Boundary-condition model in two- and three-particle problems

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A unified method for solving two- and three-particle problems in the boundary-condition model is considered. The method is based on the use of specific properties of the two-particle off-shell wave functions. It is shown that the Schrödinger equation for a system of three identical bosons whose interactions take place only in relative states and are described by means of the boundary-condition model without external potential reduces exactly to a one-dimensional integral equation.

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INTRODUCTION

An important task of nuclear physics is to establish the form of the nucleon—nucleon potential. The most consistent method for solving this problem is to obtain the nucleon—nucleon potential on the basis of the meson theory of nuclear forces. However, this theory cannot yet be regarded as completed or noncontradictory. Therefore, to find the nucleon—nucleon potential wide use is made of a phenomenological approach based on experimental data on the nucleon—nucleon interaction, namely, the phase shifts of elastic scattering of nucleons (up to a laboratory energy $E \approx 400$ MeV) and data on the deuteron. As an example of the "realistic" potentials obtained in this way, we may mention the Reid potentials,[1] which contain in addition to the OPEP potential, which is well justified in meson theory, certain components that describe phenomenologically the interactions at short distances. In particular, the interaction at very short distances ($r \lesssim 0.4$ F) is simulated by either a hard core or a strong repulsive potential with short range.

In the construction of phenomenological potentials, one of the simplest ways of allowing for the short-range forces is by the introduction of the boundary-condition model. In accordance with this model,[2–4] the interaction region is divided into an external ($r > c$) and an internal ($r < c$) region. In the external region, the interaction is described by a comparatively simple potential, and the effect of the short-range forces, which may have a very complicated nature, is taken into account by introducing at $r = c$ a boundary condition for the logarithmic derivative of the wave function. The radius c of the boundary conditions and the value of the logarithmic derivative f at $r = c$ are phenomenological parameters of the model that must be determined from experimental data. Note that a hard-core potential is a special case of the boundary-condition model with $f \to \infty$. For the interpretation of the nucleon—nucleon interaction a simple boundary-condition model without external potential[2,3,5,6] and a boundary condition model with a potential in the external region[7,8] have been used. In particular, in Ref. 8 the potential in the external region was determined on the basis of meson theory.

The introduction of phenomenological nucleon—nucleon potentials involves an uncertainty due to the fact that the experimental data are in principle compatible with many potentials of different form containing rather a lot of parameters. An additional criterion for choosing such potentials is the comparison of experimental and theoretical data for a system of three nucleons. The correct equations for a system of three strongly interacting particles were obtained by Faddeev.[9] In the case of two-particle interactions, the kernels of the three-particle integral Faddeev equations can be expressed in terms of the two-particle off-shell T matrices. Therefore, the problem arises of the correct determination of the two-particle T matrix for potentials taking into account short-range forces in accordance with the boundary-condition model since in this case the matrix cannot be obtained from the ordinary Lippmann—Schwinger equation. There exist various ways for circumventing this difficulty.

One of the methods for obtaining the T matrix in the boundary-condition model is to introduce a pseudopotential that reproduces for the wave function the necessary value of the logarithmic derivative at $r = c$.[6,10] Another approach is developed in Ref. 11: An energy-independent logarithmic derivative of the wave function at $r = c$ can be obtained by means of a certain limiting procedure applied to a potential of special form acting in the internal region. In this case, both the on- and off-shell two-particle wave functions vanish in the internal region. This last circumstance can be used as the point of departure for obtaining the off-shell T matrix in the boundary-condition model. The introduction of a boundary condition at $r = c$ for the logarithmic derivative of the off-shell wave function and the condition of vanishing of this function in the internal region is completely sufficient to obtain the off-shell T and K matrices in the boundary-condition model.[12,13] Such an approach can be regarded in a certain sense as the "pure" boundary condition method since, in complete agreement with the boundary-condition model, it does not require the introduction of pseudopotentials or the use of limiting procedures for potentials of special form acting in the internal region.

The use of the off-shell two-particle T matrices corresponding to the boundary-condition model directly in

the three-particle Faddeev equations leads to certain difficulties due to the fact that these equations do not have unique solutions.[14,15] This last circumstance is due to the specific properties of the two-particle T matrices in the boundary-condition model and, on the basis of these properties, the Faddeev equations can be modified and reduced to equations that have unique solutions.[15] However, to achieve complete uniqueness of the modified Faddeev equations, it is necessary to introduce an arbitrary three-particle parameter, which has the meaning of energy and for which the channel functions satisfy the same boundary conditions as the two-particle wave functions. The introduction of such a parameter is also needed for the simplest case of the problem of three identical bosons.[16] In the special case of the boundary-condition model without external potential, the modified Faddeev equations[14,15] are integral equations for functions of one vector variable. A different approach to the solution of the three-particle problem when the two-particle interactions are described by means of the boundary-condition model has been developed in Refs. 17–20. The total three-particle wave function must satisfy boundary conditions analogous to the conditions for the off-shell two-particle wave functions.[17,18] This circumstance enables one in the general case to reduce the three-particle Schrödinger equation for the boundary condition model without external potential to a system of equations for functions of one vector variable. In the special case of a system of three identical bosons with two-particle interactions only in the s states and for zero total angular momentum, the determination of the three-particle wave function reduces to the solution of a one-dimensional integral equation for a single function.[19,20]

Concerning the one-dimensional equations of Refs. 19 and 20 it is necessary to make the following remarks: First, these equations are obtained directly from the three-particle Schrödinger equation with allowance for boundary conditions that follow from the conditions for the off-shell two-particle wave functions in the boundary condition model (in this respect, their derivation differs significantly from the derivation of the one-dimensional modified Faddeev equations[14,15]); second, the equations obtained in Refs. 19 and 20 uniquely determine three-particle characteristics such as the binding energy E_0 and the transitions amplitudes. The problem of nonuniqueness arises only in the determination of the total wave function, and for systems of three particles whose wave functions have a definite type of permutational symmetry, the problem can be solved without the introduction of additional parameters if the boundary-condition model is regarded as a limiting case of a potential of special form[11]; third, the numerical solution of the one-dimensional equation for a system of three identical bosons gives the value $E_0 = 7.7$ MeV for the binding energy,[20] which differs strongly from the result $E_0 = 12.69$ MeV obtained in Ref. 21 from the solution of the Faddeev equations. We note finally that the structure of the equations obtained in Ref. 19 and 20 enables one in a simple manner to take into account approximately the possible contribution of three-particle effects, for example, three-particle forces.

Below, we shall set forth the main results of Refs. 12, 13, and 17–20, which represent a unified approach to the solution of two- and three-particle problems in the boundary condition model based on the use of boundary conditions for the off-shell two-particle wave functions.

1. OFF-SHELL TWO-PARTICLE T AND K MATRICES IN THE BOUNDARY CONDITION MODEL

A. Boundary condition model without potential in the external region

To obtain some necessary relations, a two-particle potential $V(r)$ satisfying conditions under which Lippmann—Schwinger equations hold was considered in Refs. 12 and 13. In this case, the partial-wave l components of the T and K matrices are defined by

$$T_l(p, k, Z) = -\int_0^\infty r^2\, dr j_l(pr)\, V(r)\, \psi_{lk}(r, Z); \tag{1}$$

$$K_l(p, k, E) = -\int_0^\infty r^2\, dr j_l(pr)\, V(r)\, \psi_{lk}(r, E), \tag{2}$$

where $E > 0$ in (2) and $Z = E \pm i\epsilon$ in (1) for $E > 0$. The components T_l and K_l are normalized by the conditions

$$T_l(k, k, k^2 + i0) = \exp(i\delta_l) \sin\delta_l / k; \tag{3}$$

$$K_l(k, k, k^2) = \operatorname{tg}\delta_l / k, \tag{4}$$

where δ_l is the partial-wave phase shift.

The off-shell wave functions ψ_{lk} in (1) and (2) satisfy the Lippmann—Schwinger equations

$$\psi_{lk}(r, Z) = j_l(kr) - \int_0^\infty r'^2\, dr' \mathcal{G}_l(r, r', Z)\, V(r')\, \psi_{lk}(r', Z); \tag{5}$$

$$\psi_{lk}(r, E) = j_l(kr) - \int_0^\infty r'^2\, dr' \mathcal{G}_l^{(0)}(r, r', E)\, V(r')\, \psi_{lk}(r', E). \tag{6}$$

In Eqs. (5) and (6) the corresponding Green's functions have the form

$$\mathcal{G}_l(r, r', Z) = \frac{2}{\pi}\int_0^\infty p^2\, dp \frac{j_l(pr)\, j_l(pr')}{p^2 - Z} = i\sqrt{Z}\, j_l(\sqrt{Z} r_<)\, h_l^{(1)}(\sqrt{Z} r_>); \tag{7}$$

$$\mathcal{G}_l^{(0)}(r, r', E) = \frac{2}{\pi}\int_0^\infty p^2\, dp \frac{j_l(pr)\, j_l(pr')}{p^2 - E} = -\sqrt{E}\, j_l(\sqrt{E} r_<)\, n_l(\sqrt{E} r_>), \tag{8}$$

where $j_l(x)$, $h_l^{(1)}(x)$, and $n_l(x)$ are, respectively Bessel, Hankel, and Neumann spherical functions. Note that in (7) $\sqrt{Z} = \pm\sqrt{E}$ for $Z = E \pm i\epsilon$, $E > 0$, $\epsilon \to 0$, $\sqrt{Z} = i\gamma$ for $Z = -\gamma^2 < 0$, and in (8) $E > 0$. From Eqs. (5) and (6) and definitions (1), (2) and (7), (8) it follows directly that for short-range potentials $V(r)$ the off-shell wave functions have the asymptotic form

$$\psi_{lk}(r, Z) \underset{r\to\infty}{\sim} j_l(kr) + i\sqrt{Z}\, T_l(\sqrt{Z}, k, Z)\, h_l^{(1)}(r\sqrt{Z}); \tag{9}$$

$$\psi_{lk}(r, E) \underset{r\to\infty}{\sim} j_l(kr) - \sqrt{E}\, K_l(\sqrt{E}, k, E)\, n_l(r\sqrt{E}), \tag{10}$$

and the matrices T_l and K_l can be expressed in terms of the Fourier components $\psi_{lk}(p, Z)$ and $\psi_{lk}(p, E)$ of the corresponding off-shell wave functions:

$$T_l(p,\ k,\ Z)=(p^2-Z)\left[\psi_{lk}(p,\ Z)-\frac{\pi}{2p^2}\delta(p-k)\right]; \tag{11}$$

$$K_l(p,\ k,\ E)=(p^2-E)\left[\psi_{lk}(p,\ E)-\frac{\pi}{2p^2}\delta(p-k)\right]. \tag{12}$$

In the derivation of the last relation (12), the definition (8) is assumed, as usual, to be equivalent to the following:

$$\mathcal{G}_l^{(0)}(r,\ r',\ E)=(1/2)\lim_{\varepsilon\to 0}[\mathcal{G}_l(r,\ r',\ E+i\varepsilon)+\mathcal{G}_l(r,\ r',\ E-i\varepsilon)]. \tag{13}$$

Note further two facts. First, from the integral equations (5) and (6) there follow the differential equations

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+Z-V(r)\right]\psi_{lk}(r,\ Z)=(Z-k^2)\,j_l(kr); \tag{14}$$

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}+E-V(r)\right]\psi_{lk}(r,\ E)=(E-k^2)\,j_l(kr) \tag{15}$$

respectively, with the boundary conditions (9) and (10) as $r\to\infty$. Second, from Eqs. (5) and (6) and the definitions (1), (2) and (7), (8) one can obtain a connection between the matrices T_l and K_l (Ref. 22):

$$K_l(p,\ k,\ s^2)=T_l(p,\ k,\ s^2+i0)-is\frac{T_l(p,\ s,\ s^2+i0)\,T_l(s,\ k,\ s^2+i0)}{1+isT_l(s,\ s,\ s^2+i0)}. \tag{16}$$

The relations given above are completely adequate to find the correct expressions for the off-shell T and K matrices in the boundary-condition model. For this we shall assume in accordance with Refs. 12 and 13 that the following two propositions hold:

1) for the model off-shell wave functions $\psi_{lk}^{(c)}$ and the model $T_l^{(c)}$ and $K_l^{(c)}$ matrices the relations (9)—(12), which do not contain a potential explicitly, hold;

2) the model off-shell functions $\psi_{lk}^{(c)}$ satisfy the boundary conditions

$$\psi_{lk}^{(c)}=0,\quad r<c; \tag{17}$$

$$c\left[\frac{d}{dr}r\psi_{lk}^{(c)}\right]_{r=c_+}=f_l\,[r\psi_{lk}^{(c)}]_{r=c_+}, \tag{18}$$

where $c_+=c+\epsilon$, $\epsilon\to 0$; c, the boundary condition radius, may depend on l; f_l is a real model parameter, and c and f_l do not depend on the energy. The limit $f_l\to\infty$ corresponds to a model of a hard core of radius c. Such an approach is very simple and logical and accords completely with the spirit of the boundary-condition model, in the framework of which one should not consider the specific form of the interaction in the internal region $r<c$. From this point of view, it differs fundamentally from the method developed in Ref. 11 on the basis of the use of equations (5), (6) and (14), (15) for a potential of special form acting in the region $r\leq c$.

