Kinematics of microscopic nuclear theory

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Microscopic nuclear theory is examined from the point of view of the general kinematic properties of the nuclear Hamiltonian and the special choice of spatial variables describing the collective and internal degrees of freedom of the nucleus.

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INTRODUCTION

In this paper, we shall interpret the kinematic aspects of the results of Refs. 1–3, which were presented in the review Ref. 4. The papers Refs. 1–3 introduced a microscopic analog of the Bohr–Mottelson model of collective variables, making it possible to formulate correctly in the coordinate representation the problem of separating out the collective degrees of freedom of the nucleus. We shall also describe partly the results obtained in Refs. 5–8, in which the theory of the microscopic analog of the Bohr–Mottelson collective model was proposed and described in some detail; in addition, a computational technique for it was developed and for the first time a kinematic justification given for the microscopic theory of the nucleus by means of three simple requirements, two of which follow in an obvious manner from the properties of the nuclear Hamiltonian. The plan of the present review is laid out in accordance with these results: The exposition of the material begins with the most general formulations and ends with their successive particularization.

The microscopic approach to nuclear theory presented here rests essentially on the theory of induced representations of Lie groups, and we must therefore consider some aspects of this theory. So as not to make this review difficult to understand for readers acquainted with the fundamentals of group theory for only the rotation group and the symmetric group, we shall as far as possible avoid explicit use of the algebraic formalism of induced representations.

Let us now briefly describe the premises that enable us to pose and solve problems associated with the kinematic foundations of microscopic nuclear theory. If it is assumed that protons and neutrons are particles of the same kind i.e., if one uses the isospin concept, then the nucleus can be regarded as an isolated system of identical Fermi particles. Such a system has properties that distinguish it advantageously from the other mixed quantum systems existing in nature, in which fermions coexist with particles of a different type. Examples of mixed systems are well known: free atoms, molecules, or crystals. In mixed systems, the presence of particles of various kinds leads to the appearance of spatial structure, and hence, distinguished points of three-dimensional space. In the atom, such a point is the nucleus, which ensures the existence of the atom as a stable quantum system and largely determines the behavior of the electrons by binding them to the natural force center produced by the nucleus. In its turn, the force center enables one to introduce electron configurations, i.e., classify states by means of irreducible representations of the three-dimensional rotation group.

The spatial structure is revealed even more clearly in molecules and crystals. It is reasonable for the spatial symmetry, the existence of which is an experimental fact that cannot be proved by considering the general form of the Hamiltonian of such systems. Nuclei with inert electron shells arranged appropriately in molecules and crystals generate external (in a certain sense) fields, in which the electrons move—as a Fermi system of identical particles. The presence of spatial structure leads to the appearance of spatial symmetry and therefore the corresponding groups whose irreducible representations classify the states of this Fermi system in the zeroth approximation.

In nuclei, there are no distinguished force points of three-dimensional space, but paradoxically this circumstance can provide the point of departure for the clarification of the kinematic aspects of nuclear structure. In the description of the bound states of such a Fermi system of identical particles by means of nonrelativistic quantum mechanics, the three-dimensional space is not, in contrast to the above examples, distinguished *a priori* over the multidimensional space determined by the number of spatial degrees of freedom of the nucleus. If the three unimportant degrees of freedom associated with the center of mass motion of the nucleus are excluded, the n-nucleon nucleus is described by $3(n-1)$ spatial degrees of freedom. It is therefore natural at the beginning to regard the spatial wave function of the nucleus as a function of the point in the $3(n-1)$-dimensional space, and then project "our" three-dimensional space from this multidimensional space.

This program is implemented in practice by the choice of special variables of the nuclear wave function that enable one to induce transformations of the multidimensional space and, using these transformations, effectively sort the nuclear wave functions according to definite criteria determined by the general kinematic requirements imposed on the nuclear wave function. Group theory does not appear here in its traditional role. As a rule, the groups considered here are not symmetry groups of the nuclear Hamiltonian. The irreducible spaces of such groups provide a concrete realization of the abstract multidimensional Hilbert space, and we show that this realization is optimal and uniquely determined as far as the kinematics is concerned. Moreover, it is not necessary to know the par-

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ticular form of either the Hamiltonian or the nucleon–nucleon interaction potential. It is not essential to assume pairing nuclear forces or other experimentally established facts such as the short range and saturation property of nuclear forces, etc. The only important thing is that the Hamiltonian satisfies the general kinematic requirements of quantum mechanics; with this, we begin the next section of the review.

1. DETERMINATION OF THE KINEMATICALLY CORRECT NUCLEAR WAVE FUNCTIONS AND CRITICAL REVIEW OF EXTREME NUCLEAR MODELS

In a choice of definite variables of the wave function, it is convenient to start with a discussion of the collective and internal variables of the nucleus. The collective degrees of freedom of the nucleus have been studied in many papers, which have differed by the problems posed and the mathematical formalisms used. A certain clarity can be achieved by classifying these studies according to the mathematical formalism used; in particular, one can distinguish investigations based on the method of second quantization from those that use wave functions defined in the coordinate representation. Since we are concerned with problems that can be solved in the framework of nonrelativistic quantum mechanics, these two methods of description are in principle equivalent, although it is very difficult or, for certain reasons, even impossible in practice to trace the connection between the two methods.

These remarks have been made to underline the fact that here we shall use only the second method of description, i.e., we shall use nuclear wave functions defined in the coordinate representation. This restricts the class of questions considered. Nevertheless, the number and variety of investigations in this second direction still remain so extensive that one cannot review them satisfactorily. But this is unnecessary because the problem of separating the collective and internal degrees of freedom is treated here from a definite point of view, which can be formulated on the basis of simple and natural conditions imposed on an arbitrary nuclear wave function.

Let us now formulate these conditions.

1. *Microcausality*. If it is supposed that the observable properties of nuclei can be described by a wave function which is a solution of the Schrödinger equation, then the wave function must be microscopic, i.e., it must depend on all the variables of the nucleus. These variables consist of $3n$ spatial coordinates, say, the $3n$ components of the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$, which are specified in the original coordinate system and, in addition, we have the spin variables of all the protons and all the neutrons, if these particles are assumed to be different (in which case we shall speak of a proton-neutron nucleus), or the spin-isospin variables of all the nucleons, if it is assumed that protons and neutrons are two states of particles of the same species—nucleons (and we shall then speak of a nucleon nucleus).

It is especially important that the number of spatial variables of the microscopic wave function be exactly $3n$; for if there are fewer variables the function is no longer completely microscopic and if there are too many we shall be burdened by redundant variables.

2. *Translational invariance*. This requires that the wave function of a nucleus subject to no external forces be expressible as the product of the plane wave describing the center-of-mass motion of the nucleus and a translationally invariant function describing the internal properties of the nucleus.

The wave function may be microscopic but not translationally invariant or, conversely, translationally invariant but not microscopic. Of course, these two conditions can be combined and we then have the concept of the microscopic, translationally invariant wave function of a free nucleus.

3. *Microscopic translational invariance*. This can be guaranteed by choosing the $3n$ spatial variables of the nucleus in such a way as to distinguish three variables, for example, the vector ρ_0 specifying the position of the center of mass in the original coordinate system, from the $3(n-1)$ variables, for example, the vectors $\rho_1, \ldots, \rho_{n-1}$, that determine the remaining translationally invariant spatial variables of the nucleus. If the spin or spin–isospin variables are added, the wave function in these variables automatically satisfies the conditions of microcausality and translational invariance. In the case of a free nucleus, the wave function factorizes into the product $\Psi_0\Psi$, where Ψ_0 is the plane wave describing the center-of-mass motion and Ψ is any translationally invariant wave function, in particular, a solution of the Schrödinger equation for a translationally invariant Hamiltonian depending arbitrarily on the spatial, spin, or spin–isospin variables of the nucleus.

4. *Exact integrals of the motion*. It is well known that the Hamiltonian of a nucleus conserves to a high degree of accuracy the total angular momentum J of the nucleus, its projection M_J, spatial parity, the species and number of particles, and, finally, the permutational (with respect to all variables) symmetry of the nuclear wave function. We therefore require that the wave function of the nucleus be characterized by these five integrals of the motion. The last requirement can appear in two forms: either in the form of antisymmetry with respect to the variables of the protons and neutrons separately (for proton–neutron nuclei), or antisymmetry with respect to the variables of all the nucleons (for nucleon nuclei). Note also that we have included conservation of the species and number of particles in the integrals of the motion. This integral of the motion makes it possible to fix the number and nature of the nuclear degrees of freedom.

We now consider what these conditions mean. They are obviously necessary in the sense that the true wave function of the nucleus must certainly satisfy them, but they are insufficient in the sense that by no means all functions satisfying them are true nuclear wave functions. If one succeeds in constructing functions satisfying conditions 3 and 4 they will not in general satisfy the exact Schrödinger equation for the nucleus. However, they will reflect some properties of this equation, namely the kinematic properties of the nuclear Hamiltonian. To emphasize this, we shall say that any

functions satisfying conditions 3 and 4 (but which are not necessarily solutions of the exact Schrödinger equation) are kinematically correct functions. Note also that the concept of "kinematic properties" is here used in a generalized sense, embodying not only the usually understood requirement 4, associated with the integrals of the motion, but also requirement 3 of microscopic translational invariance.

We now verify the kinematic correctness of some model nuclear wave functions defined in the coordinate representation. To avoid a lengthy examination of all modifications of different models and in order to get to the essence of the matter, we shall consider two extreme models of nuclear structure, one based on the concept of independent particles and the other which regards nuclei as self-deforming systems in which the complex motions of the nucleons are described by a few collective variables.

The first of these two ideas is realized in the various forms of the shell model. The wave function of this model is specified by means of configurations made up of linear combinations of products of single-nucleon wave functions depending on the single-nucleon variables $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Condition 1 is already satisfied by the use of these variables. For functions of shell type, the condition 4 can also be satisfied comparatively easily, so that only the problem of translational invariance remains. The single-particle variables are not translationally invariant, containing three variables describing the center-of-mass motion of the nucleus in a form not corresponding to a plane wave; condition 2 therefore strongly contradicts the shell concept. Strictly speaking, the shell picture can be believed only if one gives up the undoubtedly correct requirement 2. The physical reason for this is obvious: The only distinguished point of space is the center of mass of the free nucleus, so that one cannot construct correct variables defined in the laboratory coordinate system.

It should be emphasized that the difficulties which arise from condition 2 in the construction of functions of shell type are primarily of principle; in actual calculations, one can frequently determine, exactly or approximately, the function describing the center-of-mass motion in one definite state and thus avoid the appearance of spurious unphysical states of the nucleus. But this does not eliminate the above objections since the replacement of free motion of the center of mass by a bound state is possible only if physical external fields are present.

In the extreme collective models it is assumed that because of certain as yet not fully understood reasons the overwhelming majority of the variables in the nuclear wave function are "frozen" and the properties of the low-lying levels for, at least, selected nuclei are governed by just a few degrees of freedom of the nucleus. As a direct consequence of such assumptions, one uses only a few collective variables, introduced intuitively on the basis of semiclassical arguments, to construct a model wave function. In the collective variables, the single-particle variables lose their meaning, so there can be no talk of satisfying condition 1 or verifying condition 2. The collective variables do not contain a mechanism for permuting the nucleons, so that one cannot talk about the Pauli principle, and condition 4 can at the best be satisfied only partly. Therefore, the collective models in all their modifications are, considered kinematically, far less correct than shell models. Nevertheless, for nuclei in definite ranges of mass numbers the experimental data frequently favor them and therefore confirm the opinion that a number of characteristic properties of atomic nuclei have been correctly guessed in the collective models. This may explain the considerable efforts made during the last decade to justify such models.

We have just described two extreme views of nuclear structure. Their premises are so contradictory as to seem mutually exclusive. The experimental data favoring the two approaches are just as contradictory. This is made all the more strange by the fact that the contradictions even appear in nuclei that have nearly the same number of nucleons; for example, the nucleus ^{17}O, which has a clearly expressed single-particle spectrum, and the nuclei ^{20}Ne and ^{24}Mg, whose spectra exhibit series of levels of rotational type. It is hard to believe that the addition of a few nucleons has such a drastic effect on the structure of the nuclei that a completely opposite physical picture is required for its understanding.

The dilemma can be resolved by assuming that the contradictions are not so deep as they first appear. Could it be that the single-particle approach has a predisposition to collective motion or, coming the other way, the collective models admit microscopic resolution, i.e., they contain hidden single-particle features? Does there, perhaps, exist an approach that combines the single-particle and the collective aspects, these being manifested in the spectra of particular nuclei in a pure form only because of fortuitous conjunction of particular circumstances? Starting with single-particle variables, can one demonstrate how collective degrees of freedom arise?

In connection with the last question, let us consider briefly the attempts made to give a microscopic justification of collective motions in nuclei by means of the following artificial device. One first constructs wave functions that depend on all the microscopic single-particle variables $\mathbf{r}_1, \ldots, \mathbf{r}_n$ and on a definite number of collective variables, for example, the variables $\xi_1, \ldots, \xi_k$. Functions of shell type are usually used to construct functions of such form. Then following a definite procedure (usually by means of certain operators that depend on the variables $\xi_1, \ldots, \xi_k$) collective variables are injected. Even without going into the details of the device, one can see immediately that the very formulation of the problem betrays its superficiality; there is no mechanism for the formation of the collective degrees of freedom from the single-particle variables; the two types of variables exist, as it were, independently of each other, so that one is confronted from the very beginning with the problem of eliminating spurious variables. It should not therefore occasion surprise that the successes achieved in this direction are not commensurate with the efforts invested. Looking at this problem from our point of view, we obviously see that

such functions are not correct in the kinematic sense for at least two reasons: because there is no translational invariance, and because of the redundant variables, whose presence, as we especially emphasized in formulating condition 1, is more than microcausality can bear.

To resolve this problem of reconciling two extreme views of nuclear structure we must seek other ways. As a start, it is helpful to attempt the answer to some definite questions. Here they are. How can collective variables be rigorously defined in the sense of quantum mechanics? How many collective variables exist and what are the remaining variables of the nucleus? Can one introduce a microscopic analog of variables describing surface vibrations of quadrupole, octupole, etc., type and, if yes, then does such a series terminate at a particular multipolarity or can it be continued as far as one wishes? Can one find a functional connection between the collective and single-particle variables?

