Separable representation method in problems of nuclear physics

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The application of the separable representation method in the two-body problem and in many-channel theory is reviewed. The results of numerical calculations are given, from which one can conclude that there is fairly rapid convergence. The connection between the method and variational principles is pointed out. The possibility of studying the corrections to the diffraction approximation is considered.

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INTRODUCTION

In recent years, refined methods for solving the Faddeev equations have been developed by approximating the two-particle t matrix $t(k, k', z)$ by series with separable terms:

$$t^{(N)}(k, k', z) = \sum_{i,j=1}^{N} [C^{-1}(z)]_{ij} \eta_i(k) \eta_j(k'). \quad (1)$$

In this case, the system of two-dimensional integral equations (for states with given total angular momentum) reduces to a one-dimensional system amenable to numerical solution. The separable representation (1) is obtained if the Fourier transform of the potential is factorized:

$$V_l^{(N)}(k, k') = \sum_{i,j=1}^{N} [d^{-1}]_{ij} \eta_i^l(k) \eta_j^l(k'). \quad (2)$$

Potentials of this type, which were introduced into theory by Wigner, were subsequently applied by Yamaguchi[1] to the two-body problem, and are now successfully used to investigate the properties of the nucleus. For example, Brown and Bolsterli[2] have proposed a model for treating particle-hole states in nuclei using the separable approximation

$$\langle ph | V | p'h' \rangle = \langle ph | D | 0 \rangle \langle 0 | D | p'h' \rangle \gamma$$

where $|ph\rangle$ is the particle–hole state and D is the operator of the dipole moment. In Ref. 3, Bauer and Prats consider the problem of dipole photoabsorption in nuclei with allowance for continuum states.

In the theory of integral equations, the approximation (2) is known as replacement of the kernel by a degenerate kernel. The essence of the separable representation method is the following. Consider the identity

$$V = VV^{-1}V = V|i\rangle \langle i|V^{-1}|j\rangle \langle j|V \quad (3)$$

(the sets $|i\rangle$ and $|j\rangle$ are different). Terminating in (3) the summation over the complete sets, we obtain the separable approximation[4,5]

$$V^{(N)} = \sum_{i,j=1}^{N} V|\eta_i\rangle d_{ij}^{(N)} \langle \zeta_j|V; \quad [d^{-1}]_{ij} = \langle \zeta_i|V|\eta_j\rangle. \quad (4)$$

The convergence of the expansion (4) is guaranteed by the well known theorem which states that if V is a completely continuous operator it can be approximated arbitrarily accurately in the norm by an operator $V^{(N)}$ of finite rank.

Thus, the problem is solved by replacing V by $V^{(N)}$, $|V - V^{(N)}|$ characterizing the error. In this sense, the separable representation method is close to Lanczos's[6] τ process, which is based on a fairly simple idea. Usually, it is difficult to find the error of a numerical solution to a problem, but it is easy to say for which similar problem the calculated answer is an exact solution.

For problems with a discrete spectrum, the separable representation method is similar to the method of intermediate problems used by Weinstein[7] to determine the eigenfrequencies of a clamped rectangular plate, since in some cases the approximation (2) in conjunction with Ritz's method gives two-sided estimates of the eigenvalues. Subsequently, Weinstein's method was developed by N. Aronszajn [Proc. Oklahoma. Symp. Spectral Theory and Differential Problems, p. 179, 1951 (unpublished)], Bazley, Fox *et al.*[8–11]

In this paper, we consider the application of the separable representation method to the quantum-mechanical problems of two or a few bodies. Essentially, we analyze the continuum case. In the development of approximate methods, one should keep in mind the quantum-mechanical many-body problem since there is little point in constructing refined approximations to the partial two-body problem for which the corresponding equation can always be integrated numerically. However, different approximate methods of solution of the two-body problem are convenient because of their occurrence in the construction of methods of solution of much more complicated problems. In the two-body problem it can be proved that the majority of the methods of approximation (1) described in the literature fit into the framework of Eq. (4). The convergence of the methods is investigated. The separable representation method is generalized to the case of the nuclear many-body problem. This generalization is used to solve explicitly the equation of the method of strong coupling of channels, and it proves possible to take into account virtual transitions in the continuum. It is shown that this procedure can be applied not only to problems of nuclear physics but also to the collision of charged particles with atoms.

The connection between the separable representation method and variational methods is investigated. It is shown that for scattering problems and bound-state problems the approximation (4) leads to a convergent

sequence of variational principles whose first member is identical with Schwinger's variational principle. We consider the possibility of constructing an amplitude that describes scattering at low and high energies equally well.

We investigate the iterative process to which separation leads. We prove the uniform and exponential convergence of the process in two-body problems and many-channel problems. The numerical convergence is studied for the examples of the two-body problem and the scattering of positrons on hydrogen atoms. We construct a method for finding two-sided estimates of the physical observables in the nuclear two- and three-body problem and for problems with continuous and discrete spectrum. A study is made of scattering at high energies and we investigate the corrections to the Born approximation and the eikonal approximation in the two-body problem. On the basis of the separable representation method, we study the possibility of finding the Fresnel corrections and the nonadiabaticity corrections in the Glauber approximation. We investigate nonstationary problems, which are of particular interest in connection with the development of laser technology. It is clear that an unambiguous interpretation of the observables is possible if the dynamical equations are solved without recourse to any additional assumptions about the properties of the system, i.e., in the model-free method of solution of the equations. At the same time, the approximation procedure must guarantee convergence of the approximation process and be sufficiently universal.

1. THE TWO-BODY PROBLEM

Let us consider application of the separable representation method to the two-body problem.

The Lippman–Schwinger equation for the two-particle t matrix $T(E+i\epsilon)$ has the form

$$T=V+VG_0T=V+TG_0V, \tag{5}$$

where $G_0=1/(E-H_0+i\epsilon)$. Substituting $V^{(N)}$ from (4) into (5), we obtain

$$\begin{aligned} T^{(N)}&=(V\,|\,\eta_i\rangle\, d_{ij}^{(N)}\,\langle\zeta_j\,|\,V)\\ &+(V\,|\,\eta_i\rangle\, d_{ij}^{(N)}\,\langle\zeta_j\,|\,VG_0T^{(N)}). \end{aligned} \tag{6}$$

We seek a solution of (6) in the form

$$T^{(N)}=V\,|\,\eta_i\rangle\, C_{ij}^{(N)}\,\langle\zeta_j\,|\,V, \tag{7}$$

and for $C_{ij}^{(N)}$ we then have the expression

$$[C^{-1}]_{ij}=\langle\zeta_i\,|\,V-VG_0V\,|\,\eta_j\rangle. \tag{8}$$

For the Schrödinger equation

$$|\,\Psi\rangle=|\,\varphi\rangle+C_0V\,|\,\Psi\rangle \tag{9}$$

the approximation for the potential (4) leads to a solution in the form

$$\left.\begin{aligned} |\,\Psi^{(N)}\rangle&=|\,\varphi\rangle+G_0V\,|\,\eta_i\rangle\, c_i;\\ c_i&=C_{ij}^{(N)}\,\langle\zeta_j\,|\,V\,|\,\varphi\rangle. \end{aligned}\right\} \tag{10}$$

To obtain the representation (2), the expansions most often used are those based on the Bubnov–Galerkin,[12] Hilbert–Schmidt,[13] and Bateman[14] methods. In the Bubnov–Galerkin and Hilbert–Schmidt methods, the functions η have an explicit form only for some simple potentials. For the majority of realistic potentials they can be constructed numerically, which greatly complicates the problem. Bateman's method can be applied to a larger class of potentials. This expansion is the most convenient if $V_l(k,k')$ has an explicit form; for example, for potentials that are a Yukawa superposition. We shall show that the above expansions fit into the framework of Eq. (4).

Consider the Hilbert–Schmidt problem

$$\left.\begin{aligned} &\left[H_0+\frac{1}{\mu_n(E)}V\right]|\,\psi_n(E)\rangle=E\,|\,\psi_n(E)\rangle;\\ &\langle\Psi_n\,|\,V\,|\,\Psi_m\rangle=-\mu_n\delta_{mn}. \end{aligned}\right\} \tag{11}$$

Set $\eta_i=\zeta_i=\Psi_i$ in (4); then

$$\langle k\,|\,V^{(N)}\,|\,k'\rangle=-\sum_{i=1}^{N}\frac{1}{\mu_i(E)}\,\langle k|\,V\,|\,\Psi_i\rangle\,\langle\Psi_i\,|\,V\,|\,k'\rangle,$$

but

$$V\,|\,\Psi_i\rangle=\mu_i(E)\,(E-H_0)\,|\,\Psi_i\rangle,$$

so that

$$\left.\begin{aligned} &\langle k\,|\,V^{(N)}\,|\,k'\rangle=-\sum_{i=1}^{N}\mu_i(E)\,a_i(k,E)\,a_i^*(k',E);\\ &a_i(k,E)=(k^2/2m-E)\,\Psi_i(k,E). \end{aligned}\right\} \tag{12}$$

Equation (12) gives the well-known Hilbert–Schmidt expansion for the Fourier transform of the potential.

In the Bubnov–Galerkin method, the Fourier transform of the potential can be represented in the form

$$V^{(N)}(k,k')=\sum_{i=1}^{N}M_i(k)\,M_i(k'), \tag{13}$$

where $M_n(k)=\langle k\,|\,V\,|\,\Phi_n\rangle$, and the system of basis functions Φ_n satisfies the orthonormality condition with weight of the potential

$$\langle\Phi_m\,|\,V\,|\,\Phi_n\rangle=\delta_{mn}.$$

It is obvious that if we set $\eta_i=\zeta_i=\Phi_i$ in (4) we obtain the expansion (13).