In this model, there is no potential in the external region, and therefore Eqs. (9) and (10) determine the form of the off-shell wave functions for $r>c$:

$$\psi_{lk}^{(c)}(r,\ Z)=j_l(kr)+i\sqrt{Z}\,T_l^{(c)}(\sqrt{Z},\ k,\ Z)\,h_l^{(1)}(r\sqrt{Z}); \tag{19}$$

$$\psi_{lk}^{(c)}(r,\ E)=j_l(kr)-\sqrt{E}\,K_l^{(c)}(\sqrt{E},\ k,\ E)\,n_l(r\sqrt{E}), \tag{20}$$

whence, with allowance for the boundary condition (18), there follow directly expressions for the half-on-shell matrices $T_l^{(c)}$ and $K_l^{(c)}$:

$$T_l^{(c)}(\sqrt{Z},\ k,\ Z)=\frac{i}{\sqrt{Z}}\frac{g_l(kc,\ f_l)}{D_l^{(1)}(c\sqrt{Z},\ f_l)}; \tag{21}$$

$$K_l^{(c)}(\sqrt{E},\ k,\ E)=\frac{1}{\sqrt{E}}\frac{g_l(kc,\ f_l)}{D_l(c\sqrt{E},\ f_l)}; \tag{22}$$

where

$$g_l(x,\ f_l)=xj_{l-1}(x)-(l+f_l)\,j_l(x); \tag{23}$$

$$D_l^{(1)}(x,\ f_l)=xh_{l-1}^{(1)}(x)-(l+f_l)\,h_l^{(1)}(x); \tag{24}$$

$$D_l(x,\ f_l)=xn_{l-1}(x)-(l+f_l)\,n_l(x). \tag{25}$$

The off-shell elements of the matrices $T_l^{(c)}$ and $K_l^{(c)}$ can be found from the expressions (11) and (12), respectively. With allowance for the conditions (17), the Fourier components $\psi_{lk}^{(c)}(p,Z)$ and $\psi_{lk}^{(c)}(p,E)$ will have the form

$$\psi_{lk}^{(c)}(p,\ Z)=\int_c^\infty r^2\,dr j_l(pr)\,\psi_{lk}^{(c)}(r,\ Z); \tag{26}$$

$$\psi_{lk}^{(c)}(p,\ E)=\int_c^\infty r^2\,dr j_l(pr)\,\psi_{lk}^{(c)}(r,\ E), \tag{27}$$

where the off-shell functions $\psi_{lk}^{(c)}$ are determined by the expressions (19) and (20). To calculate (26), it is convenient to represent the spherical Hankel function $h_l^{(1)}(r\sqrt{Z})$ in (19) for $r>c$ by means of (7) as follows:

$$h_l^{(1)}(r\sqrt{Z})=-\frac{i}{\sqrt{Z}}\frac{\mathcal{G}_l(r,\ c,\ Z)}{j_l(c\sqrt{Z})}, \tag{28}$$

and to find the Fourier component $\psi_{lk}^{(c)}(p,Z)$ (26) it is necessary to calculate the integral

$$I(Z)=\int_c^\infty r^2\,dr j_l(pr)\,\mathcal{G}_l(r,\ c,\ Z),$$

which, as follows from (7), has the form

$$I(Z)=\frac{j_l(pc)}{p^2-Z}-i\sqrt{Z}\,F_l(p,\sqrt{Z})\,h_l^{(1)}(c\sqrt{Z}), \tag{29}$$

where

$$F_l(p,\ k)=\int_0^c r^2\,dr j_l(pr)\,j_l(kr). \tag{30}$$

For positive energies ($E>0$) the Fourier component $\psi_{lk}(p,E)$ in (12) has a singularity at $p^2=E$, and one therefore needs definite rules for calculating some of the integrals when $\psi_{lk}^{(c)}(p,E)$ (27) is found. By analogy with the foregoing, we represent the Neumann function $n_l(r\sqrt{E})$ in (20) by means of (8) for $r>c$ in the form

$$n_l(r\sqrt{E})=-\frac{1}{\sqrt{E}}\frac{\mathcal{G}_l^{(0)}(r,\ c,\ E)}{j_l(c\sqrt{E})}. \tag{31}$$

Thus, to determine the Fourier component (27), it is necessary to find the integral

$$I_0(E)=\int_c^\infty r^2\,dr j_l(pr)\,\mathcal{G}_l^{(0)}(r,\ c,\ E).$$

Taking into account the relation (13), we can regard this integral as the following limiting expression:

$$I_0(E)=\frac{1}{2}\lim_{\varepsilon\to 0}[I(E+i\varepsilon)+I(E-i\varepsilon)].$$

Using the expression (29) for $I(Z)$ and noting that

$\sqrt{E \pm i\epsilon} = \pm\sqrt{E}(\epsilon \to 0)$, $h_l^{(1)}(-x) = (-1)^l h_l^{(2)}(x)$, $h_l^{(1)}(x) - h_l^{(2)}(x) = 2in_l(x)$, we can readily find the final result:

$$I_0(E) = j_l(pc)/(p^2 - E) + \sqrt{E}\, F_l(p, \sqrt{E})\, n_l(c\sqrt{E}). \quad (32)$$

From these expressions and Eqs. (11) and (12) the off-shell matrices $T_l^{(c)}$ and $K_l^{(c)}$ can be readily determined. Thus, it follows from (26), (19), and (29) that

$$T_l^{(c)}(p, k, Z) = -(p^2 - Z) F_l(p, k) + \frac{T_l^{(c)}(\sqrt{Z}, k, Z)}{j_l(c\sqrt{Z})} [j_l(pc) - i\sqrt{Z}(p^2 - Z) F_l(p, \sqrt{Z}) h_l^{(1)}(c\sqrt{Z})], \quad (33)$$

and Eqs. (27), (20), and (32) lead to the result

$$K_l^{(c)}(p, k, E) = -(p^2 - E) F_l(p, k) + \frac{K^{(c)}(\sqrt{E}, k, E)}{j_l(c\sqrt{E})} \times [j_l(pc) + \sqrt{E}(p^2 - E) F_l(p, \sqrt{E}) n_l(c\sqrt{E})], \quad (34)$$

where the half-on-shell values of the matrices $T_l^{(c)}$ and $K_l^{(c)}$ and $F_l(p, k)$ are given by (21), (22), and (30). It is easy to see that the expressions (33) and (34) for $T_l^{(c)}$ and $K_l^{(c)}$ are symmetric with respect to the momenta p and k and that the matrix $T_l^{(c)}$ in (33) satisfies the unitarity condition, and for positive energies the relation (16) is satisfied. For negative energies $Z = -E_0$ that are solutions of the equation

$$ic\sqrt{E_0} h_{l-1}^{(1)}(ic\sqrt{E_0}) - (l + f_l) h_l^{(1)}(ic\sqrt{E_0}) = 0, \quad (35)$$

the T matrix (33) has in accordance with (24) a pole. For the triplet s state of a proton and neutron, the energy E_0 must be set equal to the deuteron binding energy $\epsilon_d = \alpha^2$, Eq. (35) in this case takes the form

$$f_0 + \alpha c = 0, \quad (36)$$

and in the region $r > c$ the deuteron wave function is

$$\varphi_d(r) = \sqrt{2\alpha} \exp[-\alpha(r - c)]/r. \quad (37)$$

If there is only one bound s state, then the appropriately normalized on-shell wave functions (19) and the wave function (37) in the region $r > c$ for constant f_1 and c form a complete orthogonal system.[18]

As was pointed out above, the expression (33) for the T matrix was obtained in Ref. 11 by a limiting procedure applied to a potential of special form acting in the region $r \leq c$:

$$V(r) = V_0\theta(c - r) - cV_1\delta(r - c), \quad (38)$$

where $\theta(x) = 1$, $x > 0$; $\theta(x) = 0$, $x < 0$. For such a potential, one can readily find in analytic form the off-shell wave functions $\psi_{lk}(r, Z)$ and the T matrix. Subsequent passage to the limit $V_0 \to \infty$, $V_1 \to \infty$ with fulfillment of the condition $c(\sqrt{V_0} - cV_1) = f_1$, where f_1 is a constant, shows that the limiting value $T_l^{(c)}(p, k, Z)$ of the T matrix is determined by the expression (33), and the limit wave functions $\psi_{lk}^{(c)}(r, Z)$ satisfy the conditions (17) and (18).

B. Boundary condition model with potential in the external region

In the more general and more realistic case when a potential $V(r)$ acts in the region outside the boundary condition radius ($r > c$), we shall assume that the off-shell wave functions ψ_{lk} satisfy boundary conditions analogous to (17) and (18) (Refs. 12 and 13):

$$\psi_{lk} = 0, \quad r < c; \quad (39)$$

$$c\left[\frac{d}{dr} r\psi_{lk}\right]_{r=c_+} = f_l [r\psi_{lk}]_{r=c_+}, \quad (40)$$

and the off-shell T_l and K_l matrices are determined, respectively, by (11) and (12). Further, we represent the wave functions ψ_{lk} in the form

$$\psi_{lk} = \psi_{lk}^{(c)} + \varphi_{lk}, \quad (41)$$

where $\psi_{lk}^{(c)}$ are the off-shell wave functions (19) and (20) in the boundary condition model without external potential and they satisfy the conditions (17) and (18). It follows from the definition (41) that in the region $r > c$ the functions φ_{lk} satisfy in accordance with (14) and (15) the differential equations

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + Z\right] \varphi_{lk}(r, Z) = V(r)\psi_{lk}(r, Z); \quad (42)$$

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + E\right] \varphi_{lk}(r, E) = V(r)\psi_{lk}(r, E) \quad (43)$$

with potential $V(r)$ that is nonzero only for $r > c$, satisfy the boundary conditions (39) and (40), and as $r \to \infty$ have in accordance with (9) and (10) the asymptotic behavior

$$\varphi_{lk}(r, Z) \sim h_l^{(1)}(r\sqrt{Z}); \quad (44)$$

$$\varphi_{lk}(r, E) \sim n_l(r\sqrt{E}). \quad (45)$$

The differential equations (42) and (43) can be represented in accordance with (41) as integral equations for the functions ψ_{lk} by introducing the Green's functions H_l and $H_l^{(0)}$ of the equations

$$\psi_{lk}(r, Z) = \psi_{lk}^{(c)}(r, Z) - \int_c^\infty r'^2 dr' H_l(r, r', Z) V(r') \psi_{lk}(r', Z); \quad (46)$$

$$\psi_{lk}(r, E) = \psi_{lk}^{(c)}(r, E) - \int_c^\infty r'^2 dr' H_l^{(0)}(r, r', E) V(r') \psi_{lk}(r', E). \quad (47)$$

The Green's functions in these equations are defined only in the region $r, r' > c$, and they can be readily found under the assumption that the boundary-condition model imitates an interaction in the region $r < c$ that does not lead to the formation of a bound state for $E < 0$, and for $E > 0$ one considers an energy region that does not contain resonances associated with this interaction. In this case, the Green's functions can be constructed by means of the two linearly independent solutions of the corresponding homogeneous equations (42) and (43). With allowance for the asymptotic behavior (44) and (45) of the functions φ_{lk}, the final expressions for the Green's functions have the form

$$H_l(r, r', Z) = i\sqrt{Z}\, \psi_{l\sqrt{Z}}^{(c)}(r_<, Z) h_l^{(1)}(r_>\sqrt{Z}); \quad (48)$$

$$H_l^{(0)}(r, r', E) = -\sqrt{E}\, \psi_{l\sqrt{E}}^{(c)}(r_<, E) n_l(r_>\sqrt{E}). \quad (49)$$

Note that in accordance with the condition (17) in the region $r < c$, $r' > c$ the Green's functions (48) and (49) are identically zero.

Alternative expressions for the Green's functions can be obtained by substituting (19) and (20) into (48) and (49), which leads to the result

$$H_l(r, r', Z) = \mathcal{G}_l(r, r', Z) - ZT_l^{(c)}(\sqrt{Z}, \sqrt{Z}, Z)\, h_l^{(1)}(r\sqrt{Z})\, h_l^{(1)}(r'\sqrt{Z}); \quad (50)$$

$$H_l^{(0)}(r, r', E) = \mathcal{G}_l^{(0)}(r, r', E) + EK_l^{(c)}(\sqrt{E}, \sqrt{E}, E)\, n_l(r\sqrt{E})\, n_l(r'\sqrt{E}), \quad (51)$$

where $\mathcal{G}_l$ and $\mathcal{G}_l^{(0)}$ are determined by (7) and (8), respectively. The expression of the Green's functions in the form (50) and (51) enables us, using the relations (28), (31) and (29), (32), to obtain the following integral representations for the region $r, r' > c$:

$$H_l(r, r', Z) = \frac{2}{\pi}\int_0^\infty p^2\, dp\, \frac{j_l(pr)\, \psi_{lp}^{(c)}(r', Z)}{p^2 - Z}; \quad (52)$$

$$H_l^{(0)}(r, r', E) = \frac{2}{\pi}\int_0^\infty p^2\, dp\, \frac{j_l(pr)\, \psi_{lp}^{(c)}(r', E)}{p^2 - E}, \quad (53)$$

by means of which, in accordance with (11), (12) and (46), (47), one can readily show that the off-shell T and K matrices have the form

$$T_l(p, k, Z) = T_l^{(c)}(p, k, Z) + T_l^{(V)}(p, k, Z); \quad (54)$$

$$K_l(p, k, E) = K_l^{(c)}(p, k, E) + K_l^{(V)}(p, k, E), \quad (55)$$

where $T_l^{(c)}$ and $K_l^{(c)}$ are, respectively, the values (33) and (34) of the T and K matrices for $V(r) = 0$. The parts $T_l^{(V)}$ and $K_l^{(V)}$ of the total T and K matrices associated with the external potential $V(r)$ are defined in analogy with (1) and (2) by the expressions

$$T_l^{(V)}(p, k, Z) = -\int_c^\infty r^2\, dr \psi_{lp}^{(c)}(r, Z)\, V(r)\, \psi_{lk}(z, Z); \quad (56)$$

$$K_l^{(V)}(p, k, E) = -\int_c^\infty r^2\, dr \psi_{lp}^{(c)}(r, E)\, V(r)\, \psi_{lk}(r, E), \quad (57)$$

and they satisfy integral equations of Lippmann—Schwinger type:

$$T_l^{(V)}(p, k, Z) = V_l^{(1)}(p, k, Z) + \frac{2}{\pi}\int_0^\infty p'^2\, dp'\, \frac{V_l^{(2)}(p, p', Z)\, T_l^{(V)}(p', k, Z)}{p'^2 - Z}; \quad (58)$$

$$K_l^{(V)}(p, k, E) = V_l^{(1)}(p, k, E) + \frac{2}{\pi}\int_0^\infty p'^2\, dp'\, \frac{V_l^{(2)}(p, p', E)\, K_l^{(V)}(p', k, E)}{p'^2 - E}. \quad (59)$$

The generalized Fourier components $V_l^{(i)}(p, k, Z)$ of the potential $V(r)$ in Eq. (58) have the form

$$V_l^{(1)}(p, k, Z) = -\int_c^\infty r^2\, dr \psi_{lp}^{(c)}(r, Z)\, V(r)\, \psi_{lk}^{(c)}(r, Z); \quad (60)$$

$$V_l^{(2)}(p, k, Z) = -\int_c^\infty r^2\, dr \psi_{lp}^{(c)}(r, Z)\, V(r)\, j_l(kr). \quad (61)$$

The Fourier components $V_l^{(i)}(p, k, E)$ in (59) are obtained from (60) and (61) by formal replacement of the variable Z by E.

