Renormalization of gauge-invariant theories

A. A. Slavnov

Steklov Mathematics Institute,
Academy of Sciences of the USSR, Moscow

Fiz. El. Chast. Atom. Yad., 5, 755-790 (July-September 1974)

A renormalization procedure is considered for gauge-invariant theories. Theories with spontaneously broken symmetry are analyzed in detail.

INTRODUCTION

During the last two years, the theory of gauge fields has turned from a rather exotic science, of interest only to a few specialists, into one of the most popular divisions of quantum field theory. The vigorous growth of interest in gauge fields started after the publication of a paper by G. t'Hooft,[1] who rescued from oblivion the articles by Higgs,[2] Kibble,[3] and Weinberg,[4] and who has shown that the technique of quantizing gauge-invariant theories, developed in the papers of V. N. Popov and L. D. Faddeev[5] and De Witt,[6] is applicable also to models with spontaneously broken symmetry, which makes it possible to describe vector fields with nonzero mass.

Since that time, many papers devoted to quantization and renormalization of models with broken symmetry have been published and have in part "rediscovered" the results well known already in the theory of gauge fields. Yet the entire theory of quantization[5–8] and renormalization,[9–11] developed for models without spontaneous symmetry breaking, can also be applied to this case with practically no changes. We therefore attempt to describe in this review, from a unified point of view, the general methods of quantization and renormalization of gauge-invariant theories, using by way of illustration models with spontaneously broken symmetry.

In the second section we discuss the role of gauge invariance in elementary-particle interactions. The third section is devoted to an exposition of the Higgs mechanism of spontaneous symmetry breaking. In the fourth section, using Kibble's model as an example, we describe a general procedure for quantizing gauge-invariant theories, in the fifth we construct a renormalization procedure, and in the sixth and final chapter we consider possible applications of gauge fields to weak and electromagnetic interactions as well as the problem of anomalous Ward identities.

1. GAUGE INVARIANCE AND ELEMENTARY-PARTICLE INTERACTIONS

The principal limitations imposed on the form of an interaction Lagrangian is that it be invariant with respect to one group or another. However, the requirements of relativistic invariance, isotopic symmetry, and other rigorously established symmetries still leave a large leeway in the choice of the Lagrangian, and it is customary to resort to phenomenology in order to eliminate this leeway. In this connection, attempts were made already long ago to postulate invariance with respect to some wider group of transformations, so as to be able to establish the type of interaction and to describe from a unified point of view all the elementary-particle interactions. It appears that the role of such a group can be assumed by an infinite-dimensional group of local gauge transformations.

As is well known, the equations of an electromagnetic field interacting with charged fields ψ are invariant with respect to the gauge transformations

$$\begin{aligned} \psi(x) &\to \exp[ie\alpha(x)]\,\varphi(x); \\ \overline{\psi}(x) &\to \exp[-ie\alpha(x)]\,\overline{\psi}(x); \\ A_\mu &\to A_\mu + \partial\alpha/\partial x^\mu. \end{aligned} \tag{1}$$

Yang and Mills[12] have generalized the concept of gauge invariance to include the case of the SU_2 group (the case of an arbitrary compact group was considered by Utigama,[13] by Glashow and Gell-Mann,[14] and by Schwinger[15]). Their argument reduced briefly to the following: The isospin conservation law is due to the invariance of the Lagrangian with respect to isotopic rotations with coordinate-independent parameters, i.e., with respect to simultaneous transformations of the fields at all space–time points. In the local theory, however, it is natural to stipulate invariance with respect to independent transformation of the fields at different space–time points. Confirming the validity of this hypothesis is the electromagnetic interaction, the Lagrangian of which is invariant against the local gauge transformations (1).

The requirement that the Lagrangian be invariant with respect to the gauge transformations

$$\delta\psi(x) = \lambda T^a \alpha^a(x)\,\psi(x) \tag{2}$$

(here α^a(x) are infinitesimally small functions, and the matrices T realize the representation of a certain compact group G) leads to the need for introducing a vector "compensating" field $A^a_\mu(x)$, which is the analog of the photon in the case of the abelian gauge group, and is transformed in accordance with the law

$$\delta A^a_\mu(x) = \lambda t^{abc} A^b_\mu \alpha^c + \partial\alpha^a/\partial x^\mu, \tag{3}$$

where t^{abc} are structure constants of the Lie algebra of the group G,

$$[T^a, T^b] = t^{abc} T^c. \tag{4}$$

We shall sometimes use also the matrices $A_\mu(x)$, the values of which at each x belong to the Lie algebra:

$$A_\mu(x) = t^a A^a_\mu(x), \tag{5}$$

where t^a are generators normalized by the conditions

$$\begin{aligned} (t^a)^{bc} = t^{abc};\ [t^a, t^b] = t^{abc} t^c; \\ Tr(t^a, t^b) = -2\delta^{ab}. \end{aligned} \tag{6}$$

A finite transformation corresponding to the infinitesimal transformation (3) takes in terms of these matrices the following form:

$$A_\mu \to \Omega A_\mu \Omega^{-1} - (1/\lambda)\,\Omega \partial_\mu \Omega^{-1}. \quad (7)$$

At each value of x, $\Omega(x)$ is an element of the group G. In particular, in exponential parametrization we have

$$\Omega(x) = \exp[\lambda t^a \alpha^a(x)]. \quad (8)$$

A gauge-invariant Lagrangian describing the interaction of a vector field A_μ with other fields ψ is constructed by replacing the usual derivative with the covariant derivative

$$D_\mu = (\partial_\mu - \lambda T^a A^a_\mu). \quad (9)$$

The invariant Lagrangian of a free (more accurately, self-acting) Yang–Mills field is expressed, just as in electrodynamics, in terms of the intensity tensor

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + \lambda[A_\mu, A_\nu], \quad (10)$$

which transforms in accordance with the law

$$F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1}, \quad (11)$$

and is given by

$$L = (1/8)\,Tr F_{\mu\nu} F_{\mu\nu}. \quad (12)$$

The construction described above admits of a beautiful geometric interpretation, in which the gauge fields A_μ play the role of connectivity coefficients of a layered space, the basis of which is four-dimensional space–time, and the layer is the group space of the gauge group. The tensor $F_{\mu\nu}$ plays here the role of the curvature tensor of this space, and the covariant derivatives D_μ have the meaning of the ordinary covariant derivatives in curved space.

