Some aspects of the theory of representations of semisimple Lie groups

A. N. Leznov and M. V. Savel'ev

Institute of High Energy Physics, Serpukhov
Fiz. Elem. Chastits At. Yadra. 7, 55–107 (January–March 1976)

The main results in the theory of representations of semisimple Lie groups are expounded compactly. These results are obtained by the asymptotic method (for noncompact groups) and by universal parametrization of compact groups in a form convenient for applications in physics.

PACS numbers: 02.20.Qs

INTRODUCTION

Group-theoretical methods of investigation are an indispensable tool in modern theoretical physics. The perspicuity and relative simplicity of the group-theoretical approach (in the cases when the corresponding mathematical formalism is sufficiently developed) and, above all, its fitness for the task distinguish it and place it ahead of the other methods of investigation. It is the fundamental laws of the physical processes expressed in various symmetries that are the basis of this distinguished role.

Indeed, the theory of group representations, especially of noncompact groups, has developed largely under the stimulus of the urgent requirements of physics. The uses of group theory in theoretical physics are very diverse: solid-state physics, the theory of relativity, classical electrodynamics, nonrelativistic quantum mechanics, relativistic quantum field theory, etc (see, for example, Ref. 1). No less rich is the spectrum of groups—one to suit each physical system, from the simplest finite groups to infinite-dimensional groups whose study has just begun.

Semisimple noncompact Lie groups[2] have found wide application in theoretical physics, especially in the physics of elementary particles. However, effective use of these groups is complicated by some important circumstances. First, the representation theory of noncompact groups has not yet achieved the definitiveness that is characteristic of compact groups. Second, and no less important, many of the results already obtained in this field are formulated in a manner and in terms that are not always convenient for physical applications. Methodologically, the study of the mathematical results of the representation theory of semisimple Lie groups is made much harder by the fact that different methods are frequently used in the derivation and actual construction of the fundamental characteristics of the different groups.

The present review is a compact exposition of some results in the representation theory of semisimple Lie groups obtained by means of the asymptotic method[20,27,40,43] The main idea of the method is borrowed from physics: A particle of mass zero has the simplest kinematics, to which infinitely large values of the hyperbolic angle of rotation correspond from the group point of view. The Regge behavior of the amplitude of potential scattering is a consequence of the power (exponential) asymptotic behavior of the Legendre functions (matrix elements of the noncompact group $SU(1,1)$ with respect to the energy.[3] These properties in fact have a general nature: The asymptotic behavior of the matrix elements of an arbitrary semisimple Lie group is exponential in the region of infinitely large values of the noncompact parameters, and the problem of constructing irreducible representations of these groups can be related to their asymptotic properties.

In this review we shall attempt to formulate the results in the most general form, in which, paradoxically, they take their simplest and most unified form. As the review is intended for physicists, we prefer simplicity and perspicuity of exposition to mathematical rigor in the statements and proofs. Because the irreducible representations and basic characteristics of semisimple Lie groups are obtained in the framework of a single method, it would confuse matters to discuss other approaches and results obtained with them; these can be found in numerous reviews and monographs on the subject (see, for example, Refs. 4–8).

The problems of representation theory most frequently encountered in physical applications get the main attention: construction of algebras of irreducible representations, the finding of their Casimir operators, problems in the harmonic analysis on semisimple groups, the finding of matrix elements of finite transformations in a definite basis, and the finding of global characteristics of representations.

Let us now mention the branches of theoretical physics in which these aspects of representation theory find application. One of the main fields of application of representation theory in particle physics is spectroscopy, i.e., the classification of the elementary particles in accordance with unitary and nonunitary representations of some group (see, for example, Refs. 2 and 9). Physical considerations suggest a particular representation (or a whole class of representations) and enable one to establish the correspondence between the parameters of the representation and the potential candidates to be identified—elementary particles with definite quantum numbers.

In various branches of theoretical physics, one encounters the problem of finding the invariant Casimir operators constructed from the generators of some group, which is assumed to fit the physical system. It suffices to mention their importance in quantum-mechanical calculations in the classification of many-particle states, the determination of the energy levels of atoms and nuclei,[1,9,10] the derivation of mass formulas for elementary particles,[2,11] etc. In conjunction with the quantum numbers of the representations, these operators form a complete set of commuting operators corresponding to the observables. Knowledge of the explicit form of the operators that form the algebra of an arbitrary operator-irreducible representation of a group is helpful in concrete realizations of algebraic approaches in field theory.

Harmonic analysis on a group (or on homogeneous spaces with a given group of motions[2,12]) has found many applications in physics. The problem of expanding a square-integrable function on a group—the amplitude of a process is generally the function—requires knowledge of explicit expressions for the Plancherel measure of the representations of the various principal series and the matrix elements of finite transformations. In some problems of quantum mechanics and field theory[13] it is convenient to expand not with respect to matrix elements but with respect to generating functions, which often have a clearer analytic structure and simple properties. In some cases, sufficient information about individual properties of a physical system is contained in the formulation of Plancherel's theorem in terms of the characters of the representations of the corresponding groups, the dependence on the quantum numbers being summed in such a formulation.

We should also mention the less simple but interesting analogy between Green's functions and the characters of irreducible representations and kernels of Hermitian operators and intertwining operators.[14] It is evidently not by chance that the characters of the irreducible representations of the Poincaré group have some formal properties in common with the Green's functions of free fields; notably, singular behavior on the light cone.[15]

The technique of projection operators has found wide application in problems of the classification of atomic and nuclear levels.[16] The universal parametrization of group elements derived in this paper may, we believe, be helpful in the study of the analytic properties of physical quantities that contain expansions with respect to a complete system of intermediate states.

The results obtained in the present review take their simplest and most compact form in the Cartan–Weyl basis; this basis is intimately related to the root technique, which enables one to treat all semisimple groups from a common point of view. In applications in physics, the tensor basis, in which it is always necessary to take into account the specific tensor structure of the corresponding group, is more usual. The final results for each type of classical complex group and the corresponding real forms are given in tables.

1. REALIZATION OF IRREDUCIBLE REPRESENTATIONS OF THE ALGEBRAS OF SEMISIMPLE LIE GROUPS. INVARIANT OPERATORS

Notation and definitions. The most impressive results of the structure theory of semisimple Lie algebras and groups are associated with the invariant root formulation and go back to the classical studies of Killing, Cartan, and Weyl. The idea that one could achieve a unified classification and description of the basic properties and characteristics of all semisimple Lie algebras and groups by a classification of the associated system of roots and an invariant root form of expression had a very fruitful influence on the study of these objects. Without going into an exposition of the main concepts and results obtained in the theory of semisimple Lie algebras and groups in the invariant root technique (see, for example, Refs. 4–7), we introduce here merely the notation and definitions needed later: G is an arbitrary semisimple Lie group; $\mathscr{K}$ is the maximal compact subgroup in G; $\mathscr{A}$ is the maximal noncompact Agelian subgroup in G; S is the centralizer of $\mathscr{A}$ in $\mathscr{K}$; W_B is the Weyl group corresponding to the group $B=\{G, \mathscr{K}, S, \ldots\}$; $\mathfrak{g}$ is the Lie algebra of G; $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$; $r_B(\zeta_B)$ is the rank (dimension) of the group $B=\{G, \mathscr{K}, \mathscr{A}, S\}$; R is the system of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$; n is the number of positive roots; ρ_0 is the half-sum of the positive roots; $X_{\pm\alpha}$ are the elements of the root space of root $\alpha, \alpha \in R$; h_i are the generators of $\mathfrak{h}$, corresponding to the simple roots π_i; $1 \leqslant i \leqslant r$.

In the Cartan–Weyl basis the elements $X_{\pm\alpha}$ and h_i satisfy the relations

$$[X_\alpha, X_\beta]_- = \begin{cases} N_{\alpha\beta} X_{\alpha+\beta}, & \alpha+\beta \in R; \\ 0, & \alpha+\beta \neq 0, \ \alpha+\beta \in R; \\ h_\alpha, & \alpha+\beta = 0; \end{cases} \quad [h_i, h_j]_- = 0; \qquad (1)$$
$$[h_i, X_{\pm\alpha}]_- = \pm \alpha(h_i) X_{\pm\alpha};$$

and for compact roots $X_\alpha^+ = X_{-\alpha}$, $h_i^+ = h_i$; where + is the symbol of Hermitian conjugation (* and T are the complex conjugate and transpose, respectively). The elements $X_{\pm\alpha}$ and h_i, which form a complete set Y_μ, $\mu = \pm\alpha, i$, can be normalized by the relations

$$\mathrm{Sp}\, X_\alpha X_\beta = \delta_{\alpha+\beta,0}; \quad \mathrm{Sp}\, h_i h_j = \delta_{ij}; \quad \mathrm{Sp}\, X_\alpha h_i = 0 \qquad (2)$$

in a definite basis, i.e., $\mathrm{Sp} Y_\mu Y^\nu = \delta_\mu^\nu$.

In what follows, in the cases when it is necessary to distinguish complex groups and their real forms, the superscripts C and R^i, respectively, will be appended to the symbol of the corresponding group. The real form G^{R_i} of the complex semisimple group G^C is determined by specifying the matrix σ of the involutary automorphism, and when $\mathscr{K}^C$ is restricted to $\mathscr{K}^{R_i}$ the elements X_α go over into $\mathbf{X}_\alpha = X_\alpha + \sigma X_\alpha \sigma$.

The main characteristics of semisimple Lie groups are listed in Table 1.

Generators of Semisimple Lie Groups. The development of a constructive theory of representations of Lie algebras as an important independent branch of modern mathematics as well as a powerful tool in the investigation of group-theoretical problems is based on their linearization and in the framework of the infinitesimal approach requires knowledge of the explicit expressions for the generators of the algebras in some parametrization. The effectiveness of the infinitesimal approach and exploitation of its advantages over the global approach in the derivation of concrete results depend on the complexity of the parametric dependences of the generators of the algebras under consideration.

TABLE I.

$G=\mathcal{K}\mathcal{A}\mathcal{K}$	r_G	ζ_G	$\sigma(X)$	$\mathcal{K}$	$r_{\mathcal{K}}$	$\zeta_{\mathcal{K}}$	Element $T(\tau)$	$r_{\mathcal{A}}$	S	r_S	ζ_S
$L(n, C)$ A	$2n$	$2n^2$	—	$U(n)$	n	n^2	$\exp\tau$	n	$\underbrace{U(1)\otimes\ldots\otimes U(1)}_{n}$	n	n
$L(n, R)$ $A_{\rm I}$	n	n^2	$\sigma(X)=X^*$	$O(n)$	$\left[\frac{n}{2}\right]$	$\frac{n(n-1)}{2}$	$\exp\tau$	n	$\underbrace{Z_2\otimes\ldots\ldots\otimes Z_2}_{n}$	—	—
$U^*(2n)$ $A_{\rm II}$	$2n$	$4n^2$	$\sigma(X)=J_nX^*J_n^{-1}$	$\mathrm{Sp}(2n)$	n	$n(1+2n)$	$\begin{pmatrix}\exp\tau & 0\\ 0 & \exp\tau\end{pmatrix}$	n	$\underbrace{SU(2)\otimes\ldots\ldots\otimes SU(2)}_{n}$	n	$3n$
$U(p, q)$ $A_{\rm III}$ $p\geqslant q$, $p+q=n$	n	n^2	$\sigma(X)=-J_{pq}X^+J_{pq}$	$U(p)\otimes U(q)$	n	p^2+q^2	$\begin{pmatrix}\cosh\tau & \sinh\tau\\ \sinh\tau & \cosh\tau\end{pmatrix}$	q	$U(p-q)\otimes\otimes\underbrace{U(1)\otimes\ldots\otimes U(1)}_{q}$	p	$(p-q)^2+q$
$O(n, C)$ B, D	$2[n/2]$	$n(n-1)$	—	$O(n)$	$\left[\frac{n}{2}\right]$	$\frac{n(n-1)}{2}$	$\begin{pmatrix}\cosh\tau & i\sinh\tau\\ -i\sinh\tau & \cosh\tau\end{pmatrix}$	$\left[\frac{n}{2}\right]$	$\underbrace{U(1)\otimes\ldots\ldots\otimes U(1)}_{[n/2]}$	$\left[\frac{n}{2}\right]$	$\left[\frac{n}{2}\right]$
$O^*(2n)$ $D_{\rm III}$	n	$n(2n-1)$	$\sigma(X)=J_nX^*J_n^{-1}$	$U(n)$	n	n^2	$\begin{pmatrix}\cosh\tau & i\sinh\tau & 0 & 0\\ -i\sinh\tau & \cosh\tau & 0 & 0\\ 0 & 0 & \cosh\tau & -i\sinh\tau\\ 0 & 0 & i\sinh\tau & \cosh\tau\end{pmatrix}$	$\left[\frac{n}{2}\right]$	$\underbrace{SU(2)\otimes\ldots\otimes SU(2)}_{k}$, $n=2k$ $\underbrace{SU(2)\otimes\ldots\otimes SU(2)}_{k}\otimes\otimes U(1)$, $n=2k+1$	k $k+1$	$3k$ $3k+1$

TABLE I. *(continued)*

$G=\mathcal{K}\mathcal{A}\mathcal{K}$	r_G	ζ_G	$\sigma(X)$	$\mathcal{K}$	$r_{\mathcal{K}}$	$\zeta_{\mathcal{K}}$	Element $T(\tau)$	$r_{\mathcal{A}}$	S	r_S	ζ_S
$O(p, q)$ $p+q=n$, $p\geqslant q$ BDI	$\left[\frac{n}{2}\right]$	$\frac{n(n-1)}{2}$	$\sigma(X)=J_{pq}X^*J_{pq}$	$O(p)\otimes O(p)$	$\left[\frac{n}{2}\right]$	$\frac{p(p-1)}{2}+\frac{q(q-1)}{2}$	$\begin{pmatrix}\cosh\tau & \sinh\tau\\ \sinh\tau & \cosh\tau\end{pmatrix}$	q	$O(p-q)\otimes\otimes\underbrace{Z_2\otimes\ldots\ldots\otimes Z_2}_{q}$	$\left[\frac{p-q}{2}\right]$	$\frac{(p-q)(p-q-1)}{2}$
$\mathrm{Sp}(2n, C)$ C	$2n$	$2n(1+2n)$	—	$\mathrm{Sp}(2n)$	n	$n(1+2n)$	$\begin{pmatrix}\exp\tau & 0\\ 0 & \exp(-\tau)\end{pmatrix}$	n	$\underbrace{U(1)\otimes\ldots\ldots\otimes U(1)}_{n}$	n	n
$\mathrm{Sp}(2n, R)$ CI	n	$n(1+2n)$	$\sigma(X)=X^*$	$U(n)$	n	n^2	$\begin{pmatrix}\exp\tau & 0\\ 0 & \exp(-\tau)\end{pmatrix}$	n	$\underbrace{Z_2\otimes\ldots\ldots\otimes Z_2}_{n}$	—	—
$\mathrm{Sp}(2p, 2q)$ $p\geqslant q$, $p+q=n$ $C\,II$	n	$n(1+2n)$	$\sigma(X)=-I_{pq}X^+I_{pq}$	$\mathrm{Sp}(2p)\otimes\otimes\mathrm{Sp}(2q)$	n	$p(2p+1)+q(2q+1)$	$\begin{pmatrix}\cosh\tau & \sinh\tau & 0 & 0\\ \sinh\tau & \cosh\tau & 0 & 0\\ 0 & 0 & \cosh\tau & \sinh\tau\\ 0 & 0 & \sinh\tau & \cosh\tau\end{pmatrix}$	q	$\mathrm{Sp}(2p-2q)\otimes\otimes\underbrace{\mathrm{Sp}(2)\otimes\ldots\otimes\mathrm{Sp}(2)}_{q}$	p	$(p-q)\times(2p-2q+1)+3q$

Note. The following notation is used in the table:

$$J_{pq}=\begin{pmatrix}I_p & 0\\ 0 & -I_q\end{pmatrix};\quad J_n=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix};\quad I_{pq}=\begin{pmatrix}+I_p & 0 & 0 & 0\\ 0 & -I_q & 0 & 0\\ 0 & 0 & +I_p & 0\\ 0 & 0 & 0 & -I_q\end{pmatrix};\qquad I_n\equiv\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}.$$

The point of departure in the determination of the explicit expressions for the generators of the group algebra is the choice of a parametrization of the group's elements. If the Cartan decomposition is used, it provides a natural possibility for investigating the basic characteristics of noncompact semisimple Lie groups on the basis of the results known for the compact case; we shall also see that it brings out a certain uniformity in the final results for the compact and noncompact cases.

We should point out that a parametrization of the noncompact groups in terms of their maximal compact subgroups is not necessary for the general investigation of representations or for the formulation or use of the asymptotic method proposed later.

Each element g in G, excluding a set of lower dimension, is parametrized by the Cartan decomposition:

$$g = k_1 T(\tau) k_2; \quad k_{1,2} \in \mathscr{K}; \quad T(\tau) \in A, \tag{3}$$

which is unique under the assumption that the parameters of $k_2(k_1)$ associated with the centralizer are included in $k_1(k_2)$. Then the basic group relation $\delta g = -F_\epsilon g \sigma\varepsilon$ can be rewritten in the form

$$k_1^{-1}\delta k_1 + \delta T \cdot T^{-1} + T\delta k_2 k_2^{-1} T^{-1} = S\delta\varepsilon; \quad S = -k_1^{-1} F_\varepsilon k_1, \tag{4}$$

where F_ϵ is some tangent infinitesimal operator and $\delta\varepsilon$ is the parameter of this transformation.

The infinitesimal operator of the left regular representation corresponding to the tangent transformation F_ϵ is determined by

$$\hat{F}_\varepsilon = \sum_{\alpha,\beta} \dot{\omega}_{\alpha\beta} \hat{\tilde{\mathscr{K}}}^{\alpha\beta}_{(1)} + \sum_k \dot{\tau}_k \cdot \partial/\partial\tau_k + \sum_{\alpha,\beta} \dot{\Omega}_{\alpha\beta} \hat{\mathscr{K}}^{(2)}_{\alpha,\beta}. \tag{5}$$

where the derivatives $\mathring{\omega} = k_1^{-1}\mathring{k}_1$, $\mathring{\Omega} = \dot{k}_2 k_2^{-1}$, $\mathring{\tau} = \dot{T}T^{-1}$ ($\delta T = \mathring{T}\delta\varepsilon$, $\delta k_{1,2} = \dot{k}_{1,2}\delta\varepsilon$, $\mathring{\omega}^+ = -\mathring{\omega}$, $\mathring{\Omega}^+ = -\dot{\Omega}$) of the group parameters in (5) are found from the system of equations (4). By $\hat{\tilde{\mathscr{K}}}^{\alpha\beta}_{(1)}$ ($\hat{\mathscr{K}}^{(2)}_{\alpha\beta}$) we denote the infinitesimal operators of the right (respectively, left) regular representation of $\mathscr{K}$; $(\mathring{T}T^{-1})_{kl} = \delta_{kl}\mathring{\tau}_l$. Similarly, we define operators $\hat{\tilde{F}}_\epsilon$ for the right regular representation of G, which are expressed in terms of $\hat{\mathscr{K}}^{(1)}_{\alpha\beta}$ and $\hat{\tilde{\mathscr{K}}}^{\alpha\beta}_{(2)}$.

