Properties of the lightest nuclei and the problem of nucleon-nucleon potentials

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The present status of the theory of the lightest nuclei in the discrete and continuous spectrum is reviewed from the point of view of the nucleon-nucleon potential problem. Models of the nucleon-nucleon potentials deduced from two-nucleon data are described. The restrictions imposed on the nucleon-nucleon potentials by the data on the properties of few-nucleon systems are discussed. Some methods of solving the many-nucleon Schrödinger equation are described.

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INTRODUCTION

Even at the beginning of the sixties, almost any problem of three or more strongly interacting particles in quantum mechanics was regarded as virtually unsolvable because of the mathematical difficulties. The question did not therefore arise of attempting to calculate the properties of nuclei and nuclear reactions directly from the many-nucleon Schrödinger equation without the introduction of any models. In the last decade radical changes have taken place; methods of solving the problem of several strongly interacting particles have been developed, and a great many theoretical and numerical calculations have been made.

The problem of the properties of three-nucleon systems was solved for the first time in 1956 by Ter-Martirosyan and Skornyakov[1] in the approximation of a point nucleon-nucleon interaction (nucleons interact with one another only when the distance between them is zero). This approximation was not without complications; some of its results proved to be physically meaningless. A way of separating out the "physical" solutions was given by Danilov.[2]

In 1958, Faddeev[3] derived and investigated equations of the three-body problem that are convenient and free of ambiguities (Faddeev equations). These equations were the basis of many numerical investigations of the properties of three-nucleon systems made in the last decade. Many important results in this direction were obtained for the first time by Kharchenko.[4] In 1967, Faddeev's approach was generalized by Yakubovskiĭ[5] to the case of n particles ($n>3$); his equations, which describe a system of n particles, are sometimes called the Yakubovskiĭ equations. They are however too cumbersome and are hardly ever used.

In 1964, Simonov and Badalyan,[6] and independently Zickendraht,[7] began to develop a new approach to many-body problems—the K-harmonic method or, as it is frequently called, the method of hyperspherical functions. As a first application of the method, Simonov, Badalyan *et al.*[8] calculated approximately the binding energy and wave function of the ^{3}H, ^{3}He, and ^{4}He nuclei for some of the simplest potentials. This method was intensively developed and it is now perhaps the most convenient for solving problems of the properties of systems of a few strongly interacting particles.

The appearance of computational methods enabling one with any preassigned accuracy to solve the Schrödinger equation for several nucleons makes it vitally important to pose the following question, which at first sight appears purely academic: Can one, at least in principle, solve all the basic problems of nonrelativistic nuclear physics without recourse to models but directly from the many-nucleon Schrödinger equation with a given nucleon-nucleon (NN) interaction? One is led to pose this question by several considerations. First, the interaction between two free nucleons can be described by means of a potential V only in the nonrelativistic limit; in the general case, the NN interaction is not a potential interaction but leads to the creation (virtual or real depending on the energy) of new particles (mesons, hyperons). In this connection, one now frequently discusses the question of whether the nucleus is not simply a "sack with nucleons" but rather a more complicated formation in which there are real mesons and the nucleons spend some of their time as hyperons. Second, even if a nucleus does consist of nucleons, the interaction V between nucleons in the nucleus can differ from the "vacuum" NN interaction $\overline{V}$ describing the interaction of two nucleons in vacuum; for the NN interaction in the modern theory of elementary particles is described as a process of exchange of different mesons between nucleons. This exchange depends on whether the two nucleons are in vacuum or are surrounded by other nucleons. In the latter case, the surrounding nucleons may actively participate in the exchange process, so that $V \neq \overline{V}$.

Both of these circumstances follow directly from the theory of elementary particles. However, the theory does not enable one to estimate reliably the corresponding effects, and the general situation may therefore be characterized as follows. If the above circumstances really are important, nuclear theory cannot be constructed in isolation from the theory of elementary particles—the nonrelativistic Schrödinger equation does not apply to nuclei, whose properties must be described by more complicated equations constructed on the basis of field theory. In this case, a model-free nuclear physics is at present not to be thought of.

There is however a different possibility. As a working conjecture one can assume that the effects associated with the structure of nucleons play only a small role in nonrelativistic nuclear physics. In this case, all the basic properties of nuclei and nuclear reactions can be

obtained by solving the many-nucleon Schrödinger equation with the vacuum interaction between the nucleons. An attractive feature of such a conjecture is, besides its logical simplicity, its constructive value. It proposes a model-free method of calculating nuclear-physics quantities and its validity or invalidity will transpire through the calculation of numerous nuclear data. Anticipating, we may point out that as yet there are no indications refuting this working conjecture.

In the present review several problems are posed. Assuming as a working conjecture that a model-free nuclear physics can be constructed on the basis of the many-nucleon Schrödinger equation, we analyze the results of the existing calculations of the properties for the lightest nuclei and simple nuclear reactions.

As will be seen subsequently, one of the main difficulties is the inadequate knowledge of nucleon-nucleon potentials. From the scattering of nucleons on nucleons one cannot determine their interaction potential uniquely, and there are more than ten phenomenological potentials which describe equally well the scattering in the nucleon energy range 0—150 MeV (cms). When these potentials are used to calculate the properties of three-, four-, etc., nucleon systems, the results differ somewhat. Therefore, at the present time the main tasks in the theory of few-nucleon nuclear systems are: a) to calculate those properties that depend weakly on the details of the NN potentials, b) to use the experimentally known properties of few-nucleon systems to obtain more precise information about the form of the NN potentials.

In the present review, we attempt only to show what basic physical problems at present exist in the theory of few-nucleon systems and how they can be solved.

1. MODELS OF NUCLEON-NUCLEON POTENTIALS

General Form of the Nucleon-Nucleon Potential. It follows from the theory of elementary particles that in the nonrelativistic energy range the interaction between two nucleons (nonpotential) can be described approximately by a potential, though field theory is not in a position to calculate this potential exactly. All that it can say is what must be the general form of this potential and how it behaves at large distances between the nucleons.

Two nucleons are characterized by the vector $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ of the relative distance, their spins $\mathbf{s}_1$ and $\mathbf{s}_2$, and isospins $\mathbf{t}_1$ and $\mathbf{t}_2$. From these vectors one can construct several independent operators: $\hat{T}$, the total isospin; $\hat{\mathbf{S}}$, the total spin; $\mathbf{r}$, the distance; $\hat{\mathbf{p}}$, the relative momentum; and $\hat{\mathbf{L}}$, the relative orbital angular momentum. Proceeding from general invariance arguments, it is not difficult to construct rather a lot of terms of different structure that could occur in the potential:

a) central potentials:

$$\hat{V}^c=\sum_{S,\,T=0,\,1}\hat{V}^c_{2S+1,\,2T+1}=\sum_{S,\,T=0,\,1}V^c_{2S+1,\,2T+1}(r)\,P^S P^T, \tag{1}$$

where $P^S(P^T)$ is the operator of projection onto the state of a pair of particles with spin S (isospin T):

$$P^{(S=1)}=\hat{S}^2/2,\quad P^{(S=0)}=1-P^{(S=1)};$$
$$P^{(T=1)}=\hat{T}^2/2,\quad P^{(T=0)}=1-P^{(T=1)};$$

$V^c_{2S+1,2T+1}(r)$ are functions of the distance between the nucleons. In a state with given total spin S and isospin T of the pair of nucleons a characteristic central potential $V^c_{2S+1,2T+1}(r)$ acts;

b) spin-orbit potentials:

$$\hat{V}^{LS}=\sum_{T=0,\,1}\hat{V}^{LS}_{2T+1}=\sum_{T=0,\,1}V^{LS}_{2T+1}(r)\,(\mathbf{LS})\,P^T; \tag{2}$$

c) tensor potentials:

$$\hat{V}^t=\sum_{T=0,\,1}\hat{V}^t_{2T+1}=\sum_{T=0,\,1}V^t_{2T+1}(r)\,[6\,(\hat{\mathbf{S}}\mathbf{r})^2-2\hat{S}^2]\,P^T. \tag{3}$$

The potentials $\hat{V}^{LS}$ and $\hat{V}^t$ have only one index since they act only in states with $S=1$ and therefore the index $2S+1=3$ is omitted;

d) $\hat{L}^2$ potentials; they are sometimes written in the form

$$\hat{V}^{L^2}=\sum_{S,\,T=0,\,1}\hat{V}^{L^2}_{2S+1,\,2T+1}=\sum_{S,\,T=0,\,1}V^{L^2}_{2S+1,\,2T+1}(r)\,\hat{L}^2P^SP^T \tag{4}$$

or

$$\hat{V}^{QLS}=\sum_{S,\,T=0,\,1}\hat{V}^{QLS}_{2S+1,\,2T+1}=\sum_{S,\,T=0,\,1}V^{QLS}_{2S+1,\,2T+1}(r)\,\hat{L}_{12}P^SP^T.$$
$$L_{12}=(\hat{\sigma}_1\hat{\sigma}_2)\,\hat{L}^2-[(\hat{\sigma}_1\hat{\mathbf{L}})\,(\hat{\sigma}_2\hat{\mathbf{L}})+(\hat{\sigma}_2\hat{\mathbf{L}})\,(\hat{\sigma}_1\hat{\mathbf{L}})]/2, \tag{5}$$

where $\hat{\sigma}$ are Pauli matrices;

e) p^2 potentials can contain terms of two types:

$$\hat{p}^2\hat{V}+\hat{V}\hat{p}^2 \tag{6a}$$

or

$$(\hat{\mathbf{p}}\hat{\sigma}_1)\,\hat{V}\,(\hat{\mathbf{p}}\hat{\sigma}_2)+(\hat{\mathbf{p}}\hat{\sigma}_2)\,V\,(\hat{\mathbf{p}}\hat{\sigma}_1), \tag{6b}$$

where the potentials $\hat{V}$ depend on S and T.

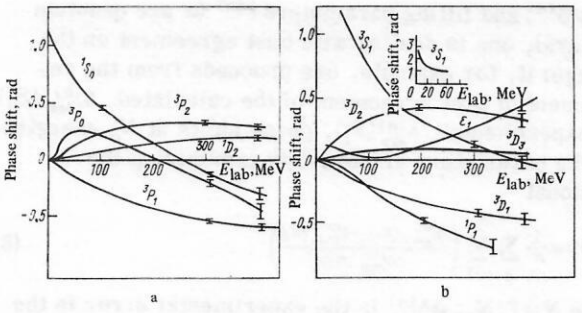
In nonrelativistic theory, the potentials cannot contain higher powers of $\hat{\mathbf{L}}$ and $\mathbf{p}$, since the differential Schrödinger equation would then contain derivatives of higher than second order and one would have difficulties due to the increase in the number of solutions and, accordingly, the need to introduce additional boundary conditions.

All the above potentials are local in the sense that when one substitutes them into the Schrödinger equation one obtains ordinary second-order differential equations. There is however the theoretical possibility of introducing nonlocal potentials $\hat{V}_{\text{nonloc}}$, whose action on the wave function Ψ is defined as

$$\hat{V}_{\text{nonloc}}\Psi=\int V(\mathbf{r},\mathbf{r}')\,\Psi(\mathbf{r}')\,d\mathbf{r}';\quad V(\mathbf{r},\mathbf{r}')\neq V(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}').$$

When they are introduced into the Schrödinger equation, an integrodifferential equation is obtained. Although such potentials are sometimes used in calculations, we shall not consider them here, since there are as yet no definite indications that they have to be introduced into the theory.

The interaction between nucleons arises as a result of the exchange of one, two, three, etc., mesons. It



a b

FIG. 1. NN phase shifts.[70]

can be shown that at large distances between the nucleons the exchange of one π-meson makes a dominant contribution to the interaction. The potential that arises as a result of this mechanism can be calculated exactly. It is called the one-pion-exchange potential (OPEP) and has the form

$$\hat{V}^{\text{OPEP}} = (g^2/12)\, m_\pi c^2 \,(m_\pi/m_N)\,(\tau_1\tau_2) \{(s_1 s_2)\, A + [6\,(\hat{S}r/r)^2 + 2\hat{S}^2]\,(1 + 3/x + 3/x^2)\}\,[\exp(-x)/x], \tag{7}$$

where g is the πN coupling constant; m_π and m_N are the masses of the pion and the nucleon; $x = \varkappa r$, where $\varkappa = m_\pi c/\hbar$. This formula describes the nucleon-nucleon potential at large distances between the nucleons. It follows from it that at large distances the NN potential consists solely of central and tensor forces. The remaining components (LS, L^2, and p^2 forces) have a shorter range.

Attempts to determine the explicit form of the NN *potential.* In the middle and end of the fifties, when the first large proton accelerators at several hundred MeV were put into operation, many hoped that detailed study of nucleon-nucleon scattering in a wide range of energies would enable one to determine completely and uniquely the NN potential. This hope was based on the theorem of Ref. 73, which states that if the phase shift is known at all energies, then the potential in the Schrödinger equation can be uniquely recovered by means of a definite mathematical procedure. One of the important problems of that period was the careful study of NN scattering and determination of phase shifts.

As the result of tremendous exertions by large teams, the *pp* and *np* phase shifts were determined in the energy range from 0 to ~200 MeV (cms). As an example, Fig. 1 shows the values of several phase shifts δ_{LS} with their experimental errors in accordance with the latest data. However, the potential could not be obtained from these data, mainly because the inverse problem of scattering theory (recovery of the potential from the phase shifts) is unstable: to determine the potential the phase shifts must be known exactly. Even a small uncertainty in their values was sufficient to lead to huge errors in the potentials calculated from the phase shifts. In addition, beginning with the energy at which production of pions becomes possible, the problem of NN scattering becomes nonpotential, and this is not taken into account in the solution of the inverse problem.

We can illustrate the situation by means of the graph (Fig. 2) of Pokrovskiĭ, who investigated the uncertainty of the potential acting in states with $S=0$ and $T=1$. He obtained a two-parameter family of potentials that describe the singlet phase shifts equally well. The hatching in Fig. 2 distinguishes the regions on the $V_c W$ plane (V_c is the value of the central part of the potential at $r=0$; W is the amplitude of the p^2 component of the potential). To each point of the hatched region there corresponds a potential which describes the singlet phase shifts excellently (description with $\chi^2 \leq 1$).

Despite the impossibility of a unique determination of the NN potentials from the NN scattering data, some qualitative conclusions about their properties could be drawn. First, it was confirmed that at large distances the NN potential is given by Eq. (7). Second, the energy dependence of the phase shifts at high energies (150—200 MeV in the center-of-mass system) indicated a strong repulsion between nucleons at distances ≤ 0.5 F. Third, it was found that the NN potential has a very complicated structure, containing besides central and tensor forces the other interactions discussed above.