The Bateman expansion for the lth harmonic of the Fourier transform of the potential:

$$V_l(k,k')=\frac{1}{2\pi^2}\int_0^\infty\frac{j_l(kr)}{k}\,\frac{j_l(k'r)}{k'}\,V(r)\,dr$$

has the form

$$\left.\begin{aligned} &V_l^{(N)}(k,k')=\sum_{i,j=1}^{N}V_l(k,s_i)\,d_{ij}^{(N)}\,V_l(s_j,k');\\ &[d^{-1}]_{ij}=V_l(s_i,s_j). \end{aligned}\right\} \tag{14}$$

Setting $\eta_i=\zeta_i=j_l(s_ir)$ in (4), we can readily see that we obtain the expansion (14). The Bateman expansion for the Fourier transform of the potential

$$\left.\begin{aligned} &V^{(N)}(\mathbf{k},\mathbf{k}')=\sum_{i,j=1}^{N}V(\mathbf{k},\mathbf{s}_i)\,V(\mathbf{s}_j,\mathbf{k}')\,d_{ij}^{(N)};\\ &[d^{-1}]_{ij}=V(\mathbf{s}_i,\mathbf{s}_j) \end{aligned}\right\} \tag{15}$$

is obtained with $\eta_i=\zeta_i=\exp(i\mathbf{s}_i\cdot\mathbf{r})$ (Ref. 15). Comparing (14) and (4) we see that the separable representation

method is a natural generalization of the Bateman method to the case of arbitrary separable functions. It is therefore interesting to consider in more detail the convergence of the expansion (14). We show that for the Bateman expansion the following theorem holds.[16]

There exists a system of nodes s_i for which:

1) if the function $V(x,y)$ is continuous on the closed interval $[a,b]$ then the expansion (14) converges to $V(x,y)$ uniformly on $[a,b]$;

2) if $V(x,y)$ is integrable on $[a,b]$, then the expansion (14) converges to $V(x,y)$ at almost all points of $[a,b]$;

3) if $V(x,y)$ is integrable on $[a,b]$ and continuous at the point (x_0,y_0), then at this point the expansion (14) converges to $V(x,y)$.

We prove the theorem for the interval $[0,1]$. Obviously, the transition to any finite interval is trivial. We introduce the system of points $\alpha_i=(i-1)/2^{m-1}$, $1\leqslant i\leqslant 2^{m-1}$ (m is integral). We consider the intervals, $l_{mi}=[(i-1)/2^{m-1},\ i/2^{m-1}]$. In the case $i=2^{m-1}$, the interval $l_{m,2^{m-1}}$ must also be assumed to be closed on the right. It is clear that $l_{m1}+l_{m2}+\cdots+l_{m2^{m-1}}=[0,1]$. We introduce the piecewise constant function

$$\Psi^m(x,y)=b_{ij}\quad\text{for}\quad \begin{matrix}x\in l_{mi},\ 1\leqslant i\leqslant 2^{m-1};\\ y\in l_{mj},\ 1\leqslant j\leqslant 2^{m-1},\end{matrix}$$

where b_{ij} is equal to the mean value of $V(x,y)$ for $x\in l_{mi}$, $y\in l_{mj}$. We construct the system of nodes $V(s_i,s_j)=b_{ij}$, $s_i\in l_{mi}$, $s_j\in l_{mj}$. It is then easy to see that

$$\left.\begin{aligned}&\Psi^m(x,y)=\sum_{i,j=1}^{2^{m-1}}\Psi^m(x,s_i)\,d_{ij}\Psi^m(s_j,y);\\ &[d^{-1}]_{ij}=V(s_i,s_j).\end{aligned}\right\}\tag{16}$$

On the other hand, $\Psi^m(x,y)$ is a section of the Fourier–Haar series for the function $V(x,y)$ and all assertions of the theorem on the convergence of $\Psi^m(x,y)$ to $V(x,y)$ hold. And, since the right-hand side of (16) converges to the Bateman expansion for $V(x,y)$, the theorem has been proved.

As can be seen from the theorem, the uniform convergence of the Bateman expansion for the potential and, therefore, for the t matrix, does not depend on the potential's being of definite sign, whereas the Hilbert–Schmidt expansion converges uniformly only for potentials of constant sign (Mercer's theorem).

We now prove that the separation (4) is unique. Suppose we have a finite-rank operator

$$\widetilde{V}=\sum_{i,j=1}^{N}|y_i\rangle\,\alpha_{ij}\,\langle z_j|.\tag{17}$$

If (17) has the properties of the expansion (4):

$$\widetilde{V}|\eta_i\rangle=V|\eta_i\rangle,\quad \langle\xi_i|\widetilde{V}=\langle\xi_i|V,\tag{18}$$

then $\widetilde{V}=V^{(N)}$.

To see this, we note that

$$\widetilde{V}=V|\eta_1\rangle\langle\xi_1|V/\langle\xi_1|V|\eta_1\rangle,$$

for $N=1$ (19)

$$\begin{aligned}&\widetilde{V}=V^{(M-1)}+\beta|y_m\rangle\langle z_m|;\quad N=M;\\ &\beta|y_m\rangle\langle z_m|=\frac{(V-V^{(M-1)})|\eta_M\rangle\langle\xi_M|(V-V^{(M-1)})}{\langle\xi_M|V-V^{(M-1)}|\eta_M\rangle}.\end{aligned}\tag{20}$$

Thus, the expansion (17) can be determined from the condition (18) uniquely.

The separable potential methods described in the literature have a serious shortcoming: In each order of the approximation one describes approximately not only the off-shell properties of the t matrix but also the properties of the amplitude on the semi-mass shell. Therefore, when the problem is posed in this way, the study of the dependence of the three-particle observables on the purely off-shell properties of the two-particle amplitudes is difficult. The expansion (4) enables one to overcome this difficulty readily. Indeed, choosing as one of the functions η the solution Ψ of the Schrödinger equation (9), for example,

$$\eta_1=\Psi,\quad \eta_2,\ \ldots\ \eta_N,\tag{21}$$

we find that

$$V^{(N)}|\Psi\rangle=V|\Psi\rangle,\tag{22}$$

i.e.,

$$T^{(N)}|k\rangle=T|k\rangle.$$

Therefore, in this case the amplitude on the semi-mass shell is described exactly in each order of the approximation.

Note that the separation in accordance with the functions (21) is a natural generalization of the Noyes–Kowalski expansion.

In studying the dependence of the three-particle observables on the semi-mass properties of the two-particle amplitudes it is helpful to have a separable approximation for the two-particle t matrix for which the amplitude is described exactly on the mass shell. For hard-core potentials such a procedure is constructed in Ref. 17. To achieve this aim, we introduce in our case the equation[18]

$$\widetilde{t}_l(k,k',z)=\widetilde{V}_l(k,k')+\frac{2}{\pi m}\int_0^\infty\frac{V_l(k,k'')\,\widetilde{t}_l(k'',k',z)\,k''^2\,dk''}{mz-k''^2+i\varepsilon}.\tag{23}$$

Equation (23) differs from the Lippmann–Schwinger equation by the inhomogeneous term. Therefore, if $\gamma_l(k,k',z)$ is the resolvent of the kernel of the Lippmann–Schwinger equation, then

$$\begin{aligned}&t_l(k,k',z)=V_l(k,k')+\int_0^\infty\gamma_l(k,k'',z)\,V_l(k'',k')\,dk'';\\ &\widetilde{t}_l(k,k',z)=\widetilde{V}_l(k,k')+\int_0^\infty\gamma_l(k,k'',z)\,\widetilde{V}_l(k'',k')\,dk''.\end{aligned}$$

As $\widetilde{V}_l(k,k')$ we choose the Bateman expansion of the lth harmonic of the Fourier transform of the potential

$$\begin{aligned}&V_l(k,k')=\sum_{i,j=1}^{N}d_{ij}^{(N)}V_l(k,s_i)\,V_l(s_j,k'),\\ &[d^{-1}]_{ij}=V_l(s_i,s_j);\\ &\widetilde{t}_l(k,k',z)=\frac{1}{2}\sum_{ij=1}^{N}d_{ij}^{(N)}\left[t_l(k,s_iz)\,V_l(s_j,k')+t_l(k',s_iz)\,V_l(s_j,k)\right].\end{aligned}\tag{24}$$

If one of the s_i's, for example, s_1, is equal to $\sqrt{mz}$, then $\widetilde{t}_l=t_l$ on the mass shell.

Let us consider the interesting method of choosing the separating functions described in Ref. 19. We choose the function $|f_n\rangle$:

$$G_0|f_n\rangle = |\eta_n\rangle. \tag{25}$$

and then

$$\left.\begin{aligned} V^{(N)} &= \sum_{n,l=1}^{N} VG_0|f_n\rangle\, d_{nl}^{(N)}\langle\xi_l|V; \\ d_{nl}^{-1} &= \langle\xi_n|VG_0|f_l\rangle; \\ T^{(N)} &= \sum_{n,l=1}^{N} VG_0|f_n\rangle\, C_{nl}^{(N)}\langle\xi_l|V; \\ [C^{-1}]_{nl} &= \langle\xi_n|(V-VG_0V)G_0|f_l\rangle. \end{aligned}\right\}$$

Further, we define the function $|h_n\rangle$:

$$|f_n\rangle = (1-VG_0)^{-1}|h_n\rangle. \tag{26}$$

In this case

$$\left.\begin{aligned} T^{(N)} &= \sum_{n,l=1}^{N} TG_0|h_n\rangle\, C_{nl}^{(N)}\langle\xi_l|V; \\ |C^{-1}|_{nl} &= \langle\xi_n|VG_0|h_l\rangle. \end{aligned}\right\} \tag{27}$$

The representation (27) has the important property

$$T^{(N)}G_0|h_n\rangle = TG_0|h_n\rangle. \tag{28}$$

Since the kernels of the Faddeev equations have the form TG_0, it follows from (28) that for their solution the most natural thing is to choose approximate solutions of the Faddeev equations as the separating functions.

2. THE NUCLEAR MANY-BODY PROBLEM

The separable representation method is used very intensively to construct approximate schemes for describing elastic and inelastic scattering in the framework of the shell model. All the approximations are based on the approximate replacement of the kernel K of the Lippmann–Schwinger equation

$$\left.\begin{aligned} \Psi_E^{a(+)} &= \chi_E^{a(+)} + (E+i\varepsilon-H_0)^{-1}V\Psi_E^{a(+)}; \\ K &= (E+i\varepsilon-H_0)^{-1}V \end{aligned}\right\} \tag{29}$$

by an operator $K^{(N)}$ of finite rank N, and the methods can be divided into two groups. In the first we have the methods in which the residual interaction V is approximated by a finite-rank operator in accordance with Eq. (4).[20–25] In the second we have methods in which the finite-rank approximation is applied to the Green's function.[26–29] In some investigations of the first group the Hilbert–Schmidt expansion is used to factorize the residual interaction between the continuum states, and perturbation theory is then developed for the residue $K^{\text{res}} = K - K^{(N)}$. In Ref. 30, the accuracy of these approaches is analyzed for the example of an exactly solvable model. The model describes S-wave scattering of a spinless particle on a target nucleus which can be in two states. For given $K^{(N)}$, one can find the corrections in the model in any perturbation order in K^{res}. The main results of the analysis are as follows:

1) the accuracy of the approximation increases with increasing N;

2) the accuracy increases with increasing perturbation order in K^{res};

3) a fairly good accuracy is achieved already for small N and low perturbation orders;

4) on the whole, the approximations of the first group give a better accuracy.