In the Cartan–Weyl basis for an arbitrary semisimple Lie group one can obtain a unified invariant root form of expression of the operators of the group algebra. We begin with the case of a complex semisimple Lie group G^C for which the tangent operators F and Φ (corresponding to compact, $\hat{F}$, and noncompact, $\hat{\Phi}$, generators, respectively) can be expressed in terms of $X_{\pm\alpha}$ and h_i as follows:

$$\left.\begin{array}{c} F^\varepsilon_\alpha = i^{\varepsilon-1} X_\alpha - (-i)^{\varepsilon-1} X_{-\alpha}; \; \Phi^\varepsilon_\alpha = i^\varepsilon X_\alpha + (-i)^\varepsilon X_{-\alpha}; \\ F_s = ih_s; \; \Phi_s = h_s; \; \varepsilon = 1, 2. \end{array}\right\} \tag{6}$$

Using the decompositions

$$M = \sum_\mu m_\mu Y_\mu; \; M \equiv \{\dot{\omega}, \dot{\Omega}, S\}; \; \dot{\tau} = \sum_i \dot{\tau}_i h_i;$$
$$m_\mu = \mathrm{Sp}\, MY^{\mu}; \; \mu = \{\pm\alpha, i\}$$

and the relation $\exp(\sum_i \tau_i h_i) X_\alpha \exp(-\sum_i \tau_i h_i) = X_\alpha \exp\alpha(\tau)$, which follows from Eqs. (1) and (2), we rewrite the system (4) in the form

$$\sum_{\alpha \gtrless 0} (\dot{\omega}_\alpha + \dot{\Omega}_\alpha \exp[\alpha(\tau)] - S_\alpha) X_\alpha + \sum_i (\dot{\omega}_i + \dot{\Omega}_i + \dot{\tau}_i - S_i) h_i = 0. \tag{7}$$

The solutions of this last system:

$$\left.\begin{array}{l} \dot{\omega}_\alpha = -\dfrac{\exp[\alpha(\tau)] S^+_{-\alpha} - \exp[-\alpha(\tau)] S_\alpha}{2\,\mathrm{sh}\,\alpha(\tau)}; \\ \qquad \dot{\Omega}_\alpha = \dfrac{S^+_{-\alpha} - S_\alpha}{2\,\mathrm{sh}\,\alpha(\tau)}; \\ \dot{\tau}_i = (S^+_i + S_i)/2; \; \dot{\omega}_i + \dot{\Omega}_i = -(S^+_i - S_i)/2 \end{array}\right\} \tag{8}$$

are a trivial consequence of the relations (2). It should be noted that all the $\alpha(\tau)$'s are nonzero since otherwise there would exist an element X_α that commutes with $\mathfrak{h}$.

The expressions obtained for the generators have a very complicated analytic dependence on the group parameters k_1, k_2, τ. However, it is easy to see that Eqs. (8) simplify appreciably in the asymptotic region of infinitely large noncompact parameters τ: the $\dot{\Omega}$'s become zero, and the dependence on the parameters τ in $\dot{\omega}$ disappears. At the same time, there is no functional dependence of the generators $\hat{F}_\epsilon(\hat{\tilde{F}}_\epsilon)$ on the parameters $k_2(k_1)$, and the τ_i's occur in them only in the form of the trivial derivatives $\partial/\partial\tau_i$. This circumstance formed the basis of the asymptotic method of constructing representations of semisimple Lie groups; the essence of the method is to go over from the shift generators on a noncompact group to the asymptotic values of the noncompact parameters. In this connection, it is helpful to mention the interesting analogy between this method and the theory of potential scattering. It is well known (see, for example, Ref. 3) that all the necessary information about the dynamical nature of potential theory is contained in the S matrix, which is the ratio of Jost functions—the preexponential factors in the asymptotic expression for the Schrödinger wave function. In the theory of representations of noncompact semisimple Lie groups there is an analogous situation, and the role of the Jost functions is played by the principal terms in the asymptotic expansion of the matrix element of the corresponding representation.

Thus, in the asymptotic region the noncompact infinitesimal operators $\hat{\Phi}$ of the left regular representation of the complex semisimple Lie group G^C have the form

$$\begin{array}{l} \hat{\Phi}^C = \sum_\alpha \dot{\omega}^{(a)}_\alpha \hat{\tilde{X}}^\alpha + \sum_i (\dot{\tau}_i \partial/\partial\tau_i + \dot{\omega}_i \hat{\tilde{h}}^i); \\ \omega^{(a)}_\alpha = -\theta(\alpha) S^+_{-\alpha} + \theta(-\alpha) S_\alpha. \end{array} \tag{9}$$

where $\hat{\tilde{X}}^\alpha$ and $\hat{\tilde{h}}^i$ are the generators of the right shifts on $\mathscr{K}^C$ corresponding to the compact element X_α of the root space and the generator h_i of the subalgebra $\mathfrak{h}$, respectively. The compact infinitesimal operators $\hat{F}$ of G are identical with the generators $\hat{Y}_\mu$ of the left shifts on $\mathscr{K}$ irrespective of the limiting procedure and in accordance with (8) (which, incidentally, is obvious) are related to the generators $\hat{\tilde{Y}}^\nu$ of right shifts on $\mathscr{K}$ by the matrix of the adjoint representation:

$$\hat{Y}_\mu = -\sum_\nu \mathrm{Sp}\,(k^{-1} Y_\mu k Y^\nu)\, \hat{\tilde{Y}}^\nu. \tag{10}$$

Note that if we consider the action of the last relation on the highest vector of weight $\{l\}$ with respect to the quantities with a tilde, Eq. (10) takes the form

$$-\hat{Y}_\mu = \sum_i \mathrm{Sp}\,(k^{-1} Y_\mu k h^i)\, l_i + \sum_{\alpha>0} \mathrm{Sp}\,(k^{-1} Y_\mu k X_\alpha)\, \hat{\tilde{X}}^{-\alpha}. \tag{11}$$

The infinitesimal operators of the right regular representation of G can be expressed by an equation analogous to (9) with the obvious replacement of k_1 by k_2 and $\hat{\tilde{X}}^\alpha$, $\hat{\tilde{h}}^i$ by $\hat{X}^\alpha$, $\hat{h}_i$. Because the operators $\partial/\partial\tau_i$ and $\hat{\tilde{h}}^i$ obviously commute with all generators of left shifts on G^C, they can be replaced by C-number parameters ρ_i and s_i characterizing representations of the constructed group algebra. In what follows, we require the infinitesimal operators $\hat{F}_\pm = (\hat{F} \pm i\hat{\Phi})/2$, for which there is the simple expression

$$\hat{F}_\pm = \pm \sum_i \mathrm{Sp}\,(k^{-1} F k h_i) \frac{\rho_i \pm s_i}{2} + \sum_{\alpha>0} \mathrm{Sp}\,(k^{-1} F k X_{\pm\alpha})\, \hat{\tilde{X}}^{\mp\alpha}. \tag{12}$$

We emphasize that these results for the generators are valid, as follows from their construction, not only for the complex classical series but also for the exceptional Cartan groups G_2, F_4, E_6, E_7, E_8.

TABLE II.

G	Infinitesimal operators	$k \in \mathscr{K}$
$L(n, C)$	$\Phi_{ij} = -\sum_1^n n_i^{*k} n_j^k \partial/\partial\tau_k + \sum_{k,l} \operatorname{sgn}(l-k)\, n_i^{*k} n_j^l \widetilde{F}^{kl}$	$n \in U(n)$
$L(n, R)$	$\Phi_{ij} = -\sum_1^n n_i^k n_j^k \partial/\partial\tau_k + \sum_{k,l} \operatorname{sgn}(l-k)\, n_i^k n_j^l \widetilde{F}^{kl}$	$n \in O(n)$
$U^*(2n)$	$\Phi_{ij} = -\sum_{-n}^{n} \left[k,\ i;\ {k \atop j}\right] \partial/\partial\tau_k$ $+1/2 \sum_{\lvert k\rvert, \lvert l\rvert} + \operatorname{sgn}(\lvert l\rvert - \lvert k\rvert) \left[k,\ i;\ {l \atop j}\right] \widetilde{F}_l^k;\ \left[k,\ i;\ {l \atop j}\right]$ $\equiv n_{ki} n_j^l - n_{kj} n_i^l$	$n \in \mathrm{Sp}(2n)$
$U(p, q)$ $p \geqslant q$	$\Phi_{\alpha j}^{\mu} = i^{\mu-1} A_{\alpha j} - (-i)^{\mu-1} B_{\alpha j};\ \mu = 1, 2;\ 1 \leqslant \alpha \leqslant p;$ $1 \leqslant j \leqslant q$ $A_{\alpha j} = -1/2 \sum_1^q p_\alpha^{*k} q_j^k (\partial/\partial\tau_k + \widetilde{\mathscr{F}}^{kk} - \widetilde{Q}^{kk})$ $- \sum_{p \geqslant \beta > l \geqslant 1} p_\alpha^{*\beta} q_j^l \widetilde{\mathscr{F}}^{\beta l} + \sum_{1 \leqslant k < l \leqslant q} p_\alpha^{*k} q_j^l \widetilde{Q}^{kl},\ B_{\alpha j} = A_{\alpha j}^*$	$p \in U(p)$ $q \in U(q)$
$O(n, C)$	$n = 2k$: $\Phi_{ij}^{2k} = \sum_1^k \left[{2\alpha \atop i};\ {2\alpha \atop j}\right] \partial/\partial\tau_{2\alpha}$ $+ \sum_{\alpha > \beta} \left\{ \left[{2\beta - 1 \atop i};\ {2\alpha \atop j}\right] \widetilde{F}^{2\beta-1,\,2\alpha} \right.$ $+ \left[{2\beta \atop i};\ {2\alpha - 1 \atop j}\right] \widetilde{F}^{2\beta-1,\,2\alpha-1}$ $- \left[{2\beta - 1 \atop i};\ {2\alpha - 1 \atop j}\right] \widetilde{F}^{2\beta,\,2\alpha-1}$ $\left. - \left[{2\beta - 1 \atop i};\ {2\alpha \atop j}\right] \widetilde{F}^{2\beta,\,2\alpha} \right\}$	$n \in O(n)$

TABLE II. *(continued)*

G	Infinitesimal operators	$k \in \mathscr{K}$
$\mathrm{Sp}(2n, R)$	$\Phi_{kl}^1 = \widetilde{F}^{lk} - \widetilde{F}^{kl} - \sum_1^n \left\{{i \atop l};\ {\overset{*}{i} \atop k}\right\} \partial/\partial\tau_i + \sum_1^n \left[{i \atop l};\ {\overset{*}{i} \atop k}\right]$ $\times \widetilde{F}^{ii} + \sum_{i>j} \left\{{i \atop l};\ {j \atop k}\right\} \widetilde{F}^{ij} - \sum_{i<j} \left\{{\overset{*}{i} \atop l};\ {\overset{*}{j} \atop k}\right\} \widetilde{F}^{ij},$ $\left[{i \atop k};\ {\overset{*}{i} \atop l}\right] = n_k^i n_l^i - n_k^{*i} n_l^{*i}$ $\Phi_{kl}^2 = -\mathrm{i} \sum_1^n \left[{i \atop k};\ {\overset{*}{i} \atop l}\right] \partial/\partial\tau_i - \mathrm{i} \sum_1^n \left\{{i \atop l};\ {\overset{*}{i} \atop k}\right\} \widetilde{F}^{ii}$ $-\mathrm{i} \sum_{i>j} \left\{{i \atop l};\ {j \atop k}\right\} \widetilde{F}^{ij} - \mathrm{i} \sum_{i<j} \left\{{i^* \atop l};\ {j^* \atop k}\right\} \widetilde{F}^{ij},$ $\left\{{i \atop l};\ {j \atop k}\right\} = n_l^i n_k^j + n_k^i n_l^j$ $\left\{{i \atop l};\ {i^* \atop k}\right\} = n_k^i n_l^i + n^{*i} n_l^{*i}$	$n \in U(n)$
$\mathrm{Sp}(2p, 2q)$ $p \geqslant q$	$\Phi_{\lambda k} \equiv \Phi_{k\lambda} = -1/4 \sum_{-q}^{q} \left[i,\ \alpha;\ {i \atop k}\right] \partial/\partial\tau_i$ $-1/4 \sum_{-q}^{q} \left[\mu,\ \lambda;\ {\nu \atop k}\right] (\widetilde{\mathscr{P}}_\nu^\mu - \widetilde{Q}_\nu^\mu)$ $-1/4 \sum_{-q}^{q} \operatorname{sgn}(\lvert\mu\rvert - \lvert\nu\rvert) \left[\mu,\ \lambda;\ {\nu \atop k}\right]$ $\times (\widetilde{\mathscr{P}}_\nu^\mu + \widetilde{Q}_\nu^\mu); \left[\mu,\ \lambda;\ {\nu \atop k}\right] \equiv p_{\mu\lambda} q_k^\nu - q_{\mu k} p_\lambda^\nu$	$p \in \mathrm{Sp}(2p)$ $q \in \mathrm{Sp}(2q)$
$O(n, C)$	$n = 2k+1$: $\Phi_{ij}^{2k+1} = \Phi_{ij}^{2k} - \sum_\alpha \left\{ \left[{2\alpha \atop i};\ {2k \atop j}\right] \widetilde{F}^{2\alpha-1,\,2k} \right.$ $\left. - \left[{2\alpha - 1 \atop i};\ {2k - 1 \atop j}\right] \widetilde{F}^{2\alpha,\,2k+1} \right\};$ $\left[{\theta \atop i};\ {\varphi \atop j}\right] = n_i^\theta n_j^\varphi - n_i^\varphi n_j^\theta$	$n \in O(n)$
$O^*(2n)$	$n = 2k$: $\Phi_{ij}^{2k} = \sum_1^k \left[{2\alpha - 1 \atop i};\ {2\alpha \atop j}\right] (\partial/\partial\tau_{2\alpha}$ $- i\widetilde{F}^{2\alpha-1,\,2\alpha-1} - i\widetilde{F}^{2\alpha,\,2\alpha})$ $-\mathrm{i} \sum_{\alpha > \beta} \left\{ \left[{2\alpha \atop i};\ {2\beta \atop j}\right] \widetilde{F}^{2\alpha-1,\,2\beta-1} \right.$ $+ \left[{2\alpha \atop i};\ {2\beta \atop j}\right] \widetilde{F}^{2\alpha-1,\,2\beta}$ $- \left[{2\alpha + 1 \atop i};\ {2\beta - 1 \atop j}\right] \widetilde{F}^{2\alpha,\,2\beta-1}$ $\left. - \left[{2\alpha - 1 \atop i};\ {2\beta \atop j}\right] \widetilde{F}^{2\alpha,\,2\beta} \right\}$	$n \in U(n)$
	$n = 2k+1$: $\Phi_{ij}^{2k+1} = \Phi_{ij}^{2k} - \sum_1^k \left\{ \left[{2\alpha \atop i};\ {2k \atop j}\right] \widetilde{F}^{2\alpha-1,\,2k} \right.$ $\left. - \left[{2\alpha - 1 \atop i};\ {2k - 1 \atop j}\right] \widetilde{F}^{2\alpha,\,2k+1} \right\}$	$n \in U(n)$
$O(p, q)$ $p \geqslant q$	$\Phi_{\alpha i} = -\sum_1^q p_\alpha^k q_i^k \partial/\partial\tau_k - \sum_{p \geqslant \beta > k \geqslant 1} p_\alpha^\beta q_i^k \widetilde{\mathscr{F}}^{\beta k}$ $+ \sum_{1 \leqslant k < l \leqslant q} p_\alpha^k q_i^l \widetilde{Q}^{kl},\ 1 \leqslant \alpha \leqslant p,\ 1 \leqslant i \leqslant q$	$p \in O(p)$ $q \in O(q)$
$\mathrm{Sp}(2n, C)$	$\Phi_{ab} = -\mathrm{i} \sum_1^{2n} \left\{c,\ a;\ {c \atop b}\right\} \partial/\partial\tau_c + \mathrm{i} \sum_{c,\,d=1}^{2n} \operatorname{sgn}(d-c)$ $\times \left\{c,\ a;\ {d \atop b}\right\} \widetilde{F}_d^c,\ \left\{c,\ a;\ {d \atop b}\right\} = n_{ca} n_b^d - n_{cb} n_a^d$	$n \in \mathrm{Sp}(2n)$

Note. In the table we give the noncompact operators of left shifts for the classical semisimple Lie groups expressed in terms of the operators of the right regular representation of the maximal compact subgroups $(\widetilde{F}, \widetilde{\mathscr{P}}, \widetilde{Q})$ and its matrix elements (n, p, q) in accordance with Eqs. (9) and (13).

By a direct calculation similar to the one above we can readily show that the generators of the real form G^{R_i} of the complex semisimple Lie group G^C can be obtained from the generators of the latter by restricting $\mathscr{K}^C$ to $\mathscr{K}^{R_i}$. Then, as in the complex case, it is convenient to choose as parameters defining the constructed representations of the algebra g^{R_i} the eigenvalues ρ_i of the operators $\partial/\partial\tau_i$, $1 \leqslant i \leqslant r_{\mathscr{A}}$ and the weights $\{s\}$ of the irreducible representation of the subgroup S. The generators of the real form obtained in this manner in the asymptotic region have the form

$$\hat{\Phi}^{R_i} = \sum_{i=1}^{r_{\mathscr{A}}} \mathrm{Sp}(k^{-1}Fkh_i)\,\rho_i + \sum_{\xi=1}^{r_s} \mathrm{Sp}(k^{-1}Fkh_i)\, s_i - \sum_\alpha \varepsilon(\alpha)\, \mathrm{Sp}(k^{-1}FkX_\alpha)\, \hat{\tilde{X}}^{-\alpha}. \qquad (13)$$

It follows from comparison of (9), (11), and (13) that the generators of the complex semisimple Lie group can be obtained from the corresponding generators of the compact group by simple replacement of the highest weights l_i by arbitrary complex numbers $(\rho_i \pm s_i)/2$, whereas the generators of the real form are obtained by subsequent restriction to the maximal compact subgroup of the real form.

Note that until the representations are particularized, i.e., we require of them pseudounitarity, unitarity, etc., the parameters $\{\rho\}$ are arbitrary complex numbers. Imposition of the requirement of pseudounitarity and unitarity leads to restrictions on the real and imaginary parts of these parameters. Because the generators (9) and (13) are shift operators on the space of square integrable functions defined on the maximal compact subgroup $\mathcal{K}$ of G, the representations of G for which the set $\{s\}$ is integral are single-valued. As will be shown in what follows, the representations are operator-irreducible and the separation of the topologically irreducible components from them is based on the study of the analytic properties of the kernels of the intertwining operators.

Explicit expressions for the generators of the algebras of all types of classical semisimple Lie groups in the tensor basis are given in Table 2.

