

# Quasiclassical theory of nuclear shell structure

V. M. Strutinskii and A. G. Magner

*Institute of Nuclear Research, Academy of Sciences of the Ukrainian SSR, Kiev  
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The quasiclassical approximation for the density of single-particle levels based on Feynman path integrals is extended to the case of degenerate classical motion. It is shown that the appreciable fluctuations (gross shells) in the density of the single-particle levels correspond to quasiclassical quantization of the motion in multidimensional periodic orbits. Formal estimates are obtained for the scale and period of the oscillations of the single-particle level density, the level density of an excited nucleus, and the shell corrections to the binding energies and deformation energies of nuclei. The connection between the classical theory and the method of numerical averaging is discussed.

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## INTRODUCTION

The shell structure of the nucleus is one of the most important concepts in the modern theory of complex nuclei. However, despite a long historical development, only recently, as a result of numerical calculations for strongly deformed nuclei, have we discovered<sup>1,2</sup> that single-particle levels are grouped in regularly distributed bunches—gross shells. This phenomenon is due neither to spherical symmetry nor the assumed proximity of the nuclear nucleon potential to an harmonic oscillator's. Earlier, this was known only for spherical nuclei, for which the properties of magic nuclei were explained in this way. In deformed nuclei, the gross shells are manifested just as strongly, and they are more numerous. They are associated not only with anomalies in the nuclear masses but also fundamental phenomena such as nonsphericity of the shape of the nuclei of the rare earth elements and actinides, and the existence of a quasistationary intermediate state in nuclear fission.

Thus, the shell structure of deformed nuclei is one of the most important factors determining the properties of heavy nuclei at not too high excitation.<sup>1-4</sup> Examples of the distribution of the level density in a Woods-Saxon potential with very arbitrary shape of the surface are shown in Fig. 1. The appreciable variations in the number of levels over the interval 2-3 MeV used in the calculations cannot be explained by random fluctuations. A similar picture is observed in other cases, and we are probably confronted here with a general feature of the distribution of single-particle levels in a three-dimensional potential well. Therefore, the problem of the origin of the gross shells and the clarification of the physical conditions under which the shell nonuniformity of the spectrum is particularly pronounced cannot be confined to the framework of the shell model. To an equal extent, the problem of gross shells exists in all self-consistent theories which admit almost independent motion of quasiparticles in a common potential. The physical adequacy of such a picture has now been established. However, the absence of a consistent theory capable of explaining the origin of structures like gross shells has resulted in a curious "lack of recognition" of their importance in nuclear theory and the putting forward of erroneous qualitative interpretations based, as a rule, on the example of an harmonic oscillator. In order to develop further the quantitative theory of com-

plex nuclei and its applications, deeper understanding of shell phenomena is needed.

The first consistent theoretical explanation of the concentrations of levels, based on the quasiclassical solution of the problem of the distribution of eigenfrequencies in a cavity of arbitrary form, was given by Balian and Bloch (Ref. 5, see also Refs. 6 and 7). In Ref. 5(d), they obtained a quasiclassical solution of the quantum problem for a spherical, infinitely high potential on the basis of the concept of multiple reflection for the Green's function of a problem with reflecting walls.<sup>5(b)</sup> They found that the oscillating components in the level densities correspond to certain orbits of periodic motion in the cavity. In Ref. 5(e) they attempted to extend these results to the general case of the Schrödinger equation. An important qualitative conclusion of Ref. 5(e) is the assertion that the structure of the gross shells of the single-particle spectrum and the gross structure resonance of the scattering amplitude which are similar to the resonances of the optical model have a common origin. The paper Ref. 5(e) also contains a discussion of

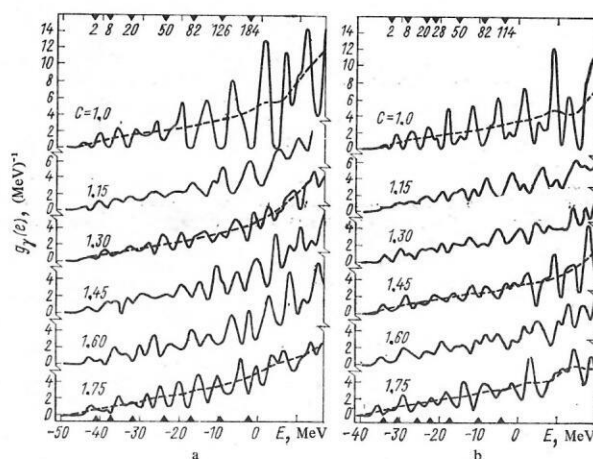


FIG. 1. Density of single-particle levels  $g_1(e)$  in Woods-Saxon potentials of the nucleus  $^{240}\text{Pu}$  with different shapes of the surface that correspond approximately to the change in the shape of the nucleus in the fission process. (a) Neutrons; (b) protons; the deformation parameter  $c$  is the length of the semimajor axis in units of  $R$ ; the continuous curves are obtained with a parameter  $\gamma = 2$  MeV; the dashed curves with  $\gamma = 7$  MeV. The upper and lower parts of the figure show the magic number of nucleons for spherical shape (Ref. 2).

the role of spin variables, which were not taken into account in the other papers. Unfortunately, the solution of a particular problem and the method of solution developed specially for a particular example do not answer many important questions related to the general characteristics of the phenomenon of shell structure. Attempts to obtain a solution not related to the specific properties of the problem with an infinite well were made in other papers of Balian and Bloch on the basis of a general iterative expansion for the single-particle Green's function in which the component  $G_0$  of the Green's function corresponding to the shortest classical path between the points  $\mathbf{r}'$  and  $\mathbf{r}''$  is the propagator.<sup>5(c)</sup> The component  $G_0(\mathbf{r}', \mathbf{r}'', e)$  corresponds to the Thomas—Fermi approximation for the particle distribution and in accordance with Ref. 5(c) must be the principal term of the iterative series in the quasiclassical limit. In fact, such a representation is not convenient for obtaining the quasiclassical approximation because the terms of the series associated with complicated paths with one or more reflection points contain a contribution of at least the same order in  $\hbar$  as  $G_0$ . For Hamiltonians of high symmetry, the contribution of more complicated orbits to the Green's function may even be of lower order in  $\hbar$  than the Thomas—Fermi component.

The general quasiclassical solution of the problem can be obtained in the framework of the method of stationary phase from the Feynman path integral for the propagator  $K(\mathbf{r}'t', \mathbf{r}''t'')$ , from which, in its turn, the Green's function is obtained by a simpler Fourier transformation.<sup>8-11</sup> The distribution of the single-particle levels and the spatial distribution of the particles are obtained by integrating the Green's function with respect to the spatial or energy variable. In this approach, one does not encounter the difficulty associated with specific degeneracy of finite classical motion: Because of the singularity at the turning points, the solution of the classical problem is not unique. This lack of uniqueness does not affect the solution of the evolution problem since all such paths differ in the time of motion. The additional degeneracy which may yet remain is due solely to the physical properties of the Hamiltonian. Other advantages of the solution based on Feynman integrals are its generality and perspicuity.

## 1. QUASICLASSICAL GREEN'S FUNCTION

*Basic theory.* The probability amplitude for transition of a particle from the point  $\mathbf{r}'t'$  to the point  $\mathbf{r}''t''$  can be represented by the Feynman path integral<sup>12,13</sup>

$$K(\mathbf{r}''t'', \mathbf{r}'t') = \lim_{n \rightarrow \infty} \prod_{k=0}^{n-1} [M/2\pi i \hbar (t_{k+1} - t_k)]^{3/2} \int \dots \int d\mathbf{r}_1 \dots d\mathbf{r}_{n-1} \exp \left\{ \frac{i}{\hbar} \sum_{k=0}^{n-1} R(\mathbf{r}_{k+1}t_{k+1}, \mathbf{r}_k t_k) \right\}, \quad (1)$$

where  $M$  is the mass of the particle;  $t_0 = t'$ ,  $t_1, t_2, \dots, t_{n-1}, t_n = t''$  are the successive time intervals that divide the interval  $t'' - t'$  into  $n$  small intervals  $t_1 - t', t_2 - t_1, \dots$ ;  $\mathbf{r}' = \mathbf{r}_0, \mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n = \mathbf{r}''$  are the corresponding values of the coordinates along the formal path  $\mathbf{r} = \mathbf{r}(t)$  joining  $(\mathbf{r}'', t'')$  and  $(\mathbf{r}', t')$ . For infinitesimally small intervals,  $R$  can be expressed in terms of the classical Lagrange function  $L(\mathbf{v}, \mathbf{r}, t)$ :

$$R(\mathbf{r}_{k+1}t_{k+1}, \mathbf{r}_k t_k) = (t_{k+1} - t_k) L((\mathbf{r}_{k+1} - \mathbf{r}_k)/(t_{k+1} - t_k), \mathbf{r}_k, t_k). \quad (2)$$

The quasiclassical expression for  $K$  was obtained in Refs. 9 and 11 from (1) by applying the method of stationary phase in each of the partial integrations. If we set  $\mathbf{r}_i = \mathbf{r}_i^* + \delta \mathbf{r}_i$  for  $i = 1, \dots, n-1$ , then

$$R = R^*(\mathbf{r}''t'', \mathbf{r}'t') + \sum_{i,j=1}^{n-1} (\partial^2 R / \partial \mathbf{r}_i \partial \mathbf{r}_j)^* \delta \mathbf{r}_i \delta \mathbf{r}_j + \dots,$$

where  $\mathbf{r}_i^*$  are the stationary points. The matrix  $(\partial^2 R / \partial \mathbf{r}_i \partial \mathbf{r}_j)^*$  is then reduced to principal axes, which leads to  $3(n-1)$  Fresnel integrals, each of which is proportional to  $\hbar^{1/2}$  and is classically a small quantity. As a result, we obtain

$$K^*(\mathbf{r}''t'', \mathbf{r}'t') \approx \sum_{\alpha} (2\pi\hbar)^{-3/2} |\mathcal{Y}_{\alpha}(\mathbf{p}'; \mathbf{r}')|^{1/2} \times \exp \left\{ \frac{i}{\hbar} R_{\alpha}^*(\mathbf{r}''t'', \mathbf{r}'t') + i\chi_{\alpha} \right\}. \quad (3)$$

The stationary values are indicated by asterisks and the sum contains all real classical paths ( $\alpha$ ) which begin at  $\mathbf{r}'$  when  $t = t'$  and end at  $\mathbf{r}''$  when  $t = t''$ ;  $R_{\alpha}^*$  is the classical, time-independent, action integral for path  $\alpha$ , and the  $3 \times 3$  determinant

$$\mathcal{Y}_{\alpha}(\mathbf{p}'; \mathbf{r}') = \det \|\partial^2 R_{\alpha}^* / \partial \mathbf{r}' \partial \mathbf{r}'\| = \det \|\partial \mathbf{p}' / \partial \mathbf{r}'\|_{\alpha} \quad (4)$$

is the Jacobian of the transformation from the initial momenta at  $\mathbf{r}'$ , defined by

$$\mathbf{p}' = -\partial R^*(\mathbf{r}''t'', \mathbf{r}'t') / \partial \mathbf{r}' \quad (5)$$

to the final coordinates  $\mathbf{r}''$ . The phases  $\chi_{\alpha}$  in (3) are determined by the number of negative eigenvalues of the matrix formed by the second partial derivatives of  $R_{\alpha}$  at each of the intermediate stationary points  $\mathbf{r}_i^*, \dots, \mathbf{r}_{n-1}^*$ .

Fourier transformation of  $K$  gives the stationary Green's function

$$G(\mathbf{r}'', \mathbf{r}', e) = \frac{1}{i\hbar} \int_0^{\infty} K(\mathbf{r}''t'', \mathbf{r}'0) \exp \left( \frac{i}{\hbar} e t \right) dt, \quad (6)$$

which is obtained in the quasiclassical approximation in the following form:

$$G^*(\mathbf{r}'', \mathbf{r}', e) = -\frac{1}{2\pi\hbar^2} \sum_{\alpha} \left\{ |\mathcal{Y}_{\alpha}(\mathbf{p}'t_{\alpha}; \mathbf{r}'e)|^{1/2} \times \exp \left[ \frac{i}{\hbar} S_{\alpha}^*(\mathbf{r}'', \mathbf{r}', e) + i\nu \right] \right\}_{\alpha}. \quad (7)$$

Here,  $S_{\alpha}$  is the classical action integral

$$S_{\alpha}^*(\mathbf{r}'', \mathbf{r}', e) = \int_{\mathbf{r}'}^{\mathbf{r}''} p_{\alpha} d\mathbf{l}_{\alpha} \quad (8)$$

along the  $\alpha$ th classical path joining  $\mathbf{r}'$  and  $\mathbf{r}''$  for given energy  $e$ ; and the  $4 \times 4$  determinant

$$\mathcal{Y}_{\alpha}(\mathbf{p}'t_{\alpha}; \mathbf{r}'e) = \det \left\| \frac{\partial \mathbf{p}' / \partial \mathbf{r}'' \partial \mathbf{p}' / \partial e}{\partial t_{\alpha} / \partial \mathbf{r}'' \partial t_{\alpha} / \partial e} \right\|_{\alpha}, \quad (9)$$

where

$$t_{\alpha}(\mathbf{r}'', \mathbf{r}', e) = \partial S_{\alpha}^*(\mathbf{r}'', \mathbf{r}', e) / \partial e = \int_{\mathbf{r}'}^{\mathbf{r}''} \frac{1}{|\dot{\mathbf{r}}|} d\mathbf{l}_{\alpha} \quad (10)$$

is the classical time of motion between the points  $\mathbf{r}'$  and  $\mathbf{r}''$  for path  $\alpha$ .

The determinant  $\mathcal{Y}_{\alpha}$  can be regarded as the Jacobian of the transformation from the variables  $(\mathbf{p}'t_{\alpha})$  to the variables  $(\mathbf{r}''e)$  connected by the relation (10) and the equation

$$\mathbf{p}' = -\partial S_{\alpha}^*(\mathbf{r}'', \mathbf{r}', e) / \partial \mathbf{r}'. \quad (11)$$

The phases  $\nu_\alpha$  are determined by the singularities of the matrix

$$\|\partial^2 S / \partial r_i \partial r_j\|^{-1}, \quad 1 \leq i, j \leq n-1, \quad (12)$$

whose eigenvalues are the coefficients in the expansion of the action integral when the Feynman integral is calculated by the method of stationary phase. The points of the path at which the eigenvalues (12) vanish are called caustic points. At these points, it is necessary to take into account the third (or higher) derivatives of the action integral and consider the limit of a small but finite eigenvalue.<sup>14</sup> Singularities of another type are associated with turning points, at which one or several components of the derivative of the classical momentum with respect to the coordinates of these points become infinite. In the neighborhood of a turning point, the integrals can be reduced to singularities of the first type by a transformation of the variables in the complex plane  $w = -i^{3/4} \lambda x$ , where  $x$  is the coordinate normal to the surface of reflection and  $\lambda = (\partial^2 S / \partial x^2)^* \rightarrow \infty$  on such a surface. The corresponding eigenvalue (12) is small in the new representation. Another method of solution is based on transforming the Green's function to the momentum representation.<sup>10</sup> The results of both methods for the additional phase in (7) agree and lead to  $\nu_\alpha = -\pi(c_\alpha + c_t)/2$ , where  $c_\alpha$  is the number of all singular points along the path between the initial and final points with allowance for the multiplicity of the singular points;  $c_t$  corresponds to the number of singularities of  $\partial R^2 / \partial t^2$  for  $t = t^*$ . In ordinary cases,  $c_t$  is equal to zero or unity depending on the sign of  $(\partial^2 R / \partial t^2)^*$ , where  $t^*$  is the stationary point of integration with respect to the time in (6).

In the quasiclassical formula (7), the sum is taken over all classical paths joining the points  $\mathbf{r}'$  and  $\mathbf{r}''$  for given energy  $e$ . These paths differ above all by the number and nature of the singular points of the matrix (12). They include one singular path whose length  $l_\alpha$  becomes zero as  $\mathbf{r}''$  approaches  $\mathbf{r}'$ . For this path of "zero length" we find in the limit  $\mathbf{r}'' \rightarrow \mathbf{r}'$  the expression

$$G_0^*(\mathbf{r}', \mathbf{r}', e) = -\frac{M}{2\pi\hbar^2 |\mathbf{r}' - \mathbf{r}'|} \exp \left\{ \frac{i}{\hbar} |\mathbf{r}' - \mathbf{r}'| p(\mathbf{r}') \right\}, \quad (13)$$

where  $\mathbf{r} = (\mathbf{r}'' + \mathbf{r}')/2$ . This component of the expansion (7) ensures the singularity required by the boundary condition for the Green's function. It corresponds to the Thomas-Fermi components in the distributions of the spatial density and level density.

To calculate the level density, we shall require an expression for the Green's function for nearly equal values of the coordinates, and also for the other components of the Green's function. At a short distance between the points ( $|\mathbf{r}'' - \mathbf{r}'| \lesssim \lambda$ ), there is a singularity in the quasiclassical approximation, this requiring an important modification of the calculation in many important cases. The point is that in our derivation we have assumed that all the paths joining the points  $\mathbf{r}'$  and  $\mathbf{r}''$  for a given energy (or corresponding to a given time  $t'' - t'$  of the motion) are isolated, so that any variation of the classical path for given conditions leads to a variation of the action integral. However, if the initial and the final point on the path coincide, then in some cases a family of infinitesimally close orbits with the same

action passes through the given point. Formula (7) cannot be used in these cases since the action integral is stationary under the variations of the classical orbit that reduce to transition from one path of the family to another. An obvious example is a spherical potential, for which a continuum of orbits of definite type pass through a given point  $\mathbf{r}$ ; these are obtained from one another by a rotation through the angle  $\Phi$  with respect to the axis  $(0, \mathbf{r})$ , where  $0$  is the center of symmetry. The angle  $\Phi$  is a free parameter, and in this sense one can speak of degeneracy of the family of periodic orbits passing through the point  $\mathbf{r}$ ; the order of degeneracy  $\mathcal{D}$ , or the number of free parameters, of these paths is equal to unity in the given case. Degeneracy of the classical orbits increases the relative contribution of the complicated periodic paths in the quasiclassical expansions since the number of classically small integrations (Fresnel integrals) is reduced.

As will be shown in Sec. 8, the degeneracy of the classical orbits is associated with the existence of independent single-valued integrals of the motion, besides the energy integral, their existence being not necessarily due to spatial symmetry. The maximal degree of degeneracy of the periodic paths is bounded by the number of these integrals, and in a given potential there may exist simultaneously several continuous families of orbits with different degeneracy. The maximal value of  $\mathcal{D}$  for a three-dimensional problem is two. Families of orbits with  $\mathcal{D} = 2$  are possible in the case of the so-called complete degeneracy of the classical motion; for example, in a degenerate harmonic oscillator or in a Coulomb potential. In an arbitrary spherical potential  $\mathcal{D} = 1$ , but there may also exist a family with  $\mathcal{D} = 0$ . This is formed by rectilinear diametral orbits, whose rotation about the  $(0, \mathbf{r})$  axis does not have any meaning.

The amount by which the amplitude of the Green's function increases depends upon the degree of degeneracy, which we have defined above as the number  $\mathcal{D}$  of independent parameters of the family of paths passing through a fixed point at given energy with one and the same action integral. The magnitude of the integral with respect to the coordinates in the expression of the level density in terms of the Green's function is determined not only by the degree of degeneracy  $\mathcal{D}$  of the orbits passing through a given point, but also by the dimension ( $\mathcal{N}$ ) of the part of the classically accessible space filled by the orbits of the given type. This factor is taken into account in calculations of the level density and of the associated physical manifestation of shell structure.

