Diagram-summation method in the theory of scattering of three and four nonrelativistic particles

V. V. Komarov and A. M. Popova

Nuclear Physics Institute, Moscow State University
Fiz. El. Chast. Atom. Yad., 5, 1075-1117 (October-December 1974)

Integral equations are derived by a diagram-summation method for the scattering amplitudes of three and four nonrelativistic particles. The method is based on the rearrangement properties of the expansion of the scattering operator in a series given in the interaction representation. By defining the interaction operator in second-quantization space it is possible to represent the series of the scattering-operator matrix elements in the form of nonrelativistic diagrams.

INTRODUCTION

An integral-equation formalism for the determination of the amplitudes of the transitions of a system from one definite asymptotic state to another is being successfully developed presently in the theory of the scattering of several nonrelativistic particles. Ways of solving problems involving the scattering of three particles, with positive and negative energies, have by now been indicated, and similar results are expected in the very near future for the solution of the four-body problem. The integral method of solving problems in scattering of three and more particles was heretofore confined to the Lippman–Schwinger equation, which is known to be unsuitable for a numerical solution of such problems. The nonintegrable δ functions in the kernel of this equation, which result from the mixing of the asymptotic channels when the equation is written down, were the principal obstacles to the application of integral equations in the theory of many-particle scattering.

New possibilities in the integral approach to the solution of several-body problems were proposed by G. V. Skornyakov and K. A. Ter-Martirosyan,[1] who considered the reaction of elastic scattering of a nucleon by a deuteron under the assumption of two-particle forces with zero effective radius. It was shown that if the amplitudes of processes in which three particles take part are represented as sums of three amplitudes, each of which corresponds to the transition of the system to one of the possible asymptotic states, i.e., to states where one of the three possible particle pairs interacts, then it is possible to get rid of the kinematic δ functions in the kernels of the integral equations with these amplitudes.

The method proposed in ref. 1 for the derivation of the equations is valid only in the case of a zero nuclear-force radius. However, the idea of obtaining a system of mutually coupled equations for the amplitudes of transitions to various reaction channels with three particles laid the ground work for further development of the theory of integral equations for scattering amplitudes.

The next important contribution to this theory were papers of L. D. Faddeev,[2] who obtained integral equations for three-body scattering amplitudes, within the framework of the formalism of three-dimensional Green's functions. These equations were generalizations of the equations of ref. 1 to include a large class of two-particle potentials.

Subsequently, various methods of writing down the integral equations for the four-body problem were proposed independently and almost simultaneously,[3–12] and also for the case of N-body interaction. Most of the authors of the cited papers used the formalism of three-dimensional Green's functions. In this approach it is necessary to separate the first two-particle interaction (assuming two-particle forces), even if several independent particle pairs can interact at the initial instant of time. Thus, the amplitudes for which the systems of integral equations are written down are amplitudes of transitions from asymptotic states with the first interaction separated.

The diagram-summation method[4,14–16] of writing down the integral equations for the scattering amplitudes differs in principle from the above-mentioned methods. In this method, the equations are obtained by rearranging and summing an infinite series of matrix elements of four-dimensional perturbation theory for the scattering operator, under the condition that the particle interaction is defined in second-quantization space.

In this approach, the matrix elements of the scattering operator can be represented as ladder-type diagrams. Inasmuch as in the nonrelativistic case the number of particles is conserved in the interaction process, the direction of the particle propagation lines in the diagrams coincides with the direction of the time. The amplitude of the transition from one asymptotic state to another is therefore determined by an infinite series of contributions from diagrams that illustrate exactly the scattering process.

The diagrams pertaining to the amplitude of a definite transition can be summed into a graphic equation, i.e., what is actually effected is the summation of the entire infinite series of the matrix elements of four-dimensional perturbation theory. Since all the elements of the diagrams have mathematical equivalence, it is possible, on the basis of the graphic equation, to write down an integral equation for the probability amplitude of a given process. This method of writing down the integral equations of scattering theory was called the diagram-summation method. Subsequently, the results obtained by the diagram-summation method were confirmed on the basis of an operator formalism.[17,18]

In the case of the three-body problem, the integral equations for the interaction amplitudes, obtained in the diagram-summation method, agree fully with the integral equations of ref. 2, and in this sense the two methods give identical results. The diagram-summation method, however, turned out to be more useful in the investigation of problems with four and more bodies and, importantly, for application to nuclear-physics problems. The point is that the sequence of the interactions occurring in independent particle pairs is immaterial in the elements of the four-dimensional perturbation-theory series, in agreement with experiment. Therefore the amplitudes for which the equations are written down are amplitudes of transitions from real asymptotic states of the system. The potentials are completely excluded from the equations. The number of equations is determined by the number of the physical asymptotic values, the iterations of the equations can be carried out directly, and they were already used[13-16] to explain the mechanisms of the scattering process.

1. FORMALISM OF THE METHOD OF SUMMATION OF NONRELATIVISTIC DIAGRAMS

Formulation of Problem of N-Particle Scattering

We consider a system with several nonrelativistic particles. It is assumed that all particles are different, interact only pairwise, and have neither spin nor isospin.

The Hamiltonian of this system is

$$H = H_0 + V, \tag{1}$$

where H_0 is the free-particle kinetic-energy operator; V is the potential-energy operator and, assuming two-particle interaction, takes the form

$$V = \sum_{i<j}^{N} V_{ij}; \tag{2}$$

V_{ij} is the operator of the interaction of the pair of particles i and j; N is the number of particles in the system.

We introduce an operator T connected with the scattering matrix by the relation

$$S = 1 + T. \tag{3}$$

T will henceforth be called the operator of scattering of N particles in the system, since the matrix element of the operator T is, by definition, the scattering amplitude. In the diagram-summation method, the solution of the problem of the scattering of N particles reduces to finding the integral equations for the amplitude of scattering of particles in a given system.

To this end we introduce the scattering operator T in the interaction representation

$$2\pi i T = \sum_{n=1}^{\infty} (-i/\hbar)^n (1/n!) \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n P\{V(t_1) \ldots V(t_n)\}, \tag{4}$$

where P is the time-ordering operator, and the interaction operator $V(t) = \sum_{i<j}^{N} (t)$ is defined in second-quantization space:

$$V_{ij}(t) = \sum_{\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_i', \mathbf{k}_j'} a_i^+(\mathbf{k}_i, t)\, a_j^+(\mathbf{k}_j, t)\, V_{ij}(\mathbf{k}_i - \mathbf{k}_j; \mathbf{k}_i' - \mathbf{k}_j')$$
$$\times a_i(\mathbf{k}_i', t)\, a_j(\mathbf{k}_j', t)\, \delta(\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_i' - \mathbf{k}_j'). \tag{5}$$

Here $a_i^+(\mathbf{k}_i, t)$ and $a_j^+(\mathbf{k}_j, t)$ are the operators for the production of the particles i and j at the instant of time t, with momenta $\mathbf{k}_i$ and $\mathbf{k}_j$, respectively; $a_i(\mathbf{k}_i', t)$ and $a_j(\mathbf{k}_j', t)$ are the operators for the annihilation of the particles i and j at the instant t with momenta $\mathbf{k}_i'$ and $\mathbf{k}_j'$, respectively; the interaction $V_{ij}(\mathbf{k}_i - \mathbf{k}_j; \mathbf{k}_i' - \mathbf{k}_j')$ is the Fourier transform of the interaction potential function:

$$V_{ij}(\mathbf{k}_i - \mathbf{k}_j; \mathbf{k}_i' - \mathbf{k}_j') = \int d\mathbf{r}_{ij} V(\mathbf{r}_{ij}) \exp\{\mathbf{r}_{ij}(\mathbf{k}_i - \mathbf{k}_j - \mathbf{k}_i' + \mathbf{k}_j')\}. \tag{6}$$

The time-dependent particle-creation and annihilation operators are connected with the time-independent operators by the relations

$$\left. \begin{aligned} a_i(\mathbf{k}_i, t) &= a_i(\mathbf{k}_i) \exp(-iE_{k_i} t); \\ a_i^+(\mathbf{k}_i, t) &= a_i^+(\mathbf{k}_i) \exp(iE_{k_i} t), \end{aligned} \right\} \tag{7}$$

where $E_{ki} = k_i^2/2m_i$.

The operators $a_i(\mathbf{k}_i)$ and $a_i^+(\mathbf{k}_i)$ are defined only for the wave function of particle i in the following manner:

$$\left. \begin{aligned} a_i(\mathbf{k}_i)\, |\mathbf{k}_i\rangle &= |0\rangle; \\ a_i^+(\mathbf{k}_i)\, |0\rangle &= |\mathbf{k}_i\rangle. \end{aligned} \right\} \tag{8}$$

Therefore the following permutation relations hold for the particle-creation and annihilation operators:

$$\left. \begin{aligned} &[a_i(\mathbf{k}_i)\, a_j^+(\mathbf{k}_j)] = \delta(i, j)\, \delta(\mathbf{k}_i - \mathbf{k}_j); \\ &[a_i(\mathbf{k}_i)\, a_j(\mathbf{k}_j)] = 0; \\ &[a_i^+(\mathbf{k}_i)\, a_j^+(\mathbf{k}_j)] = 0. \end{aligned} \right\} \tag{9}$$

The rule for determining the average, over the vacuum state, of the product P of the time-dependent particle-creation and annihilation operators is

$$\langle 0 | P\{a_i^+(\mathbf{k}_i, t)\, a_j(\mathbf{k}_j, t)\} | 0\rangle = \langle 0 | N[a_i^+(\mathbf{k}_i, t)\, a_j(\mathbf{k}_j, t)] | 0\rangle + \delta(i, j)\, \delta(\mathbf{k}_i - \mathbf{k}_j)\, g_i(t_0, t_1), \tag{10}$$

where

$$g_i(t_0, t_1) = \frac{1}{2\pi i} \int dE_i \exp\{-iE_i(t_1 - t_0)\} (E_i - E_{k_i} + i\tau)^{-1}$$

is the pairing or convolution of the creation and annihilation operators of the particle of type i with momentum $\mathbf{k}_i$ and energy E_i. Introduction of the second-quantization formalism makes it possible to represent the matrix elements of the scattering operator T in the form of diagrams that illustrate exactly the particle interaction process in the given system.

Calculation of the Matrix Elements of the Scattering Operator

The matrix elements of the scattering operator T, specified in the form of an expansion in the order of interaction in four-dimensional perturbation theory [Eq. (4)], contain time integrals of the interaction operators (5). The technique for calculating these integrals is known (see, e.g., ref. 19), but it is important to present several examples of the calculation of the integrals in the second-order matrix element for the case of scattering in a system of N different particles. These examples will help introduce the notation that will be subsequently encountered.

Thus, we consider an integral in the form

$$\langle i \mid J \mid f \rangle = \langle i \mid \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 P\left\{V_a(t_1) V_b(t_2)\right\}, \tag{11}$$

where $\langle f \mid$ and $\mid i \rangle$ are the final and initial states of a system of N different free particles; $V_a(t_1)$ and $V_b(t_2)$ are the two-particle interaction operators.

We calculate the integral (11) for the following three products of two-particle interaction operators:

a) Let $V_a(t_2) = V_{ij}(t_1)$ and $V_b(t_2) = V_{ij}(t_2)$; then the integral (11) takes the form

$$\langle i \mid J_a \mid f \rangle = \langle i \mid \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 P\left\{V_{ij}(t_1) V_{ij}(t_2)\right\}. \tag{12}$$

We shall henceforth designate the integral (12) symbolically by $J_a = V_{ij} c V_{ij}$. Here c is an operator product such that the matrix element of this product is

$$\langle i \mid J_a \mid f \rangle = \frac{i}{2\pi} \int d\varepsilon_i' \, d\mathbf{k}_i' V_{ij}(\mathbf{k}_{ij}, \mathbf{k}_{ij}') \, g_i(\varepsilon_i', \varepsilon_{k_i'})$$
$$\times g_j(E - \varepsilon_i; \varepsilon_{k_j'}) V_{ij}(\mathbf{k}_{ij}', \mathbf{k}_{ij}^{(1)}) \, \delta(\mathbf{p}_{ij} - \mathbf{p}_{ij}') \, \delta(\mathbf{p}_{ij}' - \mathbf{p}_{ij}^{(1)})$$
$$\times \delta(\varepsilon_{k_i} + \varepsilon_{k_j} - \varepsilon_{k_{1i}} - \varepsilon_{k_{1j}}) \, \delta(\varepsilon_{k_i} + \varepsilon_{k_j} - \varepsilon_{k_i'} - \varepsilon_{k_j'}),$$

where

$$g_i(\varepsilon_i', \varepsilon_{k_i'}) = (\varepsilon_i - \varepsilon_{k_i'} + i\tau)^{-1};$$
$$\mathbf{k}_{ij}^{(1)} = (\mathbf{k}_{1i} - \mathbf{k}_{1j}); \quad \mathbf{k}_{ij} = (\mathbf{k}_i - \mathbf{k}_j); \quad \mathbf{k}_{ij}' = (\mathbf{k}_i' - \mathbf{k}_j');$$
$$\mathbf{p}_{ij}^{(1)} = (\mathbf{k}_{1i} + \mathbf{k}_{1j}); \quad \mathbf{p}_{ij} = (\mathbf{k}_i + \mathbf{k}_j); \quad \mathbf{p}_{ij}' = (\mathbf{k}_i' + \mathbf{k}_j');$$

$E_{ij} = \varepsilon_{k_i} + \varepsilon_{k_j}$; E_{ij} is the total energy of the system of two particles i and j.

b) Let $V_a(t_1) = V_{ij}(t_1)$ and $V_b(t_2) = V_{im}(t_2)$; then the integral (11) takes the form

$$\langle i \mid J_\delta \mid f \rangle = \langle i \mid \frac{1}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 P\left\{V_{ij}(t_1) V_{im}(t_2)\right\};$$
$$J_\delta = V_{ij} c V_{im}/2 + V_{im} c V_{ij}/2. \tag{13}$$