Casimir operators of semisimple Lie Groups. The Casimir operators in the general case of an arbitrary semisimple Lie algebra $\mathfrak{g}$ are elements of the center of its universal enveloping algebra, the center being isomorphic to the algebra of all polynomials over $\mathfrak{h}$ that are invariant under the Weyl group[17]; the Casimir operators can be expressed in terms of the traces of successive powers of the matrix of generators of the corresponding representation.[18] (For groups in which there is a completely antisymmetric tensor, the complete system of Casimir operators also includes an additional operator which is a generalization of the Pauli—Lubański pseudoscalar for the Lorentz group. The presence of this operator enables one, for these groups, to establish a one-to-one connection between the weight vector, which completely determines the irreducible representation, and the set of Casimir operators.) For operator-irreducible representations of $\mathfrak{g}$ the Casimir operators reduce to C-number functions of their highest weights. In the case of the classical series, the calculations can, because of the presence of a matrix realization, be performed in the tensor and the Cartan basis, whereas for the exceptional Cartan groups the form of the generators is known only in the Cartan—Weyl basis, and the first possibility is therefore eliminated. In Ref. 19, Perelomov and Popov used the traditional method (application to the highest vector) to consider the Casimir operators of semisimple Lie groups that have at least one finite-dimensional nondegenerate representation (nontrivial), and they calculated explicitly their eigenvalues for compact groups of the type A_r, B_r, C_r, D_r, G_2.

The explicit expressions obtained in the previous section for the shift operators on a semisimple Lie group enable one to solve the eigenvalue and eigenfunction problem for Casimir operators constructed from the generators for finite values of the parameters $\{\tau\}$ and calculate the eigenvalue spectrum of the Casimir operators for an arbitrary operator-irreducible representation of these groups irrespective of whether they are degenerate or nondegenerate and finite- or infinite-dimensional. Leaving out the detailed calculations, which are made in Ref. 20, we list only the main stages for the complex case, which are not significantly modified on the transition to the real forms of the complex groups. By virtue of their commutation properties, the algebra of the generators $\hat{F}_{\pm}$ (12) is the direct sum of the algebras of the corresponding compact group $\mathcal{K}^C$. Therefore, the system of Casimir operators is divided into two sets of Casimir operators $C_p^{\pm}$ of the corresponding compact group constructed with the generators $\hat{F}_{\pm}$. The system Y_μ being complete, the matrix of generators of the representation, $G_{\pm}=\sum_\mu (\hat{F}_\mu)_{\pm} Y_\mu$, can be expressed by

$$G_{\pm} = \pm \sum_i (kh_ik^{-1})\, d_{\pm i} + \sum_{\alpha>0} (kX_{\pm\alpha}k^{-1})\, \tilde{X}^{\mp\alpha}; \quad d_{\pm i} = (\rho_i \pm s_i)/2. \quad (14)$$

To calculate the successive powers of this matrix, it is convenient to rewrite it in the form (to be specific we take G_+)

$$G_+ = k\, [\sum_i h_id_i + \sum_{\alpha>0} (X_\alpha X_{-\alpha} + X_\alpha \tilde{X}^{-\alpha})]\, k^{-1},$$

whence

$$\begin{aligned} C_p^+ \equiv \operatorname{Sp} G_+ = \operatorname{Sp}(k\Phi^p k^{-1}); \quad \Phi = \Psi + \sum_{\alpha>0} X_\alpha \tilde{X}^{-\alpha}; \quad \Psi \\ \equiv \sum_i h_id_i + \sum_{\alpha>0} X_\alpha X_{-\alpha}. \end{aligned} \quad (15)$$

We represent the result of commuting the matrices Φ and k^{-1} by

$$\begin{aligned} \Phi_{ab}\,(k^{-1})_c^i = (k^{-1})_c^i\, \Phi_{ab} + \sum_d (k^{-1})_d^i\, \theta_{ab;\,cd}; \quad \theta_{ab;\,cd} \equiv \\ - \sum_{\alpha>0} (X_\alpha)_{ab}\, (X_{-\alpha})_{cd}. \end{aligned} \quad (16)$$

Since the trace of a product containing an arbitrary number of elements of the space of positive roots at the same time as matrices of $\mathfrak{h}$ is zero,

$$C_p^+ = \sum_{a_1,\, b_1;\, a_2,\, b_2} [\sum_i h_id_i + \sum_{\alpha>0} h_\alpha]_{a_1b_1} D_{a_1b_1;\, a_2b_2} [\sum_i h_id_i]_{a_2b_2}, \quad (17)$$

where

$$D_{a_1,\, b_1;\, a_2,\, b_2} = \Psi_{a_1a_2}\delta_{b_1b_2} + \theta_{a_1a_2;\, b_1b_2}. \quad (18)$$

Realizing the action of the matrix D on a "column" with two indices, in the general case of an arbitrary complex semisimple Lie group we obtain

$$\left.\begin{aligned} C_p^+ = \sum_{i,\, j} [\sum_k h_kd_k + \sum_{\alpha>0} h_\alpha]_{ii} A_{ij}^{p-2} [\sum_s h_sd_s]_{jj}; \\ A_{ij} = \sum_k \operatorname{Sp}(S_iS^jh_k)\, d_k \\ + \sum_{\alpha>0} [\operatorname{Sp}(S_iS^jX_\alpha X_{-\alpha}) + \operatorname{Sp}(X_\alpha S_i X_{-\alpha} S^j)], \end{aligned}\right\} \quad (19)$$

where $\{S_i\}$ is a complete "orthonormal" system of matrices $\operatorname{Sp} S_iS^j = \delta_i^j$ that commute with $\mathfrak{h}$.

In the case of the classical complex semisimple Lie groups, the last equation can be rewritten in the form

$$C_p^+ = \sum_{a,\, b} (A^p)_{ab}; \quad A_{ab} = \Psi_{ab} - \theta_{ab;\, ba}. \quad (20)$$

The expression (20) is also valid for finite-dimensional unitary representations (including degenerate ones) of compact groups, and the eigenvalue spectrum of the Casimir operators in this case is the same as the one obtained by a different method (which does not apply for degenerate representations h_i, $X_{\pm\alpha}$) by Perelomov and Popov.[19] Thus, the Casimir operators for infinite-dimensional representations with weights $d_{\pm i}$ of complex semisimple Lie groups are obtained by the formal replacement of the integral weights l_i by $d_{\pm i}$ in the Casimir operators of the corresponding compact groups.

For the real forms of the complex semisimple Lie groups the difference resides in the restriction of the matrix of generators $\tilde{X}^\alpha$ to $\mathcal{K}^{R_i}$, and the trace of the successive powers of the matrix is determined, as in the complex case [Eq. (19) in the general case and Eq. (20) for the classical series], by the sum of the matrix elements of the numerical matrix A. The explicit form of the matrix A for each of the classical complex semisimple Lie groups and their real forms is given in Table 3.

TABLE III.

G	Matrix A	m
$L(n, C)$	$A_{ab}=\delta_{ab}A_{aa}-\theta(a, b)$, $A_{aa}=m_a+n-a$, $1\leqslant a\leqslant n$	$m_{\pm a}=(\rho_a\pm\varkappa_a)/2$
$L(n, R)$		$m_a=\rho_a$
$U^*(2N)$ $n=2N$		$m_{2k}=(\rho_k-\varkappa_k)/2$, $m_{2k-1}=(\rho_k+\varkappa_k)/2$, $1\leqslant k\leqslant N$
$U(p, q)$ $p\geqslant q$, $p+q=n$		$m_a=(\rho_a+\varkappa_a)/2$, $1\leqslant a\leqslant q$, $m_a=l_{p-a+1}$, $q+1\leqslant a\leqslant p$, $m_a=(\rho_{n-a+1}-\varkappa_{n-a+1})/2$
$O(N, C)$ $N=2n$	$A_{ab}=\delta_{ab}A_{aa}-\theta(a, b)(1-\delta_{a, 2n-a+1})$, $A_{ii}=m_i+n(\varepsilon_i+1)-(i+1)$, $i=\begin{cases} a, & a\leqslant n \\ a-2n-1, & a\geqslant n\end{cases}$	$m_i^{\pm}=\rho_i\pm\varkappa_i$, $m_{-i}\equiv -m_i$, $1\leqslant i\leqslant n$
$O^*(2n)$ $n=2k+(1)$		$m_{2i}=(\rho_i-\varkappa_i)/2$, $m_{2i-1}=(\rho_i+\varkappa_i)/2$, $1\leqslant i\leqslant k$ $m_{-i}\equiv -m_i$ $(m_{k+1}=\varkappa_{k+1}$, $n=2k+1)$
$O(p, q)$ $p-q=2k$ $p+q=n$		$m_i=\rho_i$, $1\leqslant i\leqslant q$, $m_i=l_{n-i+1}$, $q+1\leqslant i\leqslant n$, $m_{-i}\equiv m_i$
$O(N, C)$ $N=2n+1$	$A_{ab}=\delta_{ab}A_{aa}-\theta(a, b)\times(1-\delta_{a, 2n-a+2})$, $i=\begin{cases} a, & a\leqslant n \\ a-2n-2, & a>n-1\end{cases}$ $A_{ii}=m_i+(n-1/2)\times(\varepsilon_i+1)-(i+1)$,	$m_i^{\pm}=(\rho_i\pm\varkappa_i)/2$, $m_{-i}\equiv m_i$, $1\leqslant i\leqslant n$, $m_0=0$
$O(p, q)$ $p-q=2k+1$ $p+q=n$		$m_i=\rho_i$, $1\leqslant i\leqslant q$, $m_i=l_{i-q}$, $q+1\leqslant i\leqslant (n-1)/2$
$\mathrm{Sp}(2n, C)$	$A_{ab}=\delta_{ab}A_{aa}-\theta(a, b)(1+\delta_{a, 2n-a+1})$; $A_{ii}=m_i+(n-1)\times(\varepsilon_i+1)-i$; $i=\begin{cases} a, & a\leqslant n, \\ a-2n-1, & a\geqslant n;\end{cases}$ $\theta(a, b)=\begin{cases} 1, & a<b, \\ 0, & a\geqslant b;\end{cases}$ $\varepsilon_i=\begin{cases} 1, & i>0; \\ -1, & i<0\end{cases}$	$m_i^{\pm}=\rho_i\pm\varkappa_i$, $m_{-i}\equiv m_i$, $1\leqslant i\leqslant n$
$\mathrm{Sp}(2n, R)$		$m_i=\rho_i$, $m_{-i}\equiv m_i$, $1\leqslant i\leqslant n$
$\mathrm{Sp}(2p, 2q)$ $p\geqslant q$, $p+q=n$		$m_i=\begin{cases} m_{2k}=(\rho_k-\varkappa_k)/2, & 1\leqslant k\leqslant q \\ m_{2k-1}=(\rho_k+\varkappa_k)/2, & \\ m_{n-1-j}=l_j, & 1\leqslant j\leqslant p-q\end{cases}$

2. REALIZATION OF IRREDUCIBLE REPRESENTATIONS OF SEMISIMPLE LIE GROUPS. CHARACTERS

Some general relations and matrix elements of finite transformation. To obtain representations of a Lie group G on the basis of its algebra, it is necessary to integrate the algebra (when possible); in other words, solve the system of Lie equations. Knowledge of the transformation law of the group parameters under an arbitrary element g of G is sufficient to construct important characteristics of a group representation such as the matrix elements of finite transformations, the characters of the representations, the Plancherel measure, etc.

As we have already said, the representations of the group constructed in this manner are realized on the space $D^{\{\rho\}}$ of functions defined on the subgroup $\mathscr{K}$ of G:

$$T_g^{\{\rho\}} f(k)=\prod_{1}^{r_{\mathcal{A}}}[R_i(g, k)]^{\rho_i} f(\tilde{k});\quad R_i\equiv\exp(\tilde{\tau}_i-\tau_i), \tag{21}$$

where $\tilde{k}$ and $\tilde{\tau}$ are obtained from the original k and τ under the action of the transformation g of G. The relations connecting the original and the transformed parameters are found from the solution of the Lie equations with the generators (12) and (13).

The representation (21) is reducible since its generators commute with transformations in the subgroup S. To separate out operator-irreducible components, we decompose functions f of $D^{\{\rho\}}$ with respect to matrix elements of the centralizer. We represent an element k in $\mathscr{K}$ in the form $k=k_{-s}\circ s$, where $s\in S$, $k_{-s}\in\mathscr{K}/S$. Invariance of $T_g^{\{\rho\}}$ under the subgroup S means that under an arbitrary transformation of G the element undergoes a shift, $\tilde{s}=sN(k_{-s}g)$, where the matrix $N\in S$ does not contain the parameters s. Bearing this in mind, we obtain

$$T_g^{\{\rho, s\}} f_{\{m\}}(k_{-s})=\prod_{1}^{r_{\mathcal{A}}}\{R_i(g, k_{-s})]^{\rho_i}\sum_{\{m'\}} D^{\{s\}}_{\{m\}, \{m'\}}(N) f_{\{m'\}}(\tilde{k}_{-s}), \tag{22}$$

where $D^{\{s\}}_{\{m\}, \{m'\}}$ is the matrix element of the irreducible representation of weight $\{s\}$ of the group S between the states with the quantum numbers $\{m\}$ and $\{m'\}$. The expression (22) is an integral form of the realization of the representation $\{\rho, s\}$ of G corresponding to the algebra constructed earlier with the generators (12) and (13). The operator irreducibility of the representation $\{\rho, s\}$ defined by (22) will be proved below in connection with the general investigation of the reducibility properties on the basis of the study of the kernels of the interlacing operators. (For infinite-dimensional representations, operator irreducibility and the absence of invariant subspaces in the representation space are not, in general, equivalent requirements.)

In accordance with the realization (22), the operator $T_g^{\{\rho, s\}}$ can be regarded as an integral operator with singular kernel:

$$T^{\{\rho, s\}}_{\{m\}, \{m'\}}(k_{-s}, k'_{-s})=\prod_{1}^{r_{\mathcal{A}}}[R_i(g, k_{-s})]^{\rho_i} D^{\{s\}}_{\{m\}, \{m'\}}(N)\,\delta(k'_{-s}\tilde{k}_{-s}^{-1}), \tag{23}$$

where $\delta(k'_{-s}\circ\tilde{k}_{-s}^{-1})$ is the δ function on $\mathscr{K}/S$, and it is defined by

$$\int dk'_{-s} f(k'_{-s})\,\delta(k'_{-s}\cdot k_{-s}^{-1})=f(k_{-s}).$$

Proceeding from the definition of the character $\pi^{\{\rho, s\}}(g)$ as the trace of the operator of the representation on the class of generalized functions[21,22] and from Eq. (23), we have

$$\pi^{\{\rho, s\}}(g)=\int dk_{-s}\,\delta(k_{-s}\tilde{k}_{-s}^{-1})\prod_{1}^{r_{\mathcal{A}}}[R_i(g, k_{-s})]^{\rho_i}\,\pi^{\{s\}}(N), \tag{24}$$

where dk_{-s} is the measure on $\mathscr{K}/S$; $\pi^{\{s\}}(N)$ is the character of the irreducible representation $\{s\}$ of the centralizer S. [The question of the necessity of introducing a smoothing function in order to define the character correctly as a generalized function of the eigenvalues of the matrix g has been rather fully studied in the mathematical literature (see, for example, Ref. 21) and therefore does not require additional discussion.]

In accordance with the decomposition (3) of the element g of the semisimple group G, it is convenient to take as basis in the space $D^{\{\rho, s\}}$ of the representation the matrix elements $D^{\{L\}}_{\{M\};\{N\}}(k)$ of the irreducible unitary representations of the subgroup $\mathscr{K}$ between the states with quantum numbers $\{M\}$ and $\{N\}$, which are eigenvectors of the subgroup S. From (21) there then follows an integral representation for the matrix elements $D^{\{\rho, s\}}(g)$ of the representation $\{\rho, s\}$ of G:

$$D^{\{\rho, s\}}_{\{L \atop MN\};\{L' \atop M'N'\}}(g)=N_L^{-1}\int dk\prod_{1}^{r_{\mathcal{A}}}[R_i(k, g)]^{\rho_i}\times\overset{*}{D}{}^{\{L'\}}_{\{M'\}, \{N'\}}(\tilde{k})\,D^{\{L\}}_{\{M\}, \{N\}}(k). \tag{25}$$

where N_L is the dimension of the representation $\{L\}$ of the group $\mathscr{K}$; $dk = dk_{-s}ds$ is an invariant measure on $\mathscr{K}$ normalized to unity. The number of continuous parameters of this discrete basis is exactly equal to the number of quantum numbers, including the weights of the representation.

Note that formal calculation of the trace in Eq. (25), i. e.,

$$\sum_{\{L,M,N\}} D^{\{\rho,s\}}_{\{L\atop MN\}}; \{{L \atop M\,N}\}(g),$$

leads to the expression (24) for the character of the representation, although the reversal of the order of summation and integration requires additional investigation in a correct definition of such operations in the class of generalized functions.

For the further reduction and particularization of the expressions we have obtained for the matrix elements and characters of operator-irreducible representations of G, we need to know the explicit dependence of the transformed parameters $\tilde{k}$ and $\tilde{\tau}$ on g, k, and τ. We begin by considering the complex groups. It follows from comparison of (11) and (12) that

$$\exp\left(\sum_i \tau_i l_i\right) D^{\{l\}}_{\{m\},\{l\}}(k) \equiv E^{\{l\}}_{\{m\}}(k,\ \tau),$$
$$(\tilde{\hat{X}}^{\alpha} E^{\{l\}}_{\{m\}} = 0,\ \alpha > 0:\ \tilde{\hat{h}}^{i} E^{\{l\}}_{\{m\}} \equiv l_i E^{\{l\}}_{\{m\}})$$

transforms in accordance with a finite-dimensional nonunitary representation of G^C, i. e.,

$$E^{\{l\}}_{\{m\}}(\tilde{k},\ \tilde{\tau}) = \sum_{\{m'\}} D^{\{l\}}_{\{m\},\{m'\}}(g)\, E^{\{l\}}_{\{m'\}}(k,\ \tau). \tag{26}$$

Use of the orthogonality condition of the matrix elements of $\mathscr{K}$ leads to the following equations, which are deduced from (26):

$$\prod_1^{r_{\mathscr{A}}} [R_i(k,\ g)]^{l_i} = \Big[\sum_{\{m\}} |D^{\{l\}}_{\{m\},\{l\}}(gk)|^2\Big]^{1/2};\ D^{\{l\}}_{\{l\},\{l\}}(k^{-1}\tilde{k})$$
$$= D^{\{l\}}_{\{l\},\{l\}}(k^{-1}gk)\Big/\Big[\sum_{\{m\}} |D^{\{l\}}_{\{m\},\{l\}}(gk)|^2\Big]^{1/2}. \tag{27}$$

Taking $\{l\}$ to be the system of fundamental representations of the corresponding group, we can readily obtain relations connecting the transformed $\tilde{k}$ and $\tilde{\tau}$ to the original parameters k and τ. In the asymptotic region of infinitely large values of the noncompact parameters ε_i in g, the right-hand sides of both equations (27) for $g = T(\varepsilon) \in \mathscr{A}$ simplify considerably and can be expressed in terms of the highest vectors $\xi^{\{l\}}(k) = D^{\{l\}}_{\{l\}\{l\}}(k)$ of the corresponding representations of the group $\mathscr{K}$; namely $\exp(\sum_i \varepsilon_i l_i)$, $|\xi^{\{l\}}(k)|$, and $|\xi^{\{l\}}(k)|$, respectively. The explicit form of the highest vectors $\xi^{\{l\}}(k)$ for compact semisimple Lie groups in the universal parametrization is given in the Appendix [Eq. (A.5)].