Note that the expressions (54)—(61) hold not only for the boundary-condition model but also when a certain potential $V_c(r)$ acts in the internal region $r < c$. In this case, the expression (54) for the total T matrix and Eq. (58) are well known from the theory for two non-overlapping potentials.[23] Because of the nonlinear connection (16) between the K and T matrices, the relation (55) and Eq. (59) do not appear completely obvious. Strictly speaking, Eq. (59) was obtained only for a potential $V_c(r)$ that does not lead to resonances in the considered energy range $E > 0$. This restriction is not fundamental, but otherwise one would need a more accurate construction of the Green's function (49). In the special case of s-wave scattering on a potential $V(r)$ with hard core ($f_0 \to \infty$) we obtain from the above relations for the K matrix

$$K_0(k, k, k^2) = \frac{1}{k}\operatorname{tg}\delta = \frac{1}{k}\left(\operatorname{tg}\delta_c + \frac{1}{\cos^2\delta_c}\,\frac{\operatorname{tg}\delta_V}{1 - \operatorname{tg}\delta_c \operatorname{tg}\delta_V}\right) = \frac{1}{k}\operatorname{tg}(\delta_c + \delta_V),$$

i.e., we obtain the well known result of Ref. 24: $\delta = \delta_c + \delta_V$, where δ is the total s-wave phase shift; $\delta_c = -kc$ is the phase shift for scattering on a hard core of radius c; δ_V is the phase shift for scattering on the shifted potential $V(r + c)$.

As a concrete example, in Ref. 13 a study was made of nucleon—nucleon s-wave interaction with a potential of exponential form in the external region:

$$V(r) = -V_0 \exp[-\beta(r - c)], \quad r > c. \quad (62)$$

The potential (62) is based on the consideration that as $r \to \infty$ it is damped exponentially and for $l = 0$ the scattering problem for such a potential has an exact analytic solution[25]; the potential (62) with hard core ($f_0 \to \infty$) was considered in Refs. 26—28.

Allowance for short-range forces by means of a more general boundary-condition model enables one to introduce into the expressions for the s-wave phase shifts an additional parameter f_0 not present in the hard-core case, which enables one, with given functional form of the potential in the region $r > c$, to vary the value of c without significantly worsening the agreement between the theoretical and the experimental phase shifts. In Table I, we give the values of the parameters f_0 and the parameters of the potential (62) obtained in Ref. 13 for some values of c for the triplet and singlet states of two nucleons, respectively. The parameters were determined by the least-squares method using the exact expressions for the s-wave phase shifts and the experimental values[29] of the $^3S_0$ and $^1S_0$ phase shifts up to a laboratory energy 360 MeV, which corresponds to the

TABLE I. Values of f_0 and the parameters V_0 and β of the potential (62) for the s state of two nucleons as a function of c and also the values of the scattering lengths a, the effective range r_0, and χ^2.

c, F	f_0	V_0, F^{-2}	β, F^{-1}	a, F	r_0, F	$29\,\chi^2$
			Triplet state			
0.2	1090.54	6.108	1.734	5.407	1.781	217.51
0.4	1100.31	10.528	2.406	5.396	1.762	8.97
0.6	5.13	7.176	2.486	5.393	1.771	2.37
0.7	17.87	23.008	4.177	5.426	1.726	61.41
0.8	23.22	48.972	6.356	5.435	1.696	84.49
		Experiment (Ref. 30)		5.396±0.011	1.726±0.014	
			Singlet state			
0.2	1109.22	4.468	1.810	−23.692	2.499	3.71
0.4	4.61	3.039	1.798	−23.668	2.506	2.67
0.6	7.69	6.151	2.460	−23.620	2.527	38.45
0.7	6.57	7.631	2.876	−23.334	2.480	66.89
0.8	20.08	18.906	4.028	−23.722	2.512	106.4
	Experiment (Ref. 30)			−23.678±0.028	2.51±0.11	

apppearance of the first resonance associated with the parameters f_0 and c. Use was also made of the experimental values of the scattering length a and the effective range r_0 (Ref. 30). Table I gives the values of χ^2, which was calculated from 29 points and whose minimum was used to choose the parameters. In Ref. 13, the adequacy of the relations (55) and (57) and Eq. (47) was tested. On the one hand, we obtained an analytic expression for the s-wave component of the total off-shell T matrix for the potential (62) with allowance for the boundary conditions (39) and (40) and, using Eq. (16), we found the exact value of the off-shell K matrix. On the other hand, we developed an effective method of approximate solution of Eq. (47) based on the method of moments.[31] The good agreement between the exact values and the approximate values of the K matrix obtained by means of (55) and (57) indicates the correctness of these relations and of Eq. (47), and therefore also of (59).

2. THREE-PARTICLE EQUATIONS IN THE BOUNDARY CONDITION MODEL

C. Schrödinger and Faddeev equations for a system of three identical particles

In Refs. 17–20, we considered the simplest variant of the three-particle problem with two-particle interactions: a bound state of three spinless identical particles. This restriction is not fundamental and at the same time, without superfluous algebraic complications, enables one to demonstrate the method for obtaining three-particle one-dimensional equations when the interactions are described by the boundary-condition model without external potential. As in the case of two particles, we consider first the problem under the assumption that the two-particle interactions are determined by potentials $\hat{V}(\mathbf{r})$ for which two-particle Lippmann—Schwinger equations and three-particle Faddeev equations hold. The expression of the potentials in the form $\hat{V}(\mathbf{r})$ presupposes a dependence of the interactions on the relative orbital angular momentum of a pair of particles.

In the center-of-mass system, we introduce Jacobi coordinates:

$$\mathbf{r}_1=\mathbf{R}_2-\mathbf{R}_3;\ \boldsymbol{\rho}_1=-\mathbf{R}_1+(\mathbf{R}_2+\mathbf{R}_3)/2, \tag{63}$$

where $\mathbf{R}_i$ is the radius vector of particle i. Besides the coordinates (63), in what follows we shall use the coordinates $\mathbf{r}_2, \boldsymbol{\rho}_2$ and $\mathbf{r}_3, \boldsymbol{\rho}_3$, which can be obtained from (63) by cyclic permutations of the indices. The wave function of a system of three identical bosons $\Psi(\mathbf{r}_1, \boldsymbol{\rho}_1)$ is symmetric with respect to any permutations of the particles:

$$\Psi(\mathbf{r}_1, \boldsymbol{\rho}_1)=\Psi(-\mathbf{r}_1, \boldsymbol{\rho}_1)=\Psi(\mathbf{r}_2, \boldsymbol{\rho}_2)=\Psi(\mathbf{r}_3, \boldsymbol{\rho}_3) \tag{64}$$

and can be expressed in terms of the channel function $\psi(\mathbf{r}, \boldsymbol{\rho})$ (in the given case a single one) as follows:

$$\Psi(\mathbf{r}_1, \boldsymbol{\rho}_1)=\psi(\mathbf{r}_1, \boldsymbol{\rho}_1)+\psi(\mathbf{r}_2, \boldsymbol{\rho}_2)+\psi(\mathbf{r}_3, \boldsymbol{\rho}_3). \tag{65}$$

In the momentum representation, the channel function $\psi(\mathbf{k}, \mathbf{q})$ satisfies the integral Faddeev equation

$$\psi(\mathbf{k}, \mathbf{q})=\left[2\pi^2\left(k^2+\tfrac{3}{4}q^2-E\right)\right]^{-1}\int d\mathbf{q}'\left[T\left(\mathbf{k}, \tfrac{1}{2}\mathbf{q}+\mathbf{q}', E-\tfrac{3}{4}q^2\right)\times\psi\left(-\mathbf{q}-\tfrac{1}{2}\mathbf{q}', \mathbf{q}'\right)+T\left(\mathbf{k}, -\tfrac{1}{2}\mathbf{q}-\mathbf{q}', E-\tfrac{3}{4}q^2\right)\times\psi\left(\mathbf{q}+\tfrac{1}{2}\mathbf{q}', \mathbf{q}'\right)\right], \tag{66}$$

where $E<0$ is the total energy of the system (binding energy); $T(\mathbf{k}, \mathbf{p}, Z)$ is the two-particle off-shell T matrix, which in accordance with (11) is related to the Fourier component $\psi_{\mathbf{p}}(\mathbf{k}, Z)$ of the two-particle off-shell wave function $\psi_{\mathbf{p}}(\mathbf{r}, Z)$:

$$T(\mathbf{k}, \mathbf{p}, Z)=(1/4\pi)(k^2-Z)[\psi_{\mathbf{p}}(\mathbf{k}, Z)-(2\pi)^3\delta(\mathbf{p}-\mathbf{k})]. \tag{67}$$

Substitution of (67) into (66) leads to the relation[17,18]

$$\Psi(\mathbf{r}, \mathbf{q})=\frac{1}{(2\pi)^3}\int d\mathbf{p}\psi_{\mathbf{p}}\left(\mathbf{r}, E-\tfrac{3}{4}q^2\right)\times\left[\psi\left(-\tfrac{1}{2}\mathbf{p}+\tfrac{3}{4}\mathbf{q}, -\mathbf{p}-\tfrac{1}{2}\mathbf{q}\right)+\psi\left(-\tfrac{1}{2}\mathbf{p}-\tfrac{3}{4}\mathbf{q}, \mathbf{p}-\tfrac{1}{2}\mathbf{q}\right)\right], \tag{68}$$

which relates the $\mathbf{r}$ dependence of the Fourier component $\Psi(\mathbf{r}, \mathbf{q})$ with respect to the variable $\boldsymbol{\rho}$ of the three-particle wave function (65) to the $\mathbf{r}$ dependence of the two-particle off-shell function $\psi_{\mathbf{p}}(\mathbf{r}, E-3q^2/4)$.

Further, for simplicity, we shall assume, as in Refs. 19 and 20, that the particles interact only in s states and that the total orbital angular momentum of the three-particle wave function $\psi(\mathbf{r}, \boldsymbol{\rho})$ depends only on r, ρ and on the angle between $\mathbf{r}$ and $\boldsymbol{\rho}$ and has the form

$$\Psi(\mathbf{r}_1, \boldsymbol{\rho}_1)=\psi(r_1, \rho_1)+\psi(r_2, \rho_2)+\psi(r_3, \rho_3), \tag{69}$$

and the result of applying $\hat{V}(\mathbf{r})$ to $\Psi(\mathbf{r}, \boldsymbol{\rho})$ is determined by the relation

$$\hat{V}(\mathbf{r})\Psi(\mathbf{r}, \boldsymbol{\rho})=V(r)\frac{1}{4\pi}\int d\Omega_{\mathbf{r}}\Psi(\mathbf{r}, \boldsymbol{\rho})=V(r)\Psi_0(r, \rho). \tag{70}$$

The channel function $\psi(r, \rho)$ in (69) is related to $V(r)\Psi_0(r, \rho)$ by

$$\psi(r, \rho)=\int_0^\infty r'^2 dr'\int_0^\infty \rho'^2 d\rho' \mathcal{G}_0(r, \rho; r', \rho'; E)V(r')\Psi_0(r', \rho'), \tag{71}$$

where

$$\mathcal{G}_0(r, \rho; r', \rho'; E)=-\frac{4}{\pi^2}\int_0^\infty k^2 dk\int_0^\infty q^2 dq\frac{j_0(kr)j_0(kr')j_0(q\rho)j_0(q\rho')}{k^2+3q^2/4-E}. \tag{72}$$

Thus, it follows from (69) and (71) that the channel function $\psi(r, \rho)$ and the total three-particle function $\Psi(\mathbf{r}, \boldsymbol{\rho})$ are uniquely determined if the product $V(r)\Psi_0(r, \rho)$ is known. This is very important for obtaining one-dimensional three-particle equations in the boundary-condition model without external potential.[19,20]

The equation for the s-wave component $\Psi_0(r, \rho)$ of the wave function $\Psi(\mathbf{r}, \boldsymbol{\rho})$ follows directly from the three-particle Schrödinger equation and with allowance for the symmetry (64) and Eq. (70) has the form

$$\frac{1}{r_1\rho_1}\left(\frac{\partial^2}{\partial r_1^2}+\frac{3}{4}\frac{\partial^2}{\partial\rho_1^2}+E\right)r_1\rho_1\Psi_0(r_1, \rho_1)=V(r_1)\Psi_0(r_1, \rho_1)+\frac{1}{4\pi}\int d\Omega_{r_1}[V(r_2)\Psi_0(r_2, \rho_2)+V(r_3)\Psi_0(r_3, \rho_3)]. \tag{73}$$

It follows from this equation that for the Fourier component $\Psi_0(r, q)$ with respect to the variable ρ of $\Psi_0(r, \rho)$ we have the equation

$$\frac{1}{r_1}\left[\frac{d^2}{dr_1^2}+E_q-V(r_1)\right]r_1\Psi_0(r_1, q)=S(r_1, q), \tag{74}$$

where

$$E_q = E - 3q^2/4; \tag{75}$$

$$S(r_1, q) = \frac{1}{4\pi}\int d\Omega_{r_1} d\rho_1 \exp[-iq\rho_1] \times [V(r_2)\Psi_0(r_2, \rho_2) + V(r_3)\Psi_0(r_3, \rho_3)]. \tag{76}$$