Just as in the preceding case, the symbol c stands for an operator product with matrix element

$$\langle i \mid \frac{1}{2} V_{ij} c V_{im} \mid f \rangle = \frac{1}{2} V_{ij}((\mathbf{k}_{ij}, \mathbf{k}_{ij}'^{(1)}) \, g_i(\varepsilon_{ij} - \varepsilon_{k_{1j}}, \varepsilon_{k_i'})$$
$$\times V(\mathbf{k}_{im}', \mathbf{k}_{im}^{(1)}) \, \delta(\mathbf{p}_{ij} - \mathbf{p}_1') \, \delta(\mathbf{p}'' - \mathbf{p}_{ij}^{(1)})$$
$$\times \delta(\varepsilon_{k_i} + \varepsilon_{k_j} + \varepsilon_{k_m} - \varepsilon_{k_i} - \varepsilon_{k_{1j}} - \varepsilon_{k_{1m}}),$$

where

$$\mathbf{k}_{ij}'^{(1)} = \mathbf{k}_i - \mathbf{k}_{1j}; \quad \mathbf{k}_i' = (\mathbf{k}_i + \mathbf{k}_j - \mathbf{k}_{1j}); \quad \mathbf{k}_{im}' = \mathbf{k}_i' - \mathbf{k}_m;$$
$$\mathbf{k}_{im}^{(1)} = \mathbf{k}_{1i} - \mathbf{k}_{1m}; \quad \mathbf{p}_1' = k_{i'} + k_{1j}, \quad \mathbf{p}'' = \mathbf{k}_i' + \mathbf{k}_m, \quad \mathbf{p}^{(1)} = \mathbf{k}_{1i} + \mathbf{k}_{1m};$$

E_{ij} is the energy of the system of particles i and j.

c) If the operators $V_a(t_1)$ and $V_b(t_2)$ pertain to different particles, i.e., $V_a(t_1) = V_{ij}(t_1)$, $V_b(t_2) = V_{mn}(t_2)$, then the integral (11) becomes

$$\left.\begin{aligned} &\langle i \mid J_b \mid f \rangle = \langle i \mid \frac{1}{2} \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} dt_2 P\left\{V_{ij}(t_1) V_{mn}(t_2)\right\} \mid f \rangle; \\ &J_b = \frac{1}{2} V_{ij} \otimes V_{mn} + \frac{1}{2} V_{mn} \otimes V_{ij}, \end{aligned}\right\} \tag{14}$$

where the symbol ⊗ denotes an operator product such that the matrix element of the operator J_b is

$$\langle i \mid J_b \mid f \rangle = 2\pi^2 V_{ij}(\mathbf{k}_{ij}, \mathbf{k}_{ij}^{(1)}) V_{mn}(\mathbf{k}_{mn} - \mathbf{k}_{mn}^{(1)}) \, \delta(\mathbf{p}_{ij} - \mathbf{p}_{ij}^{(1)})$$
$$\times \delta(\mathbf{p}_{mn} - \mathbf{p}_{mn}^{(1)}) \, \delta(\varepsilon_{k_i} + \varepsilon_{k_j} - \varepsilon_{k_{1i}} - \varepsilon_{k_{1j}}) \, \delta(\varepsilon_{k_n} + \varepsilon_{k_m} - \varepsilon_{k_{1n}} - \varepsilon_{k_{1m}}),$$

where

$$\mathbf{k}_{mn} = \mathbf{k}_m - \mathbf{k}_n; \; \mathbf{k}_{mn}^{(1)} = \mathbf{k}_{1m} - \mathbf{k}_{1n};$$
$$\mathbf{p}_{mn} = \mathbf{k}_m + \mathbf{k}_n; \quad \mathbf{p}_{mn}^{(1)} = \mathbf{k}_{1m} + \mathbf{k}_{1n}.$$

Fundamental Theorems of the Formalism of the Diagram-Summation Method

The diagram-summation method employs certain important regrouping properties of the series (4) for the scattering operator T in the system of N particles. These properties are reflected in the following two theorems.

Theorem 1. Let the interaction operator $V_1(t)$ of the system particles be represented in the form of a sum

$$V_1(t) = \sum_a^{n_1} V_a(t), \tag{15}$$

where $V_a(t)$ is a certain sum of pair interaction operators and is part of the total interaction operator of the system particles. The terms $V_a(t)$ satisfy the following conditions: The same particles can participate in interactions that are included in different $V_a(t)$; different $V_a(t)$ do not have equal pair-interaction operators.

In a system of particles whose interaction is given by the operator $V_1(T)$, the scattering operator $T^{(1)}$ is then a sum in the form

$$T^{(1)} = \sum_a^{n_1} T_a, \tag{16}$$

and to determine T_a we have the system of equations

$$T_a = t_a + t_a c \sum_{c \neq a}^{n_1} T_c, \tag{17}$$

where t_a is the operator for the scattering of the particles whose interaction is specified by the operator $V_a(t)$.

Proof. Consider the series (4) for the scattering operator. We substitute in it the operator $V_1(t)$, defined in the form (15):

$$T^{(1)} = \sum_{n=1}^{\infty} \left(\frac{-i}{\hbar}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n P\left\{\sum_a^{n_1} V_a(t_1) \ldots \sum_a^{n_1} V_a(t_n)\right\}. \tag{18}$$

Expanding the products of the sums in the integrand of (18), we obtain the infinite series

$$T^{(1)}=\sum_{a}^{n_1}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_a(t_1)\ldots V_a(t_n)\}\Big]$$
$$+\Big(\sum_{a}^{n_1}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_a(t_1)\ldots V_a(t_n)\}\Big]\Big)$$
$$\times c\Big(\sum_{b\neq a}^{n_1}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_b(t_1)\ldots V_b(t_n)\}\Big]\Big)$$
$$+\Big(\sum_{a}^{n_1}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_a(t_1)\ldots V_a(t_n)\}\Big]\Big)$$
$$\times c\Big(\sum_{b\neq a}^{n_1}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_b(t_1)\ldots V_b(t_n)\}\Big]\Big)$$
$$\times c\Big(\sum_{c\neq b}^{n}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_c(t_1)\ldots V_c(t_n)\}\Big]\Big)$$
$$+\ldots \tag{19}$$

We denote by t_a the scattering operator of the N particles whose interaction is defined by the operator $V_a(t)$. Then the following representation holds true for the operator t_a:

$$t_a=\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar!}\Big)^n\frac{1}{n}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_a(t_1)\ldots V_a(t_n)\}. \tag{20}$$

Bearing this in mind, we rewrite the series (19):

$$T^{(1)}=\sum_a^{n_1}t_a+\sum_a^{n_1}t_ac\sum_{b\neq a}^{n_1}t_b+\sum_a^{n_1}t_ac\sum_{b\neq a}^{n_1}t_bc\sum_{c\neq b}^{n_1}t_c+\ldots \tag{21}$$

Obviously, the series (21) can be represented as the sum of series:

$$T^{(1)}=\sum_a^{n_1}T_a, \tag{22}$$

where

$$T_a=t_a+t_ac\sum_{b\neq a}^{n_1}t_b+t_ac\sum_{b\neq a}^{n_1}t_bc\sum_{c\neq b}^{n_1}t_c+\ldots \tag{23}$$

the series (23) can be contracted to form the expression

$$T_a=t_a+t_ac\sum_{b\neq a}^{n_1}T_b. \tag{24}$$

Theorem 2. Let the interaction operator $V_2(t)$ of the N particles in the system be a sum of operators:

$$V_2(t)=\sum_i^{n_2}V_i(t), \tag{25}$$

where $V_i(t)$ is a certain sum of pair-interaction operators; the operators $V_i(t)$ describe the interaction of the particles in independent n_2 subsystems made up of the N particles.

Then the operator $T^{(2)}$ for the scattering of particles whose interaction is given by the operator of type (25) is the sum

$$T^{(2)}=\sum_i^{n_2}t_i+\sum_{i<j}^{n_2}t_i\otimes t_j+\sum_{i<j<k}^{n_2}t_i\otimes t_j\otimes t_k+\ldots+t_1\otimes t_2\otimes t_3\otimes\ldots\otimes t_{n_2}, \tag{26}$$

where t_i is the scattering operator of the particles whose interaction is given by the operator $V_i(t)$.

Proof. Consider the series (4) for the operator T. We substitute in it the operator $V_2(t)$ defined in the form (25):

$$T^{(2)}=\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\Big\{\sum_i^{n_2}V_i(t_1)\ldots\sum_i^{n_2}V_i(t_n)\Big\}. \tag{27}$$

Expanding the product of the sums of the integrand in (27), we obtain a sum in the form

$$T^{(2)}=\Big(\sum_i^{n_2}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_i(t_1)\ldots V_i(t_n)\}\Big]\Big)$$
$$+\Big(\sum_{i<j}^{n_2}\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_i(t_1)\ldots V_i(t_n)\}\Big]$$
$$\otimes\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_j(t_1)\ldots V_j(t_n)\}\Big]\Big)+\ldots$$
$$+\Big(\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_1(t_1)\ldots V_1(t_n)\}\Big]$$
$$\otimes\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_2(t_1)\ldots V_2(t_n)\}\Big]\otimes\ldots$$
$$\otimes\Big[\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_{n_2}(t_1)\ldots V_{n_2}(t_n)\}\Big]\Big). \tag{28}$$

We define the operator t_i in series form:

$$t_i=\sum_{n=1}^{\infty}\Big(\frac{-i}{\hbar}\Big)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_i(t_1)\ldots V_i(t_n)\}. \tag{29}$$

The sum in the right-hand side of (28) can be then expressed in the form

$$T^{(2)}=\sum_i^{n_2}t_i+\sum_{i<j}^{n_2}t_i\otimes t_j+\ldots+t_1\otimes t_2\otimes t_3\otimes\ldots t_{n_2}. \tag{30}$$

2. EQUATIONS FOR THE THREE- AND FOUR-BODY SCATTERING AMPLITUDES

We proceed to an investigation of systems of three and four pairwise-interacting particles. We assume that the problem of the particle scattering has been solved if a system of integral equations is obtained for the amplitudes of the transitions of a given system from all the possible asymptotic states. In the diagram-summation method it is proposed to find these equations on the basis of a rearrangement and subsequent summation of the series (4) for the scattering operator of the particles in the system.

The general rules for transforming the series (4) into a system of equations for operators whose matrix elements are the sought amplitudes consists in the following:

1) The series corresponding to the scattering of two particles with definite indices are first separated and summed in the elements of the series (4). As a result of this action, the series (4) breaks up into several series, each of which corresponds to a certain scattering operator. These operators determine the transition from states in which the particles are distributed in independent systems of two interacting bodies. We call these auxiliary operators. The series corresponding to the auxiliary operators can be contracted, on the basis of Theorems 1 and 2, into a system of equations which will also be called auxiliary.

2) During the second stage, the auxiliary scattering operators are assigned to those types of operators whose matrix elements are the amplitudes of transitions from all possible asymptotic states of the system or from all possible channels. These will be called the channel operators. For the channel operators one can construct a system of equations on the basis of the auxiliary equations.

Equations for the Operators for the Scattering of Three Pairwise-Interacting Particles

We shall show how to analyze the problem of three-body scattering on the basis of the operator formalism of the diagram-summation method. Assuming that only two-particle forces exist, the three-particle interaction operator V(t) takes the form

$$V_{123}(t)=V_{12}(t)+V_{23}(t)+V_{13}(t). \tag{31}$$

To solve the problem it is necessary to represent the operator T_{123} in the form of a sum of channel operators, and then obtain a system of equations for the channel operators in accordance with the scheme described above. Following the algebra rules of the diagram-summation method, we sum the infinite series corresponding to the scattering operators of each of the three pairs of particles (12), (23), and (31) in the elements of the series (4) for the three-particle scattering operator.

Let us examine the structure of the interaction operator $V_{123}(t)$. In the case of the three-body problem, only one pair of particles can interact in a definite instant of time. This means that each of the terms of the operator $V_{123}(t)$ in (31) characterizes a certain state of the system, wherein the interaction is turned on only between particles that are joined into independent pairs at a fixed instant of time.

Hence, as follows from Theorem 1, the operator T_{123} of the three-particle scattering amplitude should be the sum of three operators:

$$T_{123}=T_{12}+T_{23}+T_{13}. \tag{32}$$

The matrix elements of each of the operators T_{12}, T_{23}, and T_{31} corresponds to the amplitude of a transition from a state in which only one pair of particles interacts. Consequently these are auxiliary operators. According to Theorem 1, we have for the determination of the auxiliary operators T_{12}, T_{23}, and T_{31} the following system of equations:

$$\left.\begin{aligned} T_{12}&=t_{12}+t_{12}c\,[T_{13}+T_{23}];\\ T_{23}&=t_{23}+t_{23}c\,[T_{12}+T_{13}];\\ T_{13}&=t_{13}+t_{13}c\,[T_{12}+T_{23}]. \end{aligned}\right\} \tag{33}$$

Here t_{12}, t_{23}, and t_{31} are the two-particle scattering operators.

In the three-body problem, the operators T_{12}, T_{23}, and T_{31} turn out to be channel operators. Therefore the system (33) is the desired system of equations for the channel operators.

If we introduce operators whose matrix elements are amplitudes for the transition from one two-particle channel of a reaction with three bodies into another, then we have instead of the three operators $T_{12,\,12}$, $T_{12,\,23}$, $T_{12,31}$, $T_{23,12}$, $T_{23,23}$, $T_{23,31}$, $T_{31,12}$, $T_{31,23}$, and $T_{31,31}$.