In the case of the real forms of the complex groups, the basic relations (26) and (27) keep their form. The corresponding quantities $E^{\{l\}}_{\{m\}}(\tau, \mathbf{k})$ transform in accordance with finite-dimensional nonunitary representations of the real form, $\mathbf{k}$ belongs to the restriction of $\mathscr{K}^C$ to $\mathscr{K}^{R_i}$, and not all the parameters $\{\tau\}$ are linearly independent and nonzero.

The relations we have obtained solve the problem of integrating the system of Lie equations with the generators (12) and (13) in the case of an arbitrary semisimple Lie group. For the classical groups which admit matrix realization, these relations can be written down in the tensor basis. The explicit expressions connecting $\tilde{\tau}$ and $\tilde{k}$ to k and τ and $g = T(\varepsilon)$ in the asymptotic region with respect to $\{\varepsilon\}$ are given in Table 4 for all types of the classical Lie groups in the tensor basis.

Characters of operator-irreducible representations. The characters of the constructed representations for an arbitrary semisimple Lie group G are expressed by the general formula (24); the transformed $\tilde{k}$ and $\tilde{\tau}$ in the integrand of (24) are determined by Eqs. (27). In the case of the classical semisimple Lie groups, for which matrix realization is known, the calculation of the characters of the representations $\{\rho, s\}$, in the sense of finding their explicit analytic dependence on the eigenvalues of the matrix g of G and the weights of the representation, can be carried out in accordance with a unified scheme. Omitting the direct calculations for the different types of groups (see, for example, Refs. 40–42), we describe the main stages of this scheme in the case of the matrix g conjugate to the elements of the maximal noncompact Cartan subgroup, and we give a unified expression for the characters of an arbitrary semisimple group, this being the invariant root form of expression of the corresponding formulas for the particularized types of groups given in Table 5.

Because the integral expression (24) contains the δ function with respect to the set of parameters of $\mathscr{K}/S$, it is equal to the ratio of the integrand (after the δ symbol has been removed) to the Jacobian J of the transition from the variables of the argument $k_{-s}\tilde{k}^{-1}_{-s}$ of the δ function to the variables of integration k_{-s}, the ratio being taken at the "points" of the support of the δ function and summed over them. The possibility of replacing integration over $\mathscr{K}/S$ by integration over a multidimensional Euclidean space[37] appreciably simplifies all the calculations, and it enables one to take them to the stage of concrete expressions for the characters as functions of the eigenvalues λ of the matrix g of G. Denoting by $\{k^0_{-s}\}$ the set of elements of $\mathscr{K}/S$ that lie in the support of $\delta(k_{-s}\tilde{k}^{-1}_{-s})$, we obtain for $\pi^{\{\rho,s\}}(g)$

$$\pi^{\{\rho,s\}}(g) = \sum_{\{\mathring{k}_{-s}\}} \Big|\prod_1^{r_{\mathscr{A}}} [R_i(g,\ k_{-s})]^{\rho_i}\, \pi^{\{s\}}(N)\, J\,[k_{-s}\tilde{k}^{-1}_{-s};\ k_{-s}]\Big|\Big|_{\{\mathring{k}_{-s}\}}. \tag{28}$$

Thus, to calculate the character of this expression, we must determine at the "points" of $\{\mathring{k}_{-s}\}$ the values of the multipliers R_i and the Jacobian J of the transition and relate the eigenvalues $\exp(i\theta_j)$ of the matrix N, in terms of which the characters $\pi^{\{s\}}(N)$ are expressed by means of Weyl's well known formula[6]:

$$\pi^{\{s\}}(N) = \sum_{\omega} \det\omega \exp\Big[\mathrm{i}\sum_j \theta_j (s_j + \rho_{0j})_{\omega}\Big]\Big/\sum_{\omega} \det\omega$$
$$\times \exp\Big[\mathrm{i}\sum_j \theta_j (\rho_{0j})_{\omega}\Big];\ \det\omega = \pm 1;\ \omega \in W_S$$

to the eigenvalues of the matrix g of G. The previously obtained relations connecting the parameters $\tilde{k}_{-s}$ and k_{-s} enable one to calculate all the quantities in Eq. (28) explicitly. The final expressions given in Table 5 for the characters of the operator-irreducible representations of each type of classical semisimple Lie groups in the invariant root form can be written in the unified manner

$$\pi^{\{\rho,s\}} = \sum_{\omega} \exp\sum_j \rho_j \tau^{\omega}_j \Big/ \prod_{\alpha>0}' |1 - \exp[-\alpha(\tau)]|^{P_\alpha} \pi^{\{s\}}(\varphi_\omega). \tag{29}$$

The summation in is extended over all permutations ω in the factor group W_G/W_S; $\alpha(\tau)$ is determined from the

TABLE IV.

G	$\widetilde{k}^{(a)}$	$\prod_1^{\alpha} [R_l^{(a)} \exp(-\varepsilon_l)]$	$k \in \mathscr{K}$
$L(n, C)$	$(\widetilde{n}^{(a)})_\alpha^\alpha = \frac{\det_\alpha n}{\lvert\det_\alpha n\rvert} \frac{\lvert\det_{\alpha-1} n\rvert}{\det_{\alpha-1} n}$	$\lvert\det_\alpha n\rvert$	$n \in U(n)$
$L(n, R)$	$(\widetilde{n}^{(a)})_\alpha^\alpha = \mathrm{sgn}\left(\frac{\det_\alpha n}{\det_{\alpha-1} n}\right)$	$\lvert\det_\alpha n\rvert$	$n \in O(n)$
$U^*(2n)$	$(\widetilde{n}^{(a)})_{\pm s \pm 1}^{\pm s \pm 1} = \frac{\det_{2s+1}\left\{\begin{matrix} 1, -1; 2, -2; \ldots \pm s \pm 1 \\ 1, -1; 2, -2; \ldots \pm s \pm 1 \end{matrix}\right\}}{\left\lvert\det_{2s+2}\left\{\begin{matrix} 1, -1; 2, -2; \ldots; s+1, -s-1 \\ 1, -1; 2, -2; \ldots; s+1, -s-1 \end{matrix}\right\}\right\rvert^{1/2}}$	$\left\lvert\det_{2\alpha} \times \left\{\begin{matrix} 1, -1; 2, -2; \ldots \\ \ldots; \alpha, -\alpha \\ 1, -1; 2, -2; \ldots \\ \ldots; \alpha, -\alpha \end{matrix}\right\}\right\rvert^{1/2}$	$\left\{\begin{matrix} a_1 \ldots a_k \\ a_1, \ldots, a_k \end{matrix}\right\} \equiv \{n_{a_1}^{a_1} - n_{a_k}^{a_k}\}$ $n \in \mathrm{Sp}(2n)$
$U(p, q), p \geqslant q$	$(\widetilde{p}^{(a)})_s^s = \frac{\det_{p-s+1}(p+q) \det_{p-s}^*(p+q)}{\lvert\det_{p-s+1}(p+q) \det_{p-s}(p+q)\rvert}$; $(\widetilde{q}^{(a)})_s^s = \frac{\det_s(p+q) \det_{s-1}^*(p+q)}{\lvert\det_s(p+q) \cdot \det_{s-1}(p+q)\rvert}$, $p-q+1 \leqslant s \leqslant p$; $\sum_{k=1}^{p-q} (\widetilde{p}^{(a)})_\beta^k p_\gamma^{*k} = \delta_{\beta\gamma} - \sum_{i, j=1}^{q} n_\beta^{p+1-i} \times [(1+n)^{-1}]_{p+1-i, p+1-j} n_\gamma^{*p+1-j}$, $1 \leqslant \beta, \gamma \leqslant p-q$	$\lvert\det_\alpha(p+q)\rvert$	$p \in U(p)$, $q \in U(q)$ $n \equiv q_{(1)}^{-1} p$ $q_{(1)} = \begin{pmatrix} q & 0 \\ 0 & I_{p-q} \end{pmatrix}$
$O(n, C)$	$(\widetilde{a}^{(a)})_s^s = \frac{\det_s a}{\lvert\det_s a\rvert} \frac{\lvert\det_{s-1} a\rvert}{\det_{s-1} a}$, $2a_\beta^\alpha = n_{-\beta}^{-\alpha} + n_\beta^\alpha + i(n^{-\alpha}{}_\beta - n^\alpha{}_{-\beta})$	$\lvert\det_\alpha a\rvert$	$n \in O(n)$
$O^*(2n)$	$(\widetilde{n}^{(a)})_{\pm s \pm 1}^{s+1} = \frac{\det_{2s+1}\left\{\begin{matrix} 1, -1; \ldots; s+1 \\ 1, -1; \ldots; \pm s \pm 1 \end{matrix}\right\}}{\left\lvert\det_{2s+2}\left\{\begin{matrix} 1, -1; \ldots; s+1, -s-1 \\ 1, -1; \ldots; s+1, -s-1 \end{matrix}\right\}\right\rvert^{1/2}} = \frac{\det_{2s+1} a}{\lvert\det_{2s+2} a\rvert^{1/2}}$, $a = n + Jn^*J$	$\left\lvert\det_{2\alpha} \times \left\{\begin{matrix} 1, -1; \ldots; \alpha, -\alpha \\ 1, -1; \ldots; \alpha, -\alpha \end{matrix}\right\}\right\rvert^{1/2} = \lvert\det_{2\alpha} a\rvert^{1/2}$	$n \in U(n)$
$O(p, q)$ $p \geqslant q$	$(\widetilde{p}^{(a)})_s^s = \mathrm{sgn}\left(\frac{\det_{p-s+1}(p+q)}{\det_{p-s}(p+q)}\right)$; $(\widetilde{q}^{(a)})_s^s = \mathrm{sgn}\left(\frac{\det_s(p+q)}{\det_{s-1}(p+q)}\right)$; $p-q+1 \leqslant s \leqslant p$; $\sum_{k=1}^{p-q} (\widetilde{p}^{(a)})_\beta^k p_\gamma^k = \delta_{\beta\gamma} - \sum_{i, j=1}^{q} n_\beta^{p+1-i} [(1+n)^{-1}]_{p+1-i, p+1-j} \times n_\gamma^{p+1-j}$, $1 \leqslant \beta, \gamma \leqslant p-q$	$\lvert\det_\alpha(p+q)\rvert$	$p \in O(p)$, $q \in O(q)$ $n = q_{(1)}^{-1} p$ $q_{(1)} = \begin{pmatrix} q & 0 \\ 0 & I_{p-q} \end{pmatrix}$
$\mathrm{Sp}(2n, C)$	$(\widetilde{n}^{(a)})_s^s = \frac{\det_s n}{\lvert\det_s n\rvert} \frac{\lvert\det_{s-1} n\rvert}{\det_{s-1} n}$	$\lvert\det_\alpha n\rvert$	$n \in \mathrm{Sp}(2n)$
G	$\widetilde{k}^{(a)}$	$\prod_1^{\alpha} [R_l^{(a)} \exp(-\varepsilon_l)]$	$k \in \mathscr{K}$
$\mathrm{Sp}(2n, R)$	$(\mathrm{Re}\, \widetilde{n}^{(a)})_s^s = \mathrm{sgn}\left(\frac{\det_s(\mathrm{Re}\, n)}{\det_{s-1}(\mathrm{Re}\, n)}\right)$	$\lvert\det_\alpha(\mathrm{Re}\, n)\rvert$	$n \in U(n)$
$\mathrm{Sp}(2p, 2q)$ $p \geqslant q$	$(\widetilde{\mathbf{p}}^{(a)})_k^k = \frac{\det_{p-k+1}(\mathbf{p}+\mathbf{q}) \det_{p-k}^*(\mathbf{p}+\mathbf{q})}{\lvert\det_{p-k+1}(\mathbf{p}+\mathbf{q}) \det_{p-k}(\mathbf{p}+\mathbf{q})\rvert}$, $(\widetilde{\mathbf{q}}^{(a)})_k^k = \frac{\det_k(\mathbf{p}+\mathbf{q}) \det_{k-1}^*(\mathbf{p}+\mathbf{q})}{\lvert\det_k(\mathbf{p}+\mathbf{q}) \det_{k-1}(\mathbf{p}+\mathbf{q})\rvert}$ $p-q+1 \leqslant k \leqslant p$ $\sum_{k=1}^{p-q} (\widetilde{\mathbf{p}}^{(a)})_\beta^k \mathbf{p}_j^{*k} = \delta_{\beta\gamma} - \sum_{i, j=1}^{q} \mathbf{n}_\beta^{p+1-i} [(1+\mathbf{n})^{-1}]_{p+1-i, p+1-j} \times \mathbf{n}_\gamma^{*p+1-j}$, $1 \leqslant \beta, \gamma \leqslant p-q$	$\lvert\det_\alpha(\mathbf{p}+\mathbf{q})\rvert$	$\mathbf{p} \in \mathrm{Sp}(2p)$ $\mathbf{q} \in \mathrm{Sp}(2q)$

Note. Here we give asymptotic expressions for the transformed parameters $\widetilde{k}^{(a)}$ (the nonvanishing elements of these matrices) and $R_i = \exp(\widetilde{\tau}_i - \tau_i)$ for the classical semisimple Lie groups; $\det_i$ are the principal minors of the corresponding matrices.

TABLE V.

G	$A^{\{\rho\}}(\lambda)$	$B(\lambda)$	$\pi^{\{s\}}(\lambda)$
$L(n, C)$	$\prod_j \lvert\lambda_j\rvert^{\rho_j+2(n-j)}$	$\prod_{i>j} \lvert\lambda_i-\lambda_j\rvert^2$	$\prod_{j=1}^{n} (\lambda_j/\lvert\lambda_j\rvert)^{\varkappa_j}$
$L(n, R)$	$\prod_j \lvert\lambda_j\rvert^{\rho_j-(n-j)}$	$\prod_{i>j} \lvert\lambda_i-\lambda_j\rvert$	$\prod_{j=1}^{n} (\operatorname{sgn}\lambda_j)^{\xi_j}$, $\xi_j=0, 1$
$U^*(2n)$	$\prod_j \lvert\lambda_j\rvert^{\rho_j-4(n-j)}$	$\prod_{i>j} (\lambda_i-\lambda_j)(\lambda_i-\lambda_j^*)$	$\prod_{j=1}^{n} \pi_{SU(2)}^{\{l_j\}}$
$U(p, q)$ $p \geqslant q$ $p-q=n$	$\prod_{j=1}^{q} [\lvert\lambda_j\rvert^{\rho_j-2(n-2j-1)}+\lvert\lambda_j\rvert^{-\rho_j}]$.	$\prod_1^q [1-\lvert\lambda_j\rvert^2] \prod_{q\geqslant i>j\geqslant 1} \lvert\lambda_i-\lambda_j\rvert^2$ $\times \lvert\lambda_j-\lambda_i^{*-1}\rvert^2 \cdot \prod_{\substack{q\geqslant i\geqslant 1\\ p\geqslant\alpha\geqslant q-1}} \lvert\lambda_i-\lambda_\alpha\rvert^2$	$\prod_{j=1}^{q} (\lambda_j/\lvert\lambda_j\rvert)^{\varkappa_j}$ $\cdot\pi_{U(p-q)}^{\{l\}}$
$O(n, C)$	$\prod_{j=1}^{'[n/2]} [\lvert\lambda_j\rvert^{\rho_j}$ $+\lvert\lambda_j\rvert^{-\rho_j-2\Delta_j-4\varkappa_j} (\lambda_j/\lvert\lambda_j\rvert)^{-2\varkappa_j}]$, $'\Pi$ contains only even powers of λ_j for $n=2k$; $\Delta_j=\begin{cases} 2(n-j), & n=2k \\ n-2j, & n=2k+1 \end{cases}$	$\prod_{i>j} \lvert 1-\lambda_i\lambda_j^{-1}\rvert^2 \cdot \lvert 1-\lambda_i^{-1}\lambda_j^{-1}\rvert^2$ $\times\begin{cases} 1, & n=2k \\ \prod_j \lvert 1-\lambda_j^{-1}\rvert^2, & n=2k-1 \end{cases}$	—
$O^*(2n)$	$\prod_j [\lvert\lambda_j\rvert^{\rho_j}+\lvert\lambda_j\rvert^{-\rho_j-2(2n-2j+1)}]$	$\prod_j \lvert 1-\lambda_j^{-2}\rvert \lvert 1-\lambda_j^{*-2}\rvert \prod_{i>j} \lvert 1-\lambda_i\lambda_j^{-1}\rvert^2$ $\times \lvert 1-\lambda_i\lambda_j^{*-1}\rvert^2 \cdot \lvert 1-\lambda_i^{-1}\lambda_j^{-1}\rvert^2$ $\times \lvert 1-\lambda_i^{-1*}\lambda_j^{-1}\rvert^2$ $\times\begin{cases} 1, & n=2k \\ \prod_j \left\lvert \frac{\lambda}{\lvert\lambda\rvert}-\lambda_j\right\rvert^2 \left\lvert\frac{\lambda}{\lvert\lambda\rvert}-\lambda_j^{-1}\right\rvert^2, & n=2k+1 \end{cases}$	$\prod_{j=1}^{k} \pi_{SU(2)}^{\{l_j\}}$ $\times\begin{cases} 1, & n=2k \\ \left(\frac{\lambda}{\lvert\lambda\rvert}\right)^{\varkappa}, & n=2k+1 \end{cases}$
$O(p, q)$ $p \geqslant q$ $p+q=n$	$\prod_{j=1}^{q} [\lvert\lambda_j\rvert^{\rho_j}+\lvert\lambda_j\rvert^{-\rho_j-n+2j}]$,	$\left\lvert \prod_{i<j,\ i<q} (1-\lambda_j\lambda_i^{-1})(1-\lambda_i^{-1}\lambda_j^{-1}) \right\rvert$	$\prod_{j=1}^{q} (\operatorname{sgn}\lambda_j)^{\xi_j}\pi_{O(p-q)}^{\{l\}}$
$\mathrm{Sp}(2n, C)$	$\prod_{j=1}^{n} [\lvert\lambda_j\rvert^{\rho_j}+\lvert\lambda_j\rvert^{-\rho_j-4(n-j+1)-4\varkappa_j}$ $\times\lambda_j^{4\varkappa_j}] \left(\frac{\lambda_j}{\lvert\lambda_j\rvert}\right)^{-2\varkappa_j}$	$\prod_{j=1}^{n} \lvert 1-\lambda_j^{-2}\rvert^2 \cdot \prod_{i=j+1}^{n} \lvert 1-\lambda_i\lambda_j^{-1}\rvert^2$ $\times \lvert 1-\lambda_i^{-1}\lambda_j^{-1}\rvert^2$	—
$\mathrm{Sp}(2n, R)$	$\prod_{j=1}^{n} [\lvert\lambda_j\rvert^{\rho_j}+\lvert\lambda_j\rvert^{-\rho_j-2(n-j+1)}]$	$\prod_{j=1}^{n} \lvert 1-\lambda_j^{-2}\rvert \cdot \prod_{i=j+1}^{n} \lvert 1-\lambda_i\lambda_j^{-1}\rvert$ $\times \lvert 1-\lambda_i^{-1}\lambda_j^{-1}\rvert$	$\prod_{j=1}^{n} (\operatorname{sgn}\lambda_j)^{\xi_j}$
G	$A^{\{\rho\}}(\lambda)$	$B(\lambda)$	$\pi^{\{s\}}(\lambda)$
$\mathrm{Sp}(2p, 2q)$ $p \geqslant q$ $p+q=n$	$\prod_{j=1}^{q} [\lvert\lambda_j\rvert^{\rho_j}+\lvert\lambda_j\rvert^{-\rho_j-2(2n-2j+3)}]$	$\prod_{j=1}^{q} \lvert 1-\lambda_j^{-2}\rvert \cdot \lvert 1-\lambda_j^{*-2}\rvert \cdot \lvert 1-\lvert\lambda_j\rvert^{-2}\rvert$ $\times \prod_{i=j+1}^{q} \lvert 1-\lambda_i\lambda_j^{-1}\rvert^2 \cdot \lvert 1-\lambda_i^*\lambda_j^{-1}\rvert^2$ $\times \lvert 1-\lambda_i^{-1}\lambda_j^{-1}\rvert^2 \lvert 1-\lambda_i^{*-1}\lambda_j^{-1}\rvert$ $\times \prod_{k=1}^{p-q} \lvert\lambda_k'-\lambda_j\rvert^2 \lvert\lambda_k^{*\prime}-\lambda_j\rvert^2$	$\prod_{j=1}^{q} \pi_{\mathrm{Sp}(2)}^{\{l_j\}} \cdot \pi_{\mathrm{Sp}(2p-2q)}^{\{\varkappa\}}$