The results obtained from examining the extreme models have not given comforting results: They are incorrect in the kinematic respect. Questions such as the following arise: How can one construct kinematically correct nuclear models? Which of these are the simplest and in what direction do they generalize the incorrect models? Are there models that combine single-particle, collective, and intermediate aspects of the motion of nucleons in the nucleus?

2. PARAMETRIZATION OF ORTHOGONAL GROUPS AND CHANGE OF VARIABLES IN MULTIDIMENSIONAL SPACES

We are concerned primarily with choosing variables that automatically satisfy microscopic translational invariance. These variables are well known—they are the normalized Jacobi coordinates associated with the single-particle vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$ by means of the relations

$$\rho_i = \frac{1}{\sqrt{i(i+1)}}\left(\sum_{t=1}^{i} \mathbf{r}_t - i\mathbf{r}_{i+1}\right), \tag{1}$$

where $i = 1, 2, \ldots, n-1$. If the spin or spin—isospin variables are denoted by Q, our problem is to study the class of wave functions

$$\Psi(\Gamma \mid \rho_1, \ldots, \rho_{n-1}; Q), \tag{2}$$

which undoubtedly satisfy condition 3. In (2), Γ is an arbitrary but necessarily complete set of quantum numbers of the nucleus, and the completeness must here ensure decomposition of $\hat{T}\Psi(\Gamma)$ in terms of the original set of functions $\Psi(\Gamma)$, where $\hat{T}$ are certain "good" operators. Below, we give a detailed description of the operators $\hat{T}$, from which it can be seen that they are indeed "good" operators.

As long as the quantum numbers Γ are unspecified, the set of functions (2) is highly arbitrary. It would seem at first that for these functions one cannot make constructive assertions permitting the separation of their dependence on the collective variables of the nucleus. But in reality this is not so: There exists a mathematical formalism by means of which one can formulate an effective method of separating the collective function of the nucleus without any essential additional assumptions about the form of the function (2).

Above, we have criticized the attempts in which collective degrees of freedom have been introduced as redundant variables; their limited success is due to an incorrect formulation of the problem at the very start. In a correct approach, one does not use single-particle and collective degrees of freedom simultaneously; rather, one must make an ordinary change of the spatial variables, i.e., take $3(n-1)$ functions f_i^s and assume that

$$\rho_i^s = f_i^s(\xi_1, \ldots, \xi_k; q_{k+1}, \ldots, q_{3(n-1)}), \tag{3}$$

where ρ_i^s are the three Cartesian components of the vector $\rho_i (s = x, y, z;\ i = 1, 2, \ldots, n-1)$; $\xi_1, \ldots, \xi_k$ are k collective variables and $q_{k+1}, \ldots, q_{3(n-1)} - 3(n-1)$ are the remaining k variables of the nucleus. For brevity, we shall frequently denote the sets $\xi_1, \ldots, \xi_k$ and $q_{k+1}, \ldots, q_{3(n-1)}$ by ξ and q and call them, respectively, the collective and internal variables of the nucleus. The first task is to choose the functions (3) sensibly.

To clarify the meaning of the functions (3), we need to know an appropriate parametrization of the group of proper rotations of an r-dimensional space. In this space there are $r(r-1)/2$ planes which can be labeled by two indices p and t if it is assumed that $p = 2, 3, \ldots, r$ and $t = 1, 2, \ldots, p-1$. Rotation in the plane tp can be defined by an operator T_{tp}, which is realized as an r-dimensional real matrix which depends on the parameter ϑ_{tp} with matrix elements $(T_{tp})_{ij}$ that differ from the matrix elements of the unit matrix only by the two diagonal elements $(T_{tp})_{tt} = (T_{tp})_{pp} = \cos\vartheta_{tp}$ and the two nondiagonal elements $(T_{tp})_{tp} = -(T_{tp})_{pt} = \sin\vartheta_{tp}$. It is obvious that any rotation can be represented as a product of one-parameter rotations $T(\vartheta_{tp})$. It is important to note that the order in which these operators are arranged in their product is essentially unimportant because the T_{tp}'s form a complete set of different operators of rotation, i.e., operators that each implement a rotation in its corresponding plane. This parametrization of the general rotation of the r-dimensional space—the proper rotation group O_r^+—is convenient in the infinitesimal respect since, using it in the space of parameters ϑ_{tp} in the neighborhood of the identity element, one obtains one-parameter curves to which the various infinitesimal operators correspond.

However, this parametrization is too cumbersome for global methods, i.e., methods based on integration with respect to a group. There is another way of specifying rotation of r-dimensional space by means of operators that implement a rotation only in planes with $t = p-1$. The total number of parameters of the group must remain the same, and therefore for each plane $p-1$, p it is necessary to introduce v parameters, where $v = 2, 3, \ldots, p$; with each of these parameters there is associated an operator, so that instead of T_{tp} we have the operators $T^{(\vartheta)}_{p-1,p}$ defined by the r-dimensional matrices

$$T^{(\vartheta)}_{p-1\,p} = \begin{array}{c|ccc} & p-1 & & p \\ \hline & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} & 0 & 0 \\ p-1 & & c_p^{(\vartheta)} \quad s_p^{(\vartheta)} & \\ & 0 & & 0 \\ p & & -s_p^{(\vartheta)} \quad c_p^{(\vartheta)} & \\ \hline & 0 & 0 & \begin{matrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{matrix} \end{array} \qquad (4)$$

In (4), we have introduced the abbreviated notation $c_p^{(v)} = \cos\vartheta_{p-1\,p}^{(v)}$ and $s_p^{(v)} = \sin\vartheta_{p-1\,p}^{(v)}$. All the parameters ϑ with $p > 2$ take values in the range $0 \leq \vartheta_{p-1\,p}^{(v)} < \pi$, while the parameters ϑ with $p = 2$ take them in range $0 \leq \vartheta_{12}^{(v)} < 2\pi$.

Any proper rotation G_r^+ of r-dimensional space can be specified by a product of the operators $T_{p-1\,p}^{(v)}$, though now their order is important. Indeed, one cannot place operators with the same p next to one another because their product contains, not the two parameters $\vartheta_{p-1\,p}^{(v')}$ and $\vartheta_{p-1\,p}^{(v'')}$, but only their sum, i.e., one of these parameters is lost. It is therefore necessary to determine a definite correct order of multiplication of the operators $T_{p-1\,p}^{(v)}$. The requirement that operators with the same p's must not occur next to each other leaves a great freedom of choice, which can be partly removed by using a recursive construction. Suppose we have already chosen a definite order for constructing the operator G_{r-1}^+ of the proper rotation of the $(r-1)$-dimensional space and we need to make it into the operator G_r^+ of a rotation of r-dimensional space. To be specific, we shall do this "from the right", i.e., we set

$$G_r^+ = G_{r-1}^+ g_r^+, \qquad (5)$$

and fix g_r^+ by assuming that

$$g_r^+ = \prod_{p=r}^{2} T_{p-1\,p}^{(r)}. \qquad (6)$$

To analyze the structure of Eq. (5), let us write down in the case $r = 4$: $G_4^+ = T_{12}^{(2)} T_{23}^{(3)} T_{12}^{(3)} T_{34}^{(4)} T_{23}^{(4)} T_{12}^{(4)}$. It can be seen from this example that on the transition from the $(r-1)$-dimensional to the r-dimensional rotation only one essentially new rotation, $T_{r-1\,r}^{(r)}$, appears, its subscript r adding the new dimension. On the left of it, there are all the rotations of the space of $r-1$ dimensions; on the right only some. It is now clear that this parametrization is convenient in the global, integral respect because the number of essentially new rotations appearing on the transition from the group O_{r-1}^+ to O_r^+ is minimal. However, if we attempt to use this parametrization to find infinitesimal matrices, we face disappointment since differentiation with respect to the parameters gives only $r-1$ basis infinitesimal operators $I_{12}, I_{23}, \ldots, I_{r-1\,r}$. To determine the remaining $r(r-1)/2 - (r-1)$ operators, one must have recourse to their commutation relations.

The elements of the orthogonal group O_r^+, whose parametrization we have just obtained, implement a proper rotation of the r-dimensional space. There is a weighty reason that calls for the extension of this group in connection with the Pauli principle. Suppose that the variables $x_1, \ldots, x_r$, the objects to which a transformation of the group O_r^+ is applied, are the variables of particles that satisfy a definite statistics. To guarantee the permutation properties of functions that depend on these variables it is necessary to use permutation operators whose matrices have a determinant equal to -1; but the determinant of matrices of proper rotations is 1. We conclude that permutations are exterior operations with regard to proper rotations, i.e., the permutation operators act outside the framework of the algebraic formalism of the orthogonal groups O_r^+. It is therefore necessary to extend the group of orthogonal transformations, including in it not only operators of proper rotations but also permutation operators. This extension can be made by using a single numeral orthogonal matrix with determinant equal to -1 since its product with the matrices of proper rotations exhaust, because of the group properties of the latter, all the real orthogonal matrices of improper rotations as well. We take this matrix to be the diagonal matrix s_r which differs from the unit matrix e_r by only the sign of the rth element of the principal diagonal. We now introduce a group of reflections σ_r consisting of two elements $\sigma_r \equiv \{e_r, s_r\}$ in the form of r-order matrices and multiply, for example, these elements from the left by elements of the group of proper rotations. The elements G_r of the complete orthogonal group O_r can now be specified by means of the matrices $G_r = G_r^+ \sigma_r$, where σ_r are the elements of the reflection group. For what follows, it is necessary to perform also a recursive (with respect to r) construction of the elements of G_r. Taking into account (5), we have $G_r = G_{r-1}^+ g_r^+ \sigma_r = G_{r-1}^+ \sigma_{r-1} \sigma_{r-1}^{-1} g_r^+ \sigma_r$, and therefore

$$G_r = G_{r-1} g_r \quad (g_r = \sigma_{r-1}^{-1} g_r^+ \sigma_r). \qquad (7)$$

In mathematics, a set of elements like the elements g_r of the group O_r are usually called a factor space and denoted symbolically by $g_r = G_r / G_{r-1}$. We can write down the same thing for the group of proper rotations: $g_r^+ = G_r^+ / G_{r-1}^+$. Above, we have already encountered one further factor space, $\sigma_r = G_r / G_r^+$, which arose when O_r^+ was extended to O_r.

If we have an arbitrary group G and a subgroup of it H, then we can introduce the factor space $g = G/H$. Under additional conditions, when H is invariant under inner automorphisms (this means that $gHg^{-1} = H$ for all g, i.e., H is a normal divisor of G), the factor space becomes a factor group. In our case, the sets G_r/G_{r-1} or G_r^+/G_{r-1}^+ are only factor spaces, whereas G_r/G_r^+ is a factor group.

The concept of a factor space and the convenient notation for it enable us to write the formulas for the transition to a spherical coordinate system in the multidimensional space in a compact form. In accordance with (6) and taking into account the reflection operators, we obtain for the r-dimensional matrix $D^{(1r)}$, which depends on the elements of the factor space G_r/G_{r-1},

$$D^{(1r)}(g_r) = \sigma_{r-1}^{-1} \prod_{p=r}^{2} T_{p-1\,p}^{(r)} \sigma_r. \qquad (8)$$

The last formula shows that the elements g_r of this factor space are defined by means $r-1$ continuous variables $\vartheta_{12}^{(r)}, \vartheta_{23}^{(r)}, \ldots, \vartheta_{r-1\,r}^{(r)}$ and the two discrete variables σ_{r-1} and σ_r.

We now give an equation that brings out the algebraic essence of the transition to the spherical coordinate system. Let D^{1r}_{ri} be the matrix elements of the last row of the matrix (8), and $x_1, \ldots, x_r$ the Cartesian coordinates of a vector of r-dimensional space. Then the transition to new angular and discrete variables of the multidimensional spherical coordinate system is made by means of the relation

$$x_i = \rho D^{(1r)}_{ri}(g_r), \tag{9}$$

where $i = 1, 2, \ldots, r$ and ρ is the multidimensional (global) radius $\rho = (\sum_i x_i^2)^{1/2}$. Direct multiplication in (8) enables us to prove readily that the last row of the matrix (8) does not depend on σ_{r-1}, so that only the reflection group σ_r participates in the change of variables (9). For clarity, we give the expanded formula (9) for $r = 3$. If the operators in (8) are written out in the form of matrices and they are multiplied, we readily find that

$$x_1 = \rho \sin \vartheta^{(3)}_{23} \sin \vartheta^{(3)}_{12}; \quad x_2 = -\rho \sin \vartheta^{(3)}_{23} \cos \vartheta^{(3)}_{12}; \quad x_3 = (-1)^{\sigma_3} \rho \cos \vartheta^{(3)}_{23}, \tag{10}$$

where $(-1)^{\sigma_3} = 1$ for $\sigma_3 = e_3$ and $(-1)^{\sigma_3} = -1$ for $\sigma_3 = s_3$. We reverse the signs of the two angles ϑ in (10), which is the same as choosing opposite signs of the nondiagonal elements of the rotation matrices (4), and we set $\sigma_3 = e_3$ in (10). Then this change of variables leads to the usual formula for the transition to a spherical coordinate system.

In order to understand why we can write the change of variables in the form (9), we must recall that the matrix elements of matrices of irreducible representations of the orthogonal group form a complete system of functions that depend on $r(r-1)/2$ continuous variables. Because of the irreducible properties of the coordinates $x_1, \ldots, x_r$, we can, when choosing the functions for the transition to the new variables, take only those matrix elements of D, of the representation $(1r)$, that depend on only $r-1$ continuous parameters. In the case of our parametrization, this condition is satisfied by the functions that form the last row of the matrix $D^{(1r)}$. It is they that occur in Eq. (9), in which the O_r-invariant variable ρ is chosen for obvious reasons as the rth new variable.

Note also that in the case of an arbitrary group G and subgroup H of it the formulas for the change of variable by means of matrix elements of matrices defined on the factor space G/H enable one to penetrate to the algebraic essence of the connection between the different coordinate systems used in mathematical physics. The true meaning of this assertion can be understood only on the basis of the connection between the theory of special functions and the algebraic formalism of induced representations.

3. COLLECTIVE AND INTERNAL VARIABLES OF THE NUCLEUS

Formula (9), which at first glance has nothing remarkable about it, provides the algebraic tool for answering the questions posed earlier. We now know in which class of functions it is necessary to seek the functions (3) of the transition from the Jacobi coordinates to new translationally invariant variables of the nucleus: We have the product of the variable ρ and the elements of the matrices of the irreducible representation $D^{(1r)}$ of the orthogonal group O_r for $r = 3(n-1)$.

