fluctuations of the matter density ρ, and $\Delta\rho_0$ is the amplitude of the fluctuations of this quantity. Then $\langle(\Delta\rho)^2\rangle = (\Delta\rho_0)^2 N[4\pi R_0^3/(3V)]$. Since $\delta\gamma \approx \operatorname{Re} b\,\Delta\rho$ (for $\operatorname{Im} b = 0$), we obtain for $\varkappa R_0 \ll 1$

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_1 = (\operatorname{Re} b)^2\frac{\langle(\Delta\rho)^2\rangle}{\langle\rho\rangle^2}\langle\rho\rangle^2\frac{4\pi R_0^3}{3}\frac{V}{N_{\text{nuc}}}$$
$$= (\operatorname{Re} b)^2\frac{\langle(\Delta\rho^2)\rangle}{\langle\rho\rangle^2}N_{\text{nuc}}^2\frac{4\pi R_0^3}{3V}\frac{1}{N_{\text{nuc}}}.$$

The ratio $4\pi R_0^3/(3V)$ is the fraction of the matter that scatters coherently, and the appearance of the factor $N_{\text{nuc}}^2 4\pi R_0^3/(3V)$ has a transparent physical origin. Thus, we can write finally

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle_1 = (\operatorname{Re} b)^2\left\langle\left(\frac{\Delta\rho}{\langle\rho\rangle}\right)^2\right\rangle\langle\rho\rangle\frac{4\pi R_0^3}{3}$$
$$= (\operatorname{Re} b)^2\left\langle\left(\frac{\Delta\rho}{\langle\rho\rangle}\right)^2\right\rangle\frac{4\pi R_0^3}{3v_0}, \tag{122}$$

where v_0 is the volume per scattering nucleus.

In the approximation we are considering ($\varkappa R_0 \ll 1$) the scattering is isotropic and the total cross section of fluctuation scattering per scattering nucleus is

$$\langle\sigma\rangle_1 = 4\pi(\operatorname{Re} b)^2\left\langle\left(\frac{\Delta\rho}{\langle\rho\rangle}\right)^2\right\rangle\left[\frac{4\pi R_0^3}{3v_0}\right]. \tag{123}$$

The possibility of experimental investigation of the structure of the inhomogeneities by measuring the transmission coefficient of very cold neutrons through matter layers has been discussed by Steyerl.[4] It follows from his analysis that such an approach gives the same information about the structure of the target as a measurement of the angular distribution in the case of scattering through small angles in the region of thermal energies. Measurement of the total cross section has a number of methodological advantages over measurement

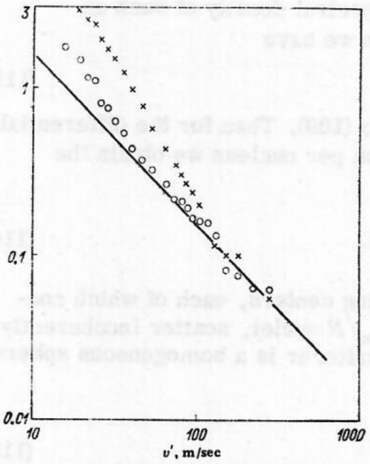

FIG. 1. Total macroscopic scattering cross section of neutrons in $Al_{0.92}Zn_{0.08}$ as a function of the velocity[4]: the open circles are for material annealed at 250 °C; the crosses are for a material after 44-hour aging at 110 °C; the straight line is the $1/v'$ law that holds in a homogeneous medium; v' is the velocity of the neutrons in the material of the target.

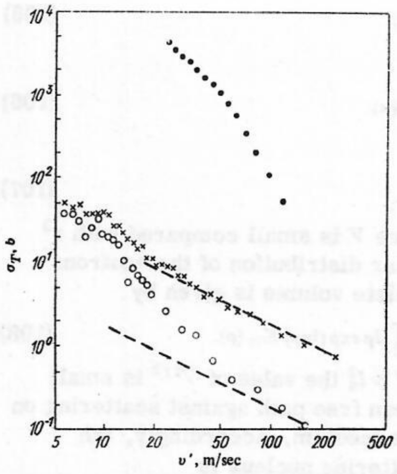

FIG. 2. Total cross section of scattering of neutrons in graphite as a function of the velocity. The black circles are electrographite at 295 °K; the crosses are for pyrolytic graphite at 295 °K; the open circles are for the same material at 100 °K; the dashed lines correspond to the law $1/v'$ (Ref. 5).

of the angular distribution.[4] The energy interval for the incident neutrons is determined by the sizes of the particles that one intends to investigate [see the discussion of (111)]. Experimentally, one has observed an appreciable deviation of the cross section from the law $1/v_n$ for the case of passage of neutrons through pyrolytic graphite, electrographite, and $Al_{0.92}Zn_{0.08}$. This result was interpreted as manifestation of an appreciable contribution of fluctuation scattering on an inhomogeneous target[4,5] (Figs. 1 and 2). Using (123), let us find the mean free path l_s of very cold neutrons in an inhomogeneous medium:

$$l_s=\frac{1}{4\pi(\mathrm{Re}\,b)^2}\left\langle\left(\frac{\Delta\rho}{\langle\rho\rangle}\right)\right\rangle^{-2}\frac{3v_0^2}{4\pi R_0^3}. \qquad (124)$$

In the case of ultracold neutrons, the concept of a mean free path is meaningless. The wave number of the very cold neutrons acquires an imaginary correction $\mathrm{Im}k$, and $\mathrm{Im}k=1/l_s$. If the source of the neutrons and the region in which the flux of scattered neutrons is detected are outside the volume V containing the scattering impurities, it is necessary to take into account the reflection and refraction of the neutron wave on the boundary of this volume.[18]

Above, we have considered scattering on a system of randomly placed inhomogeneities. Strictly speaking, there are always correlations in the positions of such scattering centers, and the scattering is partly coherent. Let us consider the contribution of this coherent scattering. The intensity of the scattered neutron wave if the volume V contains N identical scatterers is proportional to

$$I(\varkappa)=f^2\left\{N+\sum_{k\neq j=1}^{N}\exp\left[i\varkappa(\mathbf{R}_j-\mathbf{R}_k)\right]\right\}, \qquad (125)$$

where f is the amplitude of scattering on an individual center.

Averaging this expression over the distribution of the coordinates $\{\mathbf{R}_j\}$, we obtain

$$\overline{I(\varkappa)}=f^2\left\{N+N(N-1)\int_V\int\frac{d\mathbf{R}_j\,d\mathbf{R}_k}{V^2}\times W(\mathbf{R}_j-\mathbf{R}_k)\exp\left[i\varkappa(\mathbf{R}_j-\mathbf{R}_k)\right]\right\}. \qquad (126)$$

Here, $W(\mathbf{R}_j-\mathbf{R}_k)d\mathbf{R}_k/V$ is the probability of finding the scattering element near the point $\mathbf{R}_k$ if a further impurity is already situated at the point $\mathbf{R}_j$. It is convenient to write the expression (126) in the form

$$\overline{I(\varkappa)}=f^2\left\{N-N(N-1)\times\int_V\int\frac{d\mathbf{R}_j\,d\mathbf{R}_k}{V^2}\exp\left[i\varkappa(\mathbf{R}_j-\mathbf{R}_k)\right]\left[1-W(\mathbf{R}_j-\mathbf{R}_k)\right]+N(N-1)\int_V\exp(i\varkappa\mathbf{R}_j)\frac{d\mathbf{R}_j}{V}\int_V\exp(-i\varkappa\mathbf{R}_k)\frac{d\mathbf{R}_k}{V}\right\}. \qquad (127)$$

The first term describes purely incoherent scattering on N randomly distributed centers. The second term takes into account the correlations in the position of the different inhomogeneities and describes the contribution of the coherent part of the scattered neutron wave. The last term is nonzero in the region of small solid angle near the direction of the incident neutron beam ($\sim\lambda/L$). It describes the wave that is coherently scattered by the complete volume $V=L^3$. In cases of practical interest, when the volume V is macroscopic, this term can be omitted.

As an example, let us consider a system of $N(N\gg 1)$ spherical scattering particles of radius R_0. Since these particles cannot penetrate each other, $W(\mathbf{r})=0$ for $r\leqslant 2R_0$. Otherwise, their distribution over the volume V is arbitrary, i.e., $W(\mathbf{r})=1$ for $r>2R_0$. In this case,

$$\overline{I(\varkappa)}=Nf^2\left[1-8N\delta\tilde{\gamma}_0(\varkappa)\right]. \qquad (128)$$

The definition of $\delta\tilde{\gamma}_0(\varkappa)$ is given by (118) for $\Gamma^0=1/V$. The fraction of the coherently scattered wave in the intensity $\tilde{I}(\varkappa)$ decreases with decreasing density N/V of the scattering centers.