Sakurai[16] proposed to consider the requirement of local gauge invariant as a fundamental physical principle, according to which each conservation law should correspond to a gauge or "compensating" field, the interaction with which is described by a Lagrangian that is invariant with respect to transformations of the type (2) and (3). This principle establishes a natural and elegant connection between symmetry properties and dynamics, and leads to a number of important physical consequences, and in particular to universality of the interaction.

An essential shortcoming of the Sakurai hypothesis has been the fact that the Lagrangian (12) describes a massless field, whereas all the known (and also postulated) vector particles with the exception of the photon have nonzero mass.

Introduction of a mass term in the form $(m^2/2)A_\mu A_\mu$ violates the gauge invariance. This difficulty can be circumvented with the aid of the generalized Stuekelberg formalism.[17] Stuekelberg proposed a gauge-invariant formulation of the theory of a neutral massive vector field, in which the invariance is ensured by introducing an auxiliary scalar field that transforms in accordance with the law

$$B \to B + m\alpha(x). \quad (13)$$

The gauge-invariant Lagrangian can be expressed in the form

$$L = -\frac{1}{4} f_{\mu\nu} f_{\mu\nu} + \frac{m^2}{2}\left(A_\mu - \frac{1}{m}\cdot\frac{\partial B}{\partial x^\mu}\right)^2 + \bar{\psi}\,[\gamma^\mu(\partial_\mu - ieA_\mu) + M]\,\psi. \quad (14)$$

The vector field is described by a five-component quantity (A_μ, B). However, just as in electrodynamics, gauge invariance decreases the number of degrees of freedom by two, and as a result we are left with three degrees of freedom, in accordance with the three possible polarization states of a massive vector field.

The Stuekelberg formalism can be generalized to include the case of a nonabelian gauge group by introducing an essentially nonlinear field $\Pi(x)$, which is an element of the group G for each value of x. If we stipulate that the fields Π transform under gauge transformations (7) in accordance with the law

$$\Pi(x) \to \Omega(x)\,\Pi(x), \quad (15)$$

then, without loss of gauge invariance, we can introduce a mass term in the form

$$(m^2/4)\,Tr\,(A_\mu - L_\mu)^2, \quad (16)$$

where

$$L_\mu(x) = (1/\lambda)\,\Pi\,(\partial \Pi^{-1}/\partial x^\mu), \quad (17)$$

is a vector field and, as follows from (15), transforms under gauge transformations in the same manner as A_μ:

$$L_\mu \to \Omega L_\mu \Omega^{-1} - \frac{1}{\lambda}\,\Omega \partial_\mu \Omega^{-1}. \quad (18)$$

The aforementioned gauge-invariant formulation of the theory of a massive vector field is described in detail in the present author's review.[18] Referring those interested in details to this review, we indicate only that an advantage of this formulation is the absence of physical supplementary particles (scalar fields Π can be eliminated by a gauge transformation and are absent from the asymptotic states), but owing to the essential monlinearity of the field $\Pi(x)$ the corresponding theory is nonrenormalizable in the usual sense.

Faddeev[19] has recently proposed a model of weak and electromagnetic interactions, based on a certain modification of the formalism described above (in this model, two components of the vector field have a mass, while the third, identified with the photon, remains massless). This model has the advantage of economy (there is no need to introduce any as yet unobservable quantities, with the exception of charged vector mesons), but the possibilities of its practical utilization are limited for the time being, owing to the absence of sufficiently reliable calculation methods in the nonrenormalizable theories.

Another possibility of gauge-invariant description of massive vector fields was indicated by Higgs.[2] In the Higgs model, the vector-field mass is the result of spontaneous symmetry breaking. Unlike the model described above, the Higgs–Kibble model is renormalizable, but at the expense of the need for postulating the existence of additional scalar particles, which in principle should be observed in experiment. In the next section we describe in detail the Higgs–Kibble mechanism and present nonrigorous but physically lucid arguments in favor of the possibility of constructing a renormalizable unitary S matrix in these models.

2. THE HIGGS MECHANISM

Attempts to obtain particle mass as a result of spontaneous symmetry breaking were made long ago,[20,21] but they encountered serious difficulties, connected in part with the problem of ultraviolet divergences, and in part with the fact that, according to the Goldstone theorem,[22] spontaneous symmetry breaking is accompanied by the appearance of massless scalar particles, the existence of which is not confirmed by experiment.