Note. Here we give the characters of the operator-irreducible representation of the classical semisimple Lie groups; $\pi^{\{\rho, s\}}(g) = \sum_\omega A^{\{\rho^\omega\}}\pi^{\{s\}}/B$.

relation $[\tau, X_\alpha]_- = \alpha(\tau)X_\alpha$, $\tau \equiv \sum_j h_j \tau_j$. The product $\Pi'_{\alpha>0}$ in the denominator of (29) is extended to all positive roots of G except for the roots of S, all the $\alpha(\tau)$'s being different; p_α is the multiplicity of the corresponding positive root. In the special case of the complex group ($W_{G^C}/W_S = W_{G^C}$, $p_\alpha = 2$), Eq. (29) has the form

$$\pi^{\{\rho,s\}} = \sum_\omega \exp \sum_j (\rho_j\tau_j + is_j\varphi_j)_\omega / \prod_{\alpha>0} |1 - \exp[-\alpha(\tau)]|^2 \qquad (30)$$

and it is the same as the expressions for the characters of the complex groups obtained earlier in Refs. 21 and 22 for the principal continuous series and the complementary series. For the real forms G^{R_i} of the complex classical groups, the expression (29) can also be written in the form

$$\begin{aligned}\pi^{\{\rho\}}(\boldsymbol{\tau}) = \sum_\omega \delta\omega \exp \mathrm{Sp}\,(\boldsymbol{\rho} - \rho_0, \boldsymbol{\tau}^\omega) / \sum_\omega \delta\omega \\ \times \exp \mathrm{Sp}\,(\rho_0, \boldsymbol{\tau}^\omega);\ \rho \equiv \sum_i h_i \rho_i,\end{aligned} \qquad (31)$$

where $\delta\omega = -1$ for each permutation ω of $W_{G^{R_i}}$ of the compact roots of the matrix $g(|\lambda_j| = 1)$, while $\delta\omega = +1$ for the noncompact roots ($|\lambda_j| \neq 1$).

It should be noted that if the operator-irreducible representations $\{\rho, s\}$ are topologically reducible, Eq. (29) gives the sum of the characters of the irreducible components.

Generating functions and integral representation for the characters. In the investigation of the analytic properties of matrix elements, establishment of recursion relations between them, solution of the problem of distinguishing the invariant subspaces, etc, the method of generating functions is very convenient. In particular, this method is effective in various applications of the group-theoretical approach associated with expansions with respect to the system of generating functions (see, for example, Ref. 13).

In the general case, the generating function $F^{\{\rho,s\}}(g; k_1, k_2)$ of the representation $\{\rho, s\}$ of G depends on the group element g and the choice of the parameters k_1 and k_2 in $\mathscr{K}$. The matrix elements $D^{\{\rho, s\}}_{\{n\};\{m\}}(g)$ are connected to the expansion of $F^{\{\rho,s\}}$ with respect to some complete orthonormal system of function $\varphi_{\{n\}}(k)$:

$$F^{\{\rho,s\}}(g; k_1, k_2) = \sum_{\{m\},\{n\},\{n'\}} G^{\{\rho,s\}}_{\{n\}\{n'\}} D^{\{\rho,s\}}_{\{n'\}\{m\}}(g)\, \varphi_{\{n\}}(k_1)\, \overset{*}{\varphi}_{\{m\}}(k_2). \qquad (32)$$

From (25) there follows one of the expressions for the generating function of the matrix elements with fixed right quantum numbers; namely,

$$\begin{aligned}\Phi^{\{\rho,s\}}_{\left\{\begin{smallmatrix}L\\NM\end{smallmatrix}\right\}}(g; k) = \prod_1^{r_{\mathcal{A}}} [R_i(g, k)]^{\rho_i} D^{\{L\}}_{\{N, M\}}(\tilde{k}) \\ = \sum_{\left\{\begin{smallmatrix}l\\nm\end{smallmatrix}\right\}} D^{\{\rho,s\}}_{\left\{\begin{smallmatrix}l\\nm\end{smallmatrix}\right\};\left\{\begin{smallmatrix}L\\NM\end{smallmatrix}\right\}}(g)\, D^{\{l\}}_{\{n\},\{m\}}(k),\end{aligned} \qquad (33)$$

from which, by the Peter–Weyl theorem, we obtain

$$\begin{aligned}F^{\{\rho,s\}}(g; k_1, k_2) = \prod_1^{r_{\mathcal{A}}} R_i^{\rho_i}\, \delta(\tilde{k}_1 k_2^{-1}) \\ = \Sigma N_{\{L\}} D^{\{\rho,s\}}_{\left\{\begin{smallmatrix}l\\nm\end{smallmatrix}\right\};\left\{\begin{smallmatrix}L\\NM\end{smallmatrix}\right\}} g D^{\{l\}}_{\{n\},\{m\}}(k_2)\, \overset{*}{D}{}^{\{L\}}_{\{N\},\{M\}}(k_1).\end{aligned} \qquad (34)$$

Generating functions of this type are associated with asymptotic values of the characters of semisimple Lie groups. To elucidate this, we must use the fact that the character is an invariant eigendistribution on G (see Ref. 22); in other words, it satisfies the system of equations

$$\begin{aligned}(\hat{F}_i + \hat{\tilde{F}}_i)\,\pi(g) = 0; \\ [\hat{C}_\alpha - c_\alpha(\rho, s)]\,\pi(g) = 0;\ 1 \leqslant \alpha \leqslant r_G.\end{aligned} \qquad (35)$$

where $\hat{F}_i(\hat{\tilde{F}}_i)$ are the generators of left (respectively, right) shifts on G, and $\hat{C}_\alpha$ are Casimir operators. Not all solutions of (35) determine characters of representations; to separate out these from the complete set of solutions we require additional selection rules.[22] An analogous system can also be written down in the asymptotic region in which all the generators in (35) are replaced by their limiting expressions. We denote the functions then obtained by $\pi_{(a)}$. The connection between the functions $\pi(g)$ in the finite and the asymptotic region is given by the integral representation

$$\pi(g) = \int \mathcal{M}(g, g_0)\, \pi_{(a)}(g_0)\, d\mu_{(a)}(g_0), \qquad (36)$$

where the kernel $\mathcal{M}(g, g_0)$ satisfies the system of equations

$$[\hat{F}_i - \hat{F}^T_{i(a)}]\,\mathcal{M}(g, g_0) = 0;\ \ [\hat{\tilde{F}}_i - \hat{\tilde{F}}^T_{i(a)}]\,\mathcal{M}(g, g_0) = 0. \qquad (37)$$

Equation (36) is in a certain sense a generalization of the Gel'fand–Graev transformation[23] to the case of arbitrary semisimple Lie groups. The asymptotic expression for the character is given by the principal term of the expansion of $\pi^{\{\rho,s\}}(g)$:

$$\begin{aligned}\pi^{\{\rho,s\}}(g) \Rightarrow \sum_\omega \exp\left(\sum_j \tau_j \tilde{\rho}_{\omega(j)}\right) \pi^{\{\rho,s\}\omega}_{(a)}(k_1, k_2); \\ g = k_1 T(\tau) k_2;\ \ \omega \in W_G.\end{aligned} \qquad (38)$$

It follows from this analysis that the function $\mathcal{M}(g, g_0) \equiv \mathcal{M}^{\{\rho,s\}}(k_2 g k_1)$, which is a solution of the system (37), in which the operators $\hat{F}_{i(a)}$ and $\hat{\tilde{F}}_{i(a)}$ are related to the set of parameters $\{\rho, s; k_1\}$ and $\{\rho, s; k_2\}$, respectively, is a generating function. The system of equations (37) has the same form as the corresponding system for the kernels $B^{\{\rho,s\}}(k)$ of the intertwining operators. Therefore, its solutions can be obtained from the latter by the formal replacement of k by $k_2 g k_1$.

3. INTERTWINING OPERATORS AND HERMITIAN FORM

Connection between the intertwining operators and questions of reducibility, equivalence, and unitarity. The theory of unitary representations of real semisimple Lie groups, despite the important results obtained in this field, still cannot pretend to completeness. The reason for this is to be found in the characteristic features of the representations of the real groups (which are not present in the complex case); in particular, in the appearance of several different types of principal series of unitary representations, this having an intimate connection to the fact that these groups have nonisomorphic Cartan subgroups and representations realized in spaces of analytic functions of many complex variables with a complicated topology.[8,24]

The methodological difficulties in the construction of unitary representations of the real semisimple Lie groups are due, in particular, to the fact that in this case, in contrast to the complex, the method of induced and holomorphically induced representations[4,6,25] does not permit one to obtain all unitary representations.

Among the various approaches used to construct unitary representations of semisimple Lie groups, the most constructive, in our opinion, are investigations of the of the asymptotic properties of the matrix elements and the kernels of the Hermitian forms. Both these tasks can be related to the study of the analytic properties of the intertwining operators in the space of the representation weights. In addition, investigation of the reducibility and equivalence of representations also leads to the intertwining operators.

The point of departure for the construction of explicit expressions for the intertwining operators $\hat{B}$ and the elucidation of their relation to the problems of representation theory listed above is the intertwining property of these operators:

$$\hat{B}^{\{\omega,\Lambda\}}T_g^{\{\Lambda\}}=T_g^{\{\Lambda'\}}\hat{B}^{\{\omega,\Lambda\}} \tag{39}$$

for the representations $T_g^{\{\Lambda\}}$ and $T_g^{\{\Lambda'\}}$ which are determined by the weights $\{\Lambda\}$ and $\{\Lambda'\}$ connected by the transformations of the Weyl group W_G, $\{\Lambda'\}=\omega\{\Lambda\}$.

In the case of infinite-dimensional representations (in contrast to finite-dimensional ones) of the group G, reducibility includes, because of the possible presence of invariant subspaces in the representation space,[6,8,23] two inequivalent requirements—operator and topological reducibility. Both these requirements can be connected to the analytic properties of the intertwining operators in the space of weights that are solutions of Eq. (39). To clarify this assertion, it is convenient to realize $\hat{B}$ as an integral operator on functions in the space of the corresponding representation on $\mathscr{K}$:

$$\hat{B}^{\{\omega,\Lambda\}}f(k_1)=\int dk_2 B^{\{\omega,\Lambda\}}(k_1,k_2)f(k_2), \tag{40}$$

whose kernel is, generally speaking, a generalized function. Then operator irreducibility (in the sense of there being no operator that commutes with the representation $T_g^{\{\Lambda\}}$ except for a multiple of the identity operator) entails in the language of $B^{\{\omega,\Lambda\}}(k_1,k_2)$ that for the identity transformation $\omega=e$, $\{\Lambda'\}=\{\Lambda\}$, the kernel is proportional to $\delta(k_2^{-1}/k_1)$. For integral values of the weights $\{\Lambda\}$ (or certain linear combinations of them) the representation $T_g^{\{\Lambda\}}$ is, in general, topologically reducible. The possibility of separating out the completely irreducible representations in the framework of this approach is based on an investigation of the analytic properties of the intertwining operators in the space of the weights of the representation $T_g^{\{\Lambda\}}$ of the group G in the basis of the quantum numbers of the representation $T_k^{\{l\}}$ of its maximal compact subgroup $\mathscr{K}$:

$$B_{\{l\}}^{\{\omega,\Lambda\}}=\int B^{\{\omega,\Lambda\}}(k)D^{*\{l\}}(k)\,dk \tag{41}$$

(the kernel $B^{\{\omega,\Lambda\}}(k_1,k_2)$, as we shall see below, depends only on $k_2^{-2}k_1\equiv k$). Note that to investigate the explicit analytic dependence of the intertwining operators on the weights of the representation it is convenient to go over to the operator form of the expression (41), i.e.,

$$\hat{f}^{\{\omega,\Lambda\}}=\int B^{\{\omega,\Lambda\}}(k)\,\hat{k}\,dk, \tag{42}$$

in which the compact operators $X_\alpha+X_{-\alpha}$ and h_α in $\hat{k}$ [see Eq. (A.1)] have a well defined integral spectrum of eigenvalues.

The problem of the equivalence of representations with weights $\{\Lambda\}$ and $\{\Lambda'\}$ reduces to establishing the condition for the mapping of the representation space $D^{\{\Lambda\}}$ onto the space $D^{\{\Lambda'\}}$ under the influence of the intertwining operator $\hat{B}^{\{\omega,\Lambda\}}$ to be isomorphic, $\{\Lambda'\}=\omega\{\Lambda\}$, and is ultimately determined by the normalization constant of this operator. In the case of nonisomorphic mapping, the operators $\hat{B}$ are operators of partial equivalence.

As we have already said, the intertwining operators also occur in an important characteristic of the representations of noncompact groups such as the asymptotic behavior of the matrix elements, which is operator form can be written as follows:

$$T_g^{\{\Lambda\}}\underset{\{\tau\}\to\infty}{\Rightarrow}\sum_{\omega}\hat{B}^{\{\omega,\Lambda\}}\exp\sum_j\tau_j\rho_{\omega(j)}\hat{B}^{-1\{\omega,\Lambda\}}\hat{B}^{\{\omega,\Lambda\}}, \tag{43}$$

where ω corresponds to complete permutation of all the weights $\{\Lambda\}$.

The asymptotic expansion of the matrix element of the operator of the representation $T_g^{\{\Lambda\}}$ between states with fixed values of the quantum numbers (including the weights) $T_k^{\{l\}}$ obviously follows from Eq. (43). The coefficients of the exponentials can be expressed in terms of the functions (41) and contain complete information about the unitary components of the representation $\{\Lambda\}$, whose matrix elements decrease in the asymptotic region in a definition manner. In particular, representations of the various principal series correspond to matrix elements that are square-integrable with the measure $D(\tau)$; $D(\tau)\xrightarrow[\{\tau_j-\tau_{j+1}\}\to\infty]{}\exp 2\sum_j\Delta_j\tau_j$. All terms of the series (43) contribute to the asymptotic behavior for the principal continuous series ($\rho_j=-\Delta_j+i\sigma_j$, $-\infty<\sigma_j<\infty$), whereas for the semidiscrete and discrete series (if they exist) the absence of terms in the sum (43) that decrease slower than $\exp(-\sum_j\Delta_j\tau_j)$ (or increase) is ensured by the requirement that the corresponding functions B vanish. The condition of vanishing of the corresponding functions is directly determined by their analytic properties (by the position of the zeros in the weight space). It is here that we can see most clearly the analogy mentioned in Sec. 1 between the asymptotic method in the representation theory of noncompact groups and potential scattering theory, in which the Jost functions $f(\lambda,k)$ play the role of B. The investigation of the analytic properties (poles and zeros) in the complex space of $\{\rho\}$, which distinguishes the different completely irreducible and unitary representations, is similar to the study of bound states, resonances, etc., on the basis of the analytic properties of the Jost functions in the complex k plane and their physical interpretation.[3]

The problem of constructing the kernels of bilinear invariant Hermitian forms, which enables one in the cases when they exist to distinguish pseudounitary and unitary representations, can be reduced to the finding of the kernels of the corresponding intertwining operators. This question will be considered in more detail below.

Thus, to make effective use of the intertwining operators in these problems of the theory of representations of semisimple Lie groups, one requires, first, knowledge of their explicit analytic expressions and, second, diagonalization of the functions $B_{\{l\}}^{\{\omega,\Lambda\}}$, which are finite-dimensional matrices. The first problem is solved below, where we obtain unified expressions for the kernels of the intertwining operators of semisimple Lie groups and the operator-valued functions (42) in root form. However, we have not succeeded in diagonalizing matrices of the form (41) in the general case. In individual particular cases in which this procedure has been implemented, for example,[36] for the group $U(m,1)$, the technique of intertwining operators has made it possible to obtain a complete solution of the listed problems.

Construction of intertwining operators. We now turn to the direct construction of the intertwining operators for the representations of semisimple Lie groups. To this end, we rewrite Eq. (39) in infinitesimal form, and then, with allowance for the realization (40), we have

$$({}_1\hat{F}_i^{\{\Lambda'\}} - {}_2\hat{F}_i^{T\{\Lambda\}})\, B^{\{\omega,\Lambda\}}(k_1, k_2) = 0, \tag{44}$$

where the subscripts 1 and 2 of $\hat{F}_i$ indicate the variables (k_1 or k_2) to which these operators are applied. It follows from the subsystem related to the generators corresponding to compact transformations in G that the kernel depends only on $k \equiv k_2^{-1}k_1$. Then, using the expressions for the noncompact generators $\hat{F}_i$ of the group G in the asymptotic region obtained in Sec. 1, we can rewrite the remaining Eqs. (44) explicitly. In the case of a complex group G^C, the system of equations for the kernel $B(k)$ has the form

$$\left[\sum_i \mathrm{Sp}\left(k^{-1}\begin{Bmatrix} h_j \\ X_{-\beta} \\ X_\beta\end{Bmatrix} kh_i\right) d_i + \sum_{\alpha\in R_+} \mathrm{Sp}\left(k^{-1}\begin{Bmatrix} h_j \\ X_{-\beta} \\ X_\beta\end{Bmatrix} kX_\alpha\right)\tilde{X}^{-\alpha} + \begin{Bmatrix} -d_j' - \Delta_j \\ X_{-\beta} \\ 0\end{Bmatrix}\right]_{\beta\in R} B(k) = 0;\quad \rho_0 = 1/2\sum_{\alpha\in R_+}\alpha = \sum_j \Delta_j\varepsilon_j \tag{45}$$

(ε_j is the canonical basis in the root space[6]), while, in accordance with (9), for real groups the system is obtained from Eqs. (45) by simple restriction of the maximal compact subgroup of G^C to the maximal compact subgroup of its corresponding real form G^{R_i} and the replacement of $\{\Lambda\}_{G^C}$ by $\{\Lambda\}_{G^{R_i}}$.

