Relativistic wave equations with internal degrees of freedom and partons

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For many years, investigations have been made of relativistic wave equations with internal degrees of freedom (internal variables). In the majority of cases, these equations have "defective" solutions, but some equations are known with additional conditions that have a reasonable mass spectrum. On the other hand, it is difficult to introduce an interaction into such equations. In view of a possible description of the parton model by means of these equations, we give the relevant data on the relativistic equations with internal variables.

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INTRODUCTION

The investigation of relativistic wave equations with a definite, and finite, spin was begun in 1926–1928 by Klein and others[1] for a particle with spin zero and by Dirac[2] for a particle with spin $\frac{1}{2}$. Subsequently, equations for particles with higher spin[3–8] and with several mass and spin values[9–14] were considered. The first infinite-component relativistic equation with a mass spectrum was obtained in 1932 by Majorana[15] (for many years this paper remained little known and hardly accessible). In 1947, I. E. Tamm and one of the authors published Ref. 16, in which particles with a spectrum of masses and spins were described by means of internal variables and the wave function was of the type $\Psi(x_\mu, u_\mu)$, where in the simplest case Ψ is a scalar, x_μ are the coordinates of the particle ($\mu = 0, 1, 2, 3$), and u_μ is a four-vector corresponding to the internal degrees of freedom. To a considerable extent, such an approach is equivalent to the bilocal theory of Yukawa,[17] in which u_μ is the difference between the coordinates $x_\mu^{(1)}$ and $x_\mu^{(2)}$ of two bound particles that form a single observable particle.

Relativistic wave equations with higher spins and equations with internal degrees of freedom have since been considered in many other papers.[18–46] To consider this problem, one can also use a matrix representation (instead of introducing new four-variables of the type u_μ); the most important thing is not the means of expression but the use of infinite-dimensional representations of the Lorentz group (and other groups) or finite-dimensional but irreducible representations of this group with several values of the spin of the particle. The equations for particles with definite spin but greater than 0 and $\frac{1}{2}$ are rather complicated and have specific features. Therefore, much attention has been devoted to the study of particles with spin equal to 1, 3/2, 2, etc. (see, for example, Refs. 47–52).

From the point of view of physics, the stimulus for investigating relativistic equations with internal degrees of freedom has been the desire to solve the problem of the mass spectrum; in other words, to obtain an equation that determines the possible masses and all quantum numbers corresponding to the observed elementary particles or, at least, individual families of them. It would seem such a program can be solved only if the particle to be described by the equations interacts sufficiently weakly with all fields. It is only under this condition that the mass spectrum of the original unperturbed equation can have a real significance, just as the calculation of the levels in an atom is meaningful only if they have a sufficiently small width and shift due to interactions not taken into account in the approximation under consideration.

The theory of relativistic wave equations with internal degrees of freedom has not led, or at least not yet, to any concrete or even fairly promising results. However, in recent years interest has been reawakened in a space-time description of the structure of elementary particles in connection with the problem of the internal symmetry of particles, the quark model of hadrons,[53–56] and the parton model.[56] The questions that then arise bear some relation to the problems encountered in the study of relativistic equations with internal degrees of freedom. We therefore feel it is appropriate to consider these equations.

1. SOME RELATIVISTIC WAVE EQUATIONS WITH A MASS SPECTRUM

To illustrate the possibilities and difficulties presented in the approach, let us give some well known examples of relativistic wave equations with mass spectrum.

We shall assume that the particle is described by a scalar wave function $\Psi(x_\mu, u_\mu)$. Proceeding from the model that generalizes the model of a nonrelativistic top to the relativistic case, we make this function satisfy the equation[16]

$$\left[\frac{\partial^2}{\partial x_\mu\,\partial x^\mu} - \varkappa^2 + \frac{\beta}{2} M_{\mu\nu} M^{\mu\nu}\right]\Psi = 0, \tag{1}$$

where $M_{\mu\nu} = u_\mu\,\partial/\partial u^\nu - u_\nu\,\partial/\partial u_\mu$; β and $\varkappa$ are constants, β being analogous to the top's moment of inertia. To solve Eq. (1) in this case, it is necessary to separate the variables x_μ and u_μ and factorize the wave function $\Psi(x_\mu, u_\mu)$:

$$\Psi(x_\mu;\, u_\mu) = \Psi(x_\mu)\,\Phi(u_\mu). \tag{2}$$

The function $\Phi(u_\mu)$ is an eigenfunction of the operator

$$L\Phi = M_{\mu\nu} M^{\mu\nu}\Phi/2 = \lambda\Phi. \tag{3}$$

Then the function $\Psi(x_\mu)$ satisfies the equation

$$\left[\frac{\partial^2}{\partial x_\mu\,\partial x^\mu} - \varkappa^2 + \lambda\beta\right]\Psi(x_\mu) = 0. \tag{4}$$