It would appear that the required formula for the change of variables (3) can be obtained from (9) for $r = 3(n-1)$, and that this will give the transition from ρ_i^s to the new spatial variables of the nucleus. To see whether this is true or not, we require constructive assertions on the meaning of the new variables. What criteria can one use to sort the microscopic variables into collective variables and internal variables of the nucleus? Can one carry out a sorting in the set of variables ρ, $\vartheta^{(3(n-1))}_{12}, \vartheta^{(3(n-1))}_{23}, \ldots, \vartheta^{(3(n-1))}_{3(n-1)-1,3(n-1)}$, which must be chosen on the basis of the above parametrization of the factor space g_r with $r = 3(n-1)$, or must one find a different parametrization more natural for our purposes? In order to understand what is meant by a natural parametrization of the group $O_{3(n-1)}$ for the choice of new variables of the nuclear wave function, we must first of all answer the first of these questions.

We begin by determining the collective variables. We shall say that the collective variables are those selected variables, constructed from the microscopic variables of the nucleus, which, generally speaking, can exist independently of the other variables. This definition, which is as yet extremely vague, can be given a rigorous meaning by recalling that it is the Pauli principle that mixes the variables of a quantum system of identical particles. The requirements of the Pauli principle can be guaranteed by means of permutation operators, i.e., by means of elements of the symmetric group S_n. Therefore, variables which "do not feel" the Pauli principle, i.e., are invariants of the symmetric group S_n, are the only variables which can be independent of the others. By definition, we shall call them the collective variables ξ. We shall say that the remaining variables q are the internal variables of the nucleus.

In order to stay within a kinematically correct theory, these rigorous definitions must be used exclusively on the basis of conditions 3 and 4 formulated earlier. Indeed, proceeding, for example, from r variables $x_1, \ldots, x_r$, we can always construct r functions that are invariant under transformations of the group S_r, and thus introduce the collective variables of the nucleus corresponding to them. However, the wave functions that depend on such variables will necessarily be S_r-invariant functions, so that such a construction precludes fulfillment of the Pauli principle, and therefore condition 4. It follows that for given n the number of collective variables of the nucleus must be strictly bounded. In other words, the Pauli principle can be fulfilled only by means of a definite number of internal variables of the nucleus, and the problem consists of finding a minimal set of these, minimality being understood here in the sense that a further restriction precludes fulfillment of the Pauli principle.

To find the minimal set of internal variables of the nucleus, it is necessary to consider more closely the significance of the symmetric group for sorting the variables into collective and internal variables. And to test the "reaction" of the variables to the action of operators of the symmetric group S_n, it is necessary

to understand how this group is contained in the orthogonal group $O_{3(n-1)}$. The operators of the symmetric group permute the single-nucleon vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Using the expressions (1), one can find the transformation of the Jacobi coordinates induced by permutation of the variables $\mathbf{r}_i$. The permutations act on the indices i of the vectors $\mathbf{r}_i$, from which it follows that the elements of the symmetric group S_n act only on the subscripts i of the Jacobi vectors ρ_i, generating on the basis of these vectors an irreducible representation of S_n (see Ref. 9 for details). If transformations of the orthogonal group O_{n-1} are applied to the vectors $\rho_1, \ldots, \rho_{n-1}$, it becomes obvious that any transformation of the symmetric group can be represented as a product of proper rotations of $(n-1)$-dimensional space and reflection operations; it was because of this possibility that the orthogonal group of proper rotations was extended to the complete orthogonal group. In other words, the symmetric group S_n, which permutes the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$, is a subgroup of the orthogonal group O_{n-1} transforming the vectors $\rho_1, \ldots, \rho_{n-1}$, where ρ_i and $\mathbf{r}_i$ are related by Eqs. (1). This assertion is written in abbreviated form as $O_{n-1} \supset S_n$, i.e., the symmetric group S_n is contained in the orthogonal group O_{n-1} in accordance with the reduction $O_{n-1} \supset S_n$. Note that the rank of the orthogonal group is one less than the symmetric group's, so that S_n is contained very densely in O_{n-1}.

It is now not difficult to see how the symmetric group S_n is contained in the orthogonal group $O_{3(n-1)}$; this is suggested by the indices of the Jacobi coordinates ρ_i^s. The presence of the two indices $s = x, y, z$ and $i = 1, 2, \ldots, n-1$ means that the labeling is adapted to the direct product of the 3×3 matrix which rotates "our" three-dimensional space and the $(n-1) \times (n-1)$ matrix which rotates the abstract $(n-1)$-dimensional space of the Jacobi vectors. The direct product of such matrices corresponds to the direct product of the groups O_3^+ and O_{n-1}, and it is therefore natural to introduce the reduction $O_{3(n-1)} \supset O_3^+ \times O_{n-1}$, which means that the elements of the orthogonal group $O_{3(n-1)}$ are restricted to the direct product of elements of O_3^+ and O_{n-1}. In this restriction, it is assumed that the matrix of reflection of the $3(n-1)$-dimensional space is replaced by the matrix of reflection of the $(n-1)$-dimensional space; this replacement is justified since to allow for reflection one need use only a single numerical matrix with determinant equal to -1.

Let us now particularize formula (3). We have shown above that in this formula the factor space cannot be understood as the factor space g_r with $r = 3(n-1)$, which is parametrized by the method described above; a parametrization of this kind is suited to the canonical reduction $O_{3(n-1)} \supset O_{3(n-1)-1} \supset \cdots \supset O_1$, whereas the symmetric group is contained in the densest way possible only in the physical reduction $O_{3(n-1)} \supset O_3^+ \times O_{n-1} \supset O_3^+ \times S_n$. To find the meaning of formula (3) for such an imbedding, we write the elements $g_{3(n-1)}$ of the group $O_{3(n-1)}$ in the form $g_{3(n-1)} = \tilde{g} G_3^+ q_{n-1}$, where G_3^+ are the parameters of the rotation group O_3^+ (the Eulerian angles); q_{n-1} are parameters of the group O_{n-1}, whose meaning we have yet to clarify, and $\tilde{g}$ are the remaining parameters that augment the set $G_3^+ q_{n-1}$ to a complete set of $3(n-1)-1$ variables. The formula (9) of the change of variables now takes the form

$$\rho_i^s = \rho D^{(1_{3(n-1)})}_{s_0 i_0,\, si}(\tilde{g} G_3^+ q_{n-1}), \tag{11}$$

where $s_0 i_0$ denotes the last row of the $3(n-1)$-dimensional matrix $D^{(1_{3(n-1)})}$; ρ is the global radius of the nucleus.

To find the meaning of the parameters of the matrix $D^{(1_{3(n-1)})}$, we must write it as a product of three matrices depending, respectively, on $\tilde{g}$, G_3^+, and q_{n-1}, and use the well known properties of the matrix elements of matrices of irreducible representations. This question is considered in detail in Ref. 8, and we give here only the final result. Equation (11) has the expanded form[5]

$$\rho_i^s = \sum_{s'} \rho^{(s')} D^{(1_3)}_{s's}(G_3^+)\, D^{(1_{n-1})}_{s'i}(q_{n-1}), \tag{12}$$

where $D^{(1_3)}$ and $D^{(1_{n-1})}$ are a three-dimensional and an $(n-1)$-dimensional matrix of irreducible representations of O_3^+ and O_{n-1};

$$\rho^{(s')} = \rho D^{(1_{3(n-1)})}_{s_0 i_0,\, s's'}(\tilde{g}). \tag{13}$$

In (12), $q_{n-1} = g_{n-3} g_{n-2} g_{n-1}$, which can also be written in the form $q_{n-1} = \sigma_{n-1}^{-1} g_{n-3}^+ g_{n-2}^+ g_{n-1}^+ \sigma_{n-1}$, where the g^+'s are determined by the expression (6). It is not difficult to show that q_{n-1} depends on $3(n-3)$ continuous parameters In accordance with the remark made after Eq. (8), the factor spaces g_{n-3}^+, g_{n-2}^+, and g_{n-1}^+ are defined by $n-2$, $n-3$, and $n-4$ continuous parameters, whose sum gives $3(n-3)$. If we also take into account the three parameters of the group O_3^+ and the global radius ρ, then in the case $n > 3$ the overall balance of the variables, $3(n-1) - 3(n-3) - 3 - 1 = 2$, shows that the remaining factor space $\tilde{g} = G_{3(n-1)}/G_3^+ \times G_{n-1}$ is specified by just two continuous parameters, which we denote by ϑ_1 and ϑ_2, where $0 \leq \vartheta_1$, $\vartheta_2 < -\pi/2$. When $n = 3$, we have the variable ρ, the three Eulerian angles, and the single parameter of O_2, and the balance of the variables shows that the factor space $\tilde{g}$ is defined solely by the single variable ϑ_1. The first thing which must be noted about (12) is the correlation between the rows of the matrix $D^{(1_{n-1})}$ and the rows of the matrix $D^{(1_3)}$, this being due to the properties of the elements of the matrix $D^{(1_{3(n-1)})}(\tilde{g})$, which has the form

si \ $s'i'$	$x1 \ldots$	$x\ n-3$	$y\ n-2$	$z\ n-1$
$x1$	$\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}$	0	0	0
$x\ n-3$	0	C_2	S_2	0
$y\ n-2$	0	$-C_1 S_2$	$C_1 C_2$	S_1
$z\ n-1$	0	$S_1 S_2$	$-S_1 C_2$	C_1

(14)

In (14) we have introduced the notation $c_2 = \cos\vartheta_2$, etc. It can be seen directly from this matrix that its elements are nonzero only when $s' = i'$, which dictates the distinctive labeling of the rows of the matrix $D^{(1_{n-1})}$: Eq. (12) contains the matrix elements of the last

three rows of the matrix $D^{(1_{n-1})}$, and for $n>3$ its rows

$$s'=x',\ s'=y',\ \text{and}\ s'=z'$$

correspond to the rows

$$i'=n-3,\ i'=n-2,\ \text{and}\ i'=n-1$$

of the matrix $D^{(1_{n-1})}$. If $n=3$, then in (14) we must set $\vartheta_2=0$, and we then find that the values $s'=y'$ and $s'=z'$ are correlated with the first and the second row of the matrix $D^{(1_2)}$ of the representation of O_2.

We have already pointed out that the operations of the symmetric group are internal operations of the group O_{n-1}. The variables $\rho^{(s)}$ and G_3^+ are not affected by the operators of the group O_{n-1}, i.e., all these six variables are invariants of S_n. However, in accordance with our definition, S_n-invariant variables are called collective variables of the nucleus, and the results presented above therefore show that for $n>3$ there exist at least six collective variables of a nucleus: the three Eulerian angles and the three variables $\rho^{(s)}(s=x,y,z)$ of radial type, which take values in the range $0\leqslant\rho^{(s)}<\infty$; for $n=3$, the group O_{n-1} degenerates into the group O_2, and instead of the three $\rho^{(s)}$ there remains only the two variables $\rho^{(s)}(s=y,z)$. For completeness, we can also consider the almost trival case $n=2$. Then the group O_{n-1} degenerates into the discrete reflection group O_1, for whose parametrization no continuous variables are required, in connection with which all the three variables of the two-nucleon problem ($\rho^{(z)}$ and the two Eulerian angles) are collective variables.

Do there exist other collective variables of the nucleus? We now know that this question is equivalent to the following: Does there exist a further continuous group G_0 that can be inserted between the groups O_{n-1} and S_n, i.e., does there exist a reduction $O_{n-1}\supset G_0\supset S_n$ with nontrivial continuous group G_0?

If the answer to this question is affirmative, then the additional parameters which appear on the extension of G_0 to the group O_{n-1} will be S_n-invariant, i.e., collective variables of the nucleus. One can show that such a group G_0 does not exist. Without going into the formal aspect of this proof, we sketch only the stages. The group O_{n-1} is compact, and therefore G_0 must also be compact. All compact groups have been classified and are well known, and therefore this assertion can be readily verified for small n, for example, $n=2,3,4$, and then, by induction, the proof can be found for any n.[1)]

Thus, we have established the maximal possible number of collective variables of the nucleus. We formulate the final result in the form of the following theorem.

Theorem 1. Without violating microscopic translational invariance and antisymmetry of the wave function of a nucleus consisting of n nucleons, one can introduce six collective variables for $n>3$, five collective variables for $n=3$, and three collective variables for $n=2$. The remaining $3(n-3)$ continuous variables with reflection for $n>3$, the single continuous variable with reflection for $n=3$, and the reflection for $n=2$ are essentially noncollective (internal) variables of the nucleus.

It should be noted that in the formulation of the theorem we speak about nucleons, and this, as was agreed earlier, means that the protons and neutrons are distinguished by means of the isospin projection. In other words, the condition of the theorem requires antisymmetry with respect to all variables of the nucleus, so that the "watchman" which prevents our finding more than six collective variables is the symmetric group S_n.

Let us consider the other variant of the theory in which protons and neutrons are treated as different particles, so that there is no need for antisymmetry under permutation of protons with neutrons. Suppose the nucleus consists of n_1 protons and n_2 neutrons and that $n_1+n_2=n$. We introduce the orthogonal groups O_{n_1-1} and O_{n_2-1} and using the reductions $O_{n_1-1}\supset S_{n_1}$ and $O_{n_2-1}\supset S_{n_2}$ imbed the symmetric groups S_{n_1} and S_{n_2} in them to fulfil the Pauli principle for protons and neutrons separately. In accordance with Theorem 1, the proton system is described by $3(n_1-3)$ internal variables and the neutron system by $3(n_2-3)$. The group O_{n-1} contains $3(n-3)$ variables, some of which, namely $3(n_1+n_2-3)-3(n_1-3)-3(n_2-3)=9$, now become collective variables of the nucleus. From this we conclude that for the proton–neutron system the following theorem holds.

Theorem 2. Without violating the requirements of microscopic translational invariance and antisymmetry of the wave function of a nucleus consisting of n_1 protons and n_2 neutrons ($n_1>3$ and $n_2>3$), one can introduce 15 collective variables.

