Physical origin of the anomalous behavior of the nuclear rotation frequency (back bending)

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The physical nature of back bending in nuclear rotational spectra is analyzed. It is shown that the phenomenon is due to the quasi-intersection of rotational bands with different moments of inertia. The moment of inertia is studied as a function of the pairing correlation, quadrupole deformation parameters, and the distribution of the nucleons over one-particle states. In the region of nuclei with stable deformation, the upper energy band is a two-quasiparticle excitation which can be interpreted as a broken pair with angular momenta of the particles directed along the axis of rotation. In the region of nuclei that are soft with respect to deformation, the ground-state band can intersect a band of levels with large deformation, this being a possible explanation of back bending.

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INTRODUCTION

The rotational spectra of nuclei are of greater interest on account of their universality and the possibility of comparatively simple interpretation. They are characterized by a regular sequence of spins, and the energies and probabilities of transitions between levels are determined by a small number of parameters that describe the collective motion.[1] In the case of rotation of the nucleus, i.e., on the transition to states with high spins, these parameters and the wave function of the internal motion of the nucleons change smoothly. Analysis of the collective parameters and their variations leads to definite conclusions about the structure of the nucleus and enables one to test different models.

In recent years in experiments made by means of beams of heavy ions in (α, xn, γ) (Ref. 2) or (HI, xn, γ) (Refs. 3–5) reactions, electric quadrupole transitions have been observed which have made it possible to establish the energies and spins of highly excited states of deformed even–even nuclei up to $I=22$ and of odd nuclei up to 45/2. It has been established that in the rotational bands with the sequence of spins $I=0^+$, 2^+, 4^+, ... the energy differences $\Delta E_{I,\,I-2}=E_I-E_{I-2}$ not only depart from the rule $E_I=I(I+1)/2J$ corresponding to the description of a nucleus as a rigid axial rotator, where J is the constant moment of inertia, but in some cases may even decrease with increasing I (Refs. 6–19). Above all, this indicates that the moment of inertia J depends on the spin of the nucleus. Initially, attempts were made to take into account this dependence in the form of an expansion of the energy of the rotational band in a series in $\hat{I}^2=I(I+1)$, i.e., to regard the rotation as a small perturbation:

$$E_I=AI(I+1)-BI^2(I+1)^2+CI^3(I+1)^3+\dots, \qquad (1)$$

where A, B, C are constants. It was demonstrated experimentally in Ref. 3 that an expansion of E_I in a finite series does not describe the rotational spectra. In Ref. 20, it was shown that the expansion parameter in the series (1) is $Ij_F/\Delta J$. The condition of convergence of the series (1), $I\ll\Delta J/j_F$ (see below), is violated already for $I\approx 6$–8 in rare-earth elements and at $I\approx 8$–10 in heavy elements. Therefore, for the theoretical evaluation of energy spectra it is convenient to introduce a variable moment of inertia $J_I=J(I)$ on the basis of the relation $E_I=I(I+1)/2J_I$. Here J_I is given by

$$J_I=\frac{1}{2}\left(\frac{dE_I}{d\hat{I}^2}\right)^{-1}=\frac{1}{2}\left[\frac{dE_I}{dI(I+1)}\right]^{-1}=\frac{2I-1}{\Delta E_{I,\,I-2}}, \qquad (2)$$

and the rotation frequency is

$$\omega_I=\frac{dE_I}{dI}=\frac{\Delta E_I}{\Delta I}=\frac{E_I-E_{I-2}}{2}=\frac{I-1/2}{J_I}. \qquad (3)$$

As was already indicated in Refs. 6–19, ω_I as a function of I exhibits a strange behavior. The value of J_I, which usually increases with increasing I, begins to decrease at a certain moment of inertia, and then again increases, giving the dependence of J_I on ω (or ω^2) the form of an S-shaped curve (Fig. 1). The phenomenon has therefore been called back bending. Since J_I and ω_I are related, it is better to speak of the behavior of one of them, for example, ω_I, as a function of the spin of the state, which is a monotonically increasing quantity. The essence of the phenomenon is not therefore a capricious behavior of the moment of inertia as a function of the frequency (as is usually shown in figures), but rather a nonmonotonic behavior of the frequency and moment of inertia as functions of the spin, these having discontinuities of the derivatives

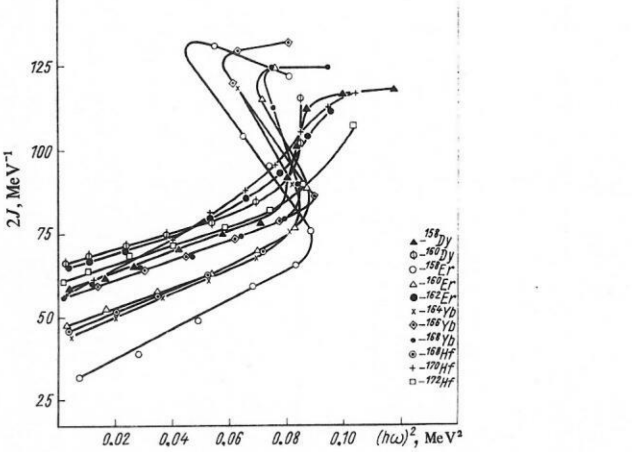

FIG. 1. Moment of inertia of yrast lines in nulcei of rare-earth elements as a function of ω^2 (Ref. 21).

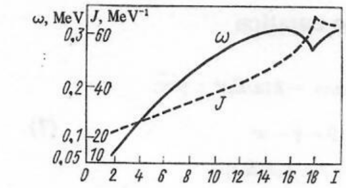

FIG. 2. Dependence of ω and J on the spin for the yrast line of ^{156}Dy.

with respect to I at the point of back bending (Fig. 2). It would therefore appear to be more natural to refer to the phenomenon as anomalous behavior of the rotation frequency, or the frequency anomaly. Possible reasons for the frequency anomaly are discussed in the reviews Refs. 22 and 23. However, many new experimental data have since been published, and these make a more definite analysis of the phenomenon possible.

The difficulties in the mathematical treatment of the frequency anomaly are of a computational nature and reside in the need to take into account simultaneously the effects of deformation of the self-consistent field, the pairing correlation, and the rotation when they are quantities of the same order. In the region of small spins $I \ll \Delta J/j_F = I_1$ (j_F is the mean angular momentum on the Fermi surface), the rotation can be treated in perturbation theory by simultaneously diagonalizing the pairing correlation and the deformed self-consistent field. At very large spins, $I \gg I_1$, the pairing correlation is small and can be treated in perturbation theory, and the self-consistent field and the Coriolis interaction are diagonalized. In the region of the frequency anomaly $\omega j_F/\Delta \approx 1$, i.e., $I \approx I_1$, the Coriolis interaction is of the order of the energy of the pairing correlation and of the order of the distance between levels coupled by the operator j_x, which is equivalent to the characteristic energy of the deformed self-consistent field.

Hitherto, a quantitative theory of the anomalous behavior of the rotation frequency has not been constructed but, using model calculations, one can draw qualitative conclusions about the nature of this interesting phenomenon. One of the most important problems in the structure of low excited states was to determine the origin of the smooth growth of the moment of inertia with increasing angular momentum I. There were two possible candidates for the cause: variation of the pairing correlation and variation of the deformation. The phenomenological models[24] take into account only the latter. However, calculations based on the microscopic approach[25,26] have shown that in the region of strongly deformed nuclei a reduction of the pairing correlation is the main factor, and these calculations have therefore made it possible to choose between the two possibilities. A similar situation has now arisen with regard to the explanation of the frequency anomaly, for which several mechanisms have also been proposed. Only a detailed investigation of the microscopic models of the nucleus can establish the origin of the phenomenon.

The characteristic features of experiments with heavy ions enable one to observe cascades of electric quadrupole ($E2$) γ's between states with lowest energy at given spin I. These levels are called yrast states, and the cascade of γ's between them an yrast line. The anomalous behavior of the rotation frequency is observed in an yrast line or in rotational bands near it such as, for example, the bands based on β-vibrational levels. Table 1 gives the energies of the known states and the corresponding values of ω_I and J_I in the well studied nucleus ^{156}Dy for the yrast line, the ground-state band, and a β-vibrational band. The belonging of levels to the ground or the β band is established by the condition that the reduced probability of a transition within one band should exceed the probabilities of transitions between the levels of different bands. The energy bands and the observed γ transitions are shown in Fig. 3. In the yrast band, the frequency anomaly is observed at $I = 16$, and in the β band at $I = 10$.

TABLE 1. Energies of levels, energy differences, frequencies, and moments of inertia for the yrast line, the ground state, and the β-vibrational state in ^{156}Dy.

I	$^{156}Dy_{yrast}$				$^{156}Dy_{gr}$				$^{156}Dy_{\beta}$			
	E, keV	ΔE, keV	ω, keV	J_I, MeV^{-1}	E, keV	ΔE, keV	ω, keV	J_I, MeV^{-1}	E, keV	ΔE, keV	ω, keV	J_I, MeV^{-1}
0	0	—	—	—	0	—	—	—	674	—	—	—
2	137.8	137.8	68.9	21.8	137.8	137.8	68.9	21.8	828.9	154.9	77.5	19.4
4	404.2	266.4	133.2	26.2	404.2	266.4	133.2	26.2	1088.5	259.3	129.6	27
6	770.4	366.2	183.1	30	770.4	366.2	183.1	30	1437.4	348.8	174.4	31.5
8	1215.7	445.3	222.6	33.7	1215.7	445.3	222.6	33.7	1858.7	421.5	210.7	35.7
10	1724.8	509.1	254.5	37.5	1724.8	509.1	254.5	37.5	2315.5	456.8	228.4	41.5
12	2285.5	560.8	280.4	40.5	2285.5	560.8	280.4	40.5	2706.9	391.4	195.7	59
14	2887.5	601.9	300.9	45	2887.5	601.9	300.9	45	3066.0	359.2	179.6	75
16	3498.5	611.0	305.5	50.8	3523	535.7	317.8	48.5	3498.5	432.5	216.2	72
18	4024	526	263	66.5	4177	654	327	53.5	4024	526	263	66.5
20	4633	609	304.5	64	4859	682	341	57.5	4633	609	304.5	64

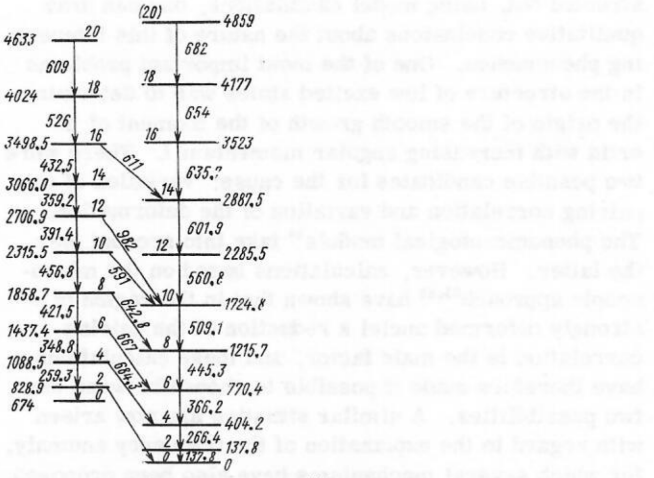

FIG. 3. Scheme of levels in ^{156}Dy (Ref. 15).