In the parametrization (A.1), the kernel of the intertwining operator for the group G^C contains δ functions with respect to the parameters $\theta_\beta, \varphi_\beta, \beta \in \Sigma_- \cap \omega(\Sigma_-)$. To within the factors corresponding to these parameters, the kernel has the form

$$B^{\{\omega,\Lambda\}_C}(k) \sim \prod_{\alpha\in\Sigma_+\cap\omega(\Sigma_-)} (\cos\theta_\alpha/2)^{2(\alpha\rho)/(\alpha\alpha)} \times \exp[i\varphi_\alpha(\alpha s)/(\alpha\alpha)] \prod_j \exp[i\psi_{j/2}(\pi_j s)]_{\substack{s\equiv\{s_j\} \\ \rho\equiv-\{\rho_j+2\Delta_j\}}}. \tag{46}$$

Substituting the expressions (A.1), (A.3), and (46) into Eq. (42) and taking into account the δ-function factors in (46) corresponding to the roots $\beta\in\Sigma_-\cap\omega(\Sigma_-)$, we obtain

$$\hat{J}^{\{\omega,\Lambda\}_C} = \prod_{\alpha\in\Sigma_+\cap\omega(\Sigma_-)} \delta[h_\alpha - (\alpha s)]\, J_\alpha[X_{\pm\alpha}, \rho] \prod_j \delta[h_j - (\pi_j s)], \tag{47}$$

where

$$J_\alpha[X_{\pm\alpha}, \rho] = \int d\theta_\alpha \sin\theta_\alpha/2\, (\cos\theta_\alpha/2)^{2\frac{(\alpha,\rho+2\rho_0)}{(\alpha\alpha)}-1} \exp\left[i\theta_\alpha \frac{X_\alpha + X_{-\alpha}}{\sqrt{2(\alpha\alpha)}}\right]. \tag{48}$$

Note that in the case $\omega(\Sigma_-) = \Sigma_+$, Eq. (47) has the same form as the expression (A.10) for the projection operator onto the highest vector of the irreducible representation of the compact group. By virtue of the reduction nature of (47), it can be represented in the form

$$\hat{J}^{\{\omega,\Lambda\}_C} = \hat{J}^1 J_{(-2)}^{\{\omega,\Lambda\}_C};\quad \hat{J}^1 = \prod_{\alpha\in\Sigma_+^1\cap\omega(\Sigma_-)} \delta[h_\alpha - (\alpha s)]\, J_\alpha \prod_j \delta[h_j - (\pi_j s)], \tag{49}$$

where $\hat{J}_{(-2)}^{\{\omega,\Lambda\}_C}$ is the corresponding operator for the group of rank $r_G - 2$, whose set of positive roots $\Sigma_+^{(-2)} = \Sigma_+/\Sigma_+^1$ does not contain π_1 and π_{r_G} (in other words, it is obtained by deleting the two simple roots π_1 and π_{r_G} from the Dynkin diagram for the group G^C of rank r_G). The intertwining operators of real groups from which $r_s = 0$ can be expressed by formulas similar to (47) and (49) of the corresponding complex case.

As an example of the intertwining operators for an arbitrary real semisimple Lie group ($r_s \neq 0$), we give their expressions for the real form of the classical type AIII, or more precisely, for the reductive group $U(p,q)$, $p \geqslant q$, which differs from $SU(pq)$ only by the center. As is shown in Ref. 27, the integral formula (42) in this case leads to the factorized form

$$\hat{J}_{U(p,q)}^{\{\omega,\Lambda\}} = {}_1\hat{J}_L \hat{J}_{U\,2}\hat{J}_L \hat{J}_{U(p-1,q-1)}^{\{\omega,\Lambda\}}, \tag{50}$$

where ${}_\mu\hat{J}_L$ are the expressions (49) analogous to $\hat{J}^1_{L(n,c)}$ with the obvious replacement of Σ_+^1 by ${}_\mu\Sigma_+$ and $X_{\pm\alpha}$ by $X_{\pm\alpha} + X_{\pm\bar{\alpha}}$ ($\bar{\pi}_k = \pi_{p+q-k}$, $1 \leqslant k \leqslant q-1$); $\hat{J}_U$ is the operator corresponding to the subgroup $U(p-q+1,1)$, and $\hat{J}_{U(p-1,q-1)}^{\{\omega,\Lambda\}}$ is the corresponding operator for the subgroup $U(p-1,q-1) \subset U(p,q)$, and it does not contain quantities with index 1. Therefore, in the case of the pseudounitary group $U(p,q)$ the problem reduces to finding the operator $\hat{J}_U$ for the group $U(m,1)$, $m = p-q+1$, of rank 1. This group occupies a distinguished position among the pseudounitary groups for the following reason: When an irreducible representation of it is restricted to the maximal compact subgroup $U(m)\otimes U(1)$ the irreducible representations of the latter occur not more than once, which enables one to diagonalize the kernel of the intertwining operator in the form (41) and on this basis obtain a complete solution of the problems mentioned above.

Let us consider in more detail the group $U(m,1)$ and illustrate in this example the general discussion from the preceding section. The kernel of the intertwining operator of the representation $\{\rho, \varkappa; p_1, \ldots, p_{m-1}\}$ ($\{p_1, \ldots, p_{m-1}\}$ and $\varkappa$ weight of the representations of the subgroups $U(m-1)$ and $U(1)$ of the centralizer $S = U(m-1)\otimes U(1)$) in the case of nontrivial Weyl substitution has the form

$$\begin{gathered} B^{\{d_\pm, p\}} = (\exp(i\varphi) - m_1^1)^{-(d_- + m)}(\exp(-i\varphi) - \overset{*}{m}{}_1^1)^{-(d_+ + m)} \times \prod_1^{m-1} (\det_j \tilde{m})^{p_j - p_{j+1}}; \\ \tilde{m}_\beta^\alpha = m_\beta^\alpha + \frac{m_1^\alpha m_\beta^1}{\exp(i\varphi) - m_1^1};\quad 2 \leqslant \alpha, \beta \leqslant m;\ p_m \equiv 0; \\ d_\pm = (\rho \pm \varkappa)/2;\ m \in U(m);\ \exp[i\varphi] \in U(1), \end{gathered} \tag{51}$$

where $\det_j m$ are the "principal" minors of the matrix $m - \det_1 m = m_m^m$, $\det_2 m = m_m^m m_{m-1}^{m-1} - m_m^{m-1} m_{m-1}^m$, etc. Note that the weights L_α of the representation are connected to the parameters $d_\pm$ and p_i by the relations $L_1 = d_+ + m$, $L_{i+1} = p_i + m - i$, $1 \leqslant i \leqslant m-1$, $L_{m+1} = -d_-$. In accordance with (51), the matrix element (41) that is diagonal with respect to the quantum numbers of S:

$$B_{\{l\}}^{\{\rho, \varkappa; p\}} = N_{\{l\}}^{-1} \int dm\, d\varphi\, B^{\{d_\pm, p\}}(m, \varphi) \overset{*}{D}{}_{\{p\}, \{p\}}^{\{l\}}(m) \times \exp(-ir\varphi), \quad \sum_1^m l_i - r = \varkappa + \sum_1^{m-1} p_i \tag{52}$$

has the form

$$B_{\{l\}}^{\{\rho, \varkappa; p\}} = \frac{(-)^{\sum_1^m l_\alpha}}{N_{\{l\}}} \frac{\Gamma(-d_+ - d_- - m)}{\Gamma(-d_- - l_m)\Gamma(+d_- - l_m - m - 1)} \times \prod_1^{m-1} (-d_+ + p_\alpha - \alpha)^{-1} \frac{B(-d_- - p_\alpha - m + \alpha - d_+ + p_\alpha - \alpha + 1)}{B(-d_- - l_\alpha - m + \alpha - d_+ + l_\alpha - \alpha + 1)}.$$

(A detailed derivation of Eqs. (51) and (52) is given in Ref. 36.) The analytic properties of the expression (52) are directly determined for integral $d_\pm$ by the signs of the arguments of the Γ functions which occur in it, which enables one to list all the completely irreducible components, including the various "strange" series. In the particular case of the group $SU(2,1)$ the spaces of all the completely irreducible representations described earlier in Ref. 33 are exactly reproduced by the structure function (52) for $m=2$, just like the unitary series for arbitrary m classified in Refs. 24 and 34. The matrix form of expression of the asymptotic expansion (43) for the transformation $g \exp\Phi_{mm+1}\tau$ of the group $U(m,1)$ in the case of maximal occupation with respect to the quantum numbers of S has the form

$$D_{\{l\},\{l\}}^{\{\rho, s\}}(\tau) \underset{\{\tau\}\to\infty}{\Rightarrow} \exp(\tau\rho) B_{\{l\}}^{\{-\rho-2m, s\}} \theta(\operatorname{Re}\rho + m) + \exp[-\tau(\rho + 2m)] (-)^{\sum_1^m l_\alpha} B_{\{l\}}^{\{\rho, s\}} \theta(-\operatorname{Re}\rho - m),$$

$$\theta(-\operatorname{Re}\rho - m), \quad \theta(x) = \begin{cases} 1, & x \geqslant 0, \\ 0, & x < 0, \end{cases}$$

where the coefficients $B_{\{l\}}^{\{\rho, s\}}$ of the principal terms of the asymptotic expansion are determined by Eq. (52). This enables one to separate out all the unitary representations on the basis of the structure of the arguments of the Γ functions.

The factorizability of the expressions (49) and (50), which reduce to a product of the corresponding operators for the groups of rank 1, is a concrete realization of Schiffman's suggestion[28] that one could reduce the problem of the investigation of the intertwining operators of a group of arbitrary rank to groups of rank 1. However, in contrast to the abstract form of expression of the intertwining operators in the form of a convolution[29] (in the general case multiple) of the corresponding operators for the simple reflections, we have obtained an explicit expression for an arbitrary transformation of the Weyl group. At the same time, the operators (42) are represented as products of known functions of the type (48) of the generators of the compact subgroups, which have a known integral spectrum of eigenvalues.

We note finally that operator irreducibility of the representations $\{\rho, s\}$ constructed in the previous chapters follows directly from our analysis. To see this, choosing the identity transformation for the transformation of the Weyl group, i.e., $\{\Lambda'\} = \{\Lambda\}$, we find that the kernel $B^{\{\Lambda\}}(k)$ has a δ-function form since in this case $\Sigma_+ \cap \omega(\Sigma_-) = \phi$ if the weights $\{\Lambda\}$ do not satisfy additional constraint relations. (This last case, which corresponds to degenerate representations, is investigated in detail in Ref. 26.)

Bilinear invariant Hermitian form. The problem of distinguishing unitary representations can be related to that of finding a bilinear invariant Hermitian form (in the cases when it exists) and the subsequent investigation of whether it is positive definite. The existence of a bilinear invariant Hermitian form means that one can introduce into the representation space $D^{\{\Lambda\}}$ a scalar product, whose most general form in the realization used is

$$(f_1, f_2)^{\{\Lambda\}} = \int dk_1\, dk_2\, f_1^*(k_1) K^{\{\Lambda\}}(k_1, k_2) f_2(k_2), \tag{53}$$

and on which the following conditions are imposed:

1) Hermiticity

$$(f_1, f_2)^{\{\Lambda\}} = (f_2, f_1)^{*\{\Lambda\}}; \tag{54}$$

2) invariance

$$(f_1, \hat{F}_i f_2)^{\{\Lambda\}} = (\hat{F}_i f_1, f_2)^{\{\Lambda\}}. \tag{55}$$

In general, the kernel $K^{\{\Lambda\}}$ is a generalized function.

Representations $\{\Lambda\}$ satisfying these conditions are called pseudounitary. The additional condition

3) of positive definiteness:

$$(f_1, f_2) \geqslant 0 \tag{56}$$

leads to unitary representations. In addition, if the functions f in the representation space $D^{\{\Lambda\}}$ are square-integrable, the representation belongs to one of the principal series and occurs, in general, in the decomposition of the regular representation; otherwise, the representation belongs to one of the complementary series. Equations (54) and (55) can be rewritten in the form of the following conditions on the kernel:

$$K^*(k_1, k_2) = K(k_2, k_1) \tag{57}$$

and

$$({}_1\hat{F}_i^t - {}_2\hat{F}_i^T) K(k_1, k_2) = 0, \tag{58}$$

and it follows from the subsystem (58), which is related to the generators of the compact transformations of G, that the kernel depends only on $k \equiv k_2^{-1} k_1$, so that

$$K^*(k) = K(k^+). \tag{59}$$

The condition of invariance

$$T_g^{+\{\Lambda\}} \hat{K}^{\{\Lambda\}} T_g^{\{\Lambda\}} = \hat{K}^{\{\Lambda\}}, \tag{60}$$

whose infinitesimal form is Eq. (58), can be written in the case of a nondegenerate operator $\hat{K}$ in the form

$$T_{g^{-1}}^{+\{\Lambda\}} = \hat{K}^{\{\Lambda\}} \hat{T}_g^{\{\Lambda\}} \hat{K}^{\{\Lambda\}^{-1}}, \tag{61}$$

from which it follows, in particular, that the representations $T_{g^{-1}}^{+\{\Lambda\}}$ and $T_g^{\{\Lambda\}}$ have the same characters, i.e.,

$$\overset{*}{\pi}{}^{\{\Lambda\}}(\lambda_\alpha^{-1}) = \pi^{\{\Lambda\}}(\lambda_\alpha). \tag{62}$$

If the explicit form of the characters of the irreducible representations is known, this enables one to calculate the restrictions imposed by the invariance condition on the weights of the representation.

In the general case of an arbitrary semisimple Lie group G, there follows from the invariance condition (58) in the infinitesimal form in conjunction with the explicit form of the generators (12) and (13) a system of equations for the kernel $K^{\{\Lambda\}}(k)$ of the type (45), the solvability condition of this system being $\{\rho, s\}_\omega = \{-\rho^* - 2\Delta, s\}$, where $\omega \in W_G$. At the same time, the solution of the system for $K^{\{\Lambda\}}(k)$ is the kernel of the corresponding interlacing operator. To see this, it is sufficient to compare the systems (44) and (58). Therefore, the expressions obtained above enable one to separate out the pseudounitary components from all completely irreducible representations of the groups considered. The problem of finding unitary representations by effective use of the condition of positive definiteness in the form (56) entails the construction of a basis in the representation space that diagonalizes the corresponding bilinear form.

Let us illustrate what we have said above for the example of the group $U(m,1)$, for which the solution of this problem in the proposed approach can be found completely.[36] It follows from the solvability condition of the corresponding system that for pseudounitary representations either $\rho = -m, \ +i\sigma, \ \sigma = \sigma^*$, or ρ is real. In the first case, which corresponds to the principal continuous series, the kernel is simply the δ function and the positive definiteness of the form is obvious. In the second case, proceeding from the explicit expression (52) and imposing the condition of positive definiteness of the corresponding form, we can obtain restrictions on the weights $(l_1, \ldots, l_m)$ of the irreducible unitary representations of the compact subgroup $U(m)$ that occur in the decomposition of the representations $(L_1, \ldots, L_{m+1})$ of the group $U(m,1)$ with integral weights, and we can therefore completely describe the discrete series of unitary representations of this group.

The condition of definiteness of the invariant Hermitian form with respect to the parameter l_α leads to the inequality $f_\alpha \leqslant \min(L_1, L_{m+1}) - 1$ or $f_\alpha \geqslant \max(L_1, L_{m+1})$, $f_\alpha \equiv l_\alpha + m - \alpha$, for which the residue of the corresponding Γ function in the denominator of the expression (52) ensures that the sign function $(-)^{l_\alpha}$ disappears from the form. From the total number of 2^m possible sets of inequalities, only $m+1$ are compatible with the chain $f_1 > f_2 > \cdots > f_m$; namely $\min(L_1, L_{m+1}) > f_1 > \cdots > f_m$; $f_1 > \cdots > f_m \geqslant \max(L_1, L_{m+1})$; $f_1 > \cdots > f_{k-1} \geqslant \max(L_1, L_{m+1}) \geqslant \min(L_1, L_{m+1}) > f_k > \cdots > f_m$, $2 \leqslant k \leqslant m$. With allowance for the Gel'fand–Tseitlin inequalities for the unitary group $f_1 > L_2 \geqslant f_2 > L_3 \geqslant f_3 > \cdots > L_m \geqslant f_m$, we finally obtain for the parameters of the discrete series of nondegenerate unitary representations of the group $U(m,1)$

$$\left.\begin{array}{l} \min(L_1, L_{m+1}) > f_1 > L_2 \geqslant f_2 > \ldots > L_m \geqslant f_m; \\ f_1 > L_2 \geqslant f_2 > \ldots > L_m \geqslant f_m \geqslant \max(L_1, L_{m+1}); \\ f_1 > L_2 \geqslant f_2 > \ldots > L_{k-1} \geqslant f_{k-1} \\ \geqslant T(L_1, L_k, L_{m+1}) > f_k > L_{k+1} \geqslant \ldots > L_m \geqslant f_m, \end{array}\right\} \tag{63}$$

where T is the ordering symbol.

Thus, with respect to representations of the subgroup $U(m)$ the representation space (of the discrete series) of the group $U(m,1)$ has been split into a sum of pairwise inequivalent irreducible invariant subspaces $D(l_1, \ldots, l_m)$ determined by the inequalities (63), on each of which the bilinear invariant Hermitian form is determined uniquely (to within an unimportant factor) and is positive definite.

In the general case of an arbitrary semisimple Lie group the complete solution of the problem of separating the unitary components from the pseudounitary representations on the basis of the condition of positive definiteness of a form requires further investigation.

4. PLANCHEREL MEASURE OF THE PRINCIPAL CONTINUOUS SERIES OF UNITARY REPRESENTATIONS

General expression for the Plancherel measure. We calculate the Plancherel measure of the principal continuous series of unitary representations of the complex classical Lie groups and their real forms. To do this, using the results of Sec. 2, we calculate the explicit form of the normalization constant $F(\sigma, s)$ of the matrix elements of the principal continuous series $(\rho_j = -\Delta_j + i\sigma_j, \ \sigma_j^* = \sigma_j)$, with which the weight function $\omega(\sigma, s)$ of the Plancherel measure is related by an inverse proportionality.