To avoid excessive pedantry, we have not included in the formulation of Theorem 2 the cases $n_1=1,2,3$ and $n_2=1,2,3$ and, in addition, we have not mentioned the number of internal variables of the nucleus. This theorem does not give recommendations how the collective variables should be chosen, because their explicit form depends on the particular physical problem to be solved. If one needs to separate the degrees of freedom of the nucleus responsible for the relative motion of the proton and neutron subsystems, then the collective variables must be the six S_{n_1}-invariant proton variables, the six S_{n_2}-invariant neutron variables, and added to them the three $S_{n_1}\times S_{n_2}$-invariant variables specifying the relative position of the centers of mass of the two subsystems. However, from the point of view of kinematics, these 15 variables are not optimal since they do not include the six truly collective S_n-invariant variables that in accordance with Theorem 1 can certainly be introduced for a nucleon nucleus, and *a fortiori* for a proton–neutron nucleus. Therefore, if it is necessary to have more than six collective variables, it is more attractive from the theoretical point of view to add to the six true variables nine additional variables that are invariants of only the group $S_{n_1}\times S_{n_2}$. The explicit connection between these two different

[1)] In the search for a rational method of classifying repeated representations of the symmetric group S_n, such a proof was given in Ref. 10 on the basis of the reduction $O_{3(n-1)}\supset S_n$. Here it is sufficient to consider the simpler reduction $O_{n-1}\supset S_n$.

sets of 15 collective variables can be established by means of Eq. (27) of Ref. 5.

The second of these sets enables one to introduce in a natural manner the concept of approximate collective variables of a nucleon nucleus, the use of which leads to breaking of the exact conservation of the nuclear isospin quantum number. The possibility cannot be excluded that it is this path which will make it possible to understand the microscopic meaning of the collective degrees of freedom of a nucleus of higher multipolarities used in phenomenological theories.

4. GENERAL METHOD OF PROJECTING COLLECTIVE AND INTERNAL WAVE FUNCTIONS OF THE NUCLEUS

To advance further in the study of the collective and internal degrees of freedom of the nucleus, it is necessary to give a rigorous mathematical meaning to the concept, frequently used in phenomenological nuclear theory, of the nuclear (moving) and original (fixed) coordinate system and, in passing, recall shift operators on a group. It can be seen from the expression (12) that the functions by means of which the transition is made from the Jacobi coordinates to the new variables of the nucleus depend on parameters that specify the elements of rotation groups. The elements of the matrix $D^{(1_3)}$ depend on all the parameters of the group O_3^*, whereas the elements of the matrix $D^{(1_{n-1})}$ depend only on the parameters of the factor space $q_{n-1} = G_{n-1}/G_{n-4}$. In mathematics, functions of the first type are usually called functions defined on the group G, while functions of the second type are said to be functions defined on the factor space $g = G/H$. When H is the trivial subgroup consisting of the identity element, the factor space coincides with the complete group G and therefore, without loss of generality, one can always speak of functions defined on the factor spaces g constructed for the given group G by means of all possible subgroups H of G.

Equation (12) contains the matrix elements of matrices of irreducible representations. To clarify the concept of an induced representation, the property of irreducibility is not important, and therefore, generalizing the class of functions considered, we shall speak of functions defined on the factor space g, these being elements of the matrix of some (not necessarily irreducible) representation of G. Such matrix elements no longer have a superscript, and we therefore denote them by $B_{i'i}(g)$.

The reaction of these functions to the action of the rotation operators $\hat{T}(G_r)$ of the orthogonal group O_r [in the general case, the operators $\hat{T}(G)$ defined on an arbitrary group G] can be tested by giving a meaning to the expression $\hat{T}(G)B_{i'i}(g)$. In the physics literature it is as a rule assumed that application of $\hat{T}$ to $B_{i'i}$ yields a function of the new argument $G^{-1}g$. In mathematics, operators of this type are usually called operators of a left shift on the group G and are denoted by the letter $\hat{L}$. Decomposing $\hat{L}B$ in terms of the original set of functions, we obtain

$$\hat{L}(G)\,B_{i'i}(g) = B_{i'i}(G^{-1}g) = \sum_{i''} B_{i'i''}(G^{-1})\,B_{i''i}(g). \tag{15}$$

This last formula shows that the operators of a left shift transform the rows of elements of the matrix B, and this transformation does not depend on the number of the column.

We now give a different meaning to the expression $\hat{T}(G)B_{i'i}(g)$. Suppose that by applying $\hat{T}$ to $B_{i'i}$ we obtain a function of the new argument gG. We shall say that operators of this type are operators of a right shift on the group G and denote them by $\hat{R}$. Decomposing the resulting function in terms of the original functions, we obtain

$$\hat{R}(G)\,B_{i'i}(g) = B_{i'i}(gG) = \sum_{i''} B_{i'i''}(g)\,B_{i''i}(G). \tag{16}$$

It can be seen from (16) that the operators of a right shift transform the columns of the elements of the matrix B, and this transformation does not depend on the number of the row. Using $\hat{L}$ and $\hat{R}$, we can transform the rows and columns of the matrix B, and therefore the operators of left and right shifts form a complete set of operators that transform functions defined on the factor space g.

The operators $\hat{L}$ and $\hat{R}$, which are adapted to the neighborhood of the identity element for an infinitesimally small shift in accordance with correctly chosen single-parameter subgroups of the group G, give left and right infinitesimal operators. In the formulation of the kinematics of microscopic nuclear theory one also requires global (defined on the group) infinitesimal operators. Frequently encountered examples of the latter are the projections of the operator of three-dimensional rotation onto the axes of the original and the moving coordinate system. It is well known that the commutation relations of the operators in the moving coordinate system differ by their sign from the commutation relations of the operators in the original coordinate system. This difference between their algebras reflects the fact that they are either right or left infinitesimal operators of rotation acting on the left or right indices of the matrix elements $D^L_{KM}(G_3^*)$.

Let us consider the behavior of the variables ρ_i^s under the action of the operators of a right shift. By means of (16), we readily obtain

$$\hat{R}(G_{03}^+)\,\hat{R}(G_{n-1})\;\rho_i^s = \sum_{s'} \rho^{(s')} D^{(1_3)}_{s's}(G_3^+ G_{03}^+)\,D^{(1_{n-1})}_{s'i}(q_{n-1}G_{n-1}). \tag{17}$$

Suppose, in particular, G_{n-1} is the identity element of the group O_{n-1}, and $G_{03}^+ = (G_3^*)^{-1}$. Then (17) gives

$$R((G_3^+)^{-1})\,\rho_i^s \equiv \overset{0}{\rho}{}_i^s = \rho^{(s)} D^{(1_{n-1})}_{si}(q_{n-1}). \tag{18}$$

What is the meaning of this last formula? Suppose that in three-dimensional space we have the original and the moving coordinate system and the moving system is specified with respect to the original by the Eulerian angles of G_3^*. The operators $\hat{R}((G_3^*)^{-1})$ define the rotation from the original to the moving system, and therefore if ρ_i^s are variables of the nucleus defined in the original system, $\overset{0}{\rho}{}_i^s$ are the variables in the moving system. To check this assertion once more, let us compare the symmetric second-rank tensors $\Pi^{ss'} = \sum_i \rho_i^s \rho_i^{s'}$ and $\overset{0}{\Pi}{}^{ss'} = \sum_i \overset{0}{\rho}{}_i^s \overset{0}{\rho}{}_i^{s'}$, in the two coordinate systems. Using (12) and (18), we can readily show that $\overset{0}{\Pi}{}^{ss'} = \delta(ss')(\rho^{(s)})^2$, i.e., as must be, the transition to the moving coordinate system reduces $\Pi^{ss'}$ to its principal axes.

The introduction of collective variables of the nucleus and the associated reduction to principal axes of the tensor $\Pi^{ss'}$, whose components are made up of single-nucleon variables averaged with respect to the single-particle indices, was considered in Ref. 11. Equation

(18) in the case of the factor space without reduction, $q^+_{n-1}=g^+_{n-3}g^+_{n-2}g^+_{n-1}$, was obtained for the first time explicitly in Ref. 2 (see also Ref. 4), which was the starting point for further work in this direction.

The transition to the principal axes of the tensor $\Pi^{ss'}$ is instructive in a further respect: It enables one to understand the mathematical essence of the transition to the moving coordinate system. Indeed, if one starts with components of $\Pi^{ss'}$ that are functions depending on the parameters G^+_3 of the group O^+_3 (for further generalizations it is better to say parameters of the factor space G^+_3/G^+_1, where G^+_1 is a trivial group) and also the other variables $\rho^{(s)}$, then one can obtain new functions $\overset{0}{\Pi}{}^{ss'}$ defined in the moving coordinate system. This system, in its turn, is determined by the factor space G^+_3/G^+_1, and the tensors $\Pi^{ss'}$ and $\overset{0}{\Pi}{}^{ss'}$ are related by $\overset{0}{\Pi}{}^{ss'}=\hat{R}((G^+_3/G^+_1)^{-1})\Pi^{ss'}$. We conclude from this that the transition to the moving coordinate system is made by means of the operator of a right shift $\hat{R}$ taken at the "point" $(G^+_3/G^+_1)^{-1}$.

It is now clear how, generalizing this example, we can formulate in group language the rule for the transition to a "moving" coordinate system in the case of any group G and a subgroup H of it. Suppose we have a function F defined on the factor space g, which, perhaps, also depends on the set of other variables ξ. Then the function $\overset{0}{F}$ in the coordinate system that is "moving" with respect to the factor space $g=G/H$ is related to the function F by

$$\overset{0}{F}(\xi)\equiv\hat{R}(g^{-1})F(\xi, g)=F(\xi, gg^{-1}). \tag{19}$$

Let us continue our example. Using the operator of a right shift we can not only obtain the tensor $\overset{0}{\Pi}{}^{ss'}$ from the tensor $\Pi^{ss'}$, but we can even realize a decomposition of $\Pi^{ss'}$ in terms of $\overset{0}{\Pi}{}^{ss'}$. For this, it is sufficient to write out the identity $\Pi^{ss'}\equiv\hat{R}\hat{R}^{-1}\Pi^{ss'}$, where $\hat{R}^{-1}$ is the operator which is the inverse of $\hat{R}$, i.e., $\hat{R}^{-1}$ is $\hat{R}$ taken at the "point" G^+_3/G^+_1. Applied to $\Pi^{ss'}$, the operator $\hat{R}^{-1}$ transforms the components of this tensor in accordance with (16). Therefore,

$$\hat{R}^{-1}\Pi^{ss'}=\hat{R}(G^+_3)\Pi^{ss'}=\sum_{s_0s_0'}\Pi^{s_0s_0'}B_{s_0s_0', ss'}(G^+_3). \tag{20}$$

Using (12) and the explicit form of $\Pi^{ss'}$ we can readily verify that the reducible matrix B is the direct product $D^{(1_3)}\times D^{(1_3)}$. Applying further the operator $\hat{R}$ to (20), we project $\overset{0}{\Pi}{}^{s_0s_0'}$ from $\Pi^{s_0s_0'}$, obtaining as a result

$$\Pi^{s's}=\sum_{s_0}\overset{0}{\Pi}{}^{s_0s_0}D^{(1_3)}_{s_0s}(G^+_3)D^{(1_3)}_{s_0s_0'}(G^+_3). \tag{21}$$

In deriving (21), we have also used the diagonality of the tensor $\overset{0}{\Pi}{}^{s_0s_0}$.

The generalization of this example for a set of any functions F_γ defined on the factor space g leads to the formula

$$F_\gamma(\xi, g)\equiv\hat{R}(g^{-1})\hat{R}(g)F_\gamma(\xi, g)=\sum_{\gamma'}\overset{0}{F}_{\gamma'}(\xi)B_{\gamma'\gamma}(g). \tag{22}$$

It is readily verified that the matrices B form a reducible representation of the group G. The decomposition (22) is possible only in the case when the set of functions F_γ is complete with respect to the operators of a right shift, i.e., when F_γ forms a complete basis for the matrix representation of the operator $\hat{R}(G)$. It is in this sense that we spoke earlier about completeness of the class of wave functions (2).

Expansions of the type (22) are encountered in many branches of theoretical physics. We have, for example, the expansion of a plane wave in terms of a product of spherical functions and Bessel functions, expansions of vector fields in terms of vector spherical harmonics, expansion of a central potential in terms of Legendre polynomials, and so forth.

Thus, we have approached closely the problem of separating the collective and internal components of the wave function (2). It is solved very simply by means of the formula (22). We write down the expression (12) for the variables ρ^s_i in the coordinate system that is moving with respect to the factor space $q_{n-1}=G_{n-1}/G_{n-4}$:

$$\hat{R}(q^{-1}_{n-1})\rho^s_i=\begin{cases}0 & \text{for } i=1, 2, \ldots, n-4\\ \rho^{(i)}D^{(1_3)}_{is}(G^+_3) & \text{for } i=n-3,\ n-2,\ n-1\end{cases} \tag{23}$$

and use the expansion (22) for the wave function (2). We obtain

$$\Psi(\Gamma|\rho^s_1, \ldots, \rho^s_{n-1}; Q)=\sum_{\Gamma'}\overset{0}{\Psi}(\Gamma'|\xi Q)B_{\Gamma'\Gamma}(q_{n-1}). \tag{24}$$

Here and in what follows, ξ stands for the six collective variables $\rho^{(s)}G^+_3$, and $\overset{0}{\Psi}$ is the original wave function, in which the arguments ρ^s_i are replaced by their values given by (23). We now understand the meaning of the term "good" operators discussed earlier—they are operators $\hat{R}$ of the compact and therefore unquestionably "good" group O_{n-1}. The remark following Eq. (22) also explains what is meant by completeness of the set of functions (2).

Equation (24) is unsatisfactory in that the collective function $\overset{0}{\Psi}$ in it depends on the spin—isospin variables of the nucleus. The formalism of orthogonal groups, which is well suited to study of spatial variables, in no way affects the spin coordinates, and therefore they remain in the final expression (24) in the place they were from the very start—in the wave function $\overset{0}{\Psi}$. Since we are studying spatial degrees of freedom of the nucleus, we ought to exclude the variables Q from the very start when formulating the problem. Such a possibility exists because of the supermultiplet scheme proposed in the thirties by Wigner.[12]

Let us briefly recall the essence of the supermultiplet scheme. Suppose S_n is the symmetric group that permutes the spatial variables of the nucleus, and S'_n is the symmetric group that permutes either the spin (for a proton—neutron nucleus) or the spin—isospin (for a nucleon nucleus) variables of the nucleus. To separate the orbital variables from the spin—isospin variables, we must assume that the quantum numbers of the wave functions contain irreducible representations of the group S_n and S'_n, which are usually called Young patterns.