A state with definite mass corresponds to the solution of this equation with the form

$$\Psi(x_\mu) = \text{const}\exp(-imt). \tag{5}$$

This corresponds to a transition to the rest frame (momentum $\mathbf{p}=0$), and for the mass of the particle we obtain

$$m^2 = \varkappa^2 - \beta\lambda. \tag{6}$$

Thus, the mass spectrum described by Eq. (1) is given by the eigenvalue of the invariant operator L. In Eq. (1), the four-vector u_μ is spacelike. This choice is dictated by the desire to have a discrete spectrum of λ values and therefore a discrete mass spectrum (6). Going over to spherical coordinates on a single-sheeted hyperboloid:

$$\left.\begin{array}{l} u_0 = r\sinh\Psi;\ u_1 = r\cosh\Psi\sin\theta\cos\varphi; \\ u_2 = r\cosh\Psi\sin\theta\sin\varphi;\ u_3 = r\cosh\Psi\cos\theta; \\ r^2 = \mathbf{u}^2 - u_0^2; \\ -\infty < \Psi < \infty;\quad 0 \leqslant \theta \leqslant \pi;\quad 0 \leqslant \varphi \leqslant 2\pi, \end{array}\right\} \tag{7}$$

we can readily show that the operator L is given in differential form by the expression

$$L = -\frac{1}{\cosh^2\Psi}\frac{\partial}{\partial\Psi}\left(\cosh^2\Psi\frac{\partial}{\partial\Psi}\right) + \frac{\Delta_{\theta,\varphi}}{\cosh^2\Psi}, \tag{8}$$

where $\Delta_{\theta,\varphi}$ is the angular part of the three-Laplacian:

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{9}$$

Note that

$$\frac{\partial^2}{\partial u_\mu\,\partial u^\mu} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\right) + \frac{L}{r^2}. \tag{10}$$

If we seek only solutions of the eigenvalue equation (3) that have finite norm, i.e., are square-integrable on the single-sheeted hyperboloid, we obtain a discrete spectrum of λ; since the variables Ψ and θ, φ separate, i.e., the function $\Phi(\Psi, \theta, \varphi)$ can be factorized:

$$\varphi(\Psi, \theta, \varphi) = Y_{lm}(\theta, \varphi)\,R(\Psi). \tag{11}$$

Here, $Y_{lm}(\theta, \varphi)$ are ordinary spherical functions; the function $R(\Psi)$ satisfies the equation

$$\left[\frac{1}{\cosh^2\Psi}\frac{\partial}{\partial\Psi}\left(\cosh^2\Psi\frac{\partial}{\partial\Psi}\right) + \frac{l(l+1)}{\cosh^2\Psi} + \lambda\right]R(\Psi) = 0. \tag{12}$$

Normalized solutions of this equations are the Legendre polynomials $P_l^j(\tanh\Psi)$, where j is an integer. We introduce a new variable and new function:

$$z = \tanh\Psi;\quad P = \sqrt{1-z^2}\,R. \tag{13}$$

Then Eq. (12) can be rewritten as

$$(1-z^2)\frac{\partial^2 P}{\partial z^2} - 2z\frac{\partial P}{\partial z} + l(l+1)P + \frac{\lambda-1}{1-z^2}P = 0. \tag{14}$$

This is the standard equation for the Legendre functions.[56] It has the Legendre polynomials $P_l^j(z)$, where l and j are integers, as normalizable solutions. The polynomials are normalized by the condition

$$\int_{-1}^{1}|P_l^j(z)|^2\,dz = \frac{2}{(2l+1)}\frac{(j+l)!}{(j-l)!}. \tag{15}$$

The eigenvalue λ is discrete and related to the number j:

$$\lambda = 1 - j^2;\quad j = 1, 2, 3, \ldots \tag{16}$$

The number l ranges over the values $l = j,\ j+1,\ j+2, \cdots$. The variable r does not occur in the final expressions since one takes a function on the single-sheeted hyperboloid, so that $\partial\Psi/\partial r = 0$. The spacelike vector u_μ means that one must choose $r^2 = \mathbf{uu} - \mathbf{u}_0^2 > 0$.

If one takes a timelike vector u_μ, the eigenvalue equation for the mass does not have normalizable solutions, and there exists only a continuous mass spectrum.[16] In the case of a spacelike vector u_μ, one must also take into account[57] the continuous part of the spectrum of the operator L. The point is that this operator is like the Hamiltonian for the hydrogen atom, which has a continuous and a discrete spectrum of energies. The eigenfunctions of the discrete spectrum are normalizable, while those of the continuum are not. But if one works in the Hilbert space of square-integrable functions, it is necessary to take into account the complete spectrum (discrete and continuous) in order to have a complete system of functions in this space. At the same time, in such a space one can choose a subspace in which the system of functions belonging to only the discrete spectrum is complete.

This result is also clear when expressed in group language. The point is that the mass operator L is the Casimir operator $C_1 = M_{\mu\nu}M^{\mu\nu}/2$ (Ref. 16 and 12). The second Casimir operator is $C_2 = -i\varepsilon_{\mu\nu\rho\sigma}M^{\mu\nu}M^{\rho\sigma}/2$. The spectrum of the Casimir operators for irreducible representations of the Lorentz group is well known (see, for example, Refs. 12, 16, and 58):

$$\begin{array}{l} \lambda_1 \equiv \lambda = -(j^2-1) + \alpha^2; \\ \lambda_2 = -\alpha j. \end{array} \tag{17}$$

Here, j is integral or half-integral; α is a real positive number. In the case of the realization $M_{\mu\nu} = u_\mu\,\partial/\partial u^\nu - u_\nu\,\partial/\partial u^\mu$ on the single-sheeted hyperboloid (1), $\alpha = 0$ and j is integral if one requires that the eigenfunctions of L be square-integrable. For a spacelike vector u_μ, the problem of the eigenvalues of the Casimir operators is equivalent to the problem of constructing a complete system of functions on the single-sheeted hyperboloid, i.e., on the homogeneous manifold defined by the stationary group of the vector $(0, 1, 0, 0)$. This is the same problem as arises in the decomposition of the quasiregular representation on a homogeneous manifold into irreducible components. For a two-sheeted hyperboloid [timelike vector u_μ, stationary subgroup of the vector $(1, 0, 0, 0)$], the problem was solved in Ref. 58; for a single-sheeted hyperboloid, in Ref. 59 and then in more detail in Ref. 57.