All this transpired when the NN potential began to be determined by the method of trial and error, in which a definite functional form is chosen for the NN potential with a fair number of variable parameters, which are then determined by the condition of best agreement of the phase shifts calculated for these potentials with the experimental phase shifts. Calculations of this type are very laborious and require large computers. Nevertheless, there are now already several tens of investigations by different authors on this subject and a corresponding number of fairly strongly differing sets of NN potentials, which more or less equally well describe experimental phase shifts. The most widely known NN potentials are listed below. Recall that we have already restricted ourselves to local NN potentials, so that we exclude investigations aimed at finding nonlocal ones. This restriction is solely for the sake of simplicity. If in the future it transpires that local NN potentials cannot describe the complete set of experimental data, it

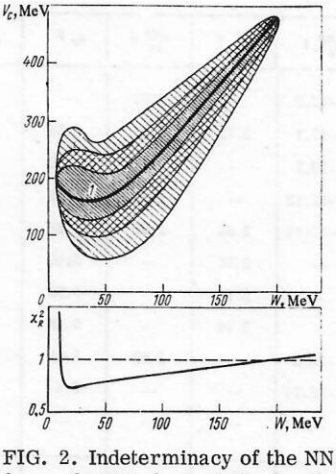


FIG. 2. Indeterminacy of the NN potential V_{13} determined from the NN phase shifts for the energy range $E_{lab}=0$ to 425 MeV (dense hatching), to 330 MeV (intermediate hatching), to 210 MeV (low-density hatching); below, the value of χ^2_R corresponding to curve 1.

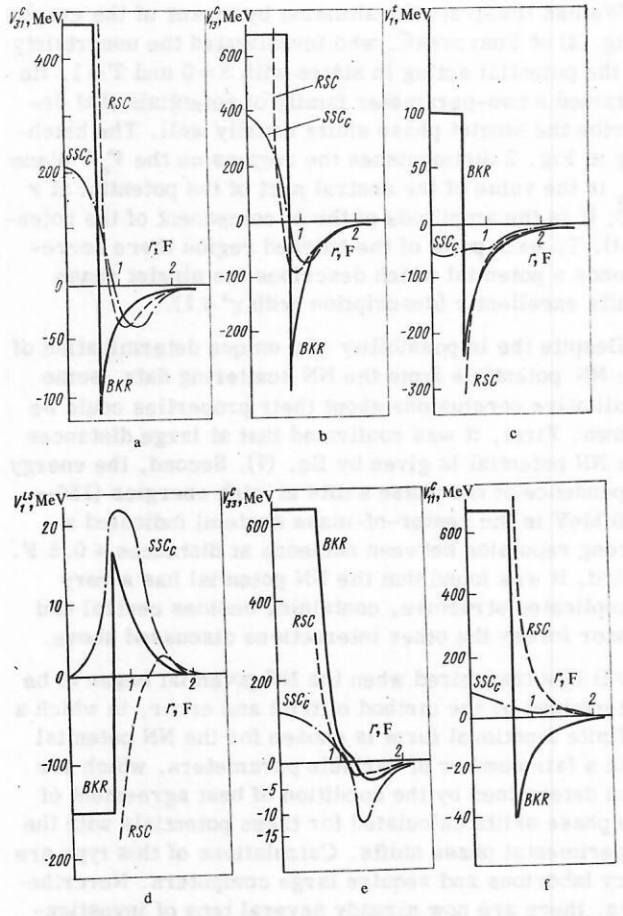


FIG. 3. Radial dependence of the NN potentials of Reid (RSC), Bressel-Kerman-Rouben (BKR), and of the potential with hypersoft core (SSC_C) (see Table 1).

will be necessary to add nonlocal terms to the local ones in the NN potential.

Characteristics of some NN *potentials*. When the parameters of NN potentials are determined from the condition of best agreement with the experimental phase shifts $\delta^{(\alpha)}$ and fitting parameters $\epsilon^{(\alpha)}$ (α are quantum numbers), one is dealing with best agreement on the average: if, for example, one proceeds from the requirement of best agreement of the calculated, $\delta^{(\alpha)}_{\text{theor}}(E_i)$, and experimental, $\delta^{(\alpha)}_{\text{exp}}(E_i)$, phase shifts at N_α energies E_i, the parameters are found by minimizing the functional

$$\chi^2 = \frac{1}{N}\sum_{\alpha}\sum_{i=1}^{N_\alpha}\left(\frac{\delta^{(\alpha)}_{\text{theor}}(E_i)-\delta^{(\alpha)}_{\text{exp}}(E_i)}{\Delta\delta^{(\alpha)}_{\text{exp}}(E_i)}\right)^2, \qquad (8)$$

where $N=\sum_\alpha N_\alpha$; $\Delta\delta^{(\alpha)}_{\text{exp}}$ is the experimental error in the measured phase shift. The smaller the value achieved by minimization, the better this set of NN potentials agrees with the experimental values on which the calculation is based. Apart from the value of $\chi^2_{\min}$, the quality of the NN potentials is also determined by the extent to which they reproduce the properties of the deuteron, the triplet and singlet scattering lengths a_t and a_s, and the effective radii r_{0t} and r_{0s}.

The first carefully selected sets of NN potentials obtained in the above manner appeared in 1962. These were the so-called Hamada-Johnston (HJ) potentials[9] and the Yale (Y) potential.[10] Both include central, tensor, LS, and L^2 components. The repulsion between nucleons at short distances is described in both cases by the introduction of a "hard core" (when $r<r_c$ the potential tends to $+\infty$, where $r_c=0.4852\hbar/(m_\pi c)$ and $0.35\hbar/(m_\pi c)$ for the first and second NN potential). Subsequently, the introduction of a hard core was eschewed and in all the NN potentials which have appeared since 1962 the repulsion is "softer".

From the beginning of the sixties, more and more new NN potentials appeared continuously: the Reid[11] potential (RSC) in 1968, the Bressel-Kerman-Rouben[12] potential (BKR) in 1969, the Gogny-Pires-de Tourreil[13] potential (GPT) in 1970, the Eikemeier-Hackenbroich[14] potential (EH) in 1971, several variants of the de Tourreil-Sprung[15] potentials (SSC_A, SSC_B, SSC_C) in 1973, and so on. All these potentials have a complicated structure and contain central, tensor, LS, and L^2 components.

TABLE 1. Some properties of the two-nucleon system calculated with realistic NN potentials.

Potential	$\chi^2_{\min}$	a_s^{pp}, F	a_s^{np}, F	r_{0s}^{pp}, F	r_{0s}^{np}, F	a_t, F	r_t, F	$\lvert E_d\rvert$, MeV	Q_d, F²	Experiment
HJ [9]	4	—	−17.0	—	2.83	—	—	2.269	0.285	6.97
RSC [11]	2	−7.78	−17.1	2.72	2.80	5.39	1.72	2.2246	0.2796	6.47
BKR [12]	2	−7.825	−23.7	—	—	5.44	—	2.224	0.291	6.53
GPT [13]	8.4		−22.12	—	2.83	5.24	1.93	2.6	0.271	3.8
EH [14]		−7.79	−17.14	2.84	2.91	5.37	1.71	2.226	0.273	6.215
SSC_A [15]	$\leqslant 3$	−7.82	—	2.71	—	5.50	1.85	2.224	0.262	4.43
SSC_B [15]	$\leqslant 3$	−7.82	—	2.71	—	5.50	1.86	2.228	0.262	4.25
SSC_C [15]	$\leqslant 2$	−7.82	—	2.71	—	5.48	1.83	2.224	0.279	5.45
BS [16]	10.8	—	−23.3	—	2.62	5.45	1.65	2.1	—	—
SW [17]	2.2	−7.7	−23.74	—	—	5.4	—	—	—	—
IP [18]	—	—	—	—	—	—	—	2.2245	0.289	5.5
Experiment	—	−7.82± ±0.01	−23.715± ±0.015	2.80± ±0.01	2.73± ±0.03	5.41± ±0.01	1.75± ±0.004	2.2246± ±0.0005	0.2796	3.9

From the point of view of describing the properties of a two-nucleon system, all these NN potentials are more or less equally good. This can be seen from Table 1, which for each of the potentials gives the values of $\chi^2_{\min}$ obtained for them and for the low-energy parameters a_s, a_t, r_{0s}, and r_{0t}, the deuteron binding energy E_d, the deuteron quadrupole moment Q_d, and the D-wave admixture P_D in the deuteron.

Although the general structure of these potentials is the same, their magnitudes and radial dependences differ fairly strongly. This is demonstrated by Fig. 3, which shows the V^c_{31}, V^c_{13}, V^c_{11}, V^c_{33}, V^t_1, V^{LS}_1 components of the potentials for some of the above NN potentials.

Alongside these phenomenological NN potentials, others of a somewhat different type began to appear in the literature—the so-called one-boson-exchange potentials (OBEP). In the construction of these potentials it is assumed that the interaction between nucleons can be adequately described by summing up only the simplest one-meson graphs but with allowance for the exchange of not only pions but also the other, heavier η, ω, ρ,... mesons. In this case the variable parameters are the number of allowed for mesons, the masses of the hypothetical exchanged mesons, and, partly, the meson-nucleon coupling constant.

Potentials of this second type have the same general form as the phenomenological NN potentials, i.e., they include central, tensor, and LS components. The only difference is that instead of L^2 terms they have p^2 terms.

Several such potentials have been proposed: the Bryan-Scott[16] potential (BS), the Scotti-Wong[17] potential (SW), the Ingber-Potenza[18] potential (IP), the Ueda-Green[74] potential. These potentials describe the two-nucleon data about as well as the purely phenomenological NN potentials (see the last rows of Table 1).

Conclusions and Problems They Pose

Conclusions. From the presently existing data on two-nucleon systems at nonrelativistic energies it is impossible to determine uniquely the interaction potential between nucleons. We have in mind only the interaction of nucleons in states with $L \leqslant 3$—4, since higher partial waves do not in practice exist at nonrelativistic energies and to allow for their influence one can use the asymptotic part of the potential V^{OPEP} [see Eq. (7)].

Analysis of the experimental data establishes only the general properties of the NN potential: a) its complicated structure (LS and L^2 or p^2 components in addition to the central and tensor components); b) at short distances it must lead to strong repulsion; c) at long distances it must go over into V^{OPEP} [see Eq. (7)].

Problems. These conclusions directly pose the following questions:

1) Can one, staying in the range of nonrelativistic energies (which is the only range in which the interaction between nucleons can be described by a potential), determine this potential uniquely?

2) Can one now, when only the qualitative form of the NN potential is known, hope for success of a microscopic theory of the nucleus in which the properties of nuclear states and amplitudes of the nuclear reactions are calculated without a model but directly from the many-nucleon Schrödinger equation containing the NN potentials?

The two questions are related; for the class of allowed NN potentials is restricted somewhat by more accurate experimental values of the NN phase shifts. But elimination of the ambiguity of the NN potential cannot be completely achieved in this way, requiring, as it does, knowledge of the phase shifts in an infinite energy range, whereas we have at our disposal only the nonrelativistic energies. Therefore, apart from the NN scattering data, it is necessary to invoke other data on the properties of nuclei and nuclear reactions for which the nonrelativistic condition is satisfied. And here we come right up against the second question. If the properties of nuclei depend on only the NN scattering amplitude on the mass shell but not on the actual form of the NN potential, this potential cannot be determined uniquely, though a microscopic theory of the nucleus can be constructed on the basis of the many-nucleon Schrödinger equation (into which one can substitute any reasonable NN potential). But if the concrete form of the NN potential is important, calculation of the properties of nuclei and reactions and their confrontation with experiment provide the additional conditions needed to fix the NN potential uniquely.

The real situation is only now beginning to become clear, and to understand the picture which is now being delineated we must dwell in more detail on the results of the presently existing calculations of the properties of the lightest nuclei and the simplest nuclear reactions.

2. CALCULATIONS OF THE PROPERTIES OF THE LIGHTEST NUCLEI (^{3}H, ^{3}He, ^{4}He)

Many theoretical investigations have been made of the ^{3}H and ^{3}He nuclei. The problem of a bound state of three nucleons is solved under different assumptions about the form of the NN potential and by different methods:

a) by solving the Faddeev equations;

b) by application of a direct variational procedure;

c) by expanding the three-nucleon wave function $\Psi(1,2,3)$ in a series with respect to a complete set of certain three-body functions and solving the secular equation then obtained from the Schrödinger equation of the three-nucleon system.

Here we shall not discuss the methods of solution (see Appendix 1). We merely mention that the most reliable results are obtained from the third of the methods when a basis of hyperspherical functions (K harmonics) and hyper-radial functions are used as a complete set.

Binding Energies and Wave Functions of the ^{3}H *and* ^{3}He *Nuclei*. In these nuclei $J^P = 1/2^+$, and their wave function can be written down in the form of four terms:

$$\Psi(1,2,3) = \Psi_S + \Psi_{S'} + \Psi_P + \Psi_D, \tag{9}$$

where Ψ_S and $\Psi_{S'}$ correspond to values of the total orbi-

TABLE 2. Properties of ^{3}H and ^{3}He with realistic NN potentials.