To determine these operators we can write on the basis of (33) a system of nine equations. To simplify the notation, we designate by α or β a certain pair of particles interacting in the initial or final state. Obviously, α and β can assume the values (12), (23), or (31). Then the nine operators defined above are designated by the symbol $T_{\alpha,\,\beta}$, and the system of equations for them takes the form

$$\left.\begin{aligned} T_{\alpha,\,\alpha}&=t_\alpha+t_\alpha c\sum_{\gamma\neq\alpha}T_{\gamma,\,\alpha};\\ T_{\alpha,\,\beta}&=t_\alpha c\sum_{\gamma\neq\alpha}T_{\gamma,\,\beta},\\ \beta&\neq\alpha. \end{aligned}\right\} \tag{34}$$

If we write down integral equations for the kernels of the operators $T_{\alpha,\,\alpha}$ and $T_{\alpha,\beta}$ on the basis of (34), and integrate over the energy of the intermediate particles, we obtain a system of coupled equations equivalent to the Faddeev equations.[2] This is understandable, for in the case of the three-body problem, when only one pair of particles can interact at a definite instant of time, a description of the free motion of the particles by the Green's function of the entire system or by a product of Green's functions of individual particles does not lead to different results, as is the case in problems where four or more particles take part.

The Four-Body Problem. Equations for the Auxiliary Amplitudes

We now consider a system consisting of four different pairwise-interacting particles. The interaction operator of this system is a sum in the form

$$V_{1234}(t)=V_{12}(t)+V_{13}(t)+V_{14}(t)+V_{23}(t)+V_{24}(t)+V_{34}(t). \tag{35}$$

It is required to find the system of integral equations for the channel operators. Following the algebra rules of the diagram-summation methods, it is first necessary to substitute V_{1234} in the form (35) in the elements of the series for the operator T, and then sum all the possible infinite series corresponding to the two-particle scattering operators.

Since two independent pairs of particles that interact within each pair can exist simultaneously in the four-particle system, it is necessary to sum in parallel two infinite series corresponding to the operators of scattering of two particles with fixed indices.

The interaction operator of a system of four particles in a state when the particles are grouped into two independent pairs, say (12) and (34), is given by

$$V_{12,34}(t)=V_{12}(t)+V_{34}(t). \tag{36}$$

It is easy to show, by cyclic permutation of the particles, that two other similar states can exist for a system of four pairwise-interacting particles, and in these states the particle-interaction operators are

$$\left.\begin{aligned} V_{13,24}(t)&=V_{13}(t)+V_{24}(t);\\ V_{14,23}(t)&=V_{14}(t)+V_{32}(t). \end{aligned}\right\} \tag{37}$$

Consequently, the interaction operator $V_{1234}(t)$ breaks up into a sum of four terms:

$$V_{1234}(t)=V_{12,34}(t)+V_{13,24}(t)+V_{14,23}(t). \tag{38}$$

Hence, by Theorem 1, the four-particle scattering operator T can be expressed as a sum of three operators:

$$T=T_{12,34}+T_{13,24}+T_{14,23} \tag{39}$$

or

$$T=\sum_{ij,\,i'j'} T_{ij,\,i'j'}, \tag{40}$$

where the subscripts ij and i'j' obviously take on the values (12, 34), (13, 24), and (14, 23).

To determine the operators $T_{ij,\,i'j'}$, we can, as follows from Theorem 1, write down a system of equations in the form

$$T_{ij,\,i'j'}=t_{ij,\,i'j'}+t_{ij,\,i'j'}c\sum_{\substack{mn,\,m'n'\\ mn\neq ij\\ mn\neq i'j'}} T_{mn,\,m'n'} \tag{41}$$

[the subscripts mn and m'n' can assume the values (12, 34); (13, 24,) and (14, 23)]. In (41), the operator $t_{ij,\,i'j'}$ corresponds to the four-particle scattering amplitude under the condition that their interaction is given by an operator $V_{ij,\,i'j'}$ in the form $V_{ij,\,i'j'}(t)=V_{ij}(t)+V_{i'j'}(t)$. This means that in accordance with Theorem 2 the operator $t_{ij,\,i'j'}$ can be written with the aid of the two-particle scattering operators t_{ij} and $t_{i'j'}$, using the equation

$$t_{ij,\,i'j'}=t_{ij}t_{i'j'}+t_{ij}\otimes t_{i'j'}. \tag{42}$$

Equation (41) for the operator $T_{ij,\,i'j'}$ can then be rewritten in the form

$$\begin{aligned} T_{ij,\,i'j'}&=(t_{ij}+t_{i'j'}+t_{ij}\otimes t_{i'j'})\\ &+(t_{ij}+t_{i'j'}+t_{ij}\otimes t_{i'j'})\,c\sum_{\substack{mn,\,m'n'\\ mn\neq ij\\ mn\neq i'j'}} T_{mn,\,m'n'}. \end{aligned} \tag{43}$$

Let us examine the free term of this equation. The two terms t_{ij} and $t_{i'j'}$ in this equation are the operators for the scattering of four particles in the case when the interaction of only one pair of particles, ij or i'j', is considered. The term $(t_{ij}\otimes t_{i'j'})$ is the four-particle scattering operator when the particles interact simultaneously in the two independent pairs. In other words, the operators t_{ij} or $t_{i'j'}$, and the operator $t_{ij}\otimes t_{i'j'}$ determine two fundamentally different processes of four-particle scattering. We therefore introduce in place of the operator $T_{ij,\,i'j'}$ two four-particle scattering operators T_{ij} and $T_{ij,\,i'j'}$, the matrix elements of which correspond to the amplitudes of the transition of a system of four particles from a state in which only one pair of particles interacts or two independent pairs of particles interact, respectively. The newly introduced operators will be called auxiliary operators.

If we specify the system of equations for the auxiliary operators T_{ij} and $T^{(1)}_{ij,\,i'j'}$ in the form

$$\left.\begin{aligned} T_{ij}&=t_{ij}+t_{ij}c\sum_{\substack{mn\neq ij,\ m'n'\neq ij\\ mn\neq i'j',\ m'n'\neq i'j'}}(T_{mn}+T_{m'n'}+T^{(1)}_{mn,\,m'n'});\\ T^{(1)}_{ij,\,i'j'}&=t_{ij}\otimes t_{i'j'}+(t_{ij}\otimes t_{i'j'})\\ &\times c\sum_{\substack{mn\neq ij,\ m'n'\neq ij\\ mn\neq i'j',\ m'n'\neq i'j'}}(T_{mn}+T_{m'n'}+T_{mn,\,m'n'}), \end{aligned}\right\} \tag{44}$$

then, bearing (43) in mind, it can be shown that $T_{ij,\,i'j'}=T_{ij}+T_{i'j'}+T^{(1)}_{ij,\,i'j'}$.

Consequently, the scattering operator T can be represented as a sum of auxiliary operators: $T=\sum_{ij,\,i'j'}(T_{ij}+T_{i'j'}+T^{(1)}_{ij,\,i'j'})$, and the system (44) can be used to find the auxiliary operators.

Equations for the Channel Operators of Four-Particle Scattering

In the four-body problem, the system (44) for the auxiliary operators is not the final one, as was the case for the three-body problem. The kernel of the equation for the operator T_{ij} contains nonintegrable δ functions. To solve the four-body problem it is necessary to go over from the auxiliary operators to channel operators.

In the present problem the channel operators are those whose matrix elements are the amplitudes of transitions from three fundamentally different asymptotic states: first from states where there are two independent pairs of particles (states of the first type); second, from states where there is a system of three pairwise-interacting particles and one free particle (states of the second type); third, from states where there is a system of two interacting particles and two free particles (states of the third type). The operators $T_{ij,\,i'j'}$ obtained above are the channel operators of the first type. Consequently, our problem is to determine the channel operators of the second and third types. To this end we turn to the operators T_{ij} and $T_{i'j'}$. The free term in Eqs. (44) for these operators obviously belongs to the channel operators of the third type. The kernel of the equations for the operators T_{ij} and $T_{i'j'}$ shows that these operators should include also operators whose matrix elements are the amplitudes of transitions from states of the first type. It follows therefore that T_{ij} is a certain sum of channel operators. To separate them, we integrate Eq. (44) once for T_{ij}:

$$T_{ij}=t_{ij}+t_{ij}c\sum_{\substack{kl\neq ij\\ kl\neq i'j'}}t_{kl}+t_{ij}c\sum_{\substack{kl\neq ij\\ kl\neq i'j'}}t_{kl}c\sum_{\substack{mn\neq kl,\ m'n'\neq kl\\ mn\neq k'l',\ m'n'\neq k'l'}}(T_{mn}+T_{m'n'}+T^{(1)}_{mn,\,m'n'})+t_{ij}c\sum_{\substack{mn\neq ij,\ m'n'\neq i'j'\\ mn\neq i'j',\ m'n'\neq i'j'}}T^{(1)}_{mn,\,m'n'}. \tag{45}$$

We shall show that the free term $t_{ij}c\sum_{kl\neq ij,\ kl\neq i'j'}t_{kl}$ and the kernel in Eq. (45) are sums of the second iterations of the equations for the three-particle scattering operators $T_{ij}(\eta_1)$ and $T_{ij}(\eta_2)$. The indices η_1 and η_2 denote here three-particle systems made up of a total of four particles in such a way that both systems contain two identical particles i and j.

Indeed, the scattering operator of the three particles making up the system η_1 or η_2, when the particles i and j interact in the initial state, satisfies the equations

$$\left.\begin{aligned} T_{ij}(\eta_1)&=t_{ij}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_1}T_{kl}(\eta_1);\\ T_{ij}(\eta_2)&=t_{ij}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_2}T_{kl}(\eta_2).\end{aligned}\right\} \tag{46}$$

We integrate equations (46) once:

$$\left.\begin{aligned} T_{ij}(\eta_1)&=t_{ij}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_1}t_{kl}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_1}t_{kl}c\sum_{mn\neq kl}^{mn\subset\eta_1}T_{mn}(\eta_1);\\ T_{ij}(\eta_2)&=t_{ij}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_2}t_{kl}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_2}t_{kl}c\sum_{mn\neq kl}^{mn\subset\eta_2}T_{mn}(\eta_2).\end{aligned}\right\} \tag{47}$$

Comparing the free terms in (45) and (47), we can show that

$$t_{ij}c\sum_{\substack{kl\neq ij\\ kl\neq i'j'}}\equiv t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_1}t_{kl}+t_{ij}c\sum_{kl\neq ij}^{kl\subset\eta_2}t_{kl}.$$

We can therefore substitute the operator T_{ij} in the form of a sum of three operators:

$$T_{ij}=T^{(1)}_{ij}+T^{\eta_1}_{ij}+T^{\eta_2}_{ij}, \tag{48}$$

which satisfy the equations

$$\left.\begin{aligned} T^{(1)}_{ij}&=t_{ij}+t_{ij}c\sum_{\substack{mn\neq ij,\ m'n'\neq ij\\ mn\neq i'j',\ m'n'\neq i'j'}}T^{(1)}_{mn,m'n'};\\ T^{\eta_\nu}_{ij}&=t_{ij}c\sum_{\substack{kl\neq ij\\ kl\neq i'j'}}^{kl\subset\eta_\nu}t_{kl}+t_{ij}c\sum_{\substack{kl\neq ij\\ kl\neq i'j'}}^{m\subset\eta_\nu}\\ &\times c\sum_{\substack{mn\neq kl,\ mn\neq k'l'\\ m'n'\neq kl,\ m'n'\neq k'l'}}(T_{mn}+T_{m'n'}+T_{mn,\,m'n'}),\end{aligned}\right\} \tag{49}$$

where $\nu=1$ or 2.

The equations for $T^{\eta_1}_{ij}$ and $T^{\eta_2}_{ij}$ can be rewritten in the form

$$T^{\eta_1}_{ij}=\sum_{kl\subset\eta_1}M^{\eta_1}_{ij,\,kl}+\sum_{kl\subset\eta_1}M^{\eta_1}_{ij,\,kl}c\times\sum_{\substack{mn\neq kl,\ mn\neq k'l',\ m'n'\neq kl,\ m'n'\neq k'l'\\ mn\not\subset\eta_1\\ \mu\neq\eta_1}}(T^{(1)}_{mn}+T^{\mu}_{mn}+T^{(1)}_{mn,\,m'n'});$$

$$T^{\eta_2}_{ij}=\sum_{kl\subset\eta_2}M^{\eta_2}_{ij,\,kl}+\sum_{kl\subset\eta_2}M^{\eta_2}_{ij,\,kl}c\times\sum_{\substack{mn\neq kl,\ mn\neq k'l',\ m'n'\neq kl,\ m'n'\neq k'l'\\ mn\not\subset\eta_2\\ \mu\neq\eta_2}}(T^{(1)}_{mn}+T^{\mu}_{mn}+T^{(1)}_{mn,\,m'n'}), \tag{50}$$

where $M^{\eta_\nu}_{ij,\,kl}$ satisfies the equation

$$M^{\eta_\nu}_{ij,\,kl}=t_{ij}ct_{kl}+t_{ij}c\sum_{\substack{mn\subset\eta_\nu\\ mn\neq ij}}M_{mn,\,kl}. \tag{51}$$

Thus, we obtain three types of equations for the determination of the channel operators in a system of four particles:

$$\left.\begin{aligned} T^{(1)}_{ij,\,i'j'}&=t_{ij}\otimes t_{i'j'}+t_{ij}\otimes t_{i'j'}c\sum_{\substack{mn\neq i'j'\\ mn\neq ij}}(T^{(1)}_{mn}+T^{\mu}_{mn}+T^{(1)}_{mn,\,m'n'});\\ T^{(1)}_{ij}&=t_{ij}+t_{ij}c\sum_{\substack{mn\neq ij,\ m'n'\neq ij\\ mn\neq i'j',\ m'n'\neq i'j'}}T^{(1)}_{mn,\,m'n'};\\ T^{\eta_\nu}_{ij}&=\sum_{kl\subset\eta_\nu}M^{\eta_\nu}_{ij,\,kl}\\ &+\sum_{kl\subset\eta_\nu}M^{\eta_\nu}_{ij,\,kl}c\sum_{\substack{mn\neq kl,\ mn\neq k'l'\\ m'n'\neq kl,\ m'n'\neq k'l'\\ mn\not\subset\eta_\nu,\ \mu\neq\eta_\nu}}(T^{(1)}_{mn}+T^{\mu}_{mn}+T^{(1)}_{mn,\,m'n'}).\end{aligned}\right\} \tag{52}$$

Equations (52) can be reduced to equations with compact kernels, by subtracting from the operators $T^{(1)}_{ij}$, $T^{(1)}_{ij,\,i'j'}$, and $T^{\eta_\nu}_{ij}$ the amplitudes of the partially-coupled processes, i.e., by subtracting the operators that are free terms in the equations.