Let us now discuss the mass spectrum which is obtained. As can be seen from Eqs. (6) and (17), the square of the mass is discrete, but each value is infinitely degenerate with respect to the number l and with respect to $m = l_z$, the projection of the angular momentum onto the z axis (the mass of the state depends only on the number j). Such states can be regarded as states with indeterminate spin (the mass spectrum has the property that for given number j the mass does not depend on the spin, and there is infinite degeneracy).

The infinite degeneracy can be eliminated by imposing the following relativistically invariant additional condition:

$$\left[M_{\mu\sigma}M^{\rho\sigma}\frac{\partial^2}{\partial x_\mu\,\partial x^\rho}-(j_0+1)\frac{\partial^2}{\partial x_\mu\,\partial x^\mu}\right]\Psi(x_\mu,\,u_\mu)=0. \quad (18)$$

where j_0 is a fixed number. On the transition to the rest frame, we obtain the condition

$$[M_{0\sigma}M^{0\sigma}-(j_0+1)]\,\Psi(x_\mu,\,u_\mu)=0. \quad (19)$$

It can be seen that the additional condition (19) imposes on the numbers l and j the following constraint: $l^2+l+j^2-j_0=0,\ l\geqslant j,\ j\neq 0$. For integral l and j, this constraint has solutions only for integral j_0. Thus, the additional condition (18) lifts the infinite degeneracy of the mass spectrum with respect to l.

One can also find a mass spectrum corresponding to a relativistic equation in which there are mixed terms depending on the center-of-mass coordinates x_μ and the internal coordinates u_μ:

$$\left[\frac{\partial^2}{\partial x_\mu\,\partial x^\mu}-\varkappa^2+\frac{1}{2}\beta M_{\mu\nu}M^{\mu\nu}+\varepsilon M_{\mu\sigma}M^{\rho\sigma}\frac{\partial^2}{\partial x_\mu\,\partial x^\rho}\right]\Psi(x_\mu,\,u_\mu)=0. \quad (20)$$

For the rest energy [a solution of the type $\exp(-imt)$] we obtain from this equation

$$m^2=[\varkappa^2-\beta(\alpha^2-j^2+1)]/[1+\varepsilon(l^2+l-j^2+\alpha^2+1)]. \quad (21)$$

We again consider only eigenfunctions that have unit norm.

The relativistic equation (1) was constructed on the basis of a physical model of a top, or a relativistic rotator. Relativistic equations with continuous internal variables have also been constructed on the basis of a model of a relativistic oscillator.[17,18]

In particular, in Ref. 17, Yukawa replaced the mass in an equation of the Klein–Gordon type $[\partial^2/\partial x_\mu\,\partial x^\mu-\varkappa^2]\Psi=0$ by the following scalar operator, which depends on the internal variables:

$$\hat{\varkappa}^2=\lambda^2[-\partial^2/\partial u_\mu\,\partial u^\mu+u_\mu u^\mu/\lambda^4]^2/2, \quad (22)$$

where λ is a constant with the dimensions of a length. In this case, the wave function $\Psi(x_\mu,u_\mu)$ can again be factorized. For the factor that depends only on the internal variables, we find the wave function

$$\Psi^{n_1n_2n_3n_0}=H_{n_1}(u_1/\lambda)\,H_{n_2}(u_2/\lambda)\,H_{n_3}(u_3/\lambda)\,H_{n_0}(u_0/\lambda)\,\exp[-(u_0^2+\mathbf{u}^2)/2\lambda^2] \quad (23)$$

which has unit norm (here H_n are Hermite polynomials). The mass spectrum corresponding to the mass operator (22) has the form

$$m^{n_1n_2n_3n_0}=(\sqrt{2}/\lambda)\,(n_1+n_2+n_3-n_0+1). \quad (24)$$

Here n_0, n_1, n_2, n_3 are non-negative integers.

The mass spectrum (24) has zero rest mass and is infinitely degenerate since the sign in front of n_0 is negative (the number n_0 corresponds to the quantum number of vibrations along the time axis). As in the case of the top, one can eliminate the infinite degeneracy by means of an equation with mixed terms, for which the internal coordinates and the cms coordinates do not separate trivially. Consider, for example, the equation

$$\left\{\frac{\partial^2}{\partial x_\mu\,\partial x^\mu}-\lambda^2\left[-\frac{\partial^2}{\partial u_\mu\,\partial u^\mu}+u_\mu u^\mu/\lambda^4\right]^2/2+\varepsilon\lambda^2[-(\partial^2/\partial x_\mu\,\partial u^\mu)^2+(u_\mu\partial/\partial x_\mu)^2/\lambda^4]\right\}\Psi=0, \quad (25)$$

which has the mass spectrum

$$m^{n_1n_2n_3n_0}=\frac{\sqrt{2}}{\lambda}\,\frac{(n_1+n_2+n_3-n_0+1)}{[1-2\varepsilon\,(n_0+1/2)]^{1/2}}. \quad (26)$$

At the same time, one imposes the condition that the eigenfunction be normalizable in the space of the internal variables u_μ. The treatment is carried through for a timelike momentum vector $\mathbf{p}_0^2>\mathbf{p}^2$ and one goes over to the rest frame $\mathbf{p}=0$. By choosing the parameter ε one can eliminate large values of n_0 since otherwise an imaginary mass appears. We shall discuss the use of such a requirement below. But in Ref. 17 Yukawa does use it, and for an example he takes $\varepsilon=1/2$. Then

$$m^{n_1n_2n_3n_0}=2\,(n_1+n_2+n_3+1)/\lambda \quad (27)$$

and the infinite degeneracy is lifted.