We now consider the interaction of the neutron wave with an impurity implanted in the target when this interaction is fairly strong and the Born approximation does not apply. Problems of this kind were considered in

$$n^2 - 1 = -(v_{gr}/v)^2 [1 - i(u/v_{gr})^2], \tag{78}$$

where

$$v_{gr} = (2\hbar/m)\sqrt{\pi N \operatorname{Re} b}; \; u = \sqrt{(\hbar/m) v \sigma_{rem}(v) N}; \; \sigma_{rem} = \sigma_{abs} + \sigma_{inel}. \tag{79}$$

These expressions and formulas, which are analogous to the well known Fresnel formulas in optics, enable us to find the coefficient of reflection of neutrons from a planar boundary of the scattering medium. This reflection coefficient in the case of neutrons impinging at angle θ from the vacuum has the form[25,29,30]

$$R(\theta) = \frac{\cos\theta - i\sqrt{-(n^2-1) - \cos^2\theta}}{\cos\theta + i\sqrt{-(n^2-1) - \cos^2\theta}} = \frac{\cos\theta - i\sqrt{(v_{gr}/v)^2[1 - i(u/v_{gr})^2] - \cos^2\theta}}{\cos\theta + i\sqrt{(v_{gr}/v)^2[1 - i(u/v_{gr})^2] - \cos^2\theta}}. \tag{80}$$

In the case of scattering of neutrons on small volumes of matter, the cross section can be found by solving the Schrödinger equation with an optical potential. The cross section for the removal of neutrons for this volume is determined by means of the relation[11]

$$\Sigma_{rem} = -[2/(\hbar v)] \operatorname{Im} [(\varphi_p^+, U_0 \varphi_p)]. \tag{81}$$

Here, φ_p is the neutron wave function in the optical potential U_0; v is the velocity of the neutron in vacuum. We emphasize that the cross section (81) is calculated for the complete scattering volume. The removal cross section per nucleus has only an ancillary value in intermediate calculations. This parameter can be found by extrapolating the values of the removal cross section at energies $\varepsilon \gg \langle U_0 \rangle$ to the region of very cold and ultracold neutrons.

We express the coefficient of "absorption" of neutrons in terms of the removal cross section Σ_{rem}:

$$W = \Sigma_{rem}/S_0, \tag{82}$$

where S_0 is the area of the surface normal to the incident beam.

In the case when the scattering medium fills the half-space $z \leq z_0$, the wave function of the optical model has the form

$$\varphi_p(x, z) = [1 + R(\theta)] \exp\left[ik_0 \sin\theta - ik_0 \int_{z_0}^{z} \sqrt{n^2 - \sin^2\theta}\, dz'\right]. \tag{83}$$

Assuming $\sqrt{n^2 - \sin^2\theta} \approx i\sqrt{(v_{gr}/v)^2 - \cos^2\theta}$, we obtain from (81)–(83) the following expression for the coefficient of absorption of ultracold neutrons[2]:

$$W = (u/v_{gr})^2 [2v\cos\theta/\sqrt{v_{gr}^2 - v^2\cos^2\theta}]. \tag{84}$$

In the case of propagation of very cold neutrons through a thick layer of matter, when the damping of the neutron wave is due to the removal of neutrons, the neutron density decreases into the reflector in accordance with the law

$$|\varphi_p(x, z)|^2 = |1 + R(\theta)|^2 \exp[k_0 (z - z_0)(u/v)^2/\sqrt{\cos^2\theta - (v_{gr}/v)^2}]. \tag{85}$$

In the case of normal incidence of the neutrons, the argument of the exponential in (85) is equal to $v\sigma_{rem}(v)N(z - z_0)/\sqrt{v^2 - v_{gr}^2}$ (see also Ref. 6). It is easy to show that the coefficient of absorption of neutrons in a thick layer of matter when (85) is satisfied is

$$W = \frac{4\cos^2\theta}{(\cos\theta + \sqrt{\cos^2\theta - (v_{gr}/v^2)})^2} \frac{\sqrt{\cos^2\theta - (v_{gr}/v)^2}}{\cos\theta} \tag{86}$$

and in this approximation it does not depend on the removal cross section $[\cos^2\theta > (v_{gr}/v)^2]$. The correction $\sim u^4$ arises as a result of the series expansion of $1 + R(\theta)$ and $\sqrt{n^2 - \sin^2\theta}$ in (81).

If the extinction length $d(v)$ of the neutron wave due to any cause is less than the damping length due to removal, the absorption coefficient has the form

$$W = \frac{4\cos^2\theta}{[\cos\theta + \sqrt{\cos^2\theta - (v_{gr}/v)^2}]^2} \frac{mu^2}{\hbar v} \frac{d(v)}{\cos\theta}. \tag{87}$$

In the cases when a, the target thickness, is short compared with the absorption length of very cold neutrons, $l_{abs} = \sqrt{v^2 - v_{gr}^2}/v\sigma(v)N$, the absorption coefficient W_{vcn} is proportional to u^2. This enables one to express W_{ucn} in terms of W_{vcn}. For simplicity, we restrict ourselves to normal incidence of a neutron wave on a bottle wall, and we obtain in the approximation linear in u^2 the result

$$W_{ucn} = \frac{2v/v_{gr}}{\sqrt{1 - (v/v_{gr})^2}} \cdot \frac{\hbar v_1}{4mav_{gr}^2} \times \frac{(1 + k'/k_1)^4 + (1 - k'/k_1)^4 - 2\cos 2k'a\,[1 - (k'/k_1)^2]^2}{(1 + k'/k_1)^2 + (1 - k'/k_1)^2 - \frac{\sin 2k'a}{k'a}[1 - (k'/k_1)^2]} W_{vcn}. \tag{88}$$

Here we have adopted the following notation: $k = mv/\hbar$ is the wave number of the ultracold neutrons in the bottle; $k_1 = mv_1/\hbar$ and $k' = (1/\hbar)\sqrt{2m(E_1 - U)}$ are the wave numbers of the very cold neutrons in vacuum and in the wall of the bottle, respectively; $v_{gr} = \sqrt{2U_0/m}$. For $v_1 \gg v_{gr}$, we again obtain from (88) Eq. (83). The restriction $l_{abs} \gg a$ presupposes $a \ll 0.1$ cm for E_1 near V_0. In the energy range of very cold neutrons, $E_1 \gg V_0$, this restriction on the target thickness is weakened since l_{abs} increases with increasing neutron energy. For $E_1 \approx U_0$ and $l_{abs}/a \ll 1$, we have[6]

$$W_{vcn} \approx 2\sqrt{2}\, u/v_1.$$

This behavior of W_{vcn} encompasses a larger energy range near U_0 the stronger the absorption. This result is also valid for $E_1 > U_0$ near the angle of total reflection.

The coefficient W is the probability of absorption of a neutron in a single collision with the wall; in conjunction with n_0, the mean number of collisions of a neutron with the bottle wall in unit time, W determines the neutron lifetime in the bottle[2]:

$$\tau = [n_0 W + \lambda_{decay}]^{-1};$$

$$\lambda_{decay} \approx 10^{-3} \text{ sec}^{-1}.$$

Here, the motion of the neutron is assumed to satisfy the laws of classical mechanics. An analogous result can be obtained by regarding the neutron bottle as a multiwave resonator (the size of the resonator exceeding the wavelength of the radiation by many times) or by calculating the relaxation constant of the neutron density by means of the theory of damping of a quasistationary state of a quantum-mechanical system (see, for example, Ref. 31).