The reduced probability of transition from the level $I = 16$ ($E_{16} = 3498.5$ keV) to the level with $I = 14$ ($E^*_{14} = 3066.6$ keV) is five times greater than the probability of transition to the level $I = 14$ ($E_{14} = 2887.6$ keV), which indicates that the state E^*_{14} belongs to an excited band.

Thus, the frequency anomaly in the yrast band at $I = 16$ is due to intersection of levels.[14,15] In this case, the reduction of the frequency is a simple reflection of the fact that the moment of inertia is different for different bands. Experimentally, it has not yet been established whether there exists a third rotational band responsible for the bending in the β band. However, there are arguments that show that in this case too the frequency anomaly is due to the intersection of levels of different bands.

In reality, there is always an interaction between levels that "pushes the levels apart." The exact wave function of each level is a superposition of the wave functions of the original noninteracting states, and when the angular momentum of a level changes there is a smooth transition from one "unperturbed" physical state to another. Let us call this behavior of the levels "quasi-intersection." Of course, an argument like this presupposes a weak interaction since otherwise one cannot speak about individual bands.

In order to represent the phenomena that occur in the frequency anomaly schematically, let us consider a simple model of two levels that are the bases of rotational bands with moments of inertia J_1 and J_2 and distance E_0 between the bases. Then $E_1^{(0)}(I) = I(I+1)/2J_1$ and $E_2^{(0)} = E_0 + I(I+1)/2J_2$, and we take the interaction in the form $|V_{12}|^2 = a^2 I(I+1)$. The energies of the system are

$$E_{1,2} = E_0\{1 + AI(I+1) \mp [|-1 + I(I+1)\alpha|^2 + I^2(I+1)^2\beta]^{1/2}\}/2, \qquad (4)$$

and the corresponding frequencies and their derivatives are

$$\omega_{1,2} = \frac{dE_{1,2}}{dI} = AE_0 I\left\{1 \pm \frac{\alpha - \gamma I^2}{A[1 - 2\alpha I^2 + \gamma I^4]^{1/2}}\right\}; \qquad (5)$$

$$\frac{d\omega_{1,2}}{dI} = AE_0\left\{1 \pm \frac{\alpha - 3\gamma I^2 + 3\alpha\gamma I^4 - \gamma^2 I^6}{|1 - 2\alpha I^2 + \gamma I^4|^{3/2} A}\right\}. \qquad (6)$$

Here, we have introduced the notation

$$\begin{aligned} &J_\pm = J_2 \pm J_1;\ J_e = \frac{2J_1J_2}{(J_1+J_2)};\quad \alpha = -2(a/E_0)^2 + \sqrt{\gamma};\\ &\gamma = (J_-/J_+)^2\frac{1}{(J_eE_0)^2};\quad \beta = \gamma - \alpha^2\\ &= -4(a/E_0)^2[(a/E_0)^2 - J_-/(J_+J_eE_0)], \end{aligned} \qquad (7)$$

and in Eqs. (5) and (6) we have assumed that $I \gg 1$.

It there is no interaction between the levels, i.e., $a = 0$, Eqs. (4)–(6) simplify considerably and take the form

$$E_1 = \begin{cases} E_0(A + 2\sqrt{\gamma})\, I(I+1)/2 = I(I+1)/2J, & 0 \leqslant I \leqslant I^*;\\ E_0 + E_0(A - 2\sqrt{\gamma})\, I(I+1)/2 = E_0 + I(I+1)/2J_2, & I > I^*; \end{cases} \qquad (8)$$

$$E_2 = \begin{cases} E_0 + I(I+1)/2J_2, & 0 \leqslant I \leqslant I^*;\\ I(I+1)/2J_1, & I > I^*; \end{cases} \qquad (8')$$

$$\omega_1 = \begin{cases} I/J_1; \\ I/J_2; \end{cases} \quad \omega_2 = \begin{cases} I/J_2 & 0 \leqslant I \leqslant I^*;\\ I/J_1 & I > I^*. \end{cases} \qquad (9)$$

Here,

$$I^* = \sqrt{E_0J_e(J_+/J_-) + 1/4} - 1/2 \approx \sqrt{J_+/J_-E_0J_e} \qquad (10)$$

is the spin at which the two rotational families $E_1^{(0)}$ and $E_2^{(0)}$ intersect. It follows from (10) that if there is to be intersection of levels in the case $a = 0$ one must have $J_- > 0$, i.e., the moment of inertia of the upper family must be greater than that of the lower ($J_2 > J_1$). In this case, whenever $I > I^*$ there will be a decrease in the frequency ω, i.e., a frequency anomaly. The frequency difference is

$$\Delta\omega = \omega_1 - \omega_2 = 2I^*J_-/J_+J_e = 2\sqrt{J_-E_0/J_+J_e}. \qquad (11)$$

The position of the point of intersection is determined by the two dimensionless parameters J_+/J_- and (E_0J_e). The frequency at which the levels intersect is

$$\omega^* = (2I^* + 1)/2J_1 = \sqrt{(J_+/J_-)(J_e/J_1)(E_0/J_1)}. \qquad (12)$$

This situation is shown in Fig. 4a.

In reality, there is always a certain interaction, $a \neq 0$, which pushes the levels E_1 and E_2 apart, as is shown in Fig. 4b. In this case, one must use Eqs. (4)–(6). The abrupt bend in the rotation frequency

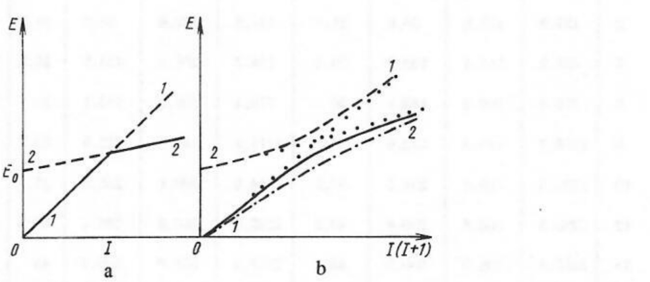

FIG. 4. Energy of intersecting levels as a function of I in the absence of interaction (a) and with interaction (b). —— yrast line; – – – second level; —·—·— behavior of yrast states in the absence of frequency anomaly in the case of strong interaction; ··· noninteracting levels.

disappears, but there is anomalous behavior of the frequency if

$$d\omega_1/dI \leqslant 0, \tag{13}$$

and the equation $d\omega_1/dI = 0$ determines the point of the anomalous behavior of the rotation frequency.

It is easy to show that the condition (13) imposes a restriction on α. The relation (13) to terms $\sim I^2$ gives an expression for the spin of the frequency anomaly: $I^{*2} = A - |\alpha|(A - |\alpha|)/3\gamma$. It follows that if a frequency anomaly is to occur one must have

$$a < a_{cr} = \sqrt{E_0/2J_1}, \tag{13'}$$

i.e., the constant of the interaction must be sufficiently small. If the interaction is too strong, and (13) is violated, there is no anomaly. The mixing of levels is called combination or hybridization,[27] and its influence on the frequency anomaly is shown schematically in Fig. 5. It can be seen from (13') that $a_{cr} = E_0(J_1E_0)^{-1/2}/\sqrt{2}$ is proportional to the square root of the nonadiaticity parameter $(J_1E_0)^{-1}$. Thus, we obtain a very natural explanation of the frequency anomaly: If a rotational band at an energy higher than another band has a large moment of inertia, then at a certain spin $I = I_1$ there is "quasi-intersection" of the levels. In each band, the frequency (and the moment of inertia) behaves monotonically, but on the transition from one band to another the frequency changes, and this is the frequency anomaly. In principle, several levels could "intersect" a given band. In the case of three levels, the quantitative analysis is more complicated,[28] although the qualitative picture established above is preserved.

The energies of rotational levels in ^{154}Gd are shown as functions of $I(I+1)$ in Fig. 6. The moments of inertia of the conjectured third band, which is a two-quasiparticle excitation with angular momentum projection $\Omega = 1$ (Ref. 28), are almost the same for different I and are near the rigid-body value. A similar situation also obtains in the nucleus ^{156}Dy. This behavior of the levels confirms the conjecture. As we shall see below, a theoretical treatment based on microscopic models leads to a similar picture.

These examples do not pretend to explain the inner nature of the frequency anomaly, but they do show that the problem of the physical investigation reduces to determining the nature and position of the initial levels (the bases) of the rotational bands, their moments of inertia as functions of I, and the strength and the nature of the interaction between the levels of different bands. The deeper is the base of a family and the greater is its moment of inertia, the earlier the frequency anomaly occurs. The nature of the low excited states in even–even nuclei is now known. They are above all β- and γ-vibrational, octupole-vibrational, pairing-vibrational, and two-particle excitations. Their positions have been well studied experimentally and theoretically[29] and are equal to the corresponding frequencies of the vibrations or the sum $E_1 + E_2$ of the energies of the quasiparticles in the case of a two-quasiparticle excitation. These states are the bases of rotational bands.

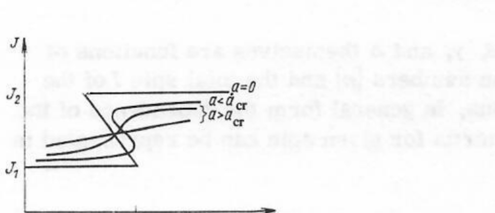

FIG. 5. Moment of inertia J as a function of ω^2 in the case of two levels for different interaction constants a.