We are interested, however, not in Eqs. (52), but in equations for operators whose matrix elements are the amplitudes of transitions from a definite initial state to a definite final state. In the four-body problem these operators are

$$T_{ij,\,i'j'/kl,\,k'l'},\ T_{ij,\,i'j'/kl},\ T_{ij,\,i'j'/kl^{\xi}},$$
$$T_{ij/kl,\,k'l'},\ T_{ij/kl},\ T_{ij/kl^{\xi}},\ T_{ij^{\eta}/kl^{\xi}},\ T_{ij^{\eta}/kl},\ T_{ij^{\eta}/kl,\,k'l'}.$$

The initial and final states are designated here by the indices ij, i'j' and kl, k'l' if there are two pairs of interacting particles; the indices ij and kl are used if there is one pair of interacting particles and two free particles; the indices ij^{η} and kl^{ξ} are used if there is a system of three pairwise-interacting particles η and ξ.

We define the indicated operators in such a way that they satisfy the equations

$$T_{ij^{\eta}/kl}=\sum_{\varkappa\zeta\subset\eta}M^{\eta}_{ij,\,\varkappa\zeta}c\Big[\sum_{\substack{mn\not\subset\eta\\ mn\neq\varkappa'\zeta'}}\Big(\sum_{\mu}T_{mn^{\mu}/kl}+T_{mn/kl}\Big)+\sum_{\substack{mn\neq\varkappa\zeta\\ mn\neq\varkappa'\zeta'}}T_{mn,\,m'n'/kl}\Big];$$

$$T_{ij,\,i'j'/kl}=t_{ij}\otimes t_{i'j'}c\Big[\sum_{\substack{mn\neq ij\\ mn\neq i'j'}}\Big(\sum_{\mu}T_{mn^{\mu}/kl}+T_{mn/kl}\Big)+T_{mn,\,m'n'/kl}\Big];$$

$$T_{ij/kl}=t_{ij}\delta(ij,\,kl)+t_{ij}c\sum_{\substack{mn\neq ij\\ mn\neq i'j'}}T_{mn,\,m'n'/kl};$$

$$T_{ij\eta/kl\xi} = M^{\eta}_{ij,\,kl}\delta(\eta,\xi) + \sum_{\varkappa\zeta\subset\eta} M^{\eta}_{ij,\,\varkappa\zeta}c\Big[\sum_{\substack{mn\neq\varkappa'\zeta'\\ mn\not\subset\eta}}\Big(\sum_{\mu} T_{mn\mu/kl\xi} + T_{mn/kl\xi}\Big) + \sum_{\substack{mn\neq\varkappa\zeta\\ mn\neq\varkappa'\zeta'}} T_{mn,\,m'n'/kl\xi}\Big];$$

$$T_{ij,\,i'j'/kl\xi} = t_{ij}\otimes t_{i'j'}c\Big[\sum_{\substack{mn\neq ij\\ mn\neq i'j'}}\Big(\sum_{\mu} T_{mn\mu/kl\xi} + T_{mn/kl\xi}\Big) + T_{mn,\,m'n'/kl\xi}\Big];$$

$$T_{ij/kl\xi} = t_{ij}c\sum_{\substack{mn\neq ij\\ mn\neq i'j'}} T_{mn,\,m'n'/kl\xi};$$

$$T_{ij\eta/kl,\,k'l'} = \sum_{\varkappa\zeta\subset\eta} M^{\eta}_{ij,\,\varkappa\zeta}c\Big[\sum_{\substack{mn\not\subset\eta\\ mn\neq\varkappa'\zeta'}}\Big(\sum_{\mu} T_{mn\mu/kl,\,k'l'} + T_{mn/kl,\,k'l'}\Big) + \sum_{\substack{mn\neq\varkappa\zeta\\ m'n'\neq\varkappa'\zeta'}} T_{mn,\,m'n'/kl,\,k'l'}\Big];$$

$$T_{ij,\,i'j'/kl,\,k'l'} = t_{ij}\otimes t_{i'j'}\delta(i_j,\,kl) + t_{ij}\otimes t_{i'j'}c\Big[\sum_{\substack{mn\neq ij\\ mn\neq i'j'}}\Big(\sum_{\mu} T_{mn\mu/kl,\,k'l'} + T_{mn/kl,\,k'l'}\Big) + T_{mn,\,m'n'/kl,\,k'l'}\Big];$$

$$T_{ij/kl,\,k'l'} = t_{ij}c\sum_{\substack{mn\neq ij\\ mn\neq i'j'}} T_{mn,\,m'n'/kl,\,k'l'}. \tag{53}$$

3. NONRELATIVISTIC DIAGRAMS AND GRAPHICAL REPRESENTATION OF INTEGRAL EQUATIONS FOR TWO-, THREE-, AND FOUR-PARTICLE SCATTERING AMPLITUDES

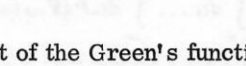
Rules for Reading Nonrelativistic Diagrams

As already discussed in the introduction, the operator formalism described above was developed on the basis of a graphic representation of the matrix elements of the scattering operator, followed by summation of the diagrams into graphic equations.

The diagram representation of iterations of integral equations, and consequently also of the amplitudes of scattering processes, is the method used to analyze many-particle reactions in nuclear and atomic physics. It is therefore important to describe here the graphical method of deriving the integral equations for the amplitudes of scattering of several particles, and demonstrate by the same token the exact correspondence between the iteration series of the considered equation and the infinite series of the contributions of the nonrelativistic diagrams.

The diagram-summation method is general because of the unified rules for the representation of individual functions that make up the matrix elements of the scattering operator written down for a system with an arbitrary number of particles. The rules are the following:

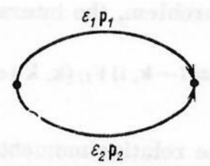
1) The Green's function $g_c(E_c, E_{p_c})$ of a particle c with momentum p_c and energy E_c, namely $g_c(E_c, E_{p_c}) = i(E_c - E_{p_c} + i\tau)^{-1}$, corresponds to the propagation line



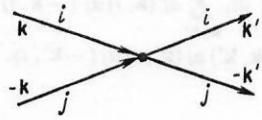
2) The product of the Green's functions of two particles comprising a system with energy E and momentum p is

$$\frac{1}{2\pi i}\int\frac{d\varepsilon_1}{(2\pi)}\int\frac{dp_1}{(2\pi)^3}\,g_1(\varepsilon_1,\,\varepsilon_{p_1})\,g_2(\varepsilon_2,\,\varepsilon_{p_2}),$$

where $p_2 = p - p_1$ and $\varepsilon_2 = E - \varepsilon_1$ are set in correspondence to the loop:



3) The functions $iV_{ij}(k, k')$, which are matrix elements of the operator V_{ij} of the interaction of two particles (i, j), are set in correspondence to the vertex of the intersection of the propagation lines of particles i and j:



4) On going from an amplitude obtained on the basis of summation of matrix elements to an amplitude whose square determines the differential cross sections, it is necessary to multiply the former by the coefficient

$$\frac{-i}{2\pi}\sqrt{\frac{dE}{dk}\cdot\frac{dE}{dk_0}},$$

where E is the total energy of the system; k_0 is the relative momentum of the particles in the initial state; k is the relative momentum of the particles after the scattering.

5) The momentum and energy conservation laws must be satisfied in each two-particle interaction vertex.

Graphical Method of Obtaining the Integral Equations

The diagram-summation method consists of representing the matrix elements of the series (4) of the four-dimensional perturbation theory by diagrams of the ladder type, on the basis of the rules given in the preceding section. The graphical representation of the matrix elements of the series (4), which determines the scattering operator, makes it possible, first, to reconstruct this series into infinite series that correspond to the channel amplitudes, i.e., to the amplitudes of transitions from one definite asymptotic state to another, and second, determine the types of connections between the diagrams, and consequently to convert the series into graphical equations on the basis of which integral equations can be derived for the channel amplitudes.

The sequence of the reconstruction of the series of diagrams of four-dimensional perturbation theory is the following. The series is first broken up into several series in which the diagrams have definite starting-point and end-point topologies. Partial summations of the diagrams are then carried out in succession in the newly obtained series, so as to make the vertices correspond to the operators of the two-, three-, (N−1)-particle scattering amplitudes. After this partial summation, the infinite series of the diagrams are summed into graphical equations.

Integral Equation for the Two-Body Problem

In the two-body problem, the interaction operator is

$$V_{ij}(t)=\sum_{\mathbf{k},\mathbf{k}'} a_i^\dagger(\mathbf{k},t)\,a_j^\dagger(-\mathbf{k},t)\,V_{ij}(\mathbf{k},\mathbf{k}')\,a_i(\mathbf{k}',t)\,a_j(\mathbf{k}',t), \quad (54)$$

where k and k' are the relative momenta of the particles before and after scattering. Our problem is to calculate the matrix elements of the series (4) for an interaction operator in the form (54) and to set them in correspondence with diagrams. The first-order matrix element is

$$\left.\begin{aligned} &M_1=\langle a|T_1|b\rangle;\\ &T_1=\frac{-\mathrm{i}}{\hbar}\int dt_1\sum_{\mathbf{k},\mathbf{k}'} a_i^\dagger(\mathbf{k},t)\,a_j^\dagger(-\mathbf{k},t)\\ &\quad\times V_{ij}(\mathbf{k},\mathbf{k}')\,a_i(\mathbf{k}',t)\,a_j(-\mathbf{k}',t).\end{aligned}\right\} \quad (55)$$

Hence

$$M_1=-2\pi\mathrm{i}V(\mathbf{k},\mathbf{k}')\,\delta(E_k+E_{-k}-E_{k'}-E_{-k'}). \quad (56)$$

The second-order matrix element in the two-body problem is

$$\left.\begin{aligned} &M_2=\langle a|T_2|b\rangle;\\ &T_2=\left(\frac{-\mathrm{i}}{\hbar}\right)^2\frac{1}{2}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2P\\ &\times\Big\{\sum_{\mathbf{k},\mathbf{p},\mathbf{k}',\mathbf{p}'}[a_i^\dagger(\mathbf{k},t_1)\,a_j^\dagger(-\mathbf{k},t_1)\,V_{ij}(\mathbf{k},\mathbf{p})\,a_i(\mathbf{p},t_1)\,a_j(-\mathbf{p},t_1)]\\ &\times[a_i^\dagger(\mathbf{p}',t_2)\,a_j^\dagger(-\mathbf{p}',t_2)\,V(\mathbf{p}',\mathbf{k}')\,a_i(\mathbf{k}',t_2)\,a_j(-\mathbf{k}',t_2)]\Big\}.\end{aligned}\right\} \quad (57)$$

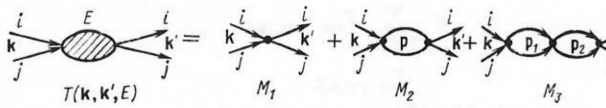
Consequently

$$\begin{aligned} M_2=2\pi\int\frac{d\mathbf{p}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{d\varepsilon_1}{(2\pi)}\,&V_{ij}(\mathbf{k},\mathbf{p})\,V_{ij}(\mathbf{p},\mathbf{k}')\,[\varepsilon_1-E_p+\mathrm{i}\tau]^{-1}\\ &\times[E-\varepsilon_1-E_{-p}+\mathrm{i}\tau]^{-1}\,\delta(E_k+E_{-k}-E_{k'}-E_{-k'})\\ &\times\delta(E_k+E_{-k}-E_p-E_{-p}),\end{aligned} \quad (58)$$

where $E_a = E_k + E_{-k} = E$; $E_b = E_{k'} + E_{-k'} = E$. The third-order matrix element is

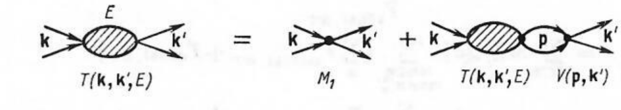
$$\begin{aligned} M_3=\mathrm{i}2\pi\int_{-\infty}^{\infty}\frac{d\varepsilon_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\varepsilon_2}{2\pi}\int\frac{d\mathbf{p}_1}{(2\pi)^3}\int\frac{d\mathbf{p}_2}{(2\pi)^3}\\ \times\{V(\mathbf{k},\mathbf{p}_1)\,V(\mathbf{p}_1,\mathbf{p}_2)\,V(\mathbf{p}_2,\mathbf{k}')\,[\varepsilon_1-E_{p_1}+\mathrm{i}\tau]^{-1}\\ \times[E-\varepsilon_1-E_{-p_1}+\mathrm{i}\tau]^{-1}[\varepsilon_2-E_{p_2}+\mathrm{i}\tau]^{-1}[E-\varepsilon_2-E_{-p_2}+\mathrm{i}\tau]^{-1}\}\\ \times\delta(E-E_{p_1}-E_{-p_2})\,\delta(E_{p_1}+E_{-p_1}-E_{p_2}-E_{-p_2})\\ \times\delta(E_{p_2}+E_{-p_2}-E)\text{, etc.}\end{aligned} \quad (59)$$

Bearing in mind the rules for reading the diagrams, we can represent the two-particle scattering amplitude in the form of a sum of contributions from an infinite number of diagrams:



(60)

The diagram series (60) can be summed into the graphical equation



(61)

on the basis of which we can obtain the integral equation

$$T_{ab}(\mathbf{k},\mathbf{k}',E)=V(k,k')+\int\frac{d\mathbf{p}}{(2\pi)^3}\cdot\frac{T(\mathbf{k},\mathbf{p},E)\,V(\mathbf{p},\mathbf{k}')}{E-p^2/m+\mathrm{i}\tau}. \quad (62)$$

This result is the well-known equation for the two-body scattering amplitude.