$\lvert E^{^3\mathrm{H}}\rvert$, MeV	$r^{^3\mathrm{H}}_{ch}$, F	$r^{^3\mathrm{He}}_{ch}$, F	$P(^3\mathrm{H})$, % S'	P	D	E_{Coul}, MeV	Method	Literature
RSC								
6.5*	1.85	—	0.52	None	8.92	—	OB	[19] (c)
6.5	—	—	1.8	None	8.1	0.63[3]*	FE	[20] (b,d)
6.80	1.77	—	1.05	0.08	9.1	—	HF	[21] (e)
6.85	1.77	1.90	1.00	0.07	9.1	0.66[3]*	HF	[21] (a)
6.7	—	1.96	1.68	None	8.6	—	FE	[22] (b)
6.8±2[2]*	—	—	—	—	—	—	HF	[23] (a)
7.0	1.65	1.9	1.8	None	9.0	0.575	FE	[24] (b)
7.75±0.5	—	—	~1	—	9.5	—	VM	[25] (a)
HJ								
6.5±2	1.85	1.90	1.8	0.03	9.0	0.547	VM	[26, 27] (a,f)
6.7	—	—	0.36	None	13.9	—	VM	[28] (a)
6.0	—	—	—	None	7.9	—	GVM	[29] (a,f)
BKR								
6.2±2	1.87	2.09	1.4	None	7.8	—	FE	[30] (b,d)
6.34	1.76	1.96	1.46	None	7.95	—	FE	[31] (b,d)
6.2	1.82	1.97	0.50	0.05	8.8	—	HF	[21] (a)
SSC$_A$								
7.64	1.64	1.85	1.5	None	6.5	0.61	FE	[24] (b)
7.47	1.78	1.94	1.0	0.04	6.4	0.63[3]*	HF	[21] (e)
SSC$_B$								
7.71	1.62	1.82	1.4	None	6.3	0.62	FE	[24] (b)
7.64	1.78	1.96	1.1	0.03	6.2	0.62[3]*	HF	[21] (a)
EH								
7.5	—	—	1.2	0.05	9.0	0.68[3]*	HF	[32] (e)
GPT								
8.25	1.69	1.84	1.17	None	4.07	—	FE	[31] (b,d)
8.61			0.8	0.02	5.0	0.66[3]*	FE	[32] (a)
Experiment								
8.48	1.70±0.05	1.87±0.05	<4		5—10	0.764	—	—

1) Extrapolated value from 6.3 MeV
2) Extrapolated value from 6.26 MeV
3) Point nucleons

Notes:
a) all terms of NN potential taken into account;
b) allowance made for only $^1S_0$, $^3S_1 - {}^3D_1$, NN interaction;
c) total potential + one-meson-exchange potential in $^3D_3$ state;
d) neglect of D wave of the third particle relative to the center of mass of the first two;
e) odd potentials not allowed for;
f) slightly modified version of HJ potential.
Some results of Refs. 21 and 32 improved by increasing the HF basis.

tal angular momentum L and total spin S equal to 0 and 1/2, respectively. These two terms differ by the symmetry properties of the coordinate part of the wave function: Ψ_S is completely symmetric under permutations of the coordinates $\mathbf{r}_i$ of the three nucleons while $\Psi_{S'}$ (the so-called mixed-symmetry function) is not.

The third and the fourth terms in (9) correspond to the admixture of the P and D states to the wave function ($L=1$, $S=3/2$, $1/2$, and $L=2$, $S=3/2$, respectively). Besides the wave function $\Psi(1,2,3)$ one usually calculates the binding energy E_T of the ^{3}H nucleus, the Coulomb energy E_{Coul} in the ^{3}He nucleus, and the rms radii R_T and $R_{^3\mathrm{He}}$ of the ^{3}H and ^{3}He nuclei. The main results are given in several tables.

Table 2 collects the calculations for different forms of the NN potential and gives the global characterization of the ^{3}H wave function—the values of the $P(S')$, $P(P)$, and $P(D)$ contributions to the normalization integral of the S', P, and D states of the three-nucleon system.

The greatest number of calculations has been made with the Reid potential (RSC). As can be seen from Table 2, the various authors working with this potential obtained slightly different results. To some extent this is due to different simplifications and approximation introduced into the computational procedure, and the uncontrollable errors then arising. This is the situation with regard to Hennell and Delves's result for the binding energy of the ^{3}H nucleus: $E_T = 7.75 \pm 0.5$ MeV, which is not confirmed by the other calculations. A comparison of the results of the different authors shows that the average (with respect to the authors) error in the calculation of the wave function is several percent: the weight of the D state (~9%) is almost the same for all of them, whereas there are large discrepancies in the weight of the S' state (~1%), which is an order of magnitude less. Considering the results selected in the table, one cannot avoid pointing out the following:

a) For all types of realistic NN potentials the binding energy of ^{3}H is the same to within an error ±0.8 MeV and close to the experimental value (less than it by 1—2 MeV);

b) the rms radii are also obtained with fully satisfactory agreement with experiment;

c) the Coulomb energy of the two protons in ^{3}He, calculated under the assumption of charge invariance of the NN potential, is approximately 100 keV less than the experimental value and does not depend on the form of the NN potentials.

When comparing the calculated binding energies E_T with the experimental values, it must be remembered that it is the difference $E_T = |\bar{V}| - \bar{T}$ of two energies of large absolute magnitude: the potential energy ($|\bar{V}|$ ~50 MeV) and the kinetic energy ($\bar{T}$ ~40 MeV). An indeterminacy in these values of order 1 MeV leads to an indeterminacy in the binding energy of order 1—2 MeV. Thus, a correction to the wave function of 1—2% is sufficient to explain the discrepancy between the calculated E_T and the experimental ($|E_{T_{\mathrm{exp}}}| \approx 8.48$ MeV). This correction could be due to the NN interaction being slightly nonpotential or to many-particle forces.

It follows from a) and b) that the basic properties of the ^{3}H and ^{3}He nuclei are almost independent of the actual form of the NN potential, since they describe well the two-nucleon data. This conclusion is underlined if one considers the contributions of the different parts of the Hamiltonian to E_T. In Table 3 we give the corresponding values for the GPT and EH potentials. The contributions of the terms of different type to E_T

TABLE 3. Averaged values of the kinetic energy T and the components of the NN potential for $^3\mathrm{H}(\psi_{L=1}=0)$ and ^{4}He.

$(\Psi \mid \ldots \mid \Psi)$	^{3}H GPT	^{3}H EH	^{4}He GPT	^{4}He EH
T	29.48	42.97	57.64	77.95
$V^c_{31} + V^c_{13}$	−27.90	−22.85	−66.74	−58.00
V^t_1	−9.61	−24.53	−17.20	−40.67
$V^{(LS)^2}_{31} + V^{(LS)^2}_{11}$	−0.31	—	−0.46	—
V^{LS}_1	−0.18	−2.78	−0.25	−2.80
$V^c_{33} + V^c_{11}$	0.02	0.08	0.11	0.42
V^t_3	−0.1	−0.20	−0.34	−0.36
$V^{(LS)^2}_{33} + V^{(LS)^2}_{13}$	0.02	—	0.06	—
V^{LS}_3	0.07	0.02	0.40	0.44
E	−8.57	−7.28	−26.76	−23.03

TABLE 4. Overlapping of the symmetry components of the wave functions (%).

Potential	SW	RSC	BKR	SSC_B	SSC_A
SW	100	—	—	—	—
RSC	85	100	—	—	—
BKR	87	97	100	—	—
SSC_B	91	93	94	100	—
SSC_A	91	93	94	98	100

Note: SW is a simple potential of rectangular form

for these potentials differ strongly from one another, but in the final result, in which all contributions are added, we obtain practically the same result. In other words, if the properties of the ^{3}H nucleus are calculated by means of different NN potentials, it is not the actual form of the potentials which is important but what they have in common. A common property of the NN potentials is their ability to describe equally well the two-nucleon data at not too high energies. This conclusion is also confirmed if we compare the dominant components of the wave function Ψ_S calculated for these potentials. As a numerical comparison measure it is convenient to introduce the "overlap" β:

$$\beta^{(1,2)} \equiv (\Psi_S^{(1)}, \Psi_S^{(2)}) \equiv \int \Psi_S^{(1)} \Psi_S^{(2)} \, dV \Big/ \int |\Psi_S^{(1)}|^2 \, dV. \qquad (10)$$

The departure of β from unity indicates how strongly the wave functions calculated with different NN potentials differ. Some values are given in Table 4, from which it follows that the total wave functions and their individual components are close to one another. The difference is a few percent, i.e., the wave function of the nucleus is also weakly dependent on the actual form of the NN potential. It appears all the more remarkable that the difference between the binding energies of the ^{3}H and ^{3}He nuclei, which is usually regarded as the average value E_{Coul} of the Coulomb interaction energy of the two protons in ^{3}He, is smaller than the experimental value. This is obtained in all cases when charge-invariant nuclear forces are used for the calculation. Such a discrepancy between calculation and experiment indicates that the NN potentials contain charge-noninvariant terms. Theoretical estimates of such terms[33] give the correct order of magnitude of the effect.

Above, we have discussed calculations made comparatively recently with realistic NN potentials. In the earlier calculations (see, for example, Ref. 8) very simple potentials were used (central potentials of rectangular, exponential, or Yukawa form), which give the correct NN scattering lengths. Of these potentials, only the rectangular reproduce the experimental data fairly well. All the others lead to a too strong coupling of three and, in particular, four nucleons. Therefore, the weak dependence of the ^{3}H and ^{3}He properties on the actual form of the NN potentials discussed above should not be taken too literally: The dependence is weak if the potentials are realistic, i.e., they reproduce the NN phase shifts in a sufficiently wide range of energies (for this one requires effective repulsion at short distances). In the case of potentials of exponential, Yukawa, or Gaussian form, conversely, the attraction increases when the distance between the nucleons decreases. As a result, in three- and even more so in four- and many-nucleon systems the coupling is found to be stronger than follows from the experiments. For a potential of rectangular form, this effect is not so strongly expressed. The binding energy is only 1—2 MeV higher than the experimental value, and the ^{3}H wave function is near (with an error of 10%) the one calculated with realistic potentials.

Electric Form Factors of ^{3}H *and* ^{3}He. The general conclusion that the ^{3}H and ^{3}He wave functions depend weakly on the form of the NN potential is also confirmed by the data on the electric form factors $F_e(q)$, which describe electron scattering on nuclei and are defined by

$$F_e(q) = \int \exp(i\mathbf{qr}) \, \rho_e(\mathbf{r}) \, d\mathbf{r}, \qquad (11)$$

where $\rho_e(r)$ is the charge density in the nucleus. Experimentally, $F_e(q)$ is known accurately for ^{3}He but poorly for ^{3}H. Therefore, we shall here discuss only ^{3}He. The experimental data and the theoretical form factors calculated for different NN potentials are shown in Fig. 4. It should be pointed out that the form factor has frequently been calculated by different authors. However, the results have not always agreed for $q^2 \gtrsim 10$ F^{-2}. In Fig. 4 we have given the data of Ref. 34, which are apparently the most accurate.

It is important that for not too large q^2 the form factor is almost independent of the form of the NN potential and describes the experiments well. Differences begin to appear only at $q^2 \approx 7$—8 F^{-2}, where $|F_e(q)|$ is small. A characteristic feature of the form factor is a minimum at $q^2 = 11.6$ F^{-2}. The position of the minimum and the

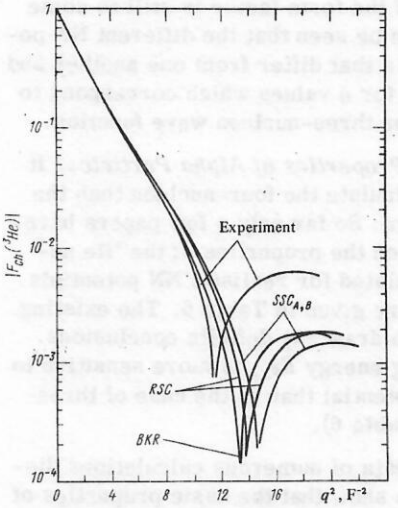


FIG. 4. Electric form factor of ^{3}He for the following NN potentials: BKR, RSC, and $SSC_{A;B}$. Experimental points from Ref. 35.

TABLE 5. Position of the diffraction minimum and height of the second maximum in $F_e(q^2)$ for realistic potentials.

Potential	^{3}He		^{3}H		Literature
	q^2_{min}, F^{-2}	$\lvert F_e(q^2_{max})\rvert 10^3$	q^2_{min}, F^{-2}	$\lvert F_e(q^2_{max})\rvert 10^3$	
HJ	12.5±0.3	1.0	13.4±0.3	1.3	[26]
RSC	13.3*	2.0	13.7	2.0	[19]
	12.8±0.3	—	—	—	[25]
	13.8	1.9	15.0	2.5	[34]
	14.0	2.0	—	—	[24]
	15.5	0.7	—	—	[22]
	15.5	—	—	—	[31]
	17.0	—	—	—	[69]
BKR	13.2	1.9	14.6	2.4	[34]
	14.4	1.0	15.6	1.0	[30]
	14.5	—	—	—	[31]
SSC_A	14.6	1.4	15.6	1.8	[34]
	14.8	—	—	—	[24]
SSC_B	14.7	1.3	15.7	1.7	[34]
	15.0	—	—	—	[24]
SSC_C	14.4	—	—	—	[24]
Experiments	11.6	6.0	—	—	[35]

* This value obtained by the authors after the introduction of some corrections (private communication by Jackson.)

further dependence of the form factor cannot be described for any of the NN potentials hitherto used.

The results of different investigations in which the position of the minimum q^2_{min} and the height of the following maximum of the function $|F_e(q^2)|$ have been calculated are given in Table 5. Not all the values given in this table are reliable. A general tendency is, however, clear: All calculations give a value that is too high for q^2_{min} and predict a too small value of the subsequent maximum in $|F_e(q^2)|$. It may be that the behavior of the form factor at large q^2 (distances ~0.3 F correspond to q^2 ~10 F^{-2}) lie outside the scope of the nonrelativistic potential approach. However, estimates of the corrections to the theoretical form factor due to some relativistic effects (see Ref. 75) and a possible influence of three-particle forces[26] only worsen the situation. Although the question of the form factor is still to some extent obscure, it can be seen that the different NN potentials lead to results that differ from one another and the experiments only for q values which correspond to very fine details of the three-nucleon wave function.

Calculation of the Properties of Alpha Particles. It is much harder to calculate the four-nucleon than the three-nucleon problem. So far only a few papers have been published in which the properties of the ^{4}He nucleus have been calculated for realistic NN potentials. Some of the results are given in Table 6. The existing data are too sparse to draw any definite conclusions. Evidently, the binding energy $E_{^4He}$ is more sensitive to the form of the NN potential than in the case of three-nucleon nuclei (see Table 6).

Conclusions. The data of numerous calculations discussed in this section show that the basic properties of the ^{3}H, ^{3}He, and, to a lesser extent, ^{4}He nuclei depend weakly on the actual form of the NN potential, since this last describes the two-nucleon data well. In the first place, this weak dependence on the NN potential refers to the nuclear wave function, which, to within an error of a few percent, is the same for all the NN potentials used. The binding energies E_T and $E_{^3He}$, which are the difference of two large values—the kinetic energy and potential energy with reversed sign—are somewhat more sensitive.

From the formal point of view the weak dependence of the ^{3}H, ^{3}He, and ^{4}He properties on the actual form of the NN potential means that it is the behavior of the two-nucleon scattering amplitudes on the mass shell which is decisive. The main properties of the lightest nuclei are found to depend weakly on the behavior of this amplitude for unphysical momenta.

On the one hand, this is very agreeable, since it means that the wave functions of the lightest nuclei can now be calculated with an error of a few percent. On the other hand, it means that from the known basic properties of the lighest nuclei one cannot extract much additional information about the form of the NN potential. Thus, the only new property of the NN potential to be reliably revealed by the calculations is the presence of charge-noninvariant terms in the potential of the internucleon interaction.