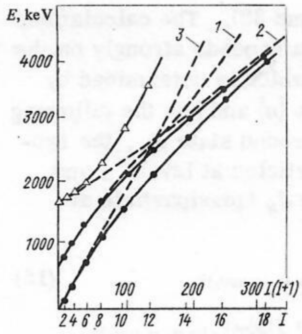

FIG. 6. Conjectured scheme of quasi-intersection of three levels in ^{154}Gd (Ref. 28). The black circles are the experimental data; the dashed curves are the unperturbed bands of the ground state (3), the β-vibrational state (1), and the two-quasiparticle state (2); the open triangles are the levels of the conjectured third band.

In order to determine the position of "quasi-intersection" of levels, it is necessary to know the moments of inertia of the corresponding bands. The occurrence of a frequency anomaly is determined by the amount by which the moment of inertia of the excited level exceeds the base level's and the strength of their interaction. Thus, for a qualitative analysis of the origin of the frequency anomaly the behavior of the moment of inertia of the intersecting levels is particularly important.

1. MOMENTS OF INERTIA OF NUCLEI

To establish the quantities on which the moment of inertia J depends, we can consider its expression in the model of constrained rotation.[30] In this case, the nucleus is represented as a system of nucleons moving on a self-consistent field which rotates with angular velocity ω and is described by the quadrupole deformation parameters β and γ. The expression for J is

$$J = 2\sum_{i,k} |(j_x)_{i,k}|^2/(\varepsilon_i - \varepsilon_k), \tag{14}$$

where j_x is the projection of the angular momentum of the nucleon onto the axis of rotation x and ε_i is the energy of the nucleon in level i. A quasiclassical calculation in accordance with Eq. (14) leads to the rigid-body value of the moment of inertia, which differs strongly from the observed value. Allowance for the residual interaction responsible for the pairing correlation modifies J (see Refs. 31 and 32). The calculations show that the moment of inertia depends strongly on the nature of the nucleon motion, which is determined by the nucleon population numbers $\{n\}$ and has the following forms, respectively, for the ground state J_{gr}, the two-quasiparticle state J_t (quasiparticles at levels α and β), and one-quasiparticle state J_N (quasiparticle at level α)[29,31–35]:

$$J_{gr}=\sum_{i,k}\frac{|(j_x)_{ik}|^2}{(E_i+E_k)}(u_iv_k=u_kv_i)^2; \tag{15}$$

$$\begin{aligned}J_t=&\sum_{i,k}\frac{|(j_x)_{ik}|^2}{(E_i+E_k)}(u_iv_k-u_kv_i)^2+\sum_{i\neq\alpha}\frac{|(j_x)_{\alpha i}|^2}{(E_i-E_\alpha)}(u_iu_\alpha+v_iv_\alpha)^2\\&+\sum_{i\neq\beta}\frac{|(j_x)_{i\beta}|^2}{E_i-E_\beta}(u_iu_\beta+v_iv_\beta)^2\\&-\frac{|(j_x)_{\alpha\beta}|^2}{E_\alpha-E_\beta}(u_\alpha v_\beta-u_\beta v_\alpha)^2=J_{gr}(\Delta_t)+\delta J_t;\end{aligned} \tag{16}$$

$$\begin{aligned}J_H=&\sum_{i\neq\alpha}\frac{|(j)_{\alpha i}|^2(u_iu_\alpha+v_iv_\alpha)^2}{(E_i-E_\alpha)}\\&+\sum_{i,k}\frac{|(j_x)_{ik}|^2}{(E_i+E_k)}(u_iv_k-u_kv_i)^2=J_{gr}(\Delta_H)+\delta J_N.\end{aligned} \tag{17}$$

Here, $E_i=\sqrt{\Delta^2+(\varepsilon_i-\varepsilon_0)^2}$; Δ is the energy of the pairing correlation; ε_0 is the energy of the Fermi surface; $|i\rangle$, $|\alpha\rangle$ is the one-particle wave function; $u_i^2=[1+(\varepsilon_i-\varepsilon_0)/E_i]/2$; $v_i^2=[1-(\varepsilon_i-\varepsilon_0)/E_i]/2$. The total moment of inertia of the nucleus is equal to the sum of the proton and neutron moments of inertia, each of which is given by Eqs. (15)–(17). In them, for simplicity, we have ignored effects associated with the change in the chemical potential and the energy of pairing correlations associated with the presence of quasiparticles.

The calculations showed (Refs. 32 and 35) that the moments of inertia of the ground states of even–even nuclei are two or three times smaller than the rigid-body value. The moments of inertia of one- and two-quasiparticle states may be 20 to 50% greater than the moments of inertia of the ground states of the neighboring even–even nuclei: first, because of the reduction of Δ in excited states due to the blocking effect, and second, because of the additional terms δJ in Eqs. (15)–(17) corresponding to interaction of the quasiparticles with the core. The influence of these terms is particularly large for neutron states with large j, this being due to the large matrix element and small energy difference E_i-E_α in the case of small projections of the angular momentum of the quasiparticle onto the symmetry axis.

Table 2 gives the moments of inertia of one-quasiparticle states and the values of $\delta J=J_H-J_{\text{even}}=J_{A+1}-(J_A+J_{A+2})/2$. In Eqs. (15)–(17) it is assumed that the coupling of the quasiparticle to the deformation axis is strong and that the states of each quasiparticle are characterized by the projection of the moment of inertia onto the symmetry axis z of the nucleus. In reality, for large I the quasiparticles may cease to be coupled to the deformation, and one can assume that their angular momenta are along the axis of rotation.[20,36] Then, although the moment of inertia itself does not change, the angular momentum of the free particles makes a large contribution to the total angular momentum. The angular momentum R of the core, which determines the energy dependence $E_R=R(R+1)/2J$, can

TABLE 2. Moments of inertia of odd nuclei.[29] The quasiparticle state is characterized by the projection Ω of the angular momentum onto the z axis and the asymptotic quantum numbers $[Nn_z\Lambda]$.

State of odd particles $\Omega[Nn_Z\Lambda]$	Nucleus	J_H, MeV^{-1}	δJ_H, MeV^{-1}
3/2 (521)	^{155}Sm	46.7	9.4
	^{155}Gd	41.3	11.3
	^{157}Gd	45.8	9.6
	^{159}Dy	44.2	10.0
	^{161}Dy	43.2	5.8
	^{163}Dy	45.7	6.0
5/2 (523)	^{157}Gd	43.5	7.3
	^{159}Gd	43.8	4.4
	^{159}Dy	41.0	6.8
	^{161}Dy	45.2	7.8
	^{163}Dy	47.2	7.5
	^{163}Er	41.6	10.0
	^{165}Er	45.1	9.3
7/2 (633)	^{165}Dy	53.7	12.0
	^{167}Er	56.7	18.8
	^{169}Er	60.2	22.0
7/2 (633)	^{169}Yb	63.3	27.7
	^{171}Yb	61.3	24.3
3/2 (411)	^{155}Eu	39.0	3.7
	^{157}Tb	41.3	8.5
	^{159}Tb	44.6	7.0
	^{161}Tb	44.2	3.0
	^{165}Ho	42.9	7.1
7/2 (523)	^{163}Ho	49.0	13.2
	^{165}Ho	47.6	11.8
	^{169}Tm	48.0	11.2
7/2 (404)	^{169}Tm	38.4	0
	^{171}Lu	37.0	2.9
	^{175}Lu	39.3	2.3
	^{177}Lu	37.3	2.5
	^{181}Ta	33.1	1.7

then be small, which leads to small ω_I.

The other quantities that determine the moment of inertia are the quadrupole deformation parameters β and γ and the pairing correlation energy Δ. A dependence of J on Δ enters directly through E_λ, u_λ, and v_λ, and on β and γ through the position of the one-particle energies and the matrix elements of j_x. In order to investigate analytically the moment of inertia as a function of the deformation parameters and Δ, it is necessary to use a simplified model for the nuclear potential, for example, the model of a deformed axial oscillator. In this case, one obtains the simple equation[32]

$$\begin{aligned}J=J_r&[1-g+g^2x_1^2/(x_1^2g+\ln 2x_2)];\\g(x)&=\sinh^{-1}x/(x\sqrt{1+x^2});\\x_1&=\omega_0\beta/2\Delta;\quad x_2=\omega_0/\Delta\end{aligned} \tag{18}$$

for $0\leqslant x_i\leqslant\infty$, where ω_0 is the oscillator frequency; J_r is the rigid-body moment of inertia, equal to $2MAR_0^2/5$; R_0 is the radius of the nucleus.

It was found that J is a monotonically increasing function of the universal parameter x_1. It follows from Eq. (18) that the moment of inertia increases with decreasing Δ and increasing β. Therefore, any effects that decrease Δ and increase β will automatically increase J.

Note that β, γ, and Δ themselves are functions of the population numbers $\{n\}$ and the total spin I of the nucleus. Thus, in general form the dependence of the moment of inertia for given spin can be represented in the form

$$J=J[\{n\},\ \beta(\{n\}),\ \gamma(\{n\}),\ \Delta(\{n\}),\ I], \tag{19}$$

and one can distinguish three main causes of variation of J, which can each in principle lead to a frequency anomaly: 1) reduction of the pairing correlation energy Δ; 2) change in the nucleon state population $\{n\}$; and 3) increase of the deformation parameters β and γ. Let us consider each factor separately.

2. VARIATION OF THE PAIRING CORRELATION

The nucleons in nuclei are coupled in pairs, like pairs of electrons in superconductors. This has the effect, in particular, that the spectrum of quasiparticle excitations in even–even nuclei begins with energy $E_1 + E_2 \geqslant 2\Delta$ and the moment of inertia is two or three times smaller than the rigid-body value J_r. One can say that the pairing correlation "couples" particles to angular momentum 0 and tends to arrange them with equal probability over states with projections Ω_i and $-\Omega_i$ of the moment of inertia onto the deformation axis, the energy gain being maximal in this case. Such a distribution corresponds to the ground state of the nucleus with $I=0$.

In a rotating nucleus, particle i is subject to Coriolis forces derived from the potential

$$\hat{V}_h = -\frac{\overline{I}\,\overline{j}\,(1)}{J} = -(\overline{\omega}\,\overline{j}\,(i)). \tag{20}$$

These forces tend to destroy the correlation, "break" the pair, and give rise to a spin $I \neq 0$. The spins of the quasiparticles tend to align along I. The larger I, the more pairs must break and the more Δ must be reduced. As long as the gain in the energy of rotation of the broken pair due to the action of the Coriolis forces is less than 2Δ, the breaking is virtual and Δ varies as a function of I fairly slowly as a result of the slight redistribution of particles over the state necessary for the formation of I. However, real breaking is subsequently possible, and this results in a stronger dependence of Δ on I. It is the reduction in Δ that is the main factor responsible for J depending on the spin I. This effect is analogous to the action of a magnetic field, which reduces the energy gap in a superconductor, and it was first considered in Refs. 37 and 38.