Let us now consider application of the separation idea directly to the equations of the method of strong coupling of channels. We consider the scattering of particle a on a nucleus, using the Hamiltonian

$$H(\mathbf{r}, \mathbf{R}_1, \ldots, \mathbf{R}_A) = K_a(\mathbf{r}) + h_A(\mathbf{R}_1, \ldots, \mathbf{R}_A) + \sum_i V_i(\mathbf{r}-\mathbf{R}_i), \tag{30}$$

where $\mathbf{r}$ are the coordinates of particle a; $\mathbf{R}_i$ are the coordinates of the nucleus; $V_i(\mathbf{r}-\mathbf{R}_i)$ is the potential of the interaction between particle a and the ith nucleon; $K_a(\mathbf{r})$ is the kinetic energy operator of the scattered particle; h_A is the Hamiltonian of the target nucleus, having the eigenfunctions φ_ν and eigenvalues ϵ_ν:

$$[h_A(\mathbf{R}_1, \ldots, \mathbf{R}_A) - \varepsilon_\nu]\varphi_\nu(\mathbf{R}_1, \ldots, \mathbf{R}_A) = 0. \tag{31}$$

Expanding the wave function Ψ of the system, which satisfies the equation

$$H\Psi = E\Psi, \tag{32}$$

in a series in $\varphi_\nu(\mathbf{R}_1, \ldots, \mathbf{R}_A)$:

$$\Psi(\mathbf{r}, \mathbf{R}_1, \ldots, \mathbf{R}_A) = \sum_\nu \Psi_{\mathbf{k}_\nu}(\mathbf{r})\varphi_\nu(\mathbf{R}_1, \ldots, \mathbf{R}_A) \tag{33}$$

and substituting the resulting expansion into (32), we obtain the following system of equations of the method:

$$\left.\begin{aligned} &[K_a(\mathbf{r}) - E + \varepsilon_\nu]\Psi_{k_\nu}(\mathbf{r}) = -V_{\nu\mu}(\mathbf{r})\Psi_{k_\mu}(\mathbf{r}); \\ &V_{\nu\mu}(\mathbf{r}) = \int \varphi_\nu^*(\mathbf{R}_1, \ldots, \mathbf{R}_A)\sum_i V_i(\mathbf{r}-\mathbf{R}_i) \\ &\quad\times\varphi_\mu(\mathbf{R}_1, \ldots, \mathbf{R}_A)\, d^3R_1, \ldots, d^3R_A. \end{aligned}\right\} \tag{34}$$

The wave function $\Psi_{\mathbf{k}_\nu}$ describes the system consisting of particle a plus the νth excited state of the target nucleus.

We now turn to the system of integral equations

$$\left.\begin{aligned} \Psi_{\mathbf{k}_\nu}^{(+)}(\mathbf{r}) &= \varphi_{\mathbf{k}_\nu}(r) + \int G_{0\nu}^{(+)}(\mathbf{r}-\mathbf{r}')V_{\nu\mu}(\mathbf{r}')\Psi_{\mathbf{k}_\mu}^{(+)}(\mathbf{r}')\, d^3r'; \\ G_{0\nu}^{(+)}(\mathbf{r}-\mathbf{r}') &= -\frac{m}{2\pi\hbar^2}\frac{\exp[ik_\nu|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}, \end{aligned}\right\} \tag{35}$$

where $\varphi_{\mathbf{k}_\nu}(\mathbf{r})$ is a plane wave in the entry channel. We shall solve Eq. (35) by the separable representation method. Note that now $V\Psi = V_{\nu\mu}\Psi_{\mathbf{k}\mu}$ and therefore Eq. (4) takes the form[1)]

$$\left.\begin{aligned} V_{\nu\mu}^{(N)}\Psi_{\mathbf{k}_\mu} &= \sum_{i,j=1}^{N} V_{\nu\mu}|\eta_\mu^i\rangle\, d_{ij}^{(N)}\langle\xi_l^j|V_{ln}|\Psi_{\mathbf{k}_n}\rangle, \\ [d^{-1}]_{ij} &= \langle\xi_\nu^i|V_{\nu\mu}|\eta_\mu^j\rangle. \end{aligned}\right\} \tag{36}$$

Using (36), we obtain a solution of Eq. (35):

$$\left.\begin{aligned} &\Psi_{\mathbf{k}_\nu}^{(+)}(\mathbf{r}) = \varphi_{\mathbf{k}_\nu}(\mathbf{r}) \\ &+\int \sum_{i,k=1}^{N} G_{0\nu}^{(+)}(\mathbf{r}-\mathbf{r}')V_{\nu\mu}(\mathbf{r}')\eta_\mu^i(\mathbf{r}')\, d^3r' C_{ik}^{(N)}\langle\xi_l^k|V_{ll'}|\varphi_{k_{l'}}\rangle, \\ &[C^{-1}]_{ik} = \langle\xi_l^i|V_{l\mu} - V_{ll'}G_{0l'}^{(+)}V_{l'\mu}|\eta_\mu^k\rangle. \end{aligned}\right\} \tag{37}$$

[1)] In (36), the summation over ν, μ, l, n includes integration over the continuum.

The expressions (37) give the complete solution of the problem since the amplitudes of the elastic and inelastic processes are determined by the expression

$$f_n^{(N)}(\theta, \varphi) = -\frac{1}{2\pi} \sum_{i,k=1}^{N} \langle \mathbf{k}_n' | V_{nm} | \eta_m^k \rangle C_{ik}^{(N)} \langle \zeta_l^k | V_{ll'} | \varphi_{\mathbf{k}_l'} \rangle. \tag{38}$$

On the transition to the momentum representation, we obtain a system of integral equations with kernels $Q_{nm}(\mathbf{k}, \mathbf{k}')$, and the separable representation method in this representation amounts to replacement of the kernel Q_{nm} by the degenerate kernel

$$Q_{nm}^{(N)}(\mathbf{k}, \mathbf{k}') = \sum_{i,j=1}^{N} d_{ij} f_n^i(\mathbf{k}) \eta_m^{*j}(\mathbf{k}'). \tag{39}$$

Thus, using the theorems in the theory of integral equations on the replacement of the kernel by a degenerate kernel, we can assert that

$$|f_n(\theta, \varphi) - f_n^{(N)}(\theta, \varphi)| \sim |Q_{nm} - Q_{nm}^{(N)}|. \tag{40}$$

Hitherto, we have not considered the choice of the separating functions, since it will be discussed throughout the paper. Here, we point out the choice of functions in the form

$$\eta_n^i = \zeta_n^i = \delta_{in} \exp(i\mathbf{s}_i \mathbf{r}). \tag{41}$$

The parameters $\mathbf{s}_i$ can be chosen from the conditions of a minimum of the functional (40). The separation (41) is none other than a generalization of the Bateman method to the many-body problem. In conclusion, we emphasize that the separable representation method can also be applied to problems of atomic physics; namely, to the scattering of particles on neutral atoms.[31]

3. VARIATIONAL PRINCIPLES AND THE SEPARABLE REPRESENTATION METHOD

We show that on the basis of the separable approximation (4) one can obtain an infinite sequence of variational principles in which Schwinger's occupies the first position.

For the partial two-body problem we set in Eq. (4)

$$N = 1 \text{ and } |\eta\rangle = |\zeta\rangle = |\chi_l\rangle. \tag{42}$$

Then

$$V^{(1)} = V|\chi_l\rangle \langle \chi_l | V / \langle \chi_l | V | \chi_l \rangle. \tag{43}$$

With the potential (43), the equation

$$\Psi_l(r) = j_l(kr) + \int_0^\infty G_l(r, r') V(r') \Psi_l(r')\, dr', \tag{44}$$

where

$$G_l(r, r') = \begin{cases} k^{-1} j_l(kr)\, n_l(kr'), & r \leqslant r', \\ k^{-1} n_l(kr)\, j_l(kr'), & r' \leqslant r, \end{cases}$$

can be solved explicitly, and for the phase shift we obtain the expression

$$\tan \delta_l = -\frac{1}{k} \langle j_l | V | \chi_l \rangle^2 / \langle \chi_l | V - V G_l V | \chi_l \rangle. \tag{45}$$

Note that the expression (45) is Schwinger's variational functional with trial function $|\chi_l\rangle$ (Ref. 32). Thus, in the case of the single-term separation (43) the separating function is the trial function. Consider the N-term approximation

$$\left.\begin{aligned} V^{(N)} &= \sum_{i,j=1}^{N} V|\chi_l^i\rangle d_{ij}^{(N)} \langle \chi_l^j | V; \\ [d^{-1}]_{ij} &= \langle \chi_l^i | V | \chi_l^j \rangle. \end{aligned}\right\} \tag{46}$$

In this case (44) can also be solved and

$$\left.\begin{aligned} \tan \delta_l &= -\frac{1}{k} \sum_{i,j=1}^{N} \langle j_l | V | \chi_l^i \rangle C_{ij}^{(N)} \langle j_l | V | \chi_l^j \rangle, \\ [C_{ij}^{-1}] &= \langle \chi_l^i | V - V G_l V | \chi_l^j \rangle. \end{aligned}\right\} \tag{47}$$

We show that (47) is a variational functional stable with respect to first-order variations with respect to each of the trial functions $|\chi_l^i\rangle$. To prove this, we substitute $\chi_l^k = \Psi_l + \delta\Psi_l$ into the functional, and then

$$\tan \delta_l = -\frac{1}{k} \sum_{i,j=1}^{N} \langle \psi_l | V - V G_l V | \chi_l^i \rangle C_{ij}^{(N)} \langle \Psi_l | V - V G_l V | \chi_l^j \rangle$$
$$= -\frac{1}{k} \langle j_l | V | \psi_l \rangle + O[(\delta\Psi_l)^2].$$

We now turn to the three-dimensional two-body problem. In (4) we set

$$\left.\begin{aligned} |\eta_l\rangle &= |\chi_{\mathbf{k}}^{i(+)}\rangle, \\ |\zeta_l\rangle &= |\chi_{\mathbf{k}'}^{i(-)}\rangle. \end{aligned}\right\} \tag{48}$$