One can assume that in all nuclei heavier than ^{4}He in which there is a spectrum of excited states, the order of the levels and the distances between them must depend strongly on the actual form of the NN potential and, in particular, on the relative weights of the different components of the potential (central, LS, L^2, etc.). Among the levels with different quantum numbers, one can find pairs of levels for which their energy difference is determined mainly by just one component of the nuclear forces (just as the difference of the ^{3}H and ^{3}He binding energies depends on only the Coulomb interaction and the charge-noninvariant components of the potential). As yet, detailed calculations of the properties of nuclei with $A > 4$ made by means of nonrelativistic NN potentials have not appeared in the literature.

3. CALCULATIONS OF THE AMPLITUDES OF THE SIMPLEST NUCLEAR REACTIONS

The first successful calculation of a nuclear reaction cross section obtained by solving the many-nucleon Schrödinger equation was published in 1956.[1] The reaction was very simple (neutron scattering on the deuteron at low energies) and the calculation itself greatly simplified by assuming a zero range of the nuclear forces.

TABLE 6. Properties of ^{4}He with realistic NN potentials.

Potential	$\lvert E\rvert$, MeV	P, %		Method	Literature
		S'	D		
EH	23.04	0.2	9.5	HF	[32] (a)
GPT	26.76	0.1	6.0	HF	[32] (a)
	26.64	—	—	Faddeev-Brueckner-Hartree-Fock	[71]
SSC_A	25.2	—	—		[72]
RSC	20.8	—	—		[72]
HJ	19.0	—	—	GVM	[29]

(a) see the note of Table 2.

Some years later the same process was calculated by means of the Faddeev equations, first for only nonlocal, so-called "separable" interaction potentials between the nucleons (their use greatly simplifies the calculation, since in general one integration is eliminated), and then, toward the end of the sixties, for nonrelativistic local NN potentials as well.

At the same time, calculations were made of more complicated reactions in a system of three, four, five, or more nucleons.[36-40] These calculations were based on variational principles developed for continuum problems[41] (Hulthén-Cohn principle, etc.). Although the computational methods used by different authors differ in details, the general approach is always the same. We describe the basic ideas below. A more detailed description of the various methods[36,42-45] can be found in Appendices 2 and 3.

Of great practical importance is the investigation of the asymptotic behavior of three-particle wave functions made by Merkur'ev (see Ref. 76 and the references there).

General description of variational methods of calculating nuclear-reaction amplitudes. Suppose we wish to consider the reaction

$$a_i + X_i \to a_j + X_j \quad (i, j = 1, 2, \ldots, N),$$

in a system of A nucleons. We shall assume that the energy is not too high, so that only the N binary channels $a_i + X_i$ $(i = 1, 2, \ldots, N)$ are open, and the remaining binary channels and channels of decay into three, four, etc., particles are closed. We denote by a_i and X_i the pairs of particles forming channel i. We assume that the internal wave functions of the particles a_i and X_i are known and are φ_{i1} and φ_{i2}, respectively.

In order to calculate the reaction cross section, we must solve the Schrödinger equation for this system of A nucleons:

$$(\hat{H} - E)\,\Psi = 0. \tag{12}$$

Here $\hat{H}$ is the total Hamiltonian, and on the wave function Ψ one must impose the usual conditions of finiteness for all coordinates of the nucleons. Altogether, there are N independent solutions $\Psi^{(i)}$ $(i = 1, \ldots, N)$, in complete correspondence to the N free parameters. These parameters are the amplitudes of the ingoing waves in the N open channels.

The general form of Ψ is known (see Appendix 2):

$$\Psi = \Psi_{\text{int}} + \Psi_{\text{ext}}. \tag{13}$$

Here, the second term describes the relative motion of the particles in the open channels and it can be written as $(i = 1, \ldots, N)$

$$\Psi^{(i)}_{\text{ext}} = \hat{A}\Big\{F_i(R_i)\,\varphi_{i1}\varphi_{i2}\mathcal{Y}_{L_iM_i}(\Omega_{R_i}) + \sum_{j=1}^{N} f_{ji} G_j(R_j)\,\varphi_{j1}\varphi_{j2}\mathcal{Y}_{L_jM_j}(\Omega_{R_j})\Big\}, \tag{14}$$

where F_i and G_i are functions which describe the relative motion (free or in a Coulomb field if the particles are charged) of the particles a_i and X_i. For these functions, it is convenient to choose the regular and irregular solution of the free (or Coulomb) equation of motion of two particles; for example, in the case of free motion

$$\left.\begin{aligned} F_i(R_i) &\sim \sin(k_iR_i - L_i\pi/2)/R_i; \\ G_i(R_i) &\sim \cos(k_iR_i - L_i\pi/2)/R_i, \end{aligned}\right\} \quad R_i \to \infty. \tag{15}$$

Here, k_i is the wave vector of the relative motion of the pair $a_i + X_i$ at infinity and R_i is the distance between a_i and X_i. The irregular solution of the free (Coulomb) equation becomes infinite as $R_i \to 0$, and it is therefore multiplied by a regularizing factor, for example, by

$$T_L(R) = 1 - \Big(\sum_{\nu=0}^{2L} \frac{(\beta R)^\nu}{\nu!}\Big)\exp[-\beta R], \tag{16}$$

which is unity everywhere except at small R, where it tends to zero in such a way as to suppress the singularity of the irregular function of free (Coulomb) motion. The wave function must be antisymmetric with respect to permutations of the coordinates of the nucleons. This is ensured by introducing the antisymmetrization operator $\hat{A}$ in (14). The matrix of constants f_{ji} is not determined in a general form and must be found from the solution of the equations of motion (see below).

With regard to $\Psi^{(i)}_{\text{ext}}$ in the form (14) one can say that (for arbitrary f_{ji}) it is an exact solution of the Schrödinger equation (2) in the (asymptotic) region of the configuration space in which the particles a_i and X_i in all the channels have moved apart to a distance greater than the range of the nuclear forces.[1)] In addition, $\Psi^{(i)}_{\text{ext}}$ is everywhere finite and satisfies the Pauli principle.

When all A nucleons of the system are close to one another and, accordingly, all the R_i are small, the function (14) no longer satisfies the Schrödinger equation (12), since (14) does not take into account the interaction between nucleons in different particles of the channels. To rectify this, one introduces into the expression (13) for the wave function an additional term $\Psi^{(i)}_{\text{int}}$, chosen such that it is nonzero only when all A nucleons are sufficiently close together; in the asymptotic region, $\Psi^{(i)}_{\text{int}}$ must vanish fairly rapidly. One can, for example, specify $\Psi^{(i)}_{\text{int}}$ in the form of an expansion with respect to the states of the A nucleons in the field of an harmonic oscillator:

$$\Psi^{(i)}_{\text{int}} = \sum_{\mu=1}^{M} C^{(i)}_{\mu}\chi_{\mu}(1, 2, \ldots, A), \tag{17}$$

where $C^{(i)}_{\mu}$ are certain coefficients; χ_μ is a corresponding set of orthonormalized functions. Other choices are possible.

The expression constructed in this way for the required wave function

$$\Psi^{(i)} = \sum_{\mu=1}^{M} C^{(i)}_{\mu}\chi_{\mu} + \hat{A}\Big\{F_i\varphi_{i1}\varphi_{i2}\mathcal{Y}_{L_iM_i} + \sum_{j=1}^{N} f_{ji}G_j\varphi_{j1}\varphi_{j2}\mathcal{Y}_{L_jM_j}\Big\} \tag{18}$$

is not of course an exact solution of the Schrödinger equation (12), though it does have the correct general form. The unknown constants C_μ and f_{ji} in (18) must be found from the condition that (18) be the best approximation to the exact solution $\Psi^{(i)}_{\text{exact}}$.

Depending on what one regards as the "best approximation," the conditions differ. But if one stays consis-

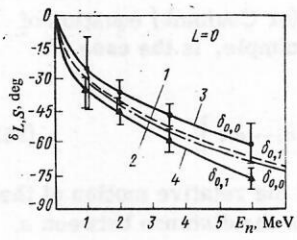


FIG. 5. Dependence of the $(n+{}^3\mathrm{H})$ phase shifts $\delta_{L,S}$ on the energy E_n of the incident neutron for $L=0$: 1) and 2) results of calculation; 3) and 4) phase-shift analysis of experiment.

tently with the variational approach, then for the scheme sketched above it leads to the following system of equations for C_μ and $f_{j_1 i}$:

$$\left.\begin{aligned} &\int d\tau \chi^*_{\mu_1}[\hat{H}-E]\,\Psi^{(i)}=0, \quad \mu_1=1,\ 2,\ \ldots,\ M;\\ &\int d\tau\,\{G_{j_1 l_1}\varphi_{j_1 1}\varphi_{j_1 2}\mathcal{Y}_{L_{j_1}M_{j_1}}\}^*[\hat{H}-E]\,\Psi^{(i)}=0, \quad j_1=1,\ 2,\ \ldots,\ N. \end{aligned}\right\} \quad (19)$$

Here, integration over the coordinates of all the nucleons and summation over their spins and isospins is understood. The total number of equations $(MN+N^2)$ is equal to the number of unknown constants C_μ and $f_{j_1 i}$.

We denote the constants found from (19) by $\bar{C}_\mu$ and $\bar{f}_{j_1 i}$. These solutions can be improved by using Kato's many-channel identity (see Appendix 2); we then obtain a simplified expression for $f_{j_1 i}$:

$$f_{j_1 i}=\bar{f}_{j_1 i}-2\int d\tau \Psi^{(j)}[\hat{H}-E]\,\Psi^{(i)} \quad (20)$$

The correction term (the second) in this expression is frequently called the variational correction. Using the matrix $\hat{f}$ of the coefficients $f_{j_1 i}$, we can easily find the $\hat{S}$ matrix as well:

$$\hat{S}=(1+i\hat{f})(1-i\hat{f})^{-1}. \quad (21)$$

The amplitudes of the different reactions in the considered system of A nucleons are expressed in the usual manner in terms of the elements of the $\hat{S}$ matrix.

Above, we have outlined the general approach used to calculate reaction cross sections, in the simplest variant. Some more precise details can be found in the Appendices.

In the main text we shall now leave aside the technical side of the calculations and concentrate on analyzing the results which are obtained.

Calculations with the simplest potentials. In the first calculations, the simplest processes were considered: elastic scattering of neutrons on the deuteron, triton, and α particle. The doublet length, a_2, and quartet length, a_4, of *nd* scattering have been calculated by many authors using the Faddeev equation. It should be noted particularly that the early calculations[4] were made when the scattering lengths a_2 and a_4 were not known from the experiments because of the uncertainties of the phase-shift analysis. The calculations led to the conclusion that $a_4 \approx 6$ F and that a_2 is much smaller, and this was subsequently confirmed experimentally.

If the number A of nucleons is greater than three, the Faddeev-Yakubovskiĭ equations cannot in practice be used. All the results for the amplitudes of reactions possible in systems of four, five, and more nucleons discussed below were obtained with one or other variants of the variational method.

This method was tested for the first time in a calculation of the scattering lengths a_2 and a_4 for an *nd* collision.[36] The result was hopeful and in Refs. 37 and 38 the same method was used to calculate the s and p phase shifts of $n+{}^3\mathrm{H}$ scattering (Fig. 5) and the p phase shift of $n+{}^4\mathrm{He}$ scattering (Fig. 6). It can be seen that the calculation correctly reproduces the behavior of the singlet and triplet s phase shifts of $n+{}^3\mathrm{H}$ scattering and the resonance behavior of the p phase shift in $n+{}^4\mathrm{He}$ scattering. In these calculations it was assumed that the interaction between the nucleons is purely central and contains only two components—singlet V^c_{13} and triplet V^c_{31} [see Eq. (1)]—of rectangular form. The parameters of these potentials were chosen so as to reproduce the NN scattering cross section at low energies. This assumption about the form of the NN potential is very crude in at least two respects: First, the potentials have a very simple form which, in particular, does not take into account the repulsion of nucleons at short distances and, second, important components of the interaction such as the tensor and spin-orbit components are taken to be zero.

The last fact may not be important in the calculation of the s phase shifts of $n+{}^3\mathrm{H}$ scattering. The qualitative effect of the LS and the tensor forces between nucleons is the appearance of effective LS forces in the interaction between a nucleon and the nucleus, and in the s states these forces are zero. However, the effective LS force must be taken into account in the calculation of the p phase shifts in $n+{}^4\mathrm{He}$ scattering. It is well known that these forces here acquire a particular importance and are manifested through the phase shifts behaving differently in the states with angular momenta $j=3/2$ and $1/2$ (see Fig. 6). In the calculation of Ref. 38 discussed earlier, no effective LS forces arise and the phase shifts $\delta_{3/2}$ and $\delta_{1/2}$ in the two states are the same. When the calculated p phase shift is compared with experiment, it lies in the region between the experimental values of the phase shifts $\delta_{3/2}$ and $\delta_{1/2}$ (see Fig. 6) and thus descibes their behavior on the average.

With regard to the simple rectangular form of the potentials used in Refs. 37 and 38, this is not such a bad approximation as it seems at the first glance; for in calculations of variational type such as we consider here the interaction between the nucleons enters in several ways. As can be seen from Eqs. (A.33) and

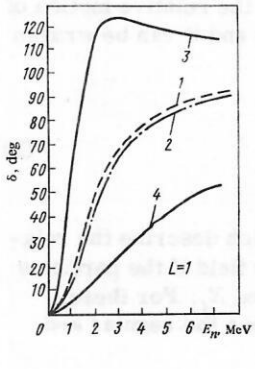


FIG. 6. Dependence of the $n+{}^4\mathrm{He}$ phase shifts on the energy E_n of the incident neutron for $L=1$: 1) and 2) results of calculation for different formulations of the variational principle; 3) phase shift for $J=3/2$; 4) phase shift for $J=1/2$.

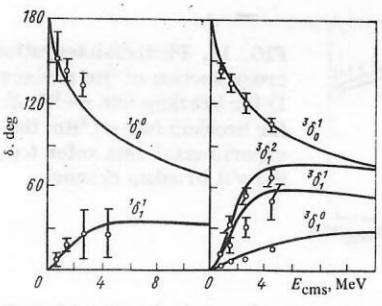


FIG. 7. Dependence of the elastic $n+^3$H phase shifts $^{2S+1}\delta_L^J$ on the cms energy: the continuous curves are calculated.