The first attempt to explain the frequency anomaly was associated with an estimate of the critical spin I_{cr} of the intersection of the energies of the normal and the superconducting state. The moment of inertia of the normal state is J_r. The pairing correlation energy is $E_0 = \rho_0 \Delta_0^2/4$, where ρ_0 is the density of levels on the Fermi surface. The equality between the energies of the superconducting and the normal state:

$$-\rho_0\Delta_0^2/4 + I_{cr}(I_{cr}+1)/2J = I_{cr}(I_{cr}+1)/2J_r \tag{21}$$

gives the critical spin $I_{cr} = 12$ for rare-earth elements; $I_{cr} = 18$ for heavy elements. At small spins $I \leqslant I_{cr}$, the superconducting state is the ground state, but when $I \geqslant I_{cr}$ the normal state is energetically more advantageous than the superconducting. The energy difference Δ is zero for $I \geqslant I_{cr}$, and the moment of inertia takes the rigid-body value.

The energy is shown as a function of $I(I+1)$ in Fig. 7. Since $J < J_r$, the energy E_s always intersects E_N at some I_{cr}. However, the perturbation theory estimate made in Ref. 37 showed that at the point I_{cr} the value of Δ is changed by about 40% and the true curve for the rotational energy passes lower than is predicted by Eq. (21) [the curve $E_s(\Delta)$ in Fig. 7]. In addition, in deriving (21) we assumed that all the pairs are the same. However, this situation does not correspond to the real structure of the nucleus. In reality, the nucleons occupy states with different angular momenta and therefore behave differently with respect to the rotation of the nucleus since the Coriolis interaction of the nucleons depends on their angular momentum j. This interaction is large for nucleons with large angular momenta. Therefore, at a certain $I = I_1$ there is a rearrangement of the particle state population and, before the pairing correlation becomes completely equal to zero in the system, two free quasiparticles arise at levels near the Fermi surface (two-quasiparticle excitations, the E_t curves in Fig. 7). A pair is broken under the influence of rotation. From the point of view of noninteracting levels, this corresponds to intersection of the ground state and the two-quasiparticle state.

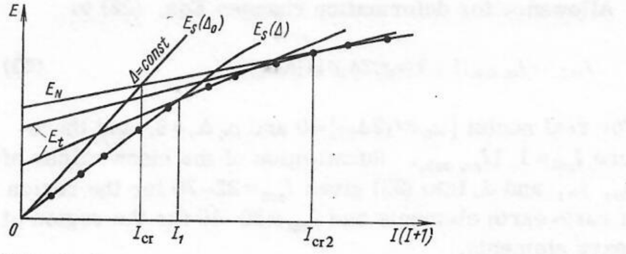

FIG. 7. Scheme of intersections of the ground E_s ($\Delta \neq 0$), normal E_N ($\Delta = 0$), and two-quasiparticle E_t levels and I_1, the spin at which the frequency anomaly of the yrast line begins (•••).

Thus, as one moves further along the yrast line, the two-quasiparticle state becomes the lowest state. This phenomenon was first discovered theoretically by the author and Larkin.[20] The moment of inertia for the states of the yrast band with $I > I_1$ is equal to J_t and is greater than J_{gr}. The rotational frequencies are accordingly lower. It is this that leads to the frequency anomaly. The remaining particles continue in the superconducting state coupled in pairs. The complete disappearance of Δ does not occur abruptly, but is extended over a certain range of spins I as a result of the successive breaking of individual pairs, and this leads to a phase transition of the second kind.[20] In this case, Δ decreases as a result of the effect of the Coriolis forces (Refs. 37 and 38) and the blocking effect caused by the broken pairs.[39] Both effects are the result of the same cause—rotation of the nucleus—since the pairs are also broken under the influence of the Coriolis forces. Note that the complete disappearance of Δ occurs when

$$I_{\text{cr. sph.}} = \begin{cases} \dfrac{e^{3/2}}{2}\dfrac{\Delta_0 J_r}{j_F} = 2.2\,\dfrac{\Delta_0 J_r}{j_F} & \text{— for the oscillator model;} \\[2ex] \dfrac{e^{7/3}}{4}\dfrac{\Delta_0 J_r}{j_F} = 2.6\,\dfrac{\Delta_0 J_r}{j_F} & \text{— for the rectangular well model.} \end{cases} \tag{22}$$

Allowance for deformation changes Eqs. (22) to

$$I_{cr2} = I_{cr.\,sph.}[1+2(\omega_0\beta/2\Delta_0)^2 \ln(\rho_0\Delta_0)/3e^3]. \quad (23)$$

For real nuclei $[\omega_0\beta/(2\Delta_0)] \approx 0$ and $\rho_0\Delta_0 \approx 8$, and therefore $I_{cr2} \approx 1.1 I_{cr.\,sph.}$. Substitution of the mean values of Δ_0, j_F, and J_r into (23) gives $I_{cr2} \approx 22-26$ for the region of rare-earth elements and $I_{cr2} \approx 30-40$ for the region of heavy elements.

The behavior of the levels with increasing angular momentum in Fig. 7 is of course idealized since it does not take into account the interaction between the levels. The interaction smooths the abrupt jumps of ω, leading to "quasi-intersections" and hybridization of the levels. Instead of an abrupt transition from a no-quasiparticle state to a two-quasiparticle state as a result of intersection at $I = I_1$, there is a continuous though fairly rapid increase in the admixture of a certain two-quasiparticle state (with large j) in the sequence of yrast levels. However, because of the weakness of the interaction, which satisfies the condition (13), the situation considered in Fig. 7 qualitatively reflects the behavior of the rotational levels of real nuclei (see Fig. 6).

Therefore, in strongly deformed nuclei the frequency anomaly is associated with breaking of pairs, i.e., with a sudden change of $\{n\}$. The state that was previously the ground state becomes an excited state. Its moment of inertia continues to change smoothly with increasing I because of the continuing decrease of Δ.

It is of interest to look for states with $\Delta = 0$ (and consequently with $J = J_r$) and attempt to detect a frequency anomaly associated with intersection with this level. If it is assumed that the levels of the yrast line in ^{156}Dy with spins 16^+, 18^+, and 20^+ belong to a normal band, this band must have the levels $E_{10}^N = 2.31$, $E_8^N = 2.02$, $E_6^N = 1.80$, $E_4^N = 1.63$, $E_2^N = 1.52$ MeV and begin somewhere near $E_0^N = 1.48$ MeV. This energy is too low for the condensation energy, so that the levels 16^+-20^+ cannot belong to the normal completely unpaired state. Thus, the frequency anomaly is not associated with complete disappearance of pairing but rather with a transition to a two-quasiparticle state with other values of Δ and J (see Ref. 36).

3. CHANGE IN THE PARTICLE-STATE POPULATION. BREAKING OF PAIRS

We have shown earlier that the moments of inertia of two-quasiparticle states can appreciably exceed the moments of inertia of the ground states. However, there are rather a lot of two-quasiparticle states above the energy 2Δ. The question therefore arises of what excited states, and at what frequencies, can intersect with the ground state or β- or γ-vibrational states to give a frequency anomaly. These questions can be answered on the basis of simple energy considerations.[36,39,40] The energy required to break a pair of particles is $E_1 + E_2$. On the other hand, as a result of the interaction of the angular momentum of the nucleon with the total angular momentum of the nucleus as a whole one can obtain a gain in the energy equal to $2\omega\sqrt{\langle j_x^2\rangle}$ (the maximal gain is $2\omega j$). At a definite rotation frequency $\omega_1 \approx (E_1 + E_2)/2\sqrt{\langle j_x^2\rangle}$ excitations arise in the nuclei whose formation does not require the expenditure of energy. With further increase in the rotation frequency, $\omega > \omega_1$, it becomes energetically more advantageous for population of particles with this two-quasiparticle excitation. In the model of three levels with angular momentum j and energy distance d between the levels, we obtain the relation

$$\omega_1(j, \Omega) = \sqrt{2(\Delta^2 + d^2)/[j(j+1) - \Omega^2]}. \quad (24)$$

It follows from this equation that a two-quasiparticle excitation will occur earliest in nuclei on whose Fermi surface there is a level with large j and small Ω for which d is also small, i.e., in the region of the start of filling of the $i_{13/2}$ shell in rare-earth elements.

For the characteristic values $\beta = 0.3$, $\Delta \approx d \approx 0.9$ MeV, $l = 6$, Eq. (24) gives $\omega_1 \approx 0.28$ MeV. This value is very near the experimentally observed angular velocity of the start of the frequency anomaly, $\omega_{1\exp} = 0.29$ MeV. In nuclei with $N = 90$, 92, 94, 96 nucleons one observes levels of the $i_{13/2}$ shell with projections $1/2^+$, $3/2^+$, and $5/2^+$ on the Fermi surface (or near it). This fact in conjunction with the small $d = \varepsilon_{3/2} - \varepsilon_{1/2}$ determines the occurrence of the frequency anomaly. If there are more nucleons, Ω_λ and d are larger, which increases ω by a factor 1.5–2. Such frequencies have not yet been observed experimentally.

In the region of heavy elements, levels of the $j_{15/2}$ shell lie near the Fermi surface, but they have relatively large projections, so that the distance d is large. Equation (20) gives $\omega_1(15/2, m_\lambda = 7/2) \approx 0.24$ MeV for this region. These frequencies should be found at $I \sim 16$.

Equation (24) and the physical arguments on which it is based can readily be extended to the frequency anomaly of vibrational states, in particular β-vibrational states. In this case, the minimal excitation energy is $E_1 + E_2 - \omega_\beta$ and for the frequency anomaly we obtain

$$\omega_{1\beta} = \frac{E_1 + E_2 - \omega_\beta}{2\langle j_x^2\rangle} = \frac{\sqrt{d^2 + \Delta^2} - \omega_\beta/2}{\sqrt{(j(j+1) - \Omega^2)/2}}. \quad (25)$$

Applying the estimate (25) to the nuclei ^{154}Gd and ^{156}Dy, for which $\omega_\beta = 0.680$ and 0.67 MeV, respectively, we find that $(\omega_{1\beta})_{teor} = 0.22$ MeV, in agreement with the experimental value $(\omega_{1\beta})_{exp} = 0.22$ MeV (Ref. 15). Of course, these relations have a qualitative nature and can be applied only to estimate ω_1.