2. PROPAGATION OF NEUTRONS OF VERY LOW ENERGIES THROUGH INHOMOGENEOUS MEDIA

We continue here the examination of the influence of inhomogeneity of the scattering medium on the propaga-

Refs. 9, 10, 33, and 34. We write down the expression for the Green's function $G(\mathbf{r}|\mathbf{r}_0)$ of the wave equation in a medium with impurity:

$$G(\mathbf{r}|\mathbf{r}_0)=G(\mathbf{r}|\mathbf{r}_0)+\int d\mathbf{r}'\int d\mathbf{r}''G_0(\mathbf{r}|\mathbf{r}')\,t(\mathbf{r}'|\mathbf{r}'')\,G(\mathbf{r}''|\mathbf{r}_0). \quad (129)$$

Here, $G_0(\mathbf{r}|\mathbf{r}_0)$ is the Green's function of the wave equation in the case of a homogeneous medium; $t(\mathbf{r}'|\mathbf{r}'')$ is the t matrix of scattering of the neutron wave on an impurity written down in the coordinate representation with respect to the neutron variables. The matrix $t(\mathbf{r}|\mathbf{r}_0)$ satisfies the Lippmann–Schwinger equation

$$t(\mathbf{r}|\mathbf{r}_0)=V(\mathbf{r}|\mathbf{r}_0)+\int d\mathbf{r}'\int d\mathbf{r}''V(\mathbf{r}|\mathbf{r}')\,G_0(\mathbf{r}'|\mathbf{r}'')\,t(\mathbf{r}''|\mathbf{r}_0), \quad (130)$$

where $V(\mathbf{r}|\mathbf{r}_0)$ is the potential of the interaction of the neutrons with the impurity. If the impurity contains a sufficiently large number of scattering nuclei, then $V(\mathbf{r}|\mathbf{r}_0)$ is the optical potential corresponding to the material of the impurity. In the cases when the neutron wavelength is long compared with the size of the impurity (we shall for simplicity assume that the impurity is a sphere of radius a), to describe s-wave scattering one can use the following model of the potential $V(\mathbf{r}|\mathbf{r}_0)$:

$$V(\mathbf{r}|\mathbf{r}_0)=U_0\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(\mathbf{r}-\mathbf{r}_1), \quad (131)$$

where $\mathbf{r}_1$ is the radius vector to the center of gravity of the impurity. For this model, a solution of Eq. (130) can be found in analytic form:

$$t(\mathbf{r}|\mathbf{r}_0)=t_0\delta(\mathbf{r}-\mathbf{r}_0)\,\delta(\mathbf{r}-\mathbf{r}_1), \quad (132)$$

where

$$t_0=U_0/[1-U_0G_0(\mathbf{r}_1|\mathbf{r}_1)]. \quad (133)$$

We augment the definition of the Green's function for coincident arguments as follows:

$$G_0(\mathbf{r}_1|\mathbf{r}_1)\equiv G_0(a). \quad (134)$$

If U_0 is real, i.e., there is no absorption in the impurity, we can readily show that

$$\operatorname{Im} t_0=|t_0|^2\operatorname{Im} G_0(\mathbf{r}_1|\mathbf{r}_1). \quad (135)$$

This relation reflects the content of the optical theorem for the given case. But if the impurity absorbs neutrons, i.e., $U_0=U_1+iU_2$, Eq. (135) takes the form

$$\operatorname{Im} t_0=|t_0|^2\{\operatorname{Im} G_0(\mathbf{r}_1|\mathbf{r}_1)+U_2/(U_1^2+U_2^2)\};\quad U_2<0. \quad (136)$$

We can express t_0 in terms of the amplitude of scattering on the impurity:

$$t_0=-(2\pi\hbar^2 f/m). \quad (137)$$

The scattering cross section is

$$\sigma=4\pi|f|^2=4\pi\,[m/(2\pi\hbar^2)]^2\,|t_0|^2. \quad (138)$$

If $G_0(\mathbf{r}_0|\mathbf{r})=-[m/(2\pi\hbar^2)]\{\exp[ik|\mathbf{r}-\mathbf{r}_0|]/|\mathbf{r}-\mathbf{r}_0|\}$ and accordingly

$$\operatorname{Im} G_0(\mathbf{r}_0|\mathbf{r}_0)=[-m/(2\pi\hbar^2)]\,k, \quad (139)$$

then

$$\operatorname{Im} f=\frac{k\sigma}{4\pi}\left[1-\frac{2\pi\hbar^2}{mk}\frac{U_2}{U_2^2+U_1^2}\right]. \quad (140)$$

It is natural to regard $[-\sigma 2\pi\hbar^2/(mk)][U_2/(U_2^2+U_1^2)]$ as the cross section of absorption of neutrons in the impurity.

If the absorbing impurity is at a depth in the material of the target, then in the case of weak absorption of neutrons and weak inelastic scattering in the material of the target the imaginary part of the Green's function, $\operatorname{Im}G_0(\mathbf{r}_1|\mathbf{r}_1)$, is small in the case of ultracold neutrons, and the optical theorem takes the form

$$\operatorname{Im} t_0=|t_0|^2\,U_2/(U_1^2+U_2^2). \quad (141)$$

Thus, in this case scattering on the impurity does not contribute to the damping of the coherent neutron wave (ultracold neutrons). A detailed analysis of Eq. (136) is contained in Ref. 10. Note that the effect of a large number of impurities can be taken into account by means of a wave equation with optical potential (see Sec. 1), for which the t matrix of an elementary scatterer must be replaced by the t matrix of the impurity. We shall consider below this approach and some physical problems that can be solved by means of it; here we analyze the interaction of neutrons with a system of scattering centers whose radius is small compared with the neutron wavelength. It is convenient to consider this question by directly applying the formulas of the theory of multiple scattering.[11,24] One of the problems of this type of practical interest is the calculation of the corrections to the optical potential due to the discrete structure of the scattering medium.[17]

For the mean scattering matrix $\langle T\rangle$ we have in the so-called quasicrystal approximation[24] the relations

$$\langle T(\mathbf{r}|\mathbf{r}_0)\rangle=N\langle Q(\mathbf{r}|\mathbf{r}_0)\rangle; \quad (142)$$

$$\langle Q(\mathbf{r}|\mathbf{r}_0)\rangle=\frac{1}{V}\int d\mathbf{R}_n\,\langle\langle Q(\mathbf{r}|\mathbf{r}_0;\ \mathbf{R}_n)\rangle\rangle; \quad (143)$$

$$\langle\langle Q(\mathbf{r}|\mathbf{r}_0;\ \mathbf{R}_n)\rangle\rangle=t(\mathbf{r}|\mathbf{r}_0;\ \mathbf{R}_n)+\sum_{m\neq n}\int\int d\mathbf{r}'\,d\mathbf{r}''\int d\mathbf{R}_m\Gamma^{(2)}(\mathbf{R}_n-\mathbf{R}_m)$$
$$\times\, t(\mathbf{r}|\mathbf{r}';\ \mathbf{R}_n)\,G_0(\mathbf{r}'|\mathbf{r}'')\,\langle\langle Q(\mathbf{r}''|\mathbf{r}_0;\ \mathbf{R}_m)\rangle\rangle. \quad (144)$$

Here, $\Gamma^{(2)}(\mathbf{R}_n-\mathbf{R}_m)d\mathbf{R}_m$ is the conditional probability that near the point $\mathbf{R}_m$ there is scatterer m if at point $\mathbf{R}_n$ scattering center n is already present; N is the total number of scattering centers in the volume V. Equation (144) is exact in the case of a completely random distribution of the scatterers and for a strictly periodic structure. By means of a Fourier transformation, this equation can be reduced to an algebraic one. It is known that the poles of the Fourier transform $T(\mathbf{p})=\int d\mathbf{r}\langle T(\mathbf{r})\rangle\times\exp(i\mathbf{p}\mathbf{r})$, as a function of the complex variable p, determine the dispersion law of the neutron wave in the scattering medium. One can show that the poles of $T(p)$ for $\operatorname{Im}p/\operatorname{Re}p\ll 1$ can be found by means of the equation (the dispersion relation)

$$K(\mathbf{p})=1. \quad (145)$$

Here

$$K(\mathbf{p})=\frac{N-1}{(2\pi)^3}V\int d\mathbf{p}_1\gamma^{(2)}(\mathbf{p}-\mathbf{p}_1)\,\tau(\mathbf{p}_1)\,g_0(\mathbf{p}_1), \quad (146)$$

where $\gamma^{(2)}(\mathbf{p})$, $\tau(\mathbf{p})$, and $g_0(\mathbf{p})$ are the Fourier transforms of the functions $\Gamma^{(2)}(\mathbf{r})$, $\langle t(\mathbf{r})\rangle$, and $G_0(\mathbf{r})$, respectively.