We shall assume that the spatial wave function, which depends on the variables ρ^s_n, is characterized by the quantum numbers $\Gamma_0LM\lambda\mu$, where L and M are the total orbital angular momentum of the nucleus and its projec-

tion; λ and μ are an S_n-irreducible representation and its basis; and Γ_0 is an arbitrary set of the remaining quantum numbers. Suppose further that the spin–isospin (or spin) function, which depends on the variables Q, is characterized by the quantum numbers $\Gamma_S S M_S \lambda' \mu'$, where S and M_S are the spin angular momentum of the nucleus and its projection; λ' and μ' are an S_n'-irreducible representation and its basis; and Γ_s are the remaining quantum numbers. It is well known (see, for example, Ref. 9) what is the meaning of the set Γ_S and how one must construct the spin–isospin function, and therefore there is no need here to go into more detail about its properties, since in what follows we shall be concerned solely with orbital functions. From the set of their quantum numbers Γ_0 we require only completeness with respect to an expansion of the type (22). The quantum numbers of the orbital and the spin–isospin wave functions necessarily contain the quantum numbers of the orbital, L, and spin, S, angular momenta and their projections M and M_S, because only when they are used can one ensure an exact integral of the motion of the nucleus—the total angular momentum J and its projection M_J.

The symmetric groups S_n and S_n' act in spaces that are independent of each other, in connection with which the total antisymmetric function of the nucleus can be constructed by coupling the representations λ and λ' by means of the Clebsch–Gordan coefficients of the symmetric group into a resultant one-dimensional antisymmetric representation a. It is well known that under this coupling λ' and μ' are uniquely correlated to λ and μ; to emphasize this, we shall not write $\lambda'\mu'$ but rather $\tilde{\lambda}\tilde{\mu}$. Using the Clebsch–Gordan coefficient of the group O_3^+, we also couple L and S into J, and we obtain the total antisymmetric wave function of the supermultiplet scheme:

$$\Psi\begin{pmatrix}\Gamma_0\Gamma_s(LS)JM_J\\(\lambda\tilde{\lambda})\,a\end{pmatrix}=\sum_{\mu}\Psi\begin{pmatrix}\Gamma_0 LM\\ \lambda\mu\end{pmatrix}\Psi\begin{pmatrix}\Gamma_s SM_s\\ \tilde{\lambda}\tilde{\mu}\end{pmatrix}C^{LSJ}_{MM_SM_J}C^{\lambda\tilde{\lambda}a}_{\mu\tilde{\mu}}. \quad (25)$$

In (25), we have deliberately used the Clebsch–Gordan coefficients of the symmetric group instead of the usually employed factor $(d_\lambda)^{-1/2}$, where d_λ is the dimension of the S_n-irreducible representation λ; this form of expression leaves a freedom in the choice of the phases of these coefficients, which may sometimes be important.

At a comparatively low price—the introduction of the approximate integrals of the motion $\Gamma_S L S_\lambda$—we have succeeded in avoiding the variables Q and transferring the problem of separating the collective variables to the spatial function in the expansion (25). The technique of projecting the collective and internal functions from it differs in no way from the technique used to derive Eq. (24). Moreover, as a result of the appearance of the quantum numbers LM among the quantum numbers of the orbital function, we obtain the possibility of separating explicitly the dependence of this function on the Eulerian angles G_3^+. We now write down the expression for the variables ρ_i^s in the moving coordinate system with respect to G_3^+ and q_{n-1}:

$$\hat{R}\left((G_3^+)^{-1}\right)\hat{R}\left(q_{n-1}^{-1}\right)\rho_i^s \equiv \overset{00}{\rho}{}_i^s = \rho^{(s)}\delta(si), \quad (26)$$

and then, using the general formula (22), we readily find that

$$\Psi\begin{pmatrix}\Gamma_0 LM\\ \lambda\mu\end{pmatrix}\Big|\rho_1,\ldots,\rho_{n-1}\Big)=\sum_{\Lambda_0}\Theta(\Lambda_0 LM\,|\,\xi)\,B_{\Lambda_0,\,\Gamma_0\lambda\mu}(q_{n-1}), \quad (27)$$

where B has the same meaning as in (24); the collective function Θ depends on the Eulerian angles as follows:

$$\Theta(\Lambda_0 LM\,|\,\xi)=\sum_K\Theta_0(\Lambda_0 LK\,|\,\rho^{(x)}\rho^{(y)}\rho^{(z)})D^L_{KM}(G_3^+); \quad (28)$$

we obtain Θ_0 from the original set of functions Ψ by means of the formula

$$\Theta_0(\Lambda_0 LK\,|\,\rho^{(x)}\rho^{(y)}\rho^{(z)})=\Psi(\Lambda_0 LK\,|\,\rho_1^s=0,\ldots,\rho_{n-4}^s=0,\ \rho_{n-3}^{(x)}=\rho^{(x)},\ \rho_{n-3}^y=0,\ldots,\rho_{n-1}^y=0,\ \rho_{n-1}^z=\rho^{(z)}). \quad (29)$$

In (28), K is the projection of the angular momentum of the nucleus onto the moving axis z. Expressions of the type (28) are usually used in phenomenological collective models of the nucleus for the expansion of the collective wave function with respect to the elements of the matrix D^L.

Note particularly the summation index in (27), in which it is denoted by a new letter. This emphasizes that the labeling of the rows of the matrix B can be specified by the set of quantum numbers Λ_0, which has nothing in common with the set $\Gamma_0\lambda\mu$. In accordance with (15) and (16), the rows and columns of B transform independently, so that there is no need to correlate the notation for them. The form of the functions B determines the dependence of the wave function of the nucleus on the internal variables, so that these functions can be called internal wave functions of the nucleus.[5]

The expression (24) in the general case and the expressions (27)–(29) in the case of supermultiplet orbital functions give the general solution of the problem posed here of separating the collective and internal functions of the nucleus. Indeed, the functions $\overset{0}{\Psi}$ in (24) are known if the set of original functions Ψ is. In turn, for known Ψ's one can in principle also find the functions B: Multiplying (24) by $\overset{0}{\Psi}{}^*$, integrating with respect to ξ, and summing over Q, we obtain a system of algebraic equations for determining the matrix elements of B:

$$\sum_Q\int d\tau(\xi)\,\overset{0}{\Psi}{}^*(\Gamma''\,|\,\xi Q)\,\Psi(\Gamma'\,|\,\rho_1^s,\ldots,\rho_{n-1}^s;\,Q) = \sum_{\Gamma'}c_{\Gamma''\Gamma'}B_{\Gamma'\Gamma}(q_{n-1}), \quad (30)$$

where the numbers $c_{\Gamma''\Gamma'}$ here are calculated in accordance with

$$c_{\Gamma''\Gamma'}=\sum_Q\int d\tau(\xi)\,\overset{0}{\Psi}{}^*(\Gamma''\,|\,\xi Q)\,\Psi(\Gamma'\,|\,\xi Q). \quad (31)$$

The expression $d\tau(\xi)$ was obtained in Refs. 1 and 2, and the volume element for the variables q_{n-1} can be found in Ref. 13 (see also Refs. 4 and 8). It should be noted that the collective functions Θ are not orthogonal with respect to Λ_0, and therefore the integral (31) is nonzero for $\Lambda_0' \neq \Lambda_0$.

5. THE KINEMATICALLY SIMPLEST ORBITAL WAVE FUNCTIONS OF A NUCLEUS

Although the formulas given above do in principle solve the problem of separating the collective and

internal wave functions, their application in the case of a set of arbitrary initial functions $\Psi(\Gamma)$ or even orbital functions $\Psi(\Gamma_0\lambda\mu LM)$ of the supermultiplet scheme may be a very difficult problem. The extreme generality of the formulas hides important properties of the collective and, expecially, internal wave functions and prevents treating practically important problems such as, for example, the complete or even partial separation of the collective and internal degrees of freedom of the nucleus. It would also be desirable to clarify the meaning of the set of quantum numbers Λ_0 in the expansion (27) and write down an orthogonality condition for the functions B, and also the analog of the orthogonality relation for the functions Θ.

If we retrace our entire path, we see that the results we have obtained are based solely on the two requirements of kinematic correctness formulated in Sec. 1 and, in addition, on the assumption of a supermultiplet structure of the nuclear wave function. All the possibilities inherent in the conditions 3 and 4 have already been exhausted, and we must now formulate a new requirement in order to open up a way to further particularization of Eqs. (27) and (29).

There exists an immense set of supermultiplet functions (25) satisfying the requirement of kinematic correctness. They include eigenfunctions of the part of the true nuclear Hamiltonian that conserves the orbital angular momentum and the Young pattern. Until the solutions of the corresponding Schrödinger equation have been studied, it is sensible to select the simplest functions from the set of orbital functions. This problem will be well posed only when we have specified the criterion of simplicity. The algebraic formalism enables one to do this readily because irreducibility is a synonym of simplicity in the theory of group representations. Indeed, by definition an irreducible quantity (space, matrix, operator) is the simplest entity that does not admit further subdivision. Therefore irreducibility is the necessary and sufficient kinematic criterion for selecting the simplest functions from the class of orbital functions.

We should point out immediately that the requirement of kinematic simplicity restricts to a very great extent the class of Hamiltonians whose eigenfunctions satisfy this criterion. We here have a secondary restriction of the Hamiltonian: At the start, we were considering functions of an arbitrary, possibly exact, nuclear Hamiltonian, and we now consider only eigenfunctions of a Hamiltonian that conserves the quantum numbers λ and L. It is as yet too early to say what type of Hamiltonian comes under the category of the kinematically simplest, but we do know one thing for certain: These Hamiltonians will differ strongly from a realistic nuclear Hamiltonian.

Henceforth, supermultiplet functions satisfying conditions 3 and 4 and the criterion of kinematic simplicity just formulated will be called the simplest kinematically correct wave functions; we now proceed to a further particularization of the results obtained in Sec. 4.

The criterion of kinematic simplicity can be applied to the transformations of any group. The form of Eq. (27) suggests that it would be expedient to apply it in the first place to transformations of the group O_{n-1}. The point is that the matrix B in (27) gives a matrix realization of the operator of a right shift of the group O_{n-1}. This can be seen immediately by applying Eq. (16) to the function (27). We require that such a representation be the simplest possible, i.e., irreducible, which is possible only when the set of quantum numbers Γ_0 contains the quantum numbers of the irreducible representations ω of the orthogonal group O_{n-1}. The symmetric group S_n is contained in the group O_{n-1} in accordance with $O_{n-1}\supset S_n$, and for the quantum numbers of the complete basis of this reduction we may require an index α to distinguish equal λ's contained in ω. Therefore, the criterion of kinematic simplicity applied with respect to the group O_{n-1} means that in (27) we must replace Γ_0 by the new set $\Gamma_0\omega\alpha$, where Γ_0 is an arbitrary set of the remaining quantum numbers. We also replace the set Λ_0 by $\Lambda_0\omega'\nu^0$, where ν^0 is a basis of the O_{n-1}-irreducible representation ω; Λ_0 is again an arbitrary set of the remaining quantum numbers. Using also the properties of the matrix elements of the matrices of irreducible representations that give $\delta(\Lambda_0\Gamma_0)\delta(\omega'\omega)$ in (27), and also taking into account (28), we obtain

$$\Psi\left(\begin{matrix}\Gamma_0 LM\\ \omega\alpha\lambda\mu\end{matrix}\middle|\,\rho_1,\ldots,\rho_{n-1}\right)=\sum_{\nu^0 K}\Theta\left(\begin{matrix}\Gamma_0 LK\\ \omega\nu^0\end{matrix}\middle|\,\rho^{(x)}\rho^{(y)}\rho^{(z)}\right)\times(d_L)^{1/2}D^L_{KM}(G^+_3)\,(d_\omega)^{1/2}D^\omega_{\nu^0,\,\alpha\lambda\mu}(q_{n-1}),\qquad(32)$$

where D^ω is the matrix of the O_{n-1}-irreducible representation ω; d_L and d_ω are the dimensions of the representations L and ω, introduced on account of the normalization. Equations (28) and (29) are modified only slightly: In them, Λ_0 must be replaced by $\Gamma_0\omega\nu^0$.

Comparison of (32) with (27) shows that we have been able to particularize them by virtue of the simplicity criterion: Instead of undetermined internal functions B of the nucleus we now have well defined functions—the matrix elements of the matrix D^ω, these depending on the main set of $3(n-3)$ variables of the nucleus. These functions are characterized by the physical set of quantum numbers $\alpha\lambda\mu$, which label the columns of the matrix D^ω, and the as yet undetermined set ν^0, which labels the rows of D^ω; about this set, we know only that it specifies the basis of the O_{n-1}-irreducible representation ω. Completely undetermined so far is the new set of quantum numbers Γ_0, which characterizes the collective function of the nucleus. The criterion of kinematic simplicity has not yet been applied to this function.

We have now obtained the possibility of proving an important property of the internal wave functions D^ω; in particular, it enables us to say much about the meaning of the quantum numbers ν^0. We begin by establishing the transformation properties of the variables (12) under transformations of the group O_{n-1}. Let G_{n-1} be an arbitrary element of this group. Then applying to (12) the left-shift operator $\hat{L}(G_{n-1})$, we obtain

$$\hat{L}(G_{n-1})\,\rho^s_i=\sum_{s'}\rho^{(s')}D^{(1^3)}_{s's}(G^+_3)\,D^{(1^{n-1})}_{s'i}(G^{-1}_{n-1}q_{n-1})$$
$$=\sum_{s'}\rho^{(s')}D^{(1^3)}_{s's}(G^+_3)\sum_{i'}D^{(1^{n-1})}_{s'i'}(G^{-1}_{n-1})\,D^{(1^{n-1})}_{i'i}(q_{n-1}).\qquad(33)$$

In (33) we now take as element of the group O_{n-1} the elements G_{n-4} of its subgroup O_{n-4} and recall that in

(33) s' denotes only the last three rows of the matrix $D^{(1_{n-1})}$. These rows do not depend on G_{n-4}, as a result of which one can take for G_{n-4} the identity element e. But then in (33) $\delta(s'i')$ appears in the summation, and we obtain $\hat{L}(G_{n-4})\rho_i^s=\rho_i^s$. This important property of the variables ρ_i^s shows that they are invariant under a left shift through an arbitrary element of the subgroup O_{n-4}. One can show (see Ref. 8 for the details) that because of this left-invariance of the variables ρ_i^s the expansion (32) contains only those rows of the matrix D^ω that are O_{n-4}-scalar rows. In other words, the basis ν^0 is labeled by means of the indices of the irreducible representations of the groups G^0 of the reduction $O_{n-1}\supset G^0\supset O_{n-4}$, where G^0 are arbitrary groups placed between the groups O_{n-1} and O_{n-4}.