Thus, the frequency anomaly in the ground state is associated with a rearrangement of the quasiparticle vacuum and the appearance of a large admixture of certain two-quasiparticle states in the wave function of the state with lower energy. In the case of a β-vibrational state, the frequency anomaly is associated with the transition from a vibrational to a two-quasiparticle state. This circumstance must be manifested experimentally in a change in the constant of the interaction of the levels of the ground-state band with the levels of the β-vibrational band on either side of the point of the frequency anomaly; this is because the wave function of the excited state after that point contains a large admixture of the two-quasiparticle state.

These features of the anomalous behavior of the rotation frequency can be obtained in a very simple model of a rotator interacting with free particles.[41,42] Krumlinde and Szymański assume that the rotator has one rotational degree of freedom and that its Hamiltonian has the form $H_r = a\hat{R}^2/2$, where $\hat{R}$ is the angular momentum and a is the reciprocal moment of inertia. The particles are described by the one-particle Hamiltonian $H_{\text{part}} = H_{\text{sc}} + H_p$, where H_p is the pairing part and H_{sc} is the self-consistent field, for which one uses the model of two Ω-fold degenerate levels. The operator of the total angular momentum of the particles is $\hat{j}$, and the total angular momentum of the system is $\hat{I} = \hat{R} + \hat{j}$. It is also assumed that the states of the system are axially symmetric and that the rotation takes place around the x axis. In the simplest case, when $H_{\text{sc}} = 0$ and the particles are on one degenerate level, the eigenvalues of the Hamiltonian

$$H = H_p + a\,(I - j_x)^2/2 \tag{26}$$

have the form

$$E_I^{Lk} = -GL\,(L+1) + a\,(I-k)^2/2. \tag{27}$$

Here, G is the constant of the pairing interaction; L is the quasispin, which is related to the seniority ν by the relation $\nu = 2(\Omega - L)$. The quasispin takes the values $L = 0, 1, 2, \ldots, \Omega$, and $-GL(L+1)$ is an eigenvalue of j_x. The superscript k is the eigenvalue of the operator j_x and is $0, 2, 4, \ldots, 2\Omega$. For the ground state $L = \Omega$ ($\nu = 0$), $k = 0$ and in the case of complete unpairing $k = 2\Omega$, $L = 0$. In the general case $k = 0, 2, \ldots, 2\Omega$; and for given k, $L = 0, 1, \ldots, (\Omega - k/2)$.

The results of calculations for the case of four particles on the level $j = 7/2$ are shown in Fig. 8. It is clear from the figure that the level of the ground state with $L = 2$, $k = 0$ interesects the level with $L = 1$, $k = 6$ at some I_1. This corresponds to the transition from the state without broken pairs to a state with one broken pair, i.e., to intersection with the two-quasiparticle

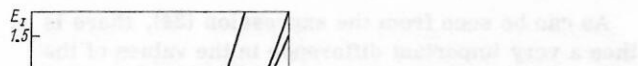

FIG. 8. Scheme of intersections of levels in the Krumlinde–Szymański model for $j = 7/2$ (four particles).

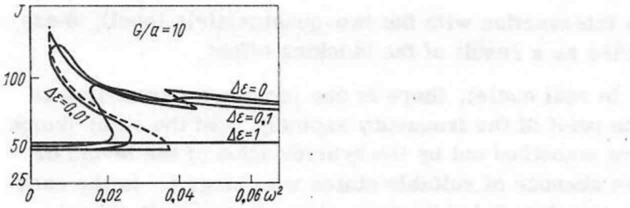

FIG. 9. Moment of inertia as a function of ω^2 in the Krumlinde–Szymański model with $j = 7/2$ (four particles); $\Delta\varepsilon$ is the distance between neighboring one-particle levels.

state. Further, at $I = I_{\text{cr}}$ there is an intersection with the level $L = 0$, $k = 8$, which corresponds to two broken pairs, i.e., to the normal, completely unpaired state in this model.

Figure 9 shows the moment of inertia as a function of ω_2 in the case $j = 7/2$. It can be seen from these figures that at the points of intersection of the levels the moment of inertia and the rotation frequency (dE/dI) change by an abrupt amount determined by the ratio of the pairing energy to the reciprocal moment of inertia. As the distance between the levels increases, the frequency anomaly gradually disappears. In the Krumlinde–Szymański model the frequency anomaly can be interpreted very simply. In this case, the rotation frequency of the nucleus is

$$\omega = d\langle H\rangle/dI = a\,(I - \langle j_x\rangle). \tag{28}$$

On the breaking of a pair, when $I = I_1$, the mean angular momentum of the free quasiparticles is aligned along the x axis, the value of $\langle j_x\rangle$ increases abruptly, and ω decreases. The angular velocity is proportional, not to the total angular momentum, but to the collective part $R = I - \langle j_x\rangle$ and $\omega = R/J$. After the alignment of the angular momentum of the pair has ended, the frequency again begins to increase with increasing I. As a rule, the following jumps in ω are very weak and virtually disappear if ε is not zero.

It is interesting to compare the dependence of the parameters of the pairing correlation for the levels of an yrast line on the spin I (Fig. 10) for the following cases: phase transition of the first kind, reduction of pairing correlation with allowance for pair breaking, and alignment and gradual phase transition of the second kind. At a pairing correlation transition with allowance for pair breaking one observes very characteristic jumps Δ when the pairs are broken (or there

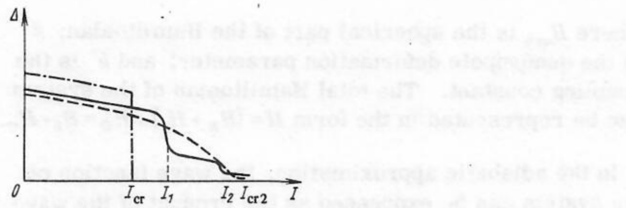

FIG. 10. Pairing correlation parameter Δ for yrast levels as a function of the spin in the case of phase transitions of the first (—·—·—) and the second (– – –) kind and a transition with allowance for pair breaking (——).

is intersection with the two-quasiparticle level); these arise as a result of the blocking effect.

In real nuclei, there is one jump in an yrast line at the point of the frequency anomaly, and the other jumps are smoothed out by the hybridization of the levels or the absence of suitable states with large j. In the case of complete hybridization, there is a smooth phase transition of the second kind.

The Krumlinde–Szymański model does not take into account the change in the moment of inertia of the core under the influence of rotation due to the change in Δ, but reflects only the pair breaking and the alignment of the angular momenta of the valence particles. It therefore has a qualitative nature and needs to be improved so as to combine both effects.

4. QUASIPARTICLE COUPLING TO A ROTATING NUCLEUS. ODD NUCLEI

As we have shown above, two-quasiparticle excitations acquire a decisive importance in the phenomenon of the frequency anomaly. It is therefore of interest to consider in more detail the behavior of a quasiparticle in a rotating nucleus. At high rotation frequencies, i.e., for large I and small moments of inertia, the strength of the Coriolis interaction increases. This region of frequencies, which requires small moments of inertia, actually is realized at small deformations. The pairing correlation energy Δ changes fairly smoothly from nucleus to nucleus and does not take on large values, which could also decrease the moments of inertia.

The problem of the spectrum of a rotating axial nucleus and a quasiparticle was considered by Stephens *et al.*[43] The operator of the total angular momentum of such a system is $\bar{I}=\bar{R}+\bar{j}$, where $\bar{R}$ and $\bar{j}$ are respectively the angular momenta of the core and the particle. The Hamiltonian of the nucleus, $H_n=(R_x^2+R_y^2)/2J$, takes the form

$$H_n=(2J)^{-1}(\hat{I}^2+\hat{j}^2-I_3^2-j_3^2)-J^{-1}(I_xj_x+I_yj_y)=\hat{H}_R+\hat{H}_C. \quad (29)$$

The second term in (29) is the Coriolis interaction:

$$H_C=-(2J)^{-1}(I_+j_-+I_-j_+),\ \text{where } I_\pm=I_x\pm iI_y,\ j_\pm=j_x\pm ij_y.$$

The Hamiltonian of the particle in the deformed nucleus is chosen in the form

$$H_p=H_{\rm sph}+k'\beta Y_{20}, \quad (30)$$

where $H_{\rm sph}$ is the spherical part of the Hamiltonian; β is the quadrupole deformation parameter; and k' is the coupling constant. The total Hamiltonian of the system can be represented in the form $H=(H_R+H_p)+H_C=H_0+H_C$.

In the adiabatic approximation, the wave function of the system can be expressed as the product of the wave functions $D^I_{M\Omega}$ of the rotator and χ^j_Ω of the particle. The diagonal part H_0 in this representation is

$$E_0=\varepsilon_j+[I(I+1)+j(j+1)](k-1/J)\Omega^2/2J, \quad (31)$$

and the constant k is given by $k=3\beta k'/[4j(j+1)]$. If $k-1/J=0$, then E_0 does not depend on Ω and as wave function one can choose a superposition of χ^j_Ω that approximately diagonalizes the Coriolis term in the Hamiltonian. For this, one introduces a function χ^j_α for a particle with total angular momentum j and projection onto the x axis of the angular momentum equal to α. The new function can be represented in the form of the expansion

$$\chi^j_\alpha=\sum_\Omega C_\Omega(\alpha)\chi^j_\Omega,$$

where $C_\Omega(\alpha)=d^j_{\alpha\Omega}(\pi/2)$ are finite-rotation matrix elements. The requirement of symmetry under reflection in the plane perpendicular to the z axis leads to the condition $I-\alpha=2n$ $(n=0,1,2\ldots)$. Then in the approximation $(I/\Omega)^2\gg1, I\gg1$ for $I>j$ (since $|\Omega|<j$) we obtain $H_C\Psi^{Ij}_{M\alpha}\simeq-2AI\Psi^{Ij}_{M\alpha}$, and the energy of the system is

$$E(I,j,\alpha)=\varepsilon_j+(2J)^{-1}[j(j+1)+I(I+1)-2I\alpha]. \quad (32)$$

It follows from Eq. (32) that the states with lowest energy have $\alpha=j$, and the band associated with them depends on the quantum numbers as follows:

$$E_R=\varepsilon_j+(2J)^{-1}(I-j)(I-j+1)=\text{const}$$
$$+R(R+1)/2J,\quad R=0,2,4\ldots \quad (33)$$

The rotational band in an odd nucleus with $\alpha=j$ is said to be decoupled and energy favored. As can be seen from Eq. (33), the odd particle does not participate in the rotation, which is determined solely by the angular momentum R of the core. The state itself is said to be aligned, since it corresponds to quantization along the axis of rotation. Thus, the levels of this band reproduce the spectrum of the even–even nucleus. The moment of inertia of the system must be determined from

$$J_R=(2R-1)/\Delta E_{R,R-2},\ \Delta E_R=(E_R-E_{R-2},\ R=2,4,\ldots), \quad (34)$$

and not from the usually employed expression

$$J_I=(2I-1)/\Delta E_{I,I-2}=(2R-1+2j)/\Delta E_{R,R-2}.$$

As can be seen from the expression (34), there is then a very important difference in the values of the moment of inertia $\mu=J_R/J_I=[1+2j/(2R-1)]^{-1}$: For $j=11/2$, $R=2$, we have $\mu=0.215$, while for $R=6$, $\mu=0.5$.