The derivation of this equation was presented only to illustrate, with a simple example, the diagram-summation method. In addition, since problems of scattering of several particles will be considered under the assumption that pair forces are present, the principal parts of the matrix elements of the amplitudes for the scattering of several particles, as will be shown later on, will be the two-particle scattering amplitudes. The graphical representation of the two-particle scattering amplitude will therefore be essential for the solution of problems with three and more particles.

Three-Particle Scattering Amplitude

On the basis of the diagram-summation method, as shown above, we can obtain integral equations for the amplitudes of scattering of three free particles. We present a graphical derivation of these equations. Assuming the presence of two-particle forces, the scattering operator of a system of free particles is

$$V_{123}(t)=V_{12}(t)+V_{13}(t)+V_{23}(t), \quad (63)$$

where the operator $V_{ij}(t)$ is defined in (5).

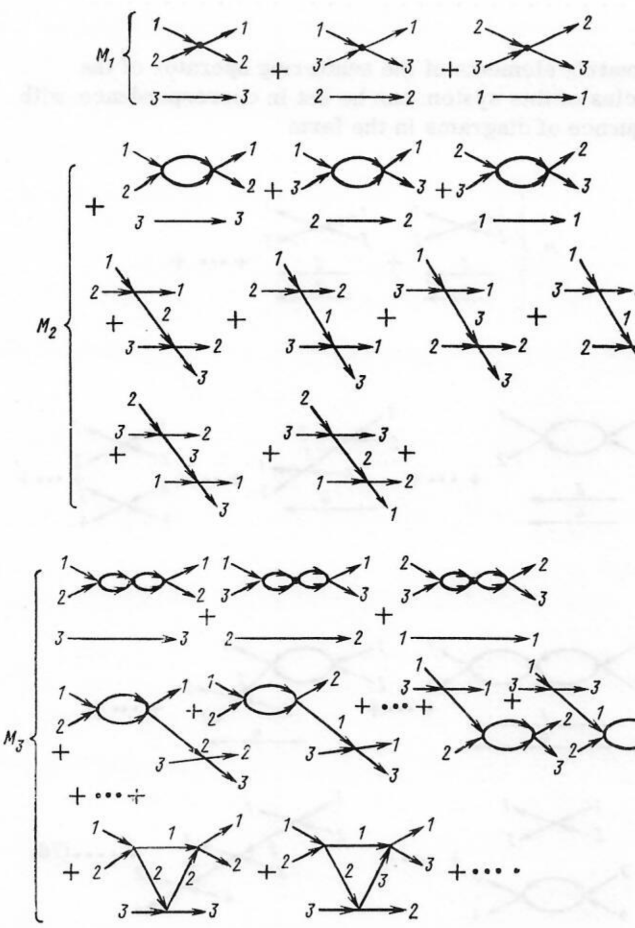
Substituting the interaction operator (62) in the elements of the series (4), we obtain a representation for the three-particle scattering operator, in the form

$$T_{123}=T_{123}^{(1)}+T_{123}^{(2)}+T_{123}^{(3)}+\ldots+T_{123}^{(n)}+\ldots, \quad (64)$$

where

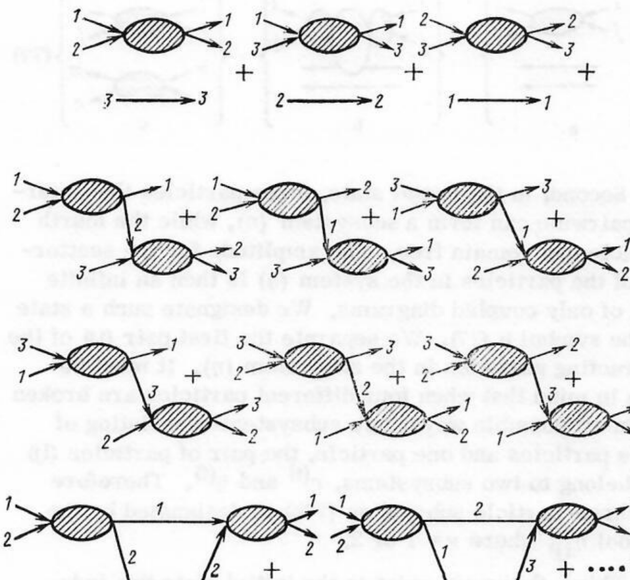
$$\left.\begin{aligned} &T_{123}^{(1)}=\frac{-\mathrm{i}}{\hbar}\int_{-\infty}^{\infty}dtV_{123}(t);\\ &T_{123}^{(2)}=\left(\frac{-\mathrm{i}}{\hbar}\right)^2\frac{1}{2}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2P\{V_{123}(t_1)\,V_{123}(t_2)\};\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &T_{123}^{(n)}=\left(\frac{-\mathrm{i}}{\hbar}\right)^n\frac{1}{n!}\int_{-\infty}^{\infty}dt_1\ldots\int_{-\infty}^{\infty}dt_nP\{V_{123}(t_1)\ldots V_{123}(t_n)\}.\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \quad (65)$$

The matrix elements of the series (65) can be represented, using the rules described above, by means of the following diagrams:



(66)

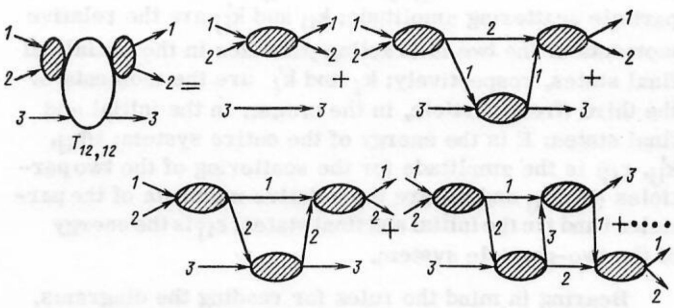
In the diagram series (66) we can sum the vertices corresponding to the scattering of two identical particles, as the result of which we obtain diagrams whose vertices correspond to two-particle scattering amplitudes: *



(69)

The diagrams of the series (69) can then be distributed in nine series, in each of which the diagrams have the same initial-point and end-point topology. The series obtained will obviously determine the channel amplitudes. We denote these amplitudes by $T_{\alpha\beta}$, where α and β stand for the interacting pair of particles in the initial and final states [α and β can assume the values (12), (23), and (31)].

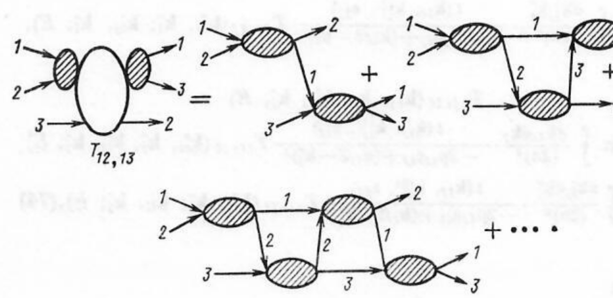
Let us consider the next sequence of diagrams for the amplitude $T_{12,\,12}$:



(70)

Obviously, an analogous structure is possessed by the sequence for the amplitudes $T_{23,\,23}$ and $T_{31,\,31}$.

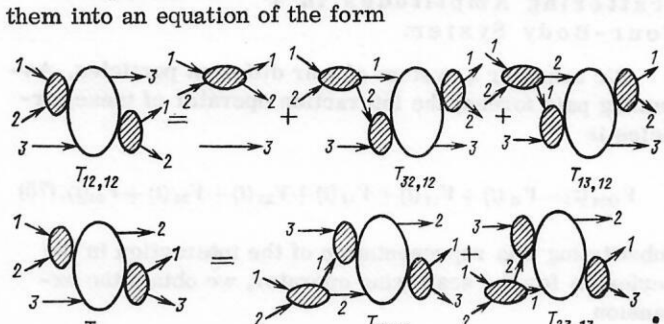
The sequence of diagrams for the amplitudes $T_{12,\,13}$ takes the following form:



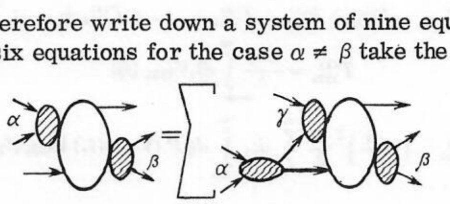
(71)

It should be noted that all six sequences of diagrams for the amplitudes $T_{\alpha,\,\beta}$, where $\alpha \neq \beta$, are similar in structure.

Comparing the sequences (70) and (71), we can sum them into an equation of the form

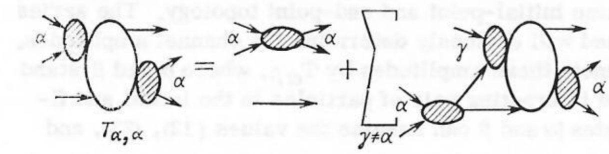


We can therefore write down a system of nine equations, in which six equations for the case $\alpha \neq \beta$ take the form



(72)

and three equations for the case $\alpha = \beta$ take the form



(73)

On the basis of the graphical equations (72) and (73) we can write down integral equations by introducing the following notation: $T_{ij,\,ij}(\mathbf{k}_{ij},\,\mathbf{k}_l;\,\mathbf{k}'_{ij},\,\mathbf{k}'_l;\,E)$ is the three-particle scattering amplitude; $\mathbf{k}_{ij}$ and $\mathbf{k}'_{ij}$ are the relative momenta of the two interacting particles in the initial and final states, respectively; $\mathbf{k}_l$ and $\mathbf{k}'_l$ are the momenta of the third (free) particle, in the c.m.s., in the initial and final states; E is the energy of the entire system; $t(\mathbf{k}_{ij},\,\mathbf{k}'_{ij},\,\varepsilon_{ij})$ is the amplitude for the scattering of the two particles (ij); $\mathbf{k}_{ij}$ and $\mathbf{k}'_{ij}$ are the relative momenta of the particles i and j in the initial and final states; ε_{ij} is the energy of the two-particle system.

Bearing in mind the rules for reading the diagrams, we write down the integral equations for the amplitudes $T_{ij,\,ij}$ and $T_{ij,\,jl}$:

$$T_{ij,\,ij}(\mathbf{k}_{ij},\,\mathbf{k}_l;\,\mathbf{k}'_{ij},\,\mathbf{k}'_l,\,E)=\frac{3}{4}t(\mathbf{k}_{ij},\,\mathbf{k}'_{ij},\,\varepsilon_{ij})\,\sigma(\mathbf{k}_l-\mathbf{k}'_l)$$
$$-4\pi\int\frac{d\mathbf{k}''_{li}d\mathbf{k}''_j}{(2\pi)^6}\cdot\frac{t(\mathbf{k}_{ij},\,\mathbf{k}^{(1)}_{ij},\,\varepsilon_{ij})}{-2\mu_{ij}\varepsilon_{ij}+(\mathbf{k}_l/2-\mathbf{k}''_j)^2}T_{li,\,ij}(\mathbf{k}''_{li},\,\mathbf{k}''_j;\,\mathbf{k}'_{ij},\,\mathbf{k}'_l,\,E)$$
$$-4\pi\int\frac{d\mathbf{k}''_{lj}\mathbf{k}''_i}{(2\pi)^6}\cdot\frac{t(\mathbf{k}_{ij},\,\mathbf{k}^{(2)}_{ij},\,\varepsilon_{ij})}{-2\mu_{ij}\varepsilon_{ij}+(\mathbf{k}_l/2-\mathbf{k}_i)^2}T_{lj,\,ij}(\mathbf{k}''_{lj},\,\mathbf{k}''_i;\,\mathbf{k}'_{ij},\,\mathbf{k}'_l;\,E),$$

$$T_{ij,\,il}(\mathbf{k}_{ij},\,\mathbf{k}_l;\,\mathbf{k}'_{il},\,\mathbf{k}'_j;\,E)$$
$$=-4\pi\int\frac{d\mathbf{k}_{li}d\mathbf{k}_j}{(2\pi)^6}\cdot\frac{t(\mathbf{k}_{ij},\,\mathbf{k}^{(1)}_{ij},\,\varepsilon_{ij})}{-2\mu_{ij}\varepsilon_{ij}+(\mathbf{k}_l/2-\mathbf{k}''_j)^2}T_{li,\,il}(\mathbf{k}''_{li},\,\mathbf{k}''_j;\,\mathbf{k}'_{li},\,\mathbf{k}'_j;\,E)$$
$$-4\pi\int\frac{d\mathbf{k}''_{lj}d\mathbf{k}''_j}{(2\pi)^6}\cdot\frac{t(\mathbf{k}_{ij},\,\mathbf{k}^{(2)}_{ij},\,\varepsilon_{ij})}{-2\mu_{ij}\varepsilon_{ij}+(\mathbf{k}_l/2-\mathbf{k}''_i)^2}T_{lj,\,il}(\mathbf{k}''_{lj},\,\mathbf{k}''_i;\,\mathbf{k}'_{il},\,\mathbf{k}'_j;\,E),\quad(74)$$

where

$$\mathbf{k}^{(1)}_{ij}=(\mathbf{k}_i+\mathbf{k}_j)/2-\mathbf{k}''_j=-\mathbf{k}_l/2-\mathbf{k}''_j;$$
$$\mathbf{k}^{(2)}_{ij}=(\mathbf{k}_i+\mathbf{k}_j)/2-\mathbf{k}''_i=-\mathbf{k}_l/2-\mathbf{k}''_i.$$