Thus, we have shown that the basis of the reduction $O_{n-1}\supset G^0\supset O_{n-4}$ gives the selection rules in the expansion (32). We now explain why the difference between the labelings of the rows and columns of the matrix B was especially emphasized in the discussion of (27). If we go back to the more particularized formula (32), we see that the method of labeling the columns of D^ω is, except for the choice of the repetition index α, determined by the Pauli principle, whereas the basis of the reduction just considered serves as an optimal basis for the rows of this matrix. In what follows we shall assume that ν^0 in (32) denotes the basis of only the reduction $O_{n-1}\supset G^0\supset O_{n-4}$, the superscript zero of the letter ν indicating that this basis is O_{n-4}-scalar.

In mathematics, the special functions studied here on the factor spaces of the orthogonal groups (see, for example, Ref. 13) are very simple compared with the functions D^ω. Therefore, considerable exertions were made to study them in detail, even as far as the development of a recursive method of construction (see Refs. 6 and 7 for details). The properties of these functions are now known in such detail that the expansion (32) can be used to investigate the properties of concrete nuclei.

We shall not present in detail the general theory of the internal wave functions D^ω, but discuss only the simplest of their properties. These functions are orthogonal with respect to the indices $\nu^0\omega\alpha\lambda\mu$ (see Refs. 6 and 7). The orthogonality of the functions D^ω gives rise to the somewhat unusual property of orthogonality of the collective wave functions Θ of the nucleus.[7,8]

The basis ν^0 determines the number of terms in (32); this number governs the extent to which the internal and collective motions of the nucleons in the nucleus are coupled. It was shown above that the intermediate groups G^0 can be chosen by discretion. From the mathematical point of view, the simplest groups are $G^0\to O_{n-2}\supset O_{n-3}$, which lead to the canonical reduction $O_{n-1}\supset O_{n-2}\supset O_{n-3}\supset O_{n-4}$, in the case of which it is convenient to replace ν^0 by the set $\bar{\omega}\bar{\bar{\omega}}\bar{\bar{\bar{0}}}$ which means that $\bar{\omega}$, $\bar{\bar{\omega}}$, and $\bar{\bar{\bar{0}}}$ are irreducible representations of the groups O_{n-2}, O_{n-3}, and O_{n-4}, respectively. Once the groups G^0 have been completely fixed, it is possible to find the number of terms in the expansion (32). This is done readily by using the rules of reduction to the canonical chain of the orthogonal groups and the tables given in Refs. 9 and 14, in which the states ω allowed by the Pauli principle are listed.

Analysis of the tables of allowed states of nuclei with $n\leq 40$ nucleons shows[8] that an expansion (32) with one term is encountered very seldom. In the overwhelming majority of cases, even for the simplest kinematically correct wave functions (32), there is more than one term, and this means that for kinematical reasons in the nucleus the internal and collective motions of the nucleons do not separate.

How many terms are there in the sum (32)? Let us take, for example, the O_{11}-irreducible state $\omega=(44)$ of the ^{12}C nucleus. Calculations show that in this case the basis ν^0 consists of 15 components. Is this many or few? On the one hand, it is enough to make it impossible to separate the collective and internal motions of the nucleons. But, on the other hand, it is very few compared with the dimension of the matrix $D^{(44)}$ of the O_{11}-irreducible representation (44). It can be shown that this dimension is 112200. Therefore, the O_{n-4}-scalar properties of the basis ν^0 are alone sufficient to make the sum (32) tractable.

If desired, one can use other intermediate groups G^0 in (32) and, for example, instead of $\bar{\omega}\bar{\bar{\omega}}$ take the basis of the reduction $O_{n-1}\supset O_3\dot{+}O_{n-4}$, where $\dot{+}$ denotes the direct sum of matrices; O_3 is the rotation group of the abstract three-dimensional space spanned by the Jacobi vectors ρ_{n-3}, ρ_{n-2}, and ρ_{n-1}. This group must not be confused with the rotation group of "our" three-dimensional space. For the $O_3\dot{+}O_{n-4}$ reduction it is convenient to denote the basis ν^0 by the set $\beta l\nu_l$, where β is the repetition index; l is an irreducible representation of O_3, and ν_l is the representation basis. The index l takes the values 0, 0*, 1, 1*, ..., where the asterisk denotes the conjugate representations. If we are not interested in reflection and making the further reduction $O_3\supset O_3^*$, then l^* is equivalent to l, and l becomes the ordinary angular momentum. This angular momentum is associated with the intrinsic rotation of the abstract three-dimensional space, and it may therefore be called the quasiangular momentum.[5]

We now give simple rules that enable us to list all possible values of the quasiangular momentum l. In nuclear theory, one encounters O_{n-1}-irreducible representations ω specified by three positive integers $\omega_1\geq\omega_2\geq\omega_3$. If we introduce the notation $E_1=\omega_1-\omega_3$ and $E_2=\omega_2-\omega_3$, then the following assertion holds: In an O_{n-4}-scalar basis of the O_{n-1}-irreducible representation $\omega\equiv(\omega_1\omega_2\omega_3)$ there are exactly the same quasiangular momenta l as there are angular momenta L in the SU_3-irreducible representation $[E_1E_2]$. The reduction rules for the case $SU_3\supset O_3^*$ are well known (see, for example, Ref. 9), so that we now also know the values of the quasiangular momentum contained in the representation $(\omega_1\omega_2\omega_3)$.[2)] If it is necessary to list the representations of the complete orthogonal group O_3, the more complicated rules of Ref. 8 must be used.

[2)]This apparently hitherto unnoted isomorphism between labelings of bases of such a different nature is interesting from the algebraic point of view. Of course, it can also be established in the general case of an O_{r_2}-scalar basis of the $O_{r_1+r_2}$-irreducible representation $(\omega_1,\ldots,\omega_{r_1})$.

We give here the simple example of the basis of the reduction $O_3 \dot{+} O_{n-4}$. Suppose $\omega = (64)$; then l takes the values 0, 2 (twice), 3*, 4 (twice), 5*, 6.

The quantum numbers of the reduction $O_{n-1} \supset O_3 \dot{+} O_{n-4}$ are less convenient from the mathematical point of view than the quantum numbers of the canonical reduction, but they are more interesting because of the distinguished role of the abstract three-dimensional space. The basis of the $\nu_l O_3$-irreducible representation l can be fixed in several ways. The first of them is by the introduction of the ordinary projection m defined by the reduction $O_3^+ \supset O_2^+$. But this is not the only possibility. The basis ν_l for the group O_3 (but not the group O_3^+) can, for example, be specified by means of the reduction $S_3 \supset S_2$ which here means that the groups S_3 permute the vectors ρ_{n-3}, ρ_{n-2}, ρ_{n-1}, and the groups S_2 the vectors ρ_{n-2} and ρ_{n-1}.

Such a basis makes it possible to sort the collective and internal functions in accordance with their behavior under geometrical permutations of the axes of the three-dimensional subspace of the $(n-1)$-dimensional space of the Jacobi vectors; the symmetric group S_3 is here imbedded in accordance with $O_3 \supset S_3$, and therefore the permutations do not directly permute the single-nucleon vectors $\mathbf{r}$.

There is one other possibility of choosing the basis ν_l by means of a maximally dense imbedding of the symmetric group S_4 in the orthogonal group O_3 in accordance with $O_3 \supset S_4$. Then the elements of the group S_4, permuting the variables $\mathbf{r}_{n-3}$, $\mathbf{r}_{n-2}$, $\mathbf{r}_{n-1}$, and $\mathbf{r}_n$, induce a transformation of the Jacobi vectors ρ_{n-3}, ρ_{n-2}, ρ_{n-1}. Further, the basis of the S_4-irreducible representation is fixed by the reduction $S_3 \supset S_2$, and S_3 and S_2, respectively, permute the vectors $\mathbf{r}_{n-2}$, $\mathbf{r}_{n-1}$, $\mathbf{r}_n$ and $\mathbf{r}_{n-1}$, $\mathbf{r}_n$.

We have here gone into a detailed description of the various possibilities of choosing the basis ν^0 since, although they are equivalent in Eq. (32), they may not be so if this series is artificially truncated.

It may be that the quantum numbers of one of these sets are sometimes "good" approximate quantum numbers, admitting approximate truncation of the series (32) and thus an approximate separation of the collective and internal degrees of freedom of the nucleus. If we attempt to draw a parallel between the quantum number K in the expansion (28) and the basis ν^0, we can say that K is the projection of the angular momentum L in the intrinsic (with respect to three-dimensional rotation) coordinate system, whereas ν^0 for any of the reductions listed above gives the "projection of the angular momentum" ω in the intrinsic [with respect to an $(n-1)$-dimensional rotation] coordinate system. Continuing this analogy, we also note that Eq. (28) separates the dependence of the collective functions on the Eulerian angles; at the same time, one must use the complete reserve of irreducible functions on the group O_3^+, whereas Eq. (32) also reveals the dependence of the microscopic wave function on the internal variables of the nucleus, it being sufficient for this purpose to use only a negligible fraction of the reserve of irreducible functions defined on the group O_{n-1}, namely, the set of O_{n-4}-scalar (under left shift) functions defined on the factor space O_{n-1}/O_{n-4}.

6. SCHRÖDINGER EQUATION FOR COLLECTIVE FUNCTIONS; THE KINEMATICALLY SIMPLEST COLLECTIVE FUNCTIONS

Equation (32) gives the orbital wave functions $D^L_{KM} D^{\omega}_{\nu^0, \alpha\lambda\mu}$ that are the kinematically simplest with respect to transformations of the group $O_3^+ \times O_{n-1}$. It is important that the quantum numbers of these functions include the supermultiplet quantum numbers LM and $\lambda\mu$, whose presence is necessary and sufficient for constructing, by means of them, the complete wave function of the nucleus. It can be seen from (25) that this function will be characterized by the exact quantum numbers JM_J and the approximate quantum numbers $\nu^0 \omega \alpha \lambda K \Gamma_S L S$.

We now take an arbitrary translationally invariant Hamiltonian $\mathcal{H}(\rho_1, \ldots, \rho_{n-1}; Q)$ of a nucleus and, using (12), perform a change of variables in it and average it with respect to the variables G_3^+, q_{n-1}, Q on the basis of the functions just described. This averaging gives the following system of differential equations with respect to the variables $\rho^{(x)}\rho^{(y)}\rho^{(z)}$:

$$\sum_{\nu^0 K} \langle \nu^{0\prime} \omega' \alpha' K' \lambda' \Gamma_S' (L'S') J \,|\, \mathcal{H} - \mathcal{E} \,|\, \nu^0 \omega \alpha K \lambda \Gamma_S (LS) J \rangle \Theta(\nu^0 K \,|\, \rho^{(s)}) = 0, \tag{34}$$

where $\mathcal{E}$ is an eigenvalue of the Hamiltonian of the nucleus.

If nuclei do in reality have a tendency to be excited in the first place with respect to the collective degrees of freedom, this system of equations will make it possible to find good collective wave functions $\Theta(\rho^{(s)})$ of the nucleus. Equations (34) are characterized by all the exact integrals of the motion formulated in condition 4, and they are therefore correct in the kinematic respect. Such a system of equations is a microscopic generalization of the equations of the Bohr–Mottelson collective model. It is infinite but "goes" to infinity only with respect to the single quantum number ω. Indeed, a fixed ω determines the sets ν^0 and $\alpha\lambda$. In turn, each λ defines a finite set of Γ_S and S, and for given J every S allows a finite set of orbital angular momenta L. Therefore, the real problem is the investigation of the system (34) in an approximation diagonal with respect to ω. The equations can be further simplified if one is content at first with a Hamiltonian that does not depend on the spin and isospin variables. The technique of averaging $\mathcal{H}$ with respect to the variables $G_3^+ q_{n-1} Q$ is given in Refs. 15 and 5–7, and in Refs. 16 and 17 examples are given of such functions (see also Ref. 4) and their properties are discussed.

In deriving the expansion (32) from the more general expansion (27) we used the requirement of kinematic simplicity with respect to the group O_{n-1}, which made it possible to go over from the undetermined internal wave functions $B_{\Delta_0, \Gamma_0 \lambda\mu}$ to the completely determined wave functions $D^{\omega}_{\nu^0, \alpha\lambda\mu}$. This principle can also be applied to other groups, in particular to $O_{3(n-1)}$. Using (13) in the functions (29) we replace $\rho^{(s)}$ by $\rho\tilde{g}$ and Γ_0 by $\Gamma_0 \omega \nu^0$; we apply to them the operator $T(\tilde{g}^{-1})T(\tilde{g})$ and, using the expansion (22), applying it to the factor space $\tilde{g}$, we obtain

$$\Theta\left(\begin{matrix}\Gamma_0 LK \\ \omega\nu^0\end{matrix}\,\middle|\, \rho\tilde{g}\right) = \sum_{\Delta} \Theta(\Delta \,|\, \rho)\, B_{\Delta, \Gamma_0 LK\omega\nu^0}(\tilde{g}). \tag{35}$$

The matrices B are reducible matrices of a representation of the group $O_{3(n-1)}$. As in (27), use of the new notation Δ as summation index means that the rows and columns of the matrix B can be labeled in different ways. The requirement of kinematic simplicity with respect to the group $O_{3(n-1)}$ means that we consider here only a class of functions (35) for which B is an $O_{3(n-1)}$ irreducible matrix. This means that the set Γ_0 is replaced by $\Gamma_0\Omega\delta$, where Ω is an $O_{3(n-1)}$-irreducible representation; δ is the index of repetition of L and ω in Ω; Γ_0 is an arbitrary new set of the remaining quantum numbers. We label the basis Δ differently: Instead of it, we shall write $\Delta\Omega'\overline{\Omega\xi}$, where $\overline{\Omega\xi}$ is an $O_{3(n-1)}$-irreducible representation and its basis; Δ are the remaining quantum numbers. As in Sec. 5, one can show that the variables ρ_i^s are invariants under a left shift of the operators of the group $O_{3(n-1)-1}$. For this reason, in complete analogy with the selection rules due to the rows of the matrix D^ω being O_{n-4}-scalar, we obtain here only $O_{3(n-1)-1}$-scalar rows of the matrix D^Ω, i.e., $\overline{\Omega}=0$ (see Ref. 8 for details), as a result of which there remains in (35) only a single term, and the collective function Θ that is kinematically simplest with respect to the factor space $\tilde{g}$ takes the form

$$\Theta\left(\begin{matrix}\Gamma_0\Omega\delta LK\\ \omega\nu^0\end{matrix}\middle|\,\tilde{g}\right)=\Theta\left(\Gamma_0\Omega\,|\,\rho\right)D^{\Omega}_{\overline{0},\,\delta LK\omega\nu^0}(\tilde{g}). \tag{36}$$

The function $D^\Omega(\tilde{g})$ is the microscopic analog of the basis functions in the Bohr–Mottelson model used to describe β and γ vibrations of the nucleus. Substituting (36) into (32) and summing over $\nu^0 K$, we obtain

$$\Psi\left(\begin{matrix}\Gamma_0\Omega\delta LM\\ \omega\alpha\lambda\mu\end{matrix}\middle|\,\rho_1,\ldots,\rho_{n-1}\right)=\Theta\left(\Gamma_0\Omega|\rho\right)D^{\Omega}_{\overline{0},\,\delta LM\omega\alpha\lambda\mu}(g), \tag{37}$$

where $g=\tilde{g}G_3^*q_{n-1}$. The function D^Ω is normalized to d_Ω, where d_Ω is the dimension of the representation Ω.