*Spherical potential.* In what follows, we shall be interested in Green's functions with nearly equal arguments ( $\mathbf{r}'' \rightarrow \mathbf{r}'$ ). The Green's function appears in this form in the determination of the level density. If the Hamiltonian is spherically symmetric and  $\mathbf{r}'' = \mathbf{r}' = \mathbf{r}$ , the action integral is stationary for rotation of the path around the symmetry axis through the given point  $\mathbf{r}$  and the center of symmetry  $0$ . The conclusion reached above becomes meaningless: If the distance between the points  $\mathbf{r}''$  and  $\mathbf{r}'$  is less than or of the order of the wavelength, one must, instead of summing over the discrete



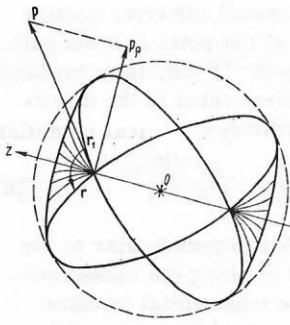


FIG. 2. Surface of stationary action for calculating the Green's function  $G(\mathbf{r}, \mathbf{r}', e)$  for a spherical potential well. The figure shows the center of symmetry 0, the point  $\mathbf{r}$ , the intermediate point  $\mathbf{r}_1$ , the momentum  $\mathbf{p}(\mathbf{r})$ , and the component  $p_\rho$  perpendicular to  $\mathbf{r}$ .

set of paths, calculate the sum of the contributions of orbits of the same type that go over continuously into each other, and after this calculate the sum over the selection of axisymmetric surfaces that correspond to the different types of orbit.

To calculate  $G(\mathbf{r}'', \mathbf{r}', e)$  in the limit  $\mathbf{r}'' \rightarrow \mathbf{r}' \rightarrow \mathbf{r}$ , we proceed as follows. We first eliminate the degeneracy by fixing an intermediate point  $\mathbf{r}_1$  in the multiple Feynman integral (1). In accordance with the rule for adding amplitudes<sup>12, 13</sup>

$$K(\mathbf{r}'', \mathbf{r}') = \int d\mathbf{r}_1 K(\mathbf{r}'', \mathbf{r}_1) K(\mathbf{r}_1, \mathbf{r}'). \quad (14)$$

We can now use the usual quasiclassical approximation for each of the two amplitudes in the integrand corresponding to the two parts of the isolated classical paths  $(\mathbf{r}'', \mathbf{r}_1)$  and  $(\mathbf{r}_1, \mathbf{r}')$ . For the integration with respect to  $\mathbf{r}_1$  in (14) it is convenient to use a cylindrical coordinate system  $(\rho, z, \varphi)$  in which the  $Oz$  axis is along the symmetry axis  $(O, \mathbf{r})$ , and the center is at the point  $\mathbf{r}$  (Fig. 2). The momentum coordinates  $p_\rho$  and  $p_z$  and the angle  $\varphi$  can be expressed in terms of the Cartesian components of the momentum,  $p_\rho = (p_x^2 + p_y^2)^{1/2}$ ,  $p_z, \varphi = \tan^{-1}(p_y/p_x)$ . For fixed  $\varphi$ , the integration with respect to the other components of  $\mathbf{r}_1$  can be made by the method of stationary phase. The stationary point  $(\rho^*, z^*, \varphi)$  belongs to the stationary surface, and in the subsequent integration with respect to  $\varphi$  the integrand does not depend on  $\varphi$ . We finally obtain

$$K^*(\mathbf{r}'', \mathbf{r}') = K_0 + \sum_\alpha \frac{1}{2\pi\hbar^2} \left\{ p_\rho \left| \Psi(p'_\rho p'_z; \rho'' z'') \right|^{1/2} \times \exp \left[ \frac{i}{\hbar} R^*(\mathbf{r}'', \mathbf{r}') + i\chi \right] \right\}_\alpha, \quad (15)$$

where  $K_0$  is the component associated with the path of zero length, for which the symmetry of the potential is not important. The summation is over the periodic paths that belong to different stationary surfaces, and it is assumed that these surfaces form a discrete set. In (15),  $\Psi_\alpha$  is the Jacobian, of lowered rank, of the transformation from the initial momenta  $(p'_\rho p'_z)$  to the final coordinates  $(\rho'', z'')$ . In the derivation of (15) it was assumed that the Jacobians of the corresponding transformations,  $|\Psi(\varphi'; \varphi)|$  and  $|\Psi(\varphi''; \varphi)|$ , are equal to unity. The Fourier transformation of (15) is performed by the method of steepest descent. We then obtain the following expression for the Green's function of the problem with spherical potential in the limit  $\mathbf{r}'' \rightarrow \mathbf{r}'$ :

$$G^*(\mathbf{r}'', \mathbf{r}', e) = G_0 - \frac{1}{(2\pi\hbar^2)^{1/2}} \sum_\alpha \left\{ p_\rho \left| \Psi(p'_\rho p'_z; \rho'' z'') \right|^{1/2} \times \exp \left[ \frac{i}{\hbar} S^*(\mathbf{r}'', \mathbf{r}') + i\chi - \frac{i\pi}{4} \right] \right\}_\alpha. \quad (16)$$

The additional phase  $-\pi/4$  is due to the fact that in the sequence of Feynman integrals (1) there is one less Fresnel integral in the given case. In (16),  $\Psi_\alpha$  is the Jacobian of lowered rank for the transformation of the two groups of variables given as arguments of  $\Psi_\alpha$  and related by the classical equations of motion for the given path  $\alpha$ ;  $t_\alpha$  is the time of motion along path  $\alpha$ , and it is determined in (10). The Jacobian in (16) can also be represented in the form

$$\Psi_\alpha = \det \begin{vmatrix} \partial p'_\rho / \partial \rho'' & \partial p'_\rho / \partial z'' & \partial p'_\rho / \partial e \\ \partial p'_z / \partial \rho'' & \partial p'_z / \partial z'' & \partial p'_z / \partial e \\ \partial t_\alpha / \partial \rho'' & \partial t_\alpha / \partial z'' & 0 \end{vmatrix}, \quad (17)$$

because the determinant of the submatrix surrounded by the wavy line is equal to zero because of the Hamiltonian equations.

Comparing (16) and (7), we see that in the case of spherical symmetry the contributions of the nonsingular terms associated with the periodic paths are increased by a factor proportional to  $\hbar^{-1/2}$ , and are quantities that, formally, are larger than  $G_0$ . This is due to the fact that the integration with respect to  $\mathbf{r}_1$  in (14) contains the classically small volume of a tube proportional to  $\hbar$  instead of the effective volume of a small neighborhood of the point  $\mathbf{r}_1$  proportional to  $\hbar^{3/2}$  in the case of isolated paths.

Equation (16) does not apply if the potential is spherical but the degree of degeneracy of the classical orbits is higher. As examples, one can mention the spherical harmonic oscillator and the hydrogen atom, in which, besides the trivial integrals of the energy and the angular momentum, there are specific additional integrals of the motion. At the same time, through a given point  $\mathbf{r}$  there passes a two-parameter family of periodic paths. The amplitude of the corresponding terms in the quasiclassical expression for the Green's function is augmented by an additional factor of order  $\hbar^{-1/2}$  compared with the spherical case and is formally a quantity of order  $\hbar^{-1}$  compared with  $G_0$ . However, this does not apply to the values of the corresponding terms in the expansions for the various physical quantities which are expressed as integrals of the Green's function. In them, the singularity of  $G_0$  for equal arguments leads to a relative increase of the Thomas—Fermi components by a formal factor  $\hbar^{-1}$ .

The derivation of analytic expressions for the Green's functions of problems with "symmetry" different from spherical requires knowledge of the concrete single-valued integrals of the motion corresponding to the symmetry. To solve these problems, one can use the device described above for separating an intermediate point of integration.

Equation (16) can also not be used to estimate the contribution of rectilinear diametral orbits passing through the center of symmetry. Even in the presence of spherical symmetry, only one such path passes through a given point  $\mathbf{r}$ . Therefore, the contributions



of such orbits to the Green's function are determined by the usual expression (7), and their order in  $\hbar$  is the same as the contribution of an ordinary isolated path, i.e.,  $\hbar^{-1/2}$  times less than the contribution of the more complicated orbits. This also applies to the calculation of the Green's function for problems with a symmetry less than spherical, for example, for an axisymmetry potential.

## 2. DENSITY OF SINGLE-PARTICLE LEVELS

*Isolated orbit.* The single-particle level density  $g(e)$  is related to the Green's function by the equation

$$g(e) = -\frac{1}{\pi} \operatorname{Im} \int dr \{G(r'', r', e)\}_{r'' \rightarrow r', r' \rightarrow r} \quad (18)$$

Using the quasiclassical expression (16) for  $G$ , we represent the level density in the form

$$g(e) = g_{TF} + g_{osc}, \quad (19)$$

where

$$g_{TF} = -\frac{1}{\pi} \operatorname{Im} \int dr \{G_0(r'', r', e)\}_{r'' \rightarrow r', r' \rightarrow r} \quad (20)$$

is the expression of the statistical Thomas—Fermi model:

$$g_{TF} = \frac{M}{2\pi^2 \hbar^3} \int dr p(r). \quad (21)$$

The components of the Green's function corresponding to paths whose length does not vanish in the limit  $r'' \rightarrow r'$  give oscillations associated with periodic paths. To see this, we compare the  $\alpha$ th component of  $g_{osc}$

$$(g_{osc})_\alpha = \frac{1}{2\pi^2 \hbar^2} \int dr \left\{ \left| \mathcal{F}_\alpha(p', r'; e) \right|^{1/2} \times \exp \left[ \frac{i}{\hbar} S_\alpha^*(r'', r', e) + i\nu_\alpha \right] \right\}_{r'' \rightarrow r', r' \rightarrow r}, \quad (22)$$

and find that in the limit  $\hbar \rightarrow 0$  the largest contributions come from the stationary points at which

$$\frac{\partial S_\alpha}{\partial r} = \left\{ \frac{\partial S_\alpha^*(r'', r', e)}{\partial r'} + \frac{\partial S_\alpha^*(r'', r', e)}{\partial r''} \right\}_{r'' \rightarrow r', r' \rightarrow r} = p'' - p' = 0. \quad (23)$$

Equality of the initial and the final momentum at a given point is the condition of periodic motion. Thus, it is the periodic paths that make the largest contribution, and in our subsequent discussion of the oscillation of the level density we shall take into account only the periodic paths, which we denote by the new superscript  $\beta$ :

$$g_{osc}(e) = \sum_\beta g_{osc}^{(\beta)}. \quad (24)$$

We also omit the asterisk above the classical quantities, since it is only these that will occur in the conclusions which follow. What we have said above does not apply to the calculation of the oscillations of the spatial density<sup>15, 16</sup>; however, in the integration of the spatial distributions, one can also use the stationarity condition, which separates out the contribution of the periodic orbits.

For potentials that have only isolated periodic paths, the oscillating component of the level density is

$$g_{osc}(e) = \frac{1}{2\pi^2 \hbar^2} \sum_\beta \operatorname{Im} \left\{ \exp \left[ \frac{i}{\hbar} S_\beta(e) + i\nu \right] \oint dl \sigma(l) \left| \frac{1}{2} \left( \frac{p}{\hbar} l_\beta, r'' \right) \right|^{1/2} \right\}_{\beta} \quad (25)$$

In (25),  $\sigma_\beta(l)$  is the classically small effective section of the tube containing the path at the point  $l$  on the path. Formally, it is proportional to  $(\hbar^{1/2})^2 = \hbar$ , in accordance with the scale of the quantum dispersion in the direction of the two transverse coordinates:

$$\sigma_\beta(l) = 2\pi i \hbar \left\{ \det \left\| \frac{\partial^2 S}{\partial \xi_i' \partial \xi_j'} + 2 \frac{\partial^2 S}{\partial \xi_i' \partial \xi_j''} + \frac{\partial^2 S}{\partial \xi_i'' \partial \xi_j''} \right\| \right\}_\beta^{-1/2}, \quad (26)$$

where  $\xi = (\xi_1, \xi_2)$  is a small vector perpendicular to the path  $\beta$ . In (25), the integration is along the closed contour of the periodic path  $\beta$ . The exponential is taken outside the integral because for periodic motion the action integral  $S_\beta$  does not depend on the coordinates.

Paths corresponding to different numbers  $m$  of revolutions appear in the sum (25) as independent components. This follows directly from the time-dependent description presented above. For a path traversed  $m$  times, the action is

$$S_\beta(e) = m S_\beta^{(1)}(e). \quad (27)$$

Accordingly, the time of motion along the path is  $t_\beta = m T_\beta$ , where  $T_\beta = dS_\beta^{(1)}(e)/de = 2\pi/\Omega_\beta$  is the period for orbit  $\beta$ . The other quantities in (25) also depend on  $m$ . In accordance with the property (27), each periodic orbit of the classical motion leads to a Fourier series in the expansion of the level density (25), the sum of this series being a periodic function of the energy  $e$  with period

$$\Delta e = \hbar \Omega_\beta. \quad (27a)$$

In accordance with (25) and (26), the oscillating component in the expansion of the level density contains a small factor proportional to  $\hbar^2$  with respect to the Thomas—Fermi term (21), although the corresponding components had formally the same order in the expansion of the Green's function.

For the case of isolated orbits, each component in (25) can be estimated as

$$(g_{osc})_\beta \approx a_\beta / \hbar \Omega_\beta, \quad (28)$$

where  $a_\beta$  is a dimensionless factor which depends on the specific properties of the path. Using the procedures of classical stability theory, we can express the coefficient  $a_\beta$  in terms of the stability index of the periodic orbit. Here, we shall not give these calculations<sup>95</sup> since in what follows the simplest case of an isolated orbit will interest us only as a convenient unit for estimating the oscillation amplitude of the level density.

*Enhancement of the oscillation amplitude.* In the general case, the magnitude of the integral of the oscillating components of the Green's function with respect to the spatial variables is determined by the dimension  $\mathcal{R}$  of the part of the region of the classically allowed motion filled by orbits of one and the same type with the same value of the action integral. The action does not change when the position of the point  $r$  in the integral (20) varies within this space because this corresponds to either displacement of the point along the orbit or transition to another periodic orbit with the same action. In the case of an isolated orbit,  $\mathcal{R} = 1$ , the action is stationary only if the point is displaced

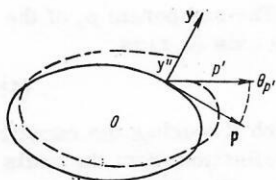


FIG. 3. Relation between the displacement of the final coordinate  $y''$  and the variation of the momentum  $p(r')$  used to calculate the Jacobian for a spherical potential well.

along the orbit. For  $\mathcal{R}=2$  and 3, the number of classically small integrations in (20) is reduced by one and two units, respectively, since in the first case the action is stationary for displacement of the point of integration over a certain surface and in the second with-in a volume of the order of the total volume of the system. The result of integration is accordingly determined by the effective volume of the classically small layer around the stationary surface or by the volume of space filled by the orbits. The amplitude  $g_{osc}$  is enhanced in (20) by a factor proportional to  $(\hbar^{-1/2})^{-1}$ , and, in conjunction with a possible amplification of the amplitude of the Green's function, the total enhancement is of order  $\hbar^{-(D+R-1)/2}$  compared with an isolated orbit.

The true dimensionless small parameter of the quasi-classical expansion, which corresponds to  $\hbar$ , is of order  $\hbar/pR$ , where  $R$  is the spatial dimension of the system. In the case of a nucleus, the parameter is approximately equal to  $0.7A^{-1/3} \approx 0.1$ , where  $A$  is the mass number (see Sec. 8). The total enhancement factor of the oscillation amplitude of the level density is of order  $A^{(D+R-1)/6}$ .

The values of  $\mathcal{D}$  and  $\mathcal{R}$  can usually be found without difficulty. In an ordinary spherical potential we have two families of orbits: one with  $\mathcal{D}=1$ ,  $\mathcal{R}=3$  and the other with  $\mathcal{D}=0$ ,  $\mathcal{R}=3$  (diameters). Accordingly, the amplitude is enhanced by factors of order  $A^{1/2}$  or  $A^{1/3}$ . In an arbitrary axisymmetric potential only one orbit ( $\mathcal{D}=0$ ) passes through a given point, and the stationary points lie on an axisymmetric surface ( $\mathcal{R}=2$ ). For the rectilinear orbits corresponding to motion along the symmetry axis,  $\mathcal{D}=0$ ,  $\mathcal{R}=1$ . A deformed harmonic potential is characterized by a diversity of cases, including orbits with the maximal degeneracy ( $\mathcal{D}=2$ ,  $\mathcal{R}=3$ ). In the following section, we derive the quasi-classical expression for the level density of an arbitrary spherical potential. The more complicated solution of the problem for a deformed oscillator will be presented separately. The results are given below in the present review.

**Spherical potential.** The expression for the oscillating component of the level density in the spherical case can be obtained by using the representation (16) for the Green's function in the form

$$g_{osc}(e) = +4(2\pi\hbar^3)^{-1/2} \text{Im} \sum_{\beta} \left\{ \exp \left[ \frac{i}{\hbar} S(e) + i\nu - i\frac{\pi}{4} \right] \times \int_{r_{\min}}^{r_{\max}} r^2 dr p_{\beta} | \Psi(p_{\beta} p'_{\beta}; p'' z'' e) |^{1/2} \right\}_{\beta}. \quad (29)$$

The radial integration is from the minimal to the maximal value of  $r$  for the trajectory of type  $\beta$ . For the periodic orbits in which we are interested, we can further simplify the expression for the Jacobian in (29)

by using the relations between certain derivatives that follow from the Hamiltonian—Jacobi equations. For this, it is convenient to introduce a two-dimensional Cartesian coordinate system  $(x, y)$  in the plane of the paths with center at the point  $r$  and abscissa  $(x)$  in the direction of the momentum  $p(r)$  (Fig. 3). Since the new coordinate system differs from the system  $(\rho, z)$  introduced above by a simple rotation about a common center, the value of the Jacobian remains the same in the new system, i.e.,

$$\mathcal{J}_{\beta} = \det \begin{vmatrix} \partial p'_x / \partial x'' & \partial p'_x / \partial y'' & \partial p'_x / \partial e \\ \partial p'_y / \partial x'' & \partial p'_y / \partial y'' & \partial p'_y / \partial e \\ \partial t_{\beta} / \partial x'' & \partial t_{\beta} / \partial y'' & 0 \end{vmatrix}.$$

Following Ref. 9(b), we calculate the partial derivatives of the identical equations  $H(p'_x p'_y, x' y') = e$ ,  $H(p''_x p''_y, x'' y'') = e$  (the unimportant angular variables are not given here) and we note that in the new coordinate system

$$dx'/dt = \partial H / \partial p'_x = p/M; \quad dy'/dt = \partial H / \partial p'_y = 0;$$

$$dx''/dt = \partial H / \partial p''_x = p/M; \quad dy''/dt = \partial H / \partial p''_y = 0.$$

Thus, we find

$$\mathcal{J}_{\beta} = \det \begin{vmatrix} 0 & 0 & -M/p \\ 0 & \partial p'_y / \partial y'' & \dots \\ M/p & \dots & \dots \end{vmatrix}_{\beta} = \left( \frac{M}{p} \right)^2 (\partial p'_y / \partial y'')_{\beta}. \quad (30)$$

The derivative in (30) determines the displacement of the orbit at the point  $r$  in the direction perpendicular to the orbit as a result of variation of the transverse component of the initial momentum, which is a characteristic of the stability of the orbit (see Fig. 3). Introducing the angle  $\theta_p \ll 1$ , as shown in Fig. 3, we can write

$$\mathcal{J}_{\beta} = \frac{M^2}{p} (\partial \theta_p / \partial y''). \quad (31)$$