Graphical Representation of the Scattering Amplitudes in a Four-Body System

We consider a system of four different particles. Assuming pair forces, the interaction operator of these particles is

$$V_{1234}(t)=V_{12}(t)+V_{13}(t)+V_{14}(t)+V_{23}(t)+V_{24}(t)+V_{34}(t).\quad(75)$$

Substituting this representation of the interaction in the series (4) for the scattering operator, we obtain the expansion

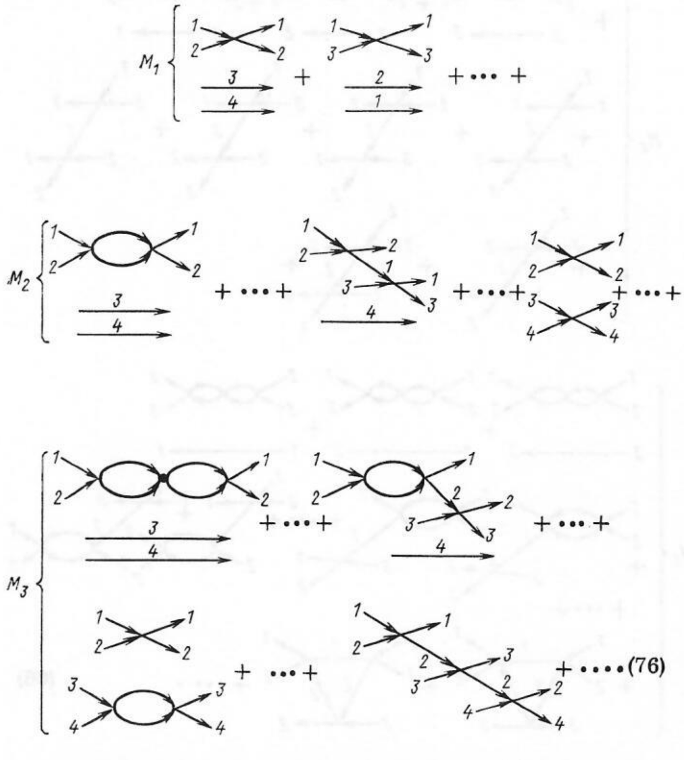
$$T_{1234}=T^{(1)}_{1234}+T^{(2)}_{1234}+T^{(3)}_{1234}+\ldots+T^{(4)}_{1234}+\ldots;$$
$$T^{(1)}_{1234}=\frac{-\mathrm{i}}{\hbar}\int\limits_{-\infty}^{\infty}dt_1V_{1234}(t);$$
$$T^{(2)}_{1234}=\left(\frac{-\mathrm{i}}{\hbar}\right)^2\frac{1}{2}\int\limits_{-\infty}^{\infty}dt_1\int\limits_{-\infty}^{\infty}dt_2P\{V_{1234}(t_1)\,V_{1234}(t_2)\};$$

. . . .

$$T^{n}_{1234}=\left(\frac{-\mathrm{i}}{\hbar}\right)^n\frac{1}{n!}\int\limits_{-\infty}^{\infty}dt_1\ldots\int\limits_{-\infty}^{\infty}dt_nP\{V_{1234}(t_1)\,\ldots\,V_{1234}(t_n)\}.$$

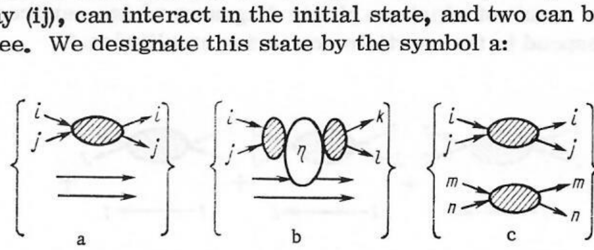
. . . .

The matrix elements of the scattering operator of the particles of this system can be set in correspondence with a sequence of diagrams in the form



(76)

The diagrams (76) can be discriminated by the topologies of their starting and end points. First, two particles, say (ij), can interact in the initial state, and two can be free. We designate this state by the symbol a:



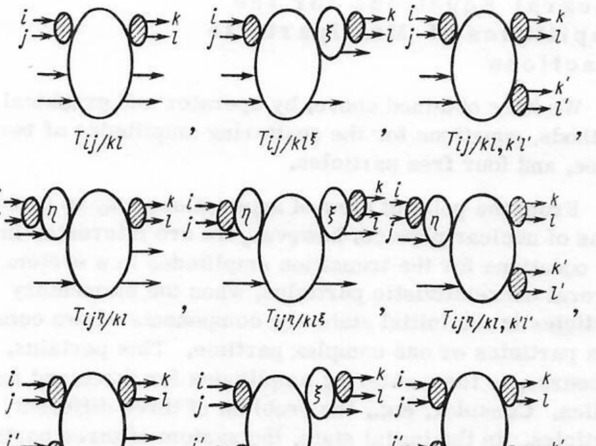
(77)

Second, in the initial state, three particles that interact pairwise can form a subsystem (η), while the fourth particle can remain free. The amplitude for the scattering of the particles in the system (η) is then an infinite sum of only coupled diagrams. We designate such a state by the symbol b (77). We separate the first pair (ij) of the interacting particles in the subsystem (η). It must be born in mind that when four different particles are broken up, in all possible ways, into subsystems consisting of three particles and one particle, the pair of particles (ij) can belong to two subsystems, $\eta^{(1)}$ and $\eta^{(2)}$. Therefore the three-particle subsystem (77b) is designated by the symbol η^{ν}_{ij}, where $\nu = 1$ or 2.

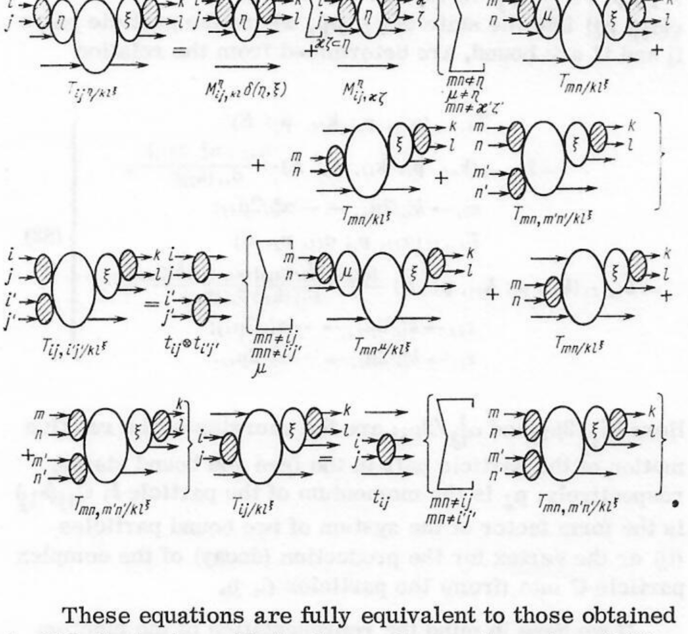
Third, there can exist in the initial state two independent particle pairs (ij) and (mn) (with ij ≠ mn). The particles within each pair interact with each other. We designate this state by the symbol c (77c). The same pertains to the final state of the diagrams, i.e., there exist three fundamentally different final states of the four particles. We designate them by the same respective symbols with primes.[1)]

Thus, nine different types of transitions can be distinguished in a system of four particles.

To describe these transitions, we introduce the following nine types of amplitudes:



From the infinite sequence of diagrams (76) we can separate a different infinite sequence of diagrams for each of the indicated types of amplitudes. By summing these sequences we can obtain the following graphical equations:



These equations are fully equivalent to those obtained by the algebraic methods above. It was shown in ref. 20 that these equations have a unique physical solution satisfying the Schrödinger equation for the analogous problem.

4. CERTAIN PROPERTIES OF THE EQUATIONS OBTAINED BY THE DIAGRAM-SUMMATION METHOD

Elementary and Complex Particles in Nonrelativistic Diagrams

We have considered above a system of several nonrelativistic particles that interact pairwise. We call these particles elementary. It was shown that the scattering amplitude of elementary particles can be represented in the form of an infinite sum of diagrams of the ladder type, which correspond to the matrix elements of the four-dimensional perturbation theory. The basis functions of the system of scattered particles are the set of plane waves of the elementary particles.

In order not to violate the completeness of the system of basis functions, only elementary particles can serve as virtual particles in the intermediate states of nonrelativistic diagrams.

In addition, in the nonrelativistic-perturbation-theory diagrams, the vertices that represent the interaction of virtual elementary particles in the intermediate state is set in correspondence with the exact scattering amplitude of these particles in the general case off the energy shell. For example, the interaction vertex of two elementary particles (ij) corresponds to the scattering amplitude $t(k_{ij}, k'_{ij}; \varepsilon_{ij})$ of these two particles, which is a function of the relative-motion momenta k_{ij} and k'_{ij} before and after the scattering, respectively, and of the energy ε_{ij} of the relative motion. It can be shown that any other representation of the intermediate-scattering vertex, such as simulation of this vertex by a complex quasiparticle, is an approximation.

Indeed, if the interacting elementary particles can form a bound state at $\varepsilon_{ij} = -\alpha_{ij}^2/2\mu_{ij}$, then the following representation holds true for the amplitude $t(k_{ij}, k'_{ij}; \varepsilon_{ij})$:

$$t(\mathbf{k}_{ij},\ \mathbf{k}'_{ij},\ \varepsilon_{ij}) = \frac{G_{ij}(\mathbf{k}_{ij})\,G_{ij}(\mathbf{k}'_{ij})}{\varepsilon_{ij}+\alpha_{ij}^2/2\mu_{ij}} + v(\mathbf{k}_{ij},\ \mathbf{k}'_{ij})$$
$$+ \int t(\mathbf{k}_{ij},\ \mathbf{q}_{ij},\ q_{ij}^2/2\mu_{ij} \pm i0)\,(q_{ij}^2/2\mu_{ij}-\varepsilon_{ij})^{-2}$$
$$\times\, t(\mathbf{k}'_{ij},\ \mathbf{q}_{ij},\ q_{ij}^2/2\mu_{ij} \pm i0)\, d\mathbf{q}_{ij}. \qquad (78)$$

The function $G_{ij}(k_{ij})$ is connected here with the eigenfunction $\psi(k_{ij})$ of the Hamiltonian of the system of two interacting particles by the relation

$$\psi(\mathbf{k}_{ij}) = G_{ij}(\mathbf{k}_{ij})/(k_{ij}^2/2\mu_{ij}+\alpha_{ij}^2/2\mu_{ij}). \qquad (79)$$

We note that for the function $G_{ij}(k_{ij})$ there exists a normalization in the form

$$\int G_{ij}(\mathbf{k}_{ij})\,(k_{ij}^2/2\mu_{ij}+\alpha_{ij}^2/2\mu_{ij})^{-2}\,G_{ij}(\mathbf{k}_{ij})\, d\mathbf{k}_{ij} = 1.$$

Further, $v(k_{ij}, k'_{ij})$ in (78) is the Fourier transform of the two-particle interaction potential; μ_{ij} is the reduced mass of these particles.

The representation (78) for the scattering amplitude $t(k_{ij}, k'_{ij}, \varepsilon_{ij})$ was obtained on the basis of an analysis of its analytic properties. The term of the type $G_{ij}(k_{ij})G_{ij}\cdot(k'_{ij})/(k_{ij}^2/2\mu_{ij} + \alpha ij/2\mu_{ij})$ in (78) is a contribution from the pole of the amplitude $t(k_{ij}, k'_{ij}, \varepsilon_{ij})$.

Consequently, replacement of the exact intermediate-state particle-scattering amplitude by its pole approximation, which is seen to be the amplitude for the production and subsequent decay of a certain new complex particle, is an unjustified operation when the contributions from the diagrams are written down. The pole approximation is valid if the energy of the elementary-particle subsystem is close to the binding energy of the complex particle. Inasmuch as in the virtual state the energy of the subsystem of the interacting elementary particles is not fixed and varies in a wide range, the choice of the pole approximation for the virtual-particle scattering amplitude is most arbitrary. It follows therefore that introduction of a complex particle consisting of several elementary particles in the intermediate state calls for a special corroboration.

It should be noted that the scattering of three and more elementary particles having indeterminate momentum and energy in the initial and final states is not encountered as a rule in a real experiment. On the contrary, we always deal with the collision of two particles at least one of which is complex, and complex particles can be produced in the final state. This raises the question of introducing the complex particle, but only in the initial or final state, without violating the formalism in which a plane-wave basis is used. It turns out that a complex particle can be introduced in the initial (or final) state because the energy of the particles that enter in the reaction, including the relative energy of the particle subsystem, can be determined exactly. For example, the relative energy of a subsystem of basis particles in the initial (or final) state can be arbitrarily close to the bound-state energy. In this case the pole approximation is perfectly valid for the scattering amplitude of elementary-particle subsystems, e.g., in the case of a subsystem of two particles. This means that the scattering amplitude is determined mainly by one pole term in the expansion (78). The denominator of (78) can be interpreted as the Green's function of the complex particle, and the numerator as the product of the decay vertex by the production vertex of the complex particle.

Vertices in Nonrelativistic Diagrams

As shown above, nonrelativistic diagrams corresponding to scattering amplitudes can contain complex-particle decay (production) vertices in the initial and final states, depending on how the transition channels are stipulated. The following rule can be derived for the determination of the vertices: The vertex $G(\mathbf{k}_1, \ldots, \mathbf{k}_N)$ for the decay (production) of a complex particle into its component elementary particles is equal to the square root of the residue at the pole of the scattering amplitude of these elementary particles, obtained under the conditions that the momenta have the same configuration in the initial and final states.