For small $\Omega\leqslant I<j$ one can show that

$$E(I,j,\varkappa)=\varepsilon_j+(2J)^{-1}[j(j+1)+I(I+1)-2j\varkappa], \quad (35)$$

where $\varkappa$ is the projection of I onto the x axis. In this case, one has the ordinary dependence of E on I $[\sim I(I+1)]$. The condition $k=1/J$, which was used in the derivation of the expressions (33) and (35), depends on the choice of the values of j, k', β, and $1/J$. For each concrete nucleus, j, k, and $1/J$ are fixed, and this condition restricts the value of β for which (33) and (35) are good approximations.

The accuracy of the approximate diagonalization of the Hamiltonian H is evident from Fig. 11, in which the level $j = 11/2$ has been used. The approximate solutions agree well with the exact ones for $\beta \approx +0.18$ (the energy difference between the state with spin I and the lowest state $I_0 = 11/2$ is plotted along the ordinate). Analysis of the region of applicability of the aligned coupling scheme shows that it applies for deformation $\beta \approx +0.2$ for particles on the Fermi surface and for holes near deformation $\beta \approx -0.2$. For hole states $k' < 0$, and the condition $k = 1/J$ is satisfied for negative β since the energies of the system for this scheme lie lower than for the strong-coupling scheme. At deformation $\beta > 0.3$ or for negative deformations the strong-coupling scheme is energetically more advantageous. Thus, when there are one-particle states with large angular momentum but small projection Ω on the Fermi surface, conditions favorable for alignment arise for small positive deformations. Similarly, alignment is advantageous for small projection of hole states in the case of negative deformations.

The model of aligned states finds very effective confirmation in the odd nuclides of La (Ref. 44), in which one observes a band coupled to a $j = 11/2$ level. The energies of the odd La nuclides and the neighboring even–even ^{56}Ba nuclei are shown in Fig. 12. The relation $\Delta E_{I\,\mathrm{even}} \approx \Delta E_{I\,\mathrm{odd}}$ holds, and therefore the moments of inertia of the core in the odd nucleus for large angular momenta are near the moments of inertia of the corresponding even–even nuclei.

An example that enables one to establish the coupling scheme of an odd particle in states with large spin $I = 21/2,\ 17/2,\ 13/2$ in the nuclei $^{157}_{68}$Er and $^{159}_{68}$Er is the measurement of the reduced probability of the quadrupole transition $B(E2)$ and the gyromagnetic ratio g (Ref. 45). In these nuclei, the experimental value $|g| = 0.05 \pm 0.05$ agrees with the alignment or weak-coupling schemes, whereas the $B(E2)$ values can be explained only by strong-coupling or alignment. Thus, the two experiments taken together favor alignment.

In the light of what we have said, the problem of frequency anomaly in odd nuclei is rather trivial. Breaking of a suitable pair under the influence of rotation occurs in odd nuclei provided the pair exists, i.e., the

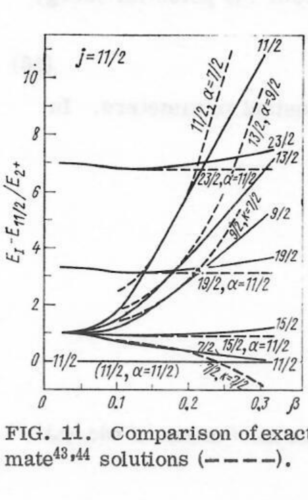

FIG. 11. Comparison of exact solutions (——) with approximate[43,44] solutions (– – – –).

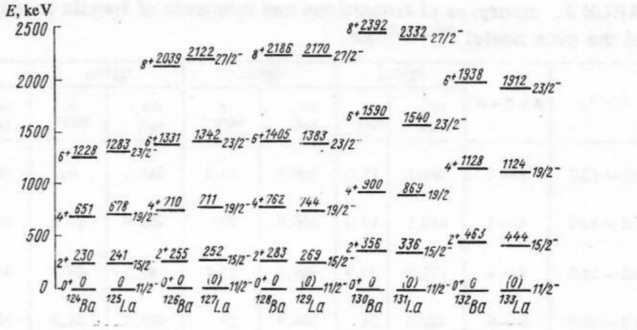

FIG. 12. Spectra of aligned states in odd La nuclides compared with neighboring even–even Ba nuclides.

corresponding state is not occupied by an odd particle. The relation (24) for the breaking of a pair remains the same if instead of I one substitutes R, the angular momentum of the rotating core. This can be verified directly[46] if conditions favorable for the aligned coupling scheme are realized in the odd nucleus.

The energies of the transitions of the favored (decoupled) band for the nuclei $_{67}$Ho$_{90,92,94}$ in comparison with ΔE_I of the neighboring even–even nuclei $^{156}_{66}$Dy and $^{158}_{68}$Er and the mean ΔE_I with respect to even nuclei are given in Table 3. A frequency anomaly is observed in all the Ho nuclei. The spins at which it appears, $I_1 = 14$, and its nature are similar to the anomaly in the neighboring even nuclei, which indicates breaking of a neutron pair in the $i_{13/2}$ shell as the origin of the frequency anomaly in Ho. The Ho spectra are particularly similar to the $^{158}_{68}$Er$_{90}$ spectrum because their moments of inertia are nearly equal at small angular momenta. In the odd Er nuclei, no frequency anomaly is observed even up to $I = 45/2$, which indicates that breaking of the neutron pair is energetically disadvantageous because the state with $\alpha = j$ is occupied by an odd particle.

On the other hand, one can investigate the frequency anomaly in different states occupied by an odd particle in one nucleus. If the odd particle occupies a nucleon state of a pair involved in breaking, the rotational spectrum based on this state should not exhibit a frequency anomaly. In all other states, the rotational bands must have a frequency anomaly. This fact was discovered[48] in the nucleus $^{163}_{70}$Yb$_{95}$. It was found that the rotational band based on the level 3/2$^-$[521] from the state with $j = h_{9/2}$ has a frequency anomaly on the transition from $I = 33/2$ to $I' = 29/2$, which corresponds to a change of R from 12 to 10. In the band based on the state of the

TABLE 3. Values of ΔE_I (keV) in Ho nuclei. For comparison ΔE_I in $^{156}_{60}$Dy$_{90}$, $^{158}_{68}$Er$_{90}$, and $\overline{\Delta E_I}$ (relative to Dy$_{90}$ and Er$_{90}$) are given.

$I_i \to I_f$	$R+2 \to R$	$^{157}_{67}$Ho$_{90}$	$^{159}_{67}$Ho$_{92}$	$^{161}_{67}$Ho$_{94}$	$^{158}_{68}$Er$_{90}$	$^{156}_{66}$Dy$_{90}$	Mean value, $\overline{\Delta E}$, keV
19/2 → 15/2	4 → 2	315.9	317.8	312.2	335.1	266.3	300.7
23/2 → 19/2	6 → 4	424.3	408.4	397.0	443.1	366.4	405
27/2 → 23/2	8 → 6	513.0	486.4	472.4	523.0	455.6	489.3
31/2 → 27/2	10 → 8	583.2	550.7	534.5	578.9	509.8	545
35/2 → 31/2	12 → 10	630.9	592.5	573.3	608.1	561.1	585
39/2 → 35/2	14 → 12	565.9	587.1	568.9	510.0	602.3	556
43/2 → 39/3	16 → 14	500.1	547.3	544.8	473.8	611.1	542
47/2 → 43/2	18 → 16	—	—	—	—	527.4	—

TABLE 4. Energies of transitions and moments of inertia in aligned states of ^{157,159}Er and the even nuclei ^{156,158,160}Er.

$I_i \to I_j$	$R+2 \to R$	$^{157}_{68}Er_{89}$ ΔE_I, keV	$^{157}_{68}Er_{89}$ J, MeV^{-1}	$^{159}_{68}Er_{91}$ ΔE_I, keV	$^{159}_{68}Er_{91}$ J, MeV^{-1}	$^{156}_{68}Er_{88}$ ΔE_I, keV	$^{156}_{68}Er_{88}$ J, MeV^{-1}	$^{158}_{68}Er_{90}$ ΔE_I, keV	$^{158}_{68}Er_{90}$ J, MeV^{-1}	$^{160}_{68}Er_{92}$ ΔE_I, keV	$^{160}_{68}Er_{92}$ J, MeV^{-1}
17/2 → 13/2	2 → 0	266.1	11.2	208.3	14,4	344.4	8,7	192	15,6	125,6	23.9
21/2 → 17/2	4 → 2	415.1	16.9	350,0	20	452.9	15.5	335.1	20.8	263.8	26.5
25/2 → 21/2	6 → 4	527.2	20.9	464.5	23.7	543.2	20.2	443.1	24.8	375.3	29.4
29/2 → 25/2	8 → 6	622.4	24	555.9	27	618.2	24.3	523.0	28.6	463.7	32,5
33/2 → 29/2	10 → 8	702.2	27.2	625.9	30.5	675	28,1	578.9	33	531.7	35.7
37/2 → 33/2	12 → 10	765.0	30	675.7	34	—	—	608.1	38	579.2	39.7
41/2 → 37/2	14 → 12	802.9	33.6	708.7	38.2	—	—	510.0	53	592.2	45.6
45/2 → 41/2	16 → 14	—	—	(738.4)	42	—	—	472.8	65.6	533.9	57,6
49/2 → 45/2	18 → 16	—	—	—	—	—	—	566.3	62	554.4	63,2

shell $j = i_{13/2}$ (7/2$^+$[633] or 5/2$^+$[642]), a frequency anomaly is not observed right up to the level $I = 41/2^+$, which corresponds to $R = 14$. This fact shows that the broken pair consists of nucleons on the $i_{13/2}$ level and not the $h_{9/2}$ level. It is interesting to note that in the even–even nulceus $^{164}_{70}Yb_{94}$ a frequency anomaly occurs when I changes from 14 to 12, i.e., at a slightly higher spin than in the neighboring odd nucleus. Evidently, this can be understood as the result of the reduction in the pairing-correlation energy due to the level blocking, which means that it is easier to break a pair of nucleons in an odd nucleus than an even one.