Substituting the expression (31) into (29), we obtain finally the oscillating component of  $g(e)$ :

$$g_{osc}(e) = 8(2\pi\hbar^3)^{-1/2} \sum_{\beta} \sum_{m=1}^{\infty} M \int_{r_{\min}}^{r_{\max}} r^2 dr (p_{\beta}/p) \times | p \partial \theta_p / \partial y'' |^{1/2} \sin [m S_{\beta}^{(1)}(e)/\hbar + \nu_{\beta m} - \pi/4]. \quad (32)$$

The factor  $4\pi$  in (32) appears after integration with respect to the angular variables. In addition, the result has been doubled to take into account not only paths that begin at  $r'$  and end at  $r''$  but also those that correspond to motion in the opposite direction. We have explicitly indicated the summation over repeated paths. The momentum is  $p(r) = \sqrt{2M(e - V(r))}$  and  $p'_{\beta}$  is a component of  $p$  in the direction perpendicular to the axis  $(0, r)$  (see Fig. 2). Equation (32) gives the solution for an arbitrary spherical potential, excepting the spherical harmonic oscillator or other cases of higher degeneracy. One can obtain a simple and fairly accurate general estimate of the magnitude of the amplitude of  $g_{osc}$  for the spherical case in order of magnitude from

$$\partial \theta_p / \partial y'' \approx q_{\beta} (r/R)/l_{\beta}, \quad (33)$$

where  $l_{\beta}$  is the length of the orbit;  $q_{\beta}$  is a dimensionless function of order unity. In (33),  $R$  is a scale factor of the order of the radius of the system. The expression (32) can be represented as

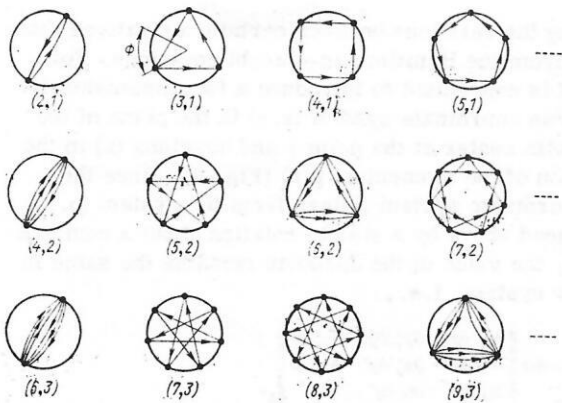


FIG. 4. Closed orbits in an infinite spherical potential well.

$$g_{osc} = 2 \frac{2MR^2}{\hbar^2} (2\pi)^{-1/2} (\bar{k}R)^{1/2} \times \sum_{\beta m} \left( \frac{4R}{l_\beta} \right)^{1/2} \sin \left[ \frac{1}{\hbar} m S_\beta^{(1)}(e) \pm v_{\beta m} \frac{\pi}{4} \right] \times \int_{r_{\min}}^{r_{\max}} \frac{r^2 dr}{R^3} [p_\beta | q(r/R) / \bar{p}_\beta ]^{1/2} ]_{\beta m}. \quad (34)$$

Here,  $\bar{p}$  is the mean value of the momentum along path  $\beta$ ;  $\bar{k} = \bar{p}/\hbar$ ;  $4R$  is the length of the shortest periodic diametral orbit. In (34), the integral is of order unity, and  $4R/l_\beta$  is also of order unity for  $m=1$ . From this we obtain an approximate estimate for the amplitude  $g_{osc}$  in the form

$$g_{osc} \approx (2MR^2/\hbar^2) \sqrt{2/\pi} (\bar{k}R)^{1/2} (4R/l_\beta)^{1/2}. \quad (35)$$

One can also give an expression in terms of the frequency  $\Omega$  of revolutions for orbit  $\beta$ ;

$$g_{osc} \approx \sqrt{2\pi} (\bar{k}R)^{3/2} (4R/l_\beta)^{3/2} / \hbar \Omega_\beta, \quad (36)$$

where

$$\Omega_\beta = 2\pi \bar{p} / M l_\beta = 2\pi / T_\beta. \quad (37)$$

It follows from these expressions that the amplitude of the oscillating component of the density of the single-particle levels in a spherical nuclear potential is of the order of a few  $\text{MeV}^{-1}$ .

### 3. SIMPLE EXAMPLES

**Infinite spherical potential well.** In this case, all the periodic orbits are regular polygons, and they can be characterized<sup>5(d)</sup> by means of two integers, namely, the number of vertices  $a$  and the number of revolutions about the center  $b$  made by a particle during one period (Fig. 4). The action integral for one period is

$$S_\beta^{(1)} = p(e) L(a, b), \quad 2a \geq b, \quad (38)$$

where  $p(e)$  is the momentum of the particle;  $L(a, b)$ , the length of the orbit, is equal to

$$L(a, b) = 2a\bar{R} \sin \phi. \quad (39)$$

Here,  $\phi = b\pi/a$ . The total length of the path after  $m$  revolutions is  $l(a, b) = m L(a, b)$ , and the period of revolution in a given orbit is

$$T(a, b) = l(a, b) / v, \quad (40)$$

where  $v = p/M$  is the velocity. The component  $p_\rho$  of the momentum perpendicular to the axis  $(0, r)$  is

$$p_\rho = (pR \cos \phi) / r. \quad (41)$$

The points of the orbit at which it touches the caustic surface and also the points of reflection from the walls of the potential well are singular. The caustic surface is the sphere of radius  $r_{\min} = R \cos \phi$ . The number of caustic points on the orbit is equal to  $a$ , and at each of them there arises an additional phase equal to  $-\pi/2$ . At the turning points, there is a double singularity since one of the eigenvalues of the matrix (12)—the one associated with the variation of  $r$  tangent to the boundary sphere and perpendicular to the plane of the path—is equal to zero, while the other, which is associated with variation of the coordinate normal to the surface, becomes infinite. Therefore, each reflection point adds  $-\pi$  to the total phase. At the end, we obtain the phase shift

$$v_\beta = -3ma\pi/2 (mb-1)\pi. \quad (42)$$

The correct value is<sup>5(d)</sup>

$$v_\beta = -3ma\pi/2 (mb-1)\pi, \quad (43)$$

but we have not been able to obtain it. The difference between (42) and (43) does not affect the results discussed in the present review.

Substituting the expressions (38)–(43) into Eq. (29), we obtain

$$g_{osc}(e) = 4(2\pi\hbar^2)^{-1/2} \text{Im} \sum_{a \geq 2, b \geq 1} pR \cos \phi \exp [ikl(a, b) + iv_\beta - i\pi/4] \times \int_{r_{\min}}^{r_{\max}} r dr \left| \frac{1}{\mathcal{H}} (p_\beta p_\rho l_\beta; p'' z'' e) \right|^{1/2}. \quad (44)$$

The Jacobian in Eq. (44) is  $|\mathcal{H}| = (M^2/p) |(\partial \theta_\rho / \partial y'')|$ , where  $\theta_\rho = \tan^{-1}(p_y/p_x)$ . The Cartesian coordinate system is chosen here in such a way that the  $X$  axis coincides with the direction along the path at the point  $r$  at which the coordinate origin is situated, and the plane  $(X, Y)$  coincides with the plane of the orbit (see Fig. 3). The conditions of reflection at the turning points determine

$$|\partial \theta_\rho / \partial y''| = R |\sin \phi| 2am (r^2 - R^2 \cos^2 \phi), \quad (45)$$

whence

$$|\mathcal{H}| = \frac{RM^2}{2amp} |\sin \phi| (r^2 - R^2 \cos^2 \phi). \quad (46)$$

We can now integrate with respect to the radial variable in Eq. (44), and we obtain

$$g_{osc}(e) = (kR)^{1/2} (\hbar^2/2MR^2)^{-1} \text{Im} \sum_{a \geq 2, b \geq 1} \sinh(2\phi) \sqrt{\frac{\sin \phi}{am\pi}} \times \exp [ikmL(a, b) + iv_\beta - i\pi/4]. \quad (47)$$

Here,  $k(e)$  is the wave number. We have omitted the terms corresponding to the diametral orbit ( $a=2b$ ).

Equation (47) agrees with the expression obtained in Ref. 5(d) by Balian and Bloch in the quasiclassical limit for the level density in a spherical potential well with infinite walls based on the idea of multiple reflections for the Green's function of the problem. For applications, the main interest attaches to the simplest orbits



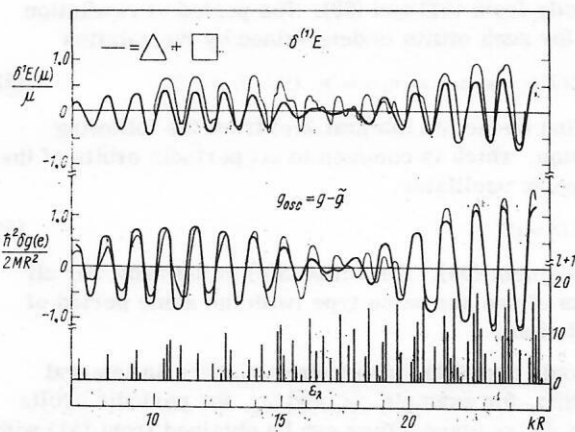


FIG. 5. Characteristics of shell structure for a spherical infinitely deep potential well. At the bottom, the eigenvalues for a spherical infinitely deep potential well and their degeneracy; in the middle, the oscillating component of the level density calculated by means of numerical averaging as described in the text (thin curve) with the averaging parameters  $\gamma = \hbar\Omega/3$  and  $\tilde{\gamma} = 2\hbar\Omega$ , and in the quasiclassical approximation (thick curve) in accordance with (48); at the top, the shell corrections  $\delta E$  to the energy found numerically in the framework of the method of shell corrections (thin curve) and in the quasiclassical approximation (thick curve) in accordance with (80).

that have sufficient degeneracy. According to Ref. 5(d), these are the orbits that have the shape of an equilateral triangle and a square, each with  $m=1$  (one orbit). They correspond to the parameters  $(a, b) = (3:1)$  and  $(4:1)$ . In particular, only such orbits contribute to the oscillation of the level density  $g_\gamma(e)$  smoothed by averaging with respect to the energy over an interval  $\gamma$  that is a few times less than or is of the order of the mean distance between the shells  $\hbar\Omega = 2\pi\hbar/T$ , where the mean period is  $\bar{T} = \bar{L}/v$ , and  $\bar{L} = [L(3,1) + L(4,1)]/2 = 5.42R$  is the mean length of these two orbits. In accordance with (47), the contribution of these orbits to  $g_{osc}$  is

$$(g_{osc})_{3+4} = -\frac{1}{2} \left( \frac{2MR^2}{\hbar^2} \right) \sqrt{\frac{kR}{\pi}} \times \left\{ \left( \frac{\sqrt{3}}{2} \right)^{1/2} \sin \left( 3\sqrt{3}kR + \frac{\pi}{4} \right) + \left( \frac{1}{\sqrt{2}} \right)^{1/2} \sin \left( 4\sqrt{2}kR + \frac{3\pi}{4} \right) \right\}. \quad (48)$$

The amplitudes and periods of the terms in (48) are very close to each other, which leads to an interesting phenomenon—pulsations of the amplitude  $g_{osc}$  as a function of the energy because of the interference of the contributions of two different classical orbits with nearly equal characteristics (Fig. 5). The expression (48) can be rewritten in the more perspicuous form

$$(g_{osc})_{3+4} = -\left( \frac{\sqrt{3}kR}{2\pi} \right)^{1/2} \left[ \cos \left( k\Delta L + \frac{\pi}{4} \right) \cos(k\bar{L}) + \frac{1}{2} \left( 1 - \left( \frac{2}{3} \right)^{1/4} \right) \cos \left( kL(4,1) + \frac{\pi}{4} \right) \right] \frac{2MR^2}{\hbar^2}. \quad (49)$$

The relative amplitude of the second term in the expression (49) is not more than 5% of the first term, and it can be ignored. In the remaining term, the first cosine represents the effect of the pulsation of the amplitude, which increases for values of  $k(e)R$  near the values  $13.7(n - \frac{1}{4})$ , where  $n=1, 2, \dots$ , which are

equal to 10.2, 14.0, etc., whereas the grouping of levels into gross shells is virtually absent for energies corresponding to  $k(e)R$  values approximately equal to  $13.7(n + \frac{1}{4}) = 3.5, 17.4, 30.8$ , etc.

The second cosine in (49) represents the ordinary grouping of levels into gross shells. Peaks appear in the level density at energies that differ by  $\hbar\Omega$ , i.e., at  $k(e)R$  values equal to  $1.16(n + \frac{1}{2})$ ,  $n=1, 2$ , etc. All these conclusions of the general theory are in excellent agreements with the numerical results obtained by evaluating a concrete spectrum of eigenvalues of the problem. The direct numerical calculations are compared with the quasiclassical analytical theory in Fig. 5. The central part of Fig. 5 shows curves of the oscillating part of the level density calculated by means of the quasiclassical expression (48) (heavy line) and the direct numerical method (thin line). In the last case, the averaging interval  $\gamma$  in the calculation of the mean level density has been taken equal to  $\hbar\Omega/3$ . The smooth value  $\tilde{g}(e)$  was obtained by numerical averaging with  $\gamma = 2.0\hbar\Omega$  (see Sec. 7). In the upper part of Fig. 5, we show the shell corrections to the binding energies (which are discussed in Sec. 6) calculated by two methods.

Our numerical results and the general expressions (34) and (47) enable us to consider the convergence of the contributions of different orbits. In the original complete expressions for  $g_{osc}$ , the convergence with respect to the number of revolutions  $m$  is very slow, the amplitude of each harmonic decreasing as  $m^{-1/2}$ . Therefore, the true shape of the oscillating component of the density differs significantly from what is shown in Fig. 5 [see Ref. 5(d)]. However, the contribution of the higher harmonics decreases exponentially in the level density averaged with respect to the energy (see Sec. 7), and then the theoretical shape of the oscillations of the averaged density approaches the sum of the contributions of the first one or two harmonics. Essentially, it is such an averaged value which is presented as the theoretical curve in Fig. 5. The contributions of the higher harmonics to the shell energy also converge rapidly. As is shown in Sec. 6, the contributions of the longer paths decrease faster than the square of the path length increases.

The contributions of orbits with a large number of reflection points  $a$  to  $g_{osc}$  decrease rapidly, despite the fact that the length of the orbit at the same time approaches the constant limit  $2\pi R$ . The convergence is due to the reduced volume of the spherical layer occupied by the orbit (32). Indeed, it can be readily seen that  $R - r_{min} \sim a^{-2}$  for  $a \gg 1$ . The main contribution to the shell variations of the density come from orbits that are not too near the surface, such as triangles and squares in the example considered above.

We give the expression for the component of the density associated with rectilinear diametral orbits, for which  $\mathcal{D}=0$ ,  $\mathcal{R}=2$ . In accordance with Ref. 5(d), for such orbits

$$g_{osc}(e) = -(\hbar^2 2MR^2)^{-1} \sum_{m=1}^{\infty} \frac{1}{2\pi m} \text{Im} \exp(4ikmR). \quad (50)$$

This expression can also be readily obtained in the

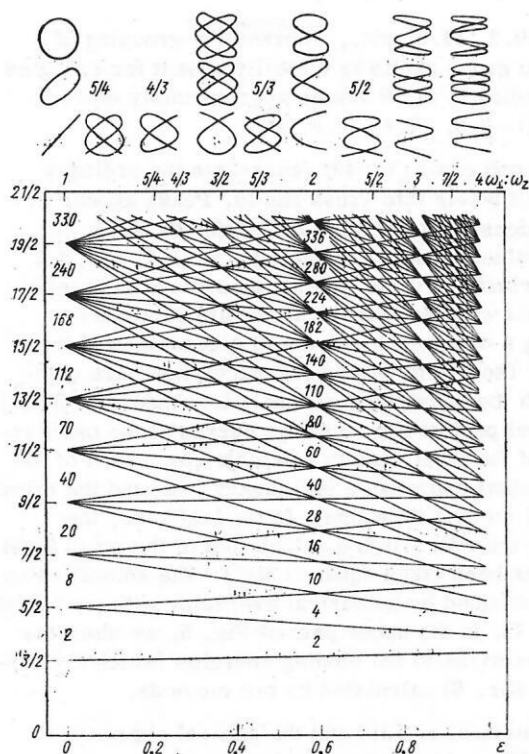


FIG. 6. Levels of an axisymmetric deformed harmonic oscillator; the energy unit is  $\hbar\omega_0(\epsilon)$ . The upper part shows examples of classical orbits [Lissajou figures in the  $xz$  plane] and the values of  $\omega_1 : \omega_2$ .

method of stationary phase. The sum of the Fourier series (50) is a saw-shaped curve, which is also characteristic of the level density in an infinite well with boundary in the shape of a planar parallelepiped. The shells are associated in this last case with rectilinear orbits—reflections from close parallel walls. The amplitude (50) is of order  $(1/kR)^{1/2} \approx A^{-1/6}$  compared with the principal term.

**Harmonic oscillator.** In a Cartesian coordinate system, the classical paths are determined by specifying five parameters of the equations of motion:

$$x = A_x \sin(\omega_x t); y = A_y \sin(\omega_y t + \varphi_y); z = A_z \sin(\omega_z t + \varphi_z), \quad (51)$$

where  $\omega_x, \omega_y$ , and  $\omega_z$  are the partial frequency parameters of the deformed oscillator potential. For any relationship between the frequencies, there exist simplest rectilinear periodic orbits corresponding to motion along each of the axes  $x, y$ , and  $z$ . Two- and three-dimensional periodic orbits are possible only in the case when two or more partial frequencies bear an integral relation to one another. When all three frequencies bear such a relation:

$$\omega_x : \omega_y : \omega_z = n_x : n_y : n_z, \quad (52)$$

there is complete degeneracy of the classical motion: All possible paths are periodic. In this case, namely, one obtains the maximally degenerate orbits, to which  $\mathcal{D}=2$  and  $\mathcal{R}=3$  correspond. Through every point of the classically allowed region there passes a doubly degenerate family of periodic orbits corresponding to one and the same energy. This can be readily seen

directly from (51) and (52). The period of revolution  $T_{xyz}$  for such orbits is determined by the relation

$$2\pi/T_{xyz} = \omega_x/n_x = \omega_y/n_y = \omega_z/n_z \quad (\mathcal{D}=2, \mathcal{R}=3). \quad (53)$$

We find the action integral  $S(e)$  from the following relation, which is common to all periodic orbits of the harmonic oscillator,

$$S(e) = eT \quad (54)$$

and formula (53). The action  $S(e)$  is the same for all orbits of one and same type (with the same period of revolution).

If only two of the parameters  $\omega_i$  bear an integral relation, for example,  $\omega_x$  and  $\omega_y$ , the periodic orbits lie in the  $xy$  plane. They can be obtained from (51) with  $A_z=0$ . The dependence on the parameter  $\varphi_z$  then disappears, i.e., we have the case  $\mathcal{D}=1, \mathcal{R}=2$ . We find the period  $T_{xy}$  of such orbits; from the relation

$$2\pi/T_{xy} = \omega_x/n_x = \omega_y/n_y \quad (\mathcal{D}=1, \mathcal{R}=2). \quad (55)$$

The planar orbits of the harmonic oscillator are the well known Lissajou figures (Fig. 6). Planar orbits also exist, of course, in the completely degenerate oscillator potential, but they form a separate family only in the case when the period  $T_x = 2\pi/\omega_x$  differs from the period  $T_{xy}$ .