Equation (37) also remains valid if the variables g are given a different meaning. Going over from the more general class of functions (32) to the simpler functions (37), we have given up a solution of the Schrödinger equation with respect to the collective variables $\tilde{g}$ and, instead of exact solutions, we have taken the kinematically simplest functions. It therefore became unnecessary to know the dependence of the elements of the matrix D^Ω on the collective and internal variables. We recall that these variables appeared as a result of the specially chosen parametrization of the group $O_{3(n-1)}$, which itself determined the complicated structure of the argument of the function D in (11). The objections against using the parameters of the factor space $O_{3(n-1)}/O_{3(n-1)-1}$ now disappear, and in the change of variables we can use Eq. (9) with $r=3(n-1)$, this formula containing the matrix elements of the last row of the matrix D parametrized in accordance with Eq. (8).

In this interpretation, the functions D^Ω in (37) are related to multidimensional spherical functions (see, for example, Ch. 9 of Ref. 13). This is because both sets of functions are matrix elements of a scalar row (or column) of matrices of irreducible representations of the rotation group. However, the multidimensional spherical functions are much simpler than the functions D^Ω, which are characterized by the noncanonical basis of the reduction $O_{3(n-1)}\supset O_3^*\times O_{n-1}\supset O_3^*\times S_n$. This circumstance makes their construction very difficult. Nevertheless, there now exists a technique developed in detail for calculation on the basis of such functions and, in particular, methods of calculation have been developed based on the idea of a fractional-parentage expansion.[18] Substituting (37) into (25), we obtain the functions of the so-called method of K harmonics (the method of hyperspherical functions).[19,20] Averaging the Hamiltonian of the nucleus with respect to all the angular and spin–isospin variables on functions of the method of K harmonics, we obtain a system of differential equations with respect to the variable ρ:

$$\langle\Omega'\delta'\omega'\alpha'\lambda'\Gamma'_S(L'S')J\,|\,\hat{H}-\mathcal{E}\,|\,\Omega\delta\omega\alpha\lambda\Gamma_S(LS)J\rangle\,\Theta\left(\Omega\,|\,\rho\right)=0. \tag{38}$$

The infinite system of equations (38), which in principle enables one to find the collective functions $\Theta(\rho)$, "goes" to infinity with respect to the quantum number Ω, and in practical calculations one must therefore restrict the possible values of Ω and frequently the other quantum numbers as well. There have been numerous investigations of solutions of the truncated system (38) in the case of few-nucleon systems and for magic or near-magic nuclei containing several tens of nucleons. Analysis of these investigations (references to many of them can be found in Ref. 21) would go beyond our present scope. Here, we merely point out that the comparatively simple averaging technique developed in the most general algebraic form in Ref. 22 enables one to write out readily the truncated system of equations (38).

Thus, we have found that the system of equations (38) can be obtained from the system (34) if the latter is additionally averaged with respect to $\tilde{g}$ on the functions $D^\Omega(\tilde{g})$. The functions depending on the variable ρ still remain undetermined, and the last thing which must be done is to impose on them the requirement of kinematic simplicity with respect to the appropriate continuous group. As this group, we take the unitary group $U_{3(n-1)}$, which is universal among the so-called compact groups (see, for example, Ref. 23). However, this universality introduces additional conditions since in the case of unitary groups it is necessary to define clearly the object of their transformation, this ultimately giving the concrete form of the functions that depend on the variable ρ.

Without going into details,[8] we here merely point out that in the case when the objects of transformation of the group $U_{3(n-1)}$ are operators of creation and annihilation of oscillator quanta,

$$(d_\Omega)^{-1/2}\Theta\left(E\Omega\,|\,\rho\right)=R_{E\Omega}(\rho), \tag{39}$$

where R is the radial function of a $3(n-1)$-dimensional harmonic oscillator and E is the number of oscillator quanta.

The choice of $\Theta(\rho)$ in the form of a radial oscillator function fundamentally restricts the space of functions to the space of functions of the discrete spectrum. Indeed, as has been frequently noted in studies of the method of K harmonics, the functions (37) give a complete n-particle basis with respect to the angular variables suitable for investigating bound states and continuum states. The variable ρ always guarantees the possibility of a nucleon or some nucleon association going to infinity. Technically, this is realized by means of a fractional-parentage re-expansion[18] that prepares the

wave function for separation of such a fragment of the nucleus. The choice of $\Theta(\rho)$ in the form (39) closes the channels of all processes that require allowance for the properties of the continuum and, strictly speaking, leaves one with the possibility of studying only bound states.

If we restrict ourselves to bound-state problems, then, substituting (39) into (36) and taking into account the factors that guarantee normalization to unity, we obtain the kinematically simplest collective functions:

$$\Theta(E\Omega\delta LK\omega\nu^0 \mid \rho\tilde{g}) = (d_\Omega/d_L d_\omega)^{1/2} R_{E\Omega}(\rho)\, D^{\Omega}_{0,\,\delta LK\omega\nu^0}(\tilde{g}). \qquad (40)$$

In (40), one can return to the variables $\rho^{(x)}$, $\rho^{(y)}$, $\rho^{(z)}$. For these variables, there exists a different and more natural basis defined by the reduction $U_{3(n-1)} \supset U_3 \times U_{n-1} \supset O_3^+ \times O_{n-1}$, the transition to this having the form[24]

$$\Theta\left(\begin{matrix} E\,[E_1E_2E_3]\,\gamma LK \\ \beta\omega\nu^0 \end{matrix} \middle| \rho^{(x)}\rho^{(y)}\rho^{(z)}\right) = \sum_{\Omega\delta} A^{(E\omega L)}_{\beta[E_1E_2E_3]\gamma,\ \Omega\delta} \times \Theta(E\Omega\delta LK\omega\nu^0 \mid \rho^{(x)}\rho^{(y)}\rho^{(z)}). \qquad (41)$$

Here, $[E_1E_2E_3]$ is an irreducible representation of the group U_3, and β and γ are the repetition indices for the reductions $U_{n-1} \supset O_{n-1}$ and $U_3 \supset O_3^+$. In many cases of practical interest, the matrix A is the identity matrix, and therefore, having the functions (41), we can readily obtain the functions (40) by the change of variables (13). Two methods of constructing the functions (41) can be found in Refs. 5 and 7.

Before we complete the description of the simplest kinematically correct wave functions of a nucleus, let us point out the circumstance that these functions now become completely definite, and therefore, as in the case of the function (37), we have once more obtained an additional freedom in the choice of their variables. If we do not intend to solve the dynamical problem with respect to particular variables but only to use simplest functions, it is no longer necessary to represent these functions in the form (37), and it is simplest of all to return in them to the original variables ρ_i^s. We then have the complete nuclear wave function (25) constructed by means of orbital functions whose quantum numbers are determined either by the reduction $U_{3(n-1)} \supset U_3 \times U_{n-1} \supset O_3^+ \times O_{n-1} \supset O_3^+ \times S_n$ or the reduction $U_{3(n-1)} \supset O_{3(n-1)} \supset O_3^+ \times O_{n-1} \supset O_3^+ \times S_n$.

Wave functions of the first type can be called functions of the unitary scheme, and those of the second type functions of the orthogonal scheme. The bases of the functions of these schemes are related by the transformation (41), and there is therefore no need to distinguish them, it being convenient to call functions of both types functions of the $U_{3(n-1)}$ scheme.[24] There exists a well developed technique of calculation with such functions[9,18] based on a fractional-parentage expansion or on the even more economic method of decomposing a density matrix in terms of group operators. This technique enables one to average physical operators on the basis of functions of the $U_{3(n-1)}$ scheme and thus use them to study the properties of bound states of nuclei. On the basis of these methods, a technique is given in Refs. 5–7 and 22 for deriving Eqs. (34) and (38); this technique enables one to replace the insoluble problem of constructing the basis functions D^{ω} and D^{Ω} by the much simpler problem of calculating the components of the corresponding density matrices.

CONCLUSIONS

The requirements of kinematic correctness and the formalism of induced representations have enabled us to replace arbitrary wave functions of the nucleus (2) by completely specified internal and collective functions, or functions of the $U_{3(n-1)}$ scheme. This algebraic formalism has in particular enabled us to formulate the requirement of kinematic simplicity, which, in conjunction with a technical matter—separation of the spatial functions from the spin–isospin functions by means of the supermultiplet scheme—leads to the simplest kinematically correct wave functions of the nucleus. Let us make the last assertion more precise. How do we know that these functions are the simplest? The theory of Lie groups guarantees the assertion. We have already pointed out that kinematic simplicity means irreducibility with respect to transformations of some continuous group. The simplest of the Lie groups are the compact groups.[23] They have been classified: We know a complete list of them, just as we do for the point or space groups used in the theory of molecules and crystals. Moreover, we know reductions that enable one to ascend from a compact subgroup to a compact group in accordance with a maximally dense reduction, i.e., a reduction in which one cannot insert any other continuous groups between the group itself and its subgroup.

Dense imbedding of the group S_n in the group O_{n-1} was used to prove the theorems formulated above. This gave us a guarantee that the group O_{n-1} is the simplest, and it enabled us to find the internal functions of the nucleus. In the present review, although this was not especially emphasized, we also used the property of the most dense imbedding on the transition from the subgroup $O_3^+ \times O_{n-1}$ to the group $O_{3(n-1)}$ and then from the subgroup $O_{3(n-1)}$ to the group $U_{3(n-1)}$; here, there is a certain ambiguity because there is a different path available—transition from the subgroup $O_3^+ \times O_{n-1}$ to the group $U_3 \times U_{n-1}$ and then to the group $U_{3(n-1)}$. However, this lack of uniqueness is unimportant since the bases determined by the subgroups of the two reductions are unitarily equivalent by virtue of the transformation (41). In fact, we have therefore shown that the functions of the unitary (or the orthogonal) scheme are indeed the simplest of all possible functions. We formulate this assertion as the following theorem.

Theorem. In the class of kinematically correct wave functions, functions of the $U_{3(n-1)}$ scheme are the simplest.

The practical value of this theorem can be readily recognized by recalling that in seeking solutions of many types of differential equations, describing diverse physical processes, one can as a rule select a natural system of basis functions. For example, for periodic processes we have exponentials when the reflection operation is unimportant, or sines or cosines when the expanded functions have a definite parity; for scattering problems in a spherically symmetric field we have the products of Bessel functions and spherical functions; for the expansion of vector fields, we have vector

spherical harmonics. All such functions are irreducible (in our terminology, they are "kinematically the simplest") bases of the corresponding groups: of the group O_2^+ without reflection or of O_2 with reflection in the case of Fourier series, of the group of motion of three-dimensional space reduced to its maximal compact subgroup O_3^+ in the case of the scattering problem and, finally, it is the irreducible basis formed by means of the Clebsch–Gordan coefficients from the direct product of spherical functions and unit vectors for the expansion of vector fields. Our theorem shows that for bound states of a translationally invariant Hamiltonian of a nucleus functions of the $U_{3(n-1)}$ scheme constitute a natural basis of this kind. As always, the problem of convergence of an expansion with respect to such bases remain open.

Because of the high rank of the unitary group, the basis of the $U_{3(n-1)}$ scheme is very complicated, with the result that a satisfactory expansion with respect to it of solutions of the Schrödinger equation is possible only for few-nucleon systems. In the case of nuclei containing several tens of nucleons, the dimension of this basis increases very rapidly. Whereas for $n<16$ and also for n near the mass numbers $n=16$, $n=40$, and $n=80$ the number of components of a function of the $U_{3(n-1)}$ scheme is still under control, for $n\approx 30$, for example, and even more so for $n\approx 60$ this number is in the hundreds or thousands for the simplest value $E=E_{\min}$ allowed by the Pauli principle. There is nothing surprising about this because expansion with respect to such a set of functions is equivalent to solution of the many-particle Schrödinger equation, and it is not difficult to conceive how hard this problem is.

It is clear from what we have said that practical calculations can be performed only on a restricted basis, and this means that the states which terminate the expansion must be given the meaning of approximate quantum numbers. If we wish to preserve the algebraic structure of the irreducible basis of the group $U_{3(n-1)}$, the first thing to be done is to make a restriction to one term with respect to E, i.e., in the calculations ignore the matrix elements that are nondiagonal with respect to E. Then the matrix of the total Hamiltonian of the nucleus splits up into blocks and the states are sorted accordingly with respect to the quantum number E.

The assumption that E is a quantum number associated with experimentally observed states of the nucleus means that we give up the unrealizable attempt at exact solution of the nuclear problem of many bodies and go over to the more modest but realistic problem of studying the properties of nuclei in a restricted subspace of the many-particle Hilbert space. Such an approximation is equivalent to introducing a model of the nucleus: By definition, we shall call the simplified picture of nuclear structure defined by the basis functions of the $U_{3(n-1)}$-irreducible representation E a model of the $U_{3(n-1)}$ scheme. In contrast to the extreme nuclear models and their modifications discussed in Sec. 1, the $U_{3(n-1)}$ scheme gives the simplest kinematically correct model of the nucleus.

It is valid to ask: How constructive is this result? Is not the model trivial? Does it not give a too primitive picture of nuclear structure? What is the picture?