The mechanism proposed by Higgs successfully gets around these difficulties. We considered using as an example a model that is invariant with respect to the group SU_2. (In his original paper[2] Higgs considers the case of an abelian gauge group. The generalization to the nonabelian case was presented by Kibble.[3])

Assume we have a scalar isotriplet φ interacting with a Yang–Mills field A_μ. The gauge-invariant Lagrangian is of the form

$$L = Tr\left\{\frac{1}{8}F_{\mu\nu}F_{\mu\nu} - \frac{1}{4}[\partial_\mu\varphi - \lambda(A_\mu, \varphi)]^2 - V(\varphi)\right\}, \qquad (19)$$

where $V(\varphi)$ is an invariant polynomial in φ. If we confine ourselves to renormalizable interactions, then we have

$$V(\varphi) = Tr\left[\pm\frac{m^2}{4}\varphi^2 + \frac{h^2}{2}(\varphi^2)^2\right] \qquad (20)$$

From the point of view of invariance, the choice of the sign of the mass term is immaterial, but the physical meaning of the theory depends essentially on this sign.

In classical theory, the ground state (vacuum) is defined by the conditions

$$\delta V/\delta\varphi = 0;\ \ \delta^2 V/\delta\varphi^2 \geqslant 0. \qquad (21)$$

If the mass term is positive, these equations have only a trivial solution and, consequently, the ground state (vacuum) is isotopically invariant. If the sign is negative, the situation changes, and a stable equilibrium state corresponds to

$$|\varphi|^2 = m^2/4h^2. \qquad (22)$$

This means that the vacuum does not have the symmetry of the initial Lagrangian, and there is a certain preferred direction in isotopic space. A symmetrical extremum corresponding to $\varphi = 0$ is unstable.

In quantum theory, the instability of a symmetrical ground state becomes manifest in the difference between the quasimean values and the ordinary mean values.[23] A quasimean value is defined as an average over the vacuum, calculated in the presence of an infinitesimally small term that breaks the symmetry. If the limit of this mean value as the symmetry-breaking term tends to zero differs from the value calculated directly in the symmetrical theory, then we have the effect of spontaneous symmetry breaking.

Since the Lagrangian (19) is even in φ, the usual vacuum mean value is $\langle\varphi\rangle = 0$. We shall show that the quasimean value is $\langle\tilde{\varphi}\rangle \neq 0$.

We introduce a symmetry-breaking term $Tr\, c\varphi$, where $c = c^a t^a$, and c^a is a constant isovector that can be regarded as directed along the third axis without loss of generality. The introduction of such a term leads already, in the tree approximation, to the appearance of transitions between vacuum and the single-particle state, and consequently to the appearance of a nonzero vacuum mean value:

$$\langle\varphi_i\rangle = \xi\delta_{i3}. \qquad (23)$$

We change, with the aid of a canonical transformation, from the fields φ to fields φ' that have zero vacuum mean values:

$$\varphi_i' = \varphi_i - \xi_i;\ \ \xi_i = \xi\delta_{i3}. \qquad (24)$$

This substitution transforms the function $V(\varphi)$ in the following manner:

$$V(\varphi') = Tr\,[(-m^2/4)(\varphi'^2 + 2\varphi_3'\xi + \xi^2) + (h^2/2)(\varphi'^4 + 4\varphi'^2\xi^2 + 4\varphi_3'^2\xi^2 + 4\varphi'^2\varphi_3'\xi + 4\varphi_3'\xi^3 + \xi^4)]. \qquad (25)$$

The terms linear in φ' lead already in the tree approximation to transitions between vacuum and the single-particle state. Inasmuch as $\langle\varphi'\rangle = 0$ by definition, the self-consistency condition requires that the sum of all the terms linear in φ' vanish:

$$(-m^2/2)\,\xi + 2h^2\xi^3 + c = 0. \qquad (26)$$

(We have written out here the self-consistency condition in the tree approximation. In the exact self-consistency condition it is necessary to take into account also the contribution of diagrams of the tadpole type.)

We see that in the limit as $c \to 0$ this equation has, besides the trivial solution $\xi = 0$, also the solution

$$\xi^2 = m^2/4h^2. \qquad (27)$$

This solution corresponds to a stable ground state and, as will be shown below, leads to a theory that is free of paradoxes of the imaginary-mass type and others.

Using the condition (27) we can easily show that the terms quadratic in φ' combine to form the group

$$(m^2/4)\,\varphi_3'^2, \qquad (28)$$

i.e., the mass terms for the first and second components vanish, and the third component acquires a real mass.

Finally, as a result of a canonical transformation, the term containing the square of the covariant derivative gives rise to a mass term for the field A_μ:

$$(\lambda^2\xi^2/2)\,(A_{\mu1}^2+A_{\mu2}^2). \tag{29}$$

Finally, a canonical transformation leads to a Lagrangian (we omit the primes from now on)

$$L=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2}(\partial_\mu\varphi-\lambda[A_\mu\times\varphi])$$
$$+\frac{M^2}{2}(A_{\mu1}^2+A_{\mu2}^2)+M(\partial_\mu\varphi_1A_{\mu1}+\partial_\mu\varphi_2A_{\mu2})$$
$$+\lambda M[A_\mu^2\varphi_3-A_{\mu3}(A_\mu\varphi)]+\frac{m^2}{2}\varphi_3^2-h^2(\varphi^2)^2-2mh(\varphi^2)^2\varphi_3. \tag{30}$$

This Lagrangian describes a triplet of vector fields, two components of which have nonzero mass, interacting with a scalar triplet φ, one of the components of which has a nonzero mass. The zero-mass components $\varphi_{1,2}$ are none other than Goldstone bosons.