(A.35) of Appendix 3, the equations for determining the scattering amplitudes contain the NN potential a) through the internal wave functions of the particles [^{3}H for $n+^3$H scattering and ^{4}He for $n+^4$He scattering]; b) through the binding energies of these particles; c) through the effective potential between the neutron and the nucleus. Now the wave functions and binding energies of the lightest nuclei can be perfectly well described by means of rectangular NN potentials provided they are chosen so as to correctly describe NN scattering at low energies. Therefore, points a) and b) need not disturb us at all. The only quantity in Eqs. (19) containing the NN potential directly is the effective interaction between the neutron and the nucleus. But the equations from which this interaction is determined are integral: In them, the NN potential is averaged over the wave functions of the particles and the result of averaging does not depend too strongly on the actual form of the potential. Therefore, the fact that the crude approximations of the quoted papers are in good qualitative agreement with experiment is not surprising. One would expect this in the variational calculation. The only question is, how well does the variational method work? Although one can never be absolutely certain about the accuracy of the results obtained by a variational calculation, some idea of the accuracy can be obtained by calculating the variational correction to the $\hat{S}$ matrix. The calculations in Refs. 37 and 38 were made for energies at which the variational correction is small, so that the calculated function is near the exact one.

The most important qualitative result of the calculations with the simplest potentials is the clarification of the tremendous importance which the Pauli principle acquires in the dynamics of nuclear processes. In some states of the relative motion of the nucleon and nucleus at not too high energies, it is the Pauli principle which prevents the nucleon's penetrating into the nucleus, imitating the strong repulsion in these states. For example, it is such an effect as this which explains the small and negative s phase shift of $n+^4$He scattering and the negative s phase shifts in $n+^3$H scattering.

Calculations with realistic potentials. The last years have seen a whole series of investigations in which more realistic interaction potentials between nucleons have been used to calculate reactions. Below we give the main results.

$n+^3$H *Scattering.*[47] The interaction potential between the nucleons was chosen in the form of the sum of a central $\hat{V}^c$, spin-orbit $\hat{V}^{LS}$, tensor $\hat{V}^t$, and Coulomb potential:

$$\hat{V}=\hat{V}^c+\hat{V}^{LS}+\hat{V}^t+\hat{V}_{\text{Coul}}; \tag{22}$$

the potential $\hat{V}^c$ being taken from Ref. 48, and $\hat{V}^t$ and $\hat{V}^{LS}$ from Ref. 14. Taking the different components of the potential from different sources is of course hardly consistent, and we shall return to this question.

The interaction (22) was used to calculate the phase shifts of nT scattering for all the lowest partial states up to $J=4^-$. Figure 7 compares the calculated lowest phase shifts $^{2S+1}\delta_L^J$ with the experiments. The differential cross section was also calculated at several energies. The agreement between calculation and experiment can be seen in Fig. 8.

$p+^3$He *Scattering.*[49] The same form of the NN potential as in the foregoing calculation was used. The wave function was again sought in the form of the sum $\Psi_{\text{int}}+\Psi_{\text{ext}}$, in which the second term describes the $p+^3$He channel and the first the behavior of the system in the region of configuration space in which all the four nucleons are close to one another and form a kind of compound nucleus. The phase shifts were calculated for all the lowest partial states up to $J=4^-$, together with the differential cross sections and the polarization of the scattered protons. Some typical results of the calculations are shown in Figs. 9 and 10. It can be seen that the experimental data agree satisfactorily with the experiments.

Photodisintegration of ^{4}He. Comparison of the theoretical and experimental phase shifts, angular distributions, and polarization in $n+^3$H and $p+^3$He scattering serves as a test of the correct calculation of the asymptotic part of the corresponding wave functions. To test the accuracy of the calculation of the wave function in the "interior" region, where all the nucleons are close to one another, one must use the physical processes which are basically determined by the form of the wave function in the interior region. One such process is the photodisintegration of ^{4}He: $^4\text{He}(\gamma,n)^3\text{He}$ and $^4\text{He}(\gamma,p)T$. The cross sections of these reactions were calculated in Ref. 50 using the wave functions of the $n+^3$He and $p+^3$H systems calculated in Ref. 51. The results of the calculations are given in Fig. 11; 1) refers to the reaction $^4\text{He}(\gamma,p)^3\text{H}$, 2) to $^4\text{He}(\gamma,n)^3\text{He}$. It can be seen that the agreement between theory and experiment is fairly good.

$n(\alpha,\alpha)n$, $n(\alpha,T)D$, $D(T,T)D$ *Reactions.*[40] The NN interaction was taken somewhat different from the one

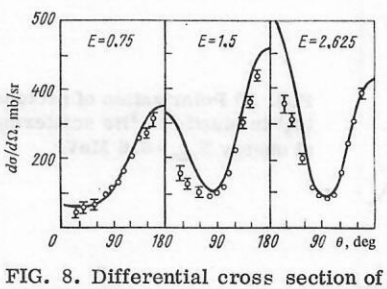


FIG. 8. Differential cross section of $n+^3$H scattering for the energies $E_{\text{cms}}=0.75$, 1.5, and 2.65 MeV.

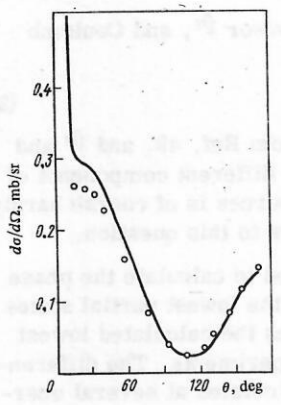


FIG. 9. Differential cross section of $p+{}^3$He scattering at the energy $E_{cms}=6.6$ MeV.

used to calculate $n+{}^3$H scattering. In fact, the resonating-group method was used, since in the expression for the wave function (13) the first term Ψ_{int} was assumed equal to zero. In the considered case of the system of five nucleons there are two channels $(n+\alpha)$ and $(D+T)$. The mass difference of these channels calculated by means of the NN potential and wave functions of the particles of the $(\alpha, D, {}^3\text{H})$ channels used in the calculation was 20 MeV (experimentally, 17.6 MeV). The phase shifts in both channels and the cross section of the transition from channel to channel were calculated. The agreement between theory and experiment was perfectly satisfactory (Figs. 12 and 13). The figures show the dependence of the calculated (continuous curves) phase shifts δ_J of $n-\alpha$ scattering and the corresponding experimental curves (dashed curves). The calculation reproduces the resonance of the p phase shift $(J=3/2^-)$ identified with the ground state of the quasistationary ^{5}He nucleus and the resonance of the d phase shift $(J=3/2^+)$ corresponding to the well known "thermonuclear" resonance in the $D+{}^3$H cross section. The position of these resonances [with respect to the thresholds of the $(n+\alpha)$ and $(D+{}^3\text{H})$ channels, respectively] and their widths are close to the experimental values.

It is important to note two things:

1) The resonance with $J=3/2^+$ in the channel $D+{}^3$H is manifested in the partial state with $L=0$, $S=3/2$, and in the channel $n+\alpha$ in the state with $L=2$, $S=1/2$. Thus, a transition from channel to channel can occur only with a change of the total orbital angular momentum L and the total spin S of the system of five nucleons.

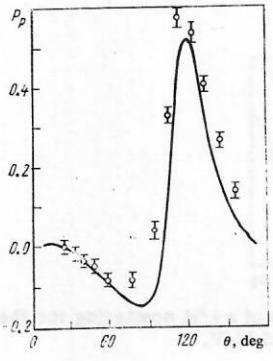


FIG. 10 Polarization of protons (P_p) in elastic $p+{}^3$He scattering at energy $E_{cms}=6.6$ MeV.

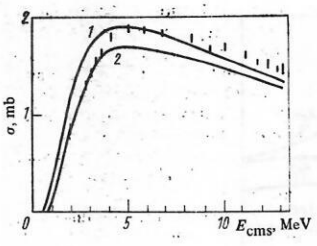


FIG. 11. Photodisintegration cross section of ^{4}He nucleus: 1) for breakup into $p+{}^3$H; 2) for breakup into $n+{}^3$He; the experimental data refer to the $p{}^3$H breakup channel.

A transition of this type can arise only because tensor forces are present in the NN potential. Therefore, a comparison of theory and experiment in the region of this resonance enables one to obtain directly additional information about the behavior of the tensor component of the NN potential. For this, more detailed calculations must be made.

2) In the region of energies near the resonance $J=3/2^+$, the two channels are strongly coupled (in the partial state) and this is perfectly understandable. But it turns out that such a coupling must also be taken into account at much lower energies. In Ref. 39, in which the phase shifts of $p+\alpha$ and $D+{}^3$He scattering (mirror system to that considered) were calculated under the same assumptions as above, the result for the phase shift of $p+\alpha$ scattering $(L=2,\ J=3/2^+)$ was compared with the same phase shift but calculated in the one-channel approximation in which the $D+{}^3$He channel is ignored from the very beginning. The corresponding curves are shown in Fig. 14, from which it can be seen that allowance for the second channel greatly improves the agreement between theory and experiment in a wide range of energies below the threshold of the $D+{}^3$He channel. At the same time, at higher energies the influence of the channels on one another is small, and the phase shifts of $D+{}^3$H scattering can be calculated in the one-channel approximation.

$D+\alpha$ Scattering.[52] In the interaction between the nucleons the spin-orbit and tensor components were ignored and the central part of the potential was taken from Ref. 53. No other channels except the $D+\alpha$ channel were included in the calculation, but at the same time an investigation was made of the sensitivity of the calculated s phase shift of $D+\alpha$ scattering to the form of the part Ψ_{int} of the wave function Ψ which describes the behavior of the nucleons within the compound nucleus. The main results of the calculations are shown in Fig. 15, in which the phase shifts are given for Ψ_{int}

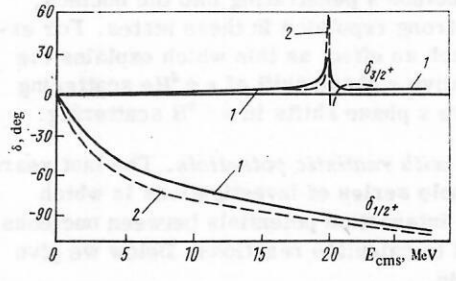


FIG. 12. Phase shifts δ_J for $n+{}^4$He scattering: 1) calculation; 2) experiment; 3) for experimental phase shift $J=3/2^+$ shifted with respect to the calculated threshold of $D{}^3$H formation.

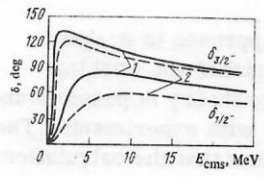


FIG. 13. Phase shifts δ_J for $n+{}^4$He scattering; 1) calculation; 2) experiment.

$\equiv 0$ and $\Psi_{\text{int}} \neq 0$. It can be seen that the introduction into the wave function of the term Ψ_{int}, which describes the behavior of the nucleons at short distances, appreciably improves the agreement between theory and experiment.

$\alpha+\alpha$ Scattering. In Ref. 54, the s, d, and g phase shifts of $\alpha+\alpha$ scattering were calculated in the one-channel ($\alpha+\alpha$) and two-channel approximations by the resonating-group method. The channel $\alpha+\alpha^*$, in which one of the α particles is in an excited 0^+ state, was taken as a second channel. The interaction between the nucleons was assumed central, with allowance for repulsion at short distances. The parameters of the NN potential were taken from Ref. 48. The calculated phase shifts are given in Fig. 16. The agreement with the experiment is good.

Conclusions. As yet, only a few nuclear-reaction amplitudes have been calculated by solving the many-nucleon Schrödinger equation. The majority of these have been described above. Although the material for analysis is as yet sparse, some preliminary conclusions can nevertheless be drawn.

1. In all the calculations described, the NN potential was specified in some form at the very beginning and then not varied. Several NN potentials were used, all of them describing fairly well NN scattering at low energies. Without exception, good qualitative agreement between the calculations and experiments was obtained. This is apparently due to the fact that the main features of the wave function of a system of nucleons, like bound states, depend comparatively weakly on the characteristic form of the NN potential provided it agrees with the NN-scattering data. Of course, this conclusion is only true to a certain extent, if for no other reason than that the amplitudes of many processes depend directly on the individual components of the NN potential. For example, the amplitude of the $n(\alpha, D)T$ reaction in the region of the resonance with

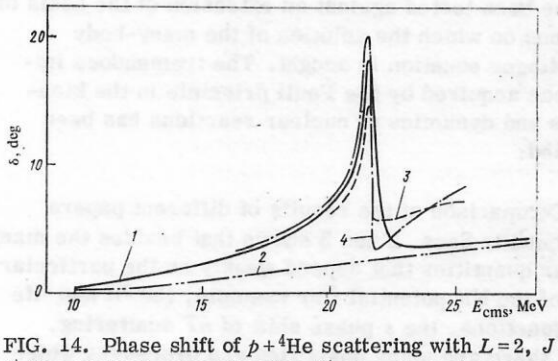


FIG. 14. Phase shift of $p+{}^4$He scattering with $L=2$, $J=3/2^+$; 1) two-channel approximation; 2) one-channel approximation; 3) and 4) experiment; experimental curves shifted toward the calculated threshold.

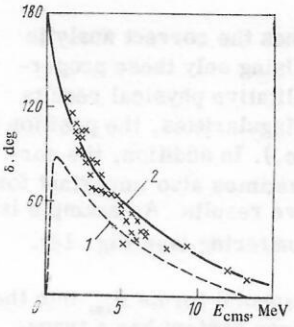


FIG. 15. Phase shift of $D+\alpha$ scattering with $L=0$: 1) without Ψ_{int}; 2) with Ψ_{int}.

$J=3/2^+$ depends on only the tensor component of the NN potential, and cannot be calculated if this is ignored. This example is very important, since it shows that there are nuclear processes from which one can directly obtain additional information about the different components of the NN potential. This information is essential for the unambiguous determination of the interaction of nonrelativistic nucleons with one another. The cross sections of the $n(\alpha, D)T$ reactions can serve as a source of information, together with finer details of the reactions such as the polarization.

2. As we have already pointed out, the interaction between nucleons enters the calculation of the wave function of a many-nucleon system and the different amplitudes both directly and indirectly: directly, through the effective interaction between the particles of the channels; indirectly, through the wave functions of the particles of the channels and their binding energies. Therefore, many properties of the nuclear forces can be taken into account phenomenologically by substituting into the equations the experimental values of the masses of the particles in the channels and using known properties of the particles such as, for example, their radii to construct their interior wave functions.