Substituting (48) into the equation $\Psi_{\mathbf{k}}^{(\pm)} = |\mathbf{k}\rangle + G_0^{(\pm)} V \Psi_{\mathbf{k}}$, we obtain for the amplitude the expression

$$\left.\begin{aligned} f(\mathbf{k}, \mathbf{k}') &= -\frac{1}{2\pi} \sum_{i,j=1}^{N} \langle \chi_{\mathbf{k}'}^{i(-)} | V | \mathbf{k} \rangle C_{ij}^{(N)} \langle \mathbf{k}' | V | \chi_{\mathbf{k}}^{j(+)} \rangle, \\ [C^{-1}]_{ij} &= \langle \chi_{\mathbf{k}'}^{i(-)} | V - V G_0 V | \chi_{\mathbf{k}}^{j(+)} \rangle. \end{aligned}\right\} \tag{49}$$

For $N=1$, the expression (49) reduces to Schwinger's variational principle for the scattering amplitude. As in the case of the partial two-body problem considered above, we can show that (49) is stationary with respect to first-order variations of each trial function. If

$$\chi_{\mathbf{k}'}^{n(-)} = \Psi_{\mathbf{k}'}^{(-)} + \delta\Psi_{\mathbf{k}'}^{(-)}, \quad \chi_{\mathbf{k}}^{n(+)} = \Psi_{\mathbf{k}}^{(+)} + \delta\Psi_{\mathbf{k}}^{(+)},$$

then

$$f(\mathbf{k}, \mathbf{k}') = -\frac{1}{2\pi} \langle \mathbf{k}' | V | \Psi_{\mathbf{k}}^{(+)} \rangle + O[(\delta\Psi)^2]. \tag{50}$$

In the two-body problem, a solution can be found exactly and the variational principles (47) and (49) are hardly useful. For a larger number of particles, when an exact solution of the Schrödinger equation cannot be found, a generalization of the variational principle like this is important.

In many-channel theory in the case of an elastic process we choose the separation

$$|\eta_n^i\rangle = |\chi_{\mathbf{k}_n}^{i(+)}\rangle, \quad |\xi_n^i\rangle = |\chi_{\mathbf{k}_n'}^{i(-)}\rangle,$$

and then the amplitude of the elastic process is determined by the expression

$$\left.\begin{aligned} f(\mathbf{k}, \mathbf{k}') &= -\frac{1}{2\pi} \sum_{i,j=1}^{N} \langle \varphi_{\mathbf{k}'_\alpha} | V_{\alpha\beta} | \chi_{\mathbf{k}_\beta}^{i(+)} \rangle C_{ij}^{(N)} \langle \chi_{\mathbf{k}'_\gamma}^{j(-)} | V_{\gamma\delta} | \varphi_{\mathbf{k}'_\delta} \rangle, \\ [C^{-1}]_{ij} &= \langle \chi_{\mathbf{k}'_\alpha}^{i(-)} | V_{\alpha\beta} - V_{\alpha\gamma} G_{0\gamma}^{(+)} V_{\gamma\delta} | \chi_{\mathbf{k}_\delta}^{j(+)} \rangle, \end{aligned}\right\} \tag{51}$$

where $|\varphi_{\mathbf{k}\alpha}\rangle$ is a plane wave corresponding to the entry channel, i.e.,

$$|\Psi_{\mathbf{k}_\alpha}^{(+)}\rangle = |\varphi_{\mathbf{k}_\alpha}\rangle + G_{0\alpha}^{(+)} V_{\alpha\beta} |\Psi_{\mathbf{k}_\beta}^{(+)}\rangle \tag{52}$$

We now show that (51) is a variational functional stationary with respect to first-order variations in any of the trial functions χ. Suppose that

$$\chi^{i(-)}_{\mathbf{k}'_\alpha} = \Psi^{(-)}_{\mathbf{k}'_\alpha} + \delta\Psi, \quad \chi^{n(+)}_{\mathbf{k}_\beta} = \Psi^{(+)}_{\mathbf{k}_\beta} + \delta\Psi. \tag{53}$$

Substituting (53) into (51), we obtain

$$f(\mathbf{k},\mathbf{k}') = -\frac{4}{2\pi}\sum_{i,j=1}^{N} \langle \Psi^{(-)}_{\mathbf{k}'_\alpha} | V_{\alpha\beta} - V_{\alpha\gamma} G^{(+)}_{0\gamma} V_{\gamma\beta} | \chi^{j(+)}_{\mathbf{k}_\beta} \rangle \times C^{(N)}_{ij} \langle \chi^{i(-)}_{\mathbf{k}'_\delta} | V_{\delta\sigma} - V_{\delta\rho} G^{(+)}_{0\rho} V_{\rho\sigma} | \Psi^{(+)}_{\mathbf{k}_\sigma} \rangle = \doteq -\frac{1}{2\pi} \langle \Psi^{(-)}_{\mathbf{k}'_\alpha} | V_{\alpha\beta} | \varphi_{\mathbf{k}_\beta} \rangle + O[(\delta\Psi)^2].$$

For inelastic processes without redistribution, we construct the separable approximation

$$\left.\begin{aligned} V^{(N)}_{\alpha\beta} &= \sum_{i,j=1}^{N} V_{\alpha\gamma} | \eta^{i(+)}_{\mathbf{k}_\gamma} \rangle d^{(N)}_{ij} \langle \zeta^{j(-)}_{\mathbf{k}'_\delta} | V_{\delta\beta}, \\ [d^{-1}]_{ij} &= \langle \zeta^{i(-)}_{\mathbf{k}'_\alpha} | V_{\alpha\beta} | \eta^{j(+)}_{\mathbf{k}_\beta} \rangle. \end{aligned}\right\} \tag{54}$$

For the amplitude we then obtain

$$\left.\begin{aligned} f_m(\mathbf{k},\mathbf{k}') &= -\frac{4}{2\pi}\sum_{i,j=1}^{N} \langle \varphi^{(m)}_{\mathbf{k}'_\alpha} | V_{\alpha\beta} | \eta^{i(+)}_{\mathbf{k}_\beta} \rangle C^{(N)}_{ij} \langle \zeta^{j(-)}_{\mathbf{k}'_\delta} | V_{\delta\sigma} | \varphi_{\mathbf{k}_\sigma} \rangle, \\ [C^{-1}]_{ij} &= \langle \zeta^{i(-)}_{\mathbf{k}'_\alpha} | V_{\alpha\beta} - V_{\alpha\gamma} G^{(+)}_{0\gamma} V_{\gamma\beta} | \eta^{j(+)}_{\mathbf{k}_\beta} \rangle. \end{aligned}\right\} \tag{55}$$

Here, $\varphi^{(m)}_{\mathbf{k}_\alpha}$ is the plane wave in the channel with number m, i.e.,

$$| \Psi^{(m)(\pm)}_{\mathbf{k}_\alpha} \rangle \doteq | \varphi^{(m)}_{\mathbf{k}_\alpha} \rangle + G^{(\pm)}_{0\alpha\alpha} V_{\alpha\beta} | \Psi^{(m)(\pm)}_{\mathbf{k}_\beta} \rangle.$$

It is easy to see that if $\zeta^{i(-)}_{\mathbf{k}'_\alpha} = \Psi^{(m)(-)}_{\mathbf{k}'_\alpha} + \delta\Psi$ and $\eta^{i(+)}_{\mathbf{k}_\alpha} = \Psi^{(+)}_{\mathbf{k}_\alpha} + \delta\Psi$, then $f_m(\mathbf{k},\mathbf{k}') = -1/2\pi \langle \varphi^{(m)}_{\mathbf{k}'_\alpha} | V_{\alpha\beta} | \Psi^{(+)}_{\mathbf{k}_\beta} \rangle + O[(\delta\Psi)^2]$.

Thus, in both the two-body problem and many-channel theory the separable representation method leads to a sequence of variational principles whose convergence is guaranteed by the convergence of the expansion (4), and the first member of the sequence is Schwinger's variation principle. It follows from this that each of these principles can be given a dynamical meaning since $|V - V^{(N)}|$ characterizes the error. And, finally, in contrast to the variations in these variational principles the variation is performed at once with respect to the set of trial functions. This leads to a unique possibility of constructing an approximate solution of quantum-mechanical equations. As a rule, the corresponding dynamical equations can be fairly accurately integrated in exceptional cases. Using these functions to separate, we obtain an amplitude that in all the exceptional cases is equal to the exact amplitude, and in the intermediate region can give an estimate of the error. For example, taking the S-wave and eikonal solutions as trial functions in the two-body problem, we find an amplitude which describes the scattering processes in a wider range than the eikonal amplitude or the S-wave solution.

We now turn to bound-state problems. We show that the expansion (4) leads in this case too to a variational principle. We consider the two-body problem and for simplicity we restrict ourselves in (4) to $N=1$, $\eta=\zeta=\chi$. In this case, the equation $\Psi_E = G_E V\Psi$ can be solved explicitly and the binding energy is determined from the expression

$$1 = \langle \chi | VG_E V | \chi \rangle / \langle \chi | V | \chi \rangle. \tag{56}$$

Suppose $\chi = \psi + \delta\Psi$, $E = E_0 + \delta E$; then

$$\frac{\langle \Psi | VG_{E_0} V | \Psi \rangle + 2\langle \delta\Psi | VG_{E_0} V | \Psi \rangle}{\langle \Psi | V | \Psi \rangle + 2\langle \delta\Psi | V | \Psi \rangle} - 1 = -\delta E \,\mathrm{const} + O[(\delta\Psi)^2],$$

or $\delta E \sim O[(\delta\Psi)^2]$, i.e., (56) gives for the binding energy a functional that is stationary with respect to a first-order variation. The separable representation method leads to a variational principle for the eigenvalues μ_n of the Hilbert–Schmidt problem.[2)] With the potential $V^{(1)} = V|\chi\rangle\langle\chi|V/\langle\chi|V|\chi\rangle$, the Hilbert–Schmidt problem $\Psi_n = G_E V\Psi_n/\mu_n(E)$ can be solved explicitly, and for the eigenvalues we find

$$\mu^{(1)}_n = \langle \chi | VG_E V | \chi \rangle / \langle \chi | V | \chi \rangle. \tag{57}$$

We show that (57) is a variational functional stable with respect to first-order variations of the trial function.