Although the Lagrangian (30) no longer has isotopic symmetry, it is invariant, as before, against local gauge transformations. The initial Lagrangian (19) was invariant to the transformations

$$A_\mu\to A_\mu+\lambda[A_\mu\alpha]+\partial\alpha/\partial x^\mu;\quad \varphi\to\varphi+\lambda[\varphi\alpha]. \tag{31}$$

Since the Lagrangian (30) was obtained from (19) by the formal substitution $\varphi\to\varphi+\xi$, it is obviously invariant to transformations obtained when φ in (31) is replaced by $\varphi+\xi$:

$$\begin{aligned}&A_\mu\to A_\mu+\lambda[A_\mu\alpha]+\partial\alpha/\partial x^\mu;\\&\varphi\to\varphi+\lambda[\varphi\alpha]+\lambda[\xi\alpha].\end{aligned} \tag{32}$$

In the case of spontaneously broken symmetry, the gauge transformation of the field φ is inhomogeneous: The components $\varphi_{1,2}$ experience a shift. Since the observed quantities should not depend on the choice of the gauge, we can use this leeway to cause the components φ_1 and φ_2 to vanish. (It is impossible to let the component φ_3 vanish, since it is homogeneously transformed.) In this gauge there are no Goldstone bosons, and the space of the asymptotic states consists only of physical vectors. Therefore the S matrix constructed in the usual manner is patently unitary. A shortcoming of this gauge is the nonobvious renormalizability. Since the Green's function of the vector field

$$D^c_{\mu\nu}\sim(g^{\mu\nu}-k^\mu k^\nu m^{-2})/(k^2-m^2) \tag{33}$$

tends to a constant at large k, a formal count of the degrees of the divergence shows that the theory described by the Lagrangian (30) is not renormalizable. However, as is well known, gauge invariance usually lowers the degree of divergence. Indeed, one can choose in place of the gauge $\varphi_{1,2}=0$, e.g., the gauge $\partial_\mu A_\mu=0$. In this gauge, the Green's function of the vector field is

$$D^c_{\mu\nu}\sim(g^{\mu\nu}-k^\mu k^\nu k^{-2})/(k^2-m^2), \tag{34}$$

and a count of the degrees of the divergence leads to a conclusion that the number of primitively diverging diagrams is finite, and consequently that the theory is renormalizable. The gauge $\partial_\mu A_\mu=0$ is not explicitly unitary, for in this case the theory contains the Goldstone bosons $\varphi_{1,2}$, and also vector-field quanta with negative energy. However, inasmuch as the S matrix should not depend on the gauge, the total probability of the transitions to all the nonphysical states should be equal to zero. In other words, by virtue of gauge invariance, the S matrix is simultaneously renormalizable and unitary. Of course, all the foregoing are merely intuitive arguments. A rigorous proof of unitarity and renormalizability will be given in succeeding chapters.

In the Higgs–Kibble model, the Goldstone bosons are unobservable and can be generally eliminated from the spectrum by suitable choice of the gauge. The Goldstone theorem does not hold in this case because its proof is based essentially on relativistic invariance. But for massless gauge fields there is no explicitly relativistic invariant formulation in Hilbert space. On the other hand, if a theory is formulated in a space with indefinite metric (the gauge $\partial_\mu A_\mu=0$), then Goldstone bosons do appear, but contribute, just like longitudinal and timelike photons, to virtual states and are absent from the asymptotic states.

Before we proceed to rigorous results, let us describe one more model in which spontaneous symmetry breaking causes all three vector-field components to acquire a mass.

The initial Lagrangian is

$$L=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2}(\partial_\mu\varphi-igA_\mu\tau\varphi)^*$$
$$\times(\partial_\mu\varphi-igA_\mu\tau\varphi)+\frac{m^2}{2}\varphi^*\varphi-h^2(\varphi^*\varphi)^2, \tag{35}$$

where φ is a non-Hermitian doublet:

$$\varphi=\begin{pmatrix}\varphi^+\\ \varphi^0\end{pmatrix}.$$

It is easy to verify that a stable ground state corresponds to $\langle\widetilde{\varphi}\rangle\neq0$. As a result of the canonical transformation $\varphi\to\varphi+\xi$, where ξ is a constant isospinor, we arrive at a theory describing a massive triplet of vector fields, interacting with a scalar field. It is convenient to express the Lagrangian obtained in this manner in terms of the fields B and σ:

$$\varphi^+=(1/\sqrt{2})\,(iB_1-B_2);\quad \varphi^0=\xi+(1/\sqrt{2})\,(\sigma+iB_3); \tag{36}$$

$$L=-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{M^2}{2}A_\mu^2+MA_\mu\partial_\mu B+\frac{1}{2}\partial_\mu B\partial_\mu B$$
$$+\frac{1}{2}\partial_\mu\sigma\partial_\mu\sigma-\frac{m}{2}\sigma^2+\frac{g}{2}A_\mu^k(\sigma\partial_\mu B^k-B^k\partial_\mu\sigma-[B\times\partial_\mu B]^k)$$
$$+\frac{Mg}{2}\sigma A_\mu^2+\frac{g^2}{8}(\sigma^2+B^2)A_\mu^2-\frac{gm^2}{4M}\sigma(\sigma^2+B^2)-\frac{g^2m^2}{32M^2}(\sigma^2+B^2)^2. \tag{37}$$

This Lagrangian is invariant with respect to the gauge transformations

$$\left.\begin{aligned}&A_\mu\to A_\mu+g[A_\mu\alpha]+\partial_\mu\alpha;\\&\sigma\to\sigma+\frac{g}{2}B\alpha;\\&B\to B+M\alpha+\frac{g}{2}[B\alpha]+\frac{g}{2}\sigma\alpha.\end{aligned}\right\} \tag{38}$$

In this model, a triplet of Goldstone bosons (B) is produced; these can be eliminated from the spectrum by a suitable choice of the gauge.