This is very important; the point is that the system of equations for finding the wave function has singularities at the thresholds of the different channels. Therefore, when we specify the position of the thresholds we also specify the correct position of the singularities in the

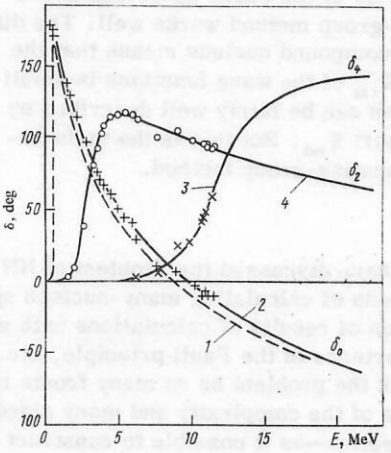


FIG. 16. Phase shifts δ_L of elastic $\alpha\alpha$ scattering; 1) one-channel approximation; 2), 3), and 4) two-channel approximation.

equations, which predetermines the correct analytic properties of the solutions. Using only these properties, we can obtain many qualitative physical results (the nature of the threshold singularities, the position of possible threshold states, etc.). In addition, the correct analytic behavior is sometimes also important for finding the correct quantitative results. An example is the d phase shift of $p + {}^4\mathrm{He}$ scattering (see Fig. 14).

3. The introduction of the special terms Ψ_{int} into the wave function of a many-nucleon system has a transparent physical meaning. These terms describe the formation of a compound nucleus when particles collide, i.e., a stage of the process in which all the nucleons are close together. From a purely formal point of view, it is necessary to introduce the terms Ψ_{int}. It is known that in some cases the resonating-group method works well (in this method, $\Psi_{\mathrm{int}} \equiv 0$). For example, in $\alpha\alpha$ scattering it is only necessary to take into account Ψ_{int} from an energy of about 15 MeV of the colliding α particles.

From the point of view of physics, there is nothing remarkable here: It is known experimentally that besides processes which proceed through a compound nucleus there are others, the so-called direct processes, in which the reaction results from a single "contact" without the formation of a compound nucleus. But how can all this be explained from the formal point of view? An exhaustive explanation hardly exists, since one can think up various reasons why it is difficult for a compound nucleus to be formed in one and the same process. For example, one could invoke the similarity of the nucleon structure in the initial and the final state. It has been shown that the Pauli principle acquires great importance. The requirement of antisymmetry of the wave function is very stringent if all the nucleons of the system are close together. Only some configurations of the nucleons are allowed, and if the colliding particles do not have too high energies or orbital angular momenta it is as a rule difficult for them to penetrate one another. There arises an effective repulsion between the particles of the channel,[56] which makes the formation of a compound nucleus difficult. It is this that happens in the majority of the cases described above when the resonating-group method works well. The difficulty of forming a compound nucleus means that the corresponding part Ψ_{int} of the wave functions is small and the whole function can be fairly well described by only its "exterior" part Ψ_{ext}. But this is the main assumption of the resonating-group method.

CONCLUSIONS

In this review we have discussed the problem of NN potentials, the methods of calculating many-nucleon systems, the comparison of results of calculations with experiments, the importance of the Pauli principle, etc. That we had to attack the problem on so many fronts is a direct consequence of the complexity and many sidedness of the basic question—is it possible to construct a model-free nuclear physics in fairly good agreement with experiment, a nonrelativistic theory of the nucleus based entirely on the Schrödinger equation with the "vacuum" interaction between nucleons? At the present time, the feasibility of such an approach to nuclear theory can only be judged if one makes the working hypothesis that such a model-free theory is possible and then compares the consequences with experiments. The data obtained in Secs. 2 and 3 show that the calculations so far made do not contradict the conjecture.

The general conclusions can be formulated as follows.

1. The very existence of a large number of investigations in which quite accurate solutions have been found to the many-nucleon Schrödinger equation for systems with 3, 4, 5, 6, and 8 nucleons is an indication that the technical difficulties associated with solving problems with several strongly interacting bodies are not insuperable. It is true that, as yet, the calculations have been made for the lightest nuclei and the simplest nuclear reactions.

2. The greatest number of investigations have been devoted to calculating the properties of $^3\mathrm{H}$ and $^3\mathrm{He}$. In all cases when the calculations were made with realistic potentials of complicated structure, perfectly satisfactory results were obtained. The wave function and the binding energy were found to depend weakly on the particular form of the NN potential. The estimates made in Sec. 2 show that the wave function is calculated with an accuracy of a few percent, although the exact form of the NN potential is unknown. Although the properties of $^3\mathrm{H}$ and $^3\mathrm{He}$ depend weakly on the form of the NN potential, the calculations do enable one to obtain additional information about the potential. In particular, they definitely indicate the presence of charge-noninvariant terms in the nuclear forces. At the same time, some data, for example, those on the behavior of the form factor at large q^2, cannot be explained in the framework of the potential approach.

3. Quite a number of calculations have been made of the amplitudes of the simplest nuclear reactions. Somewhat unexpectedly it was found to be simpler to do this than calculate the binding energy of nuclei (which is a small difference of two large quantities). Without exception, the calculations of the reactions yielded very good qualitative agreement with the experimental data although the computational procedure contains no free parameters. These results must as yet be regarded as preliminary, since only in some cases, and then incompletely, has the stability of the computational scheme been tested against an extension of the basis of functions on which the solution of the many-body Schrödinger equation is sought. The tremendous importance acquired by the Pauli principle in the kinematics and dynamics of nuclear reactions has been revealed.

4. Comparison of the results of different papers mentioned in Secs. 2 and 3 shows that besides the many nuclear quantities that depend weakly on the particular form of the NN potential (for example, the $^3\mathrm{H}$ and $^3\mathrm{He}$ wave functions, the s phase shift of nT scattering, etc.), there are many quantities and processes which are sensitive to its form [for example, the cross section of the reaction $p(\alpha, D)^3\mathrm{He}$, the spectra of nuclear levels, etc.]. This enables one to obtain additional

information about the NN potential. The question now arises of determining uniquely the NN potentials from the complete set of nuclear data.

5. The good qualitative agreement of the results of calculations with experiments obtained in the hitherto published papers does not yet enable one to make any choice between the different sets of NN potentials (to a large extent this is because the problem of determining the NN potential uniquely has not yet been seriously posed). Moreover, there are no indications that the use of realistic NN potentials leads to any serious contradictions when nuclear quantities are calculated. This means that as yet no contradiction has been found to the main conjecture—a model-free nuclear physics is possible on the basis of the nonrelativistic many-nucleon Schrödinger equation with the "vacuum" interaction between nucleons.

APPENDIX 1

We here describe briefly the main methods of solving the bound-state problem of three (or four) nucleons. Since a fairly extensive literature has been devoted to the method of Faddeev equations, we shall not describe the method but refer the reader to, for example, Refs. 4 and 64.

Oscillator-basis method (OBM). The wave function of the three-particle system as a function of the variables ξ_1 and ξ_2,

$$\xi_1=(\mathbf{r}_1-\mathbf{r}_2)/\sqrt{2}; \quad \xi_2=(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3)/\sqrt{6} \tag{A.1}$$

is represented as an expansion with respect to a complete oscillator basis for each variable ξ_1 and ξ_2:

$$|\Psi\rangle=\sum_i C_i\Phi_i \equiv \sum_{n_1l_1n_2l_2}\sum_{LS\alpha} C^{LS\alpha}_{n_1l_1n_2l_2}|(n_1l_1),\ (n_2l_2);\ LSJ,\ \alpha\rangle, \tag{A.2}$$

where L, S, and J are the total orbital, spin, and angular momenta of the system; α are the other quantum numbers characterizing the states; $|n_il_i\rangle$ are harmonic-oscillator functions for the variable ξ_i (n_i is the principal quantum number and l_i is the orbital angular momentum).[19]

Restricting ourselves in the sum over i in (A.2) to a finite number of terms, substituting the wave function (A.2) into the Schrödinger equation

$$(H-E)\Psi=(\hat{T}+\hat{V}-E)\Psi=0,$$

multiplying from the left by the basis functions Φ_i, and integrating (summing) over all variables, we reduce the problem to the solution of the secular equation

$$\sum_{j=1}^{i_{\max}}\{\langle i|T+V|j\rangle-E\delta_{ij}\}C_j=0, \quad i=1,\ 2,\ \ldots,\ i_{\max}, \tag{A.3}$$

i.e., to the solution of a system of linear algebraic equations for the eigenfunctions and eigenvalues.

The oscillator-basis method is attractive in that using it one can very simply calculate the matrix elements of pairing-interaction potentials: everything reduces to one-dimensional integrals which can be programmed quite easily for calculation on a computer. A shortcoming of the method is that the fixed form of the basis functions makes it necessary to use many of them to obtain convergent results in the case of realistic potentials (about 400—500; see Ref. 19). This is because the basis functions describe the behavior of the wave function as $|\xi_i|\to 0,\ \infty$ badly.

The basis (A.2) was used by Jackson *et al.*[19] to calculate the properties of ^{3}H and ^{3}He with the RSC potential. Nunberg *et al.*[19a] used a slightly modified set of oscillator functions, with which they investigated the Riihimaeki potential and some other potentials.

Method of hyperspherical functions for $A=3$ and 4. This method is based on the wave-function expansion

$$\Psi=\rho^{-(3A-4)/2}\sum_{K,\alpha}\chi_{K\alpha}(\rho)\Gamma_{K\alpha}(\Omega_n,\ \{\sigma_i,\ \tau_i\}), \tag{A.4}$$

where n is the number of variables in the center-of-mass system, $n=3A-3$; A is the number of particles; ρ is the hyper-radius $\rho^2=\sum_{i=1}^{A-1}\xi_i^2$, where $\{\xi_i\}$ are the Jacobi coordinates defined in the usual way; $\Gamma_{K\alpha}(\Omega_n,\ \{\sigma_i,\tau_i\})$, the basis functions, are totally antisymmetric under permutations of the particles and depend on the angles Ω_n on the hypersphere $\rho=\text{const}$ and the spin-isospin variables $\{\sigma_i,\tau_i\}$; K is a global quantum number: for states with positive parity $K=K_{\min}$, $K_{\min}+2$, $K_{\min}+4,\ldots$; $K_{\min}=0$ for $A=3,4$; for states with negative parity, K ranges over odd positive numbers; α are the remaining quantum numbers, which include the total angular momentum J and its projection M, the orbital angular momentum L and spin angular momentum S, the isospin T and its projection M_T, etc.

For the partial waves $\chi_{K\alpha}(\rho)$ we obtain the system of equations

$$\left\{-\frac{\hbar^2}{2M}\left[\frac{d^2}{d\rho^2}-\frac{\varkappa_K(\varkappa_K+1)}{\rho^2}\right]-E\right\}\chi_{K\alpha}+\sum_{K'\alpha'}W^{K'\alpha'}_{K\alpha}(\rho)\chi_{K'\alpha'}(\rho)=0; \tag{A.5}$$

$$W^{K'\alpha'}_{K\alpha}=\sum_{\{\sigma_i\tau_i\}}\langle\Gamma_{K\alpha},\ \hat{V}\Gamma_{K'\alpha'}\rangle, \tag{A.6}$$

(where $\langle\ldots\rangle$ denotes integration with respect to the angular variables Ω_n; $\hat{V}$ is the interaction operator of A particles; E is the cms energy of the system;

$$\varkappa_K=K+(3A-6)/2. \tag{A.7}$$

The basis functions $\Gamma_{K\alpha}$, which are antisymmetric with respect to permutations of any pair of particles, are formed by multiplying the angular functions $U^\gamma_{K\alpha}(\Omega_n)$ and the spin-isospin functions $\chi^\gamma_{SM_STM_S}$ (see, for example, Ref. 65); here, γ denotes the symmetry type. The functions $U^\gamma_{K\alpha}(\Omega_n)$ are called hyperspherical functions or K harmonics. They form a complete system of angular functions.

For example, for the three-body problem we have the following $\Gamma^\gamma_{K\alpha}$:

$$\begin{aligned}\Gamma^{(s)}_{K\alpha}&=[U^s_{KLM_L}X^a_{SM_STM_T}]_{JM};\\ \Gamma^{(a)}_{K\alpha}&=[U^a_{KLM_L}X^s_{SM_STM_T}]_{JM};\\ \Gamma^{(m)}_{K\alpha}&=\frac{1}{\sqrt{2}}[U^n_{KLM_L}X^n_{SM_STM_T}-U'_{KLM_L}X''_{SM_STM_T}]_{JM}.\end{aligned} \tag{A.8}$$

where $[\ldots]_{JM}$ denotes the combination of L and S into the total angular momentum J with projection M by means of Clebsch-Gordan coefficients; s is symmetric, a is antisymmetric, and $m(",')$ is mixed symmetry. To be specific, we have appended the symmetry type of the wave function to the basis functions $\Gamma_{K\alpha}$. Similar functions $\Gamma^{\gamma}_{K\alpha}$ can be written down for $A=4$ as well (see, for example, Refs. 62 and 65).

There are several functions $U^{\gamma}_{KLM_L}$ with given quantum numbers KLM_L and symmetry type γ; they are distinguished by the subscript i:

$$U^{\gamma}_{KLM;\,i}(\Omega_n) \tag{A.9}$$

(we omit the subscript L of M_L).

To achieve complete convergence, it is usually necessary to take into account all K up to about 20—30 in the sum (A.4) (see, for example, Ref. 32). With increasing K, the number of functions $U_{KLM;i}$ increases linearly for $A=3$ and in proportion to K^4(!) for $A=4$. For example, if $K=14$, the number of hyperspherical functions with $L=2$ (for the ground state of ^{3}H and ^{3,4}He they can be only of mixed symmetry) is 7 for $A=3$ and about 100(!) for $A=4$.

However, as Éfros[62] has shown, the hyperspherical functions $U^{\gamma}_{KLM;i}$ with given quantum numbers γ and KLM but different i can be constructed and ordered in such a way that their contribution to the wave function of a bound state of a system with $A=3$ or 4 decreases strongly with increasing serial number, so that the overwhelming majority of the functions will make a negligibly small contribution to the problem. We shall call this the physical basis. The first serial numbers in it are assigned to the so-called potential hyperspherical functions, whose special role was first pointed out by Fabre[63]: these are $U_{K\alpha}$ for which $W^{00}_{K\alpha}(\rho)\not\equiv 0$ [see Eq. (A.6)]. The hyperspherical functions $U_{K\alpha}$ for which $W^{00}_{K\alpha}\equiv 0$ have been called nonpotential.

The predominant role of the potential hyperspherical functions is due to the fact that the wave function in the hyperspherical-function method consists of $\gtrsim 90\%$ (in the norm) of the fundamental hyperspherical function ($K=0$). This property, as follows from all calculations hitherto made for $A=3$ and 4, is retained for all NN potentials. The remaining ten or less percent are distributed among the hyperspherical functions with $K\neq 0$, the main contribution to these 10% coming from the functions with $K=2$ and 4.