Indeed, let us designate the amplitude for the scattering of N particles, assuming identical momentum configurations in the initial and final states, by $T(\mathbf{k}_1, \ldots, \mathbf{k}_N; \mathbf{k}_1, \ldots, \mathbf{k}_N; E)$. Here $\mathbf{k}_i$ is the momentum of the i-th particle and E is the energy of the N-particle system. If the energy E of the particle system is close to the pole value E_0, then the scattering amplitude can be represented in the pole approximation:

$$\underset{E\to E_0}{T}(\mathbf{k}_2, \ldots, \mathbf{k}_N; \mathbf{k}_1, \ldots, \mathbf{k}_N; E) = G^2(\mathbf{k}_1, \ldots, \mathbf{k}_N)/(E - E_0), \tag{81}$$

where $G^2(\mathbf{k}_2, \ldots, \mathbf{k}_N)$ is the vertex for the decay or formation of a complex particle with binding energy E_0. Since the quantity $\underset{E\to E_0}{T}(\mathbf{k}_1, \ldots, \mathbf{k}_N; \mathbf{k}_1, \ldots, \mathbf{k}_N; E)\cdot(E - E_0)$ is the residue at the pole of the scattering amplitude, we can determine the decay (production) vertex of the complex particle from the relation $G(\mathbf{k}_1, \ldots, \mathbf{k}_N) = [T(\mathbf{k}_1, \ldots, \mathbf{k}_N; \mathbf{k}_1, \ldots, \mathbf{k}_N; E)(E - E_0)]^{1/2}$.

Integral Equations for the Amplitudes of Multiparticle Reactions

We have obtained above, by operator and graphical methods, equations for the scattering amplitudes of two, three, and four free particles.

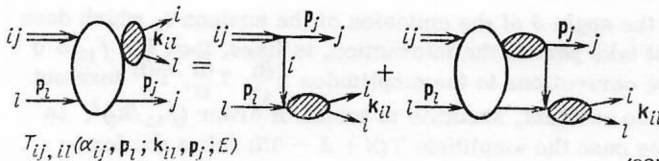
From the point of view of applications, say to problems of nuclear physics, however, we are interested in the equations for the transition amplitudes in a system of several nonrelativistic particles, when the elementary particles in the initial state are components of two complex particles or one complex particle. This pertains, of course, to the scattering amplitudes for three and more bodies. Consider, e.g., the problem of three different particles. In the initial state, the system of three particles can be broken up into two subsystems in only one manner, namely, two particles combine to form a complex particle C, and the third particle is free. In nuclear physics, e.g., such a process is the interaction of a deuteron with a nucleon.

It can be shown that in the case of the scattering of three particles i, j, and l, the amplitude $T_{ij,\,il}(\alpha_{ij}, \mathbf{p}_l; \mathbf{k}_{il}, \mathbf{p}_j; E)$ of the transition from the state $\psi(\alpha_{ij}, \mathbf{p}_l)$, in which the particles ij are bound, into the state $\psi_0(\mathbf{k}_{il}, \mathbf{p}_j)$, in which the particles are free, and the amplitude $T_{ij,\,il}\cdot(\alpha_{ij}, \mathbf{p}_l; \alpha_{il}, \mathbf{p}_j; E)$ for the transition from the state $\psi(\alpha_{ij}, \mathbf{p}_l)$ into the state $\psi(\alpha_{il}, \mathbf{p}_j)$, where the particle pairs ij and il are bound, are determined from the relation

$$\left.\begin{array}{c}
T_{ij,\,il}(\alpha_{ij}, \mathbf{p}_l; \mathbf{k}_{il}, \mathbf{p}_j; E) \\
= T_{ij,\,il}(\mathbf{k}_{ij}, \mathbf{p}_l; \mathbf{k}_{il}, \mathbf{p}_j; E)\,\dfrac{[\varepsilon_{ij} + \alpha_{ij}^2/2\mu_{ij}]}{G_{ij}(\mathbf{k}_{ij})}, \\
\varepsilon_{ij} \to k_{ij}^2/2\mu_{ij} \to -\alpha_{ij}^2/2\mu_{ij}; \\
T_{ij,\,il}(\alpha_{ij}, \mathbf{p}_l; \alpha_{il}, \mathbf{p}_j, E) \\
= T_{ij,\,il}(\mathbf{k}_{ij}, \mathbf{p}_l; \mathbf{k}_{il}, \mathbf{p}_j; E)\,\dfrac{[\varepsilon_{ij} + \alpha_{ij}^2/2\mu_{ij}][\varepsilon_{il} + \alpha_{il}^2/2\alpha_{il}]}{G_{ij}(\mathbf{k}_{ij})\,G_{il}(\mathbf{k}_{il})}; \\
\varepsilon_{ij} \to k_{ij}^2/2\mu_{ij} \to -\alpha_{ij}^2/2\mu_{ij}; \\
\varepsilon_{il} \to k_{il}^2/2\mu_{il} \to -\alpha_{il}^2/2\mu_{il}.
\end{array}\right\} \tag{82}$$

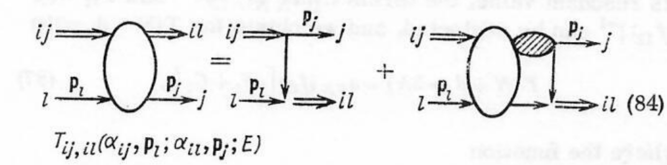
Here $k_{ij}^2/2\mu_{ij}$ and $\alpha_{ij}^2/2\mu_{ij}$ are the energies of the relative motion of the particle pair in the free and bound states, respectively; $\mathbf{p}_l$ is the momentum of the particle l; $G_{ij}(\mathbf{k}_{ij})$ is the form factor of the system of two bound particles (ij) or the vertex for the production (decay) of the complex particle C into (from) the particles (i, j).

If we have in mind the representation of the function $G_{ij}(\mathbf{k}_{ij})$ by the diagrams representing the vertex of the decay of a bound particle into two components



(83)

then the amplitudes $T_{ij,\,il}(\alpha_{ij}, p_l;\, k_{il}, p_j;\, E)$ and $T_{ij,\,il}$ $(\alpha_{ij}, p_l;\, \alpha_{il}, p;\, E)$ can be represented as infinite series of diagrams that add up to form the following equations:



(84)

Particular Solutions of the Integral Equations of the Method of Diagram Summation for the Analysis of Multiparticle Nuclear Reactions

We shall now show how the integral equations of the diagram-summation method can be used in nuclear physics to analyze differential cross sections of reactions.

We represent a multiparticle nuclear reaction as a process of scattering of n particles. Assume that in the initial state the particles are subdivided into two subsystems, each comprising a bound system. In the final state, the particles can be distributed into m subsystems, with m taking on the values from 2 to n. The amplitude of the transition of the system of particles to the final state with m subsystems (where $n \geq m > 2$) is the sum of the amplitudes corresponding to transitions to the different possible channels of the final state.[2)] There is a system of coupled integral equations for the determination of the terms of the amplitudes. At $m < n$, however, to obtain the terms of the amplitudes it is necessary to solve the system of equations for the case $m = n$. It is known that the numerical solution of the integral equations of scattering theory in the case $n > 2$ is a difficult mathematical task, so that it is of interest to find particular solutions of these equations in analytic form. In the diagram-summation method, the integral equation for the amplitude term has as a rule the following structure:

$$T_k(m) = T_k^0(m) + \sum_i T_i(n)\, CK_{ik}(m). \tag{85}$$

In this schematic representation of the equation we use the following notation: $T_k(m)$ is the amplitude term from the set of amplitudes for a definite value of m; $T_i(n)$ is the amplitude term from the set at $m = n$; $T_k^0(m)$ is the free term of the equation; $\sum_i T_i(n) CK_{ik}(m)$ is the integral term of the equation; $K_{ik}(m)$ is the kernel of the equation; C is the product of the Green's functions of all the n particles.

We integrate this equation once:

$$T_k(m) = T_k^0(m) + \sum_i T_i^0(n)\, CK_{ik}(m) + \sum_{ij} T_j(n)\, CK_{ji}(n)\, CK_{ik}(m).$$

We seek the amplitude $T_k(m)$ in the form of a sum

$$T_k(m) = T_k^{(1)}(m) + T_k^{(2)}(m),$$

where

$$T_k^{(1)}(m) = T_k^0(m) + \sum_i T_i^0(n)\, CK_{ik}(m),$$

and we can write for $T_k^{(2)}(m)$ an equation represented schematically in the form

$$T_k^{(2)}(m) = \sum_{ij} T_j^0(n)\, CK_{ji}(n)\, CK_{ik}(n) + \sum_{ij\xi} T_\xi^0(n)\, CK_{\xi j}(n)\, CK_{ji}(n)\, CK_{ik}(m) + \sum_{ij} T_j^{(2)}(n)\, CK_{ji}(n)\, CK_{ik}(m).$$

Let us explain the meaning of this solution with a concrete example. In the case of the three-body problem ($m = n = 3$) the first terms of the free member correspond to the contribution made by the diagram with two closed loops, while the second corresponds to the contribution from the diagram with three closed loops. The amplitude $T_k^{(1)}(m)$ corresponds to the sum of contributions from a pole-type diagram and from diagrams with one closed loop. These contributions can be calculated relatively simply in certain cases in analytic form. It was shown in ref. 14 that the sum of contributions from the pole diagram and the quadratic diagram is equivalent to the amplitude of the impulse approximation; thus, $T_k^{(1)}(m)$ has a clear-cut physical meaning. A similar situation arises also in the case of reactions that reduce to problems involving four and more bodies. The function $T_k^{(1)}(m)$ is as a rule equivalent to the impulse-approximation amplitude.

In the determination of $T_k^2(m)$ it is necessary to consider two cases:

1) The initial energy is large, i.e., the primary momentum k_0 in the c.m.s. is much larger than the probable momenta p_0 of the particles in the initial subsystems, and the scattering of particles with large relative momentum in the intermediate states can be regarded as free, proceeding predominantly forward or backward in a narrow cone. In this case the iteration series of equations for $T_k^{(2)}(m)$ converges[21] and the function $T_k^{(2)}(m)$ can be represented by the sum of the first iterations.

2) We consider the case of low energies. Obviously, under these conditions the terms of the iteration series for $T_k^{(2)}(m)$ do not have a small parameter, and the representation of the function $T_k^{(2)}(m)$ by several iterations is meaningless.

It is possible, however, to obtain the amplitude $T_k^{(2)}(m)$ in parametrized form, by investigating the corresponding equation.

Like the amplitude of the considered transition, the amplitude $T_k^{(2)}(m)$ is a complicated function that depends on the total energy of the system and on $(3m - 4)$ variables that characterize the magnitudes in the directions of the momenta of the m particles produced.

In the study of multiparticle processes, the main interest lies in the dependence of the differential cross sections, or transition amplitudes, on the angle or energy of one of the particles in the final state. It is therefore perfectly sufficient to represent $T_k^{(2)}(m)$ with accuracy to within a constant quantity in the form of a function of the given variable.

Scattering of Nucleons by Deuterons

To illustrate the proposed method for the analysis of amplitudes of multiparticle nuclear reactions, let us consider the decay of a deuteron under the influence of a nucleon.

Assuming two-particle forces, the amplitude $T(N + d \to 3N)$ of this reaction is the sum of the amplitudes corresponding to the interaction of each pair of nucleons (1, 2), (1, 3), (2, 3) in the final state:

$$T(N+d\to 3N)=T_{12}+T_{23}+T_{31}. \tag{86}$$

Each of these three amplitude terms T_{ij} can be determined on the basis of integral equation (83), the solution of which we seek in the form of a sum of two functions: $T_{ij} = T_{ij}^{(1)} + T_{ij}^{(2)}$. Here $T_{ij}^{(1)}$ is the sum of the free terms of the once-integrated equation for T_{ij}. Graphically, $T_{ij}^{(1)}$ is a sum of a pole diagram and a quadratic diagram. It should be noted that the sum $T_{12}^{(1)} + T_{23}^{(1)} + T_{31}^{(1)}$ corresponds to the amplitude of the reaction $N + d \to 3N$, calculated in the impulse approximation. Let us consider methods of determining the functions $T_{ij}^{(2)}$.

At high energy, when the initial momentum k_0 noticeably exceeds the momentum of the nucleons in the deuteron, and when it can be assumed that the scattering of the nucleons in the intermediate state is free, the iteration series of the equation for $T_{ij}^{(2)}$ converges rapidly. The convergence parameter is the quantity $(\beta/k_0)^2$ (ref. 21), where β is the parameter of the deuteron Hulthen wave function. Then $T_{ij}^{(2)}$ can be represented by the first iterations. At low energies, the amplitude $T_{ij}^{(2)}$ must be sought in parametric form as a function of a definite variable.

Let us consider, e.g., the dependence of the amplitude (86) of the deuteron decay under the influence of nucleons on the energy $E_{12} = f_{12}^2/m$ of the relative motion of the two nucleons (1, 2) in the region of their resonant interaction, i.e., in the region where E_{12} is close to zero. Let the initial energy be much larger than the binding energy of the nucleons in the deuteron ($E_0 = k_0^2/2m \gg \alpha^2/m$), but let E_0 be smaller than the depth of the two-nucleon interaction potential.