In concluding this section, let us formulate a theorem that enables us to identify the vector mesons that acquire mass as a result of spontaneous symmetry breaking.[3]

Let G be the invariance group of the initial Lagrangian. The vacuum average field φ differs from zero. The aggregate of the generators l_i of the group G, which cause $\langle\varphi\rangle$ to vanish, $l_i\langle\varphi\rangle = 0$, forms a subgroup S, called the small group of vacuum. Each generator of the group G can be set in correspondence with a gauge field. All the fields corresponding to the generators in the small group of vacuum remain massless. The remaining vector fields acquire a mass. In the first of the considered models, the role of the small group is played by the abelian group of charge gauge transformations, and it is natural to identify the corresponding massless vector field with the electromagnetic field. In the second model, the small group is trivial and there are no massless vector fields.

3. CANONICAL QUANTIZATION AND FEYNMAN RULES

In this section we analyze rigorously the theories described by Lagrangians of the type (30) and (37). The preceding heuristic reasoning was needed only to construct a gauge-invariant Lagrangian that describes massive vector fields. The reader who is dissatisfied with the rigor of the arguments connected with the Higgs mechanism can therefore omit them and take Eqs. (30) and (37) as the starting point. In all the constructions that follow, we use only the gauge invariance of the Lagrangians (30) and (37).

A Lagrangian describing a gauge-invariant system is degenerate. Not all the variables that enter into the Lagrangian are canonical. Some of the variables play the role of Lagrange multipliers, and the corresponding equations are constrained equations. To quantize such a system, it is necessary to solve the constraints explicitly, and, using the leeway in the gauge, to impose on the fields a condition that fixes the gauge (this condition should be compatible with the constraints). It is then necessary to express the action in terms of the remaining independent canonical variables and to apply to them a standard quantization procedure. This procedure is equally applicable to theories with and without spontaneous symmetry breaking. Referring the reader interested in details of the quantization of degenerate systems to the paper by L. D. Faddeev,[24] we illustrate this method using the Lagrangian (37) as an example.

Owing to the degeneracy of the Lagrangian (37), the canonical momentum p_0 vanishes identically. The remaining canonical momenta take the form

$$\begin{aligned} &\mathbf{p}_0 = 0;\ \mathbf{p}_i = \partial L/\partial A_i = \mathbf{F}_{i0};\\ &\boldsymbol{\pi} = \partial L/\partial\dot{\mathbf{B}} = M\mathbf{A}_0 + \partial_0\mathbf{B} + (g/2)\,\mathbf{A}_0\sigma - (g/2)\,[\mathbf{A}_0\mathbf{B}];\\ &\pi_\sigma = \partial L/\partial\sigma = \partial_0\sigma - (g/2)\,(\mathbf{A}_0\mathbf{B}). \end{aligned} \tag{39}$$

With the aid of these equations it is easy to obtain an expression for the action $A = p_i q_i - H(p, q)$. To simplify the equations that follow we fix the gauge by putting

$$\mathbf{B} = 0. \tag{40}$$

The action can then be rewritten in the form

$$A = \int \Big\{ \mathbf{p}_i\mathbf{A}_i - \tfrac{1}{2}\mathbf{p}_i\mathbf{p}_i - \tfrac{1}{2}\boldsymbol{\pi}^2 + \mathbf{A}_0\Big(\partial_i\mathbf{p}_i + g\,[\mathbf{p}_i\mathbf{A}_i] + M\boldsymbol{\pi} + \tfrac{g}{2}\sigma\boldsymbol{\pi}\Big) + F'(\sigma, \pi_\sigma\mathbf{A}_i)\Big\}\, dx. \tag{41}$$

We have not written out here explicitly the function F', which contains the canonical form for the fields σ, since this form is standard and does not lead to any complications. What matters to us is only that F' is independent of A_0, p_i, and π.

We see that the variable A_0 plays the role of a Lagrange multiplier. By varying the action (41) with respect to A_0, we obtain the equation

$$\partial_i\mathbf{p}_i + g\,[\mathbf{p}_i\mathbf{A}_i] + M\boldsymbol{\pi} + \tfrac{g}{2}\sigma\boldsymbol{\pi} = 0, \tag{42}$$

which contains no derivatives with respect to the time and consequently is a constrained equation. This equation enables us to eliminate the variables $\boldsymbol{\pi}$, after which the action is expressed only in terms of independent canonical variables A_i, p_i, σ, and π_σ. This concludes the construction of the canonical formalism, and from now on the procedure of canonical quantization can be carried out in standard fashion, by replacing the Poisson bracket with a commutator.

At this stage, it is convenient to resort to the continual integral formalism. As is well known, the generating functional for the Green's function can be expressed in the form of a continual integral

$$Z(\eta_i, \eta) = \int \exp\Big[iA + i\int(\eta_i\mathbf{A}_i + \eta\sigma)\,dx\Big]\, d\mathbf{A}_i\, d\mathbf{p}_i\, d\sigma\, d\pi_\sigma. \tag{43}$$

In our case A is given by Eq. (41), and for $\boldsymbol{\pi}$ it is necessary to substitute the solution of (42):

$$\boldsymbol{\pi} = -[\partial_i\mathbf{p}_i + g\,[\mathbf{p}\mathbf{A}_i]] \Big/ \Big(M + \tfrac{g}{2}\sigma/2\Big). \tag{44}$$

We shall make repeated use of the representation of Green's functions in the form of a continual integral. We therefore make a few remarks concerning this integral.