Equations (25) can be slightly modified[19,20]:

$$\left\{\frac{\partial^2}{\partial x_\mu\,\partial x^\mu}-\varkappa^2-\frac{\lambda^2}{2}\left(-\frac{\partial^2}{\partial u_\mu\,\partial u^\mu}+\frac{1}{\lambda^4}u_\mu u^\mu\right)^2+\varepsilon\lambda^2\left[-\left(\frac{\partial^2}{\partial x_\mu\,\partial u^\mu}\right)^2+\frac{1}{\lambda^4}\left(u_\mu\frac{\partial}{\partial u_\mu}\right)^2\right]\right\}\Psi=0. \quad (28)$$

At the same time, if $\varkappa\neq 0$, the solution with $p_\mu=0$ is eliminated since the following mass spectrum is obtained:

$$m^2=\frac{\varkappa^2+2\,(n_1+n_2+n_3-n_0+1)^2/\lambda^2}{1-2\varepsilon\,(n_0+1/2)}. \quad (29)$$

Depending on the sign of the coefficient ε, we obtain here, as in (21), either a decreasing branch of the spectrum, $m\to 0$, or an imaginary mass. A number of equations of oscillator type have been proposed by Markov.[18] The first of these equations for a bispinor $\Psi(p_\mu,u_\mu)$, where $\Psi(p_\mu,u_\mu)$ is the Fourier transform of the wave function $\Psi(x_\mu,u_\mu)$, is

$$\left\{\gamma_\mu p_\mu+\varkappa+a\left[-\frac{\partial^2}{\partial u_\mu\,\partial u^\mu}+u_\mu u^\mu+2\,\frac{(p_\mu\partial/\partial u_\mu)^2-(p_\mu u^\mu)^2}{p_\mu p^\mu}\right]\right\}\Psi=0. \quad (30)$$

In the rest frame $\mathbf{p}=0$, the mass operator in this case has the form of the Hamiltonian for a four-oscillator with the same signs for all four oscillators:

$$\hat{m}=\varkappa+a\,[\mathbf{u}^2+u_0^2-\partial^2/\partial u_0^2-\partial^2/\partial\mathbf{u}^2], \quad (31)$$

from which we obtain the mass spectrum

$$m^{n_1n_2n_3n_0}=\varkappa+2a\,(n_1+n_2+n_3+n_0+2). \quad (32)$$

The ground state $n_1=n_2=n_3=n_0=0$ is not degenerate. To describe particles with integral spin, Markov[18] proposed the analogous equation

$$\left\{p_\mu p^\mu+\varkappa^2+a^2\left[-\frac{\partial^2}{\partial u_\mu\,\partial u^\mu}+u_\mu u^\mu+2\,\frac{(p_\mu\partial/\partial u_\mu)^2-(p_\mu u^\mu)^2}{p_\mu p^\mu}\right]\right\}\Psi=0. \quad (33)$$

Since the square of the four-momentum occurs in the denominator in Eqs. (30) and (33), we have essentially gone over to equations of higher order in the derivatives $\partial/\partial x_\mu$. The infinite degeneracy in the mass spectrum (24) is due to the presence of the negative term n_0. In order to "prohibit" vibrations along the time axis, i.e., set $n_0=0$, in a relativistically invariant manner, Markov[18] proposed the following system of equations for the function Ψ:

$$\left\{\gamma_\mu \frac{\partial}{\partial x_\mu}+\varkappa+a\left[-\frac{\partial^2}{\partial u_\mu \partial u^\mu}+u_\mu u^\mu\right]\right\}\Psi=0;\qquad \frac{\partial}{\partial x_\mu}\left(u_\mu-\frac{\partial}{\partial u^\mu}\right)\Psi=0. \tag{34}$$

In the rest frame $\mathbf{p}=0$, the second of these equations, which we shall call the additional condition, takes the form

$$(u_0+\partial/\partial u_0)\,\Psi=0. \tag{35}$$

The operator $(u_0+\partial/\partial u_0)$ is an annihilation operator, which, acting on the corresponding function of the oscillator ground state, gives zero. Therefore, among all the solutions corresponding to the first equation (34), there remain only solutions of the form

$$\Psi_{n_1n_2n_3}=H_{n_1}(u_1)\,H_{n_2}(u_2)\,H_{n_3}(u_3)\exp\left[-(\mathbf{u}^2+u_0^2)\right], \tag{36}$$

which are solutions of both equations (34). In an arbitrary coordinate system, the normalized ground state of the system is described by the function

$$\Psi_0=\exp\{(p^\mu u_\mu)^2/(p_\mu p^\mu)-u_\mu u^\mu,2\}\,\pi. \tag{37}$$

After the group approach had begun to be intensively developed in the theory of elementary particles, which led to the classification of elementary particles on the basis of the unitary group SU_3 (see Ref. 53), the theory of relativistic equations with internal variables was developed further.[21–23] Namely, in the framework of the same equations of the type (1) or (34) the symmetry properties, which play the role of internal symmetry and are related to the internal quantum numbers of the elementary particles, were taken into account. We shall illustrate this for the example of the relativistic oscillator model.[23]