The solution of Eq. (74) by means of the Green's function $H(r, r', E)$ can be written formally in the form

$$\Psi_0(r, q) = \int_0^\infty r'^2 dr' H(r, r', E_q) S(r', q). \tag{77}$$

The Green's function has the form

$$H(r, r', E) = \begin{cases} -i\sqrt{E}\psi_{\sqrt{E}}(r, E)\varphi(r', E), & r < r'; \\ -i\sqrt{E}\varphi(r, E)\psi_{\sqrt{E}}(r', E), & r > r', \end{cases} \tag{78}$$

if it assumed that we know two linearly independent solutions of the homogeneous equation (74), $\psi_{\sqrt{E}}(r, E)$ and $\varphi(r, E)$, with the following asymptotic behavior as $r \to \infty$:

$$\left.\begin{aligned} &\psi_{\sqrt{E}}(r, E) \sim j_0(r\sqrt{E}) + i\sqrt{E}T(\sqrt{E}, \sqrt{E}, E)h_0^{(1)}(r\sqrt{E}); \\ &\varphi(r, E) \sim h_0^{(1)}(r\sqrt{E}), \end{aligned}\right\} \tag{79}$$

where $T(\sqrt{E}, \sqrt{E}, E)$ is the s-wave component of the two-particle T matrix (1) on the mass shell.[1)]

For potentials $V(r)$ with finite range c ($V(r) = 0$ for $r > c$) it follows from the expressions (77)—(79) that the product $V(r)\Psi_0(r, q)$ needed to determine the three-particle wave function can be represented in a form that does not explicitly contain the potential $V(r)$:

$$V(r)\Psi_0(r, q) = g(r, E_q)Y(q) + \theta(c-r)\int_0^\infty \rho^2 d\rho j_0(q\rho) D(r, \rho), \tag{80}$$

where $\theta(x) = 1$, $x > 0$; $\theta(x) = 0$, $x < 0$;

$$Y(q) = i\sqrt{E_q}\int_c^\infty r^2 dr h_0^{(1)}(r\sqrt{E_q}) S(r, q); \tag{81}$$

$$g(r, E) = \frac{2}{\pi}\int_0^\infty p^2 dp j_0(pr) T(p, \sqrt{E}, E). \tag{82}$$

D. Boundary-condition model in three-particle problem

The relations (80)—(82) serve as the basis for obtaining three-particle equations in the boundary-condition model without external potential. Indeed, in the boundary-condition model the half-on-shell two-particle T matrix in (82) has, with allowance for the symmetry condition $T^{(c)}(p, \sqrt{E}, E) = T^{(c)}(\sqrt{E}, p, E)$, the completely definite value (21). In addition, we shall assume[19,20] that the relation (80) also holds in the case when the two-particle interactions of the particles are described by the boundary-condition model. Thus, the problem of finding the three-particle wave function in the boundary-condition model reduces to determining the two-dimensional function $D(r, \rho)$ and the one-dimensional function $Y(q)$ in (80). Equations for these functions can be obtained on the basis of the relation (68) and the boundary conditions (17) and (18) for the s-wave component of the off-shell two-particle wave function, in accordance with which we have for the s-wave component of the three-particle wave function

$$\Psi_0(r, q) = 0, \quad r < c; \tag{83}$$

$$c\left[\frac{d}{dr} r\Psi_0(r, q)\right]_{r=c_+} = f(E_q)[r\Psi_0(r, q)]_{r=c_+}. \tag{84}$$

A possible dependence of the logarithmic derivative f on the energy E_q (75) is indicated explicitly in the condition (84). In principle, the parameter f in the condition (18) for two particles can be assumed to depend on the energy E. However, it is readily seen in this case that the on-shell wave functions ($E = k^2$) corresponding to physical states of two nucleons at different energies will be nonorthogonal if the condition (17) is preserved. Nevertheless, for energies $E < -\epsilon_d$ (ϵ_d is the deuteron binding energy) f can certainly be regarded as an arbitrary function of E because the on-shell wave functions in this case are not related to real states of two nucleons and the violation of the orthogonality condition for them does not have any particular physical meaning. The possibility of introducing a dependence of f on E for $E < -\epsilon_d$ will be discussed below in more detail in connection with approximate allowance for three-particle effects.

The condition (83) can be written down in the coordinate representation, and then in the region $r_1 < c$ we shall have

$$V(r_1)\Psi_0(r_1, \rho_1) + \frac{1}{4\pi}\int d\Omega_{r_1}[V(r_2)\Psi_0(r_2, \rho_2) + V(r_3)\Psi_0(r_3, \rho_3)] = 0. \tag{85}$$

If the condition (85) is satisfied for $r_1 < c$, this means that $\Psi_0(r_1, \rho_1) = 0$ in the region $r_1 < c$, since a nontrivial solution of the homogeneous equation (73) that is finite as $\rho_1 \to 0$ is incompatible with the boundary condition as $\rho_1 \to \infty$. The first equation for the functions $D(r, \rho)$ and $Y(q)$ follows from (85) if $V(r)\Psi_0(r, \rho)$ is expressed by means of (80) and use is made of the explicit expression for $g(r, E)$, which, in accordance with (82) and (21), has the form

$$g(r, E) = -\frac{1}{r}\frac{\exp[-ic\sqrt{E}]}{f(E) - ic\sqrt{E}}[f(E)\delta(r-c) + c\delta'(r-c)]. \tag{86}$$

Finally, we arrive at the equation

$$\theta(c-r_1)D(r_1, \rho_1) + \theta(c-r_1)\frac{1}{4\pi}\int d\Omega_{r_1}[\theta(c-r_2)D(r_2, \rho_2) + \theta(c-r_3)D(r_3, \rho_3)] = -\theta(c-r_1)F(r_1, \rho_1), \tag{87}$$

where

$$F(r_1, \rho_1) = \frac{1}{2\pi^2}\int d\Omega_{r_1}\int_0^\infty q^2 dq [g(r_2, E_q) j_0(q\rho_2) + g(r_3, E_q) j_0(q\rho_3)] Y(q). \tag{88}$$

In deriving (87), we have used the relation $\theta(c - r)g(r, E) = 0$, which follows from the fact that in the boundary-condition model it is necessary, in accordance with (82), (21), and (18), to regard c in the expression (86) for $g(r, E)$ as $c_+ = c + \epsilon$, $\epsilon \to 0$. Note also that in the derivation of (87) and (84) there is no question at all of

[1)] In (78) and (79) we have for simplicity omitted the index $l = 0$, which characterizes the orbital angular momentum of two particles. In what follows, we shall also omit it throughout.

introducing a definite form of the potential $V(r)$ in the region $r<c$; rather, one uses only the boundary conditions (17) and (18) for the off-shell two-particle wave functions.

The second equation for $D(r,\rho)$ and $Y(q)$ is obtained from the boundary condition (84) if the total wave function is expressed in terms of the channel function in accordance with (69). In accordance with (80), it follows from the expressions (71) and (72) that in the momentum representation the channel function $\psi(k,q)$ has the form

$$\psi(k,\ q)=-\frac{4\pi c}{k^2+\frac{3}{4}q^2-E}\,\frac{\cos kc-f(E_q)\,j_0(kc)}{f(E_q)-ic\sqrt{E_q}}\times\exp\left(-ic\sqrt{E_q}\right)Y(q)$$
$$-\frac{1}{4\pi\left(k^2+\frac{3}{4}q^2-E\right)}\int d\mathbf{r}\,d\boldsymbol{\rho}\exp\left(i\mathbf{k}\mathbf{r}+i\mathbf{q}\boldsymbol{\rho}\right)\theta(c-r)\,D(r,\ \rho). \quad (89)$$

The use of (69) and (89) and the condition (84) leads to the equation

$$\exp\left(-ic\sqrt{E_q}\right)Y(q)-\frac{2}{\pi}\int_0^\infty q'^2\,dq'\,T_1(q,\ q')\exp\left(-ic\sqrt{E_{q'}}\right)Y(q')$$
$$+\frac{1}{cL(E_q)\left[f(E_q)-ic\sqrt{E_q}\right]}\left\{c\left[\frac{d}{dr}r\Psi_0^{(1)}(r,\ q)\right]_{r=c+}-f(E_q)\left[r\Psi_0^{(1)}(r,\ q)\right]_{r=c+}\right\}=0, \quad (90)$$

where

$$T_1(q,\ q')=\frac{c}{L(E_q)\left[f(E_q)-ic\sqrt{E_q}\right]\left[f(E_{q'})-ic\sqrt{E_{q'}}\right]}$$
$$\times\int_{-1}^{+1}dx\,\frac{\left[\cos p_1c-f(E_{q'})\,j_0(p_1c)\right]\left[\cos p_2c-f(E_q)\,j_0(p_2c)\right]}{q^2+q'^2-qq'x-E}; \quad (91)$$

$$p_1^2=q^2+q'^2/4-qq'x;\ p_2^2=q^2/4+q'^2-qq'x;$$
$$L(E)=\left[1/(2ic\sqrt{E})\right]\left\{\left[f(E)+ic\sqrt{E}\right]/\left[f(E)-ic\sqrt{E}\right]-\exp\left(2ic\sqrt{E}\right)\right\};$$
$$\Psi_0^{(1)}(r,\ q)=-\frac{1}{(4\pi)^2}\int d\mathbf{r}_1\,d\boldsymbol{\rho}_1K_0(r,\ r_1,\ E_q)\,j_0(q\rho_1)$$
$$\times\left[\theta(c-r_1)\,D(r_1,\ \rho_1)+\theta(c-r_2)\,D(r_2,\ \rho_2)+\theta(c-r_3)\,D(r_3,\ \rho_3)\right]; \quad (92)$$

$$K_0(r,\ r',\ E)=\frac{2}{\pi}\int_0^\infty k^2\,dk\left[j_0(kr)\,j_0(kr')/(k^2-E)\right].$$

An alternative second equation for the functions $D(r,\rho)$ and $Y(q)$ can be obtained from (81) if, as before, $V(r)\Psi_0(r,\rho)$ is expressed by means of (80) and (76) is used for $S(r,q)$. This equation has the form

$$\exp\left(-ic\sqrt{E_q}\right)Y(q)-\frac{2}{\pi}\int_0^\infty q'^2\,dq'\,T_2(q,\ q')\exp\left(-ic\sqrt{E_{q'}}\right)Y(q')$$
$$-\frac{i\sqrt{E_q}}{(4\pi)^2}\exp\left(-ic\sqrt{E_q}\right)\int d\mathbf{r}_1\,d\boldsymbol{\rho}_1\theta(r_1-c)\,h_0^{(1)}(r_1\sqrt{E_q})\,j_0(q\rho_1)$$
$$\times\left[\theta(c-r_2)\,D(r_2,\ \rho_2)+\theta(c-r_3)\,D(r_3,\ \rho_3)\right], \quad (93)$$

where

$$T_2(q,\ q')=\frac{c}{f(E_{q'})-ic\sqrt{E_{q'}}}$$
$$\times\int_{-1}^{+1}dx\,\frac{\left[\cos p_1c-f(E_{q'})\,j_0(p_1c)\right]\left[\cos p_2c-ic\sqrt{E_q}\,j_0(p_2c)\right]}{q^2+q'^2-qq'x-E}. \quad (94)$$

E. One-dimensional equation

Equations (90) and (93) contain the unknown two-dimensional function $D(r,\rho)$, which, in its turn, satisfies Eq. (87). In Ref. 32[2)] it was shown that in the case of a hard core one can find an exact solution in analytic form of this equation. Using the method proposed in Ref. 32, one can solve Eq. (87) analytically as well for the more general case of the boundary-condition model.[19] This circumstance in conjunction with the expression (88) enables one to reduce, without any approximation, the two-dimensional equations (90) and (93) to one-dimensional equations for the single function $Y(q)$, which is a characteristic feature of the three-particle problem in the case of two-particle interactions described by means of the boundary-condition model.[14-16]

In accordance with (86) and (88), the right-hand side of Eq. (87) has the form

$$\theta(c-r_1)\,F(r_1,\ \rho_1)=-\theta(c-r_1)\,\chi(R)/(r_1\rho_1)$$
$$-\left[c/(r_1\rho_1)\right]\theta(c-r_1)\left[\delta(r_1/2+\rho_1-c)-\delta(|r_1/2-\rho_1|-c)\right]\varphi(R), \quad (95)$$

where

$$\chi(R)=\frac{4}{\pi}\int_0^\infty q^2\,dqY(q)\,b(E_q)\,F_0(q,\ R); \quad (96)$$

$$\varphi(R)=\frac{4}{\pi}\int_0^\infty q^2\,dqY(q)\,b(E_q)\,j_0(q\rho_0); \quad (97)$$

$$b(E)=\exp\left(-ic\sqrt{E}\right)/\left[f(E)-ic\sqrt{E}\right];$$
$$F_0(q,\ R)=f(E_q)\,j_0(q\rho_0)-(3qc^2/4\rho_0)\,j_1(q\rho_0); \quad (98)$$
$$R^2=r_1^2+4\rho_1^2/3;\ \rho_0^2=3(R^2-c^2)/4,$$

and the function $F_0(q,R)$ is nonzero only in the region

$$|r_1/2-\rho_1|<c<r_1/2+\rho_1. \quad (99)$$

In Refs. 19 and 20 it is shown that, in accordance with (95), the solution of Eq. (87) can be represented in the form

$$\theta(c-r)\,D(r,\ \rho)=\theta(c-r)\frac{1}{r\rho}A(r,\ \rho)$$
$$+\theta(c-r)\frac{c}{r\rho}\left[\delta(r/2+\rho-c)-\delta(|r/2-\rho|-c\right]\varphi(R), \quad (100)$$

where the function $A(r,\rho)$ in the region $r<c$ satisfies the integral equation

$$A(r,\ \rho)+2\int_{|r/2-\rho|}^{r/2+\rho}\frac{dr'}{|\rho'}\,\theta(c-r')\,A(r',\ \rho')=\chi(R)-2c\varphi(R)\,\mathcal{G}(r,\ \rho); \quad (101)$$

$$\mathcal{G}(r,\ \rho)=\int_{|r/2-\rho|}^{r/2+\rho}\frac{dr'}{\rho'}\,\theta(c-r')\left[\delta(r'/2+\rho'-c)-\delta(|r'/2-\rho'|-c)\right];$$
$$R^2=r^2+4\rho^2/3;\quad \rho'^2=3(R^2-r'^2)/4, \quad (102)$$

which will be one-dimensional if R is regarded as a fixed parameter. If new variables α and α' are introduced in accordance with the definitions

$$\left.\begin{aligned} r&=R\sin\alpha,\ \rho=(\sqrt{3}/2)\,R\cos\alpha;\\ r'&=R\sin\alpha',\ \rho'=(\sqrt{3}/2)\,R\cos\alpha',\end{aligned}\right\} \quad (103)$$