Indeed, suppose $\chi = \Psi + \delta\Psi$; then

$$\mu^{(1)}_n = \frac{\langle \Psi | VG_E V | \Psi \rangle + 2\langle \delta\Psi | VG_E V | \Psi \rangle}{\langle \Psi | V | \Psi \rangle + 2\langle \delta\Psi | V | \Psi \rangle} + O[(\delta\Phi)^2] = \mu_n + O[(\delta\Phi)^2].$$

Taking N terms in (4), we also obtain a sequence of variational principles. The resulting variational principles are currently of considerable interest in connection with the new approach to the theory of resonances based on the Hilbert–Schmidt method.[34]

We find that the variational principles (49), (51), and (53) can be obtained directly from Schwinger's variational principle:

$$f(\mathbf{k},\mathbf{k}') = -\frac{1}{2\pi}\frac{\langle \chi^{(-)}_{\mathbf{k}'} | V | \mathbf{k} \rangle \langle \mathbf{k}' | V | \chi^{(+)}_{\mathbf{k}} \rangle}{\langle \chi^{(-)}_{\mathbf{k}'} | V - VG^{(+)}_0 V | \chi^{(+)}_{\mathbf{k}} \rangle}. \tag{58}$$

Indeed, we choose the trial functions in the form

$$\chi^{(-)}_{\mathbf{k}'} = \sum_{i=1}^{N} a_i \chi^{i(-)}_{\mathbf{k}'}; \quad \chi^{(+)}_{\mathbf{k}} = \sum_{i=1}^{N} b_i \chi^{i(+)}_{\mathbf{k}}. \tag{59}$$

Substituting (59) into (58) and finding the coefficients a_i and b_i, we obtain (49) from the condition $\partial f/\partial a_i = \partial f/\partial b_i = 0$. In Ref. 35, Hufner and Lemmer use an expansion of the type (59):

$$\chi^{(-)}_{\mathbf{k}'} = \sum_{i=1}^{N} a_i \varphi_i; \quad \chi^{(+)}_{\mathbf{k}} = \sum_{i=1}^{N} b_i \varphi_i. \tag{60}$$

For the amplitude, one obtains the so-called variational approximation, which is not stationary.

Although (49), (51), and (53) can be obtained from (58) for a definite choice of the trial functions, we shall show that there is not a single variational principle with different trial functions but rather a sequence of variational principles. To prove this, we consider the case when the given potential is already an operator of finite rank N, which can always be represented in the form

$$\left.\begin{aligned} V &= \sum_{i,j=1}^{N} W | y_i \rangle d^{(N)}_{ij} \langle y_j | W, \\ [d^{-1}]_{ij} &= \langle y_i | W | y_j \rangle. \end{aligned}\right\} \tag{61}$$

Replacing V by $V^{(N)}$ in accordance with Eq. (4), we obtain

$$\left.\begin{aligned} V^{(N)} &= \sum_{\alpha,\beta=1}^{N} V | \eta_\alpha \rangle D^{(N)}_{\alpha\beta} \langle \zeta_\beta | V, \\ [D^{-1}]_{\alpha\beta} &= \langle \zeta_\alpha | V | \eta_\beta \rangle. \end{aligned}\right\} \tag{62}$$

Substituting (61) into (62), we can readily show that $V^{(N)} \equiv V$. Therefore, if the original potential is an opera-

[2)] A variational principle for $\mu_n(E)$ is considered in Ref. 33.

tor of finite rank N, the variational principle obtained from an N-term separation leads to the exact value for any trial functions. This property of the Schwinger variational principle (58) for trial functions in the form of plane waves was pointed out in Ref. 36.

4. ITERATION–SEPARABLE REPRESENTATION METHOD

To solve quantum-mechanical problems, one frequently uses the method of iteration; several iteration schemes are possible.[37] One of the general shortcomings of the majority of the known iteration procedures is that they converge under the assumption of a "weak interaction."

In Refs. 31 and 38 an iteration–separable representation method is proposed for solving quantum-mechanical problems and this converges for arbitrary short-range potentials. In this section, we apply the method to the two-body problem and the theory of many-channel scattering.

The essence of the method is this: We replace the operator of the potential V by the separable

$$V^{(1)} = V|\chi\rangle\langle\chi|V/\langle\chi|V|\chi\rangle, \tag{63}$$

and as $|\chi\rangle$ we choose any approximate solution $|\chi_0\rangle$ of the equation

$$(H_0+V)|\Psi\rangle = E|\Psi\rangle. \tag{64}$$

Then, with the separable potential (63) Eq. (64) can be solved explicitly and we find $\Psi_0=\Psi(\chi_0)$. After this, setting $|\chi_1\rangle=\Psi_0$ in (63) we again solve the equation and find $\Psi_1=\Psi(\chi_1)$, and then $|\chi_2\rangle=\Psi_1$, etc.

Note that if the iteration process

$$(E-H_0)|\Psi^{(N)}\rangle = V|\Psi^{(N-1)}\rangle\langle\Psi^{(N-1)}|V|\Psi^{(N)}\rangle/\langle\Psi^{(N-1)}|V|\Psi^{(N-1)}\rangle \tag{65}$$

converges, then $\Psi^{(N)}$ converges to the exact solution. Indeed, in this case $\Psi^{(N)}\sim\Psi^{(N-1)}$ and $(E-H_0)|\Psi^{(N)}\rangle \sim V|\Psi^{(N)}\rangle$. We now consider the convergence of the iteration–separable representation method for the two-body problem. We show that for partial-wave scattering the following assertion holds: the iteration–separable representation method for the phase shift of scattering on a short-range potential converges exponentially and uniformly.

To prove the convergence of the process

$$\left.\begin{aligned} |\Psi_l^{(N)}\rangle &= |j_l\rangle + G_lV|\Psi_l^{(N-1)}\rangle c^{(N-1)};\\ c^{(N-1)} &= \langle j_l|V|\Psi_l^{(N-1)}\rangle/\langle\Psi_l^{(N-1)}|V-VG_lV|\Psi_l^{(N-1)}\rangle \end{aligned}\right\} \tag{66}$$

we must find a way of introducing a metric. Clearly, the most felicitous choice is the case when the norm corresponds to the required physical quantity. A norm suitable for us can be introduced as follows:

$$\|\Psi\| = \sup_k |\langle j_l|V|\Psi\rangle|, \tag{67}$$

i.e., if Ψ_l is an exact solution of Eq. (44), then

$$\|\Psi_l\| = \sup_k |\tan\delta_l(k)|. \tag{68}$$

When $\delta_l(k)\sim\pi/2$, the definition of the metric must be augmented.

Suppose that after some step N we obtain $\|\Psi_l-\Psi_l^{(N)}\| \sim\epsilon\gg 1$. Then

$$\|\Psi_l - c^{(N)}\Psi_l^{(N)}\| = \sup_k |\langle j_l|V|\Psi_l\rangle - \langle j_l|V|\Psi_l^{(N)}\rangle\langle\Psi_l^{(N)}|V|j_l\rangle/\langle\Psi_l^{(N)}|V-VG_lV|\Psi_l^{(N)}\rangle|.$$

Using the results of Sec. 3, we obtain $\|\Psi_l - c^{(N)}\Psi_l^{(N)}\| \sim O(\epsilon^2)$. And, therefore,

$$\|\Psi_l - \Psi_l^{(N+1)}\| = \|G_lV(\Psi_l - c^{(N)}\Psi_l^{(N)})\| \sim O(\epsilon^2),$$

if G_lV is a completely continuous operator. Thus, we have proved the assertion.

It follows from this that $|\tan\delta_l - \tan\delta_l^{(N)}|\sim|c^{(N-1)}-1|^2$ for $|c^{(N-1)}-1|\ll 1$. Thus, the error can be estimated at each step.

We now consider the numerical convergence of the method. In Table 1 we give the scattering lengths for the case of the potential

$$V(r) = g\theta(1-r)/r. \tag{69}$$

The exact value of the scattering length a for this potential can be readily calculated[37]:

$$a = -\sum_n g^n/[(n+1)!(n-1)!]/\sum_n g^n/n! \tag{70}$$

It can be seen from Table 1 that, as we asserted, in the region of g values in which the Born series converges the first approximation of the iteration–separable representation method is not worse than the second Born approximation, the second approximation of the iteration–separable representation method is not worse than the fourth Born approximation, the third is not worse than the eighth Born approximation, etc.

For the three-dimensional two-body problem, the iteration–separable representation method can be generalized as follows:

$$\left.\begin{aligned} \Psi_{\mathbf{k}}^{(N)(+)} &= |\mathbf{k}\rangle + G_0^{(+)}V|\Psi_{\mathbf{k}}^{(N-1)(+)}\rangle c^{(N-1)};\\ c^{(N-1)} &= \langle\Psi_{\mathbf{k}'}^{(N-1)(-)}|V|\mathbf{k}\rangle/\langle\Psi_{\mathbf{k}'}^{(N-1)(-)}|V-VG_0^{(+)}V|\Psi_{\mathbf{k}}^{(N-1)(+)}\rangle. \end{aligned}\right\} \tag{71}$$

In the case of many-channel problems, the iteration–separable representation method is generalized similarly. To illustrate the method, let us consider the collision of positrons with hydrogen atoms. Since the positron mass is small compared with proton mass, the motion of the nucleus in the collision process is unimportant and can be ignored.

TABLE I

g	a_1	a_2	a_3	a
20	−0.697674416	−0.423862662	−0.632448315	−0.789290815
15	−0.681818179	−0.530973777	−0.618485048	−0.759095221
10	−0.652173914	−0.615034185	−0.617338575	−0.709979757
5	−0.576923072	−0.592755220	−0.596135705	−0.606672674
2	−0.428571425	−0.436187397	−0.436745792	−0.436821375
1	−0.300000000	−0.302182432	−0.302224215	−0.302225343
0.5	−0.187500000	−0.187956839	−0.187959047	−0.187959058
0.1	−0.046875000	−0.046881022	−0.046881025	−0.046881023
−0.1	0.053571429	0.053579508	0.053579510	0.053579510
−0.5	0.374999996	0.377094965	0.377105295	0.377105247
−1.0	1.499999985	1.574344038	1.575890094	1.575920298
−1.3	4.875000357	6.144098341	6.198497951	6.209247526
−1.4	10.500000715	20.370762348	21.150523662	21.180144786

Note: a is the scattering length for the potential (69); a_1, a_2, a_3 are the first, second, and third iteration–separable representation method approximations, respectively.