Investigation of odd nuclei enables one to establish experimentally the origin of the frequency anomaly. If one studies an aligned band in an odd nucleus, then from the transition energy one can readily deduce the moment of inertia of the nucleus. If the frequency anomaly is associated with vanishing of the pairing correlation, then in odd nuclei it must occur earlier than in even–even nuclei, since the odd particle decreases the pairing because of the blocking effect. The experiment of Ref. 47 showed that in the nuclei ^{157}Er and ^{159}Er the frequency anomaly occurs later (at larger I), if at all. This precludes explanation of the frequency anomaly by vanishing of the pairing correlation.

In Table 4, the transition energies and the corresponding moments of inertia of ^{157,159}Er and $^{156,158,160}_{68}$Er are given. The absence of a jump of the ω_I and J_I values in the odd isotopes is obvious. The ratios of the moments of inertia in odd and even nuclei are compared in Fig. 13. The moment of inertia of the odd nuclei increases smoothly, which is a very weighty indication that the anomaly is due to the breaking of a neutron pair (or intersection of the ground state with a two-quasiparticle state). This is also confirmed by the fact that removal of a proton (transition to Ho nuclei) does not remove the frequency anomaly.

Thus, investigation of the anomaly in odd nuclei gives additional information about the nature of this phenomenon.

5. CHANGE IN THE DEFORMATION OF A NUCLEUS

The simplest examples of changes in the deformation of a nucleus in different states and the associated change in the moments of inertia are the cases of coexistence in one nucleus of levels with equilibrium $\beta_1 = 0$ and $\beta_2 \neq 0$ in the magic nulcei $^{16}_{8}$O and ^{40}Ca (Fig. 14) and the neighboring nuclides $^{18}_{8}$O, $^{42,44,46,48}_{20}$Ca, and the nuclei at the ends of the transition between strongly β-deformed and spherical or γ-deformed nuclei. Figures 15 and 16 give examples of the frequency anomaly in the nuclei $_{80}$Hg (Refs. 16–19) and $_{76}$Os (Ref. 12), in which states with different deformation can coexist.

A complete quantitative treatment of the problem of equilibrium deformation is very difficult since it requires the calculation of, first, the potential energy of the nucleus, which depends on β, γ, and I, second, calculation of the mass coefficients and, third, solution of the equation for the collective Hamiltonian with allowance for strong anharmonicity and several minima in the potential energy.

As an example, let us consider the potential energy in the form

$$W(\beta) = C(\beta - \beta_0)^2/2 - B\beta^3 + D\beta^4, \quad (36)$$

where C, B, D, and β_0 are constant parameters. In

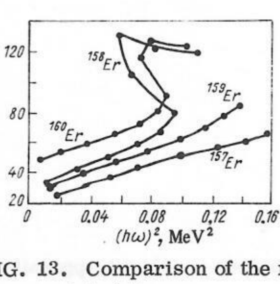

FIG. 13. Comparison of the moments of inertia of odd and even nuclei of Er.

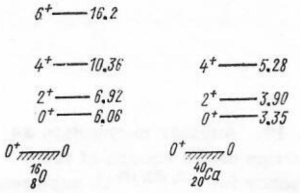

FIG. 14. Examples of coexistence of spherical and deformed states in ^{16}O and ^{40}Ca.

this case, there are two potential wells with different equilibrium deformations β_1 and β_2 and, therefore, different moments of inertia $J(\beta_1)$ and $J(\beta_2)$. For each equilibrium state of the nucleus there are corresponding rotational bands. The spin at which the levels of these bands intersect can be estimated from Eq. (10) ($a=0$):

$$I_1(I_1+1)=2E_0 J_1 J_2/(J_2-J_1). \quad (37)$$

The problem is treated in more detail in Refs. 49 and 50, in which the observed frequency anomaly is explained by a certain parametrization of the Hamiltonian with allowance for the rotational energy and γ deformation. For example, in Ref. 50 the Hamiltonian has the form

$$E_I(J,\gamma)=(2J)^{-1}\hat{R}^2(\gamma)+C(J-J_0)^2/2+K\gamma^2/2+D(J-J_0)\gamma, \quad (38)$$

where

$$\hat{R}^2=\frac{3}{4}\sum_{i=1}^{3} I_i^2/\sin^2(\gamma-2\pi i/3),$$

and C, K, D, J_0 are constants.

In this Hamiltonian, a second minimum with deformation $\gamma\neq 0$ appears when the angular momentum increases.

However, the problem of theory is to obtain a collective Hamiltonian from microscopic theory because only this approach enables one to establish the actual mechanism of the frequency anomaly. In Ref. 51, a successful attempt was made to explain the frequency anomaly in the nuclei $^{184,186,188}_{80}Hg$. The theoretical investigation of Ref. 52–54 shows that the deformation energy of the Hg nuclides has minima of approximately equal depth at quadrupole deformation parameters corresponding to oblate ($\beta=-0.13$) and prolate ($\beta=+0.25$) shapes, as is

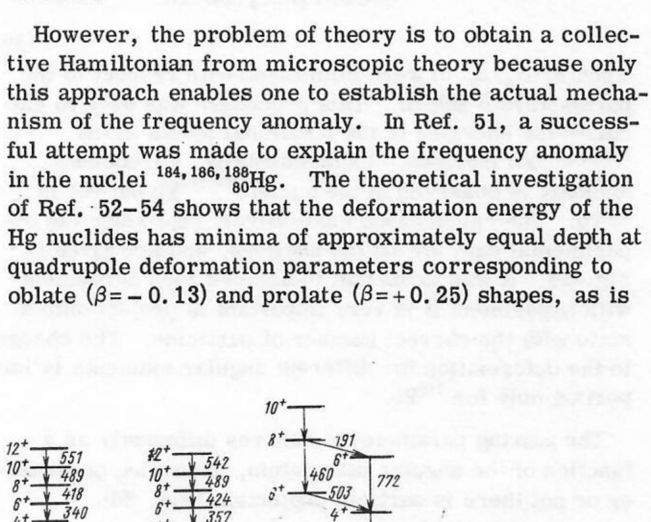

FIG. 15. Frequency anomaly in the light Hg nuclides associated with a charge in the deformation.

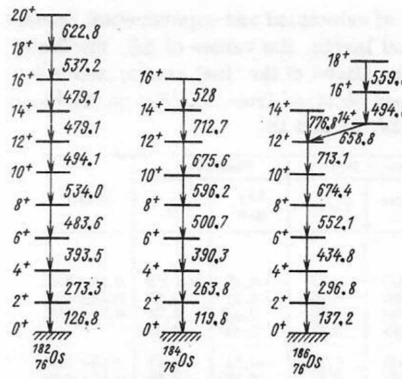

FIG. 16. Frequency anomaly in Os nuclides.

shown in Fig. 17 for the odd Hg nuclides.

The deformation energy E_I of the levels of the even–even Hg nuclides was taken in the form of the sum of the energies of the liquid drop, U_{LDM}, of the shell correction δU (Ref. 55), and the rotational energy $I(I+1)/2J$. The moment of inertia was calculated by Eq. (15) for each deformation separately. The total potential energy for the ^{184}Hg nucleus is shown in Fig. 18. Calculations of the deformation energy for the even–even Hg nuclides showed that in the region $190>A>180$ the ground states are oblate, but the difference between the minima of the deformation energy for the oblate and prolate shapes, $\Delta E_I=E_{I\,\mathrm{obl}}-E_{I\,\mathrm{prol}}$, in very small (about -0.2–0.8 MeV). The 4^+ and 6^+ states have small oblate deformation, whereas the 4^+ and 6^+ states have a relatively large prolate deformation.

The calculated and experimental transition energies corresponding to the equilibrium deformations and ΔE_I are given in Table 5. The calculations confirm the deformation nature of the frequency anomaly in the case of light Hg nuclides. The question of whether there is a barrier between the minimum with respect to the γ deformation was not investigated. However, evaluation of the experimental data on the transition probabilities shows that $\bar{\beta}_{exp}$ varies weakly and this corresponds to the absence of a barrier and high deformability with

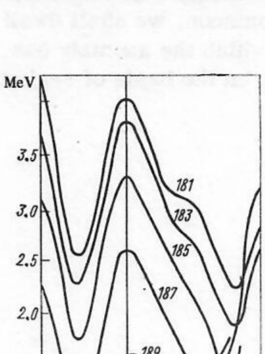

FIG. 17. Deformation energy of odd Hg nuclides.

TABLE 5. Comparison of calculated and experimental transition energies of rotational levels, the values of ΔE, the theoretical equilibrium deformations of the final states, and the experimental deformations obtained from $B_{exp}(E2)$ on the basis of the rotor model of Refs. 17 and 18.

Transition	Energy, MeV		Final state		$\bar{\beta}_{exp}$
	Experiment	Theory	ΔE_I, MeV	β_{eq}	
^{184}Hg					
$2^+ \to 0^+$	0,367	0.38	—0,29	—0,13	0.15±0,02
$4^+ \to 2^+$	0,288	0.25	—0,01	—0,13	0,22±0.01
$6^+ \to 4^+$	0.340	0.36	0.57	0,25	0.28±0.05
$8^+ \to 6^+$	0,418	0,48	1.35	0,25	
^{186}Hg					
$2^+ \to 0^+$	0,405	0.37	—0,54	—0,13	0.13±0.01
$4^+ \to 2^+$	0,403	0.54	—0.29	—0.13	0.16±0.03
$6^+ \to 4^+$	0.357	0.39	0.25	0,23	0,27±0.05
$8^+ \to 6^+$	0,424	0.52	0.95	0.24	—
^{188}Hg					
$2^+ \to 0^+$	0,413	0,37	—0,097	—0.13	—
$4^+ \to 2^+$	0.591	0.79	—0.69	—0,13	—
$6^+ \to 4^+$	0,504	0.43	—0.11	—0.14	—
$8^+ \to 6^+$	0,460	0.43	0.67	0.23	—

respect to the γ direction for the states $I = 2^+$ and 4^+.