Eq. (101) and the expression (102) take the form

[2)] The author of Ref. 32, V. Efimov (Institute of Nuclear Physics, Leningrad), is a namesake of one of the authors (V. N. E.) of the present paper.

$$A(R,\alpha)+\frac{4}{\sqrt{3}}\int_{|\pi/3-\alpha|}^{\min(\pi/3+\alpha,\,2\pi/3-\alpha)}d\alpha'\theta(c-R\sin\alpha')A(R,\alpha')=\chi(R)-2c\varphi(R)\mathcal{G}(R,\alpha); \quad (104)$$

$$\mathcal{G}(R,\alpha)=\frac{2}{\sqrt{3}}\int_{|\pi/3-\alpha|}^{\min(\pi/3+\alpha,\,2\pi/3-\alpha)}d\alpha'\theta(c-R\sin\alpha')\{\delta[R\theta(\pi/6-\alpha')\times\sin(\alpha'+\pi/3)+R\theta(\alpha'-\pi/6)\sin(2\pi/3-\alpha')-c]-\delta[R\sin|\pi/3-\alpha'|-c]\}. \quad (105)$$

Depending on the value of the parameter R, the region $r<c$ is divided in accordance with (103) and (99) into the series of regions shown in Fig. 1, in which the limits of integration in (104) and the function $\mathcal{G}(R,\alpha)$ (105) have different values. The explicit form of the integral equations for $A(R,\alpha)$ in the regions 1–6 shown in Fig. 1 is given in Ref. 19. These equations differ from those of Ref. 32 only by the right-hand sides, and for their solution one can definitely use the method proposed in Ref. 32, which leads to the following solutions of Eq. (104) for the regions 1–5 (see Fig. 1):

$$\left.\begin{aligned}&A_1(R,\alpha)=0;\quad A_2(R,\alpha)=B(R)\sin(\gamma-\pi/4);\\&A_3(R,\alpha)=C(R)\sin(\gamma-\pi/4);\\&A_4(R,\alpha)=A_5(R,\alpha)=D(R)\sin4\alpha,\end{aligned}\right\} \quad (106)$$

where

$$B(R)=-\chi(R)/\sin(\gamma_0+\pi/4);$$

$$C(R)=\frac{\sqrt{3}}{2\sqrt{2}}\frac{1}{A(R)}\left[\chi(R)\sin4\varkappa_0+\frac{4c}{R\cos\alpha_0}\varphi(R)\cos4\varkappa_0\right];$$

$$D(R)=-\frac{\sqrt{3}}{2}\frac{1}{Q(R)}\left[\chi(R)\sin\frac{4}{\sqrt{3}}\varkappa_0+\frac{4c}{\sqrt{3}R\cos\alpha_0}\varphi(R)\cos\frac{4}{\sqrt{3}}\varkappa_0\right];$$

$$Q(R)=\frac{\sin(4+4/\sqrt{3})\varkappa_0}{4+4/\sqrt{3}}-\frac{\sin(4-4/\sqrt{3})\varkappa_0}{4-4/\sqrt{3}};$$

$$\gamma=(4/\sqrt{3})(\alpha-\pi/6);\quad \gamma_0=(4/\sqrt{3})(\alpha_0-\pi/6);$$

$$\alpha_0=\arcsin(c/R);\quad \varkappa_0=\pi/2-\alpha_0.$$

In region 6 the solution $A_6(R,\alpha)$ of Eq. (104) has the form

$$A_6(R,\alpha)=F(R)\sin4\alpha, \quad (107)$$

where $F(R)$ is an arbitrary function of R. This arbitrariness corresponds to the fact noted in Refs. 15 and 16 that in the boundary-condition model the three-particle wave function is not determined uniquely. In the very simple case of the three-particle system considered here—three identical bosons—it follows from the condition (64) of symmetry of the total wave function that, in accordance with Ref. 32, it is necessary to set $F(R)\equiv0$ in (107) if it is assumed that the boundary-condition model corresponds to the limiting form of the potential (38). Thus, in contrast to Ref. 16, in which use is made of the equations obtained in Ref. 15, Eqs. (90) and (93) in conjunction with (89) uniquely determine the wave function (69) of the system of three identical bosons without the introduction of any additional condition containing an arbitrary three-particle parameter. Similar considerations also apply to a system of three nucleons, for which the components of the wave function in the case of charge invariance of the nuclear forces have a definite permutational symmetry.

The nonuniqueness mentioned above of the solution (107) in the region 6 is quite unimportant if the binding

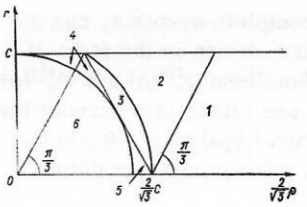

FIG. 1. Domain of definition of the function $A(R,\alpha)$.

energy is calculated by means of the alternative equation (93), since this region does not occur in the integral of the expression (81), from which Eq. (93) is obtained. A similar situation also exists in the calculation of transition amplitudes, since they are determined by integrals of the type (81). The nonuniqueness must be eliminated only if (69) and (89) are used to find the three-particle wave function. Note, finally, that Eqs. (90) and (93) can be expressed by means of (92) and (100) and the expressions (106) for the function $A(R,\alpha)$ in the explicit one-dimensional form

$$\psi(q)-\frac{2}{\pi}\int_0^\infty q'^2\,dq'\,[T_1(q,q')+V_1(q,q')]\psi(q')=0; \quad (108)$$

$$\psi(q)-\frac{2}{\pi}\int_0^\infty q'^2\,dq'\,[T_2(q,q')+V_2(q,q')]\psi(q')=0, \quad (109)$$

where $\psi(q)=\exp(-ic\sqrt{E_q})Y(q)$; $T_1(q,q')$ and $T_2(q,q')$ are determined, respectively, by (91) and (94). We do not give the expressions for $V_1(q,q')$ and $V_2(q,q')$, which are cumbersome, but merely point out that they can be found in Ref. 20.

F. Results for the energy-independent parameters f and c

In Ref. 20, Eqs. (108) and (109) were solved for constant parameters f and c corresponding to the triplet s state of two nucleons. In this case, f and c are determined from the deuteron binding energy $\epsilon_d=2.225$ MeV in accordance with the relation (36) and from the triplet scattering length $a_t=5.41$ F, and they have the values

$$f=-0.253,\quad c=1.095\ \text{F}. \quad (110)$$

The numerical solution of Eqs. (108) and (109) entails definite mathematical difficulties due to the oscillatory nature and slow decrease of the kernels $V_1(q,q')$ and $V_2(q,q')$ of these equations, whose dependence on q' is determined by the expressions (96)–(98). This behavior of the kernels prevents one solving Eqs. (108) and (109) by the standard method of transformation of the infinite interval of integration into a finite one with subsequent use of Gaussian nodes. In Ref. 20, Eqs. (108) and (109) were solved by means of two different methods. The first method is based on an expansion of $j_0(q\rho_0)$ and $j_1(q\rho_0)$ in (97) and (98) in series with respect to a complete system of orthogonal functions $\varphi_n^{(l)}(\rho_0)$ $(l=0,1)$. The parameter ρ_0 varies in the range $0\leq\rho_0\leq a$, $a=3c/2$, and therefore, to obtain the necessary expansions, it is necessary to introduce functions $\varphi_n^{(l)}(\rho)$ satisfying the orthogonality and normalization condition

$$\int_0^a\rho^2\,d\rho\,\varphi_n^{(l)}(\rho)\,\varphi_{n'}^{(l)}(\rho)=\delta_{nn'}. \quad (111)$$

Such functions, which form complete systems, can always be constructed if they are chosen in the form of polynomials. In particular, functions $\varphi_n^{(0)}(\rho)$ and $\varphi_n^{(1)}(\rho)$ satisfying the condition (111) and taking into account the parity and behavior of $j_0(q\rho)$ and $j_1(q\rho)$ as $\rho \to 0$ can be expressed in terms of Jacobi polynomials and determined as follows:

$$\varphi_n^{(0)}(\rho) = \frac{(4n-1)^{1/2}}{2^{n-1}a^{3/2}} \sum_{\nu=1}^{n} \alpha_{n\nu}^{(0)} \left(\frac{\rho}{a}\right)^{2\nu-2}, \quad n=1, 2, 3, \ldots; \tag{112}$$

$$\varphi_n^{(1)}(\rho) = \frac{(4n+1)^{1/2}}{2^{n-1}a^{3/2}} \sum_{\nu=1}^{n} \alpha_{n\nu}^{(1)} \left(\frac{\rho}{a}\right)^{2\nu-1}, \quad n=1, 2, 3, \ldots, \tag{113}$$

where

$$\alpha_{n\nu}^{(0)} = (-1)^{n+\nu} \frac{(2n+2\nu-3)!!}{[(2\nu-1)!!\,(\nu-1)!\,(n-\nu)!}\,;$$

$$\alpha_{n\nu}^{(1)} = (-1)^{n+\nu} \frac{(2n+2\nu-1)!!}{(2\nu+1)!!\,(\nu-1)!\,(n-\nu)!}\,.$$

The expansions of $j_0(q\rho)$ and $j_1(q\rho)$, respectively, with respect to the functions (112) and (113) have the form

$$\left.\begin{aligned} j_0(q\rho) &= \sum_n C_n^{(0)}(q)\,\varphi_n^{(0)}(\rho); \\ j_1(q\rho) &= \sum_n C_n^{(1)}(q)\,\varphi_n^{(1)}(\rho), \end{aligned}\right\} \tag{114}$$

where

$$C_n^{(0)}(q) = (-1)^{n+1}a^{3/2}\sqrt{4n-1}\,\frac{1}{x}\,j_{2n-1}(x);$$

$$C_n^{(1)}(q) = (-1)^{n+1}a^{3/2}\sqrt{4n+1}\,\frac{1}{x}\,j_{2n}(x); \quad x = qa.$$

The use of the expansions (114) with a finite number N of terms in (96)—(98) enables one to reduce the kernels $V_1(q,q')$ and $V_2(q,q')$ of Eqs. (108) and (109) to a degenerate form ($2N$ separable terms). For such kernels, one can readily find resolvents, and, as a result, one can replace the original equations (108) and (109) by new equations with renormalized kernels, to which the ordinary method using Gaussian nodes for a finite integration interval can be applied.

The second method of solving (108) and (109) is based on a splitting of the infinite interval of integration into two subintervals $(0, q_0)$ and (q_0, ∞) and the introduction of a new variable $x = q'^2 - q_0^2$ and formal weight $\exp(-\alpha x)$, as a result of which (108) and (109) take the form

$$q\psi(q) = \frac{2}{\pi}\int_0^{q_0} qq'\,dq'\,[T(q,q')+V(q,q')]\,q'\psi(q')$$

$$+\frac{1}{\pi}\int_0^{\infty} q\exp(-\alpha x)\,dx\,[T(q,q')+V(q,q')]\,q'\psi(q'). \tag{115}$$

For finite α, to solve (115) one can use the standard method using Gaussian nodes [in the second integral, using nodes for the weight $\exp(-\alpha x)$], and the solutions (108) and (109) will correspond to the approximately determined limit of solutions of equations of the type (115) as $\alpha \to 0$.

The two above methods were used in Ref. 20 to solve Eqs. (108) and (109), the latter being solved by the first method and the former by the second. The equations were solved with the parameters f and c (110), and for the binding energy of three identical bosons we obtained the value $E_0 = 7.70$ MeV, which differs strongly from the value $E_0 = 12.69$ MeV found in Ref. 21 for the same model problem from the Faddeev equation for the same values of the parameters f and c. As was noted in Ref. 20, it is an important fact that the two alternative Eqs. (108) and (109) lead to one and the same value of E_0 when two different methods of solution are used. Actually, there is a spread $\Delta E = \pm 0.15$ MeV, which is attributed to the inadequate accuracy of the calculations associated with the use of a bounded number of terms ($N \leq 8$) in the expressions (114) and the bounded values of α ($\alpha \geq 0.01$) in (115). The agreement of the results to within this spread may serve as an indication of the adequacy of (108) and (109) and the numerical methods employed for their solution.