TABLE II

Method	Statistical potential	$1S-2S$	$1S-2S-2P$	$1S-2S-2P-3S$	$1S-2S-2P-3S-4S$
Iteration–separable representation method	0.561	0.541	0.365	0.356	0.355
Exact calculation	0.582	0.564	—	—	—

We assume also that the annihilation takes place through the channel in which positronium is formed, and then the corresponding system of equations has the form

$$\left.\begin{aligned}&(h^2\Delta/2\mu+E-E_n)\,F_n(\mathbf{r})=V_{nm}(\mathbf{r})\,F_m(\mathbf{r});\\&V_{nm}=\int\Psi_n^*(\mathbf{r}')\,(1/r-1/|\mathbf{r}-\mathbf{r}'|)\,\Psi_m(\mathbf{r}')\,d\mathbf{r}',\end{aligned}\right\}\tag{72}$$

where Ψ_n is the system of eigenfunctions of the hydrogen atom. In Table 2 we give the e^+H scattering lengths (in the atomic system of units) for scattering of positronium on the hydrogen atom calculated by the iteration–separable representation method.

For comparison, we also give the results of exact calculations on a statistical potential and with allowance for strong coupling of the states $1S-2S$ (Ref. 39). It can be seen that the iteration–separable representation method gives scattering lengths that are close to the exact values in the problem with a restricted number of channels. As one would expect, the greatest contribution to the scattering length results from allowance for the virtual P state of the hydrogen atom. It can be seen from Table 2 that the series with respect to the virtual S states converges rapidly.

Thus, the iteration–separable representation method is a rapidly converging universal procedure which can be used to calculate nuclear collisions and also collisions between charged particles and atoms.

5. TWO-SIDED ESTIMATES IN THE NUCLEAR TWO- AND THREE-BODY PROBLEM

Although the variational principles for scattering processes express a condition of stationarity rather than of a minimum or maximum, the problem of obtaining one- and two-sided bounds on the scattering parameters is not hopeless. In this direction, a considerable number of investigations have already been made.[40–42]

We shall consider here the application of the separable representation method to construct two-sided estimates for the physical observables in the nuclear two- and three-body problem.[43,44] The essence of the method is as follows: Instead of a given problem with the Hamiltonian H we consider two others with intermediate Hamiltonians H_- and H_+, for which solutions can be found simply, and $H_-\leqslant H\leqslant H_+$. As a result, we obtain two-sided estimates. Thus, the problem consists of constructing two potentials V_- and V_+ such that $V_-\leqslant V\leqslant V_+$ with which the Schrödinger equation can be solved exactly. We consider first the case when the interaction does not contain repulsion: $V=-W$, where the operator W is positive, i.e., $\langle\Psi|W|\Psi\rangle\geqslant 0$ for any Ψ. We show that the expansion (4) with $|\eta_i\rangle=|\zeta_i\rangle$ leads to V_+. Indeed,

$$J=\Big(\langle\Psi|+\sum_i c_i^*\langle\eta_i|\Big)\,W\Big(\sum_j c_j|\eta_j\rangle+|\Psi\rangle\Big)\geqslant 0\tag{73}$$

for all c_i. We choose the c_i's from the condition $\partial J/\partial c_i=\partial J/\partial c_i^*=0$, and then

$$J=\langle\Psi|W|\Psi\rangle-\sum_{i,j=1}^{N}\langle\Psi|W|\eta_i\rangle\,d_{ij}^{(N)}\,\langle\eta_j|W|\Psi\rangle,$$

or

$$\langle\Psi|V|\Psi\rangle\leqslant\langle\Psi|V^{(N)}|\Psi\rangle\tag{74}$$

for all functions η_i and any N, so that $V^{(N)}=V_+$. To construct V_-, we go over to the momentum space. We consider the identity

$$\left.\begin{aligned}&W(k,k')\equiv W^{(N)}(k,k')+W_1(k,k');\\&W_1(k,k')=W(k,k')-W^{(N)}(k,k').\end{aligned}\right\}\tag{75}$$

Since $W_1(k,k')\geqslant 0$, using the Hölder inequality, we can readily show that

$$W_1(k,k')\leqslant\sqrt{W_1(k,k)\,W_1(k',k')}.\tag{76}$$

On the basis of (76),

$$\widetilde{W}^{(N)}(k,k')=W^{(N)}(k,k')+\sqrt{W_1(k,k)\,W_1(k',k')}$$

or

$$\begin{aligned}\widetilde{V}^{(N)}(k,k')&=V^{(N)}(k,k')\\&+\sqrt{(V^{(N)}(k,k)-V(k,k))\,(V^{(N)}(k',k')-V(k',k'))}.\end{aligned}\tag{77}$$

The approximation (77) has the following properties:

1) $\widetilde{V}^{(N)}(k,k)=V(k,k)$;
2) $\widetilde{V}^{(N)}(k',k')=V(k',k')$;
3) $\widetilde{V}^{(N)}(k,k')\leqslant V(k,k')$,

i.e., $\widetilde{V}^{(N)}=V_-$.

For potentials that contain repulsion, the construction of the operators V_+ and V_- can be generalized as follows[44]: Suppose $V=V_{\text{rep}}-V_{\text{att}}$, where $V_{\text{rep}}\geqslant 0$ and $V_{\text{att}}\geqslant 0$; then $V_-=V_{\text{rep}}^{(N)}-\widetilde{V}_{\text{att}}^{(M)}$, $V_+=\widetilde{V}_{\text{rep}}^{(N)}-V_{\text{att}}^{(M)}$. To obtain $V^{(N)}$, it is natural to choose the Bateman expansion.

We now show that the intermediate potentials V_+ and V_- lead to two-sided estimates. For this, we use the Feynman inequalities.[4] Suppose the potential is a function of some parameter λ. Then for a bound state $(H_0+V)\Psi_n=-|E_n|\Psi_n$ we have

$$\frac{\partial|E_n|}{\partial\lambda}=-\left\langle\Psi_n\left|\frac{\partial V}{\partial\lambda}\right|\Psi_n\right\rangle,\tag{78}$$

and for the scattering problem

$$\frac{\partial\delta_l}{\partial\lambda}=-2m\left\langle\Psi\left|\frac{\partial V}{\partial\lambda}\right|\Psi\right\rangle\alpha;\tag{79}$$

$$\frac{\partial a}{\partial\lambda}=2ma^2\left\langle\Psi\left|\frac{\partial V}{\partial\lambda}\right|\Psi\right\rangle;\tag{80}$$

$$\frac{\partial(\tan\delta_l)}{\partial\lambda}=-2m\left\langle\Psi\left|\frac{\partial V}{\partial\lambda}\right|\Psi\right\rangle.\tag{81}$$

In Eqs. (79)–(81), the Ψ's are solutions of the partial-wave Schrödinger equation

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{l(l+1)}{2mr^2}+V(\lambda)-E\right)\Psi=0\tag{82}$$

TABLE III.

Type of potential		a_s, F	a_t, F	$\lvert\epsilon_d\rvert$, MeV	$\lvert E_t\rvert$, MeV
exact	approximate				
1	V_+	−23.6	5.450	2.2	8.96
	V_-	−25.0	5.424	2.3	9.02
2	V_+	−23.6	5.450	2.2	9.22
	V_-	−25.4	5.33	2.4	9.227

with different boundary conditions; a is the scattering length; $\alpha=\cos^2\delta_l$.

In order to find the limits of applicability of the expressions (79)—(81), let us consider briefly the derivation of these relations. For this, we differentiate Eq. (82) with respect to λ:

$$\left(-\frac{1}{2m}\frac{d^2}{dr^2}+\frac{l(l+1)}{2mr^2}+V(\lambda)-E\right)\frac{\partial\Psi}{\partial\lambda}=-\frac{\partial V}{\partial\lambda}\Psi. \tag{83}$$

We multiply (82) by $\partial\Psi/\partial\lambda$ and (83) by Ψ, subtract one from the other, and integrate with respect to r from 0 to ∞. If Ψ and $\partial\Psi/\partial\lambda$ have asymptotic behavior in the form of standing waves:

$$\Psi\sim\sin(kr-l\pi/2)+\tan\delta_l\cos(kr-l\pi/2);$$
$$\frac{\partial\Psi}{\partial k}\sim\frac{\partial\tan\delta_l}{\partial\lambda}\cos(kr-l\pi/2),$$

then as a result we obtain

$$\frac{\partial\tan\delta_l}{\partial\lambda}=-2m\left\langle\Psi\left|\frac{\partial V}{\partial\lambda}\right|\Psi\right\rangle.$$

In the derivation, it was assumed that at all points Ψ is a differentiable function with respect to λ. Therefore, (81) is violated if δ_l passes through the resonance values $(2n+1)\pi/2$ when the potential varies, and the relation (80) is violated if a bound state appears or disappears when the potential changes. In all the remaining cases, even if the difference between V_+, V_-, and V is not small, $\tan\delta_l^{(\pm)}$, $\delta_l^{(\pm)}$, $a^{(\pm)}$, $E_n^{(\pm)}$ can serve as upper and lower bounds of the corresponding quantities.

We introduce the parameter λ as follows[4]:

$$V_1=V_-+\lambda(V-V_-),\quad V_2=V_++\lambda(V-V_+),$$

and then

$$\left.\begin{aligned}
&\frac{\partial|E_n^{(1)}|}{\partial\lambda}=-\langle\Psi_n|V-V_-|\Psi_n\rangle;\\
&\frac{\partial|E_n^{(2)}|}{\partial\lambda}=-\langle\Psi_n|V-V_+|\Psi_n\rangle;\\
&\frac{\partial\delta_l^{(1)}}{\partial\lambda}=-2m\langle\Psi|V-V_-|\Psi\rangle\alpha;\\
&\frac{\partial\delta_l^{(2)}}{\partial\lambda}=-2m\langle\Psi|V-V_+|\Psi\rangle\alpha;\\
&\frac{\partial a^{(1)}}{\partial\lambda}=2m(a^{(1)})^2\langle\Psi|V-V_-|\Psi\rangle;\\
&\frac{\partial a^{(2)}}{\partial\lambda}=2m(a^{(2)})^2\langle\Psi|V-V_+|\Psi\rangle;\\
&\frac{\partial\operatorname{tg}\delta_l^{(1)}}{\partial\lambda}=-2m\langle\Psi|V-V_-|\Psi\rangle;\\
&\frac{\partial\operatorname{tg}\delta_l^{(2)}}{\partial\lambda}=-2m\langle\Psi|V-V_+|\Psi\rangle.
\end{aligned}\right\} \tag{84}$$

We integrate (84) with respect to λ from 0 to unity, and then

$$\left.\begin{aligned}
&|E_n^{(+)}|\leqslant|E_n|\leqslant|E_n^{(-)}|;\\
&\delta_l^{(+)}\leqslant\delta_l\leqslant\delta_l^{(-)};\\
&a^{(-)}\leqslant a\leqslant a^{(+)};\\
&\tan\delta_l^{(+)}\leqslant\tan\delta_l\leqslant\tan\delta_l^{(-)}.
\end{aligned}\right\} \tag{85}$$

In Table 3 we give the results of calculation[43] of the deuteron binding energy $|\epsilon_d|$, the tritium binding energy $|E_t|$, the singlet scattering length a_s, and the triplet scattering length a_t with potentials of two types:

1) $V(r)=V\exp(-\alpha r^2)$;
2) $V(r)=V\exp(-\beta r)$.