If the inhomogeneity of the medium is due solely to its microscopic structure, i.e., each scattering center is an atomic nucleus, then

$$\tau(\mathbf{p})=(2\pi\hbar^2/m)\,b\,(1/V). \quad (147)$$

In the case of a completely disordered system of nuclei,

$$\gamma^{(2)}(\mathbf{p})=[(2\pi)^3/V]\,\delta(\mathbf{p}). \quad (148)$$

We substitute the expressions (145) and (147) into (148). As a result, we obtain

$$1=(N-1)\,g_0(\mathbf{p})\,(2\pi\hbar^2/m)\,b\,(1/V) \tag{149}$$

and

$$p^2=k^2\,[1-(U_0/\varepsilon)\,(1-1/N)], \tag{150}$$

where $k=\sqrt{2m\varepsilon}/\hbar$ and $U_0=(2\pi\hbar^2/m)b(N/V)$. If the nuclei form a strictly periodic structure, then

$$\gamma^{(2)}(\mathbf{p})=\frac{1}{N-1}\sum_{\rho_l}{}'\exp[i\mathbf{p}\rho_l]. \tag{151}$$

The prime of the summation sign over the lattice sites means that one must exclude the term with $\rho_l=0$. The dispersion law in this case can be written in the form

$$1=-b\sum_{\rho_l}{}'\exp(i\mathbf{p}\rho_l)\exp(ik\rho_l)/\rho_l. \tag{152}$$

The replacement in (152) of summation by integration again leads to the result corresponding to the random distribution of nuclei (150). The correction terms due to the periodicity in the distribution of the scattering nuclei are in order of magnitude $(d/\lambda_{gr})^2\sim 2\pi\ \mathrm{Re}b/d \lesssim 10^{-3}$, where d is the constant of the crystal lattice.

When the volume occupied by the scattering medium is sufficiently large ($V^{1/3}\gg l_s$), multiple scattering becomes important. Above, we have already considered the dispersion relations of a coherent neutron wave for a system of small impurities ($a\ll\lambda_n$). We now analyze the propagation of a coherent neutron wave through a volume occupied by an inhomogeneous scattering medium with arbitrary structure. Incoherent scattering giving rise to damping of a coherent neutron wave can be due to fluctuations in the density of the scattering matter, the presence of several isotopes, and a nonvanishing spin of the scattering nuclei. Let us consider first the influence of inhomogeneity of the scatterer.

In the theory of propagation of waves in inhomogeneous media (see, for example, Ref. 32) it is shown that the coherent wave field $\langle\psi_c\rangle$ satisfies the Dyson equation

$$\langle\psi_c(\mathbf{r})\rangle=\psi_0(r)+\int\int dr'\,dr''G_0(\mathbf{r}\,|\,\mathbf{r}')\,M(\mathbf{r}'\,|\,\mathbf{r}'')\,\langle\psi_c(\mathbf{r}'')\rangle. \tag{153}$$

Here, $M(\mathbf{r}'|\mathbf{r}'')$ is the "mass" operator (the analog of the optical potential considered in Sec. 1), which takes into account the contribution of the fluctuations of the potential:

$$\delta U=U(\mathbf{r})-\langle U\rangle=(2\pi\hbar^2/m)\,b\,[\rho(\mathbf{r})-\langle\rho\rangle]; \tag{154}$$

$G_0(\mathbf{r}|\mathbf{r}')$ is the Green's function of the Schrödinger equation with the averaged potential $\langle U\rangle$. In the case of very cold neutrons propagating in an unbounded medium

$$G_0(\mathbf{r}\,|\,\mathbf{r}')=-\frac{m}{2\pi\hbar^2}\frac{\exp[ik\,|\,\mathbf{r}-\mathbf{r}'\,|]}{|\,\mathbf{r}-\mathbf{r}'\,|}; \tag{155}$$

$$k=\sqrt{2m\,[\varepsilon-U]}/\hbar. \tag{156}$$

The mass operator $M(\mathbf{r}|\mathbf{r}')$ can be written in the form of a perturbation series. In the first nonvanishing approximation in δU we have the relation[10, 15, 32]

$$M(\mathbf{r}\,|\,\mathbf{r}')\approx M_1(\mathbf{r}\,|\,\mathbf{r}')=\left[\frac{2\pi\hbar^2}{m}\,b\right]^2 G_0(\mathbf{r}\,|\,\mathbf{r}')\,K_{\rho\rho}(\mathbf{r}\,|\,\mathbf{r}'). \tag{157}$$

Here

$$K_{\rho\rho}(\mathbf{r}\,|\,\mathbf{r}')=\langle[\rho(\mathbf{r})-\langle\rho\rangle]\,[\rho(\mathbf{r}')-\langle\rho\rangle]\rangle \tag{158}$$

is the correlation function of the density fluctuations. We can express $M_1(\mathbf{r}|\mathbf{r}')$ in terms of the scattering cross section in the Born approximation [Eq. (108)]:

$$M_1(\mathbf{r}\,|\,\mathbf{r}')=\left(\frac{2\pi\hbar^2}{m}\right)^2\frac{G_0(\mathbf{r}\,|\,\mathbf{r}')}{V}\int\frac{d\varkappa}{(2\pi)^3}\exp[-i\varkappa(\mathbf{r}-\mathbf{r}')]\left\langle\frac{d\sigma}{d\Omega}\right\rangle, \tag{159}$$

where $\varkappa$ is the change in the wave vector on scattering.

We omit for the time being the imaginary part of the coherent scattering length b. Then from (157),

$$\mathrm{Im}\,M_1(\mathbf{r}\,|\,\mathbf{r}')=\left(\frac{2\pi\hbar^2}{m}\,\mathrm{Re}\,b\right)^2K_{\rho\rho}(\mathbf{r}\,|\,\mathbf{r}')\,\mathrm{Im}\,G_0(\mathbf{r}\,|\,\mathbf{r}'). \tag{160}$$

Remembering that in the case when the volume V is large,

$$\mathrm{Im}\,G_0(\mathbf{r}\,|\,\mathbf{r}')=\begin{cases}-\dfrac{m}{2\pi\hbar^2}\dfrac{\sin k\,|\,\mathbf{r}-\mathbf{r}'\,|}{|\,\mathbf{r}-\mathbf{r}'\,|} & \varepsilon>\mathrm{Re}\langle U\rangle;\\ 0 & \varepsilon<\mathrm{Re}\langle U\rangle,\end{cases} \tag{161}$$

we obtain from (160)

$$\mathrm{Im}\,M_1(\mathbf{r}\,|\,\mathbf{r}')=\begin{cases}-\dfrac{\mathrm{Re}\,b}{|\,\mathbf{r}-\mathbf{r}'\,|}\sin k\,|\,\mathbf{r}-\mathbf{r}'\,|\,\mathrm{Re}\langle U\rangle\dfrac{K_{\rho\rho}(\mathbf{r}\,|\,\mathbf{r}')}{\langle\rho\rangle} & \varepsilon>\mathrm{Re}\langle U\rangle;\\ 0 & \varepsilon<\mathrm{Re}\langle U\rangle.\end{cases} \tag{162}$$

Allowance for the presence of a boundary means that $\mathrm{Im}G_0(\mathbf{r}|\mathbf{r}')\neq 0$ when $\varepsilon<\mathrm{Re}\langle U\rangle$ as well (ultracold neutrons), since there is a flux from the source within the medium along the direction to the boundary with the vacuum. In this case, $\mathrm{Im}M_1(\mathbf{r}|\mathbf{r}')$ is also nonzero, but is damped out as one goes into the medium. We shall discusss this question in more detail below. For $\varepsilon>\mathrm{Re}\langle U\rangle$ (very cold neutrons) the fluctuation elastic scattering makes a contribution to the imaginary part of the optical potential that does not depend on the spatial coordinates (within the scattering medium).

If l, the correlation range of the density fluctuations, is short compared with λ_n, then in the case of very cold neutrons the expression (162) takes the form

$$\mathrm{Im}\,M_1(\mathbf{r}\,|\,\mathbf{r}')=-k\,\mathrm{Re}\,b\,\mathrm{Re}\langle U\rangle\,K_{\rho\rho}(\mathbf{r}\,|\,\mathbf{r}')/\langle\rho\rangle. \tag{163}$$

For $l\ll\lambda_n$, the correlation range l determines the scale of the spatial nonlocality of the "mass" operator $M_1(\mathbf{r}|\mathbf{r}')$, i.e., $\mathrm{Im}\int M_1(\mathbf{r}|\mathbf{r}')\,d\mathbf{r}'\approx l^3\,\mathrm{Im}M_1(\mathbf{r}|\mathbf{r}')$.