In the case of the principal continuous series, the condition of orthogonality of the matrix elements $D^{\{\rho, s\}}_{\{N_1\}, \{N_2\}}(g)$ has the form

$$\begin{aligned} &\int \overset{*}{D}{}^{\{\rho, s\}}_{\{N_1\}, \{N_2\}}(g)\, D^{\{\rho', s'\}}_{\{M_1\}, \{M_2\}}(g)\, dg \\ &= \delta_{\{N_1\}, \{M_1\}} \delta_{\{N_2\}, \{M_2\}} \delta_{\{s\}, \{s'\}} F(\qquad \prod_1^{\mathscr{A}} \delta(\sigma_j - \sigma_j'), \end{aligned} \tag{64}$$

where dg is an invariant measure of G; $\delta_{\{N\}, \{M\}}$ are Kronecker deltas with respect to all quantum numbers of the basis vectors. To calculate the function $F(\sigma, s)$, we use the method employed by Fock[30] to calculate the normalization constant of the continuous spectrum of the hydrogen atom. The essence of this method is the possibility of going over to the asymptotic behavior for the functions in the integrand, for which it is known that it has a δ-function nature with respect to the continuous parameters. After this, it remains to remains to integrate only with respect to the parameters of the maximal compact subgroup, which significantly facilitates all the calculations, and enables one to carry them through to concrete expressions. In accordance with the decomposition (3) used for the elements of the group

$$\int f(g)\, dg = \int_{T(\tau)} D(\tau)\, d\tau \int_{\mathscr{K}} dk_1\, dk_2 f(k_1 T(\tau) k_2); \quad d\tau = \prod_1^{r_{\mathscr{A}}} d\tau_j, \tag{65}$$

and in the asymptotic region, specifying a definite ordering with respect to τ_j, for example, $\tau_j - \tau_{j+1} \to \infty$, $1 \leqslant j \leqslant r_{\mathscr{A}}$, $\tau_{r_{\mathscr{A}}+1} = 0$, we have $D(\tau) \Rightarrow \exp 2\sum_j \Delta_j \tau_j$. The invariant volumes on $\mathscr{K}$ are taken to be normalized to unity. Since the normalization constant $F(\sigma, s)$ in (64) does not depend on the quantum numbers $\{N_1\}$ and $\{N_2\}$ of the representation, to find it is convenient to take some definite set $\{N_1\} = \{N_2\} = \{{L \atop MN}\}$, satisfying the necessary requirements that follow from the presence of the centralizer. Namely, as matrix element $D^{\{L\}}_{\{M\}, \{N\}}(k)$ of the maximal compact subgroup in the integral representation (25) for $D^{\{\rho, s\}}_{\{N_1\}, \{N_2\}}(g)$, we choose the matrix element between the highest vectors of the corresponding representation, i. e., $\{{L \atop MN}\} \equiv \{{s \atop ss}\}$. We therefore realize the possibility of using the explicit expressions (A. 5) for the highest vectors of the irreducible representations of the compact groups to calculate integrals of the type (64) in the asymptotic region.

The asymptotic expression of the matrix elements $D^{\{\rho, s\}}_{\{s\}, \{s\}}(g)$ can be found from Eq. (25) by the usual meth-

TABLE VI.

G	Weight functions of the Pancherel measure of the principal continuous series of unitary representations
$L(n, C)$	$\prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$, $\rho_s=-(n-2s+1)+i\sigma_s$
$L(n, R)$	$\prod_{j>j} (\sigma_i-\sigma_j) \tanh \frac{\pi}{2} (\tilde{\sigma}_i-\tilde{\sigma}_j)$, $\tilde{\sigma}_s=\sigma_s+i\xi_s$, $\xi_s=0, 1$; $\rho_s=-\frac{n-2s+1}{2}+i\sigma_s$
$U^*(2n)$	$\prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2] [(\sigma_i-\sigma_j)^2+(\varkappa_i+\varkappa_j+1)^2]$, $1/2\rho_s=-(n-2s+1)+i\sigma_s$
$U(p, q)$ $p \geqslant q$	$\prod_1^q \sigma_i \tanh \frac{\pi}{2} \tilde{\sigma}_i \prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2] [(\sigma_i+\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$ $\times \prod_{s=1}^{q} \prod_{\alpha=1}^{p-q} [\sigma_s^2+(\varkappa_s-l_\alpha)^2]$, $\rho_s=-(n-2s+1)+2i\sigma_s$, $p+q=n$
$O(n, C)$	$n=2k$. $\prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2] [(\sigma_i+\sigma_j)^2+(\varkappa_i+\varkappa_j)^2]$, $\rho_s=-2(n-s)+i\sigma_s$
	$n=2k+1$. $\prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i+\sigma_j)+(\varkappa_i+\varkappa_j)^2] \cdot \prod_s [\sigma_s^2+\varkappa_s^2]$, $\rho_s=-n+2s+i\sigma_s$

TABLE VI. *(continued)*

G	Weight functions of the Pancherel measure of the principal continuous series of unitary representations
$Sp(2n, C)$	$\prod_{i>j} [(\sigma_i+\sigma_j)^2+(\varkappa_i+\varkappa_j)^2] [(\sigma_i-\sigma_j)^2-(\varkappa_i-\varkappa_j)^2] \cdot \prod_1^n (\sigma_s^2-\varkappa_s^2)$, $\rho_s=-2(n-s+1)+i\sigma_s$
$Sp(2n, R)$	$\prod_1^n \sigma_i \tanh \frac{\pi}{2} \tilde{\sigma}_i \prod_{i>j} (\sigma_i+\sigma_j) \tanh \frac{\pi}{2} (\tilde{\sigma}_i-\tilde{\sigma}_j) (\sigma_i-\sigma_j) \tanh \frac{\pi}{2} (\tilde{\sigma}_i-\tilde{\sigma}_j)$, $\rho_s=-(n-s+1)+i\sigma_s$
$Sp(2p, 2q)$ $p \geqslant q$	$\prod_1^q \sigma_j \tanh \frac{\pi}{2} \tilde{\sigma}_j \cdot [\sigma_j^2+(\varkappa_j-1/2)^2] \cdot \prod_{i>j} [(\sigma_i-\sigma_j)^2-(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i-\sigma_j)^2+(\varkappa_i+\varkappa_j-1)^2] [(\sigma_i-\sigma_j)^2-(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i+\sigma_j)^2+(\varkappa_i+\varkappa_j-1)^2] \cdot \prod_{s,\alpha} [\sigma_s^2+(\varkappa_s-1/2-l_\alpha)^2]$ $\times [\sigma_s^2+(\varkappa_s+1/2-l_\alpha)^2]$.

ods, and it has the form

$$D^{\{\rho, s\}}_{\{s\}; \{s\}}(g) \Rightarrow \exp\left[-\sum_j \tau_j \Delta_j\right] \sum_\omega \exp\left[i \sum_j \sigma_{\omega(j)} \tau_j\right]$$
$$\times \int dk \prod_1^{r_{\mathscr{A}}} [R_i^{(a)}]^{\rho_i} \overset{*}{D}^{\{s\}}_{\{s\}, \{s\}}(k) D^{\{s\}}_{\{s\}, \{s\}}(\tilde{k}^{(a)}), \quad (66)$$

where the sum is extended over all permutations ω of the Weyl group W_G and the superscript a of R_i and $\tilde{k}$ indicates the asymptotic value of these quantities. Explicit expressions for $R^{(a)}$ and $\tilde{k}^{(a)}$ have been given above. Substituting into (64) the asymptotic expressions (66) of the matrix elements and integrating with allowance for (65) with respect to $\{\tau\}$, which leads to δ functions on the right-hand side of (64), and with respect to the parameters k_1 and k_2 in $\mathscr{K}$ for $F(\sigma, s)$, we obtain

$$F(\sigma, s) = (2\pi)^{r_{\mathscr{A}}} \int dk\, dk' \prod_1^{r_{\mathscr{A}}} [R_i^{(a)}(k)]^{\rho_i} [R_i^{(a)}(k')]^{\overset{*}{\rho}_i}$$
$$\times \overset{*}{D}^{\{s\}}_{\{s\}, \{s\}}(k'^{-1}k) D^{\{s\}}_{\{s\}, \{s\}}(\tilde{k}'^{(a)^{-1}} \tilde{k}^{(a)}). \quad (67)$$

G	Weight functions of the Pancherel measure of the principal continuous series of unitary representations
$\overset{*}{O}(2n)$	$n=2k$. $\prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i-\sigma_j)^2+(\varkappa_i+\varkappa_j+1)^2] [(\sigma_i+\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i+\sigma_j)^2+(\varkappa_i+\varkappa_j+1)^2] \cdot \prod_1^k \sigma_s \tanh \frac{\pi}{2} \tilde{\sigma}_s$, $\rho_s=-(2n-4s+1)+2i\sigma_s$
	$n=2k+1$. $\prod_{i>j} [(\sigma_i-\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i-\sigma_j)^2+(\varkappa_i+\varkappa_j+1)^2] [(\sigma_i+\sigma_j)^2+(\varkappa_i-\varkappa_j)^2]$ $\times [(\sigma_i+\sigma_j)^2+(\varkappa_i+\varkappa_j+1)^2] \prod_1^k \sigma_s \tanh \frac{\pi}{2} \tilde{\sigma}_s$ $\times [\sigma_s^2+(\varkappa_s-\varkappa+1/2)^2] [\sigma_s^2+(\varkappa_s+\varkappa+1/2)^2]$, $\rho_s=-(2n-4s+1)+2i\sigma_s$
$O(p, q)$ $p \geqslant q$	$p-q=2k$. $\prod_{i>j}^q (\sigma_i+\sigma_j) \tanh \frac{\pi}{2} (\tilde{\sigma}_i+\tilde{\sigma}_j)$ $(\sigma_i-\sigma_j) \tanh \frac{\pi}{2}$ $\times (\tilde{\sigma}_i-\tilde{\sigma}_j) \cdot \prod_{s,\alpha} \left[\sigma_s^2+\left(l_\alpha+\frac{p-q}{2}-\alpha\right)^2\right]$, $\rho_s=-1/2(n-2s)+i\sigma_s$ $p+q=n$
	$p-q=2k+1$. $\prod_{i>j}^q (\sigma_i+\sigma_j) \tanh \frac{\pi}{2} (\tilde{\sigma}_i+\tilde{\sigma}_j) (\sigma_i-\sigma_j) \tanh \frac{\pi}{2}$ $\times (\tilde{\sigma}_i-\tilde{\sigma}_j) \prod_{s,\alpha} \left[\sigma_s^2+\left(l_\alpha+\frac{p-q}{2}-\alpha\right)^2\right] \cdot \prod_1^q \sigma_s \tanh \frac{\pi}{2} \tilde{\sigma}_s$, $\rho_s=-1/2(n-2s)+i\sigma_s$ $p+q=n$

Further simplification of the expression for $F(\sigma, s)$ involves using the properties of the maximally occupied matrix elements $D^{\{s\}}_{\{s\}, \{s\}}$ and the presence of a subgroup of the centralizer. Expanding $D^{\{s\}}_{\{s\}, \{s\}}(k'^{-1}k)$ with respect to a complete system and integrating with respect to the parameters of the centralizer, we obtain from (67)

$$F(\sigma, s) = \frac{(2\pi)^{r_{\mathscr{A}}}}{N_{\{s\}}} \left| \int_{\mathscr{K}, S} dk \prod_1^{r_{\mathscr{A}}} [R_i^{(a)}(k)]^{\rho_i} \overset{*}{D}^{\{s\}}_{\{s\}, \{s\}}(\tilde{k}^{(a)^{-1}} k) \right|^2, \quad (68)$$

where $N_{(s)}$ is the dimension of the representation $\{s\}$ of the centralizer, expressed in terms of the weights s_j by means of (A. 8).

Thus, the problem of calculating the Plancherel measure of the principal continuous series of unitary representations of semisimple Lie groups reduces to calculating integrals of the type

$$J(\sigma, s) = \int_{\mathscr{K}, S} dk \prod_1^{r_{\mathscr{A}}} [R_i^{(a)}(k)]^{\rho_i} \overset{*}{D}^{\{s\}}_{\{s\}, \{s\}}(\tilde{k}^{(a)^{-1}} k). \quad (69)$$

Note that the integrals in Eq. (69), which determine the weight function of the Plancherel measure of the principal continuous series of G for $\rho_j = -\Delta_j + i\sigma_j$, are obtained at the convergence limit, and they must therefore by understood in the generalized sense.

In the case of complex semisimple Lie groups, Eq. (69) admits further simplification because of the particular simplicity of the expressions for $R_i^{(a)}$ and

$D^{\{s\}}_{\{s\},\{s\}}(k\overset{-1}{l}_{\alpha}k)$ [see (27)]. For such groups, the integral (69) can be rewritten as

$$J(\sigma, s) = \int dk \prod_{1}^{r,\mathcal{A}} |\det_{\alpha} k|^{i(\sigma_{\alpha}-\sigma_{\alpha+1})-(s_{\alpha}-s_{\alpha+1})}. \qquad (70)$$

The integrand in Eq. (70) is simply the analytic continuation of the square of the modulus of the matrix element of $\mathcal{K}$ (between the highest vectors) in the region of complex values $(s_{\alpha}+i\sigma_{\alpha})/2$ of the weights of the corresponding representation. Therefore, the weight function of the Plancherel measure has the form

$$\omega(\sigma, s) = (2\pi)^{-r\mathcal{A}} |N_{\{l\}}|^2 \Big|_{\{l\} \equiv \left\{\frac{s_{\alpha}+i\sigma_{\alpha}}{2}\right\}}, \qquad (71)$$

where $N_{\{l\}}$ is the dimension of the irreducible representation of the corresponding compact group $\mathcal{K}$ determined by Eq. (A.8). For the complex classical Lie groups the weight function in (71) was obtained by other methods in Refs. 21, 28, and 31.

Explicit form of the weight function for different types of classical semisimple Lie groups. Without dwelling on the actual calculations of the Plancherel measure of the principal continuous series of unitary representations of different types of real semisimple classical Lie groups contained in Refs. 41 and 43, we give the final explicit expressions of the weight functions for these groups in Table 6. For simplicity, we omit the numerical factors. They can be readily recovered from the general expression for the weight function of an arbitrary semisimple Lie group given below.

Root form of expression of the weight functions. The expressions given in Table 6 for the weight function of the Plancherel measure of the principal continuous series of unitary representations of the classical semisimple Lie groups can be expressed in a unified manner in the invariant root form. Namely,

$$\left.\begin{aligned} &\omega(\sigma, s) = (2\pi)^{-r\mathcal{A}} N_{\{s\}} |J(\sigma, s)|^{-2};\\ &J(\sigma, s) = \prod_{\alpha>0}{}' B[p_{\alpha} 2, \alpha(\rho+\rho_0)]\, B[p_{\alpha} 2, \alpha(\rho_0)]. \end{aligned}\right\} \qquad (72)$$

where the product is extended over all positive roots α (except for the centralizer's) for which all $|\alpha(\rho+\rho_0)|$ are different; $B(x, y)$ is the beta function; p_{α} is the multiplicity of the corresponding root.

Note that for representations of class 1 (in the case of symmetric Riemannian spaces of nonpositive curvature) the expression (72) for the weight function was obtained earlier in Ref. 32. The expression (72) is also valid for an arbitrary (not necessarily classical) complex semisimple Lie group.

The expressions found for the weight functions of the Plancherel measure of the principal continuous series of unitary representations of the classical semisimple Lie groups are analytic functions of the representation parameters ρ, which have poles of not higher than first order (because of the hyperbolic tangents). The position of the poles and their number are uniquely related to the number of principal series of unitary representations of the groups. The Plancherel measure of the various

TABLE VII.

G	Highest roots of the simple complex Lie algebras		
A_n, $n>1$	$\pi_1, \pi_1+\pi_2, \ldots, \pi \equiv \sum_{j=1}^{r} \pi_j$	B_n, $n \geqslant 1$	$\pi, \pi+\pi_r, \pi+\pi_{r-1}+\pi_r, \ldots, 2\pi-\pi_1$
C_n, $n \geqslant 1$	$\pi, \pi+\pi_{r-1}, \pi+\pi_{r-2}+\pi_{r-1}, \ldots, 2\pi-\pi_r$		
D_n, $n>2$	$\pi-\pi_{r-1}, \pi-\pi_r, \pi, \pi+\pi_{r-2}, \pi+\pi_{r-3}+\pi_{r-2}, \ldots, 2\pi-\pi_{r-1}-\pi_r-\pi_1$		
G_2	(13), (23)	F_4	(1232), (1242), (1243), (2243)
E_6	(101212), (112211), (111212), (112212), (112312), (112322)	E_7	(1112322), (1212322), (1212313), (1212323), (1212423), (1213423), (1223423)
E_8	(12324524), (12323534), (12324534), (12324634), (12324635), (12424635), (13424635), (23424635)		

Note. In the table, we use the parametrization of the positive roots in the form of Ref. 6, the highest roots being arranged in order of ascending height. To simplify the notation, only the coefficients of the decomposition with respect to the simple roots are given for the highest roots of the exceptional Cartan algebras.

semidiscrete series and the one discrete series, whose existence is due to the real semisimple groups having nonisomorphic Cartan subgroups, is evidently determined by the residues at the poles of the weight function for the principal continuous series considered in the complex plane of the parameters $\{\sigma\}$.

The existing examples of real roots for which the Plancherel measure has been completely calculated,[35] i.e., for all the types of principal series which they have, do not contradict our conjecture. If this fact is confirmed in the general case and proved independently, then the expression (72) completely solves the problem of decomposing the regular representation into irreducible components (or the problem of expanding square-integrable functions with respect to the matrix elements of the various principal series of unitary representations of semisimple Lie groups).

The remarkable simplicity of the final result (72) indicates that there should be a purely algebraic derivation for the weight function of the Plancherel measure of semisimple groups. This is indicated by the presence of the reduction procedure, which enables one in concrete calculations of the measure[41,43] to carry out a restriction to subgroups of lower dimension and factorize the integrals that arise in this way.

CONCLUSIONS

At the present stage of the development and application of the group-theoretical approach in applications in physics the tensor basis undoubtedly predominates. The present investigation of various aspects of the representation theory of semisimple Lie groups on the basis of the asymptotic method and the universal parametrization of the elements of the compact groups makes essential use of the invariant root formulation for these groups, in which all the proofs and the final results take on their simplest and most unified form. In our opinion, use of the root technique in the framework of the group-theoretical approach makes it possible to simplify considerably the calculations needed in applications and reduce the majority of them to simple automatic operations.

Let us list the problems in the representation theory of semisimple Lie groups whose solution can be obtained in the framework of the method developed in this review and has great interest.

1. The separation of all the completely irreducible and unitary representations of an arbitrary semisimple Lie group. Knowledge of the explicit form of the intertwining operators, in terms of which the kernels of the Hermitian forms and the principal terms in the asymptotic expansion of the matrix elements are expressed, enables one to reduce this problem to diagonalization of the corresponding finite-dimensional matrices of the type (41) with subsequent verification of their positive definiteness and analytic properties.

2. Calculation of the Plancherel measure for all the principal series and verification on this basis of the conjecture made earlier.

3. Investigation of the number of representations of a group that can be obtained by means of the asymptotic method. This is an essentially mathematical problem and intimately related to that of the calculation of all irreducible representations of semisimple Lie groups.