The calculation was made for the following values of the parameters of the potentials:

$V_0^s=-32.348\ F^{-2}$; $\quad V_0^t=-77.022\ F^{-2}$;
$\alpha_s=0.315\ F^{-2}$; $\quad \alpha_t=0.480\ F^{-2}$;
$V^s=-103.343\ F^{-2}$; $\quad V^t=-179.249\ F^{-2}$;
$\beta_s=1.364\ F^{-1}$; $\quad \beta_t=1.442\ F^{-1}$.

To construct $V^{(N)}$, the Bateman expansion with $N=3$ was used and the expansion nodes were determined from a minimum of χ^2:

$$\chi^2=\int|V(k,k')-V^{(3)}(k,k')|^2\times dk\,dk'\Big/\int|V(k,k')|^2\,dk\,dk'.$$

In all cases, $\chi^2\sim10^{-3}$. Note that to obtain tighter bounds the nodes should be chosen from the conditions of minimum of $|\delta_l^{(+)}-\delta_l^{(-)}|$, $|a^{(+)}-a^{(-)}|$, $\|E_n^{(+)}|-|E_n^{(-)}\|$, etc.

A one-sided bound for the phase shift in the case of an exponential potential[45] is shown in Fig. 1. The Bateman expansion with $N=4$ was used to construct $V^{(N)}$ and one of the nodes was $s_4=k$ ($k^2/2\mu$ is the energy).

6. HIGH ENERGIES

Successful application of the diffraction theory of multiple scattering (the Glauber approximation, Ref. 46) in the range of energies and momentum transfers in which the present theory cannot work has drawn attention to the problem of corrections. In the framework of the Glauber approximation, no allowance is made for: a) nonadiabatic effects; b) corrections associated with the deviation from geometrical optics (Fresnel corrections); c) shadow effects, etc.

Some of the effects have been discussed in the literature. The Fresnel corrections were considered in Ref. 48, nonadiabatic effects in Ref. 49, and shadow effects in Ref. 50.

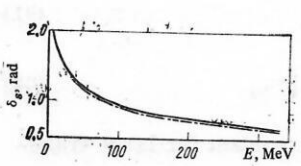

FIG. 1. $^3S_1$ phase shift for exponential potentials. The continuous curve is the exact calculation and the dashed curve is $\delta_s^{(+)}$.

Here, some of these corrections are investigated on the basis of the separable representation method.[47] We consider first the two-body problem. With decreasing energy of the particles or increasing scattering angle the importance of the corrections associated with the deviation from geometrical optics increases.

We recall that the high-energy approximation corresponds to solution of the Schrödinger equation with the eikonal Green's function

$$G_E(\mathbf{r}) = \frac{m}{(2\pi)^3}\int \frac{d^3p \exp(i\mathbf{p}\boldsymbol{\rho})}{(\mathbf{k}-\mathbf{p})\,\mathbf{k}+i\varepsilon} = \frac{m}{ik}\exp(ikz)\,\theta(z)\,\delta(\boldsymbol{\rho}),$$

where $\mathbf{k}=(0,0,k)$, $\boldsymbol{\rho}\perp\mathbf{k}$. Solutions of an equation with such a Green's function are known:

$$\left.\begin{aligned} |\Psi_{\mathbf{k}}^{(+)}\rangle &= \exp\left(ikz - \frac{im}{k}\int_{-\infty}^{z} V(\boldsymbol{\rho}, z')\,dz'\right);\\ \langle\Psi_{\mathbf{k}}^{(-)}| &= \exp\left(-ikz - \frac{im}{k}\int_{-\infty}^{z} V(\boldsymbol{\rho}, z')\,dz'\right). \end{aligned}\right\} \tag{86}$$

Using these functions to separate the potentials, i.e., setting in Eq. (4) $N=1$, $|\eta\rangle = |\Psi_{\mathbf{k}}^{(+)}\rangle$, $\langle\xi| = \langle\Psi_{\mathbf{k}'}^{(-)}|$, we obtain the amplitude in the form

$$\begin{aligned} f_{ES} &= f_E(\mathbf{k},\mathbf{k}')\,C(\mathbf{k},\mathbf{k}');\\ C(\mathbf{k},\mathbf{k}') &= f_E(\mathbf{k},\mathbf{k}')\Big(f_E(\mathbf{k},\mathbf{k}')\\ &- \int f_E(\mathbf{k},\mathbf{k}-\boldsymbol{\eta})\,f_E(\mathbf{k}-\boldsymbol{\eta},\mathbf{k}')\,g(\mathbf{k},\boldsymbol{\eta})\,\frac{d^3\eta}{(2\pi)^3}\Big)^{-1}. \end{aligned} \tag{87}$$

Here $f_E(\mathbf{k},\mathbf{k}')$ is the eikonal amplitude; $g=G_0-G_E$ is the difference between the exact and the eikonal Green's function. It is obvious that the Fresnel corrections are taken into account in the amplitude f_{ES}. From the explicit expression for

$$g(\mathbf{k},\boldsymbol{\eta}) = m\eta^2/[(\mathbf{k}\boldsymbol{\eta}+i\varepsilon)(2\mathbf{k}\boldsymbol{\eta}-\eta^2+i\varepsilon)] \tag{88}$$

it follows that: a) $C\to 1$ as $E\to\infty$; b) C increases with increasing scattering angle.

As an example, let us consider scattering on the Gaussian potential

$$V(r) = -A\exp(-\alpha^2 r^2), \tag{89}$$

restricting ourselves to the lowest term of the expansion in A/E in the correction factor C. We obtain

$$C^{-1} = 1 - J, \tag{90}$$

where

$$J = m\int \frac{d^3\eta}{(2\pi)^3}\,\frac{\eta^2}{(\mathbf{k}\boldsymbol{\eta}+i\varepsilon)(2\mathbf{k}\boldsymbol{\eta}-\eta^2+i\varepsilon)}\,\frac{V_{|\Delta+\eta|}V_{|\eta|}}{V_{|\Delta|}};$$
$$\Delta = \mathbf{k}-\mathbf{k}';\quad \Delta\mathbf{k}\sim 0;\quad V_{|\Delta|} = \pi^{3/2}\exp[-\Delta^2/(2\alpha)^2]/\alpha^3.$$

In the limiting cases a) $\Delta/\alpha \ll 1$ and b) $\Delta/\alpha \gg 1$ the expression for J takes the fairly simple form

$$\text{a)}\quad J = \frac{m}{2\pi}\,\frac{A}{8k^2}\left(1+\frac{\Delta^2}{2\alpha^2}+\dots\right); \tag{91}$$

$$\text{b)}\quad J = -\frac{m}{2\pi}\,\frac{A}{8k^2}\exp(\Delta^2/8\alpha^2)\,\Delta/(2\sqrt{2\alpha}), \tag{92}$$

i.e., the correction may be important for large transfers.

Because (87) is none other than the Schwinger variational functional with eikonal trial function, it is necessary to have quantitative estimates of the proximity of the obtained amplitude to the exact one. In this connection, an attempt was made in Ref. 51 to obtain such estimates.

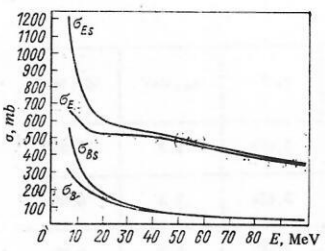

FIG. 2. Total cross section for the potential (89). σ_B and σ_E are the total cross sections in the Born and eikonal approximations; σ_{Bs} and σ_{Es} are the cross sections obtained by separation with respect to plane waves and from the eikonal solution, respectively.

Choosing in (4) $N=1$, $|\eta\rangle = |\xi\rangle = |\Psi_{\mathbf{k}}^{+}\rangle$ or $|\eta\rangle = |\xi\rangle = |\mathbf{k}\rangle$, we obtain, respectively,

$$\left.\begin{aligned} \tilde{f}_{E_s}(\mathbf{k},\mathbf{k}') &= f_E(\mathbf{k},\mathbf{k}')\,\tilde{C}_E;\\ f_{B_s}(\mathbf{k},\mathbf{k}') &= f_B(\mathbf{k},\mathbf{k}')\,\tilde{C}_B, \end{aligned}\right\} \tag{93}$$

where f_B is the scattering amplitude in the Born approximation. The total cross sections calculated in the approximation (93) for the potential (89) are shown in Fig. 2. The parameters A and α are such that (89) reproduces the data of neutron–neutron scattering at low energies. It can be seen that at high energies $\tilde{f}_{B_s}$ is nearer to the eikonal amplitude than f_B.

Since f_{B_s} satisfactorily describes the scattering at low energies (the numerical calculations show that in this energy region $\tilde{f}_{B_s}$ is more accurate than the second Born approximation) and at high energies f_{E_s} is closer to the exact amplitude, it is natural to take in (4) $N=2$, $|\eta_1\rangle = |\mathbf{k}\rangle$, $|\eta_2\rangle = |\Psi_{\mathbf{k}}^{+}\rangle$. The amplitude then has the form $f_s(\mathbf{k},\mathbf{k}') = f_B(\mathbf{k},\mathbf{k}')C_1 + f_E(\mathbf{k},\mathbf{k}')C_2$.