Let us compare the contribution $\mathrm{Im}M_1$ to the optical potential with the contribution $\mathrm{Im}\langle U\rangle$ due to inelastic scattering and absorption of neutrons. As a rule, $\mathrm{Im}\langle U\rangle=\alpha\,\mathrm{Re}\langle U\rangle$, where $\alpha=10^{-3}$–10^{-5}, and we obtain

$$\frac{l^3\,|\,\mathrm{Im}\,M_1\,|}{|\,\mathrm{Im}\,\langle U\rangle\,|}=\frac{1}{\alpha}\,k\,\mathrm{Re}\,b\,l^3\,\langle\rho\rangle\,\langle(\Delta\rho/\langle\rho\rangle)^2\rangle. \tag{164}$$

In the case of equilibrium density fluctuations

$$\langle(\Delta\rho/\langle\rho\rangle)^2\rangle=k_BT\beta_T/V_0, \tag{165}$$

where β_T is the isothermal compressibility; k_B is Boltzmann's constant; $V_0\sim l^3$. The characteristic values of β_T for metals lie near 10^{-12} cm/dyn. Setting $T=300\,^\circ\mathrm{K}$, $\mathrm{Re}b=10^{-12}$ cm, $k=10^6$ cm^{-1}, $\langle\rho\rangle=10^{23}$ cm^{-3}, we obtain from (164)

$$\frac{l^3\,|\,\mathrm{Im}\,M_1\,|}{|\,\mathrm{Im}\,\langle U\rangle\,|}\leqslant 10^{-4}\ \text{for}\ \ \varepsilon>\mathrm{Re}\langle U\rangle. \tag{166}$$

Similarly, one can obtain an estimate for $l \ll \lambda$ in the case of weak static density fluctuations (due to the inhomogeneity of the medium). Indeed, for $\langle(\Delta\rho/\langle\rho\rangle)^2\rangle \approx 10^{-4}$ we have

$$\frac{l^3\,|\,\mathrm{Im}\,M_1\,|}{|\,\mathrm{Im}\,\langle U\rangle\,|} \leqslant 10^{-2}. \tag{167}$$

If $l \gtrsim \lambda$, the estimate of the contribution $|\mathrm{Im}M_1|$ is changed. The nonlocality region of $M_1(\mathbf{r}\,|\,\mathbf{r}')$ in this case is determined by the damping of the function $G_0(\mathbf{r}\,|\,\mathbf{r}')$. For estimates, we shall assume that the characteristic scale of the spatial nonlocality of $M_1(\mathbf{r}\,|\,\mathbf{r}')$ is equal to λ_n. Then from (164) for $\varepsilon > \mathrm{Re}\langle U\rangle$ we obtain

$$\frac{\lambda_n^3\,|\,\mathrm{Im}\,M_1\,|}{|\,\mathrm{Im}\,\langle U\rangle\,|} \approx \frac{2\pi}{\varepsilon}\,\lambda_n^2\,\mathrm{Re}\,b\,\langle\rho\rangle\left\langle\left(\frac{\Delta\rho}{\langle\rho\rangle}\right)^2\right\rangle. \tag{168}$$

Hence, in the case of equilibrium density fluctuations,

$$\frac{\lambda_n^3\,|\,\mathrm{Im}\,M_1\,|}{|\,\mathrm{Im}\,\langle U\rangle\,|} \leqslant 10^{-4}. \tag{169}$$

In the case of static inhomogeneities for $\langle(\Delta\rho/\rho)\rangle^2 \approx 10^{-4}$ we have

$$\frac{\lambda_n^3\,|\,\mathrm{Im}\,M_1\,|}{|\,\mathrm{Im}\,\langle U\rangle\,|} \leqslant 1. \tag{170}$$

Thus, incoherent scattering on large-scale inhomogeneities may lead to a singificant damping of a coherent wave of very cold neutrons. The estimates have been obtained in the first nonvanishing approximation of perturbation theory. In order to establish the region of applicability of the M_1 approximation, it is necessary to estimate the following term of the perturbation series for the mass operator (see, for example, Ref. 32). A quasiclassical calculation of the propagation of a scalar wave through an inhomogeneous medium for $l \gg \lambda_n$ is made in Ref. 35.

We now include in the treatment the imaginary part $\mathrm{Im}b$ of the coherent scattering length. This leads to the appearance of an additional term

$$\Delta\,\mathrm{Im}\,M_1(\mathbf{r}\,|\,\mathbf{r}') = -\frac{2\,\mathrm{Re}\,b}{|\,\mathbf{r}-\mathbf{r}'\,|}\,\mathrm{Im}\,\langle U\rangle\,\frac{K_{\rho\rho}(\mathbf{r}\,|\,\mathbf{r}')}{\langle\rho\rangle} \times \begin{cases} \cos k\,|\,\mathbf{r}-\mathbf{r}'\,| & \varepsilon > \mathrm{Re}\,\langle U\rangle, \\ \exp\left[-\varkappa_0\,|\,\mathbf{r}-\mathbf{r}'\,|\right] & \varepsilon < \mathrm{Re}\,\langle U\rangle, \end{cases} \tag{171}$$

where

$$\varkappa_0 = \sqrt{2m\,[\mathrm{Re}\,\langle U\rangle - \varepsilon]}/\hbar. \tag{172}$$

The term $\Delta\,\mathrm{Im}M_1$ takes into account the influence of a "block effect" in the absorption of the neutrons: The absorption of neutrons is attenuated when there are fluctuations of the neutron field due to the inhomogeneity of the scattering medium. The possibility of such an effect was pointed out in Ref. 6. The magnitude of this correction is small.[10] Allowance for $\mathrm{Im}\langle U\rangle$ in the calculation of G_0 in the mass operator leads to an additional correction of the same order.

The expressions (162) and (171) correspond to the case when the scattering medium fills all space. We now estimate the influence of the boundary of the medium, restricting ourselves for simplicity to a one-dimensional problem. The mass operator M_1 in the case of a short correlation length l in the medium has the form

$$M_1'(z\,|\,z_0) = \begin{cases} -\langle(\Delta U)^2\rangle\,l\,\frac{2m}{\hbar^2}\,\frac{1}{2\varkappa_0} \times \left[1 + \frac{\varkappa_0 + \mathrm{i}k}{\varkappa_0 - \mathrm{i}k}\exp(-2\varkappa_0 z)\right]\delta(z - z_0); & z,\ z_0 > 0 \\ 0; & z,\ z_0 < 0. \end{cases} \tag{173}$$

The boundary of the medium is placed in the plane $z = 0$ and ΔU is the amplitude of the fluctuations in the potential of the medium. This form of the mass operator enables one to find a solution of the Dyson equation (153) in analytic form, namely, for $z > 0$ in the region occupied by the scattering medium

$$\langle\psi_c(z)\rangle = CJ_q(c\exp(-\varkappa_0 z)),$$

where

$$c^2 = (1 - q^2)\,[(\varkappa_0 + \mathrm{i}k_0)/(\varkappa_0 - \mathrm{i}k_0)];$$

k_0 is the ucn wave equation in the trap;

$$q^2 = 1 - \frac{l\,\mathrm{Re}\,\varkappa_0}{2}\left\langle\left(\frac{\Delta U}{\langle U\rangle}\right)^2\right\rangle\left[1 - \frac{\varepsilon}{\mathrm{Re}\,\langle U\rangle}\right]^{-2} \times \left[1 + \mathrm{i}\,\frac{1}{2}\,\frac{\mathrm{Im}\,\langle U\rangle}{\mathrm{Re}\,\langle U\rangle}\,\frac{1 - 4\varepsilon/\mathrm{Re}\,\langle U\rangle}{1 - \varepsilon/\mathrm{Re}\,\langle U\rangle}\right].$$

Using standard boundary conditions on the surface of the medium ($z = 0$), we can find the amplitude of the wave reflected from this surface and the reflection coefficient R. It has the form[9)]

$$R = SQ,$$

where

$$S = 1 - \left\langle\left(\frac{\Delta U}{\langle U\rangle}\right)^2\right\rangle\frac{l\,\mathrm{Re}\,\varkappa_0}{1 - \varepsilon/\mathrm{Re}\,\langle U\rangle}\,\frac{\varepsilon}{\mathrm{Re}\,\langle U\rangle} \times \left[1 + \frac{1}{2}\,\frac{\mathrm{Im}\,\langle U\rangle}{\mathrm{Re}\,\langle U\rangle}\,\frac{(\varepsilon/\mathrm{Re}\,\langle U\rangle)^{1/2}}{[1 - \varepsilon/\mathrm{Re}\,\langle U\rangle]^{3/2}}\right]$$

and

$$Q = 1 + 2\,\frac{\mathrm{Im}\,\langle U\rangle}{\mathrm{Re}\,\langle U\rangle}\left[\frac{\varepsilon/\mathrm{Re}\,\langle U\rangle}{1 - \varepsilon/\mathrm{Re}\,\langle U\rangle}\right]^{1/2} \times \left[1 + \frac{l\,\mathrm{Re}\,\varkappa_0}{2}\left\langle\left(\frac{\Delta U}{\langle U\rangle}\right)^2\right\rangle\frac{\varepsilon/\mathrm{Re}\,\langle U\rangle}{[1 - \varepsilon/\mathrm{Re}\,\langle U\rangle]^2}\right].$$