To find the physical meaning of the model of the $U_{3(n-1)}$ scheme, we must realize the orbital functions in terms of functions of a definite Hamiltonian, which we shall call the model Hamiltonian of the nucleus. It is well known that the quantum numbers of irreducible representations are given by the eigenvalues of the set of Casimir operators; with respect to the group $U_{3(n-1)}$, the model functions of the nucleus are characterized by the single quantum number E, and therefore, to determine the explicit form of these functions, we must solve the eigenvalue equation for the first Casimir operator of the group $U_{3(n-1)}$. The form of this operator depends on the method of realization of the objects of transformation of the group $U_{3(n-1)}$. Nor should one forget that every transformation of the space on which the unitary group acts induces a transformation of the dual space associated with it on which the so-called contragradient representation of this group acts (see, for example, Ref. 9). This representation, which is usually denoted as $D^{[0\ldots0-1]}\equiv D^{[-1]}$, is inequivalent to the original representation $D^{[10\ldots0]}\equiv D^{[1]}$, and one must therefore specify two systems of objects of transformation: one in the space $[1]$ and the other in the space $[-1]$. Suppose that the objects of transformation in the representation space $[1]$ of the group $U_{3(n-1)}$ are the components of the Jacobi vectors ρ_i^s, while in the dual space $[-1]$ they are the momenta $\hat{q}_i^s$. It is easy to verify that the direct product $\rho_i^s\hat{q}_i^s$ of these objects is given by $(3(n-1))^2$ operators that are infinitesimal operators of the group $U_{3(n-q)}$. The first Casimir operator for these infinitesimal operators obviously has the form $\sum_{is}\rho_i^s\hat{q}_i^s$.

It can be now seen that we have made an unfortunate choice of the objects of transformation—the Casimir operator is too dissimilar to the Hamiltonian, if for no other reason than that it contains only the first derivatives. Let us attempt to raise the order of the equation. For this we introduce the Cartan combination of coordinates and momenta $\sqrt{2}\xi_i^s=\rho_i^s-i\hat{q}_i^s$ and $\sqrt{2}\overset{+}{\xi}{}_i^s=\rho_i^s+i\hat{q}_i^s$, which give Bose operators of creation and annihilation. Now suppose that ξ_i^s and $\overset{+}{\xi}{}_i^s$ transform in accordance with the $U_{3(n-1)}$-irreducible representations $[1]$ and $[-1]$, respectively. Then the algebra of the unitary group is given by the infinitesimal operators $\xi_i^s\overset{+}{\xi}{}_i^s$, to which there corresponds the first Casimir operator $H_0=\sum_{is}\xi_i^s\overset{+}{\xi}{}_i^s$. The operator H_0 takes the following form if we go back to the variables ρ_i^s in it:

$$H_0+\frac{3}{2}(n-1)=-\frac{1}{2}\sum_{is}\left(\frac{\partial^2}{(\partial\rho_i^s)^2}-(\rho_i^s)^2\right). \tag{42}$$

This operator is the Hamiltonian of a $3(n-1)$-dimensional isotropic oscillator expressed in dimensionless units. Its eigenvalue is E. If desired, one can introduce the parameter of the oscillator frequency in (42).

The realization obtained for the Casimir operator of the group $U_{3(n-1)}$ enables us to conclude that the simplest wave function of the $U_{3(n-1)}$ scheme can be interpreted as a function whose spatial part is constructed from functions of a $3(n-1)$-dimensional isotropic harmonic oscillator. In other words, the spatial picture of the

motion of the nucleons is represented by an ensemble of potentially noninteracting quasiparticles, quasiparticles here meaning nucleon formations described by Jacobi coordinates.

Independent quasiparticles can also arise in a system of strongly bound particles. The classical example of such systems is a crystal: The motion of the localized atoms fixed to their positions in the ideal crystal lattice is described in the harmonic approximation by a system of independent quasiparticles—phonons. By means of this analogy, we arrive at the conclusion that the simplest correct model of the nucleus recalls to a certain extent a "crystal", albeit a very odd one. This crystal is isotropic, has a finite size, and a not necessarily spherically symmetric shape. The real particles of which it is formed—the nucleons—are not fixed at certain sites, but execute intricate motions described by the Jacobi variables. In accordance with this picture, the nucleus is a Bose system with respect to the quasiparticles, which does not prevent it being a Fermi system with respect to the nucleons (or protons and neutrons). The Fermi properties in the Bose operators are hidden in the structure of the variables ρ_i formed in accordance with Eq. (1) from the variables $\mathbf{r}_i$.

The "crystal" structure of the nucleus does not presuppose that there is no interaction in the zeroth approximation between the nucleons, nor does it assert that the nucleons move almost independently in the nucleus. Rather the opposite: Like atoms in a crystal lattice, the nucleons can interact in the zeroth approximation, even strongly in the case of a large oscillator frequency; the only important thing is that the interaction leads to a system of potentially noninteracting quasiparticles.

We must now explain why, when speaking of the system described by the zeroth model Hamiltonian H_0, we do not use the generally adopted term "noninteracting quasiparticles" but rather potentially noninteracting quasiparticles. When we construct the spatial function of the scheme $U_{3(n-1)}$ from the eigenfunctions of the Hamiltonian (42) we do indeed proceed from a product of functions of independent quasiparticles, but then, because of the need to ensure the integrals of the motion, we take definite linear combinations of functions of such a basis. At the end, the function of the complete system is not a product of functions of the particles forming the system, and this means that the particles of the system "feel" each other, i.e., interact with one another in a certain manner. This kinematic interaction is not produced by a potential, and we therefore speak of potentially noninteracting quasiparticles. One should not underestimate the importance of the kinematic interaction, since one cannot compare theoretical results and experiments if it is not taken into account.

Returning to a function of the $U_{3(n-1)}$ scheme, we can also say that the above interpretation is not unique, this being due to the well known fact that the transformation properties do not uniquely determine the explicit form of the functions. There is a "degeneracy of interpretation" and this circumstance partly explains the existence of contradictory views of nuclear structure. From this one can also conclude that the transformation properties of the wave function are more important than its actual realization.

Is the model of the $U_{3(n-1)}$ scheme capable of explaining the contradiction between the concepts of strongly collectivized and almost free motion of nucleons in the nucleus, as discussed in Sec. 1? It is already known that, using an appropriate expansion in the wave function of the $U_{3(n-1)}$ scheme, one can reveal the collective and the internal structure of the function. It can be shown that there exists an expansion of a different type, which enables one to represent this function in a form that reveals its shell properties. Such an expansion is obviously possible only in the single-nucleon variables $\mathbf{r}_1, \ldots, \mathbf{r}_n$, the set of which is equivalent to the Jacobi vectors $\rho_1, \ldots, \rho_{n-1}$ if we add to it the variable ρ_0, where $\sqrt{n}\rho_0 = \sum_i \mathbf{r}_i$, which describes the center-of-mass motion of the nucleus. Therefore, the first thing which must be done is to augment the function of the $U_{3(n-1)}$ scheme with the variable ρ_0. In order to see how this must be done, we write the model Hamiltonian H_0 in the variables $\mathbf{r}_1, \ldots, \mathbf{r}_n$:

$$H_0 + \tfrac{3}{2}(n-1) = -\frac{1}{2n}\sum_{i=1}^{n}[(\hat{\mathbf{p}}_i - \hat{\mathbf{p}}_j)^2 - (\mathbf{r}_i - \mathbf{r}_j)^2], \tag{43}$$

where $\hat{p}_i^s$ are the single-particle momenta. We give this operator the single-particle form

$$H_0 + \tfrac{3}{2}(n-1) + H_{\text{cms}} + \tfrac{3}{2} = -\frac{1}{2}\sum_i [(\hat{\mathbf{p}}_i)^2 - (\mathbf{r}_i)^2], \tag{44}$$

where $H_{\text{cms}} = \sum_s \xi_0^{s\dagger} \xi_0^s$ is the oscillator Hamiltonian describing the center-of-mass motion of the nucleus. The right-hand side of the last expression is the model Hamiltonian of the shell model of the nucleus; here we are referring to oscillator shells. The shell Hamiltonian is equal to the sum of H_0 and H_{cms}, and this means physically that the shell picture of the nucleus can be introduced only by placing the nucleus in a well, which cannot be produced by the nucleon–nucleon interaction. The common well for the motion of the nucleons described by the single-particle variables is a well produced by external factors, which are meaningless in the case of a free nucleus. If the free center-of-mass of the nucleus is replaced by a bound oscillator state, the translational invariance of the problem is lost, i.e., we violate the requirement of kinematic correctness, and it is only at this price that we can introduce a shell picture of nuclear structure. From this we conclude that the shell model of the nucleus is incompatible with the requirements of kinematic correctness. This means that the shell representations of nuclear structure are incorrect, i.e., in the nucleus there cannot exist potentially noninteracting particles, and in the best case one can only speak of potentially noninteracting quasiparticles. This conclusion follows from the requirement of translational invariance, and it is therefore universal and applies in the case of arbitrary functions of the nucleus.

These problems have frequently been discussed in the literature, in at least two aspects: One of them is associated with the problem of eliminating the center-of-mass motion of the nucleus; it was considered in

Ref. 25, and then in several other papers. This problem is usually treated as the problem of overcoming a mathematical incorrectness that does not preclude the possible existence in the nucleus of a single-particle field as a physical reality. In fact, the situation is just the reverse, and to emphasize this, we formulate the following theorem.

Theorem. In a kinematically correct theory of the nucleus one cannot introduce the concept of a single-particle field.

It follows from this theorem, which may seem obvious to some readers, that even in the zeroth approximation in nuclear theory one cannot introduce a self-consistent field of the particles, i.e., strictly speaking the Hartree–Fock method does not apply to the nucleus. In this aspect, such a problem has arisen in the literature in connection with the dependence of the average field on the properties of the potential of the nucleon–nucleon interaction. Here, the same conclusion follows from kinematic requirements, and, therefore, does not depend on the particular form of the nuclear Hamiltonian. The physical reason for this lies in the properties of a fermion system of identical particles, as discussed in the introduction: Because there are no force centers, the concept of free motion of nucleons in the nucleus logically contradicts the very existence of the nucleus as a stable quantum system.

Does any of the talk about single-particle levels, outer nucleons, single-particle transitions, and so forth having anything in common with reality? Has not the shell picture of nuclear structure disappeared without trace? From the fundamental point of view, the answer is yes, but from the computational point of view one can say the following: There are cases when the results of calculations on the basis of shell functions and functions of the $U_{3(n-1)}$ scheme agree, and this explains why the shell picture of nuclear structure does sometimes become meaningful.

These cases can be found by expanding a function of the $U_{3(n-1)}$ scheme in terms of shell functions. It was shown above that one can expand thus a function of the $U_{3(n-1)}$ scheme only if it is multiplied by the oscillator function describing the center-of-mass motion of the nucleus. Let us fix the simplest, vacuum, state of the center of mass of the nucleus and write down the expansion

$$2\pi^{-1/4}\exp(-\rho_0^2/2)\,\Psi(E\tau\,|\,\rho_1,\ldots,\rho_{n-1};\,Q) = \sum_{\chi}\Psi(E\chi\,|\,\mathbf{r}_1,\ldots,\mathbf{r}_n;\,Q)\,c_{\chi\tau}. \tag{45}$$

Here, τ is a basis of the unitary or the orthogonal scheme; χ are all the quantum numbers of the shell configurations; and c are expansion coefficients. The actual form of the set of quantum numbers χ depends on the meaning given to the concept of configuration. One can speak of O_3^+ configurations defined by the quantum numbers of the orbital angular momenta of the nucleus. Then the radial functions remain arbitrary and, in particular, as in the theory of atomic spectra, they can be found as solutions of the Hartree–Fock equation. One can also speak of SU_3 configurations (the Elliott model), defined by the oscillator energies of the nucleons. Then the radial functions are oscillator radial functions.

To be specific, we shall assume that τ in (45) stands for quantum numbers of the unitary scheme and χ for quantum numbers of the SU_3 configurations, and we shall attempt to establish when there is just one term in (45). To find the number of terms in (45), it is necessary to list all the SU_3 configurations with the same number of quanta E. Let us consider first the case of the smallest $E=E_{\min}$ allowed by the Pauli principle. It can be shown that there is then only one possible configuration, and therefore there is only one term in the expansion (45) when $E=E_{\min}$ for all n. These are the cases when the motion of the potentially noninteracting quasiparticles is rearranged by the external oscillator well into the motion of potentially noninteracting particles. The SU_3 configuration can be decomposed further in terms of O_3^+ configurations. It happens that in this expansion there is also only one term (this is always for the case for $n\leqslant 16$) and then, using the generally adopted terminology, we can say that the nucleons move in spherical orbits: It is these cases that correspond to the classical shell model of the nucleus.

For states $E=E_{\min}+1$ of the $U_{3(n-1)}$ scheme in the expansion (45) there are two terms; for $E=E_{\min}+2$, there are four,[9] etc. Therefore, when $E>E_{\min}$ the motion of the quasiparticles cannot, as a rule, be rearranged into motion of particles. Nevertheless, there are cases when some of the coefficients of the expansion (45) become, for different reasons, equal to zero and again there remains only one SU_3 configuration, which occasionally degenerates to a single O_3^+ configuration. In other words, even in the case $E>E_{\min}$ one sometimes encounters $U_{3(n-1)}$ states that in the common oscillator well can be rearranged into states of shell type; an analysis of these states in the case $E=E_{\min}+1$ for $6\leqslant n\leqslant 16$ can be found in Ref. 26.

Summarizing, we can say that the $U_{3(n-1)}$ scheme admits as an exception states of shell type in the above sense. However, on the transition from $E=E_{\min}$ to $E>E_{\min}$, and also from light to heavier nuclei, such states are encountered evermore rarely, as a result of which it is impossible, for many states of the $U_{3(n-1)}$ scheme, to rearrange the "crystal" structure of the nucleus into a shell structure by means of an external potential well.

Let us summarize. From the kinematic properties of the nuclear Hamiltonian one can obtain in explicit form all the group characteristics of model functions and, using them, give a very simple physical picture of nuclear structure. One cannot assert that this picture corresponds exactly to reality: The nucleus is too complicated an object for us to understand its structure by means of simple and lucid representations. However, one can say that our picture is the simplest of all possible pictures. The model constructed in this manner contains, despite its simplicity, single-particle, collective, and also intermediate aspects of the motion of nucleons in the nucleus. Working with wave functions defined in the coordinate representation, we must begin to study the properties of nuclei with the model of the $U_{3(n-1)}$ scheme; this model can be ramified and therefore improved, but it cannot be simplified without violating general kinematic principles.

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