In nuclear physics, one usually considers a deformed oscillator with "conserved volume", so that

$$\omega_0^3 = \omega_x \omega_y \omega_z = \text{const.} \quad (56)$$

When the oscillator model is used in nuclear theory,  $\omega_0$  is found from the condition of constancy of the particle density. Thus, one finds that  $\hbar\omega_0 = \mu(3A)^{-1/3}$ , where  $\mu$  is the chemical potential. This relation can be conveniently used to estimate the magnitude of the various quantities. Using (56), (55), (53), and (52), we can express the revolution periods  $T_{xyz}$  and  $T_{xy}$  and the associated revolution frequencies  $\Omega_{IV}$  and  $\Omega_{II}$  in terms of the frequency  $\omega_0$  of a spherical oscillator with the "equal volume". For an axisymmetric potential, which will be considered below, we obtain

$$\hbar\Omega_{IV} = 2\pi\hbar/T_{xyz} = \hbar\omega_0 n_x^{-2/3} n_z^{-1/3}, \quad (57)$$

$$\hbar\Omega_{II} = 2\pi\hbar/T_{xy} = \hbar\omega_0 n_x = n_x \hbar\Omega_{IV}. \quad (58)$$

Here

$$\omega_{\perp} = \omega_x = \omega_y; \quad n_{\perp} = n_x = n_y;$$

$n_x$  and  $n_z$  are relatively prime integers that occur in (53) or (55). The Roman numerals indicate the degree of degeneracy of the orbits, which is equal to  $\mathcal{D} + \mathcal{R} - 1$ .

The distances between the peaks of the level density, the gross shells associated with orbits of these two types, are equal, in accordance with (27a) and (54), to (57) and (58). The distances between the shells due to rectilinear orbits are equal to  $\hbar\omega_i$ , where  $i=x, y, z$ , but these orbits make a separate contribution to  $g_{osc}$  only if they are isolated.

On the basis of the above general considerations, the amplitude of the oscillating components of the level density associated with the three-dimensional orbits must be enhanced by a factor of order  $(\hbar^{-1/2})^4 \approx A^{2/3}$

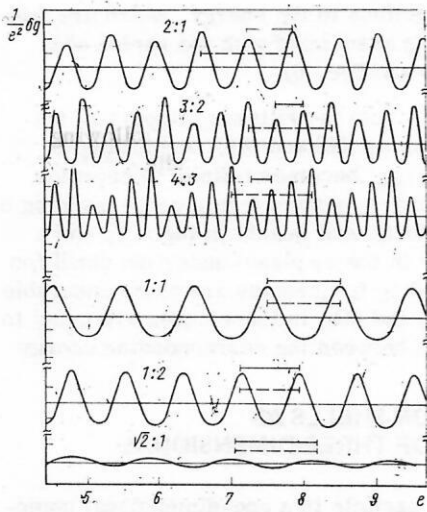


FIG. 7. Oscillating component  $\delta g = g_\gamma - \bar{g}$  of the level density of an axisymmetric harmonic oscillator for some values of the frequency ratio  $\omega_1 : \omega_2$ , which are given in the figure. The energy unit is  $\hbar\omega_0$  [see (56)]. The horizontal lines show the distance between the shells  $\hbar\Omega_{IV}$  calculated in accordance with (57). The averaging parameters are  $\gamma = \hbar\omega_0/3$ ,  $\tilde{\gamma} = 2\hbar\omega_0$ . The horizontal dashed segments correspond to  $\hbar\Omega_{II}$  for planar orbits with lower degeneracy ( $\ell = 1$ ,  $\mathcal{R} = 2$ ).

and is a quantity of the same order as the Thomas—Fermi component. The enhancement factor for planar orbits is of order  $(\hbar^{-1/2})^2 \sim A^{1/3}$ , and for rectilinear orbits the amplitude  $g_{osc}(e)$  is of minimal order, i.e., it is  $A^{2/3}$  times smaller than the Thomas—Fermi component. In a completely degenerate oscillator, orbits of all three types contribute. In an oscillator which is partly degenerate, the planar and rectilinear orbits may interfere.

We give without proof expressions for the oscillating components of the level density of a deformed harmonic oscillator. For three-dimensional orbits in an arbitrary oscillator potential whose parameters satisfy the relation (53),  $\mathcal{D} = 2$ ,  $\mathcal{R} = 3$ , and for this case we find

$$g_{osc}^{(IV)} = 2e^2/(\hbar\omega_0)^3 \sum_{m=1}^{\infty} \sin(m\epsilon T_{xyz}/\hbar + m\nu(n_x, n_y, n_z) - \pi/2); \quad (59)$$

where  $\omega_0$  is determined in (56). According to (59), the amplitude  $g_{osc}$  differs only by a numerical factor from the smooth component of the density for the harmonic oscillator:

$$g_{TF}(e) = [(e/\hbar\omega_0)^2 - 1/4]/2\hbar\omega_0, \quad (60)$$

as follows from the general estimate.

For planar orbits ( $\mathcal{D} = 1$ ,  $\mathcal{R} = 2$ )

$$g_{osc}^{(II)} = \sqrt{2} \hbar\omega_0 e/(\hbar\omega_0)^3 \sum_{m=1}^{\infty} \sin(m\epsilon T_{xy}/\hbar + \nu_m). \quad (61)$$

Here,  $\omega_x$  and  $\omega_y$  are any partial frequencies satisfying the relation (54);  $\nu_m$  is the phase. For rectilinear paths ( $\mathcal{D} = 0$ ,  $\mathcal{R} = 1$ )

$$g_{osc}^{(I)} = \sum_{i=1}^3 (\hbar\omega_i)^{-1} \sum_{m=1}^{\infty} [1 + \cos(m\omega_j T_i) \cos(m\omega_l T_i)/2]^{-1/2} \times \sin(m\epsilon T_i/\hbar + \nu_m). \quad (62)$$

Here  $i = x, y, z$ ;  $\omega_j$  and  $\omega_l$  are the partial frequencies for the two directions perpendicular to the axis  $i$ . In the case of complete degeneracy (53)

$$g_{osc} = g_{osc}^{(IV)} + g_{osc}^{(II)} + g_{osc}^{(I)}. \quad (63)$$

In the case of partial degeneracy of the oscillator ( $\mathcal{D} = 1$ ,  $\mathcal{R} = 2$ ) it is necessary to take into account the sum of only the last two terms.

The terms of the sum (63) decrease in proportion to an increasing power of the small parameter  $\hbar\omega_0/e \approx A^{-1/3}$ . This is one of the specific features of an harmonic oscillator and an important difference from the general case. For example, in a spherical potential the relative order of the contribution of orbits with lower degeneracy is determined by  $A^{-1/6}$ . The principal term of the quasiclassical expression (63) for the level density of the oscillator is  $A^{1/6}$  times larger and the following term is  $A^{1/6}$  times smaller than the principal component  $g_{osc}$  in the spherical case.

The qualitative conclusions of our quasiclassical theory can be verified by comparing them with the results of numerical evaluation of the spectrum. Figure 7 shows several examples of the oscillating component of the level density of an axisymmetric oscillator ( $\omega_x = \omega_y = \omega_1$ ) obtained by numerical averaging of the level distribution over an interval  $\gamma$  and subtraction of the smooth function  $\bar{g}(e)$  from such a distribution. Thus,  $g_{osc} = g_\gamma(e) - \bar{g}(e)$ . The interval  $\gamma$  was taken equal to  $\hbar\omega_0/3$  for orbits with the maximal degeneracy for the given ratio of the frequencies in order to obtain approximately the same reduction factor associated with averaging over a finite interval [see Eq. (90) in Sec. 8]. The values of  $\hbar\Omega_{IV}$  were determined from Eq. (57). Those are shown in Fig. 7 by the continuous horizontal lines. It can be seen that the amplitude of  $g_{osc}$  is indeed proportional to the square of the energy in the case of complete degeneracy and is proportional to the energy in the case of partial degeneracy ( $\omega_1 : \omega_z = \sqrt{2}$ ). The amplitude of the oscillations is approximately the same in all cases of complete degeneracy. In the case  $n_1 > 1$ , one can clearly see modulation of the spectrum with a period greater than the fundamental period equal to  $\hbar\Omega_{IV}$ , this being due to the contribution of orbits with lower degeneracy. In the present case these are orbits in the plane  $z = 0$  for which the period  $\hbar\Omega_{II} = \hbar\Omega_1$  is  $n_1$  times greater than  $\hbar\Omega_1$  [see Eqs. (57) and (58)]. In Figs. 7 and 8, the value of  $\hbar\Omega_1$  is indicated by the dashed segments. The modulation by planar orbits is insignificant in the cases when  $n_1 = 1$  since the two frequencies are then equal. In Figs. 7 and 8, this occurs for  $\omega_1 : \omega_z = 1, 2$ .

In addition, the contribution of planar orbits to  $g_{osc}$  should not be taken into account as an individual term in the spherically symmetric case. This is due to the fact that in the derivation of the expression (61) for  $g_{osc}^{(II)}$  we assumed that in the immediate proximity of the plane  $z = 0$  there are no three-dimensional orbits with the same period. Therefore, Eq. (61) does not represent the contribution of the planar orbits in the spherical case, when these orbits are special paths of the family of three-dimensional orbits with the same period and the same action integral. For the



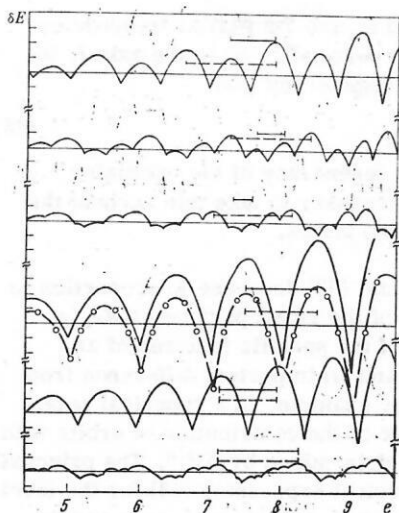


FIG. 8. Shell corrections  $\delta E$  calculated by means of the numerical method of shell corrections for the same values of the deformation parameters as in Fig. 7. The dashes are the result of calculation with more accurate determination of  $\tilde{E}_{i,p}$ ; the open circles are the values of  $\delta E$  in accordance with the quasiclassical theory.

same reason, the contribution of planar orbits passing through the symmetry axis is not a separate term of the sum (63) if  $\omega_x = \omega_y$ . The period of nearly similar three-dimensional orbits and their contribution is taken into account in the corresponding integrals for the orbits with greater degeneracy.

The lower curves in Figs. 7 and 8 show the shell structure of the spectrum of a partially degenerate oscillator. The ratio of the amplitude  $g_{osc}$  to the amplitude in the case of complete degeneracy approximately corresponds to the formal estimate, i.e., it is a quantity of order  $\hbar\omega_0/e \approx A^{-1/3}$ . The contribution of isolated rectilinear orbits is small in all cases and does not appear in the numerical calculations.

All of these features are virtually indiscernible in the level spectrum of the oscillator shown in Fig. 6. There one can see only the trivial shells due to exact quantum degeneracy of the levels corresponding in the quasiclassical picture to orbits with maximal degeneracy. None of the remaining manifestations of shell structure of the spectrum were previously noted. At the same time, their role is not restricted to the qualitative theory. It can be shown, for example, that for a not too large value of the ratio  $e/\hbar\omega_0 \approx A^{1/3}$  and  $n_1 > 2$  the contribution of  $g_{osc}^{(II)}$  to the shell oscillations of the total single-particle energy exceeds the contribution of orbits with the maximal degeneracy.

Unfortunately, one cannot make a more accurate quantitative comparison of the quasiclassical expressions for  $g_{osc}$  with numerically calculated values. The latter depend on the choice of the averaging interval  $\gamma$ , and if there is interference of components with appreciably differing frequencies the ratio of their contributions is changed by averaging: The amplitude of the  $g_{osc}$  component with smaller  $\hbar\Omega$  decreases more strongly (see Sec. 7). A more detailed comparison is possible

for the shell corrections to the energy, which are free of the arbitrariness associated with the choice of  $\gamma$  (see Figs. 7 and 8 and Sec. 6).

In conclusion, we note the following important feature of the expressions (59) and (61). According to these expressions,  $g_{osc}$  becomes infinite at energies corresponding to exact quantum levels corresponding to the excitation of vibrational quanta in the  $x$ ,  $y$ , and  $z$  directions (or only in the  $xy$  plane) under the condition that the corresponding frequencies are commensurable. The values of  $\hbar\Omega_{IV}$  and  $\hbar\Omega_{II}$  in these cases are equal to the exact distances between the corresponding energy levels.

#### 4. GROSS NUCLEON SHELLS AS QUANTIZATION OF THREE-DIMENSIONAL MOTION

The motion of a particle in a one-dimensional potential well is always periodic. The Bohr–Sommerfeld quantization rule determines a sequence of energy values which approximately represent the energy spectrum in the quantum problem. The approximate energies of the levels can be found from the condition that the action integral for one period be

$$S^{(n)}(e_N) = \oint p \, dl = 2\pi N \hbar - \nu \hbar. \quad (64)$$

Here,  $N$  is an integer;  $\nu = -\pi/2$ . Generally speaking, classical motion in a three-dimensional potential well is not periodic: having left a certain point, the particle cannot return to it with momentum equal to the original momentum. In this case, there is no simple analogy with the one-dimensional problem. However, if there is a periodic trajectory in the multidimensional motion, the Bohr–Sommerfeld rule again applies, although it acquires a new and more general content. For each of the possible periodic orbits, the condition (64) determines the positions where the energy levels are denser, but not the energies of the individual levels. Thus, the meaning of the quantization condition (64) in the multidimensional case is radically changed, although the form remains the same. The contour of integration now coincides with the multidimensional periodic orbit  $\beta$ , and each such orbit leads to the appearance of an oscillating component of the level density with distance  $\hbar\Omega_\beta = 2\pi\hbar/T_\beta$  between the gross shells. Here,  $T_\beta$  is the period of revolution in orbit  $\beta$ ;  $\Omega_\beta$  is the corresponding frequency (see Sec. 2). The number  $N$  in (64) is now the serial number of the peak in the level density (shell) in the sequence of shells associated with orbit  $\beta$ .

A difference from the one-dimensional case is also associated with the fact that the quantum-mechanical uncertainty principle does not admit motion along the classical orbit. There is unavoidably a probability distribution packet in the directions perpendicular to the orbit, a phenomenon that does not have an analog in the one-dimensional problem. Classical properties can be discerned only in the averaged quantum distribution. The properties of the classical orbits responsible for a definite sequence of “shells” in the spectrum of quantum states are not associated with the concrete properties and quantum numbers of the single-particle levels that belong even to one and the same gross

shell. The classical properties are distributed over many quantum states, and, in its turn, the strength of each state is distributed over all the classical orbits. Therefore, it is incorrect to use the word "orbit" to designate individual quantum states of the particle, as is frequently done in nuclear physics. The qualitative relations are well illustrated by the simple example of the quantum problem of motion of a particle in an infinitely high spherical potential well [see Ref. 5(d) and Figs. 4 and 5]. The exact solution of this problem is well known. The radial wave functions are Bessel functions with half-integral indices, and the energy levels are

$$\varepsilon_\lambda = \hbar^2 k_\lambda^2 / 2M, \quad (65)$$

where  $k_\lambda R$  is the  $\lambda$ th root of the Bessel function  $J_{l+1/2}(kR)$ ;  $R$  is the radius of the well. The distribution of the eigenvalues of the problem, and also of the quantum number  $l$  for each level, are shown in the lower part of Fig. 5. The value of  $l$  also characterizes the quantum degeneracy of the levels, each of which corresponds to a subshell in nuclear theory. With allowance for degeneracy, the levels form clearly visible groups—gross shells. The minima in the level density correspond to "filled gross shells". Each shell includes many single-particle levels with different values of the orbital angular momentum and parities, which, obviously, are not related to the properties of the simplest orbits in the form of a triangle and square responsible for the observed shell structure of the eigenvalues.<sup>1)</sup>

It is also instructive to consider the example of the shell structure associated with partially degenerate classical orbits in a deformed harmonic oscillator. For  $\omega_x = \omega_y = \omega_\perp$ , and for an irrational value of  $\omega_\perp / \omega_z$ , the classical periodic orbits lie exclusively in the plane perpendicular to the  $z$  axis. They lead to a relatively weaker but clearly expressed shell structure of the spectrum. Although in this case the gross shells are due solely to planar orbits that correspond to zero component of the classical momentum in the direction of the  $z$  axis, they include quantum levels associated with the excitation of vibrational quanta in the  $z$  direction. The centers of the shells determined by Eq. (64) coincide in this case with the degenerate levels of the excited quanta of vibrations in the directions of the  $x$  and  $y$  axes.

The better known shells of the deformed harmonic oscillator correspond to strongly degenerate quantum states. They arise in the case when all three frequencies bear integral ratios. It is well known that such shells are made up of states of the same parity. This also applies to the classical example of the shells of a hydrogen-like atom. Both examples are exceptional cases of complete degeneracy in classical mechanics, when any path is periodic irrespective of the initial conditions. In this respect, there is an analogy with the

one-dimensional case, and one can assume that the unique correspondence between the energy values determined from the quantization condition (64) and the quantum energy levels that are strongly degenerate in the multidimensional case occurs only when the Hamiltonian corresponds to a completely degenerate classical problem. The familiar example of one-dimensional motion can be regarded as a special case of this general rule (see also Sec. 8). One can assume that in the case of complete degeneracy of the classical motion in the amplitude  $g_{osc}$  becomes infinite at the energies that correspond to the position of the degenerate levels—the gross shells (see the example of the harmonic oscillator in Sec. 3). But if the classical motion is not completely degenerate, the condition (64) determines only the positions of the peaks in the level density. Complete degeneracy of classical motion is not a general property, but it is also a necessary condition for the occurrence of appreciable shell structure. Unfortunately, the very special properties of the shells of a completely degenerate harmonic oscillator are sometimes taken as an explanation and as a necessary condition for the structure of gross shells in realistic nuclear potentials.

With regard to the justification for using the name "shell" to designate the grouping of quantum levels, we should point out here that the shells of the hydrogen atom with their "fortuitous" degeneracy with respect to  $l$  represent a special case of the gross-shell structure considered here under conditions of complete degeneracy of the classical motion. Since the shell concept was introduced into nuclear physics on the basis of the analogy with the atom, it is perfectly natural to use this name to describe the gross shells.

The quasiclassical description given here for the integral characteristics of the single-particle level distribution can be compared with the corresponding quasiclassical approximation for the Schrödinger equation (see Refs. 18 and 19 and also the review Ref. 20). In the latter case, a condition of the type (64) follows from the requirement of uniqueness of the wave function and it determines approximate eigenvalues. The similarity of the quantization conditions is only superficial. The differences lie in the paths of integration in (64), which do not always coincide with the classical paths. The exact or approximate description of the energies of the single-particle spectrum which is possible in some cases does not solve the problems of the general properties of the level distribution and of a structure of the gross-shell type. This applies equally to the relationship between the individual wave functions and the shell oscillations of the total particle density.<sup>15,16</sup>

Thus, in the quoted papers on the quasiclassical theory a quite different problem is posed. The quasiclassical approximation for the Green's function has the advantage that the Green's function is directly related to the level distribution and the spatial distribution of the particles. The quasiclassical approximation for the integral distributions is formally appreciably simpler, and the connection with the properties of the classical motion is more definite. The reason for this is perfectly obvious: The classical properties appear

<sup>1)</sup>In the nuclear shell model, one frequently uses the name shell to designate individual levels that are degenerate with respect to the projection of the angular momentum. To avoid confusion with "gross shells" we shall not use this expression here.

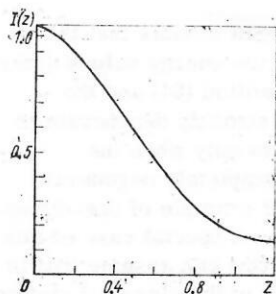


FIG. 9. The function  $I(z)$ , which arises in the quasiclassical expressions for the entropy.

when one considers packets of quantum states. Note also that the quasiclassical quantization of the spectrum in the form of gross shells is not related to the possibility of separating the variables in the Schrödinger equation or with quasiclassical quantization for each of the partial problems.