The factor $S - 1$ is due to the influence of the boundary of the scattering medium, whereas the correction $Q - 1$ is also present in the approximation (162) and (171). It can be seen that the two terms are of the same order. It follows from these results that the influence of inhomogeneity of the medium, in the case of a small scale of the inhomogeneity on the reflection coefficient of the neutron wave is small. The only possible exception is in the region of energies $\varepsilon \approx \langle U_0\rangle$, for the investigation of which one must use other methods since the corrections are large and perturbation theory does not apply. In the general case, the coefficient of reflection (absorption) can be calculated by means of the formulas of the optical model (81) and (82) if the wave function $\langle\psi_c\rangle$ is known.

Finally, we discuss the influence of spin and isotopic incoherence in the scattering of neutrons of very low energies. Since spin incoherence is manifested in scattering on an individual nucleus, this part of the incoherent scattering cross section must be included in the imaginary part of the scattering length b for a single nucleus. In the considered energy range $\sigma^{\mathrm{incoh}} \ll \sigma^{\mathrm{inel}}$

9)This result was obtained together with A. V. Shelagin.

since σ^{inel} is proportional to $1/v_n$, and $\sigma^{\text{incoh}} \approx$ const. Isotopic incoherence is similar to the static fluctuations in the density of the scattering medium[10)] considered above and it can be taken into account by including an appropriate term in the mass operator:

$$M^{\text{isotopic}}(\mathbf{r}|\mathbf{r}') \approx M_1^{\text{isotopic}}(\mathbf{r}|\mathbf{r}') = (2\pi\hbar^2/m)^2 K_{bb}(\mathbf{r}|\mathbf{r}') \langle\rho\rangle^2 G_0(\mathbf{r}|\mathbf{r}'). \tag{174}$$

Here, $K_{bb}(\mathbf{r}|\mathbf{r}')$ is the correlation function of the fluctuations of the coherent scattering length $b(\mathbf{r})$. Obviously, the above result can be used to obtain estimates of $\text{Im}M_1^{\text{isotopic}}(\mathbf{r}|\mathbf{r}')$. This term can make an appreciable contribution to the damping of a coherent wave only if the distribution of the isotopes is correlated and the correlation length satisfies $l \gtrsim \lambda_n$.

We now consider the influence of inhomogeneity of the medium on the dispersion law of a neutron wave: $\mathbf{k}=\mathbf{k}(\varepsilon)$ ($\mathbf{k}$ is the neutron wave vector and ε the neutron energy). The inhomogeneity of the medium need not necessarily be due to thermal fluctuations of the density or the implantation of random impurities. The calculation made above enables one to consider the propagation of a coherent neutron wave through a periodic structure of thin films, etc. Fourier transforming the homogeneous Dyson equation (153), we obtain the dispersion relation in the form

$$\frac{\hbar^2 k^2}{2m} = \varepsilon - \langle U_R\rangle - \text{Re}\, m(\text{E}, \mathbf{k}), \tag{175}$$

where $\langle U_R\rangle$ is defined by (2), and

$$m(\text{E}, \mathbf{k}) = \int d(\mathbf{r}-\mathbf{r}') M(\text{E}, \mathbf{r}-\mathbf{r}') \exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}')]. \tag{176}$$

In the M_1 approximation,

$$m_1(\text{E}, \mathbf{k}) = \frac{1}{(2\pi)^3} \int d\mathbf{q} k_{\gamma\gamma}(\mathbf{q}) / [\text{E} - \langle U_R\rangle - (\hbar^2/2m)(\mathbf{k}-\mathbf{q})^2 + i\eta], \tag{177}$$

where

$$k_{\gamma\gamma}(\mathbf{q}) = \int d(\mathbf{r}-\mathbf{r}') \exp[i\mathbf{q}(\mathbf{r}-\mathbf{r}')] K_{\gamma\gamma}(\mathbf{r}-\mathbf{r}'), \tag{178}$$

$K_{\gamma\gamma}(\mathbf{r})$ is the correlation function of $(2\pi\hbar^2/m)b(\mathbf{r})\rho(\mathbf{r})$. In the cases when the deviations from homogeneity in the medium are due to the presence of a periodic structure, the Fourier transform of the correlation function of the fluctuations $\gamma(\mathbf{r})$ has the form

$$k_{\gamma\gamma}(\mathbf{q}) = [(2\pi)^3/V_{\text{uc}}^2] \sum_{\mathbf{s}} |A_\gamma(\mathbf{s})|^2 \delta(\mathbf{q}-2\pi\mathbf{s}), \tag{179}$$

where

$$A_\gamma(\mathbf{s}) = \int_{V_{\text{uc}}} \exp[2\pi i\mathbf{s}\mathbf{r}][\gamma(\mathbf{r}) - \langle\gamma\rangle]\, dr. \tag{180}$$

Accordingly, from (177) we obtain

$$\text{Re}\, m_1(\text{E}, \mathbf{k}) = \frac{1}{V_{\text{uc}}^2} \sum_{\mathbf{s}} \frac{|A_\gamma(\mathbf{s})|^2}{\text{E} - \langle U_R\rangle - \frac{\hbar^2}{2m}(\mathbf{k}-2\pi\mathbf{s})^2}. \tag{181}$$

If the period l of the structure is small compared with the neutron wavelength λ_n, then $2\pi|\mathbf{s}| \gg 2\pi/\lambda_n = k$ and

$$\text{Re}\, m_1(\text{E}, \mathbf{k}) \approx -\frac{1}{V_{\text{uc}}^2} \frac{2m}{\hbar^2} \sum_{\mathbf{s}} \frac{|A_\gamma(\mathbf{s})|^2}{|2\pi\mathbf{s}|^2}. \tag{182}$$

It follows from (175) and (182) that when $l \ll \lambda_n$ a periodic disturbance of the homogeneity of the medium reduces the potential barrier (or deepens the potential well) of the scattering substance. This change of the optical potential is small and equal in order of magnitude to $\langle U_R\rangle\langle(\Delta\gamma/\gamma)^2\rangle(\langle b\rangle/l)$. A more important change in the dispersion law occurs if the period of the structure is comparable with the neutron wavelength, i.e., $l \sim \lambda_n$. In particular, there is an interesting situation if

$$k^2 \approx (\mathbf{k} - 2\pi\mathbf{s}_0)^2 \text{ (Laue–Bragg condition)}. \tag{183}$$

In this case, the treatment of a single wave with a definite wave vector is insufficient and one must take into account the interaction of waves with wave vectors $\mathbf{k}$ and $\mathbf{k}=2\pi\mathbf{s}_0$. The dispersion law can then be written in the form

$$\text{E} - \langle U_R\rangle - \frac{\hbar^2 k^2}{2m} = \frac{1}{V_{\text{uc}}^2} |A_\gamma(\mathbf{s}_0)|^2 \left[\text{E} - \langle U_R\rangle - \frac{\hbar^2}{2m}(\mathbf{k}-2\pi\mathbf{s}_0)^2\right]^{-1}. \tag{184}$$

In particular, it follows from (184) that there are forbidden energy bands in the transmission spectrum of the periodic structure (see, for example, Refs. 10 and 36).

Solving the dispersion relation with allowance for multiple scattering of waves on inhomogeneities of the medium, we can find the effective wave number of a neutron wave in the medium and the refractive index of the medium (which is in general complex). Having these parameters at our disposal, we can calculate the coefficients of reflection and transmission of the neutron wave (and the scattering cross section) for finite volumes of matter. In order to be able to ignore the boundedness of the volume in the calculation of the correlation function $K_{\gamma\gamma}$, we must have $V^{1/3} \gg l$. This is a comparatively weak restriction in cases of practical interest. In the case of weak scattering on inhomogeneities of the medium one can use the results of the Born approximation of perturbation theory.