For elements heavier than ^{188}Hg, the value of ΔE becomes very large and the nature of the frequency anomaly changes, going over to pair breaking on the $(h_{11/2}^{-2})$ level.[19] It is interesting to note that the Pt nuclides with the same number of neutrons as Hg were found to be "prolate" already in the ground state, and their moments of inertia correspond to the moments of inertia of Hg in the excited "prolate" state.[51] From the data of Ref. 51 one can find that for the ^{184}Hg nucleus $E_0 = 0.28$ MeV, $J_I = 7.9$ MeV^{-1}, $J_2 = 27.2$ MeV^{-1}. Then a calculation in accordance with Eq. (37) gives $I^* = 2$, which agrees with experiment. Thus, the light Hg nuclides are an example of the deformation mechanism of the frequency anomaly. Probably, one could find more complicated forms of the frequency anomaly associated with a change in the equilibrium values of β and γ.

6. CALCULATIONS OF THE FREQUENCY ANOMALY BY MEANS OF REALISTIC MODELS

Since our main aim was to investigate qualitatively the physical origin of the phenomenon, we shall dwell here only briefly on papers in which the anomaly has been investigated quantitatively on the basis of realistic nuclear models.

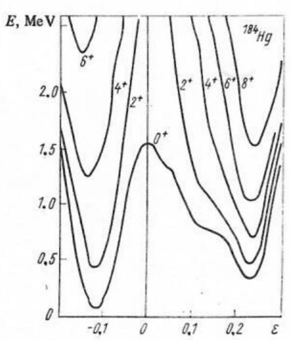

FIG. 18. Deformation energy of rotational states in ^{184}Hg. The equilibrium deformation becomes positive (and large) with increasing spin I.

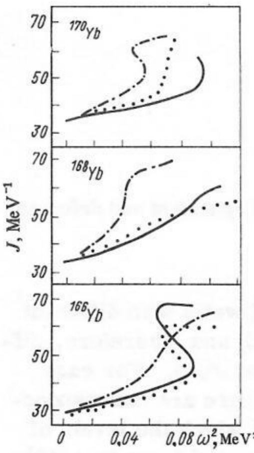

FIG. 19. Angular momentum as a function of the square of the frequency for $^{166,168,170}Yb$ nuclides. —— experimental data; ····· results of calculation; —·—·— calculations without allowance for particle number conservation for neutrons (—·—·—) and protons (····).

In Refs. 56 and 57 Faessler *et al.* calculated the mean values of the nuclear Hamiltonians by means of wave functions for Nilsson's potential with allowance for pairing correlation effects (BCSN) corrected by the introduction of the projection operators P_I, P_n, P_p. The projection procedure guaranteed fulfillment of the conservation laws for the angular momentum I and the number of neutrons (n) and protons (p). The Hamiltonian consisted of spherical one-particle states in the shells $N = 4$ and 5 for protons, $N = 5$ and 6 for neutrons, and terms corresponding to pairing and quadrupole–quadrupole interactions with parameters taken from Ref. 58. In addition, from the condition of consistency of the position of the 2^+ level, one free parameter was introduced—the moment of inertia of the core, which took into account the presence of other shells. The energies of states with given I, Δ_n, Δ_p and β were calculated in the form

$$E_I(\Delta_n, \Delta_p, \beta) = \frac{\langle \mathrm{BCSN}(\Delta_n, \Delta_p, \beta) | \hat{H} P_I P_n P_p | \mathrm{BCSN}(\Delta_n \Delta_p, \beta) \rangle}{\langle \mathrm{BCSN} P_I P_n P_p | \mathrm{BSCN} \rangle}. \quad (39)$$

Then $E_I(\Delta_n, \Delta_p, \beta)$ were minimized with respect to the parameters β and Δi. This procedure was used to calculate the energies of the rotational levels in the $^{166,168,170}Yb$ isotopes. Experimentally, a frequency anomaly is observed in the nuclei $^{166,170}Yb$ but not in ^{168}Yb. The calculations qualitatively reproduce the experimental data for all the nuclides, and are given in Fig. 19. It was found that to achieve good agreement with experiment it is very important to project onto a state with the correct number of particles. The change in the deformation for different angular momenta is important only for ^{170}Yb.

The pairing parameter Δ behaves differently as a function of the angular momentum, depending on whether or not there is particle projection (Fig. 20).

In ^{166}Yb, there is a phase transition for neutrons if allowance is made for projection of particles; without it, there is not. In ^{168}Yb, we have the opposite situation, while in ^{170}Yb a phase transition is obtained with and without allowance for projection of the particles, but it occurs at different spins. Thus, the results of the calculations are very sensitive to the quantum

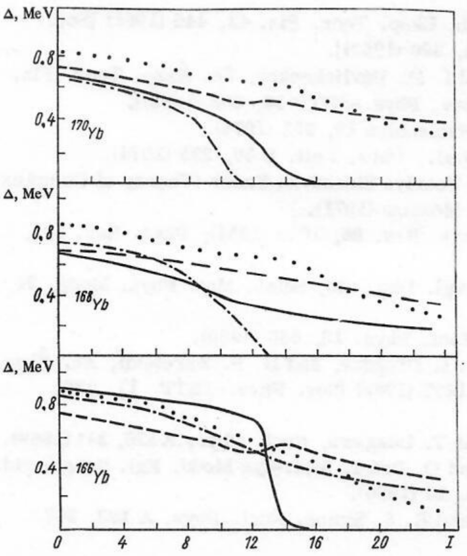

FIG. 20. Pairing-correlation energy Δ as a function of the angular momentum for neutrons (——) and protons (– – – –). Value of Δ calculated without allowance for particle number conservation for neutrons (—•—•—) and protons (•••••).

numbers of the one-particle levels and their position relative to the Fermi surface and favor a more or less abrupt phase transition as the origin of the frequency anomaly. However, the great sensitivity to the methods of calculation forces us to treat the conclusions with discretion. In addition, the projection method does not take into account the important rearrangement in the position of the one-particle levels in a strong rotation field. The excitation energy of such a state for large I (or ω) differs from the ordinary expression $E=\sqrt{(\varepsilon_\lambda-\varepsilon_0)^2+\Delta^2}$ not only by the change in Δ but also by the appearance of additional terms in E that depend on I. The calculations were restricted by the choice of axially symmetric internal wave functions. The Coriolis forces destroy this symmetry since they couple states with different Ω and they therefore induce a nonaxial deformation.

Thus, the pair-breaking phenomenon essentially was not taken into account in Refs. 56 and 57. To clarify the true mechanism of the frequency anomaly, it is also necessary to take into account the change in the energy and the fact that the one-quasiparticle states are not axial. Calculations taking into account these effects were made in Refs. 59 and 60 by the method of varying the energy after the functions have been projected onto a state with given angular momentum. The calculations were made for the nuclei $^{162}_{68}$Er, in which an anomaly is observed, and $^{168}_{70}$Yb, in which it is not. It was shown that in the ^{162}Er nucleus there is a gradual decrease in the pairing correlation due to the Coriolis forces. However, in the range $0.2<\omega<0.21$ of the rotation frequency the angular momentum of certain pairs of conjugate states is very strongly aligned along the rotation axis. The contribution from one pair to the total angular momentum can increase strongly for a small change in ω. The correlation function $\Delta_{\Omega\bar{\Omega}}$ for this pair changes much more strongly than the mean variation of Δ. Expansion of the wave function of this pair in spherical functions shows that the $i_{13/2}$ component is predominant in it. One can therefore say that two neutrons in the $1i_{13/2}$ shell break their coupling under the influence of rotation and align their angular momenta along the axis of rotation, as we have considered above. The absence of an anomaly in ^{168}Yb is explained by the absence of levels with large j near the Fermi surface. Unfortunately, conservation of the number of particles was not taken into account in these calculations.

Thus, if one works with realistic models one must make calculations in which the conservation laws for the angular momentum and the particle number are satisfied and also take into account significant changes in the one-particle functions and energies that arise in the rotational system. Once such calculations have been made, one should be able to settle finally the nature of the frequency anomaly and to calculate quantitatively the rotation spectra in each concrete case. However, the qualitative treatment shows that the frequency anomaly is essentially a pair-breaking phenomenon.

CONCLUSIONS

Summarizing, we can say that the anomalous behavior of the rotation frequency is due to the existence in a nucleus of levels with different moments of inertia. This fact reflects the different behavior in a rotating nucleus of nucleons in different states. If the moment of inertia of an upper level is greater than a lower level's, the levels intersect. It should be understood that there is always an interaction which pushes the levels apart and the intersection is formally eliminated. However, the wave functions of levels obtained with allowance for interaction go over smoothly from one unperturbed wave function to another. In this case there is still a transition from one physical state to another ("quasiintersection"), though it does become smoother. In the region of nuclei with stable deformation the excited states are two-quasiparticle levels with large j (and small projections onto the symmetry axis of the nucleus), for which the Coriolis interaction is of decisive importance. Quasi-intersection with such a level in a continuous transition looks like alignment of the angular momenta of these particles along the axis of rotation. The alignment of the angular momenta of the particles leads to blocking of levels and reduces the pairing correlation. At present, it is not completely clear whether the pairing correlation disappears entirely in the quasi-intersection process, although the general theoretical treatment suggests that the process of vanishing of Δ is drawn out. In the nuclei that are transitional with respect to the deformation there are "intersections" of states with different deformation. Anomalous behavior of the rotation frequency occurs when there are transitions from levels that consist basically of one physical state to levels of another state. The qualitative nature of the frequency anomaly has now been clarified, although very laborious calculations are required for the quantitative description of this phenomenon in different nuclei.

For a fuller theoretical investigation of the properties of highly excited states it is necessary to know in more detail the following: the potential energy $C(\beta, \gamma, \Delta, \{n\}, I)$ calculated with a realistic nucleon–nucleon interaction; the inertial parameters $B(\beta, \gamma, \Delta, \{n\}, I)$; the rotational interaction between different states; how to solve the problem of collective motion for given C and B with allowance for strong anharmonicity, shape isomerism, and large angular momenta when the rotation can no longer be treated in the adiabatic approximation. The complexity of these problems emphasizes the importance of experimental study of the properties of highly excited states.

The experimental investigations of highly excited states enable one to establish the dependence of the inertial parameters on the angular momentum and the nucleon population, the deformation of different states, the scheme of coupling of the free quasiparticles with the rotating nucleus, the interaction between levels of different nature, the structure of excited states, etc. The study of these questions will be extremely important for the development of the theory of the structure of excited states.

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