We now turn to the corrections to the Glauber theory. Let us recall briefly the derivation of the equations of the diffraction approximation in the framework of the many-channel potential model:

$$\left.\begin{aligned} \Psi_{\mathbf{k}_\nu}^{(+)}(\mathbf{r}) &= \exp(i\mathbf{k}_\nu\mathbf{r})\,\delta_{\nu\nu_0} + \int G_{0\nu}^{(+)}(\mathbf{r}-\mathbf{r}')\,V_{\nu\mu}(\mathbf{r}')\,\Psi_{\mathbf{k}_\mu}^{(+)}(\mathbf{r}')\,d^3r';\\ G_{0\nu}^{(+)}(\mathbf{r}-\mathbf{r}') &= (-m/2\pi\hbar^2)\,(\exp[ik_\nu|\mathbf{r}-\mathbf{r}'|]/|\mathbf{r}- \end{aligned}\right\} \tag{94}$$

In the case when the scattered particle is very fast, it is sensible to use the adiabatic approximation. In this approximation, ignoring the excitation energy of the system in (34), we obtain a Green's function that does not depend on the index ν:

$$G_{0\nu(\mathrm{ad})}^{(+)}(\mathbf{r}-\mathbf{r}') = (-m/2\pi\hbar^2)\exp[ik|\mathbf{r}-\mathbf{r}'|]/|\mathbf{r}-\mathbf{r}'|. \tag{95}$$

Substituting (95) into (94) and using the eikonal approximation for the Green's function (86), we obtain the solution (94) in the form

$$\begin{aligned} \Psi_{\mathbf{k}_\nu}^{(+)}(\mathbf{r}) &= \exp(ikz)\int \varphi_\nu^*(\mathbf{R}_1,\dots,\mathbf{R}_A)\\ &\times \exp\left(-\frac{ik}{2E}\int_{-\infty}^{z}\sum_i V_i(\mathbf{r}-\mathbf{R}_i)\,dz'\right)\varphi_{\nu_0}(\mathbf{R}_1,\dots,\mathbf{R}_A)\,d\tau,\\ d\tau &= d\mathbf{R}_1\dots d\mathbf{R}_A. \end{aligned} \tag{96}$$

Using (96) in the expression, for example, for the elastic scattering amplitude:

$$f(E, \Delta) = -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}'\mathbf{r})\, V_{\nu_0\mu}(\mathbf{r})\, \Psi^{(+)}_{\mathbf{k}_\mu}(\mathbf{r})\, d^3r, \quad \Delta = \mathbf{k}-\mathbf{k}',$$

we obtain the Glauber approximation

$$f_G(\mathbf{E}, \Delta) = \frac{ik}{2\pi}\int \exp[i(\Delta\rho)]\,[1-\exp(i\delta(\rho, \mathbf{s}_1, \ldots, \mathbf{s}_A))] \times |\Psi_{\nu_0}(\mathbf{R}_1, \ldots, \mathbf{R}_A)|^2 d\tau d^2\rho, \tag{97}$$

where

$$\delta(\rho, \mathbf{s}_1, \ldots, \mathbf{s}_A) = \sum_i \delta(\rho - \mathbf{s}_i) = \sum -\frac{k}{2E}\int_{-\infty}^{+\infty} V_i(\mathbf{r}-\mathbf{R}_i)\, dz;$$

and $\mathbf{s}_i$ is the transverse component of the vector $\mathbf{R}_i$.

Thus, to obtain (97), one must take the adiabatic and the eikonal approximation.

We now turn to the corresponding corrections. To analyze the Fresnel corrections, which correspond to taking into account the effects of the deviation of the eikonal Green's function from the expression (95), it is natural to use the functions (96) to separate $V_{\mu\nu}$, and then, using that potential, find the amplitude by sovling the equation

$$\Psi^{(+)}_{\mathbf{k}_\nu}(\mathbf{r}) = \exp[i\mathbf{k}_\nu\mathbf{r}]\,\delta_{\nu\nu_0} + \int G^{(+)}_{0\nu(\mathrm{ad})}(\mathbf{r}-\mathbf{r}')\, V_{\nu\mu}(\mathbf{r}')\, \Psi^{(+)}_{\mathbf{k}_\mu}(\mathbf{r}')\, d^3r'. \tag{98}$$

A program of this type was considered in Ref. 47 for elastic πd scattering.

With regard to the nonadiabatic corrections, for their analysis by the separable representation method one must apply the approximation of a separable potential directly to Eq. (94), and at the same time the Fresnel corrections also arise naturally.

In conclusion, we point out the possibility of allowing effectively for nonadiabatic effects. We introduce a mean excitation energy q of the nucleus, so that

$$\tilde{G}^{(+)}_{0\nu}(\mathbf{r}-\mathbf{r}') = -\frac{m}{2\pi\hbar^2}\,\frac{\exp[i(k+q)|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}. \tag{99}$$

It is clear that q depends on the type of process, the scattering angle, etc. With the propagator (99), Eqs. (94) can be solved in the eikonal approximation and, using these solutions to separate the potential, we obtain the variational functional for determining q.

7. NONSTATIONARY PROBLEMS

The separable representation method can also be helpful in consideration of nonstationary problems. For example, in Ref. 52 the problem of ionization of a bound level was considered in a model of a separable potential.

Consider the nonstationary Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = H_0\Psi + V(t)\Psi. \tag{100}$$

We seek a solution of (100) in the form

$$\Psi = \sum_k c_k(t)\,\Psi_k, \tag{101}$$

where $i\hbar\partial\Psi_k/\partial t = H_0\Psi_k$, and then $i\,dc_m/dt = \langle\Psi_m(t)|V(t)|\times\Psi_n(t)\rangle c_n$, or

$$c_m(t) = c_m(t_0) + \frac{1}{i}\int_{t_0}^{t}\langle\Psi_m(t')|V(t')|\Psi_n(t')\rangle c_n(t')\,dt, \tag{102}$$

We apply to $V(t)$ the single-term separation

$$V^{(1)}(t) = V(t)|\chi(t)\rangle\langle\chi(t)|V(t)/\langle\chi(t)|V(t)|\chi(t)\rangle, \tag{103}$$

and then

$$c^{(1)}_m(t) = c_m(t_0) + \frac{1}{i}\int_{t_0}^{t}\frac{\langle\Psi_m(t')|V(t')|\chi(t')\rangle\langle\chi(t')|V(t')|\Psi_n(t')\rangle c_n(t')}{\langle\chi(t')|V(t')|\chi(t')\rangle}\,dt'.$$

We introduce

$$J^{(1)}(t) = \langle\chi(t)|V(t)|\Psi_m(t)\rangle c^{(1)}_m(t)/\langle\chi(t)|V(t)|\chi(t)\rangle. \tag{104}$$

For the function $J^{(1)}$ we obtain the equation

$$\left.\begin{aligned} J^{(1)}(t) &= \frac{\langle\chi(t)|V(t)|\Psi_m(t)\rangle c_m(t_0)}{\langle\chi(t)|V(t)|\chi(t)\rangle} + \frac{1}{i}\int_{t_0}^{t} Q(t, t')\,J^{(1)}(t')\,dt'; \\ Q(t, t') &= \frac{\langle\chi(t)|V(t)|\Psi_m(t)\rangle\langle\Psi_m(t')|V(t')|\chi(t')\rangle}{\langle\chi(t)|V(t)|\chi(t)\rangle}. \end{aligned}\right\} \tag{105}$$

The expression (105) is much simpler than (102) since in this case there is one equation and not a system. If a solution of (105) has been found, then

$$c^{(1)}_m(t) = c_m(t_0) + \frac{1}{i}\int_{t_0}^{t}\langle\Psi_m(t')|V(t')|\chi(t')\rangle J^{(1)}(t')\,dt'. \tag{106}$$

For N-term separation

$$\left.\begin{aligned} V^{(N)}(t) &= \sum_{\alpha,\beta=1}^{N} V(t)|\chi_\alpha(t)\rangle d^{(N)}_{\alpha\beta}\langle\chi_\beta(t)|V(t), \\ [d^{-1}]_{\alpha\beta} &= \langle\chi_\alpha(t)|V(t)|\chi_\beta(t)\rangle, \end{aligned}\right\} \tag{107}$$

we introduce

$$J^{(N)}_\alpha(t) = \sum_{\beta=1}^{N} d^{(N)}_{\alpha\beta}\langle\chi_\beta(t)|V(t)|\Psi_n(t)\rangle c^{(N)}_n(t). \tag{108}$$

In this case, we have a system of N equations for $J^{(N)}_\gamma$:

$$\left.\begin{aligned} J^{(N)}_\gamma(t) &= d^{(N)}_{\gamma\delta}\langle\chi_\delta(t)|V(t)|\Psi_m(t)\rangle c_m(t_0) + \frac{1}{i}\int_{t_0}^{t} Q_{\gamma\alpha}(t, t')\,J^{(N)}_\alpha(t')\,dt'; \\ Q_{\gamma\alpha}(t, t') &= d^{(N)}_{\gamma\delta}(t)\langle\chi_\delta(t)|V(t)|\Psi_m(t)\rangle \times\langle\Psi_m(t')|V(t')|\chi_\alpha(t')\rangle, \end{aligned}\right\} \tag{109}$$

and for $c^{(N)}_m$ we obtain the expression

$$c^{(N)}_m(t) = c_m(t_0) + \frac{1}{i}\int\langle\Psi_m(t')|V(t')|\chi_\alpha(t')\rangle J^{(N)}_\alpha(t')\,dt'.$$

With regard to the choice of χ_α, as separating functions we should, in the spirit of the paper, choose certain approximate solutions of (100), and use (105) and (109) as a possible improvement of the existing approximations. Note that to obtain a solution of (100) in a closed form it is also necessary to carry out separation with respect to the time. For periodic solutions, this is in principle possible, though at the present time the meaning of such a procedure is not clear.

CONCLUSIONS

Let us consider the possibilities of applying the separable representation method. In bound-state problems, this method should be used to solve the equations of the

method of K harmonics.[53] For example, choosing as separating functions the solutions obtained in the approximation K_{min} and taking into account the results of Sec. 4, one can hope that the existing approximation is significantly improved. Variational methods for the Hilbert–Schmidt problem evidently find application in investigations of resonances.

For scattering problems, one can use the iteration–separable representation method to obtain three-dimensional nucleon–nucleon amplitudes and for realistic calculations of three-particle systems above the three-particle threshold.

The sequence of variational principles may be very helpful for obtaining two-particle amplitudes that coincide with the exact amplitudes at high and low energies and to construct amplitudes that coincide at high energies with the diffraction approximation and at low energies with the amplitude obtained by solving the Faddeev equations. It is obvious that calculations of this type can be carried out on the existing computers.

Note that the results of Ref. 54, in which the separable representation method is applied directly to the kernels of the Faddeev equations, show that the present method can also be used to solve the integral equations for four nucleons. One can hope that in the future the separable representation method will also find application for problems not considered here (Hartree–Fock equations, etc.).

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