Let us discuss briefly the perturbation of the neutron field due to roughness of the boundary of the medium.[16–19, 33] We shall assume that the deviation of the neutron reflecting surface from perfect smoothness can be described by means of a random function $\zeta(x, y)$, i.e., the media interface is specified in the form

$$z = \zeta(x, y). \tag{185}$$

For simplicity, we shall assume $\langle\zeta(x,y)\rangle = 0$. Then a perfectly smooth boundary obviously coincides with the plane $z=0$.

We calculate the absorption coefficient of ultracold neutrons in a medium bounded by such a rough surface. We shall assume that the scattering material is itself homogeneous. By definition, W is the absorption coefficient of ultracold neutrons—the ratio of the flux $J_z^{(-)} = J_z^{\text{inc}} + \langle J_z^{\text{ref}}\rangle$ to the incident flux J_z^{inc}, i.e.,

$$W = J_z^{(-)}/J_z^{\text{inc}}, \tag{186}$$

Where $\langle J_z^{\text{ref}}\rangle$ is the mean reflected flux. To be specific, we assume that the oz axis is directed into the medium.

In the case of small and shallow inhomogeneities of the boundary, when $\langle|\zeta|^m\rangle \ll \lambda_n^m$ and $\langle|\nabla\zeta|^m\rangle \ll 1$, to calculate $\psi_i(\mathbf{r})$ of the neutron field in the ith medium we can use the following expansion in a perturbation series in ascending powers of $|\zeta|$ and $|\nabla\zeta|$ (Ref. 37):

$$\psi_i(\mathbf{r}) = \psi_i^{(0)}(\mathbf{r}) + \psi_i^{(1)}(\mathbf{r}) + \psi_i^{(2)}(\mathbf{r}) + \dots \tag{187}$$

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10) Here we do not take into account the influence of the isotopic composition on the frequency spectrum of the normal vibrations of the crystal.

Here $\psi_i^{(0)}$ describes the propagation of neutrons in a bottle (within it and in the walls) with a smooth boundary; $\psi_i^{(1)}$ is the correction of first order due to the inhomogeneities, etc. Substituting this expansion into the standard conditions of continuity of the wave function ψ and its normal derivative on the boundary of the media, we obtain a system of equations for finding $\psi_i(\mathbf{r})$. The boundary conditions on the rough surface can be transformed into conditions on a smooth surface containing surface sources of the neutron field. This procedure was carried through in Ref. 16 to second order (see also Refs. 17 and 18). This amount of calculation enables one to find the main contribution to the angular distribution of the scattered neutrons:

$$I_+(\theta)=\frac{1}{4}\left(\frac{kv_{gr}}{v}\right)^4 w_{\zeta\zeta}(k_{x_i}-k_{x_f},\ k_{y_i}-k_{y_f}) \tag{188}$$

and the correction to the absorption coefficient W_0:

$$\begin{aligned}W-W_0=W_0\Big\{2k\sqrt{\left(\frac{v_{gr}}{v}\right)^2-\cos^2\theta}\\ \times\Big[\frac{1}{(2\pi)^2}\iint\limits_{p_x^2+p_y^2\leq k^2} dp_x\,dp_y w_{\zeta\zeta}(p_x-k_x,\ p_y-k_y)\\ \times\left(\frac{k^2-p_x^2-p_y^2}{\sqrt{p_x^2+p_y^2+k^2[(v_{gr}/v)^2-1]}}-\frac{\sqrt{k^2-p_x^2-p_y^2}\cos\theta}{\sqrt{(v_{gr}/v)^2-\cos^2\theta}}\right)\Big]\\ +k^2(v_{gr}/v)^2\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\!\!\int dp_x\,dp_y\\ \times w_{\zeta\zeta}(p_x-k_x,\ p_y-k_y)\left[1-\frac{k\sqrt{(v_{gr}/v)^2-\cos^2\theta}}{\sqrt{[(v_{gr}/v)^2-1]k^2+p_x^2+p_y^2}}\right]\Big\}.\end{aligned} \tag{189}$$

Here,

$$W_0=(u/v_{gr})^2\frac{2\cos\theta}{\sqrt{(v_{gr}/v)^2-\cos\theta}} \tag{190}$$

is the absorption coefficient of ultracold neutrons in a medium with smooth boundary[2]; θ is the angle of incidence of the neutron wave;

$$w_{\zeta\zeta}(\varkappa_x,\varkappa_y)=\frac{1}{Q}\langle|\tilde{\zeta}(\varkappa_x,\varkappa_y)|^2\rangle; \tag{191}$$

Q is the area of the irradiated underlying surface;

$$\tilde{\zeta}(\varkappa_x,\varkappa_y)=\iint dx\,dy\exp[-i\varkappa_x x-i\varkappa_y y]\,\zeta(x,y). \tag{192}$$

The velocities v_{gr} and u are defined by Eqs. (79). If l, the correlation length of the inhomogeneities of the boundary, is small compared with the neutron wavelength, then to terms linear in l

$$W=W_0[1+(kv_{gr}/v)^2\langle(\zeta)^2\rangle]\left[1-kl\frac{\sqrt{\pi}}{2}\sqrt{(v_{gr}/v)^2-\cos^2\theta}\right]. \tag{193}$$

The averaging of this expression with respect to an isotropic angular distribution of neutrons incident on the wall and with respect to a Maxwellian velocity spectrum of ultracold neutrons leads to the result

$$\begin{aligned}\langle\langle W\rangle\rangle=\langle\langle W_0\rangle\rangle\left[1+\left(k\frac{v_{gr}}{v}\right)^2\langle\zeta^2\rangle\right]\\ -\frac{8\sqrt{\pi}}{15}\left(\frac{u}{v_{gr}}\right)^2\left(k\frac{v_{gr}}{v}\right)^3\langle\zeta^2\rangle\, l;\end{aligned} \tag{194}$$

$$\langle\langle W_0\rangle\rangle=\frac{\pi}{2}\left(\frac{u}{v_{gr}}\right)^2. \tag{195}$$

Thus, the roughness of the surface of the medium leads to a certain increase of the absorption coefficient in the case of a single collision of a neutron with the bottle wall: For $\lambda_n=10^{-5}$ cm and $\langle\zeta^2\rangle^{1/2}=5.0\times10^{-7}$ cm, we have $(W-W_0)/W_0\approx0.1$. A calculation of the absorption coefficient of ultracold neutrons in a medium whose boundary contains a low density of uncorrelated small impurities was made in Ref. 33. The influence of the reduced density of matter in the surface layer on the absorption of ultracold neutrons was considered in Ref. 17. In Ref. 19, an investigation was made of the effect of surface films on the absorption of neutrons in bottle walls. In particular, it was shown that the film, besides removing ultracold neutrons, screens the material of the wall: A film that has an attractive potential for ultracold neutrons increases the removal of ultracold neutrons while a reflecting one decreases it. An estimate of the absorption of ultracold neutrons in contaminating films shows that to explain the experimentally measured anomalously short lifetime of ultracold neutrons in a bottle the film thickness must be too great.

We now consider the correlation function of a neutron field with allowance for multiple scattering on inhomogeneities of the medium: $K_{\Psi\Psi}(\mathbf{r}_1|\mathbf{r}_2)=\langle\psi^*(\mathbf{r})\psi(\mathbf{r}_2)\rangle$. This function satisfies an equation of the Bethe–Salpeter type[15,32,38]:

$$\begin{aligned}\langle\psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle=\langle\psi^*(\mathbf{r}_1)\rangle\langle\psi(\mathbf{r}_2)\rangle\\ +\int\cdots\int d\boldsymbol{\rho}_1\,d\boldsymbol{\rho}_2\,d\mathbf{r}_1'\,d\mathbf{r}_2'\langle G(\mathbf{r}_1,\boldsymbol{\rho}_1)\rangle\langle G(\mathbf{r}_2,\boldsymbol{\rho}_2)\rangle\\ \times\mathscr{B}_{\gamma\gamma}(\boldsymbol{\rho}_1,\mathbf{r}_1';\boldsymbol{\rho}_2,\mathbf{r}_2')\langle\psi^*(\mathbf{r}_1')\psi(\mathbf{r}_2')\rangle.\end{aligned} \tag{196}$$