The continual integral

$$\int F(\varphi)\prod_x d\varphi(x)$$

is defined as the formal limit of a finite-dimensional approximation based on the choice of a lattice system of points in x space:

$$\int F(\varphi)\prod_x d\varphi(x) = \lim \int F(\varphi_i)\prod_j d\varphi_j.$$

The choice of the limitations that must be imposed on the class of functionals and on the space of the functions in order for this limiting transition to be convergent has not yet been resolved.

However, when Green's functions are calculated by perturbation theory, only integrals of a special type are encountered, namely, Gaussian integrals

$$\int \exp\left\{\frac{i}{2}\int [\varphi(x)(\square - m^2)\varphi(x)]\,dx\right\}\varphi(y)\ldots\varphi(z)\prod_x d\varphi,$$

which can be calculated exactly. They are expressed with the aid of the variational derivatives of the integrals:

$$\int \exp\left\{i\left[\frac{1}{2}\varphi(x)(\square - m^2)\varphi(x)+\eta(x)\varphi(x)\right]\right\}dx$$
$$=\exp\left[\frac{i}{2}\int \eta(x)D^c(x-y)\eta(y)\,dx\,dy\right],$$

where $D^c(x)$ is the Feynman Green's function. To obtain the rules for bypassing the pole, it is necessary to take into account the boundary conditions, which reduce to the requirement that the integration be carried out along trajectories that are asymptotically (as $t \to \pm\infty$) distributed in configuration space with a probability determined by the wave function of the vacuum.[25] In this approach, the functional integral is simply shorthand for a number of perturbation-theory diagrams, and is convenient for various combinatorial proofs.

A more consistent and general approach can be based on a Euclidean formulation of quantum field theory, first proposed in refs. 26 and 27. In this approach, the Green's functions are defined first in terms of Euclidean variables, and the transition to the pseudo-Euclidean domain is effected only in the final expressions. Analytic continuation of the Green's functions is effected by the simple substitution $p_4 \to ip_0(1+i\varepsilon)$. This procedure makes it possible to circumvent in general the bypass rules. We shall therefore not stop to discuss this question specially each time.

The integral (43) can be rewritten in a more symmetrical form by introducing integration with respect to A_0 and π:

$$Z(\eta_i,\eta)=\int \exp\Big\{i\int\Big[p_iA_i-\frac{1}{2}p_ip_i-\frac{1}{2}\pi^2+A_0\left(\partial_ip_i+g[p_iA_i]\right.$$
$$\left.+M\pi+\frac{g}{2}\sigma\pi\right)+F'(\sigma,\pi_\sigma,A_i)+\eta_iA_i+\eta\sigma\Big]dx\Big\}\det\left(M+\frac{g\sigma}{2}\right)$$
$$\times dA_i\,dp_i\,d\sigma\,d\pi_\sigma\,dA_0\,d\pi. \qquad (45)$$

Indeed, integration with respect to A_0 can be carried out in explicit form, as a result of which we obtain $\delta(\partial_i p_i + g[p_iA_i] + M\pi + g\sigma\pi/2)$. Integration with respect to π is eliminated by a δ function, which is equivalent to replacing π with the right-hand side of (44). The Jacobian that appears upon integration with respect to π is cancelled out by $\det(M+g\sigma/2)$, and as a result we return to (43).

The integrals with respect to momenta p_i, π, and π_σ are Gaussian and can be calculated explicitly. Carrying out the integration, we obtain

$$Z(\eta_i,\eta)=\int \exp\left\{i\int [L(x)|_{B=0}+\eta_iA+\eta\sigma]\,dx\right\}$$
$$\times\det\{M+g\sigma/2\}\,dA_i\,d\sigma, \qquad (46)$$

where $L(x)$ is determined by Eq. (37) with $B = 0$.

Expansion of the integral (46) in a perturbation-theory series generates a standard diagram technique, the vector-field propagator being

$$D^c_{\mu\nu}=\frac{1}{(2\pi)^4}\cdot\frac{g^{\mu\nu}-k^\mu k^\nu m^{-2}}{k^2-m^2}. \qquad (47)$$

The variational derivatives of the functional (46) determine the Green's functions of the field A_i and σ:

$$\frac{\delta Z}{\delta\eta^a_{i_1}(x_1)\ldots\delta\eta^a_{i_n}(x_n)\ldots\delta\eta(y)}\bigg|_{\eta_i\eta=0}=\langle TA^a_{i_1}(x_1)\ldots A^a_{i_n}(x_n)\ldots\sigma(y)\rangle. \qquad (48)$$

The S matrix is obtained from these Green's functions with the aid of the standard reduction formula. The gauge considered here contains no nonphysical fields whatever. The asymptotic states include only three states of the polarization of the massive vector field and one massive scalar field. Therefore the S matrix is obviously unitary (of course, after suitable renormalization).

It is usually more convenient to use fully covariant Green's functions determined by the functional (46) with a source in the form $\eta_\mu A_\mu$ (summation from 0 to 3). Since the asymptotic states satisfy the Lorentz condition, we can assume that $\partial_\mu\eta_\mu = 0$.