For given $K\geqslant 4$ there are altogether three hyperspherical functions (for $A=3$ and 4)—one symmetric ($L=0$) and two of mixed symmetry ($L=0,2$). The contribution of nonpotential hyperspherical functions to the $A=3$ wave function is ~0.1% (measured by the contribution to the normalization integral),[21,32] and this contribution is determined by only 1—3 nonpotential (for given K) harmonics following the potential harmonics. In order to achieve an accuracy in the binding energy not worse than 0.01 MeV, it is sufficient to take into account only these nonpotential harmonics, these not going beyond $K\approx 12$ ($A=3$).

Below, we describe the construction of the physical basis of hyperspherical functions for $A=3$. Convenient original functions for its construction are functions with definite orbital angular momenta with respect to the variables ξ_1 and ξ_2:

$$U^{L_1L_2}_{KLM}(\Omega_6)=N^{L_1L_2}_{K}Y^{L_1L_2}_{LM}(\hat{\xi}_1,\hat{\xi}_2)(\sin\theta)^{L_1}(\cos\theta)^{L_2}\times P^{L_1+1/2,\,L_2+1/2}_{(K-L_1-L_2)/2}(\cos 2\theta), \tag{A.10}$$

where $N^{L_1L_2}_K$ is the normalization factor

$$N^{L_1L_2}_K=\left[\frac{(2K+4)\,\Gamma\left(\frac{K+L_1+L_2}{2}+2\right)\left(\frac{K-L_1-L_2}{2}\right)!}{\Gamma\left(\frac{K+L_1-L_2}{2}+\frac{3}{2}\right)\Gamma\left(\frac{K+L_2-L_1}{2}+\frac{3}{2}\right)}\right]^{1/2}; \tag{A.11}$$

$$Y^{L_1L_2}_{LM}=\sum_{M_1+M_2=M}(L_1M_1L_2M_2\,|\,LM)\,Y_{L_1M_1}(\hat{\xi}_1)\,Y_{L_2M_2}(\hat{\xi}_2), \tag{A.12}$$

here, $(\ldots|\ldots)$ are Clebsch-Gordan coefficients; Y_{lm} are spherical functions; and $P^{a,b}_n$ are Jacobi polynomials. The hyperspherical functions $U^{\gamma}_{KLM;i}$ with given symmetry are formed by linear combination from the functions $U^{L_1L_2}_{KLM}$:

$$U^{\gamma}_{KLM;\,i}=\sum_{L_1L_2}C^{\gamma}_{KL;\,i}(L_1,L_2)\,U^{L_1L_2}_{KLM}. \tag{A.13}$$

The summation in (A.13) is over even L_1 and L_2 for symmetric functions and the functions of mixed symmetry U'' and over odd L_1 and L_2 for antisymmetric functions and functions of mixed symmetry U'. In addition, $L_1+L_2\leqslant K$ and (L_1L_2L) satisfy the triangle inequality. The coefficients $C_{K\alpha}$, which realize the physical basis, are constructed by projecting the functions $U^{L_1L_2}_{KLM}$ with the lowest L_1 and L_2 onto the space of functions with given symmetry.

For example,

$$U^{s}_{KLM(L_1L_2)}=P^{s}_{r}[U^{L_1L_2}_{KLM}]=\sum_{L_1'L_2'}C^{s}_{KL(L_1L_2)}(L_1'L_2')\,U^{L_1'L_2'}_{KLM}. \tag{A.14}$$

For such functions $L\leqslant 1$, and therefore $L_2=L_1$ (and both even). Constructing and orthonormalizing successively functions with $L_1L_2=(00)$, (22), (44), etc., we find that the hyperspherical function $U^{s}_{K00(00)}$ (and it is a potential hyperspherical function) contains $L_1'L_2'=(00)$, (22), (44), etc; $U^{s}_{K00(22)}$ contains $L_1'L_2'=(22)$, (44), etc; $U^{s}_{K00(44)}$ contains $L_1'L_2''=(44)$, etc. For the hyperspherical function $U^{s}_{K1M}(L_1L_2)$ one can have only $L_1L_2\geqslant 2$, so that these do not include potential functions.

Because the functions $\Gamma_{K\alpha}$ are antisymmetric, the calculation of the matrix elements of $\hat{V}=\sum_{i<j}\hat{V}_{ij}$ in (A.6) can be reduced to the matrix element of $\hat{V}_{12}=\hat{V}_{12}(\xi_1)$. As is well known, the two-nucleon interaction in nuclear physics is important only in the states of particle pairs with the lowest L_1, the contribution of the remaining L_1 being exponentially small. For this reason, there is a sharp decrease in the contribution of the hyperspherical functions with increasing L_1L_2: $U^{s}_{K00(00)}$, $U^{s}_{K00(22)}$, $U^{s}_{K00(44)}$, etc., this decrease being so abrupt that the contribution of the function $U^{s}_{K00(44)}$ can already be ignored (this is directly demonstrated by the calculation in Ref. 21), while the function $U^{s}_{K00(22)}$ lies at the accuracy limit ($\Delta E\approx 0.01$ MeV). A similar situation may arise with a hyperspherical function of other symmetry type and (or) other L. Details on the technique for constructing the physical basis for $A=3$ can be found in Refs. 62 and 66.

Two different methods are used to solve the system of equations (A.5): 1) direct numerical solution of the system of differential equations, which was used in the first calculations by the hyperspherical-function method (see, for example, Ref. 8) and in Refs. 67; 2) expansion of the functions $\Gamma_{K\alpha}(\rho)$ with respect to some complete set of hyper-radial functions on the interval $(0,\infty)$:

$$\chi_{K\alpha}(\rho)=\sum_n C_n^{K\alpha}R_n(\rho), \tag{A.15}$$

where

$$R_n(\rho)=N(B\rho)^{s/2}\exp(-B\rho/2)\,L_n^s(B\rho) \tag{A.16}$$

(see Refs. 32 and 68); L_n^s is a Laguerre polynomial; N is a normalization coefficient; and B and s are parameters chosen in such a way that they improve the convergence with respect to n. Substituting the expansion (A.15) into Eq. (A.5) and restricting ourselves to finitely many terms in (A.15) and (A.4), we reduce the problem to the solution of a secular equation of type (A.3).

As follows from the calculations of Refs. 21 and 32, the total number of basis functions (hyperspherical plus hyper-radial) for realistic potentials required to achieve practical convergence is 200—250 (cf the 400—500 for the oscillator-basis method).

Variational method (VM). This is one of the longest tested methods of atomic and nuclear physics, used with success over several years in the nuclear three-body problem. A major contribution in this direction was made by Delves and his collaborators,[25–27] so that we shall follow their work in describing the variational method.

In the method one selects the following coordinate system: $X_i=|\mathbf{r}_j-\mathbf{r}_k|$; $i,\ j,\ k=123,\ 231,\ 312$; and three Eulerian angles (α,β,γ) which determine the orientation of the triangle $(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)$ in space. The total wave function of the three-nucleon system is expressed as the sum

$$\Psi_{JM}^{T}=\sum_{\alpha}\mathcal{Y}_{JT\alpha}(\alpha,\beta,\gamma,\{\sigma^{(i)}\},\{\tau^{(i)}\})R_\alpha(X_1,X_2,X_3), \tag{A.17}$$

where J and M are the total angular momentum and its projection; T is the isospin; $\{\sigma^{(i)}\}\{\tau^{(i)}\}$ are the spin-isospin variables; and α are the remaining quantum numbers, which include L, S, the symmetry type, and others. The function $\mathcal{Y}_{JT\alpha}$ has a given form, and the function R_α is assumed unknown. It is chosen in the form

$$R_\alpha(X_1,X_2,X_3)=\sum_{m_1+m_2+m_3\leq Q_\alpha}C_\alpha^{m_1m_2m_3}h_\alpha^{m_1m_2m_3}(X_1,X_2,X_3), \tag{A.18}$$

where $\{m_i\}$ are non-negative integers; $h_\alpha^{m_1m_2m_3}(X_1,X_2,X_3)$ is an appropriately symmetrized function, chosen, for example, in Ref. 27 in a calculation of the properties of tritium with a Hamada-Johnston potential in the form

$$\begin{cases}X_1^{m_1}X_2^{m_2}X_3^{m_3}\prod_{i=1}^{3}\exp(-\beta X_i)(1-r_c/X_i)^p, & X\geqslant r_c;\\ 0, & X<r_c,\end{cases} \tag{A.19}$$

where β and p and nonlinear parameters; r_c is the hard-core radius; and Q_α is the number of included basis functions. A preliminary solution of the problem is found first by variation with respect to the nonlinear parameters with an incomplete set of linear parameters. The nonlinear parameters found in this way are then fixed and variation is carried out with respect to the complete set of linear parameters $\{C_\alpha^{m_1m_2m_3}\}$.

The advantage of the direct variational method is that within its framework one can comparatively easily describe correctly the asymptotic behavior of the wave functions and strong pairing correlations at short distances. So far, satisfactory results for hard-core potentials have been obtained by only this method.

Among its shortcomings, one must mention the need to calculate threefold integrals and use a nonorthogonal basis, which complicates the calculation and the estimate of the errors. In this method, the extension to four bodies is extremely complicated and as yet attempted by no one.

Recently, Akaishi *et al.*[29] have proposed a simplified variant of the variational method, which they have called the generalized variational method. It combines the ideas of the variational method and the reaction-matrix theory. The advantage of this method is the possibility of applying it to problems with $A>3$; a shortcoming is the approximate nature and the impossibility of estimating the errors.

APPENDIX 2

We describe here the Hulthén-Cohn variational principle for many-channel two-fragment nuclear reactions. We then consider variational approaches based on the hyperspherical-function method and the resonating-group method.

If only two-fragment channels are open, the exact wave function of a system of A nucleons (Ψ) at total energy $E<0$ has a fairly simple structure. The Ψ function goes over into the wave function of free motion of clusters outside a region with radius of the order of the range of nuclear forces, i.e., in the exterior region of the configuration space. Within this region, all the nucleons interact with one another, and the structure of the Ψ function of the continuum is similar to that of a compact bound state of a system of nucleons.

Accordingly, we represent Ψ in the form

$$\Psi=\Psi_{\text{int}}+\Psi_{\text{ext}}. \tag{A.20}$$

For the definition of Ψ_{int} and Ψ_{ext} see above. It should be noted particularly that Ψ_{int} and Ψ_{ext} in (A.20) are not fitted explicitly at any particular boundary in the configuration space.

Let Ψ_i^{exact} be exact solutions of the Schrödinger equation given in the channel with number n [2)] by the asymptotic behaviors

$$\Psi_i^{\text{exact}}\xrightarrow[R_n\to\infty]{}\hat{A}\left\{\sqrt{\frac{\mu_n}{\hbar^2k_n}}[\delta_{ni}\sin(k_nR_n-L\pi/2)+f_{ni}^{\text{exact}}\cos(k_nR_n-L\pi/2)]\varphi_n\right\}, \tag{A.21}$$

where $\hat{A}$ is the antisymmetrization operator; δ_{ni} is the

Kronecker symbol; R_n is the distance between the centers of mass of the pair of fragments of channel n; k_n is the wave number of the channel; φ_n are surface functions with given total angular momentum, orbital angular momentum L, and spin of the channel, determined in Ref. 57, Ch. III, Eq. (2.25); μ_n is the reduced mass of the channel; f_{ni}^{exact} is the matrix element of the generalized collision matrix (or K matrix). The expression (A.21) has been written down for the case when there is no Coulomb interaction in channel n; if there is, the Coulomb phase shifts must be added to the arguments of the sine and cosine.

Suppose we have found some trial functions Ψ_i, which are approximations to Ψ_i^{exact} and have the same asymptotic behavior (A.21) but with approximate f_{ni}. We form the Hulthên-Cohn functionals

$$\Gamma_{ji}(\Psi_j, \Psi_i) = f_{ji} + I(\Psi_j, \Psi_i), \tag{A.22}$$

where

$$I(\varphi, \Psi) = 2\int d\tau\,(\varphi, [\hat{H} - E]\Psi). \tag{A.23}$$

Varying (A.22) on Ψ_i^{exact} and Ψ_j^{exact}, we obtain

$$\Delta\Gamma_{ji} = \delta f_{ji} + I(\Psi_j^{\text{exact}}, \delta\Psi_i) + I(\delta\Psi_j, \delta\Psi_i); \tag{A.24}$$

$$\delta f_{ji} + I(\Psi_j^{\text{exact}}, \delta\Psi_i) = 0, \tag{A.25}$$

where $\delta\Psi_\Lambda = \Psi_\Lambda - \Psi_\Lambda^{\text{exact}}$, i.e., we have arrived at the Kato identities for (A.22):

$$\Gamma_{ji} - f_{ji}^{\text{exact}} = I(\delta\Psi_j, \delta\Psi_i). \tag{A.26}$$

This function shows that the Γ_{ji} in (A.22) are stationary with respect to small deviations from the true Ψ functions. Therefore, the Γ_{ji} are better approximations to f_{ni}^{exact} than the original f_{ji}, and the last terms in (A.22) are called variational corrections to f_{ji}.

The approximate $\hat{S}$ matrix is expressed in terms of the matrix $\hat{\Gamma}$ by

$$\hat{S} = (1 + i\hat{\Gamma})(1 - i\hat{\Gamma})^{-1}. \tag{A.27}$$

The exact $\hat{S}$ matrix is expressed by an analogous expression for f_{ji}^{exact}. From the unitarity and symmetry of the exact $\hat{S}$ matrix there follows the symmetry $f_{ji}^{\text{exact}} = f_{ij}^{\text{exact}}$. The approximate amplitudes f_{ji} in the trial functions Ψ_i are nonsymmetric. However, when they are made more precise in accordance with Eqs. (A.22), symmetry as well as stationarity is achieved:

$$\Gamma_{ji}(\Psi_j, \Psi_i) = \Gamma_{ij}(\Psi_i, \Psi_j). \tag{A.28}$$

This is equivalent to the assertion that the unitarity condition is satisfied exactly for the approximate $\hat{S}$ matrix (A.27).

To prove (A.28), it is only necessary to note that the operator $\hat{H} - E$ is Hermitian in the class of functions $\delta\Psi_j$ (which have in their asymptotic behavior of each channel only cosine-type terms), and therefore

$$I(\delta\Psi_i, \delta\Psi_j) = I(\delta\Psi_j, \delta\Psi_i), \tag{A.29}$$

which together with the symmetry conditions of f_{ji}^{exact} leads to (A.28).

APPENDIX 3

The hyperspherical-function method, or K-harmonic method, which has been used in the present review to calculate the properties of bound states, is also very convenient for calculating continuum wave functions on the basis of a variational principle. If in the continuum wave function one separates out explicitly the asymptotic (nondecreasing) terms (Ψ_{ext}), the remaining part of the Ψ function (Ψ_{int}) can be expanded in a rapidly converging series with respect to angular harmonics. Such a method was developed in Refs. 42 and 43 and used to calculate elastic neutron scattering on D, ^{3}H, and ^{4}He nuclei.[36–38] We deduce the main equations of this approach.