Under the foregoing assumptions, the amplitude T_{12}, which takes into account the interaction of the nucleons (1, 2) in the final state, can be factorized:

$$T_{12}=a_{NN}(f_{12})[A_{12}^{(1)}+A_{12}^{(2)}],$$

where $a_{NN}(f_{12})$ is the amplitude for the scattering of the particles (1, 2) for $f_{12} \approx 0$. Bearing this in mind, we represent the amplitude of the reaction $N + d \to 3N$ by the sum

$$T(N+d\to 3N)=a_{NN}(f_{12})[A_{12}^{(1)}+A_{12}^{(2)}]+[T_{13}^{(1)}+T_{13}^{(2)}]+[T_{23}^{(1)}+T_{23}^{(2)}].$$

If the angle ϑ of the emission of the nucleon 3, which does not take part in the interaction, is fixed, then for $f_{12} \approx 0$ the corrections to the amplitudes $A_{12}^{(1)}$, $T_{13}^{(1)}$, $T_{23}^{(1)}$ turn out to be constant, accurate to terms of order $(f_{12}/k_0)^2$. In this case the amplitude $T(N + d \to 3N)$ takes the form

$$T(N+d\to 3N)=a_{NN}(f_{12})[A_{12}^{(1)}+C_1]+[T_{23}^{(1)}+C_2]+[T_{31}^{(1)}+C_3].$$

Since the amplitude $a_{NN}(f_{12})$, as a function of f_{12}, is near its resonant value, the terms $C_2[a_{NN}(f_{12})]^{-1}$ and $C_3[a_{NN}\cdot(f_{12})]^{-1}$ can be neglected, and we obtain for $T(N + d \to 3N)$

$$T(N+d\to 3N)=a_{NN}(f_{12})\left[F_1+C_1\right], \tag{87}$$

where the function

$$F_1=A_{12}^{(1)}+\frac{T_{23}^{(1)}+T_{31}^{(1)}}{a_{NN}(f_{12})} \tag{88}$$

can be calculated. The behavior of F_1 as a function of f_{12} in the region of $f_{12} \approx 0$ depends essentially on the value of the fixed emission angle of the nucleon 3. Two limiting cases can be considered here:

1. $\vartheta_3 \approx 180°$, i.e., the momentum q_3 transferred to the nucleon 3 has a maximum value $|q_3| = |k_0 - k_3|/2 \approx k_0$. Under these conditions the function F_1, accurate to terms of order $(f_{12}/k_0)^2$, is constant. Therefore the behavior of the amplitude of the reaction $N + d \to 3N$ as a function of f_{12} is determined only by the resonant factor $a_{NN}(f_{12})$:

$$T(N+d\to 3N)=\text{const}\, a_{12}(f_{12}). \tag{89}$$

This result is confirmed by an analysis of a number of experimental data (Fig. 1).

2. $\vartheta_3 \approx 0$, i.e., the momentum transfer q_3 is very small: $|q_3| = |k_0 - k|/2 \approx 0$. In this case the function F_1 (88) is itself strongly dependent on f_{12}, and consequently the reaction amplitude $T(N + d \to 3N)$ obtained on the basis of the particular solution of the integral equations

$$T(N+d\to 3N)\approx a_{NN}(f_{12})[F_1+C], \tag{90}$$

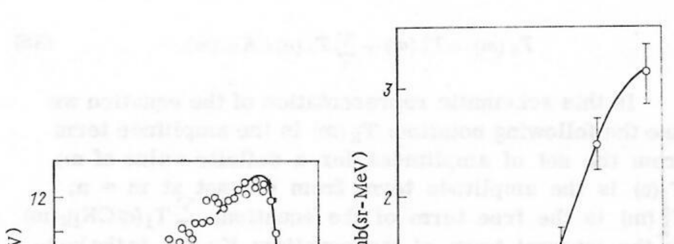


Fig. 1. Energy spectra of the neutrons from the reaction n(d, p)2n: a) incident-deuteron energy 1.86 MeV (ref. 23); b) neutron energy 14.0 MeV (ref. 13). The curves were calculated in the Migdal–Watson approximation.

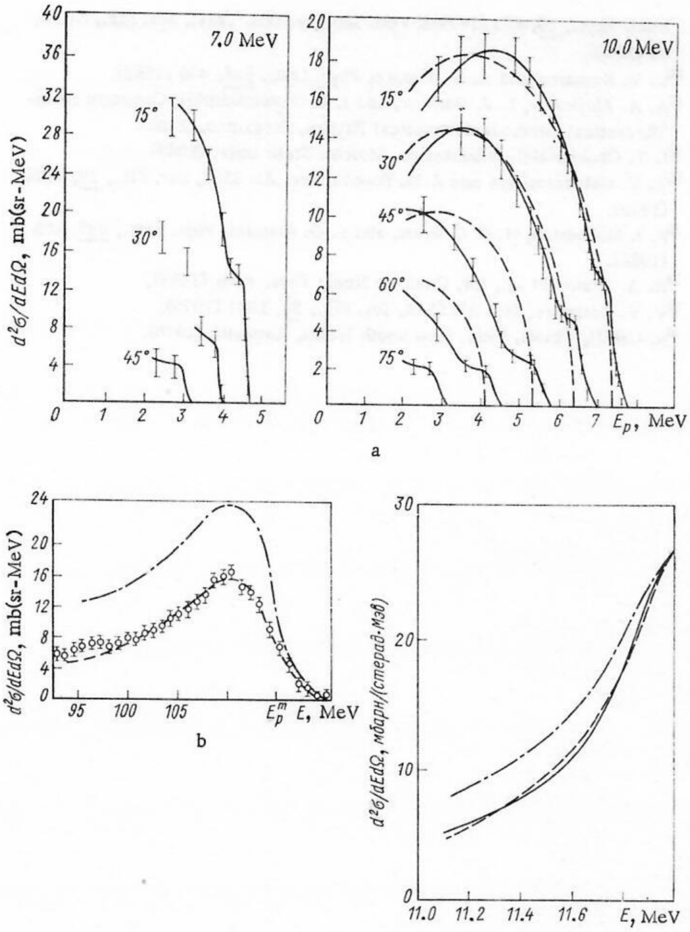


Fig. 2. Energy spectra of protons: a) from the reaction p(d, n)2p at incident-proton energies 7 and 10 MeV, for different emission angles.[24] The solid curves were calculated in the present paper; the dashed curves were calculated assuming an equally probable distribution in phase space; b) from the reaction n(d, p)2n at an incident-neutron energy 13.9 MeV for the angle 4° (ref. 13). The dashed curve was calculated in the present paper; the dash–dot curve represents calculation in the impulse approximation; c) from the reaction n(d, p)2n with 14-MeV incident neutrons at an angle 4°. The dash–dot curve shows the experimental data; the solid curve was calculated in ref. 25; the dashed curve was calculated in the present paper.

differs from the amplitude $A_{imp}(N + d \rightarrow 3N)$ calculated in the impulse approximation,

$$A_{\text{imp}}(N+d \rightarrow 3N) = a_{NN}(f_{12})\, F_1,$$

by the function $a_{NN}(f_{12}) \cdot C$.

Figure 2a shows the experimentally measured spectra of the protons from the reaction $p + d \rightarrow n + p_1 + p_2$ in the region $f_{np} \approx 0$ (ref. 24), and the curves calculated by representing the reaction amplitude in the form (90). It is seen from the figure that the proper choice of the constant C in (90) results in good agreement with the experimental data. Further, this parametrized equation for the $N + d \rightarrow 3N$ amplitude was used in the analysis of the experimental spectra of the protons from the reaction $n + d \rightarrow n_1 + n_2 + p$ at $f_{n_1 n_2} \approx 0$ and $q_p \ll f_{n_1 n_2}$, to obtain the neutron–neutron scattering length (see Fig. 2b). It was shown that only if the scattering length a_{NN} ranges from 19 F to 24 F can the calculated curves be reconciled with the experimental data by proper choice of the constant C in Eq. (90).

It is interesting to note that the function $a_{NN}(f_{NN}) \cdot F_1$, being the sum of the contributions from the first two iterations of the expressions for T_{12}, T_{23}, and T_{31}, contains, in the case of the considered particle configuration and of the initial reaction energy $E_0 \ll U_0$, the nucleon–nucleon scattering amplitudes practically on the energy shell. The quantity C in (90) is the sum of the solutions of the integral equations for the amplitude $A_{12}^{(2)}$, and therefore depends essentially on the character of the pair potentials. Consequently the absolute magnitude and the behavior of the differential cross section for the production of nucleons 3 in the region of the upper limits, under the condition $q_3 \approx 0$, obtained by numerically solving the integral equations, depends strongly on the choice of the character of the two-nucleon potentials. This is confirmed by a relative analysis of the experimentally measured spectrum of the protons from the reaction $n + d \rightarrow n_1 + n_2 + p$ with that calculated on the basis of the exact solution of the integral equations written down in the approximation of a separable nucleon–nucleon potential.[25] As seen from Fig. 2c, either spectrum can be fitted by proper choice of the quantity C in the representation (90) for the amplitude of the reaction $N + d \rightarrow 3N$.

*Publisher's note: There were no equations labelled (68) and (69) in the Russian text.

[1)]Obviously, in a system of four particles there are 21 asymptotic system states obtained as a result of all the possible permutations of the particles in the three asymptotic states indicated above.

[2)]If m = 2, there is only one transition amplitude.

[1]V. G. Skornyakov and K. A. Ter-Martirosyan, Zh. Eksp. Teor. Fiz., 31, 775 (1956) [Sov. Phys.–JETP, 4, 648 (1957)].

[2]L. D. Faddeev, Zh. Eksp. Teor. Fiz., 39, 1459 (1960) [Sov. Phys.–JETP, 12, 1014 (1961)]; Tr. Matem. Inst. AN SSSR, 69, 3 (1963).

[3]O. Ya. Yakubovskii, Yad. Fiz., 5, 1312 (1967) [Sov. J. Nucl. Phys., 5, 937 (1967)].

[4]V. V. Komarov and A. M. Popova, Zh. Eksp. Teor. Fiz., 46, 2112 (1964) [Sov. Phys.–JETP, 19, 1425 (1964); Nucl. Phys., 69, 278 (1965); A90, 625 (1967).

[5]A. N. Mitra et al., Phys. Rev., B140, 1336 (1965); J. Weyers, Phys. Rev., 145, 1236 (1966); V. A. Allesandrini, J. Math. Phys., 7, 215 (1966); N. Mishima and V. Takahashi, Progr. Theor. Phys., 35, 440 (1966).

[6]R. G. Newton, J. Math. Phys., 8, 851 (1967).

[7]R. Omnes, Phys. Rev., B165, 1265 (1968).

[8]S. Weinberg, Phys. Rev., B133, 232 (1964); R. Sugar and R. Blankenbeckler, Phys. Rev., B136, 472 (1964); L. Rosenberg, Phys. Rev., B140, 217 (1965); C. Van Winter, Kgl. Mat.–Fys. Medd. Dan. Vid. Selsk., No. 8, 2 (1964); No. 10, 2 (1965).

[9]P. Federbush, Phys. Rev., 148, 1551 (1966); R. G. Newton, Phys. Rev., 153, 1502 (1967).

[10]V. A. Pazdzerskii and V. Ya. Fainberg, Preprint FIAN SSSR [Scattering of Complex Particles) (1966).

[11]P. Grassberger and W. Sandhas, Nucl. Phys., 132, 181 (1967).

[12]A. G. Sitenko and V. F. Kharchenko, Nucl. Phys., 49, 15 (1963); V. F. Kharchenko, Ukr. Fiz. Zh., 7, 573 and 582 (1962); E. O. Alt, P. Grassberger, and W. Sandhas, Preprint 2-48, Universität, Bonn (1968).

[13]V. K. Voitovetskii, I. L. Korsunskii, and Yu. F. Pazhin, Zh. Eksp. Teor. Fiz., 47, 1612 (1964) [Sov. Phys.–JETP, 20, 1084 (1965)]; Nucl. Phys., 69, 513 (1965); Yu. I. Chernukhin and R. S. Shuvalov, Yad. Fiz., 4, 272 (1966) [Sov. J. Nucl. Phys., 4, 197 (1967); B. Kühn et al., Preprint ZFK-150, Rossendorf (1968); Ya. F. Moskalev, Yad. Fiz., 8, 1156 (1968) [Sov. J. Nucl. Phys., 8, 672 (1969)].

[14]V. V. Komarov and A. M. Popova, Zh. Eksp. Teor. Fiz., 45, 214 (1963)

[Sov. Phys.–JETP, 18, 151 (1963)]; Nucl. Phys., 54, 278 (1964).
[15]V. V. Komarov and A. M. Popova, Vest. Moskov. Gos. Univ., Ser. Fiz., No. 5, 18 (1963); No. 6, 48 (1964); V. V. Komarov, Phys. Lett., 13, 163 (1964); Proc. Phys. Soc. (Lond.), 85, 867 (1965); A. M. Popova, Problems of Elementary-Particle Physics [in Russian], Erevan, Nor-Amberd (1963); Yad. Fiz., 3, 34 (1966) [Sov. J. Nucl. Phys., 3, 23 (1966)]; Nucl. Phys., 82, 209 (1966).
[16]V. V. omarov, In: Abstracts of All-Union Conference on Nuclear Spectroscopy, Erevan, 1969 [in Russian], Leningrad, Nauka (1970); The Graph Summation Method for Multiparticle Nuclear Reactions, Invited talk, Int. Conf. on Few-Particle Problems in Nuclear Interactions, Los Angeles (1972).
[17]V. V. Komarov and A. M. Popova, Yad. Fiz., 10, 1089 (1969) [Sov. J. Nucl. Phys., 10, 621 (1970)]; Vest. Moskov. Gos. Univ., Ser. Fiz., No. 3, 34 (1969).
[18]V. V. Komarov and A. M. Popova, Phys. Lett., B28, 476 (1969).
[19]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, Quantum Field-Theoretical, Methods in Statistical Physics, Pergamon, (1965).
[20]I. T. Obukhovskii, Dissertation, Moscow State Univ. (1969).
[21]T. V. Gaivoronskaya and A.M. Popova, Izv. AN SSSR, Ser. Fiz., 37, 1120 (1973).
[22]R. J. Slobodrian, H. E. Conzett, and F. G. Resmini, Phys. Lett., B27, 405 (1968).
[23]N. A. Vlasov et al., Int. Conf. on Nucl. Phys. Paris (1958).
[24]V. V. Komarov, Izv. AN SSSR, Ser. Fiz., 34, 1691 (1970).
[25]R. Cahill, Thesis, Univ. New South Wales, Australia (1970).