The only not quite usual element in Eq. (46) is the factor $\det(M + g\sigma/2)$. Factors of this type appear in all theories in which the interaction Lagrangian contains at least two derivatives (or two vector fields). From the point of view of diagram technique, this factor is interpreted in the following manner: $\det(M+g\sigma/2)$ can be represented as the determinant of the diagonal matrix G in x space:

$$\langle x|G|x'\rangle=\delta(x-x')[M+g\sigma(x)/2]. \qquad (49)$$

Accordingly

$$\langle x|\ln G|x'\rangle=\delta(x-x')\ln[M+g\sigma(x)/2]. \qquad (50)$$

Using the equation

$$\det G=\exp Tr\ln G, \qquad (51)$$

we can rewrite $\det(M + g\sigma/2)$ in the form

$$\det(M+g\sigma/2)=\exp\left[\delta(0)\int dx\,Tr\ln(M+g\sigma/2)\right].$$

We see that from the point of view of diagram technique this factor leads to the appearance of additional diagrams that enter with weight $\delta(0)$. This expression must of course be understood in the sense of a certain regularization.) The role of these diagrams is to compensate for the most singular terms in the expansion of the integral (43), which arise in the pairings $\overline{A_\mu(x)A_\mu(x)} \sim \delta(0)$. These pairings arise because Eq. (43) does not include the operation of normal ordering. In other words, allowance for the factor $\det(M + g\sigma/2)$ means partial normal ordering of the Lagrangian contained in (43).

As already noted, owing to the slow decrease of the propagator (47), the theory with respect to the formal count of the degrees of divergence is nonrenormalizable. This conclusion, however, is in error. We shall now show that gauge invariance leads to cancellation of the strongest singularities in each order of perturbation theory, and leads as a consequence to renormalizability.

Equation (43) has only a formal meaning because the integrals that determine the individual terms of the perturbation-theory series diverge at large momenta. To make all the following arguments rigorous, we assume that the integrals have been regularized, the regularization being invariant, i.e., the regularized functional satisfies the same symmetry relations that are formally satisfied by the nonregularized functional (43). The actual method used for the invariant regularization will be considered in the next section.

We first rewrite the entire integral (43) in a more invariant form. To this end, we note that

$$\delta(B)\det(M+g\alpha/2)=\delta(B)\widetilde{\Delta}(B), \tag{52}$$

where the functional $\widetilde{\Delta}$ (B) is defined by

$$\widetilde{\Delta}(\mathbf{B})\int\delta(\mathbf{B}^{\Omega})\,d\Omega=1, \tag{53}$$

The symbol $\mathbf{B}^{\Omega}$ stands for the result of application of the gauge transformation to the field **B**. The integration is over the invariant measure on the group.

The functional $\widetilde{\Delta}$(B) is gauge-invariant in the sense that $\widetilde{\Delta}(\mathrm{B}^{\Omega}) = \widetilde{\Delta}(\mathrm{B})$. This fact follows directly from the invariance of the integration measure in (52). On the surface B = 0, only the vicinity of a single element contributes to the integral (52). Indeed,

$$\mathbf{B}^{\Omega}=\mathbf{B}+M\boldsymbol{\alpha}+(g/2)[\mathbf{B}\boldsymbol{\alpha}]+(g/2)\,\sigma\boldsymbol{\alpha}+0(\alpha^2). \tag{54}$$

Under the condition **B** = 0, the equation $\mathbf{B}^{\Omega} = 0$ becomes homogeneous and has only a trivial solution under the natural boundary conditions α (x) → 0 as x → ∞. Therefore the integral (53) can be calculated explicitly on the surface **B** = 0. Its value is

$$\widetilde{\Delta}(\mathbf{B})|_{\mathbf{B}=0}=\det[M+g\sigma(x)/2], \tag{55}$$

thereby proving our statement.

The integral (43) can be rewritten in the form

$$Z(\eta_\mu,\eta)=z^{-1}(0)\int\exp\Big\{\mathrm{i}\int[L(x)+\eta_\mu\mathbf{A}_\mu+\eta\sigma]\,dx\Big\}\widetilde{\Delta}(\mathbf{B})\,\delta(\mathbf{B})\,d\mathbf{B}\,d\mathbf{A}\,d\sigma. \tag{56}$$

Here L(x) is the gauge-invariant Lagrangian (37). To go over to an explicitly renormalizable gauge, we use the formal procedure proposed in refs. 5 and 28. We introduce a functional Δ(A) defined by the equation

$$\Delta(\mathbf{A})\int\delta(\partial_\mu\mathbf{A}_\mu^{\Omega}-\mathbf{c}(x))\,d\Omega=1. \tag{57}$$

This functional is obviously gauge-invariant. Inasmuch as the product $\Delta(\mathrm{A})\int\delta(\partial_\mu \mathrm{A}_\mu^{\Omega}-\mathbf{c}(x))\,d\Omega$ is equal to unity, we can multiply the right-hand side of (56) by this product without changing the value of the functional $Z(\eta_\mu,\eta)$. We make in the transformed integral the change of variables

$$\mathbf{A}_\mu^{\Omega}=\mathbf{A}_\mu';\ \ \mathbf{B}^{\Omega}=\mathbf{B}';\ \ \sigma^{\Omega}=\sigma'. \tag{58}$$

(we shall henceforth omit the primes). It is easy to verify that the Jacobian of this transformation is unity. By virtue of the gauge invariance of the Lagrangian and of the functionals Δ(A) and $\widetilde{\Delta}$(B), the transformation (58) leaves them invariant, and we obtain as a result

$$Z(\eta_\mu,\eta)=z^{-1}(0)\int\exp\Big\{\mathrm{i}\int[L(x)+\eta_\mu\mathbf{A}_\mu^{\Omega^{-1}}+\eta\sigma^{\Omega^{-1}}]\,dx\Big\}\times\delta(\partial_\mu\mathbf{A}_\mu-\mathbf{c}(x))\,\delta(\mathbf{B}^{\Omega^{-1}})\,\Delta(\mathbf{B})\,\Delta(\mathbf{A})\,d\mathbf{A}\,d\mathbf{B}\,d\sigma\,d\Omega. \tag{59}$$

The integration with respect to Ω can be carried out explicitly. The resultant Jacobian is cancelled by $\widetilde{\Delta}$ (B), and Ω^{-1} is expressed in terms of B through the equation $\mathrm{B}^{\Omega-1} = 0$.