For particles with half-integral spin, we use the equation

$$\left\{\gamma_\mu\frac{\partial}{\partial x_\mu}+\varkappa+\frac{1}{2}\sum_{i=1}^{r} b_i\left(-\frac{\partial^2}{\partial u_\mu^i\,\partial u^{\mu,i}}+u_\mu^i u^{\mu,i}\right)\right\}\Psi=0. \tag{38}$$

Here, u_μ^i $(i=1,2,\ldots,r)$; $u^{\mu,i}\equiv u_i^\mu$ is a set of r four-vectors. In what follows, we shall take r equal to three. This choice is due to the fact that a three-oscillator has U_3 symmetry. Therefore, the introduction of three relativistic four-oscillators enables one to take into account not only relativistic invariance but also the properties of the unitary group SU_3. The parameters b_i take the values of frequencies. A physical picture that to a certain extent corresponds to the chosen model is a system ("drop") vibrating about an equilibrium position.[23] It is these vibrations that are described by the internal variables—the four-vectors u_μ^i. To avoid infinite degeneracy of the levels, we impose on the wave function the additional conditions (34):

$$\frac{\partial}{\partial x_\mu}\left(u_\mu^i-\frac{\partial}{\partial u^{\mu,i}}\right)\Psi=0,\quad i=1,\,2,\,\ldots,\,r. \tag{39}$$

A wave function satisfying the system of Eqs. (38) and (39) can be factorized:

$$\Psi(x_\mu,u_\mu^i)=\{\exp[-i(Et-\mathbf{pr})]\}\,\Psi(u_\mu^i). \tag{40}$$

We go over to the rest frame $\mathbf{p}=0$. Then φ is an eigenfunction of the mass operator:

$$\left[\varkappa+\frac{1}{2}\sum_{i=1}^{3} b_i\left(-\frac{\partial^2}{\partial u_\mu^i\,\partial u^{\mu,i}}+u_\mu^i u^{\mu,i}\right)\right]\varphi=\lambda\varphi \tag{41}$$

and it satisfies the conditions

$$(u_0^i+\partial/\partial u_0^i)\,\varphi=0. \tag{42}$$

The eigenvalues for the rest masses have the form $(r=3)$

$$m=\varkappa+2\sum_{i=1}^{3} b_i\,(n_1^i+n_2^i+n_3^i+1). \tag{43}$$

Here, n_1^i, n_2^i, n_3^i are non-negative integers. The condition (42) leads to a zero value for the vibrations for the time coordinates of the oscillators, i.e., $n_0^i=0$. Note that the frequencies b_i in (43) are different, i.e., each of the three oscillators has its own eigenfrequency. This appreciably reduces the degree of degeneracy of the levels. The model of three relativistic oscillators reflects, in particular, some of the properties of the quark model, in which the baryons are made up of three quarks. In addition, the oscillator model also enables one to draw other important conclusions about the nature of the internal quantum numbers of the elementary particles such as hypercharge and isospin. This is due to the possibility of classifying states corresponding to internal excitations of the system by means of a unitary group. We consider the operators

$$a_\alpha^i=\frac{1}{\sqrt{2}}\left(u_\alpha^i+\frac{\partial}{\partial u^{\alpha,i}}\right);\quad a_\alpha^{i+}=\frac{1}{\sqrt{2}}\left(u_\alpha^i-\frac{\partial}{\partial u^{\alpha,i}}\right). \tag{44}$$

From the operators a_α^i and a_α^{i+} we construct the generators of the group U_3: $T_{ij}=a^{\alpha,i+}a_{\alpha,j}$. These operators satisfy the commutation relations of the infinitesimal operators of U_3 and commute with the mass operator. Therefore, the solutions of the wave equation (38) can be classified by means of the quantum numbers of U_3. In addition, from the operators a_α^i and a_α^{j+} one can construct an angular-momentum operator associated with the variables u_α^i:

$$M_{\alpha,\beta}=-i\sum_{i=1}^{3}(a_\alpha^{i+}a_\beta^i-a_\beta^{i+}a_\alpha^i).$$

The operators T_{ij} and $M_{\alpha\beta}$ commute with one another and with the mass operator; therefore, the solutions of the wave Eq. (38) can be classified by means of internal and spatial quantum numbers. Physically (but of course in the framework of this model) this means that the internal quantum numbers such as hypercharge and isospin are associated with the vibrational excitations of the oscillators which make up the elementary particle.