## 5. ENTROPY AND LEVEL DENSITY OF AN EXCITED NUCLEUS

The entropy of an nucleus "heated" to temperature  $\tau$  can be expressed in terms of the single-particle density  $g(e)$  and the Fermi distribution function  $n = n_\tau(e)$  as follows:

$$\mathcal{S}(\mu\tau) = - \int_{-\infty}^{+\infty} g(e) [n \ln n + (1-n) \ln (1-n)] de = \tau \int_{-\infty}^{+\infty} g(\mu + \tau x) F(x) dx, \quad (66)$$

where  $x = (e - \mu)/\tau$ , and the chemical potential satisfies  $\mu \gg \tau$ . The function

$$F(x) = - \frac{1}{2\pi} \int_{-\infty}^{+\infty} y \cosh^{-2}(y/2) dy, \quad F(x) = F(-x) \quad (67)$$

has a maximum at  $x=0$ . Into (66), we substitute the level density  $g(e)$  in the form of the sum of the Thomas-Fermi component and the oscillating terms. The first component leads to the well known expression of the statistical model for the entropy, which for  $\tau \ll \mu$  has the form

$$\mathcal{S}_{TF} = \pi^2 \tau g_{TF}(\mu)/3. \quad (68)$$

The second gives the oscillating shell component of the entropy as the sum of the individual terms  $\mathcal{S}_\beta$ , each of which corresponds to a definite classical orbit and contains the sum of the harmonics corresponding to a different number of revolutions  $m$  in the orbit. Substituting the expressions (25) and (29) into the integral (66), we note that the action integral  $S_\beta(e)$  is a smooth function of the energy and for  $\tau \ll \mu$  we can write

$$S_\beta(e) = S_\beta(\mu) + t_\beta(e - \mu) + \dots, \quad (69)$$

where  $t_\beta$  is the time of the classical motion along the periodic orbit  $\beta$  [see (10)]. The integration in (66) reduces to calculation of the integral

$$I(z) = - \frac{3}{2\pi^2} \int_{-\infty}^{+\infty} F(x) \exp(izx) dx, \quad (70)$$

where  $z = z_\beta^{(m)} = t_\beta/\hbar = mT_\beta/\hbar$ ;  $T_\beta$  is the period of revolution in the orbit. The other functions in the integrand depend weakly on the energy, and they can be taken in front of the integral for  $x=0$ . Thus, we obtain

$$\mathcal{S} = \mathcal{S}_{TF} + \mathcal{S}_{osc} = \frac{\pi^2}{3} \tau \left[ g_{TF}(\mu) + \sum_{\beta, m} I(z_\beta^{(m)}) g_{osc}^{(\beta m)}(\mu) \right], \quad (71)$$

where  $g_{osc}^{(\beta m)}$  is the  $(\beta, m)$  component in the expansion (25) or (29);  $I(z) = (3/\pi) [-\sinh(\pi z) + \pi z \cosh(\pi z)] / z \sinh^2(\pi z)$  (Fig. 9). In the limit of low temperatures  $\pi z \ll 1$ ,  $I(z) \approx 1$ , and

$$\mathcal{S}(\mu_0\tau) = \pi^2 \tau [g_{TF}(\mu_0) + g_{osc}(\mu_0)]/3, \quad (72)$$

where  $\mu_0$  is the chemical potential calculated from conservation of the particle number. At high temperatures,  $I(z)$  tends to zero as  $6 \exp(-\pi z)$ ,  $\pi z > 1$ , and  $\mathcal{S}$  approaches a smooth value. The critical temperature  $\tau_{SH}$  at which the shell effects disappear in the entropy corresponds to  $z \approx 2$  and

$$\tau_{SH} \approx 2\hbar/T_\beta = \hbar\Omega_\beta/\pi, \quad (73)$$

where  $\Omega_\beta$  is the frequency corresponding to the periodic trajectory. The estimate (73) is in good agreement with numerical calculations of the level density in accordance with the statistical model with use of realistic nuclear potentials. In accordance with (71), at high temperatures the contributions of the different classical orbits decrease exponentially with increasing time of motion along the periodic trajectories. This also applies to the paths that have the shortest distances, whose contribution is taken into account in the sum (71), but it quickly decreases with increasing  $m$ .

For  $\tau < \tau_{SH}$ , each component of the entropy corresponding to a definite periodic trajectory leads to a factor in the statistical expression for the nuclear level density which oscillates as a function of the number of nucleons. The level density of the nucleus has the form

$$\rho(e) = \tilde{\rho}(e) \prod_{\beta} \exp(\mathcal{S}_\beta), \quad (74)$$

where  $\tilde{\rho}$  is the usual expression of the statistical model;  $\mathcal{S}_\beta$  is the component of the entropy corresponding to a closed path  $\beta$ :

$$\mathcal{S}_\beta = \pi^2 \sum_{\beta} \sum_m I(z_\beta^{(m)}) g_{osc}^{(\beta m)}(\mu)/3. \quad (75)$$

As an illustration, we give the explicit expressions for the two most important orbits in a spherical potential with infinitely high walls, namely a triangle and a square. The sum of the oscillating components of the entropy in Eq. (71) is

$$\mathcal{S}_\beta(e) = \frac{\pi^2}{3} \frac{2MR^2\tau}{\hbar^2} \left( \frac{\sqrt{3}kR}{2\pi} \right)^{1/2} I\left( \frac{3\sqrt{3}\tau kR}{2\mu} \right) \cos\left( \frac{k\sqrt{3}L}{2} + \frac{\pi}{4} \right) \cos(k\bar{L}). \quad (76)$$

For  $\tau = 1$  MeV,  $kR = 10$ , and  $\mu = E_F = 50$  MeV, we find the coefficient  $I \approx 0.7$  (see Fig. 9). These two orbits lead to an appreciable modulation of the statistical level density considered as a function of the nucleon number. Numerically, the amplitude of the density variation in this case is determined by a factor of order  $\exp(\pm 3)$  (without allowance for spin degeneracy).

In accordance with (71), all the oscillating components are zero when  $\tau > \tau_{SH}$ . However, this does not mean that the limiting values of the entropy and the level density for large  $\tau$  are equal to the Thomas-Fermi expressions. The point is that at such a high temperature the contribution of the periodic paths is not domi-



nant in the quasiclassical expansion for the entropy. When  $\tau > \tau_{SH}$ , the stationarity condition (23) leading to the selection of periodic orbits becomes meaningless. This question is considered in more detail in Sec. 7 in connection with the properties of the averaged level density. In order to obtain the correct results, it is necessary to use the expansion (7) for the Green's function and the original expression (22) for the level density. In addition to the principal term in the Thomas—Fermi expression for  $\tau > \tau_{SH}$ , the entropy also contains smooth components proportional to the surface of the system, its curvature, etc. In the general case, these terms are of higher order in the small parameter of the quasiclassical expansion than the principal term in the Thomas—Fermi expression.

## 6. SHELL CORRECTIONS TO THE ENERGIES OF GROUND STATES

Shell effects in the energies of nuclei are associated with nonuniformity of the single-particle spectrum near the Fermi limit due to the shell structure. The shell effects in the binding energies can be conveniently represented in the form of the so-called shell correction to the averaged "drop" energy. In the first order in the deviation of the true distribution of nucleons from the statistically averaged "drop" value, the shell correction  $\delta E$  to the energy can be expressed directly in terms of the oscillating component of the single-particle level density.<sup>1,2</sup> The shell variations of the energy of second order depend explicitly on the pairing interaction of the particles and are determined by the condition of self-consistency of the single-particle potential.<sup>2,16,21-23</sup> The first-order correction does not depend on the nature of the common single-particle potential. Its variation as a function of the number of nucleons or the shape of the nucleus is the effect of redistribution of the nucleons by the shell phenomena.

An expression for  $\delta E$  in terms of  $g_{osc}(e)$  can be obtained by different but essentially equivalent ways.<sup>1,2,7,16,24,25</sup> For analytic calculations, it is convenient to define the shell correction to the energy as the difference between the true energy of the system at zero temperature and the value of the energy obtained by extrapolation to  $\tau=0$  of the asymptotic expression for the energy characteristic of temperature  $\tau \gtrsim \tau_{SH}$ , where the shell structure of the single-particle spectrum is not important. The first-order shell correction to the energy  $\delta E$  can be expressed as the difference of the exact entropy  $\mathcal{S}(\tau)$  and the value of the asymptotic expression  $\mathcal{S}(\tau) \approx \tilde{\mathcal{S}}(\tau)$  for  $\tau > \tau_{SH}$  extrapolated to lower temperatures:

$$\delta E = \int_0^\infty [\mathcal{S}(\tau) - \tilde{\mathcal{S}}(\tau)] d\tau. \quad (77)$$

The difference in the integrand is precisely the oscillating component of the entropy considered in the previous section. The integration with respect to the temperature in (77) reduces to calculation of the integral

$$\int_0^\infty \tau I(\tau t_\beta / \hbar) d\tau = 3(\hbar/t_\beta)^2 / \pi^2.$$

As a result, we obtain the shell correction to the energy

$$\delta E = \sum_{\beta m} (\hbar/t_\beta(\mu))^2 g_{osc}^{(\beta m)}(\mu). \quad (78)$$

This expression can also be obtained from the usual representation of the energy of the shell correction in the form (see Ref. 1)  $\delta E = \int^\mu (e - \mu) g_{osc}(e) de$  when the quasiclassical expansion for  $g_{osc}(e)$  with respect to the periodic orbits is substituted. Thus, in the quasiclassical limit, we obtain a very simple and direct proof of the equivalence of the two determinations of the shell correction to the energy. A more formal proof is given in Refs. 26 and 27.

It can be seen that under otherwise equal conditions, a larger contribution to the shell correction is given by the orbits corresponding to shorter periods. In particular, the contributions of repeated paths ( $m > 1$ ) decrease rapidly with increasing  $m$ . With allowance for  $t_\beta = mT_\beta$  and (78), each of the components contains a factor proportional to  $1/m^2$ . An important contribution to  $\delta E$  is made by only the lowest harmonics ( $m=1$ ) of the quasiclassical series for the level density and the simplest orbits, for which the period  $T_\beta$  is minimal.

The shell correction to the energy is proportional to the oscillating components of the level density near the Fermi energy, which has frequently been noted in numerical calculations.<sup>1,2</sup> The deepest minima of  $\delta E$  (maxima of the binding energy) correspond to the minimal density of levels around the Fermi energy. A qualitative interpretation of this result is given in Ref. 2. The special values of the nucleon number corresponding to such filling of the potential well, when  $g(\mu)$  is minimal, correspond to the so-called magic numbers for the potential of given shape. It can also be seen from Eq. (78) that the proportionality of the shell energies to  $g_{osc}(\mu)$  holds only under the condition that the shell structure in the particular problem is due predominantly to one orbit, or the periods of the main orbits differ little from one another.

For a spherical potential well, one can obtain an explicit expression, using for this the formulas of Secs. 2 and 3. We first write down (78) in the more convenient form

$$\delta E = 8\mu \sum_{\beta m} (R/mL_\beta)^2 [g_{osc}^{(\beta m)} / (\hbar^2/2MR^2)]. \quad (79)$$

As in Sec. 3, the dimensionless quantity in the square brackets is the oscillating component of the level density expressed in units of  $(\hbar^2/2MR^2)^{-1}$ ;  $L_\beta$  is the length of the  $\beta$ th periodic path;  $m$  is the number of revolutions. For convenience of comparison with the numerical calculations, we have included in (79) a factor 2, which takes into account spin degeneracy. The expression (79) is suitable for a potential of arbitrary shape that does not change significantly in the volume of the system because in the derivation of the expression we have additionally used only the relation (40) between the time and the path length.

In an infinite spherical well, the main contribution to the shell corrections to the energy come from orbits in the form of a triangle (3) and square (4) (see Sec. 3). The contribution to  $\delta E$  of these orbits is

$$\delta E_{(2+4)} = 8\mu (\sqrt{3}kR/2\pi)^{1/2} (R/L_3)^2 \{ \cos(k\Delta L + \pi/4) \cos(kL) - [1 - (2/3)^{1/4}] (L_3/L_4)^2 \cos(kL_4 + \pi/4)/2 \}. \quad (80)$$

Here, we can ignore the second term, whose relative

value does not exceed 12%; we then obtain the approximate expression

$$\delta E_{(3+4)} = (8/27) \mu (g_{osc}(\mu))_{3+4} [\hbar^2/(2M R_0^2)]. \quad (81)$$

In the upper part of Fig. 5, we compare the results of calculating the shell correction  $\delta E$  for a spherical potential with infinite walls obtained by the usual numerical method of approximation of the shell corrections<sup>1</sup> and by means of the quasiclassical formula (80). The agreement is rather close. As can be seen from the data given in Fig. 5 for the values  $\mu = E_F = 50$  MeV and  $k_F R = 10$ , which approximately correspond to the conditions of a nucleus, the amplitude  $\delta E$  is of order 20 MeV. Approximately the same amplitude of the oscillations of the shell energies is obtained for a realistic Woods-Saxon potential of spherical shape.

The analysis of an infinitely deep potential well of ellipsoidal shape will be published separately. The main difference of the problem with nonspherical potential from the case of a potential of spherical shape is due to the different degree of degeneracy of the orbits and also the fact that in a deformed potential the strongly degenerate periodic orbits are more complicated and the period  $T_\beta$  is longer than in the spherical case. This last circumstance leads to an appreciable reduction in the amplitude of the shell corrections, whose amplitude is in accordance with (78) inversely proportional to the square of  $T_\beta$ .

Qualitative estimates of the shell correction to the energy can be obtained by using the expressions given above and the estimates for the amplitude of the oscillating component of the level density. Thus, we find that in a spherical potential well the amplitude  $\delta E$  is of order  $[4/(2\pi)^2] E_F A^{1/6}$ . The estimate agrees with the results of numerical calculations.<sup>1-4</sup> Note that none of the other known attempts at a theoretical estimate of the magnitude of the shell effects in nuclear energies are based on a systematic theory; they are essentially arbitrary.<sup>1,28</sup>

We obtain the shell corrections for an harmonic oscillator by using the above analytic expressions for  $g_{osc}$  and the periods  $T$ . In accordance with (78) and (59), the component of  $\delta E$  due to the periodic orbits of maximal degeneracy has the form

$$\delta E_{IV} = \frac{2}{(2\pi)^2} \frac{\hbar^2 \Omega_{IV}^2}{\hbar \omega_0^3} \sum_{m \neq 0} \frac{1}{m^2} \sin \left[ \frac{2\pi m \mu}{\hbar \Omega_{IV}} + \nu m (n_x, n_y, n_z) - \frac{\pi \nu}{2} \right] \\ = (\mu^2 \Omega_{IV}^2 / 12 \hbar \omega_0^3) \{ -1 + 6x(1-x) \}, \quad (82)$$

where

$$x = (\mu - e_0 - \vec{p}) / \hbar \Omega_{IV}, \quad 0 \leq x \leq 1, \\ e_0 = \hbar \Omega_{IV} (n_x + n_y + n_z) / 2 \quad (83)$$

is the energy of the zero-point vibrations;  $n_x$ ,  $n_y$ , and  $n_z$  are the mutually prime numbers in (53);  $\vec{p}$  is the nearest integer to  $\mu - e_0$  that is a multiple of  $\hbar \Omega_{IV}$  and less than  $\mu - e_0$ . As in Sec. 3, the Roman numeral IV indicates the degree of degeneracy of the orbits responsible for the shell structure. The degree of filling of the shell is given by  $x$ : The shell is filled if  $x=0$  or 1. In (82), we have given the parameters on which the phase  $\nu$  depends, and we have used the representation of the Bernoulli polynomial as a Fourier series. For an

axisymmetric oscillator, we obtain from (82) and (57)

$$\delta E_{IV} = \frac{2}{12} \frac{\hbar^2 \Omega_{IV}^2}{\hbar \omega_0^3} n_z^{-4/3} n_z^{-2/3} \{ -1 + 6x(1-x) \}. \quad (84)$$

Under the "constant density" condition  $\mu / \hbar \omega_0 \approx (3A)^{1/3}$ , and  $\mu \approx E_F$  is approximately constant. Under these conditions,  $\delta E_{IV} \approx E_F A^{1/3} / 8$ , i.e., the amplitude is approximately  $A^{1/6}$  times larger than in an arbitrary spherical potential, in which the degree of degeneracy of the orbits is less by unity.

In the special case of a spherical oscillator ( $n_x = n_y = n_z = 1$ ) the expression (82) can be compared with the analytic expression for the shell correction in a spherical oscillator obtained in Ref. 7, in which  $\delta E$  was determined as the difference between the total single-particle energy of the oscillator in the limit  $A^{1/3} \gg 1$  and some "smooth" quantity. The only difference is in the last term in the square brackets in (82), which is equal to  $-1/2$  in Ref. 7 instead of  $-1$  in Eq. (82), this being due to the different definition of the smooth single-particle energy  $\tilde{E}_{i,p}$ .

According to Ref. 7, the definition of  $\tilde{E}_{i,p}$  follows from the condition of "vanishing" of shells in the deformed nuclei, which, in our opinion, is in no way justified. On the other hand, the normalization of  $\delta E$  in (82) corresponds to definition of  $\tilde{E}_{i,p}$  in accordance with the requirement formulated above that  $\delta E$  contain no smooth components of the energy like those already included in the phenomenological drop part of the total energy. Indeed, the mean value of  $\delta E$  over a gross shell is zero in accordance with (82), whereas in Ref. 7 there remains a finite value proportional to  $E_F A^{1/3}$  that depends smoothly on the number of nucleons. Such a component is present in the drop part of the energy. On the other hand, the value of  $\delta E_{IV}$  for a spherical harmonic oscillator calculated in accordance with the Bohr-Mottelson formula agrees well with the numerical value determined in the framework of the usual method of shell corrections, and also with the value which is obtained if to determine the smoothed single-particle energy one uses the exact quasiclassical expression (60) for the density of levels of an harmonic oscillator and Eq. (103) in Sec. 7.

The reason for these discrepancies is the strong degeneracy of the oscillator levels (of the order of the total number of particles  $A^{2/3}$  in the shell), which leads to a significant difference in the definition of the smooth energy as the average with respect to the number of nucleons from the definition as the mean with respect to the position of the Fermi level, which is adopted in the method of the shell correction.<sup>2)</sup> The numerical procedures of the method of shell corrections can be modified in such a way as to take into account the possibility of strong degeneracy. In Fig. 8, we give the results of numerical calculation in which  $\tilde{E}_{i,p}$  was defined as the mean with respect to the number of particles (the dashed curve for  $\omega_1 : \omega_z = 1$ ). In this case,

<sup>2)</sup> The proof given in Ref. 21 that the two definitions are identical presupposes that the Fermi level is a single-valued function of the particle number, which is not the case if there is level degeneracy.



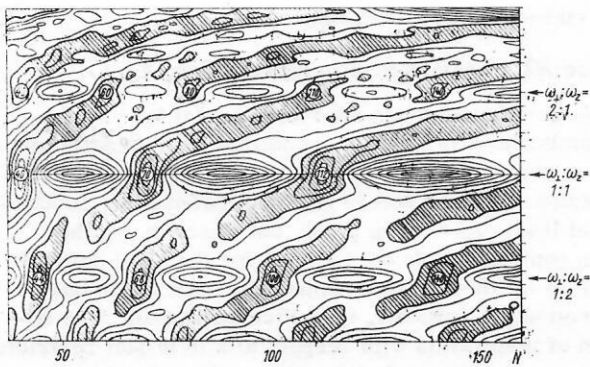


FIG. 10. Contour distribution of the shell correction for an oscillator potential as a function of the number of nucleons and the deformation parameter. The hatched regions are minima of  $\delta E$ . They correspond approximately to the minima of the level density in Figs. 7 and 8. The contours correspond to an increment of  $\delta E$  by  $0.01\mu A^{1/3}$  (Ref. 7, see also Ref. 1). The inclined strips of minima are due predominantly to a family of planar orbits.

one obtains good agreement with the quasiclassical formula. The situation is similar in the order cases of complete degeneracy of the oscillator levels.