Of Eqs. (108) and (109), the latter is of particular interest for two reasons. The first is that when this equation is used to calculate the binding energy E_0 there is no need at all to consider region 6 (see Fig. 1), which, in accordance with (107), is the only region that introduces a nonuniqueness which can be eliminated only by additional arguments. The second reason is this. If in the s-wave component of the two-particle T matrix corresponding to the boundary-condition model and determined in accordance with (33) by the expression

$$T(k,\ p,\ Z) = T(p,\ k,\ Z) = -(p^2-Z)\int_0^c r^2\,dr\,j_0(pr)\,j_0(kr)$$

$$+\,c\,\frac{\cos kc - f j_0(kc)}{f - ic\sqrt{Z}}\,[\cos pc - ic\sqrt{Z}\,j_0(pc)], \tag{116}$$

the first term is ignored, we obtain a certain factorized approximation:

$$\overline{T}(k,\ p,\ Z) = c\,\{[\cos kc - f j_0(kc)]/(f - ic\sqrt{Z})\}$$

$$\times\,[\cos pc - ic\sqrt{Z}\,j_0(pc)]. \tag{117}$$

For the approximate T matrix (117), the relations from which the nonuniqueness of the Faddeev equations follows[14] do not hold. Direct substitution of (117) into the Faddeev equation for a system of three identical bosons reduces it to a one-dimensional integral equation with the kernel (94), i.e., in fact to (109) with $V_2(q,q') = 0$. The kernel $T_2(q,q')$ (94) has a very simple structure and decreases fairly rapidly with respect to q and q', so that the solution of the resulting integral equation is trivial and leads to the value $\overline{E}_0 = 13.21$ MeV. Knowing the two-particle T matrix, we can in accordance with (11) determine the wave functions corresponding to the approximate expression (117). Thus, for the on-shell wave function, we have

$$\left.\begin{aligned} \psi_k(r,\ k^2+i0) &= 0, && r<c; \\ \psi_k(r,\ k^2+i0) &= j_0(kr) + T(k,\ k,\ k^2+i0)\exp(ikr)/r, && r>c; \end{aligned}\right\} \tag{118}$$

where $T(k, k, k^2+i0)$ is defined in accordance with (21), whereas in the region $r < c$ the off-shell function is nonzero and has the form

$$\psi_k(r,\ Z) = j_0(kr) - \exp(ic\sqrt{Z})\,[\cos kc - ic\sqrt{Z}\,j_0(kc)]\,j_0(r\sqrt{Z}). \tag{119}$$

Thus, the transition from the exact T matrix (116) to the approximate (117) corresponds to the transition from the limit form of the potential (38) to some quasipotential of the type considered in Ref. 6, which in accordance with (118), reproduces the same scattering

TABLE II. Dependence of the binding energies $\bar{E}_0$ and E_0 obtained from (109), respectively, without $V_2(q,q')$ and with allowance for $V_2(q,q')$ on the parameters f and c, which satisfy the condition $f=-c\sqrt{\epsilon_d}$ (36).

f	c, F	$\bar{E}_0$, MeV	E_0, MeV	f	c, F	$\bar{E}_0$, MeV	E_0, MeV
−0.253	1.095	13.21	7.70	−0.173	0.7489	17.96	10.61
−0.213	0.9221	15.60	9.48	−0.133	0.5757	24.8	17.20

amplitude as the boundary condition model but does not contain a repulsive core in the region $r<c$, which follows from the expressions (119) and (17). Therefore, the transition from the solution of Eq. (109) with only $T_2(q,q')$ to the solution of this equation with allowance for $T_2(q,q')$ and $V_2(q,q')$ will correspond to the transition from the quasipotential without core to the limit form of the potential (38) with infinite repulsive core for $r<c$. Obviously, such a procedure can only reduce the binding energy E_0, which is confirmed by the result obtained in Ref. 20. Note that $E_0=7.7$ MeV also differs significantly from the $E_0=18.4$ MeV of Ref. 16. This may be explained by the fact that the one-dimensional equations (108) and (109) are very different from the equation of Ref. 16. In particular, it follows from the derivation of the equation of Ref. 16 that the total wave function $\Psi(\mathbf{r},\rho)$ in the region $r<c$ vanishes, whereas (108) and (109) were derived under the condition [see (83)] of vanishing for $r<c$ of only the s-wave component of the three-particle wave function $\Psi(\mathbf{r},\rho)$. Thus, in accordance with (68), the equation of Ref. 16 corresponds to two-particle interactions with hard core acting in all the partial-wave components. In contrast to this, (108) and (109) have been obtained under the assumption that there are interactions including a hard core in only the relative two-particle s-wave states.

With regard to Ref. 16, in which the equations obtained in Ref. 15 were used, one can make the following additional remark. As was pointed out above, to achieve uniqueness of these equations it is necessary to introduce an arbitrary three-particle parameter W_0, which is a particular value of the energy at which boundary conditions analogous to the conditions (17) and (18) for the two-particle wave functions in the boundary value model are imposed independently on the channel functions. These additional, essentially three-particle conditions do not follow from the nature of the two-particle interactions and have the consequence that the equations of Ref. 15 are in fact model equations. In contrast, the one-dimensional equations (108) and (109) are not model equations at the three-particle level since their derivation uses only the boundary conditions (17) and (18) for the two-particle wave functions. Therefore, the results of Refs. 16 and 33 cannot be assumed to correspond to the "vacuum" two-particle interactions described by the boundary-condition model without external potential. The parameter W_0 determines the form of the kernels of the integral equations of Ref. 15 and 16 by means of the function $D(q)$ obtained from (24) for $l=0$ and $\varkappa=c\sqrt{W_0-3q^2/4}$, whose dependence on q is given by

$$D(q)=-ixD_0^{(1)}(x,f)=[f-ic\sqrt{W_0-3q^2/4}]\exp[ic\sqrt{W_0-3q^2/4}], \quad (120)$$

where f satisfies Eq. (36). It follows from (120) that the nature of the function $D(q)$ will depend strongly on the choice of W_0. In the region $W_0<-\epsilon_d$, $D(q)$ will be real and of constant sign; in the region $-\epsilon_d<W_0<0$ it will be real but of variable sign. The choice $W_0<0$ certainly ensures fulfilment of the unitarity condition for the three-particle T matrix,[15] but in Ref. 16 an investigation was made of only the region $W_0<-\epsilon_d$ and it was shown that there is a weak dependence of the binding energy E_0 of three bosons on the parameter W_0, whereas for completeness it would also be necessary to consider the region $-\epsilon_d<W_0<0$.

From these explicit expressions for the kernels $T_2(q,q')$ (94) and $V_2(q,q')$ (Ref. 20) of Eq. (109) it can be seen that $T_2(q,q')\sim c^0$ and $V_2(q,q')\sim c^2$. Thus, the relative contribution to the binding energy E_0 associated with the allowance in (109) for $V_2(q,q')$ must be reduced with decreasing c if $c<1$ F. In Table II, we give the results of calculations of E_0 for a number of c values; the value of f was varied and c determined by the condition $c=-f/0.231$ F, which, in accordance with (36), guaranteed, for different sets of f and c values, constancy of the deuteron binding energy $\epsilon_d=2.225$ MeV. In Table II, we also give the values of $\bar{E}_0$ obtained with allowance in (109) for only the kernel $T_2(q,q')$. It can be seen that for all values of f and c the value of E_0 is less than $\bar{E}_0$, i.e., allowance for $V_2(q,q')$ in (109) always, as we have pointed out above, reduces the binding energy, and the relative correction is significantly smaller for $c\approx 0.6$ F (last row of Table II) than for $c\approx 1$ F (first row of Table II). Note that the values of $\bar{E}_0$ given in Table II are very close to the results of Ref. 21.

G. Results for a parameter $f(E)$ that depends on the energy E

The three-particle equations (90) and (93) were obtained with the use of the condition (84) and the expression (86), which contain the parameter f in the two-particle condition (18). As we have already pointed out, if the condition (17) is preserved, one cannot assume that f in (18) depends on the energy E, since this leads to violation of the orthogonality of the wave functions for states of two nucleons of different energies. However, for "unphysical" energies $E_q<-\epsilon_d$, where E_q is determined by Eq. (75), f in (84) and (86) can be assumed to depend on the energy E_q, which should be regarded as a certain approximate method of allowing for three-particle effects such as, for example, three-particle forces. Indeed, if three-particle forces are important in a system of three particles (in this connection, see Ref. 34), then not only the Schrödinger equation (73) but also the Faddeev equation (66) must be modified to take into account these forces. In this case, one can no longer obtain on the basis of (17) and (18) boundary conditions of the type (83) and (84) for the three-particle wave function. The use of these conditions with constant f and in the same way as in (18) corresponds to complete neglect of the three-particle forces. Therefore, the introduction for $E<-\epsilon_d$ of a dependence of f on E which does not violate the orthogonality of the wave functions of real states of two nucleons can be regarded as approximate allowance for three-

TABLE III. Dependence of the binding energy E_0 on the behavior of $f(E)$ defined by the expression (121) for $E<-\epsilon_d$.

c_1	c_2	c_3	E_0, MeV	c_1	c_2	c_3	E_0, MeV
−0.506	0.253	0.3	4.30	0	−0.253	0.1	10.34
−0.506	0.253	0.2	5.16	0	−0.253	0.2	15.64
−0.506	0.253	0.1	6.50	0	−0.253	0.3	26.77
0	−0.253	0	7.70				

particle forces in the language of two-particle interactions. Such an approach is similar to the method used in the theory of finite Fermi systems, for which two types of constants: vacuum constants and interaction constant in a medium,[35] are introduce for two-particle interactions.

The dependence of f on E for $E<-\epsilon_d$ was chosen in Ref. 20 in the form

$$f(E)=c_1+c_2(-E/\epsilon_d)^{c_3}. \tag{121}$$

In Table III, we give the values of the binding energy E_0 of three identical bosons for the parameter $f(E)$. It can be seen that the values of E_0 depend strongly on the behavior of $f(E)$ for $E<-\epsilon_d$. Thus, if $f(E)\to+\infty$ as $E\to-\infty$, the resulting E_0 is less than the E_0 corresponding to $f=$ const, and, conversely, if $f(E)\to-\infty$ as $E\to-\infty$ the corresponding E_0 values are larger than those for $f=$ const. The values of E_0 given in Table III were obtained for one and the same $c=1.095$ F.

H. Energy-dependent two-particle potential

The large difference found in Ref. 20 between the binding energy of three bosons ($E_0=7.70$ MeV) and the result of Ref. 21 ($E_0=12.69$ MeV) obtained from the Faddeev equation for the same values (110) of the parameters of the boundary-condition model, together with the assertion made above relating to the model nature of the equations of Refs. 15 and 16, prompts one to look for an independent method of verification of the above facts. Such a possibility arises, in our opinion, if one introduces a boundary-condition model without external potential with logarithmic derivative f that depends on the energy E in the whole range of its variation. However, if this assumption is made and the condition (17) preserved, this leads, as we have pointed out above, to nonorthogonality of the on-shell wave functions corresponding to different physical states of two particles. Indeed, we introduce on-shell wave functions that in accordance with (19) have in the region $r>c$ the form ($Z=E\pm i0$, $E=k^2$):[3)]

$$\psi_k^{\pm}(r)=j_0(kr)+i\sqrt{Z}F^{(\pm)}(k)h_0^{(1)}(r\sqrt{Z}), \tag{122}$$

and satisfying the conditions (17) and (18) with energy-dependent f:

$$\psi_k^{(\pm)}(r)=0, \quad r<c; \tag{123}$$

$$c\left[\frac{d}{dr}r\psi_k^{(\pm)}(r)\right]_{r=c+}=f(E)\left[r\psi_k^{(\pm)}(r)\right]_{r=c+}. \tag{124}$$

An expression for the scattering amplitude $F^{(\pm)}(k)$ follows from (122) and (124):

$$F^{(\pm)}(k)=T(k,\ k,\ k^2\pm i0)=\frac{c\exp[\mp ikc]}{f(E)\mp ikc}[\cos kc-f(E)\,j_0(kc)] \tag{125}$$

and satisfies the unitarity condition

$$F^{(+)}(k)-F^{(-)}(k)=2ikF^{(+)}(k)F^{(-)}(k).$$

As is well known, the wave functions $\psi_k^{(\pm)}(r)$ must satisfy the orthogonality relation

$$\int_0^\infty r^2\,dr\psi_k^{(-)*}(r)\psi_{k'}^{(+)}(r)=(\pi/2k^2)\exp[2i\delta(k)]\,\delta(k-k'), \tag{126}$$

where $\delta(k)$ is the s-wave phase shift, determined in accordance with (125) by the expression

$$k\operatorname{ctg}\delta(k)=k[f(E)+kc\operatorname{tg}kc]/[kc-f(E)\operatorname{tg}kc]. \tag{127}$$

Substitution into (126) of the expressions (122) for $\psi_k^{(\pm)}(r)$ with allowance for (123) and (125) leads to the result

$$J=\int_c^\infty r^2\,dr\psi_k^{(-)*}(r)\psi_{k'}^{(+)}(r)=\frac{\pi}{2k^2}\exp[2i\delta(k)]\,\delta(k-k')+\frac{c\exp[-ikc-ik'c][f(E)-f(E')]}{(k^2-k'^2)[f(E)-ikc][f(E')-ik'c]}. \tag{128}$$

Thus, the conditions (123) and (124) are incompatible with the condition (126) of orthogonality of the on-shell wave functions, for the fulfillment of which one must assume that $\psi_k^{(\pm)}(r)$ in the region $r<c$ are nonzero and can be represented as follows:

$$\psi_k^{(\pm)}(r)=A_\pm(k)\chi(r), \quad r<c. \tag{129}$$

Then in accordance with (128) the condition (126) has the form

$$\int_0^\infty r^2\,dr\psi_k^{(-)*}(r)\psi_{k'}^{(+)}(r)=A_-^*(k)A_+(k')B+J; \tag{130}$$

$$B=\int_0^c r^2\,dr\chi^2(r). \tag{131}$$

The choice of the actual dependence of f on E in the form

$$f(E)=f(0)-\gamma c^2E \tag{132}$$

with arbitrary parameter $\gamma\geq 0$ ensures fulfillment of the condition $\partial f/\partial E\leq 0$, which follows from the causality principle,[4] and enables one, with allowance for (128) and (130), to determine $A_\pm(k)$ in (129) in such a way that the orthogonality condition (126) is satisfied:

$$A_\pm(k)=(\gamma c^3/B)^{1/2}\exp[\mp ikc]/[f(E)\mp ikc]. \tag{133}$$

For the wave function $\varphi_d(r)$ of the deuteron with binding energy $\epsilon_d=\alpha^2$ we shall have in analogy with (122) and (129)

$$\varphi_d(r)=A(\alpha)\chi(r), \quad r<c; \tag{134}$$

$$\varphi_d(r)=N\exp(-\alpha r)/r, \quad r>c, \tag{135}$$

here for $f(E)$ (132) the choice of $A(\alpha)$ in the form

$$A(\alpha)=N\exp(-\alpha c)(\gamma c/B)^{1/2} \tag{136}$$

leads to the condition

$$\int_0^\infty r^2\,dr\varphi_d(r)\psi_k^{(\pm)}(r)=0, \tag{137}$$

[3)] As above, we shall consider below only the s-wave state and omit the subscript $l=0$ throughout.