The operators of the hypercharge, Y, the third projection of the isospin, T_3, and the square of the isospin, T^2, are related to the generators of U_3 by

$$\left.\begin{aligned}&Y=2(T_{33}-T_{11})/3+(T_{22}-T_{33})/3;\\&T_3=(T_{33}-T_{22})/2;\\&T^2=(T_{12}T_{21}+T_{21}T_{12})/2+T_3^2.\end{aligned}\right\} \tag{45}$$

The operator of the projection of the angular momentum, M_z, is given by

$$M_z=-i\sum_{i=1}^{3}(a_1^{i+}a_2^i-a_2^{i+}a_1^i). \tag{46}$$

The operator of the square of the orbital angular momentum has the form $M^2 = M_{12}^2 + M_{23}^2 + M_{31}^2$. The operators M^2 and M_z commute with the operators (45). In group-theoretical language, one can say that states with given quantum numbers belong to one irreducible representation of the group $U(9)$. If one were to take into account vibrations along the time axis for each of the three relativistic oscillators, it would be necessary to use an irreducible representation of the noncompact group $U(9,3)$. The condition (39) leads to the transition to the group $U(9)$.

We emphasize that states belonging to one irreducible representation of $U(9)$ do not have the same mass because the oscillators have different frequencies. Thus, the group in this case is a dynamical group,[60] or a noninvariance group,[61] or an algebra which generates the spectrum.[62] These concepts are frequently related to infinite-dimensional representations of noncompact groups. In the given problem, in one representation of the compact noninvariance group $U(9)$ a finite number of states with different masses are combined. An irreducible representation of $U(9)$ is specified by the highest weight, and for the oscillator under consideration this weight is determined by the principal quantum number

$$N = \sum_{i=1}^{3} n_1^i + n_2^i + n_3^i.$$

Let us determine the values of the spin angular momenta for some of the low levels. For this, we first restrict the representation of $U(9)$ to the group $U(3) \otimes U(3)$, and we then restrict the resulting representations to the group $U(3)\otimes O(3)$. We obtain the following result. The multiplet with the highest weight (000 000 000), which corresponds to the case $N=0$, is a unitary singlet with angular momentum zero. The multiplet (100 000 000), which corresponds to $N=1$, has dimension nine and is a triplet of quarks with angular momentum unity. The next multiplet with $N=2$ and dimension 45 has the following composition: $(N=2) = (6,5) + (6,1) + (3,3)$. Here, we have used the usual notation: On the right-hand side in the brackets the left-hand number is the multiplet index of SU_3, and the number on the right is $2l+1$, where l is the angular momentum.

The next level with dimension 165 has the composition $(N=3) = (10,7) + (10,3) + (8,5) + (1,1)$. To this level there belong two unitary octets with orbital angular momenta 1 and 2 and two decaplets with angular momenta 1 and 3, and also a unitary singlet with angular momentum zero. It can be seen from these results that the relativistic equation based on the oscillator model has quarklike solutions and solutions that have the correct properties with respect to the unitary group. The lower levels correspond to quarks, and for $N=3$ we obtain an octet and a decaplet.[1)]

In Ref. 26, Dirac constructed a number of relativistic equations by introducing additional internal variables u_a $(a=1,2)$. One of these equations is particularly interesting in that it is superficially very similar to the ordinary Dirac equation but it does not have states with negative energies as solutions. The internal variables in this Dirac equation correspond to a two-dimensional oscillator, and it is an equation in which the internal variables and the cms variables are "coupled". In this respect, this equation of Ref. 26 is related, for example, to Eqs. (18) and (20). However, the internal variables u_a do not form a four-vector with respect to the Lorentz group, but transform with their derivatives in accordance with a different representation, which makes the equation rather unusual. The single-component function $\Psi(x_\mu, u_1, u_2)$ depends on six variables. The equation for it is written as follows:

$$(\partial/\partial x_0 + \alpha\partial/\partial \mathbf{x} - \beta)\, q\Psi = 0. \qquad (47)$$

Here, q (as everywhere in the present paper, $\hbar = c = 1$) is given by

$$q = \begin{pmatrix} u_1 \\ u_2 \\ -i\,\partial/\partial u_1 \\ -i\,\partial/\partial u_2 \end{pmatrix}. \qquad (48)$$

The components q_a satisfy the commutation relation

$$[q_a q_b] = i\beta_{ab}.$$

The four-element matrices β and α have the form

$$\left.\begin{array}{l} \beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};\ \alpha_1 = -\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix};\ \alpha_2 = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix};\ \alpha_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \\ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}\right\} \qquad (49)$$

and in (47) they act on the four-column $q\Psi$.

The matrices β, α_1, α_2, α_3 in (49) anticommute with one another, and $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = -\beta^2 = -1$. Thus, for the one function Ψ we have four equations. Equation (47) is, on the one hand, an equation with additional conditions, and, on the other, an equation with mixed terms. The internal variables u_a and $\partial/\partial u_a$ are multiplied by the derivatives with respect to the cms coordinates, $\partial/\partial x_\mu$. The system (47) is compatible, and the compatibility condition has the form (one sets $\varkappa^2 = 1$)

$$(\partial^2/\partial x_\mu\, \partial x^\mu - 1)\, \Psi = 0.$$

For relativistic invariance of the system (47), one needs the following rule of infinitesimal transformation of the column q on the transition from one frame of reference to another:

$$\left.\begin{array}{l} q' = (1 - iW/2)\, q\, (1 + iW/2); \\ W_{ab} = q_a\, [\omega^{\rho\sigma}\alpha_\rho\beta\alpha_\sigma/4]\, q_b \end{array}\right\} \qquad (50)$$

where $\omega^{\rho\sigma}$ is the infinitesimal matrix of a Lorentz rotation. Let us now consider the solutions of Eqs. (47) in the rest frame. Because of the relativistic invariance, one can readily obtain a solution with nonzero momentum from these solutions. The solutions are sought in the form of plane waves of the type $\exp(-ip^\mu x_\mu)\varphi(u_a, p^\alpha)$. The general case reduces to solutions of this form by means of an integral Fourier transformation. In the case $\mathbf{p} = 0$, the equations (47) for the plane waves can be written as follows:

$$(p_0 u_1 + \partial/\partial u_1)\, \Psi = 0; \quad (p_0 u_2 + \partial/\partial u_2)\, \Psi = 0. \qquad (51)$$