One can at the least assert that among the representations obtained by the asymptotic method there are all the representations of these groups described hitherto (by different methods), and they undoubtedly provide a fairly large selection for applications in physics.

The complete solution of these problems, which is known only for some special cases, would go a long way to completing the theory of representations of semisimple Lie groups.

APPENDIX

Universal parametrization of the compact groups; highest vectors of their irreducible representation. From the analysis of the structure of the system R_+ of the positive roots of the classical series A_r, B_r, C_r, D_r and the exceptional Cartan Lie algebras G_2, F_4, E_6, E_7, E_8 there follows the existence of an ordering in R_+ for which each sum root lies between its summands. An arrangement of the indices of the roots $\alpha_1, \ldots, \alpha_n$ fixed (not uniquely) in this way will be called a Σ_+ ordering.

Let us briefly clarify the meaning of this definition. In R_+ we take a collection of roots $\{\alpha\}$ and $\{\alpha'\}$ for which the maximal root s is the sum $(s=\alpha_i+\alpha_i')$ and we arrange the distinguished subset $\{\alpha, \alpha', s\} \in R_+$ in such a way that each sum root lies between its summands. In the remaining collection we order similarly. As an example, let us give a Σ_+-ordered system of roots for G_2: $(\pi_1, \pi_1+\pi_2, 2\pi_1+3\pi_2, \pi_1+2\pi_2, \pi_1+3\pi_2, \pi_2)$.

It is known that an arbitrary element $\hat{k}$ of the compact group $\mathscr{K}$ can be parametrized by the elements of its three-dimensional subgroups. Accordingly, we represent $\hat{k}$ in the form

$$\hat{k}=\prod_{\alpha>0}^{\Sigma_+} \exp\left[i\frac{h_\alpha}{(\alpha\alpha)}\varphi_\alpha\right] \exp\left[i\frac{X_\alpha+X_{-\alpha}}{\sqrt{2(\alpha\alpha)}}\right]\theta_\alpha \times \prod_{j=1}^{r} \exp[ih_j\Psi_j/2], \quad \left.\begin{matrix} 0\leqslant\theta_\alpha<\pi; \\ 0\leqslant\varphi_\alpha<2\pi; \\ 0\leqslant\Psi_j<4\pi, \end{matrix}\right\} \quad \text{(A. 1)}$$

where by the product $\Pi_{\alpha>0}^{\Sigma_+}$ we understand a Σ_+ ordering with respect to all the positive roots $\alpha>0$. Using the formula (see, for example, Ref. 5)

$$\exp\left[i\frac{X_\alpha+X_{-\alpha}}{\sqrt{2(\alpha\alpha)}}\theta_\alpha\right]=\exp\left[i\sqrt{\frac{2}{(\alpha\alpha)}}\tan\frac{\theta_\alpha}{2}X_{-\alpha}\right] \times \exp\left[\frac{2}{(\alpha\alpha)}\ln\cos\frac{\theta_\alpha}{2}h_\alpha\right]\times\exp\left[i\sqrt{\frac{2}{(\alpha\alpha)}}\tan\frac{\theta_\alpha}{2}X_\alpha\right] \quad \text{(A. 2)}$$

we can show that the invariant measure dk on $\mathscr{K}$ in the parametrization (A. 1) has the form

$$dk=\prod_{\alpha>0}(\cos\theta_\alpha/2)^{4(\alpha\rho_0)/(\alpha\alpha)-1}\sin\theta_\alpha/2\,d\theta_\alpha\,d\varphi_\alpha\prod_{j=1}^{r}d\Psi_j. \quad \text{(A. 3)}$$

To find the explicit expression for the highest vector of the irreducible representation of the group $\mathscr{K}$ in the parametrization (A. 1) we first prove the following assertion.

In the Gaussian decomposition $\hat{k}=z_-dz_+$ of the complex hull of the group $\mathscr{K}$ the element d of the maximal Abelian subgroup "factorizes" in the parameters $(\varphi_\alpha, \theta_\alpha, \Psi_j)$ and has the form

$$d=\prod_{\alpha>0}\exp\left\{i\frac{h_\alpha}{(\alpha\alpha)}[\varphi_\alpha-2i\ln\cos\theta_\alpha/2]\right\}\prod_1^r\exp(ih_j\Psi_j/2). \quad \text{(A. 4)}$$

To this end, we consider the subalgebra spanned by the elements X_α and $X_{\{-\alpha'\}}$ of the root space, where $\{-\alpha'\}$ is the collection of all negative roots, which form a subalgebra and are arranged to the right of α for the chosen method of ordering. Suppose $\alpha-\alpha_i'<0$; then $\alpha-\alpha_i'\in\{-\alpha'\}$ since a sum root must lie between its summands α and $\alpha_i'-\alpha>0$. If $\alpha-\alpha_i'>0$, then it lies to the left of α since α is a sum with respect to $\alpha-\alpha_i'>0$ and $\alpha_i'>0$. The commutators of the elements corresponding to the positive roots $\alpha-\alpha_i'$ are situated between the elements. Thus, in the subalgebra X_α, $X_{\{\alpha-\alpha'\}}$, and $X_{\{-\alpha'\}}$ one encounters no elements corresponding to roots that have equal modulus but opposite sign. Using this circumstance and Eqs. (A. 2) and (A. 1), we can prove the assertion by induction.

Proceeding from Eq. (A. 4), we obtain in the usual manner[6] the following expression for the highest vector $\xi^{\{l\}}$ of the irreducible representation of the group $\mathscr{K}$ with the highest weight $\{l\}\equiv\{l_1,\ldots,l_r\}$:

$$\xi^{\{l\}}=\prod_{\alpha>0}\exp\left[i\frac{(\alpha l)}{(\alpha\alpha)}\varphi_\alpha\right](\cos\theta_\alpha/2)^{2\frac{(\alpha l)}{(\alpha\alpha)}}\prod_1^r\exp[i\Psi_j l_j/2]. \quad \text{(A. 5)}$$

The highest vectors $\xi^{\{l\}}$ can be expressed in terms of the principal minors of the matrix $a_{\alpha\alpha'}=\mathrm{Sp}(X_{-\alpha}kX_{\alpha'}k^{-1})$ of the adjoint representation. We denote by $\Xi-\Sigma_+$ the ordered collection of "highest" roots such that $a_i+\alpha\geqslant a_{i+1}\alpha\in\Sigma_+$ and $a\in\Xi$. One of the possible systems of linearly independent "highest" roots for all the simple compact groups is given in Table 7. The following assertion is true.

The principal minors D_j of the matrix $a_{ab}, a, b\in\Xi$ measured from the maximal root $s\in\Xi$ are eigenvectors of the operators $\hat{X}_\alpha(\hat{\hat{X}}_{-\alpha})$ with zero eigenvalues and eigenvectors of the operators $\hat{h}_j(\hat{\hat{h}}_j)$ with eigenvalues $\lambda_j^{(i)}=\sum_{n=s-i+1}^{s}n_j$. The elements of the eigenspace of the αth root of the left (respectively, right) regular representation of $\mathscr{K}$ are denoted by $\hat{X}_\alpha(\hat{\hat{X}}_{-\alpha})$, and $\hat{h}_i(\hat{\hat{h}}_i)$ are the generators of $\mathfrak{h}$.

From the definition of a_{ab} and the linearity of the operator $\hat{X}_a$ it follows that

$$\hat{X}_\alpha a_{ab} = \mathrm{Sp}\,(X_{-\alpha} k [X_\alpha, X_b]_- k^{-1}) = \begin{cases} N_{ab} a_{a,\, b+\alpha}, & b+\alpha \in \Xi, \\ 0, & b+\alpha \notin \Xi. \end{cases}$$

The nilpotent nature of the action of $\hat{X}_\alpha$ on a_{ab} $(b+\alpha > b)$ makes the equation $\hat{X}_\alpha D_j(a) = 0$ obvious. The expression for $\lambda_i^{(j)}$ is a direct consequence of the formula $\hat{h}_i a_{ab} = b_i a_{ab}$. (The corresponding equations for $\hat{X}_{-\alpha}$ and $\hat{h}$ are proved similarly.) Then from the definition of the highest vector $\xi^{\{l\}}$ as the solution of the system of equations

$$\hat{X}_\alpha \xi^{\{l\}} = 0 \quad \tilde{\hat{X}}_{-\alpha} \xi^{\{l\}}; \quad \hat{h}_i \xi^{\{l\}} = l_i \xi^{\{l\}} = -\tilde{\hat{h}}_i \xi^{\{l\}} \tag{A.6}$$

there follows the formula

$$\xi^{\{l\}} = \prod_{1}^{r} [D_j(a)]^{\mu_j}, \tag{A.7}$$

where by virtue of the linear independence of the system of highest roots noted earlier the parameters μ_j can be chosen in accordance with the second pair of equations (A.6).

Knowledge of the explicit expressions for the highest vector and the invariant volume on $\mathscr{K}$ enables us to calculate the normalization constant of the highest vector and thus obtain in one more way the well known expression for the dimension $N_{\{l\}}$ of an irreducible representation of $\mathscr{K}$.[6] Normalizing the invariant measure $d\hat{k}$ to unity, we obtain

$$N_{\{l\}} = \left[\int d\hat{k} \,|\, \xi^{\{l\}} \,|^2 \right]^{-1} = \prod_{\alpha > 0} \frac{(\alpha, l + \rho_0)}{(\alpha, \rho_0)}. \tag{A.8}$$

We now give a new solution to the problem of finding the projection operator onto the highest vector; it is easy to show that this has the form

$$\hat{P}_l = N_{\{l\}} \int d\hat{k} \bar{\xi}^{\{l\}} \hat{k}, \tag{A.9}$$

where in the element $\hat{k} \in \mathscr{K}$ given by Eq. (A.1) we have the infinitesimal operators of the right (respectively, left) regular representation for $X_{\pm\alpha}$ and h_i. Substituting $\hat{k}$ from (A.1) and the expression (A.5) for $\xi^{\{l\}}$ into (A.9), we obtain the following expression for the projection operator $\hat{P}_l$:

$$\left.\begin{aligned} \hat{P}_l = N_{\{l\}} \prod_{\alpha > 0}^{\Sigma_+} \delta \left[\frac{h_\alpha - (\alpha, l)}{(\alpha \rho_0)} \right] J_\alpha \prod_{1}^{r} \delta [h_j - l_j]; \\ J_\alpha = \sum_j \frac{(-)^j}{j!} \left[\frac{2}{(\alpha\alpha)} \right]^j \\ \times \left[2 \frac{(\alpha, l + \rho_0)}{(\alpha\alpha)} \right]! \Big/ \left[2 \frac{(\alpha, l + \rho_0)}{(\alpha\alpha)} + j \right]! \, X_{-\alpha}^j X_\alpha^j , \end{aligned}\right| \tag{A.10}$$

which agrees with the form of the projection operator obtained in Ref. 39 algebraically; δ is the operator δ symbol. A mathematically more rigorous exposition of the content of this Appendix can be found in Ref. 38.

Note also that the parametrization (A.1) enables one to obtain the results of Ref. 32 on the calculation of the Plancherel measure for representations of the class 1 by an apparently more simple method.

[1] E. P. Wigner, Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, New York (1959); M. Hamermesh, Group Theory and its Application to Physical Problems, Addison Wesley, Reading, Mass (1962); Group Theory and its Applications, Academic Press, New York (1968); A. P. Cracknell, Applied Group Theory, Pergamon Press, London (1968); L. Michel and M. Schaaf, Symmetry in Quantum Physics, Russ. tr., Mir, 1974.

[2] Group Theoretical Concepts and Methods in Elementary Particle Physics, New York-London (1964); High Energy Physics and Elementary Particles, IAEA, Vienna (1965); Fizika Vysokikh Énergiĭ i Teoriya Élementarnykh Chastits (High Energy Physics and the Theory of Elementary Particles), Naukova Dumka, Kiev (1967); N. N. Bogolyubov, A. A. Logunov, and I. T. Todorov, Osnovy Aksiomaticheskogo Podkhoda v Kvantovoĭ Teorii Polya, Nauka, Moscow (1969) (Translated as: Introduction to Axiomatic Quantum Field Theory, Benjamin, New York (1975); Proc. International Conference on High Energy Physics, Kiev (1970); London (1974).

[3] V. De Alfaro and T. Regge, Potential Scattering, Amsterdam (1965).

[4] Séminaire "Sophus Lie", (Lect. Notes on Lie Algebras and Lie Groups), Paris (1955, 1957); N. Jacobson, Lie Algebras, Interscience, New York (1962); J.-P. Serre, Lie Algebras and Lie Groups, Benjamin (1965); A.A. Kirillov, Élementy Teorii Predstavleniĭ (Elements of the Theory of Representations), Nauka, Moscow (1972); D. P. Zhelobenko, Garmonicheskiĭ Analiz na Poluprostykh Kompleksnykh Gruppakh Li (Harmonic Analysis on Semisimple Complex Lie Groups), Nauka, Moscow (1974).

[5] S. Helgason, Differential Geometry and Symmetric Spaces, New York (1962).

[6] D. P. Zhelobenko, Kompleksnye Gruppy Li i ikh Predstavleniya (Complex Lie Groups and their Representation), Nauka, Moscow (1972).

[7] N. Bourbaki, Éléments de Mathématique, Vol. 34, Groupes et Algèbras de Lie, Hermann, Paris (1968).

[8] F. A. Berezin *et al.*, Usp. Mat. Nauk **11**, (Part 6), 13 (1956).

[9] G. Racah, Group Theory and Spectroscopy. Preprint JINR R-1864 (1964).

[10] L. C. Biedenharn, Lectures in Theoretical Physics, Vol. 5, New York (1963).

[11] A. Pais, Rev. Mod. Phys. **38**, 215 (1966).

[12] M. Toller, Nuovo Cimento **53**, 671 (1968); J. Strathdee *et al.*, Preprint IC/67/9, Trieste (1967); G. F. Chew and C. de Tar, Phys. Rev. **180**, 1577 (1969); M. Huszár and Ya. A. Smorodinsky, Preprint JINR E2-4225 (1968).

[13] A. M. Perelomov and V. S. Popov, Zh. Éksp. Teor. Fiz. **50**, 179 (1966) [Sov. Phys. JETP **23**, 118 (1966)]; V. G. Kadyshevskiĭ and A. N. Tavkhelidze, in: Problemy Teoreticheskoĭ Fiziki (Problems of Theoretical Physics), A Collection Dedicated to N. N. Bogolyubov on the Occasion of his 60th Birthday, Nauka, Moscow (1969); A. A. Donkov *et al.*, Preprint JINR E2-6992, Dubna (1973).

[14] W. Rühl, Preprint CERN, Geneva (1966); K. Koller, Preprint DESY 74/8, Hamburg (1974); G. Mack and I. T. Todorov, Phys. Rev. D **8**, 1764 (1973).

[15] H. Joos, J. Math. Phys. **5**, 155 (1964); F. Lurcat, Physics **1**, 95 (1964).

[16] B. F. Bayman, Lectures on Application of Group Theory in Nuclear Spectroscopy, Russian translation, Mir, 1963; R. M. Asherova and Yu. F. Smirnov, Nucl. Phys. B **4**, 399 (1968).

[17] C. Chevalley, Amer. J. Math. **77**, 778 (1955).

[18] I. M. Gel'fand, Matem. Sb. **26**, 103 (1950).

[19] A. M. Perelomov and V. S. Popov, Izv. Akad. Nauk SSSR, Ser Matem. **32**, 1368 (1968).

[20] A. N. Leznov, I. A. Malkin, and V. I. Man'ko, Preprint FIAN-139 /in Russian/, P. N. Lebedev Physics Institute (1972).

[21] I. M. Gel'fand and M. A. Naĭmark, Tr. Mat. Inst., Akad. Nauk SSSR **36** (1950).

[22] F. A. Berezin, Dokl. Akad. Nauk SSSR **110**, 897 (1956); T. Hirai, Jap. J. Math. **39**, 1 (1970); J. Kyoto Univ. **12**, 2 (1972); Harish-Chandra, Trans. Amer. Math. Soc. **119**, 457 (1965); Acta. Math. **113**, 241 (1965); **116**, 1 (1966).

[23] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, Integral'naya Geometriya i Svyazannye s neĭ Voprosy Teorii Predstavleniĭ, Fizmatgiz, Moscow (1962) [Translated as: Integral Geometry and Representation Theory, Academic Press, New York (1966)]

[24] M. I. Graev, Tr. Mosk. Matem. Obshchestva, **7**, 335 (1958).

[25] G. W. Mackey, Ann. Math. **55**, 101 (1952); **58**, 193 (1953); Bull. Amer. Math. Soc. **69**, 628 (1963); I. M. Gel'fand and

M.I. Graev, Izv. Akad. Nauk SSSR, Ser. Matem. 17, 189 (1953); M.I. Graev, Usp. Mat. Nauk 12, 179 (1957).
[26]A.N. Leznov, Dissertatsiya (Dissertation), IFVÉ, Serpukhov (1972).
[27]A.N. Leznov and M.V. Savel'ev, Preprint IFVÉ 74-46, Serpukhov (1974).
[28]G. Schiffman, C.R. Acad. Sci. Paris, Ser. A 266, 47, 859 (1968).
[29]A.W. Knapp and E.M. Stein, Ann. Math. 93, 489 (1971).
[30]V. Fock, Z. Phys. 98, 145 (1936).
[31]Harish Chandra, Amer. J. Math. 80, 241, 553 (1958).
[32]S.G. Gindikin and F.I. Karpelevich, Dokl. Akad. Nauk SSSR, 145, 252 (1962).
[33]M.I. Graev, Dokl. Akad. Nauk SSSR 113, 966 (1957); A.I. Shtern, Dokl. Akad. Nauk SSSR 179, 1289 (1968).
[34]T. Hirai, Proc. Japan Acad. 42, 907 (1966); U. Ottoson, Comm. Math. Phys. 10, 114 (1968).
[35]B.D. Romm, Izv. Akad. Nauk SSSR, Ser. Matem. 29, 1147 (1965); T. Hirai, J. Math. Soc. Japan 22, 134 (1970).
[36]A.N. Leznov, M.V. Savel'ev, and V.V. Khrushchev, Preprint IFVÉ 73-24, Serpukhov (1973).
[37]M.V. Savel'ev, Dissertatsiya (Dissertation), IFVÉ, Serpukhov (1971).
[38]A.N. Leznov and M.V. Savel'ev, Funkssional. Analiz. i Ego Prilozhen. 8, 87 (1974).
[39]R.M. Asherova, Yu. F. Smirnov, and V.N. Tolstoĭ, Teor. Mat. Fiz. 8, 255 (1971).
[40]A.N. Leznov, Preprint IFVÉ 68-20-K, Serpukhov (1968).
[41]A.N. Leznov and M.V. Savel'ev, Teor. Mat. Fiz. 2, 311 (1970); 4, 310 (1970); 8, 161 (1971).
[42]A.N. Leznov and I.A. Fedoseev, Teor. Mat. Fiz. 7, 298 (1971).
[43]A.N. Leznov, Preprint IFVÉ 71-42, Serpukhov (1971).

Translated by Julian B. Barbour