and it follows from (36) and (132) that

$$f(0)+\gamma c^2\alpha^2+\alpha c=0. \tag{138}$$

The expressions for $A_\pm(k)$ (133) and $A(\alpha)$ (136), which ensure the fulfillment of the orthogonality conditions (126) and (137), contain two arbitrary parameters γ and B, which are related, respectively, to the dependence (132) of f on E and the behavior $\chi(r)$ of the wave functions in the region $r<c$. In the last case, the simplest variant is the choice of $\chi(r)$ in the form

$$\chi(r)=\sinh\beta r/r. \tag{139}$$

The off-shell T matrix can be found by means of the method described in Sec. 1. However, in this case, for the off-shell wave functions $\psi_k(r,Z)$ in the boundary condition (18) and in the condition analogous to (129), f and A must be regarded as functions of two variables k and Z, whose form must be chosen in such a way that the on-shell wave functions are orthogonal and the off-shell T matrix is an analytic function in the plane of the complex variable Z and satisfies the symmetry and unitarity conditions. This way seems to us rather complicated, and we therefore choose here a different method of construction of the T matrix based on the introduction on the real axis $Z=E$ of a certain potential, for which the on-shell wave functions satisfy the boundary condition (124) with $f(E)$ (132) and the orthogonality conditions (126) and (137), and have the form, respectively, of (122) and (135) for $r>c$ and (129) and (134) with $\chi(r)$ (139) for $r<c$. Such a potential is the potential (38), which depends on the energy E:

$$V(r,\ E)=V_0(E)\,\theta(c-r)-cV_1(E)\,\delta(r-c), \tag{140}$$

where

$$V_0(E)=\beta^2+E; \tag{141}$$

$$c^2V_1(E)=\beta c\,\mathrm{cth}\,\beta c-f(0)+\gamma c^2E. \tag{142}$$

In order to determine the off-shell T matrix $T(p,k,Z)$, it is necessary to introduce a continuation of the potential (140) into the complex plane $Z=E\pm i\epsilon$, for which we choose the relation

$$V(r,\ Z)=V(r,\ \mathrm{Re}\,Z)=V(r,\ E). \tag{143}$$

Such a continuation is not analytic (nevertheless, the T matrix satisfies the Lippmann—Schwinger equation since in it the variable Z is regarded as a parameter,[36]) but it will define the T matrix as a nonanalytic function of Z, which is due to the definition (143) of the potential (140) in the complex Z plane. Therefore, this model is not suitable for describing effects or deriving relations for which analyticity of the T matrix as a function of Z is important. However, in what follows, we shall use the obtained T matrix only in the solution of the Faddeev equations, whose derivation does not use analyticity of the T matrix, and in which one needs the value of the T matrix on the negative real half-axis $Z=E<0$ and on the upper and lower sides $Z=E\pm i0$ of the cut along the real positive half-axis $Z=E>0$. This condition (143) and the form of the potential (140) ensure fulfillment of some important relations for the T matrix.

For the potential (140)—(143), the off-shell wave functions $\psi_k(r,Z)$ have the form[11] ($Z=E\pm i0$):

$$\psi_k(r,\ Z)=\frac{k^2-E}{k^2+\beta^2}\,j_0(kr)+A(k,\ Z)\,\frac{1}{r}\,\mathrm{sh}\,\beta r,\quad r<c; \tag{144}$$

$$\psi_k(r,\ Z)=j_0(kr)+i\sqrt{Z}\,T(\sqrt{Z},\ k,\ Z)\,h_0^{(1)}(r\sqrt{Z}),\quad r>c, \tag{145}$$

where

$$A(k,\ Z)=\frac{c}{\mathrm{sh}\,\beta c\,[f(E)-ic\sqrt{Z}]}\left\{\frac{\beta^2+E}{\beta^2+k^2}\,[\cos kc-ic\sqrt{Z}\,j_0(kc)]\right.$$
$$\left.+c^2V_1(E)\,j_0(kc)\,\frac{k^2-E}{k^2+\beta^2}\right\}; \tag{146}$$

$$T(\sqrt{Z},\ k,\ Z)=\frac{c\exp[-ic\sqrt{Z}]}{f(E)-ic\sqrt{Z}}$$
$$\times\left\{\frac{\beta^2+E}{\beta^2+k^2}\,[\cos kc-\beta c\,\mathrm{cth}\,\beta c j_0(kc)]+c^2V_1(E)\,j_0(kc)\right\}. \tag{147}$$

The explicit expressions (144) and (145) for the off-shell wave functions enables one to determine the T matrix by means of the relation (11):

$$T(p,\ k,\ Z)=-(p^2-Z)\,\frac{\beta^2+E}{\beta^2+k^2}\,F_0(p,\ k)$$
$$-(p^2-Z)\,\frac{\mathrm{sh}\,\beta c}{\beta^2+p^2}\,[\cos pc-\beta c\,\mathrm{cth}\,\beta c j_0(pc)]\,A(k,\ Z)$$
$$+\exp[ic\sqrt{Z}]\,[\cos pc-ic\sqrt{Z}\,j_0(pc)]\,T(\sqrt{Z},\ k,\ Z), \tag{148}$$

where $Z=E\pm i0$, and $F_0(p,k)$ is determined by the expression (30) for $l=0$. It follows from (146) that

$$A(k,\ k^2\pm i0)=[c\exp(\mp ikc)]/\{\mathrm{sh}\,\beta c\,[f(E)\mp ikc]\},$$

which, when (144) is combined with (129), (139), and (133) and allowance is made for (131), leads to a relation between the parameters α and β:

$$\gamma=(\sinh 2\beta c/2\beta c-1)/2\sinh^2\beta c. \tag{149}$$

This ensures orthogonality of (126) and (137) for the on-shell wave functions. However, the symmetry and unitarity of the T matrix (148) remain an open question. These properties of the T matrix are a consequence of two properties of the resolvent $R(Z)=(H-Z)^{-1}$, where H is the two-particle Hamiltonian[36]:

$$R^*(Z)=R(Z^*); \tag{150}$$

$$R(Z_1)-R(Z_2)=(Z_1-Z_2)\,R(Z_1)\,R(Z_2), \tag{151}$$

and the last relation is known as the Hilbert identity.

Since the potential (140) is real, it follows from (143) that (150) is satisfied automatically. The Hilbert identity will not be satisfied for $E_1\neq E_2$. However, in Ref. 36, for the proof of the unitarity of the T matrix, the relation (151) is used with $Z_1=E+i\epsilon$, $Z_2=E-i\epsilon$, $\epsilon\to 0$. In this case, for our model (140) and (143) the Hilbert identity (151) is valid. Thus, the T matrix (148) is symmetric with respect to p and k and satisfies the unitarity condition

$$T(p,\ k,\ E+i0)-T(p,\ k,\ E-i0)$$
$$=2i\sqrt{E}\,T(p,\ \sqrt{E},\ E+i0)\,T(\sqrt{E},\ k,\ E-i0),$$

a special case of which is the unitarity of the scattering amplitude $F^{(\pm)}(k)$ (125).

We shall regard the introduction of the model (140) and (143) as a certain auxiliary procedure that enables one to obtain in a simple and perspicuous manner the

TABLE IV. Values of the parameters of the potential (140) for a neutron and proton matched to the deuteron binding energy $\epsilon_d = 2.225$ MeV, the triplet scattering length $a_t = 5.41$ F, the singlet scattering length $a_s = -23.72$ F, and the singlet effective range $r_{0s} = 2.76$ F (Ref. 37). The values are also given for the triplet effective range r_{0t} (the experimental value if $r_{0t} = 1.75$ F).

γ	c, F	$f(0)$	β^2, MeV	r_{0t}, F	γ	c, F	$f(0)$	β^2, MeV	r_{0t}, F
Triplet state					Singlet state				
0	1.0965	−0.2540	∞	1.7788	0	1.3061	0.0522	∞	—
0.03	1.1301	−0.2638	9019	1.7788	0.03	1.3464	0.0537	6354	—
0.06	1.1653	−0.2743	2121	1.7788	0.06	1.3894	0.0553	1491.7	—
0.09	1.2020	−0.2854	885.8	1.7788	0.09	1.4354	0.0571	621.2	—
0.12	1.2403	−0.2972	464.6	1.7788	0.12	1.4845	0.0589	324.3	—
0.15	1.2803	−0.3097	272.3	1.7789	0.15	1.5373	0.0609	188.9	—

limiting case when the interaction is described by the boundary-condition model. Indeed, if $\gamma \to 0$, it follows from (132) and (149), respectively, that $f(E) \to$ const and $\beta c \to \infty$. Under these conditions, in accordance with (141) and (142), the potential (140) is equal to the limit form of the potential (38), the expressions (147) and (148) for the T matrices go over, respectively, into the expressions (21) and (33) for $l = 0$, and the off-shell functions (144) and (145) satisfy the boundary conditions (17) and (18). Thus, the case $\gamma = 0$ completely corresponds to the boundary-condition model without external potential with constant logarithmic derivative. The energy-dependent potential (140) contains four parameters: c, $f(0)$, β, and γ. It is convenient to regard the parameter γ as arbitrary, and this, in accorance with (132), leads to a different dependence on the energy E of the logarithmic derivative $f(E)$. The remaining parameters can be determined from the relations (138) (when a bound state is present), (149), and from the expressions for the scattering length a and the effective range r_0 that follow from (127). In the case of a system consisting of a neutron and a proton, to determine the parameters of the potential (140) one can use the experimental values of the deuteron binding energy $\epsilon_d = 2.225$ MeV and the scattering length $a_t = 5.41$ F for the triplet state, and for the singlet state the experimental scattering length $a_s = -23.72$ F and the effective range $r_{0s} = 2.76$ F (Ref. 37).

In Table IV, we give for different γ values the parameters of the potential (140) for the triplet and singlet states of a neutron and a proton; for the triplet state in the last column, we give the values obtained from (127) for the effective range r_{0t}, which can be compared with the experimental value $r_{0t} = 1.75$ F. It can be seen from Table IV that the collection of potentials (140) corresponding to different values $\gamma \neq 0$, and also the boundary-condition model without external potential with constant logarithmic derivative ($\gamma = 0$) describe in the same manner the s-wave phase shift of np scattering in the range of energies for which the effective-range approximation is valid. For the parameters of the potential (140) given in Table IV, the triplet T matrix (148) has only one pole at the negative energy $E = -\epsilon_d$, whereas the singlet T matrix does not have any poles for $E < 0$. In the complex plane of $k = \sqrt{E}$ the triplet scattering amplitude $F^{(+)}(k)$ (125) has a pole on the positive imaginary half-axis corresponding to the deuteron binding energy $\epsilon_d = 2.225$ MeV, and the singlet amplitude has a pole on the negative imaginary half-axis corresponding to a virtual state of the deuteron with binding energy $\epsilon_0 = -0.066$ MeV, which agrees with the experimental value.[38] In accordance with (132), the triplet and singlet amplitudes $F^{(+)}(k)$ have second poles. These are on the negative imaginary half-axis, but for the values $\gamma \neq 0$ given in Table IV they are a large distance from the real axis (from 5 to 30 F^{-1}).

The wave functions (144) in the region $r < c$ for $\gamma \neq 0$ do not vanish, i.e., in this case the necessary conditions[14] that lead to nonuniqueness of the Faddeev equations are not satisfied, and the equations can be solved directly. Then the result corresponding to two-particle interactions described by the boundary-condition model can naturally be regarded as the result of extrapolation to the point $\gamma = 0$ of the values obtained for $\gamma \neq 0$ from the solution of the Faddeev equations. Of course, it is at the same time assumed that these values, because of the simple dependence on γ of the T matrix (148), are sufficiently smooth functions of γ. The Faddeev equations can be solved by the usual method of approximate factorization of the two-particle T matrix, which with allowance for the finite number of particle-wave components reduces them to a finite system of one-dimensional integral equations.[39] The unfactorized term in the T matrix (148) corresponding to the potential (140), (143) contains the function $F_0(p, k)$, which can be conveniently factorized by means of the Bubnov—Galerkin method considered in Refs. 40 and 41. For this, it is necessary to introduce a complete system of orthonormal functions $\varphi_n(r)$ defined in the interval $(0, c)$, for which one can take $\varphi_n(r)$ in the form of polynomials of degree n. In this case, $\varphi_n(r)$, like the functions (112) and (113), can be expressed in terms of Jacobi polynomials, and they take the form

$$\varphi_n(r) = \frac{(2n+1)^{1/2}}{c^{3/2}} \sum_{\nu=1}^{n} (-1)^{n+\nu} \frac{(n+\nu)!}{(n-\nu)!\,(\nu+1)!\,(\nu-1)!} \left(\frac{r}{c}\right)^{\nu-1}, \quad n = 1, 2, 3, \ldots,$$

and for $F_0(p, k)$ we have the following satisfactorily convergent expansion[41]

$$F_0(p, k) = \sum_n M_n(p)\, M_n(k), \tag{152}$$

where

$$M_n(k) = \int_0^c r^2 \, dr j_0(kr)\, \varphi_n(r).$$

TABLE V. Dependence of the binding energy E_0 of three identical bosons and the triton binding energy E_T on the values of γ in (132). For E_0 the results correspond to the triplet potential (140) with allowance for N terms in (152) and the neglect in (148) of the term with $F_0(k, p)$ ($N = 0$). The values for $\gamma = 0$ were obtained by means of a parabolic extrapolation.