Suppose that for given E there are N open two-fragment channels. We shall look for the N linearly independent solutions of the Schrödinger equation in the form

$$\Psi_i \underset{\substack{K \leqslant K_0 \\ \alpha < \alpha_0}}{\sum} \chi_{K\alpha}(\rho)\,\Gamma_{K\alpha}(\Omega) + A[\Phi_i^{(1)}] + \sum_{j=1}^{N} f_{ji}(\rho)\,A[\Phi_j^{(2)}], \tag{A.30}$$

where $\Gamma_{K\alpha}(\Omega)$ is a hyperspherical harmonic of degree K; ρ is the variable of the K-harmonic method; $\Phi_j^{(1),(2)} = U_{Lj}^{(1),(2)}(R_j)\cdot\varphi_{nj}$, where φ_n are defined in (A.21). The functions $U_L^{(1),(2)}(R)$ are normalized by the following condition (in the absence of Coulomb interaction):

$$\left.\begin{aligned} U_{Ln}^{(1),(2)} &\xrightarrow[R_n\to\infty]{} \sqrt{\frac{\mu_n}{\hbar^2 k_n}}\begin{pmatrix}\sin\\ \cos\end{pmatrix}(k_n R_n - L\pi/2); \\ U_{Ln}^{(1),(2)} &\sim R_n^{L+1}, \quad \text{as } R_n \to 0. \end{aligned}\right\} \tag{A.31}$$

Such functions can be constructed in different ways. One such way is that the functions $U^{(1),(2)}$ are taken to be the regular and irregular (at the origin) cylindrical functions (or Coulomb functions), the latter being regularized by means of a factor $T_L(R)$, which contains certain parameters and has the properties $T_L(R) \to 1$ as $R \to \infty$ and $T_L(R) \sim R^{2L+1}$ as $R \to 0$. For example, in Refs. 49–51 this factor is chosen in the form

$$T_L(R) = 1 - \left(\sum_{\nu=0}^{2L} \frac{(\beta R)^\nu}{\nu!}\right)\exp[-\beta R]. \tag{A.32}$$

A different way, proposed in Ref. 43, proposes that one should generate $U_L^{(1),(2)}$ as solutions of the equation

$$\left[-\frac{d^2}{dR^2} + \frac{L(L+1)}{R^2} - \beta_n^{(1),(2)}\overline{V}_n(R) - k_n^2\right] U_{Ln}^{(1),(2)}(R) = 0.$$

Here, $\overline{V}$ is a short-range effective potential. Changing the constant $\beta^{(1),(2)}$, we obtain both solutions (A.31).[3] When U is defined in this way, we have a very simple expression for the action of the operator $\hat{H} - E$ on $\Phi^{(1),(2)}$:

$$(\hat{H} - E)\Phi^{(1),(2)} = \left[\hat{V}_{n;\,p,\,g} - \frac{\hbar^2}{2\mu_n}\beta_n^{(1),(2)}\overline{V}_n\right]\Phi^{(1),(2)}, \tag{A.33}$$

where $\hat{V}_{nip,g}$ is the interaction in $\hat{H}$ between the definite fragments p and g contained in φ_n; μ_n is the reduced mass in channel n.

The required quantities in the wave function Ψ_i (A.30) are $\chi_{K\alpha}(\rho)$ and the amplitude functions $f_{ji}(\rho)$. The func-

tion $\chi_{K\alpha}(\rho)$ tends to zero outside the interior region, and the $f_{ji}(\rho)$ describe the formation in the reaction of fragments of definite form and tend to the constants $f_{ji}(\infty) = f_{ji}$, the reaction amplitudes. Note that in the majority of investigations the f_{ji} are simply taken to be constants, whereas the introduction of a variable ρ does not essentially complicate the resulting equations, since ρ is symmetric under permutations of the nucleons and can therefore be taken in front of the operator $\hat{A}$, but it makes the trial function Ψ_i in (A.30) more flexible.

We obtain equations for determining the unknown $\chi_{K\alpha}$ and f_{ji} by requiring the functional Γ_{ji} in (A.22) to be stationary on the class of functions Ψ_i (A.30); this is equivalent to the requirement $I(\delta\Psi_j, \Psi_i) = 0$ for all i and j, or in an explicit form one obtains the system of ordinary differential equations

$$\left.\begin{aligned} &(\Gamma_{K\alpha},\ [\hat{H}-E]\,\Psi_i) = 0, \quad K \leqslant K_0,\ \alpha \leqslant \alpha_0; \\ &(\Phi_j^{(2)},\ [\hat{H}-E]\,\Psi_i) = 0, \quad j = 1, 2, \ldots, N. \end{aligned}\right\} \tag{A.34}$$

By (...) we understand integration with respect to the angular variables Ω on the hypersphere and summation with respect to the spin-isospin variables.

We write out the system (A.34) in more detail:

$$\left.\begin{aligned} &\hat{L}_{K,\rho}\chi_{K\alpha}(\rho) + \sum_{p=1}^{N}\Big[(\Gamma_{K\alpha},\ \Phi_p^{(2)})\,\hat{L}_\rho f_{pi}(\rho) \\ &2\,(\Gamma_{K\alpha},\ \Phi_p^{(2)\prime})\,\frac{df_{pi}(\rho)}{d\rho} - \frac{2m}{\hbar^2}\,(\Gamma_{K\alpha},\ [\hat{H}-E]\,\Phi_p^{(2)})\,f_{pi}(\rho)\Big] \\ &-\frac{2m}{\hbar^2}\sum_{K'\leqslant K_0,\ \alpha}(\Gamma_{K\alpha}\Gamma\Gamma_{K'\alpha'})\,\chi_{K'\alpha'}(\rho) = \frac{2m}{\hbar^2}\,(\Gamma_{K\alpha},\ [\hat{H}-E]\,\Phi_i^{(1)}); \\ &\sum_{K'\leqslant K_0,\ \alpha}\Big[(\Gamma_{K'\alpha'}\Phi_j^{(2)})\,\hat{L}_{K',\rho}\chi_{K'\alpha'}(\rho) + \sum_{p=1}^{N}[(\Phi_j^{(2)}\hat{A}\Phi_p^{(2)})\hat{L}_\rho f_{pi}(\rho) \\ &+2\,(\Phi_j^{(2)}\hat{A}\Phi_p^{(2)\prime})\,\frac{df_{pi}}{d\rho} - \frac{2m}{\hbar^2}\,(\Phi_j^{(2)},\ \hat{A}\,[\hat{H}-E]\,\Phi_p^{(2)})\,f_{pi}(\rho)\Big] \\ &-\frac{2m}{\hbar^2}\sum_{K'\leqslant K_0\alpha}(\Phi_j^{(2)}\hat{\Gamma}\Gamma_{K'\alpha'})\,\chi_{K'\alpha'}(\rho) \\ &= \frac{2m}{\hbar^2}\,(\Phi_j^{(2)},\ \hat{A}\,[\hat{H}-E]\,\Phi_i^{(1)}), \end{aligned}\right\} \tag{A.35}$$

where

$$\hat{L}_\rho = \frac{d^2}{d\rho^2} - \frac{3A-4}{\rho}\cdot\frac{d}{d\rho};\quad \hat{L}_{K,\rho} = \hat{L}_\rho - \frac{K(K+3A-5)}{\rho^2} - \frac{2m}{\hbar^2}\,|E|;$$

$$\Phi_p^{(2)\prime} = \frac{\partial}{\partial\rho}\,\Phi_p^{(2)}$$

m is the nucleon mass. All the coefficients of these equations depend explicitly on the energy of the relative motion of the particles in the given channel: $E_{\text{rel}}^{(i)} = |E - \epsilon_1^{(i)} - \epsilon_2^{(i)}|$, where $\epsilon_{1,2}^{(i)}$ are the binding energies of the clusters in channel i. The action of the operator $\hat{H} - E$ on $\Phi^{(i)}$ can be represented by using (A.33) in the form of a certain effective interaction potential of the particles in the given channel. The corresponding coefficients in Eqs. (A.35) are obtained then from (A.33) by averaging of this effective interaction with respect to the wave functions of the particles. Such a system of equations must be solved with the boundary conditions

$$\left.\begin{aligned} &\chi(\rho) < \infty;\quad f_{ji}(\rho) < \infty, && \text{as } \rho \to 0; \\ &\chi(\rho) \to 0;\quad f_{ji}(\rho) \to \text{const}, && \text{as } \rho \to \infty. \end{aligned}\right\} \tag{A.36}$$

One can show[43] that this system of differential equations (or algebraicodifferential equations if the f_{ji} are constant) has a unique solution for the given boundary conditions (A.36) for each fixed energy E.

Finally, we mention an important circumstance. The collective variable ρ of the K-harmonic method is the only variable invariant under nucleon permutations in the language of which one can conveniently split the configuration space into interior and exterior regions. This property of ρ was used in Refs. 58 and 59 to develop a general approach that is equally suitable for the description of bound states and different nuclear reactions.

APPENDIX 4

In its usual formulation, the well known resonating-group method[60,61] leads to integrodifferential equations of fairly complicated form. Because of the difficulty of solving systems of integrodifferential equations of several variables, this method has been used only to calculate the elastic scattering of nuclei.

Beginning in 1969, a group of West German physicists began publishing papers in which the resonating-group method was generalized to reactions; they called it the Refined Cluster Function Representation (RCFR).[44,45] In this approach the wave function of the continuous spectrum is sought in the form

$$\Psi_\varkappa = \hat{A}\Big\{\sum_{p=1}^{N}\varphi_{pL}\chi_{pL}^{\varkappa}(R_p)\Big\}, \tag{A.37}$$

where φ_{pL} is the surface function defined in (A.21); $\chi_{pL}^{\varkappa}(R_p)$ is a function of the relative motion of the given pair of clusters in the resonating-group method. In Refs. 44 and 45, $\chi(R)$ is represented in the form

$$\chi_{pL}^{\varkappa}(R) = \delta_{\varkappa,p}F_L(R) + f_{\varkappa p}G_L^R(R) + \sum_m b_{pm}^{\varkappa}H_{pm}(R). \tag{A.38}$$

Here, $F_L(R)$ is the regular Coulomb function; $G_L^R(R)$ is the irregular Coulomb function regularized at the origin by means of (A.32); $H_{pm}(R)$ is a set of ancillary functions that are rapidly damped at infinity; in particular, they could be oscillator functions.

In (A.38), the unknowns are the reaction amplitudes $f_{\varkappa p}$ and the coefficients $b_{pm}^{\varkappa}$. For such a choice of $\chi_{pL}^{\varkappa}$, the interior part of the total wave function, Ψ_{int} (A.20), corresponds to φ_{pL} multiplied by the last term in (A.38); the remaining terms in $\Psi_{\varkappa}$ (A.37) determine Ψ_{ext}. Choosing $\Psi_\varkappa$ in the form of (A.37) and substituting it into the formulation of the variational principle (A.22),

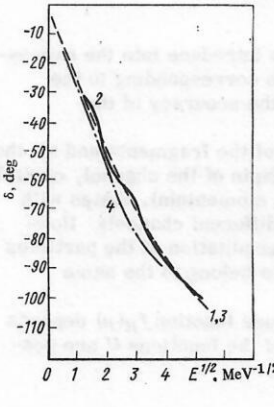


FIG. 17. Phase shift of elastic $n + {}^4\text{He}$ scattering ($\delta_{L=0}$) calculated by the resonating-group method (1) and by the RCFR-method (2, 3): the experimental phase shift is shown by the curve 4.

we obtain a system of inhomogeneous algebraic equations analogous to the system (A.34) for determining the unknown coefficients f_{ij} and b^{j}_{im}.

In Ref. 55, the ordinary resonating-group method, in which the entire $\chi(R)$ is assumed unknown, and the RCFR method are compared for the example of elastic $n\alpha$ and $\alpha\alpha$ scattering. Figure 17 shows the results of calculation of the phase shift $\delta_{L=0}$ for $n\alpha$ scattering. In Fig. 17, curve 1 shows the results by the resonating-group method; 2, the results of calculation by the RCFR method with inclusion of six oscillator functions $H_m(R)$. Note that curve 3, in which the interior part of the wave function is represented by 16 oscillator terms, hardly differs from curve 1. Thus, for a sufficiently large number of functions H_m the results of the two methods are, as one would expect, close.

In their calculations with the RCFR method, Hackenbroich and his collaborators generally used a realistic NN potential with hard cores, with tensor and *LS* forces fitted to the NN phase shifts up to 350 MeV.[14,48] The use of such a complicated realistic potential makes it necessary to use some approximations to calculate the matrix elements.[50]

When the wave function of clusters is constructed in the RCFR method, functions with maximal spatial symmetry are used. As can be seen from the results of the present review, the contribution of other components to the total wave function of the lightest nuclei is negligibly small, but, this cannot be said of the contribution of these components to the total binding energy. Therefore, in the calculation of the wave functions of clusters one uses only the central part of the NN potential, fitted separately to the phase shifts up to 350 MeV,[48] and one ignores the tensor and *LS* forces and the renormalization of the central part of the potential in the presence of the tensor and *LS* components. The tensor and *LS* interactions, taken from Ref. 14, are taken into account only in the interaction between clusters.

With allowance for what we have said above, the following values are obtained for the binding energies of the lightest nuclei: $\epsilon_{bd}(^3\text{He}) = 3.9$, $\epsilon_{bd}(^3\text{H}) = 4.6$, $\epsilon_{bd}(^4\text{He}) = 21.5$ MeV.

Since the binding energies of ^{3}H, ^{3}He, and ^{4}He are found to be lower than the experimental values, the calculated thresholds for the formation of these fragments in reactions also differ from the experimental values. Therefore, in a comparison with experiment of the calculated quantities (in the RCFR method) the energy scale is shifted in some cases so as to make the calculated and the experimental threshold coincide.

[1)] In some cases, it is convenient to introduce into the expression (18) for $\psi^{(i)}$ in addition terms corresponding to the closed channels. This increases the accuracy of the calculation.

[2)] The number n specifies the form of the fragments and all the quantum numbers of the channel (spin of the channel, orbital angular momentum, total angular momentum). States with different quantum numbers form different channels. However, states differing by only a permutation of the particles within a cluster will be assumed to belong to the same channel.

[3)] In approaches in which the amplitude function $f_{jl}(\rho)$ depends on the variable ρ, other choices of the functions U are possible (see, for example, Ref. 58).

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