For planar orbits in a degenerate oscillator, and also in the case of partial degeneracy when only the frequencies  $\omega_x$  and  $\omega_y$  are commensurable ( $\omega_x:\omega_y=n_x:n_y$ ), the shell correction to the energy is

$$\delta E_{II} = \frac{\sqrt{2}}{(2\pi)^2} \mu (\Omega_{II}^2 \omega_z / \omega_0^3) \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \left[ \frac{1}{h} m \mu T_{xy} + v_m (n_x n_y) \right], \quad (85)$$

where  $T_{xy} = 2\pi/\Omega_{II}$  is the period of revolution for the planar orbits (see Sec. 3).

Except for some cases that will be discussed below, the contribution of such orbits with degeneracy two ( $\mathcal{D}=1$ ,  $\mathcal{R}=2$ ) occurs together with  $\delta E_{IV}$ . Its relative magnitude is of order  $A^{-1/3}$ , but at not too large  $A$  it is also necessary to take into account a possible numerically large difference of the revolution periods  $T_{xyz}$  and  $T_{xy}$ . For an elongated axisymmetric oscillator, the ratio of  $\delta E_{II}$  to  $\delta E_{IV}$  is of order  $(\hbar\omega_0/\mu)n_1^2$ . For the value  $\mu/\hbar\omega_0 \approx 6-8$  usual for a nucleus, the two components of  $\delta E$  are comparable for  $n_1=3$ , while for a large value of  $n_1$  the main contribution to the shell corrections is made by the planar orbits, despite the lower degree of degeneracy and the correspondingly small amplitude of the density oscillations. This occurs, for example, for oscillator deformations  $\omega_1:\omega_z=3:2$ ,  $4:3$ , or  $3:1$ , whereas for the deformation  $\omega_1:\omega_z=2:1$  the value of  $\delta E_{II}$  is not more than 30% of the main term (see Fig. 8). The relative increase of the contribution  $\delta E_{II}$  to the shell corrections is due to the appreciably larger distance between the shells ( $\hbar\Omega_{II}$ ), which enters quadratically in (78). Compared with the distribution of  $\delta g$ , there is also an increase in the effect of modulation of  $\delta E_{II}$  as a result of addition of the contributions of orbits with different periods. This can clearly be seen by comparing Figs. 7 and 8. It is easy to see that the amplitude of the modulation of the component in Fig. 8 really is proportional to the first power of the energy  $e$ , while the component ( $\delta E_{IV}$ ) is proportional to  $e^2$ .

In an axisymmetric oscillator, a contribution of planar orbits with  $z=0$  is present for all deformations and it depends weakly on the deformation, in contrast to shells associated with completely degenerate orbits. These last arise only for definite deformations when the condition of commensurability of the frequency parameters of the potential (53) is satisfied. Since for deformations  $1 < \omega_1:\omega_z < 2:1$  and not too large ratio  $\mu/\hbar\omega_0$  the main contribution to the shell structure comes from planar orbits, the shell corrections to the energy in an harmonic oscillator at these deformations is a comparatively smooth function of the deformation and the number of filled levels. This can be clearly seen in the contour map of the  $\delta E$  distribution as a function of the number of nucleons and deformations shown in Fig. 10, in which the shells associated with planar orbits are shown in the form of continuous strips of minima and maxima of  $\delta E$  in the northeast direction. The shells of complete degeneracy appear in the form of local increases of the amplitude  $\delta E$ . In the deformed oscillator, the shell for  $\omega_1:\omega_z=2:1$  is especially marked. For different values of the deformation parameter, the contribution of these shells under normal conditions in nuclear physics is not only not predominant but, as a rule, is less than the contribution of orbits with lesser degeneracy but greater value of  $\hbar\Omega$ . This result will be discussed in Sec. 9.

## 7. AVERAGED DENSITY OF LEVELS

In numerical calculations of the shell corrections, importance attaches to the smoothed density of levels  $\tilde{g}(e)$  which is used in definitions of the variations of the single-particle distribution  $g(e)$  is found by numerical averaging of the single-particle spectrum over a sufficiently wide energy range. The qualitative properties of the averaged level density and, in particular, its behavior as a function of the averaging interval can be explained by means of the quasiclassical expansions. The distribution of the single-particle energies averaged over an interval  $\gamma$  near the energy  $e$  is

$$g_\gamma(e) = \frac{1}{\gamma} \int_{-\infty}^{+\infty} g(e') \xi\left(\frac{e'-e}{\gamma}\right) de', \quad (86)$$

where  $\xi(x)$  is a weight function normalized to unity with a maximum at  $x=0$  and width of order unity. Substituting (22) into (86), as in the derivation of (71), we expand the action  $S(\mathbf{r}''\mathbf{r}'e)$  as a series in the difference  $(e'-e) \approx \gamma$  and we note that in the averaged density of levels the terms corresponding to complicated paths appear with the coefficient

$$J(y) = \frac{1}{\gamma} \int_{-\infty}^{+\infty} de' \xi\left(\frac{e'-e}{\gamma}\right) \exp\left[-\frac{i}{\hbar} t_\alpha(e'-e)\right] \\ = \int_{-\infty}^{+\infty} dx \xi(x) \exp(ixy), \quad (87)$$

where  $y = \gamma t_\alpha/\hbar$ ;  $t_\alpha = [t_\alpha(\mathbf{r}'', \mathbf{r}', e)]_{\mathbf{r}''=\mathbf{r}'}^{\mathbf{r}''=\mathbf{r}'}$  is the time of motion along the closed but not necessarily periodic  $\alpha$ th classical path. The function  $J=J(y)$  is real and decreases exponentially for  $y \gg 1$  if  $\xi(x)$  is an even and continuously differentiable function. The special form of the integral (87) has already been encountered in the expressions for the thermodynamic characteristics.



When the averaging parameter is increased, the contributions of the longer paths to the averaged level density are eliminated exponentially, and as  $\gamma$  approaches the critical value

$$\gamma_{SH} = 2\pi\hbar/T = \hbar\Omega, \quad (88)$$

where  $T$  is the shortest period of revolution in a closed periodic orbit, there remain only the components corresponding to the shortest periodic orbits and the nonperiodic paths whose length is less than the dimension of the system. If  $\gamma > 1$ , all the oscillating components of the level density are included, and the terms of the expansion (22) corresponding to periodic orbits give the smooth quasiclassical corrections to the smoothed level density. It is impossible to obtain such corrections if, in the averaging, one uses an expansion for  $g(e)$  with respect to periodic orbits, since the derivation uses the condition of stationarity (23), which has no meaning when  $\gamma > \gamma_{SH}$ . Indeed, it is easy to see that after the substitution of (22) into the definition (86) of the averaged density  $g_\gamma(e)$  the condition of stationarity in the integral with respect to the spatial variables leads to a complex value of the momentum with imaginary part  $p_i$  of order  $\gamma/v$ , where  $v$  is the velocity. This corresponds to a damping length  $L_i$  of order  $\hbar/p_i$ . Therefore, the stationarity condition (23), which leads to the selection of periodic orbits, applies only under the condition  $L_i \ll L_\beta$ , which is equivalent to the inequality  $\gamma \ll \hbar\Omega$ .

If this condition is satisfied, we can obtain an explicit expression for  $g_\gamma(e)$  by substituting into (86) the quasiclassical expansion for  $g(e)$  with respect to closed periodic orbits. Let us consider the important case of a weight function  $\xi(x)$  of Gaussian type containing a correction polynomial of  $n$ th degree ( $n$  is even):

$$\xi(x) = \xi_n(x) = P_n(x) \exp(-x^2/\sqrt{\pi}), \quad (89)$$

where  $P_n(x) = \sum_{k=0,2,\dots,n} \alpha_k H_k(x)$ ;  $H_k$  is a Hermite polynomial;  $\alpha_k = -\alpha_{k-2}/2k$ ;  $\alpha_0 = 1$ .

The correction polynomial is necessary when one is averaging over a finite interval  $\gamma$ . It is usually chosen on the basis of the condition that the function (89) have the properties of the  $\delta$  function with respect to any smooth function: a polynomial of degree  $n$ .

Other definitions are possible, for example, from the condition of fitting of the best smooth distribution by the method of least squares (see, for example, Refs. 1-4, 16, 23, 29). The integral (87) can be calculated explicitly, and as a result we obtain

$$g_\gamma(e) = \tilde{g}(e) + \sum_{\beta} g_{osc}^{(\beta m)}(e) Q_{n/2}(\gamma_{\beta m}^2) \exp(-\gamma_{\beta m}^2), \quad (90)$$

where  $Q_{n/2}(z)$  is the sum of the first  $n/2$  terms of the power series for  $\exp(-z)$ ;  $\gamma_{\beta m} = mT_\beta\gamma/2\hbar$ ;  $T_\beta$  is the period of revolution for orbit  $\beta$ . It is assumed that in the region  $\gamma \ll e$  around the energy  $e$  the function  $\tilde{g}(e)$  can be approximated by a polynomial of degree  $n$ .

As a function of  $\gamma$ , the averaged density  $g_\gamma(e)$  remains approximately constant and is equal to the original expression for  $g(e)$  when  $\gamma \lesssim 1$ , while when  $\gamma > 1$  it approaches  $\tilde{g}$  (goes onto a plateau).

$$\exp(-\gamma_{\beta m}^2) = \exp\{-\pi^2(\gamma/\hbar\Omega_\beta)\}, \quad (91)$$

where  $\hbar\Omega_\beta$  is determined in (88) and Sec. 4.

We now consider the behavior of  $g_\gamma(e)$  for  $\gamma > \gamma_{SH}$ , assuming however that the condition  $\gamma \ll e$  is satisfied. The correction to the term of the Thomas-Fermi expression is due as before to complicated paths containing at least one turning point, but when  $\gamma > \gamma_{SH}$  the main contribution to the integral for the level density is made by short closed nonperiodic paths with one reflection, for which  $t_\alpha$  is sufficiently small. The domain of integration with respect to  $r$  is in fact restricted to an narrow layer adjoining the surface  $\Sigma$ , which bounds the region of the classical motion for the given energy. The thickness  $d$  of the surface layer that makes the main contribution is determined by the condition

$$\gamma t_\alpha/\hbar \ll 1, \quad (92)$$

where  $t_\alpha$  is the time taken by a particle moving along the shortest path from  $r$  to the turning point and then back to the point  $r$ . From (92) there follow the inequalities  $d \approx e\pi/\gamma < R$  and  $d \gg \lambda$ , which are equivalent to the conditions  $\gamma > \gamma_{SH}$  and  $\gamma \ll e$ . Here,  $R$  is the dimension of the system;  $\lambda$  is the wavelength of the particle. It follows from Eqs. (21), (22), and (86) that the averaged level density contains the Thomas-Fermi term, whose value does not change when the averaging procedure is correctly defined<sup>1-3, 16, 22, 23, 29</sup> and the quasiclassical corrections are smooth. These last are proportional to the area of the surface  $\Sigma$  defined by the condition  $e = V(r)$ , in contrast to the Thomas-Fermi component, which depends on the local density and may therefore be regarded as a volume term. For  $e \gg \gamma > \hbar\Omega$  the quasiclassical correction is

$$\tilde{g}_\gamma(e) = \frac{1}{4\pi^2\hbar^2} \int_\Sigma |Y_\Sigma(p_\alpha p_\beta; \gamma^2)|^{1/2} Y d\Sigma, \quad (93)$$

where the integration is over the surface  $\Sigma$ ;

$$Y(\Sigma) = \frac{2e}{\hbar} \int_\Sigma dx |Y_\Sigma(p_\alpha t_\alpha; x^e)|^{1/2} \times \exp\left\{\frac{i}{\hbar} S_\alpha(x^e, x', e) - [Y_\Sigma(x^e, x', e)/2\hbar]^2 - 1\right\} \Big|_{x''=x'}; \quad (94)$$

$t_\alpha(x'', x', e)$  is the time of the motion;  $Y_\Sigma$  is the  $2 \times 2$  determinant of the matrix formed by the second derivatives of the action  $S$  with respect to coordinates along the surface, and

$$Y_\Sigma(p_\alpha t_\alpha; x^e) = \det \left\| \frac{\partial^2 S}{\partial x^\alpha \partial x^\beta} \frac{\partial x^\alpha}{\partial t_\alpha} \frac{\partial x^\beta}{\partial t_\beta} \right\| = \frac{\partial p_\alpha}{\partial e} \frac{\partial p_\beta}{\partial e}. \quad (95)$$

In deriving (94), we have used the Gaussian form of the weight function  $\xi(x)$ . Near the reflection surface, the potential is

$$V(x) = V(0) + V'(0)x + V''(0)x^2/2; \quad V'(0) = dV(0)/dx, \quad (96)$$

and the normal component of the particle momentum is

$$p(x, e \pm i\delta) = \sqrt{2M(i\delta - V''(0)x)}. \quad (97)$$

Here,  $i\delta$  is a small imaginary correction to the energy variable which is set equal to zero in the final result. Using (96) and (97), we calculate the determinant (95)

and  $S$  and  $t_\alpha$  in the integrand of (94) and we represent the integral (94) in the form

$$Y = \frac{e}{a} \int_{i\frac{\delta}{\pi} - \infty}^{i\frac{\delta}{\pi} + \infty} dw w^{-1/2} \times \exp \left[ \frac{4i}{3} \left( \frac{e}{a} \right) w^{3/2} - i \left( \frac{\gamma}{a} \right)^2 w + 2 \left( \frac{\gamma}{a} \right)^2 \left( \frac{16}{e} \right)^{1/2} w^{1/2} \right], \quad (98)$$

where

$$w = (i\delta - V'(0)x)/a; \quad a = -V'(0)\hbar > 1.$$

We introduce the new variable  $u$  in accordance with the formula

$$w^{1/2} = (ia/4e)^{1/3} u - i(\gamma/e)^{1/3} (e/4a)$$

and express the integral (98) in terms of the Airy function  $\mathcal{A}i(x)$ :

$$Y = 2^{4/3} \pi^{1/3} (e/\gamma)^{1/3} |z|^{1/6} \exp \left[ -2z/3 \right] \mathcal{A}i(z^{2/3}), \quad (99)$$

where

$$z = -(\gamma/e)^3 (e/2a)^4. \quad (100)$$

The correction to the Thomas—Fermi level density is

$$\tilde{g}_\gamma(e) = \left( \frac{2}{\pi} \right)^{3/2} \frac{1}{\gamma \hbar} \operatorname{Im} i^{1/3} \int d\Sigma |\Psi_\Sigma(p'_y p'_z; y'' z'')|^{1/2} |z|^{1/6} \times \exp \left[ -2z/3 \right] \mathcal{A}i(z^{2/3}). \quad (101)$$

For  $\gamma$  values of order  $\gamma_{SH}$  or greater (but much less than the energy  $e$ ), the parameter  $z$  satisfies  $z \ll 1$ . To see this, we note that  $a \approx \hbar/d_\gamma$ , where  $d_\gamma$  is the characteristic length of variation of the potential  $V$ . For a "soft" potential for  $d_\gamma \approx R$ , and  $\lambda/R \approx A^{-1/3}$ , we find that  $z < 1$  right up to  $\gamma \approx 1.6A^{-2/9}E_F$ , i.e., for  $\gamma \lesssim 2.0-2.5\hbar\Omega$ . In other cases, the ratio  $a/e = d\hbar/dx$  is larger and the range of  $\gamma$  values in which  $z \ll 1$  increases. Thus, for not too large  $\gamma$  we can set  $z=0$  in Eq. (101), which leads to an expression for the smooth quasiclassical correction to the Thomas—Fermi density in the form

$$\tilde{g}' = (3^{-1/6} 2^{-4/3} / \Gamma(2/3) \hbar e) (e/a)^{2/3} \times \operatorname{Im} \int d\Sigma |\Psi_\Sigma(p'_y p'_z; y'' z'')|^{1/2} \exp(i\gamma y). \quad (102)$$

where  $\Gamma(x)$  is the gamma function. Formally, the ratio  $a/e$  is proportional to  $\hbar$ , and in this case the correction (102) is of order  $\hbar^{5/3}$ , and its ratio to the Thomas—Fermi term is proportional to  $\hbar^{4/3}$  and is a quantity of order  $A^{-4/9}$ . In a potential with sharp edge  $a/e \approx 1(d_\gamma \approx \hbar)$ , and the amplitude  $\tilde{g}'$  is less. In any case, the correction is approximately proportional to the area of the surface bounding the region of classical motion.

Another expression for the quasiclassical approximation is obtained for  $\gamma$  values large enough for the effective classical mean free path  $l_i$  to be of order  $\hbar$  or less, i.e., for  $\gamma \approx e$ . This case was considered by Balian and Bloch<sup>5(b)</sup> in the framework of an iterative expansion. For these large  $\gamma$  values, the approximation of classical trajectories breaks down. The correction calculated in Ref. 5(b) contains a volume integration of quantities proportional to the gradient of the potential, which also leads to a proportionality to the surface area of the system. However, the formal structure of the

correction and also its order in  $\hbar$  differ from the cases considered above. It is usually this last case that is had in mind in the usual derivation of the quasiclassical corrections to the statistical Thomas—Fermi theory in the approximation of a local density.<sup>30</sup> One further expansion is obtained for an infinitely high reflecting potential.<sup>5(a)</sup>

Summarizing, we can describe as follows the behavior of the averaged level distribution  $g_\gamma(e)$  as a function of the averaging interval  $\gamma$ . As  $\gamma$  increases from a quantity of the order of the distance between individual levels to a value of the order  $\gamma_{SH} = \hbar\Omega$  the oscillating components of the density decrease exponentially, and  $g_\gamma(e)$  approaches the smooth function  $\tilde{g}(e)$ . If  $\gamma$  is increased further, the value of  $\tilde{g}$  remains approximately constant until one reaches  $\gamma$  values that exceed  $\gamma_{SH}$  by several times. The value of  $\tilde{g}$  corresponds to the sum of the volume Thomas—Fermi term and smooth quasiclassical corrections proportional to the surface, its curvature, etc., calculated in an approximation which one can define as the approximation of the shortest reflected trajectory. If  $\gamma$  increases further and approaches the energy  $e$ , the value of the corrections and their physical meaning are changed. There is a transition to the approximation of a local density.

Except for the region of the plateau, when  $\gamma \approx \hbar\Omega$  there is not a unique value of the smooth level density  $g_\gamma$  independent of the averaging interval. This must be borne in mind, for example, in the case when the quasiclassical expressions are compared with the empirical quantities which are found from the level density of a strongly excited nucleus with temperature  $\tau \lesssim \tau_{SH}$ . In the thermodynamic description, the temperature  $\tau$  takes on a role that in a certain respect is similar to the averaging interval  $\gamma$ . It is clear from what we have said that  $\tilde{g}$  in a strongly excited nucleus may differ appreciably from the smoothed value of the single-particle level density at zero temperature.

The smoothed density  $\tilde{g}(e)$  is sometimes used to obtain the smooth total single-particle energy

$$\tilde{E}_{i,p} = \int_0^{\tilde{\mu}} \tilde{g}(e) de. \quad (103)$$

In accordance with the properties of  $\tilde{g}(e)$ , the energy  $\tilde{E}_{i,p}$  can also be divided into a sum of a volume component and quasiclassical corrections proportional to the surface and its Gaussian curvature, and one can thus define the so-called single-particle surface tension, or other such characteristics. In this connection, it must be emphasized that for a self-consistent system of interacting particles such an energy does not have an independent value and does not in any way represent the qualitative behavior or physical properties of individual components of the energy of the system.