Equations (51) have a normalized solution only for $p_0 = 1$. Therefore, the second solution of the Klein–Gordon equation $p^2 = 1$, namely, the solution $p_0 = -1$, is

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[1)]The oscillator model has recently been developed in a number of papers (see, for example, T. De, Y. S. Kim, and M. E. Noz, Nuovo Cimento 13, 1089 (1973) and the reference in it).

eliminated by the normalization condition. If $\mathbf{p}=0$, we have

$$\Psi(x_\mu, u_1, u_2)=\pi^{-1/2}\exp(-ip^\mu x_\mu)$$
$$+\exp\left\{\left[-\tfrac{1}{2}(u_1^2+u_2^2)+ip_1(u_1^2-u_2^2)-2ip_2u_1u_2\right]\Big/(p_0+p_3)\right\}. \quad (52)$$

Thus, by the introduction of internal variables (of two oscillators) Dirac has succeeded in constructing a relativistically invariant equation whose solution, chosen in the usual form of a plane wave, satisfies the normalization condition with respect to the internal variables only for one sign of the energy. Although this Dirac equation does not apparently describe any real physical object, and it is difficult to give the two internal variables u_a a perspicuous interpretation [in contrast to equations (1) and (34) and some others], this equation is nevertheless interesting from the methodological point of view and warrants further study. The same can be said about certain other equations with internal variables.

Above, we have given only a few examples that illustrate the type of equations considered. Of course, one can construct other examples.[35,37,41] We restrict ourselves however to the remark that instead of the vector u_μ one can use other different relativistic variables, for example, tensors $S_{\mu\nu}$, etc. In addition, the internal space need not necessarily be assumed to be pseudo-Euclidean; one could also consider a Riemannian internal space. Unfortunately, none of these possible generalizations have been exploited yet in practice.

2. FEATURES OF RELATIVISTIC EQUATIONS WITH A MASS SPECTRUM

All known equations with internal variables have certain shortcomings, or their use encounters as yet insuperable difficulties. Either there is a physically unacceptable mass spectrum, or one obtains solutions with imaginary masses or superluminal velocities (spacelike vector p_μ). If additional conditions are used, it is not clear how an interaction with other fields, in particular, the electromagnetic, can be introduced.

Imaginary masses and superluminal velocities can already occur for finite-dimensional equations. Thus, for the equation $(\partial^2/\partial x_\mu\,\partial x^\mu+\varkappa^2)\Psi=0$ we obtain $E^2=-\varkappa^2+\mathbf{p}^2$. Therefore, in the rest frame $\mathbf{p}=0$, $E=\pm i$ (imaginary masses), while for $\mathbf{p}^2>\varkappa^2$ we obtain spacelike solutions (velocity $|\mathbf{v}|=|\partial E/\partial\mathbf{p}|=|\mathbf{p}/\sqrt{\mathbf{p}^2-\varkappa^2}|>1$, i.e., the particle moves with a superluminal velocity. At the same time, it is well known that the equation $(\partial^2/\partial x_\mu\,\partial x^\mu-\varkappa^2)\Psi=0$ does not have such "defective" solutions. We may mention, in passing, that superluminal solutions are fairly widely discussed (in connection with tachyons), but the well known difficulties associated with the causality requirement renders their use extremely problematical. The solutions with imaginary masses are sometimes simply ignored. But this is not in order since the presence of solutions with imaginary masses imdicates an instability and the corresponding equations cannot be used (or at least, not without a renormalization, rearrangement, etc.).

Among the above equations, different cases are encountered. Thus, the spectrum (21) for $\beta>0$ and $\epsilon>0$ (and $\alpha=0$, as holds automatically when the internal variables u_μ are used) has an accumulation point at the contains the mass $m=0$; this is the case if $n_1+n_2+n_3-n_0+1=0$. The spectrum (29) does not have such a solution; in this case there are no spacelike solutions either even if solutions with imaginary mass are present.[19,20]

The Majorana equation $(\Gamma_\mu p^\mu+\varkappa)\Psi=0$, $\Gamma_0=\sum_{i=1}^2 a_i^+a_i+1$ has, as was shown in Refs. 15 and 24, both spacelike (superluminal) and lightlike (mass $m=0$) solutions. The mass spectrum of this equation is a decreasing one, $m^2=\varkappa^2/(s+\frac{1}{2})^2$. If the system of functions is to be complete, one requires all branches of solutions, as has been rigorously proved by Mukunda.[46] We give some more examples of equations[21] of a more complicated form based on the noncompact $O(4,2)$ symmetry group of the hydrogen atom, which, for this reason, have hydrogenlike mass spectra. Suppose there are generators of a representation of the Lie algebra of $O(4,2)$ of the form

$$\left.\begin{array}{l}
M_{ij}=\varepsilon_{ijk}(a^+\sigma_k a+b^+\sigma b)/2;\ \ i,j=1,2,3;\\
M_{4i}=(a^+\sigma_i a-b^+\sigma_i b)/2;\ \ M_{40}=-i(a^+Cb^++bCa)/2;\\
M_{0i}=(a^+\sigma_i Cb^+-bC\sigma_i a)/2;\\
M_{5i}=-(a^+\sigma_i Cb^++bC\sigma_i a)/2;\ \ M_{50}=(a^+a+b^+b+2)/2;\\
M_{54}=(a^+Cb^+-bCa)/2;\quad C=\begin{pmatrix}0&1\\-1&0\end{pmatrix}
\end{array}\right\}. \quad (53)$$