Here $\mathscr{B}_{\gamma\gamma}(\mathbf{x}_1,\mathbf{x}_2;\mathbf{x}_3,\mathbf{x}_4)$ is the kernel of the so-called intensity operator. In the first approximation of perturbation theory,

$$\begin{aligned}\mathscr{B}_{\gamma\gamma}(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')\approx\mathscr{B}_1(\mathbf{x},\mathbf{x}';\mathbf{y},\mathbf{y}')\\ =\delta(\mathbf{x}-\mathbf{x}')\,\delta(\mathbf{y}-\mathbf{y}')\,K_{\gamma\gamma}(\mathbf{x}|\mathbf{y})\end{aligned} \tag{197}$$

and the equation for the correlation function of the neutron field in a spatially homogeneous medium takes the form

$$\begin{aligned}\langle\psi^*(\mathbf{r}_1)\psi(\mathbf{r}_2)\rangle=\langle\psi^*(\mathbf{r}_1)\rangle\langle\psi(\mathbf{r}_2)\rangle\\ +\int d\mathbf{r}_1'\int d\mathbf{r}_2'\langle G(\mathbf{r}_1,\mathbf{r}_1')\rangle\langle G(\mathbf{r}_2,\mathbf{r}_2')\rangle\times K_{\gamma\gamma}(\mathbf{r}_1'-\mathbf{r}_2')\langle\psi^*(\mathbf{r}_1')\psi(\mathbf{r}_2')\rangle.\end{aligned} \tag{198}$$

A solution of this equation in the case of small-scale inhomogeneities ($kl\ll1$) was obtained in Tatarskiĭ's monograph, Ref. 32. We give here the result for the mean square of the fluctuations of the field $\varphi(\mathbf{r})=\psi(\mathbf{r})-\langle\psi(\mathbf{r})\rangle$:

$$\langle\varphi^*\varphi\rangle/|\langle\psi(\mathbf{r})\rangle|^2=|k|^4\varepsilon^2l^3/8\pi\,\mathrm{Im}\,k_{eff} \tag{199}$$

Here, k_{eff} is the propagation wave vector of the coherent neutron wave in the inhomogeneous medium; ε^2 is the variance of the density fluctuations of the matter. This approximation can be given the form

$$\langle\varphi^*\varphi\rangle/|\langle\psi\rangle|^2=|k|^4\varepsilon^2l^3/(8\pi\,\mathrm{Im}\,k')[n_{eff}/(2n_{eff}^2-1)], \tag{200}$$

where n_{eff} is the refractive index of the medium. The first factor in (200) is the relative value of the mean square of the fluctuations of the neutron field in the single-scattering approximation. Multiple scattering, which is taken into account by the factor $n_{eff}/(2n_{eff}^2-1)\lesssim1$, weakens the field fluctuations. Equations (199) and (200) apply for small oscillations of the neutron field: $\langle\varphi^*\varphi\rangle/|\langle\psi\rangle|^2\ll1$.

To make this solution of Eq. (198) more precise with allowance for the higher perturbation terms we must first find the average Green's function of the wave equation: $\langle G\rangle$. Note that to analyze the perturbation series the diagram method of quantum field theory is effective.[15,32] The operations discussed above must be self-consistent since the intensity operator and the mass operator M are related to each other by an equation that follows from the law of conservation of the number of particles in a nonabsorbing medium ("optical theorem"[38]):

$$M^*(x, y) - M(y, x) = \int dx' \int dy'$$
$$\times [\langle G^*(x', y')\rangle - \langle G_i(y', x')\rangle] \mathscr{B}(x', x; y', y). \quad (201)$$

In the first perturbation order,

$$\left.\begin{array}{l} M(x, y) \approx M_1(x, y) = G_0(x, y) K(x, y); \\ \mathscr{B}(x, x'; y, y') \approx \mathscr{B}_1(x, x'; y, y') \\ \quad = \delta(x-x')\,\delta(y-y')\,K(x, y) \end{array}\right\} \quad (202)$$

and the validity of (201) can be directly verified.

In the quasiclassical approximation, the Bethe–Salpeter equation for the correlation function of the neutron field reduces to a simpler transfer equation[38] that takes into account only integrally the particle balance but does not take into account interference phenomena. The transfer equation cannot of course be used to describe an essentially wave phenomenon such as total reflection, when the contribution of inhomogeneous waves is important, nor for processes that take place in systems of small volume: $V^{1/3} \lesssim 10^{-4}$ cm (Ref. 39). As an example of the use of the transfer equation in the physics of ultracold neutrons, let us consider the propagation of them in neutron guide tubes with rough walls. Usually, this problem is solved in the framework of classical mechanics by analogy with the flow of a rarefied gas along tubes.[40] As in problems of the containment of ultracold neutrons in bottles, the experimental data on the flow of neutrons along neutron guide tubes do not agree sufficiently well with the results of theoretical calculations. It would be interesting to attempt a different approach to the solution of this problem, namely, regard the motion of ultracold neutrons as propagation of a wave in a multimode waveguide. If the waveguide has rough surfaces, we can again write down for a coherent neutron wave a Dyson equation[42] taking into account the damping of the mean neutron field due to scattering on random inhomogeneities of the waveguide walls. Roughness of the waveguide surface also leads to an interaction between individual normal modes of the waveguide and changes the wavelength, group velocity, etc., from the smooth-waveguide case. The correction function of the neutron field satisfies an equation of the Bethe–Salpeter type. Ignoring the fine interference structure of the neutron field, we can restrict the investigation to the more simple transfer equation for the neutron density. In the case of large-scale inhomogeneities ($l/\lambda_n \gg 1$), this equation can be reduced to an equation of the diffusion type describing the transformation of the angular distribution of the moving neutrons: The diffusion takes place in the "space of the numbers of the normal vibrations of the neutron field" (Ref. 42). It can be shown that at a distance x from the waveguide boundary the intensity of the field is nonzero only in those modes that are separated from the initial number n_0 (at $x=0$) by $\sqrt{(\Delta n)^2}$, where $(\Delta n)^2 = 4D_{n_0}x$. The value of D_{n_0} is determined by the statistical properties of the waveguide walls, and it has the meaning of a diffusion coefficient. Accordingly, the field intensity as a function of the mode number has the form

$$I(x) = \frac{I_0}{2\sqrt{\pi D_{n_0} x}} \exp[-(n-n_0)^2/(4D_{n_0}x)], \quad (203)$$

i.e., it depends nonexponentially on the path length in the waveguide. It is possible that the properties of propagation of ultracold neutrons along neutron guide tubes are due to the above transformation of the intensity of the neutron field on the irregularities of the neutron guide tube walls. We note finally that if the macroscopic cross section of incoherent scattering on the inhomogeneities satisfies $\Sigma_{\text{incoh}} \gg \Sigma_{\text{inel}}$, then the yield of very cold neutrons from an inhomogeneous converter decreases approximately by $\sqrt{\Sigma_{\text{incoh}}/\Sigma_{\text{inel}}}$ times because of the increase in the path length of a neutron in the matter (see Ref. 2).

CONCLUSIONS

In this paper we have obtained an expression for the optical potential describing the coherent field of ultracold neutrons in matter. Our point of departure was the well known formulas of Watson's theory of multiple scattering, which take into account the possibility of excitation of the scattering system. An analogous "optical" model was proposed in order to take into account the damping of the coherent neutron wave due to scattering on inhomogeneities of the medium. The effective optical potential we have obtained enables one to find the effective wave number in a medium and calculate the refractive index of the material. This refractive index, which characterizes the propagation of a wave in an unbounded medium, can be used to solve the problem of the scattering of neutrons on bounded volumes of scattering material. The errors in such an approach in the case of very cold neutrons ($\varepsilon_n > \varepsilon_{gr}$) are small if the thickness of the scattering layer is large compared with the correlation length of the fluctuations. It should be borne in mind that Δk_{eff}, the error calculated for the modulus of the wave vector $|k_{\text{eff}}|$, imposes a restriction on the path length x traversed by the wave in the medium: $\Delta k_{\text{eff}} x \ll 1$. In the case of ultracold neutrons, allowance for only volume effects in the effective potential is insufficient and it is necessary to take into account the particular properties of the surface layer. Obviously, this conclusion is also valid for very cold neutrons when they undergo total reflection from the boundary of the medium.[43–46] At the same time, it is in principle possible to study surface vibrational modes of the medium, which are suppressed in the case of scattering of very cold neutrons by the contribution of the volume vibrations. This remark serves as an illustration of the fact that the investigation of the propagation of low-energy neutrons is of interest not only in itself (it is sufficient to recall that the anomalous escape of ultracold neutrons from bottles has not yet been explained[47]) but even more as a method of studying static and dynamic properties of matter. Of particular interest are experiments on the interaction of polarized neutrons with magnetic materials.[3]

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