The relative magnitude of the smooth components of the level density compared with the oscillating components depends on the actual Hamiltonian of the single-particle motion. However, this does not affect the fundamental possibility of unique separation of  $g_{osc}$  even in the cases when there are unallowed for smooth components exceeding  $g_{osc}$  in the general quasiclassical ex-

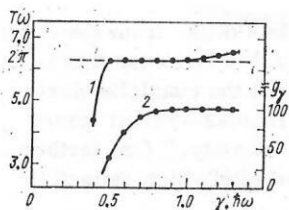


FIG. 11. Period of revolution  $T=2\pi/\omega_0$  (1) and averaged level density  $g_\gamma$  (2). The curve 1 is drawn through points found by numerical averaging of (107) for certain values of the averaging parameter. The calculation was made with correction polynomial of fourth degree for the spherical oscillator; the density  $\tilde{g}$  is obtained with  $\tilde{\gamma}=2\hbar\omega$ , and the energy is  $e=10\hbar\omega$ ; the dashed line is the theoretical value  $T\omega=2\pi$ .

pansion. Because oscillations of the density can be due solely to stationary classical paths, a further refinement of the quasiclassical approximation beyond the approximation of stationary phase considered in Sec. 2 would lead to a certain refinement of  $g_{osc}$  without a significant change in its shape. At the same time, only the oscillating component of the single-particle spectrum is of interest for the analysis of binding energies: As one can show under fairly general assumptions, the variations of the total energy do not depend explicitly on the interaction of the particles and are approximately equal to the expressions for the single-particle energy

$$\delta E \approx E_{i,p} - \tilde{E}_{i,p} \approx \int_0^{\hbar} (e - \mu) \delta g(e) de, \quad (104)$$

where

$$\delta g(e) = g - \tilde{g}; \quad (105)$$

$g(e)$  is the exact single-particle level density:

$$g(e) = \sum_k \delta(e_k - e); \quad (106)$$

$\tilde{g}(e)$  is the smooth asymptotic value. A more rigorous determination of  $\tilde{E}_{i,p}$  in (104) must correspond to the requirement that  $\delta E$  contain no smooth components, which must be included in the phenomenological drop component of the binding energy.

In giving great importance to the analytic theory for qualitative understanding of the nature and formal structure of the oscillating components of the single-particle distributions, we must point out that such a theory is of little use for practical calculations. All quantities necessary for quantitative description of shell effects in nuclei can be found more readily by means of the numerical procedures developed in the method of shell corrections. For example, the oscillating part of the single-particle level density can be determined directly from (105) for any spectrum of levels  $\epsilon_\lambda$  specified numerically in a sufficiently wide interval. For this, we take as  $\tilde{g}$  the smoothed quantity determined by means of averaging in (86) with the exact  $g(e)$  in accordance with (106),  $\gamma \gtrsim \gamma_{SH}$ , and a weight function which includes a correction for the curvature of the smoothed level distribution (89). The numerical methods developed recently in Refs. 31 and 32 enable one to find the eigenvalues for the nuclear potentials of any kind and arbitrary shape, and the numerical investigation of the distribution of energy levels is not a complicated problem. The well known results of the numerical calculations find explanation in the framework of the qualitative theory presented here.

A very important conclusion of the quasiclassical theory is that it has established a law of approach of the average density  $g_\gamma(e)$  considered as a function of the averaging parameter  $\gamma$  to the asymptotic smooth value  $\tilde{g}(e)$  for  $\gamma \approx \gamma_{SH}$  (90). This result is interesting in two respects. First, knowing the law in accordance with which the averaged level density reaches a plateau, we can more accurately determine the value of  $\tilde{g}$  by extrapolation based on formula (90) and the use of a smaller interval  $\gamma$ . It is possible that in this way one could overcome certain difficulties in the determination of  $\tilde{g}$  for potentials of finite depth by means of numerical methods.<sup>29</sup> Second, it follows from (90) that the period  $T$  of the classical orbit responsible for the shell structure can also be found from numerical calculations by studying the law in accordance with which the level density  $g_\gamma(e)$  obtained by numerical averaging of the exact value (106) reaches a plateau. (Here, the interval  $\gamma$  must not be too small.) The period of the classical orbits is one of the most important factors governing the intensity of the shell structure, but, except for a number of trivial examples,  $T$  cannot be determined by means of the analytic theory.

The revolution period  $T$  can be found numerically by solving the equation

$$\frac{d}{d\gamma^2} \ln(g_\gamma - \tilde{g}) = \frac{T^2}{4\hbar^2} / R \left( \frac{1}{4} T^2 \gamma^2 / \hbar^2 \right), \quad (107)$$

which follows from (90) if one assumes in (90) that the period  $T_\beta$  is the same for the components represented in the sum of this expression. Here,

$$R(x) = \sum_{k=0}^{n/2} \frac{(n/2)!}{k!} x^{-(n/2-k)};$$

$n$  is the degree of the correction polynomial in (89). We can find the value of the function on the left-hand side of (107) for  $\gamma \lesssim \hbar\Omega$  by means of numerical averaging of the exact level spectrum. The value of  $T$  must not depend on  $\gamma$  in a sufficiently wide range of  $\gamma$  values. The results of calculation for a spherical oscillator are given in Fig. 11. Along the ordinate, we have plotted the revolution period  $T$  found by numerical solution of Eq. (107) for the  $\gamma$  values indicated by the points. The deviation at small  $\gamma$  is due to the spectrum's being discrete, and at large values it is due to the loss of numerical accuracy because of the small differences  $g_\gamma - \tilde{g}$ . In the figure, we show the results obtained in the accuracy regime usual for such calculations.

## 8. GENERAL CHARACTERISTICS OF GROSS SHELLS

*Degeneracy of the classical orbits.* At least one periodic orbit can be found in any potential and at any energy.<sup>7</sup> However, the shell effects associated with an isolated orbit are insignificant: The strength of such shells is very low and in what follows we shall take it as the unit of the intensity characteristic of the shell structure. In accordance with the estimate made in Sec. 2, the amplitude of the oscillating component of the level density for an isolated orbit is of order  $\hbar^2$  compared with the main Thomas-Fermi term, which corresponds to

$$g_{osc} \approx A^{-2/3} \tilde{g}_{TF} \approx A^{1/3} / E_F. \quad (108)$$



Assuming that the period of revolution  $T$  in the orbit corresponds to an orbit length of the order of the length of the perimeter of the nucleus, we find from (78) the scale of variation of the energy associated with the isolated orbit:

$$\delta E \approx (\hbar\Omega/2\pi)^2 g_{osc} \approx E_F A^{-1/3}/4\pi^2 \approx 0.2 \text{ MeV} \quad (109)$$

This numerical value is for the parameters  $A=200$  and  $E_F=50$  MeV, which are typical of nuclear physics.

From the examples considered in Secs. 1 and 2, we can conclude that the amplitude of the oscillating component of the level density,  $g_{osc}$ , and its order in the quasiclassical expansion are determined by the invariance of the classical action for periodic orbits if there exist other nearly identical orbits filling a certain finite region of space. This degeneracy property of the periodic motion is due to the presence of definite integrals of the motion (conservation laws) in addition to the energy integral, and the degree of degeneracy is bounded by the total number of these conservation laws. The degree of degeneracy of a family of orbits is here found as the number  $k$  of continuous parameters  $u_1, \dots, u_k$  needed for the unique determination of a stationary orbit with given energy.

For an arbitrary potential  $k=0$ , i.e., there exist only isolated orbits. Indeed, periodic orbits are characterized by the initial and final momenta being equal at a certain point of space ( $\mathbf{p}''=\mathbf{p}'$ ,  $\mathbf{r}''=\mathbf{r}'$ ), and when written out this corresponds to the system of equations

$$\begin{aligned} \mathbf{r}(t-t_0, a_1 \dots a_5) &= \mathbf{r}(t-t_0+T, a_1 \dots a_5); \\ \mathbf{p}(t-t_0, a_1 \dots a_5) &= \mathbf{p}(t-t_0+T, a_1 \dots a_5) \end{aligned} \quad (110)$$

for the five arbitrary constants  $a_1 \dots a_5$  and the period  $T$ . Only five of Eqs. (110) are independent since the condition  $(\mathbf{p}')^2 = (\mathbf{p}'')^2$  is satisfied identically on account of the system's being conservative. Requiring that the periodicity condition be satisfied at a certain time  $t=t_0$  and the energy be equal to a given value  $e = e(a_1 \dots a_5)$ , we find the parameters  $a_1 \dots a_5$  and the revolution period  $T$  from the six equations as functions of the energy  $e$ . Note that, because there are singular points on the orbit, the orbit is not determined uniquely and there exists a discrete set of stationary paths that differ by the number and nature of the singular points (turning points and caustic points).

If besides the energy there exists an additional single-valued integral of the motion—a function  $F(\mathbf{r}\mathbf{p})$  which preserves its value for any classical path defined by the Hamiltonian  $H$ , the number of independent conditions in (110) becomes smaller by one further unit because one equation automatically follows from the identity  $F(\mathbf{r}'\mathbf{p}') = F(\mathbf{r}''\mathbf{p}'')$ . In this case, the conditions (110) admit a continuous one-parameter family of periodic orbits with given energy. If there are even more additional single-valued integrals, the number  $k$  of parameters  $u_i$  is increased accordingly, but it cannot exceed  $2n-2$ , where  $n$  is the dimension of space. This is because the total number of independent integrals of the motion [of arbitrary constants  $a_i$  in (110)] is  $2n-1$ , including the energy integral.<sup>17</sup> Thus, for the three-di-

mensional problem, the highest degree of degeneracy is four.

We now show that the action integral calculated for each of the periodic orbits  $\mathbf{r}=\mathbf{r}_{u_1 \dots u_k}(t)$  of the family of stationary orbits with given energy keeps the same value. For this, we consider two points close to each other on an orbit with parameters  $(u_1 \dots u_k)$ , writing them as

$$\mathbf{r}_1 = \mathbf{r}_{u_1 \dots u_k}(t_1), \quad \mathbf{r}_2 = \mathbf{r}_{u_1 \dots u_k}(t_2).$$

As  $t_2 \rightarrow t_1$ ,

$$\mathbf{r}_2 = \mathbf{r}_1 + \dot{\mathbf{r}}_{u_1 \dots u_k}(t_1)(t_2 - t_1),$$

where  $\dot{\mathbf{r}}$  is the velocity at the point  $\mathbf{r}_1$  for motion along the orbit  $(u_1 \dots u_k)$ . Specification of two points at fixed energy completely determines the orbit and the action integral  $S(\mathbf{r}_2, \mathbf{r}_1, e) = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{p} d\mathbf{l}$ . For a periodic orbit  $(u_1 \dots u_k)$  the action is  $S_{u_1 \dots u_k}(e) = \lim_{t_2 \rightarrow t_1} S(\mathbf{r}_2, \mathbf{r}_1, e)$ . On the transition to a nearby orbit,  $\mathbf{r}_1$  and  $\mathbf{r}_2$  are changed by  $\delta \mathbf{r}_1 = \sum_{i=1}^k (\partial \mathbf{r}_1 / \partial u_i) \delta u_i$ ,  $\delta \mathbf{r}_2 = \sum_{i=1}^k (\partial \mathbf{r}_2 / \partial u_i) \delta u_i$ , and the increment of the action is

$$\begin{aligned} \delta S &= \{(\partial S / \partial \mathbf{r}_1) \delta \mathbf{r}_1 + (\partial S / \partial \mathbf{r}_2) \delta \mathbf{r}_2\}_{t_2 \rightarrow t_1} \\ &= \{-\mathbf{P}(\mathbf{r}_1) \delta \mathbf{r}_1 + \mathbf{P}(\mathbf{r}_2) \delta \mathbf{r}_2\}_{t_2 \rightarrow t_1}. \end{aligned} \quad (111)$$

Since  $\delta S=0$  for all values of the parameters  $u_1 \dots u_k$ , it follows from Eq. (111) that the action does not depend on  $u_1 \dots u_k$ , i.e., it is constant for the given family of stationary orbits. Note that this result is intimately related to Noether's well known theorem for classical mechanics.<sup>33,34</sup> For  $k \geq 2$ , the number of arbitrary parameters of the orbits is sufficient for one to be able to determine an orbit passing through any given point of the classically accessible region. For  $k=3$  and 4 these orbits would form one- and two-parameter families. In the calculation of the level density, it was convenient to make the concept of degeneracy of orbits more precise by defining separately the parameter  $\mathcal{D}$  as the number of arbitrary degrees of freedom of the orbits passing through the given point  $\mathbf{r}$  of space and  $\mathcal{R}$  as the dimension of the space filled by the orbits of this type. The parameter  $\mathcal{D}$  determines the enhancement of the amplitude of the oscillating components of the Green's function for nearly equal values of the arguments, and  $\mathcal{R}$  determines the magnitude of the integral with respect to  $\mathbf{r}$  in the expression of  $g_{osc}$  in terms of the Green's function. The sum  $\mathcal{D} + \mathcal{R} = k+1$  does not exceed the number of single-valued integrals for the given problem, and the maximal possible value  $\mathcal{D} + \mathcal{R}$  is equal to five for three-dimensional motion, the relation  $\mathcal{R} \leq 3$  holding. The case  $\mathcal{D} + \mathcal{R} = 5$  corresponds to complete degeneracy of the classical motion. In the one-dimensional problem  $2n-1=1$ , and complete degeneracy is guaranteed by the existence of just the energy integral.

Some of the single-valued integrals are due to the obvious symmetry properties of the potentials, and in these cases the degree of degeneracy can be determined without difficulty. Thus, for an arbitrary spherical potential  $\mathcal{D} + \mathcal{R} = 4$  (the energy integral and three components of the angular momentum), and the degree of degeneracy is  $\mathcal{D} + \mathcal{R} - 1 = 3$ . In an arbitrary axisymmetric potential  $\mathcal{D} + \mathcal{R} - 1 = 1$  (energy integral and projection of the angular momentum onto the symmetry

TABLE I. Characteristics of shells.

Characteristics	$\Delta e$	$\mathcal{D}$	$\mathcal{H}$	$\mathcal{H}'$	Amp (g)	$\delta E$
Thomas—Fermi component	$E_F$	—	—	A	$A/E_F$	$AE_F$
Oscillating components						$\delta E (2\pi T/T_0)^3$
Isolated orbit	$2\pi\hbar T$	0	1	1	$A^{1/3}/E_F$	$A^{-1/3}E_F$
Axial symmetry	$2\pi\hbar T$	0	2	$A^{1/6}$	$A^{1/2}/E_F$	$A^{-1/6}E_F$
Spherical symmetry: nonsingular orbits	$2\pi\hbar/T_0$	1	3	$A^{1/2}$	$A^{5/6}/E_F$	$A^{1/6}E_F$
singular orbits (diameters)	$2\pi\hbar/T_0$	0	3	$A^{1/3}$	$A^{2/3}/E_F$	$E_F$
Harmonic oscillator: complete degeneracy	$2\pi\hbar T$	2	3	$A^{2/3}$	$A/E_F$	$A^{1/3}E_F$
planar orbits	$2\pi\hbar T$	1	2	$A^{1/3}$	$A^{2/3}/E_F$	$E_F$
rectilinear orbits	$\hbar\omega_i$	0	1	1	$A^{1/3}/E_F$	$A^{-1/3}E_F$
Separable variables	$2\pi\hbar T$	3		$A^{1/3}$	$A^{2/3}/E_F$	$E_F$
Coulomb potential	$2\pi\hbar T(e)$	5	—	—	—	—

axis). The situation is more complicated for other integrals. General methods have been developed only for systems with separable variables. It is known<sup>17</sup> that in any three-dimensional problem with separable variables there exist at least three independent single-valued integrals of the motion, including the energy ( $\mathcal{D} + \mathcal{H} = 3$ ). It is convenient to choose three partial action integrals  $S_i$ . The number of single-valued integrals is increased to four or to five if one or both ratios of the partial frequencies  $\omega_i/\omega_j$  do not depend on the integrals of the motion  $S_i$  and are equal to the ratio of integers ("fortuitous" degeneracy). Here  $\omega_i = 2\pi\partial e/\partial S_i$ ;  $e$  is the energy expressed as a function of  $S_i$ .

Some of the additional conservation laws may represent rather unexpected relations between the coordinates, momenta, and the potential. As an example, we can give the additional integral of the motion in a Coulomb potential and in the field of gravitational forces—the so-called Runge—Lenz vector:

$$\mathbf{r} \times \mathbf{M} + \alpha \mathbf{r}/r = \text{const}, \quad (112)$$

which determines the positions of the principal axes of the orbit. Here,  $V(r) = \alpha/r$  is the potential;  $\mathbf{M} = \mathbf{r} \times \mathbf{p}$  is the angular momentum. The derivative with respect to the time of (112) is zero.

In considering the examples of the harmonic oscillator and a spherical potential we noted that, besides the family with the maximal degeneracy for the given potential, there exist singular families of orbits with lower degeneracy (rectilinear diametral orbits, isolated planar orbits, and so forth). Such families appear in the cases when the orbits are characterized by fewer degrees of freedom or when, for a certain value of one of the parameters  $u_1 \dots u_k$ , the dependence on one or more of the other parameters disappears. This last can occur in a deformed oscillator.

Stationarity of the action under variations of the parameters of the orbits in a certain finite region leads to a corresponding reduction in the number of the Fresnel integrals proportional to  $\hbar^{1/2}$  in the sequence of integrals leading to the quasiclassical expression for the level density. The total enhancement of the amplitude of the oscillating component of the level density is of order  $\hbar^{-(D+R-1)/2}$ , partly because of the enhancement of the components of the Green's function ( $\hbar^{D/2}$ ) and partly because of the increase in the dimension of the space filled by the orbits ( $\hbar^{(R-1)/2}$ ). General estimates of the characteristics of  $g_{osc}$  in some simple examples are given in Table 1. This shows, in particular, the enhancement factors of the amplitude  $g_{osc}$ :

$$f' = A^{(\mathcal{D} + \mathcal{H} - 1)/6}.$$

The number  $f'$  can also be regarded as the number of particles in the gross shell if this number is estimated by

$$f' = \int_0^{\Delta e} |g_{osc}| de, \quad (113)$$

where  $\Delta e$  is the characteristic range of variation of the corresponding component of the level density as a function of the energy. For the Thomas—Fermi component, the interval  $\Delta e$  is of the order of the Fermi energy  $E_F$ , and  $f'$  is of the order of the total particle number  $A$ . For the oscillating components of the level density,

$$\Delta e = \hbar\Omega = 2\pi\hbar/T,$$

where  $T$  is the revolution period. For a spherical potential  $T_0 = L\bar{v}^{-1}$ , where  $\bar{v} = \sqrt{(2E_F/M)^{1/2}}$  is the average velocity;  $L \approx 2\pi R$  is the length of the orbit responsible for the shell structure. From this we obtain the estimate

$$\hbar\Omega_0 \approx E_F A^{-1/3}, \quad (114)$$

where we have used the relation  $k_F R \approx (9\pi A/8)^{1/3}$ ;  $k_F$  is the wave number for the Fermi energy  $E_F$ . In the third and fourth columns we have given the values of the degeneracy parameters  $\mathcal{D}$  and  $\mathcal{H}$ , except for the last two examples, for which  $\mathcal{D}$  and  $\mathcal{H}$  are not known separately. Here we have given the total number  $\mathcal{D} + \mathcal{H}$  of the single-valued integrals of the motion.