Here, σ_i are the ordinary Pauli matrices, and the expression of the type $a^+\sigma_k a$ is understood as $\sum_{i,j=1}^2 a_i^+(\sigma_k)_{ij}a_j$. Since $O(4,2)$ contains the Lorentz group $O(3,1)$ as a subgroup, the generators include Lorentz vectors and scalars. Therefore, choosing them, one can construct a number of equations, for example, $[\Gamma_\mu p^\mu+(S-\alpha)k]\Psi=0$, where k and α are constants; $\Gamma_\mu=(2M_{54}, 2M_{0i})$, $S=2M_{50}$ with the spectrum $m=\pm k[1-4\alpha^2/(N+2)^2]$, or, with a different choice ($\Gamma_\mu=2M_{5\mu}$, $S=2M_{54}$) we have the mass spectrum $m=|k|\times(1+\alpha^2/N^2)^{1/2}\,\mathrm{sgn}(k\alpha)$. On the basis of the generators (53) one can construct rather a lot of equations of this kind: They are distinguished by the possibility of describing hydrogenlike mass spectra that increase up to a certain limit. Among these solutions, one also encounters spacelike and lightlike branches of the energy spectrum.

We are inclined to believe that equations with superluminal solutions or (and) solutions with imaginary masses cannot be used directly, i.e., by simply ignoring these solutions and retaining only the "good" ones. However, such a conclusion can be drawn with complete confidence only after one has analyzed the corresponding equations with allowance for interaction.

Equations (34), (38), and (39) have a good (increasing, not infinitely degenerate) spectrum of the type (43) and no "defective" solutions. However, this is achieved by the introduction of additional conditions. As a result, one function, for example, the function $\Psi(x_\mu, u_\mu)$, satisfies two equations. Since this system of equations has nontrivial solutions, one can see no reasons to object to its use. But how can one introduce interaction with other fields into these equations? For the example of the equations with spin 3/2, 2, etc., it is well known[6-8] that introduction of an interaction with, for example, the electromagnetic field by the substitution $\partial/\partial x_\mu$

mass $m=0$. But if $\beta>0$ and $\varepsilon<0$, imaginary masses are obtained. Depending on the sign of ϵ, the spectrum (26) either has a decreasing branch ($m\to 0$), or it leads to imaginary masses. In addition, the spectrum (26) $\to \partial/\partial x_\mu - ieA^\mu$ in the equations themselves is inadmissible. An interaction can be readily introduced if one can find a Lagrange function from which the system can be deduced. But hitherto it has not been possible to find a Lagrange function for equations with internal variables and additional conditions, and in mathematics a regular method for solving such a problem is not known.[2)] The construction and analysis of equations with internal variables have not yet been sufficiently investigated—we do not know what are the possibilities with regard to the spectra, the introduction of interaction, completeness of the system of solutions, admissibility or inadmissibility of using different solutions or equations, etc.

CONCLUSIONS

What bearing do these equations have on the parton model? These equations contain variables (for example, the variables u_μ) which acquire, or may acquire, the role of the coordinates of internal "constituents" (partons) of a particle (a proton). The partons cannot exist in the free state, and, thus, the most characteristic and nontrivial feature of partons and quarks is taken into account automatically. On the other hand, on the basis of these equations even simple conclusions that follow from the idea of free partons or quarks have not yet been deduced. Therefore, the question of the connection between relativistic wave equations with internal degrees of freedom and the parton model remains completely open. Nevertheless, there is no doubt that it is worthwhile investigating further equations with internal variables; in particular, to clarify a whole series of unanswered questions. It was with this aim in view, to foster such a development, that the present paper was written.

[2)]At the present time it seems to us (but this is prior to a detailed investigation of the problem) that the difficulties could be eliminated by going over from the expression $-\partial^2/\partial u_\mu\,\partial u^\mu + u_\mu u^\mu$ in (22), (34), and (38) to expressions of the type $\partial^2/\partial u_\mu\,\partial u^\mu + u_\mu u^\mu + \delta^2(u_\mu u^\mu)^2$ or more complicated expressions. It could be that this would ensure the absence of bad solutions without the need to introduce additional conditions.

Note added in proof. We have been able to show that the mass spectrum of N relativistic oscillators (N is an arbitrary number) cannot be made discrete and finitely degenerate by introducing an interaction of the oscillators of the type

$$V=\sum_{i,j=1}^{K} u_\mu^i u^{\mu,j} A_{ij} + B_{ij} u_\mu^i \frac{\partial}{\partial u_\mu^j} + C_{ij}\frac{\partial}{\partial u_\mu^i}\frac{\partial}{\partial u^{\mu,j}}.$$

Thus, in the harmonic approximation, without the introduction of additional conditions, the mass spectrum of relativistic oscillators is either continuous or discrete, but infinitely degenerate and not bounded below.

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