γ	E_0, MeV			E_T, MeV	γ	E_0, MeV			E_T, MeV
	$N=0$	$N=1$	$N=2$	$N=1$		$N=0$	$N=1$	$N=2$	$N=1$
0.15	—	18.43	—	8.96	0.06	19.70	12.05	11.95	5.14
0.12	—	15.99	—	7.19	0.03	16.09	10.53	10.42	4.44
0.09	25.26	13.69	13.67	5.93	0	14.45	9.12	9.07	3.83

In Table V, for a number of values $\gamma \neq 0$, we give the binding energy E_0 of three identical bosons and the triton binding energy E_T obtained from the Faddeev equations. For E_0 we give the values obtained for the triplet potential (140) with allowance for N terms in the expansion (152) ($N=1,2$). It can be seen that allowance for the second term in (150) ($N=2$) gives only a small correction to the results with $N=1$, i. e., the single-term factorization of $F_0(p,k)$ in (148) by the Bubnov—Galerkin method is already a fairly good approximation. The results given in Table V for $\gamma=0$ were obtained by parabolic extrapolation of the corresponding quantities for $\gamma \neq 0$, and it is of course natural that one must consider the error associated with this procedure. To estimate this error for $\gamma \neq 0$, we solved the Faddeev equation for three bosons with the approximate T matrix obtained from (148) by neglecting the unfactorized term with $F_0(p,k)$ ($N=0$). Comparison of the corresponding extrapolated value 14.45 MeV with the value $\bar{E}_0=13.21$ MeV (see above) obtained for the approximate T matrix (117) leads to $\Delta E=1.24$ MeV, which can be regarded as the error of the extrapolation. With regard to the calculation by means of the Faddeev equations given in Table V, it is necessary to make the following remark. The Faddeev equations contain the energy $E_q=E-3q^2/4$, and therefore, to guarantee convergence of some integrals, we did not in fact use the T matrix (148) corresponding to $f(E)$ (132) and $V(r,E)$ (140), but assumed that $f(E)$ and $V_0(E)$ for $E \leqslant -\beta^2$ take the values $f(E)=f(0)+\gamma c^2\beta^2$ and $V_0(E)=0$, respectively.

The results given in Tables IV and V are interesting in two respects. First, for the neutron and proton there exists the family of potentials (140), which for the parameter values given in Table IV correspond to the experimental values of the scattering lengths and the effective ranges, and also the deuteron binding energy. For these potentials, the off-shell behavior of the T matrices (148) is different, and this circumstance leads to significantly different binding energies of three bosons, E_0, and the triton binding energy E_T (see Table V). Note that for $\gamma=0.18$ we obtained $E_0=20.73$ MeV for three bosons and found a second level $E_1=2.266$ MeV, which agrees very well with the results of Ref. 42 ($E_0=20.64$ MeV, $E_1=2.293$ MeV) for a static potential of rectangular form. Second, the value $E_0=9.07$ MeV ($\gamma=0$, $N=2$) with allowance for the extrapolation error $\Delta E=1.24$ MeV agrees with the value $E_0=7.70$ MeV obtained from the solution directly of the one-dimensional equations (108) and (109), which confirms both the adequacy of these equations and the conclusion drawn above about the three-particle model nature of the equations of Ref. 15. If it is assumed that the extrapolation error for the triton is of the same order as for three bosons ($\Delta E \approx 1$ MeV), then the value $E_T=3.83$ MeV ($\gamma=0$; see Table V) contradicts the result $E_T=7.05$ MeV of Ref. 33.

In connection with what we have said above, it is necessary to consider in more detail the large difference between the value $E_0=7.70$ MeV obtained in Ref. 20 from the solution of Eqs. (108) and (109) and the result $E_0=12.69$ MeV of Ref. 21, since in the latter the same problem was solved: namely, the calculation of the binding energy E_0 of a system of three identical bosons with zero total angular momentum interacting only in relative s states in accordance with the boundary-condition model. The value of E_0 was determined in Ref. 21 from the Faddeev equation (66) with allowance for only the s-wave component (116) of the two-particle T matrix with parameters f and c (110). Unfortunately, we cannot in detail explain the reasons for the strong discrepancy between the results of Ref. 20 and Ref. 21; we can only point out two circumstances that may lead to the difference.

The first is associated with the use in Ref. 21 of the important assumption that for the boundary condition model the Faddeev equation has the same form (66) as for "normal" potentials. Regarding the boundary condition model as a limiting form of the potential (38), we see that such an assumption will be valid if in (66) the limit of $T_\alpha \psi_\beta$ (α and β are different channels) is equal to the product of the limits of T_α and ψ_β. In our opinion, such an assertion requires a more detailed justification. The second circumstance which we wish to point out is this. In Ref. 21, using an approximate method, Kim and Tubis solve the two-dimensional Faddeev equation by reducing it to a matrix equation. It should be noted that the matrix method is rigorously justified only for integral equations whose kernels have finite norm. However, in the boundary condition model the kernels of the Faddeev equations taken in the form (66) have an infinite norm even when allowance is made for only the s-wave component of the two-particle T matrix,[44] and this is also true of potentials with hard core. Thus, even if the assumption is true that in the boundary condition model the Faddeev equations have the same form (66) as for ordinary potentials, it is necessary to make a special check whether the matrix method is valid. In Ref. 21, both assumptions were justified by reference to the results of Ref. 44, in which for two-particle potentials of rectangular form with hard core the triton binding energy E_T was determined as the limit as $V_0 \to \infty$ of the E_T values obtained from the Faddeev equations for the same potentials with a core of finite height V_0. However, it seems to us that in the case of an infinite norm of the kernels any numerical procedure for solving integral equations whose validity is proved in general form only for kernels with finite norm must be justified in each particular case in which it is applied. It would be very helpful to determine the limiting value of the binding energy E_0 of three identical bosons by using the potential (38) for a number of large but finite values of V_0 and V_1. The limiting value of E_0 given above for the potential (140) and (142) as $\gamma \to 0$ seems to confirm the result of Ref. 20 rather than Ref. 21.

CONCLUSIONS

In Refs. 12, 13, and 17—20, a unified method was proposed for solving two- and three-particle problems in the boundary condition model. The method is based on the use of the boundary conditions (17) and (18) for the off-shell two-particle wave functions and the relations (9)—(12) and (68), which follow from the Lippmann—Schwinger and Faddeev equations and do not contain potentials explicitly. For two particles, such an approach in the case of the boundary-condition model with-

out external potential is a kind of no-potential parametrization of the experimental phase shifts, and the relations (21), (22) and (33), (34) determine the off-shell T and K matrices. The first of these two circumstances is not surprising since direct verification shows that in accordance with (21) and (22) the boundary-condition model without external potential gives a good description of the experimental s-wave phase shifts of nucleon—nucleon scattering up to energies of order 40 MeV in the laboratory system, i.e., in fact in the energy region for which one can satisfactorily justify the effective-range approximation with no dependence on the shape of the potential. The method considered above can be readily generalized to the case of a tensor interaction, and in its framework the calculation of the off-shell T and K matrices will be considerably simpler than when use is made of the limiting form of a tensor potential of the form (38) (Ref. 43). In the more realistic case of the boundary-condition model with a potential in the external region, the example of the potential (62) with a simple exponential shape has shown that the experimental nucleon—nucleon s-wave phase shifts up to laboratory energies of 400 MeV can be described on the average equally well by a complete family of such potentials with different values of the boundary condition radius c. For $c=0.4$ F, the boundary condition model with the external potential (62) describes the experimental s-wave phase shifts much better than the potential (62) with a hard core of radius c.[26-28]

In the special case of the three-particle problem (system of three identical bosons interacting in s states with zero total angular momentum) we have shown that the solution of the Schrödinger equations can be reduced exactly to the solution of the one-dimensional integral equation (108) or (109), which is model-independent at the three-particle level, provided the two-particle interactions are described by the boundary-condition model without external potential. A particular feature of Eqs. (108) and (109) is their uniqueness, whereas to obtain unique three-particle equations in the boundary-condition model that are a modification of the Faddeev equations[15] it is necessary to introduce an additional three-particle condition and an arbitrary three-particle parameter, and this means that these equations are essentially model-dependent. The restriction above to a special case of the three-particle problem was made only for the sake of simplicity; the method can also be used for more realistic systems. The solution of Eqs. (108) and (109) gave for the binding energy of three bosons the value $E_0=7.70$ MeV, which differs strongly from the result $E_0=12.69$ MeV obtained in Ref. 21 on the basis of the Faddeev equations. The structure of Eqs. (108) and (109) enables one to take into account in a simple approximate manner the influence of three-particle effects on the binding energy E_0 by the introduction of a dependence of the parameter f on the energy (121). Of great interest in this connection may be calculations of the triton binding energy in order to determine the parameters in (121) and then a subsequent calculation with these parameters of the np scattering phase shifts. The introduction of an energy dependence in the np potential (140) confirmed the value $E_0=7.70$ MeV obtained from the solution of Eqs. (108) and (109) and the conclusion that the equations of Ref. 15 are model dependent and do not correspond to the "vacuum" interaction of two particles described by the boundary-condition model without external potential.

In the more realistic case of the boundary-condition model with local external potential, the method presented here does not lead to one-dimensional equations of the form (108) and (109). However, use of the well developed methods of factorization of the two-particle T matrix enables one to replace in a good approximation the local potential by a finite sum of nonlocal separable potentials, and in this case the solution of the three-particle problem reduces to the solution of a system of one-dimensional integral equations. It is of particular interest to use these equations for the family of potentials (62) to calculate three-particle parameters: the triton binding energy, the nd doublet scattering length, the momentum-transfer dependence of the triton form factor, etc. This is because the model calculations of Ref. 21 indicate a strong dependence of some of these quantities on the boundary condition radius. The method of solution of the three-particle problem in the boundary-condition model considered here may serve as the basis for correct allowance for a repulsive core when phenomenological potentials with hard core are used in three-nucleon problems.

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Barrier penetration theory in more than one dimension

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Quantum-mechanical tunneling theory in more than one dimension is reviewed. Several systems from nuclear and molecular science are considered specifically, such as, alpha decay of spheroidal nuclei, spontaneous fission, and reactive collinear collisions of hydrogen atoms with hydrogen molecules. The ranges of validity of various approximations that reduce to one-dimensional path integrals or Fröman-Nosov matrices are examined, testing where possible against fully quantum-mechanical coupled-channel solutions. The classical-equations-of-motion methods using complex variables (uniform semiclassical approximation) are explored for nonseparable fission-like model systems. Effects of variable valley widths, curving valleys, and of variable inertial tensors are delineated.

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INTRODUCTION

The problem of quantum-mechanical penetration of a two or more dimensional barrier occurs in a number of interesting physical situations. In this article we will review only a few situations, namely the ones which display the barrier penetration in the simplest and most explicit way.

The α-particle penetration problem is considered and several approaches are discussed. In addition we will deal with the barrier penetration in the spontaneous-fission problem. The various approximate treatments that have been used for this problem are described, including a recent barrier-model system for which a comparison of several of these methods is made. In the realm of molecular reactions, only the electron and the hydrogen nucleus have sufficiently low mass to undergo significant tunneling and we consider one of these problems, the collinear collision of a proton on a hydrogen molecule. This problem has been of much interest to molecular theorists. In all these examples we will, however, only consider the cases where the energy of the system is not too close to the top of the barrier.

Another place in nuclear physics where a tunneling process is involved is the tunneling of a nucleon or a few nucleons in stripping and pickup reactions. In the old semiclassical transfer theory of Breit and Ebel, [1] tunneling is displayed rather explicitly, but the more sophisticated modern versions of semiclassical transfer theory (K. Alder *et al.* [2] and R. Broglia and Aa. Winther[3]) don't show the tunneling in a simple way and we will therefore not discuss it in this paper. Electron tunneling which enters in a rather complicated way in the electron capture and loss problems from ions moving through matter also is not considered here. A review of the early quantum-mechanical treatments of this subject by Oppenheimer[4] and Brinkman and Kramers[5] has been given by Raisbeck and Yiou. [6]

The main object of this paper will be to present the various methods used in solving the multidimensional barrier-tunneling problem. The coordinates orthogonal to the barrier-penetration coordinate and the effect of a coordinate-dependent inertial tensor will be considered and an attempt will be made to isolate these different components of the problem.

1. MULTIDIMENSIONAL BARRIER PENETRATION IN ALPHA DECAY

Examination of detailed methods for two-dimensional barrier problems. We now consider in more detail the penetrability problem for anisotropic barriers in alpha decay. Beyond the range of nuclear forces we have the simple Hamiltonian for a spin-zero system

$$H=-\frac{\hbar^2}{2\mu}\nabla^2-\frac{\hbar^2}{2\mathcal{I}}J_{\theta_i}^2+\frac{2Ze^2}{r}+\frac{2e^2Q_0}{2r^3}P_2(\cos\gamma), \tag{1}$$

where the ∇^2 Laplacian operates on the coordinates of the alpha in the lab-fixed system θ, ψ, with r the center-to-center separation distance, μ the reduced mass $Mm_\alpha/M+m_\alpha$, $\mathcal{I}$ the nuclear moment of inertia, $J_{\theta_i}^2$ the square of the nuclear rotational-angular-momentum operator operating on the Eulerian angles θ_i defining the nuclear symmetry axes in the lab frame, Z is the charge and Q_0 the intrinsic quadrupole moment of the daughter nucleus, and γ is the angle between the alpha direction (θ,ψ) and the nuclear symmetry axis (θ_1,θ_2). The problem reduces to a two-dimensional one in r and γ when transformed to the body-fixed coordinate system (see Rasmussen and Segall[7]). The wave equation is transformed to a set of coupled-channel second-order radial equations by expanding the wave function as follows:

$$\Psi=\sum_{l_{\text{even}}}\frac{u_l(r)}{r}\sum_m(llm-m\,|\,00)Y_{lm}(\theta,\psi)\sqrt{\frac{2l+1}{8\pi^2}}D^l_{-m0}(\theta_i) \tag{2}$$

Left-multiplying the wave equation $(H-E)\Psi=0$ by the complex conjugate of an angular function

$$\sum_{m'}(l'l'm'-m'\,|\,00)Y^*_{l'm'}(\theta,\psi)\sqrt{\frac{2l'+1}{8\pi^2}}D^{l'*}_{-m'0}(\theta_i)$$

and integrating over all angular space θ, ψ, and θ_i gives the set of coupled-channel equations.

$$-\frac{\hbar^2}{2\mu}\frac{d^2u_{l'}}{dr^2}+\left[\frac{2Z^2}{r}+\left(\frac{\hbar^2}{2\mu r^2}+\frac{\hbar^2}{2\mathcal{I}}\right)l'(l'+1)-E\right]u_{l'}$$
$$+\frac{Q_0e^2}{r^3}\sum_l u_l\langle l'l',\,0\,|\,P_2(\cos\gamma)\,|\,ll,\,0\rangle=0. \tag{3}$$

In the total-spin-zero case (as for even-even ground-state alpha decay) the matrix elements in (3) have the simple form first given by Racah[8]

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