In the sixth column of Table 1, we have given estimates of the magnitude of the various components of the level density. To estimate the amplitude of  $g_{osc}$  in the case of an isolated orbit we have used the expression (108). Estimates of the shell corrections to the binding energy  $\delta E$  (seventh column) were obtained in accordance with Eqs. (78) and (109). For a nonspherical potential, the ratio  $T/T_0$  may be strongly different from unity. For the Coulomb potential, the characteristics of the shells are not given since  $\hbar\Omega$  depends strongly on the energy and the averaged values do not have much meaning.

For the theory of shell effects in nuclei a very important question is that of the dependence of the intensity of the shell structure of the single-particle spectrum on the shape of the potential. For the spherical case, the answer is very simple. Because of the very high degree of degeneracy and the weak dependence of the amplitude of  $g_{osc}$  and the period  $T$  on the actual form of the potential, a rather intense shell structure arises in almost any spherical potential (see Secs. 2 and 6). In a non-



spherical potential, the situation is different, since the degeneracy is less than in a spherical potential. However, if the Hamiltonian of the system contains arbitrary functions describing, for example, the shape of the nucleus, one can choose them in such a way that the degree of degeneracy of the orbits increases and the period  $T$  is decreased. Indeed, the equation

$$dF(\mathbf{r})/dt = 0, \quad (115)$$

for the integral of the motion  $F$  or the equivalent condition of vanishing of the Poisson brackets  $\{H, F\}$  can be regarded as an additional condition determining the form of a potential with higher degeneracy. What we have said applies especially to the choice of the shape of the nucleus, which can be regarded as an additional continuous degree of freedom needed to satisfy the condition (115). A well known example of such "choice" of the potential is a spherical potential of the form  $V(r) = \alpha r^\beta$ , where the additional five integrals arise if one takes  $\beta = -1$  (Coulomb potential and the Kepler problem) or  $+2$  (spherical harmonic oscillator). The nature of these additional integrals of the motion is due to the concrete properties of the Hamiltonian and cannot be foreseen. In this respect there is no substitute for numerical methods.

**Period of the orbit.** In applications to nuclear theory, we must remember that the enhancement of shells due to the presence of single-valued integrals of the motion is numerically not very large: Each new degree of degeneracy increases  $g_{osc}$  by 2–2.5 times. Therefore, degeneracy of the orbits is not always the decisive factor. This applies particularly to the calculation of the shell correction  $\delta E$ , which, as is shown in Sec. 6, is inversely proportional to the square of the period of revolution  $T_\beta$  in the orbit. The period is not related to the degree of degeneracy of the orbits and must therefore be regarded as a separate factor. The value of  $\delta E$  is enhanced by a factor of order  $A^{(2\beta+1)/6}$  due to the increase of the amplitude  $g_{osc}$  associated with the symmetry and is approximately inversely proportional to the square of the length of the orbit. In accordance with (78), orbits with shorter period of revolution make a larger contribution to the shell energies. One must also take into account the decrease in the contribution to the level density of complicated orbits that approach the perimeter of the well, which occupy a smaller volume even if there is a high degree of degeneracy. An example is provided by almost circular orbits in an infinitely deep spherical well with a large number of reflection points (see above).

The revolution period  $T$  is not a monotonic function of the shape of the potential, as a result of which the shell correction  $\delta E$  to the energy accompanying a change in the shape of the potential varies irregularly because of the change in the degree of degeneracy of the orbits and as a result of irregular variation of  $T$ . If  $A$  is of the order of a few hundreds, the possible effect of the variation of  $T$  is approximately the same as that of variation of  $\mathcal{I}$  or  $\mathcal{H}$  by one or two units. A good illustration of this situation is the example of the harmonic oscillator (see Secs. 3 and 6 and Figs. 7 and 8). Figure 10 shows a contour map of the distribution of the shell corrections in an axisymmetric harmonic oscillator as

a function of the number of occupied levels and the deformation parameter  $\omega_1: \omega_z$ .

The numerical results, like the qualitative analysis, show that the observed decrease in the amplitude of the shell oscillations of the energy in deformed single-particle potentials is not due to the disappearance of the shells when there is a departure from spherical shape, as has been frequently assumed in the literature [see, for example, Ref. 7 and the other papers quoted below]. One can say that the decrease of the amplitude  $\delta E$  is due to the increase in the length of the orbits with higher degeneracy ( $\hbar\Omega$  decreases) and also to the decrease in the degree of degeneracy of simple orbits with large  $\hbar\Omega$ . But neither factor is determined by the departure from spherical symmetry.

An important qualitative conclusion which can be drawn on the basis of our exposition is that the magnitude of the shell corrections depends irregularly on the shape of the nucleus, increasing in the cases when the conditions favor the appearance of orbits with high degeneracy and short revolution period  $T$ , i.e., shorter orbits, or, under otherwise equal conditions, orbits that fill a greater volume. The theoretical results completely explain why, in contradiction to the widely held opinion at the beginning of the sixties,<sup>35–37</sup> a clearly expressed shell structure of the single-particle spectrum is found in calculations with realistic potentials<sup>1–4</sup> that have little in common with spherical symmetry or special properties of the harmonic oscillator. At the same time, a natural explanation is provided for the characteristic periodicity with which gross shells appear in deformed nuclei.

## 9. EQUILIBRIUM DEFORMATIONS OF NUCLEI AND FISSION

The nonuniformity in the upper part of the single-particle spectrum due to shells has very great importance for the stability and especially the shape of a nucleus. Varying rapidly with the deformation, the gross shells make an appreciable contribution to the local shape stiffness of the nucleus and become a decisive factor when, under certain conditions, the much larger effect of classical forces of surface tension and Coulomb repulsion are reduced. This happens near the extrema of the component of the deformation energy that corresponds to a classical liquid drop as, for example, around spherical shape or near the so-called valley in the drop energy in strongly deformed heavy nuclei, when two large but opposite classical forces almost compensate each other.<sup>2</sup> Under these conditions, the significant reduction in the single-particle level density in the neighborhood of the Fermi energy leads to sharply expressed local minima in the deformation energy and thus determines the equilibrium shapes, spherical or deformed, of nuclei.<sup>3)</sup>

<sup>3)</sup>What we have said here does not apply to the smaller minima in the deformation energy that can indeed be due to the structure of individual single-particle states. This "fine structure" is manifested especially in the transition region between two neighboring shells; for example, in the region of the transition from a stable spherical shape to a nonspherical shape of the nucleus, the effect of the variation of the gross shells being weakened here.



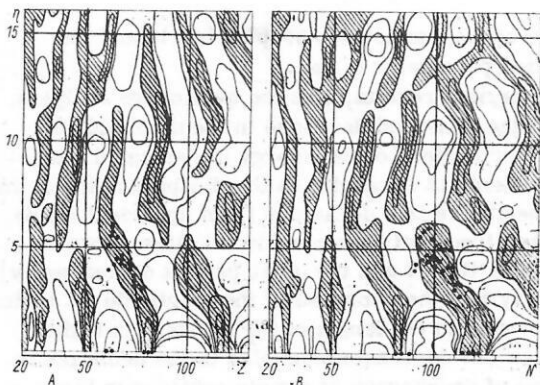


FIG. 12. Oscillating component  $\delta g = g_\gamma - \tilde{g}$  of the level density for the Nilsson model as a function of the number of neutrons  $N$ , protons  $Z$ , and deformation parameter  $\eta$ . The dots are the experimental values of the quadrupole deformation parameter for the ground state of the nucleus with the given number of nucleons.

In the strongly deformed actinide nuclei the appearance of gross shells leads to the occurrence of the relatively stable stationary states<sup>1-4,38,39</sup> responsible for the effects of intermediate structure in fission and for spontaneously fissionable isomers. These results contradict the interpretation of gross shells as due to the spherical symmetry of the nuclear potential. They depart even further from the traditional explanation of nonsphericity of the nuclear shape as due to competition between quadrupole forces and pairing forces or the deforming effects of certain definite nucleon states coupled strongly to the degrees of freedom of the shape of the nucleus. The shell nonuniformity of the spectrum includes many quantum states (see Table 1) and is not associated with the properties of individual levels or so-called subshells. The shell structure is a cooperative effect of many particles (see Ref. 2, Sec. IV.5).

The quasiclassical qualitative theory casts the problem of stability of nuclear shape in a new light. The conditions of enhanced stability of the nuclear shape can be formulated as follows<sup>2</sup>: First, the single-particle spectrum must have a clearly expressed shell structure and, second, in the region of the Fermi energy the level density must be reduced, as can be seen, in particular, from the expression (78) for the shell correction to the energy. Whereas the second condition can be satisfied by a definite choice of the number of nucleons, which is a magic number of nucleons for the given nuclear potential, the first condition requires a certain definite shape of the potential guaranteeing the existence of simple periodic or almost periodic orbits and the maximal degree of degeneracy. In this respect it is very important that the nuclear potential contain degrees of freedom of the type of nuclear shape parameters, which in the equilibrium state of the nucleus are selected by the condition that for a given number of nucleons the minimum of  $\delta E$  be as deep as possible.

The chart shown in Fig. 12 is the distribution of the oscillations of the single-particle level density as a function of the number of nucleons, which is equal to twice the number of occupied states, and the deformation parameter for one of the realistic variants of the nu-

clear potential, namely the Nilsson potential. Figure 12 shows the contours of the oscillating component of the level density near the Fermi energy:

$$\delta g = g_\gamma(\mu) - \tilde{g}, \quad (116)$$

determined by means of numerical averaging, as described in Sec. 7. The smoothed level density  $\tilde{g}$  was calculated with  $\gamma \geq \gamma_{SH}$ . For  $g_\gamma$ , the smoothing parameter  $\gamma$  was taken equal to 2–3 MeV, which is approximately one third of  $\hbar\Omega_0$ . Thus, the oscillations represent the contributions of periodic orbits whose lengths do not exceed two or three characteristic nuclear dimensions. The numerical results show convincingly that in deformed realistic nuclear potentials there does indeed exist a significant structure of gross shells in the single-particle level distribution and that it appears especially clearly for certain definite shapes of the nucleus. The increase in the binding energy follows rather closely the distribution of the minima in the level density at the Fermi energy.<sup>1,2</sup>

The same is observed in calculations with other nuclear potentials and, in particular, with the Woods-Saxon potential.<sup>2,39</sup> The experimentally known equilibrium deformations correspond closely to the minima in the level density at the Fermi energy (see Fig. 12). This result,<sup>1</sup> discovered in the framework of the method of shell corrections, can be naturally explained by means of the quasiclassical theory presented here. The most clearly expressed minima in the level density form clearly defined regions of spherical and deformed magic nuclei.

It is very important that in realistic nuclear potentials there is a region of strong shell structure at a relatively small deformation—of order 20–30%—from spherical shape (magic numbers  $N=100, 150-152$ ). It is for this reason that when a definite occupation of a spherical potential well is reached, there is an abrupt change in the deformation because the shape corresponding to a nearly filled shell in the deformed nucleus becomes energetically advantageous. If there were no shells in the deformed nucleus, the deformation would increase steadily. Since just such a possibility was assumed earlier, the observed abrupt transition from spherical to nonspherical shape was explained by the assumption that the transition is due to competition between quadrupole forces and pairing forces.<sup>41</sup> This model is not confirmed by the systematic theory and direct numerical calculations, from which it follows necessarily that there exist gross shells in deformed nuclei and that they play an important role in establishing the equilibrium shape of the nucleus.

The inadequacy of the model with competition between quadrupole forces and pairing can be confirmed directly in calculations of the equilibrium deformations with a pairing whose strength varies from zero to excessively large values (Fig. 13). According to the model of quadrupole forces, the point of transition from the spherical equilibrium shape to nonspherical shape must be a sensitive function of the pairing.<sup>41</sup> However, as can be seen from Fig. 13, except for nuclei in the immediate neighborhood of magic spherical nuclei, whose deformation in the absence of pairing is due to

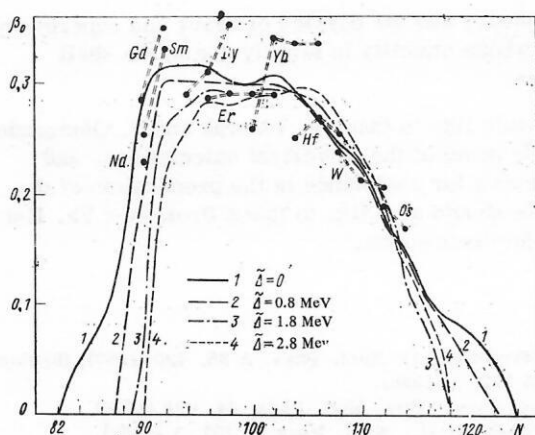


FIG. 13. Quadrupole deformation parameter  $\beta_0$  for the first minimum of the shell correction  $\delta E$  for neutrons as a function of the mass number at different intensities of the pairing correlations. Calculation with Nilsson potential, the parameter  $\Delta$  being of order of the energy gap for the uniform spectrum; the continuous curve is the calculation without pairing; the figure also shows some values of the quadrupole deformation parameter of nuclei known from Coulomb excitation experiments (Ref. 2).

the relatively weak structure of the subshells ( $j$  levels of a spherical nucleus), the value of the equilibrium deformation and the point of transition from spherical to nonspherical shape are insensitive to the pairing. The positions of the points at which the equilibrium shape changes can even be followed in calculations with pairing completely eliminated (see Fig. 13 and also Ref. 2). The change corresponds to a clear inflection of the curve of the equilibrium deformation as a function of the nucleon number for  $\Delta = 0$ .

In accordance with the new interpretation,<sup>1,2</sup> the point of transition from spherical to deformed shape of the nucleus is determined by the boundary between neighboring shells in the spherical nucleus and for deformation 20–30%. These boundaries can be clearly seen in charts like those shown in Figs. 10 and 12, and one can readily show that in calculations with realistic potentials the correct values of  $N$  and  $Z$  of nuclei in the transition regions are obtained. The shells mentioned above at a deformation of the nuclear potential of order 20–30% are responsible for the deformation of the ground states of the nuclei in the region of rare-earth elements and actinides.

Unfortunately, the connection between equilibrium deformations and the shells of deformed nuclei has not been considered in the majority of monographs, and even in the recent reviews Refs. 7 and 42 the gross shells in deformed nuclei are discussed only in connection with the quasistationary state (fission isomers) in the second potential well. The shells at smaller deformation, with which the existence of the phenomenon, in many respects more fundamental, of the existence of deformed nuclei is associated, are not even mentioned. The gross shells observed in realistic nuclear potentials at a deformation of order 20–30% do not have a direct analog in a deformed harmonic potential, and it may be that this is one of the reasons why such shells do not get due "recognition".

The shell structure corresponding to the second potential well, in contrast to the first corresponding to the ground state of a deformed nucleus, is manifested in strongly deformed nuclear potentials at a deformation corresponding approximately to the ratio  $c:a=2:1$  of the principal axes of the nucleus (see Refs. 1–4 and also Refs. 38 and 39). In Fig. 12, these are the shell minima at deformation  $\eta \approx 10$ –12. Actually, a second shell is manifested only in the heaviest strongly deformed nuclei under conditions when the surface tension is compensated by Coulomb forces near the region of the magic number  $N=144$ –146. In a deformed harmonic oscillator for  $\omega_1:\omega_2=2:1$  one also observes a clear shell structure. Therefore, after the shell effect in deformed nuclei had been discovered and analyzed by means of numerical methods, it was suggested in Refs. 7, 43, and 44 that the gross shells observed for  $c:a \approx 2$  in deformed heavy nuclei are due to the single-particle spectrum in a realistic nuclear potential being near the level spectrum of a deformed oscillator.

Attempts are made to attribute other cases of deformation of nuclei in ground and excited states to numerous shells of a slightly deformed harmonic oscillator.<sup>44</sup> It is known that for such an interpretation there is no justification. The difference between realistic potentials and a nonspherical oscillator is large from the point of view of all criteria of quantum theory and leads to a by no means small perturbation of the spectrum. The relative displacement of the levels exceeds the distance between the gross shells of the oscillator (see, for example, Ref. 7, Fig. 6.49), and the mixing of the levels of the oscillator shells increases with increasing number of nucleons.

It should however be borne in mind that the structure of the gross shells is due to the properties of the classical Hamiltonian, in particular the characteristics of the periodic orbits. Therefore, the similarity observed in certain cases can be explained by means of the following theorem of classical mechanics: If the difference between a realistic potential and a model potential which allows separation of variables is small, the characteristics of the periodic motion are nearly equal, and the degree of proximity is much greater than the relative amplitude of the difference of the Hamiltonians.<sup>17</sup>

Thus, one can understand why in realistic potentials a strong shell structure is manifested at approximately the same deformation  $c:a=2:1$  as the one at which one observes the most pronounced shells in a deformed oscillator. This agreement is not observed at small deformations, for which in an harmonic oscillator one can find many classical closed orbits with long periods that do not lead to a clearly expressed shell structure and go over into each other if the shape of the potential is slightly changed. The fact that the addition to a stationary potential of nonharmonic terms in the Nilsson model enhances the shell structure of the single-particle spectrum compared with a deformed harmonic oscillator can also be regarded as an argument against a distinguished role of oscillator symmetry: An appropriate distortion of the oscillator potential can lead to a significant simplification of the periodic orbits and an



enhancement of the shell structure, possibly at the price of a certain reduction in the degree of their degeneracy. The harmonic potential is also exceptional in that the partial frequencies  $\omega_i$  and also the revolution frequency  $\Omega$  are uniquely determined by the shape of the potential and do not depend on the constants of the motion. Therefore, orbits with a given ratio of the frequencies in the oscillator appear only at a quite definite "deformation". In the more general case, multidimensional orbits of given families of orbits of constant action appear in definite regions of deformation in the form of bands (which degenerate in the case of the harmonic oscillator into the sequence of points  $\omega_i/\omega_j = n_i/n_j$ ). The distribution of shells in realistic potentials exhibits much greater similarities with a deformed infinitely deep well of ellipsoidal shape, in which comparatively simple orbits occur at a lower deformation than for an harmonic oscillator. This interpretation readily explains the existence of both the "second" and "first" well.

When a nucleus becomes strongly deformed in the process of fission, a "neck" is formed between the future fragments. When the potential has this shape, there arise orbits that are closed within the incipient strongly deformed fragments, and the fragments acquire a shell structure. Conditions favorable for the appearance of very strong shells in the fragments arise, as a rule, for an asymmetric fission: Symmetry of the shape with respect to the plane perpendicular to the direction of fission is not necessarily a factor that enhances the shells under these conditions. Thus, one can explain qualitatively<sup>45</sup> why the shell structure of a fissile nucleus after the formation of a neck leads, as a rule, to asymmetric fission.<sup>40</sup> This conclusion of the quasiclassical theory is not restricted to the simple model considered in Ref. 45.

This rearrangement of the shells during the fission of the nucleus significantly changes not only the deformation energy but also the dynamical characteristics that are sensitive to the structure of the single-particle Fermi surface such as, for example, the inertia parameters (mass coefficients), and it is therefore a decisive factor in the fission of nuclei at not too high excitation energy.<sup>40</sup>

An important general conclusion of the quasiclassical theory of nuclear shells is that the shell structure of the nuclei is a general property of realistic nuclear potentials irrespective of the possibility of separation of variables or the particular properties of the harmonic oscillator. Overestimation of the importance of the oscillator model as well as of spherical symmetry is the explanation for some erroneous conclusions.<sup>7</sup> The required degree of degeneracy and the extremal values of the other parameters can be achieved in the case of deformed realistic potentials that have a sufficient number of degrees of freedom such as the shape of the nucleus. This conclusion not only agrees much better with the known results of numerical analysis,<sup>1-4</sup> but is also very important for adequate description of equilibrium deformations and, in particular, for the qualitative understanding of the reliability of numerical results extrapolated to relatively uninvestigated regions of

fission theory and the physics of heavy and superheavy nuclei, whose stability is largely due to the shell structure.

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