On the mass shell, the terms with the sources $\eta_\mu \mathrm{A}_\mu^{\Omega-1}$, $\eta\sigma^{\Omega-1}$ can be replaced by $\eta_\mu \mathrm{A}_\mu$, $\eta\sigma$, or more accurately

$$\prod_{i=1}^{n}(k_i^2-m_i^2)\int\limits_{k_i^2=m_i^2}\exp(\mathrm{i}k_ix_i)\langle T\mathbf{A}_{\mu_1}^{\Omega^{-1}}(x_1)\ldots\mathbf{A}_{\mu_i}^{\Omega^{-1}}(x_i)\ldots\sigma^{\Omega^{-1}}(x_n)\rangle\times dx_1\ldots dx_n=\prod_i z_i^{1/2}(k_i^2-m_i^2)\int\limits_{k_i^2=m_i^2}\exp(\mathrm{i}k_ix_i)\langle T\mathbf{A}_{\mu_1}(x_1)\ldots\times\mathbf{A}_{\mu_n}(x_n)\ldots\sigma(x_n)\rangle\,dx_1\ldots dx_n. \tag{60}$$

This statement, which is the analog of the Borhers theorem,[29] is easiest to explain in diagram language. It follows from (38) that

$$\mathbf{A}_\mu^{\Omega^{-1}}=\mathbf{A}_\mu-\partial_\mu\boldsymbol{\alpha}-g[\mathbf{A}_\mu\boldsymbol{\alpha}]+0(\alpha^2);$$
$$\sigma^{\Omega^{-1}}=\sigma-\frac{g}{2}(\mathbf{B}\boldsymbol{\alpha})+0(\alpha^2). \tag{61}$$

Since the source η_μ satisfies the condition $\partial_\mu\eta_\mu = 0$, we can omit the term $\partial_\mu\boldsymbol{\alpha}$. The terms remaining in the left-hand side of (60) correspond to the Feynman diagrams shown in Fig. 1. Diagrams of type (a) correspond to the Green's functions in the right-hand side of (60). The diagrams of type (b) and (c) are due to the next higher terms of the expansions of $\mathrm{A}_\mu^{\Omega-1}$ and $\sigma^{\Omega-1}$. When diagrams of types (c) are multiplied by $\prod_i (k_i^2-m_i^2)$, they vanish at $k_i^2 = m_i^2$, since they do not contain poles in at least one of

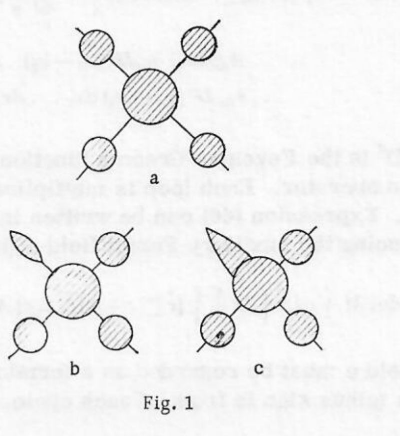

Fig. 1

the variables k_i. Diagrams of type (b) differ from the diagrams (a) for $k_i^2 = m_i^2$ only in an additional renormalization of the external lines, and this renormalization is taken into account by the factors $z_i^{1/2}$ in (60).

Consequently, if we are interested in only the matrix elements on the mass shell, we can express $Z(\eta_\mu, \eta)$ in the form

$$Z(\eta_\mu, \eta) = z^{-1}(0) \int \exp\left\{ i \int [L(x) + \eta_\mu A_\mu + \eta\sigma]\, dx \right\} \delta(\partial_\mu A_\mu - c(x)) \times \Delta(A)\, dA\, dB\, d\sigma. \quad (62)$$

Since the function $c(x)$ is arbitrary, we can integrate (61) with respect to $c(x)$ with weight $\{ i \int (2\alpha)^{-1} c^2(x)\, dx\}$. The integration yields a final expression for the generating functional of the Green's functions in an arbitrary relativistically invariant gauge:

$$Z(\eta_\mu, \eta) = z^{-1}(0) \int \exp\left\{ i \int \left[L(x) + \eta_\mu A + \eta\sigma + \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 \right] dx \right\} \Delta(A)\, dA\, dB\, d\sigma. \quad (63)$$

It remains only to calculate the explicit form of the functional $\Delta(A)$. It suffices to know its value on the surface $\partial_\mu A_\mu = c(x)$. Reasoning as in the case of the functional $\Delta(B)$, we can easily show that on this surface only the vicinity of a unit element contributes to the integral (57) that determines $\Delta(A)$. Consequently,

$$[\Delta(A)]^{-1} = \int \delta(\Box\alpha + g\partial_\mu[\alpha A_\mu])\, d\alpha = [\det \tilde{M}]^{-1},$$

where

$$\tilde{M}\alpha = \Box\alpha + g\partial_\mu[\alpha A_\mu]. \quad (64)$$

$\det \tilde{M}$ can be represented in the form of an increment to the effective action, after first separating the trivial constant factor $\det \Box$. It is convenient here to use the equality $\det \tilde{M} = \det M$, where the operator M is the adjoint of $\tilde{M}$:

$$M\alpha = \Box\alpha - g[A_\mu \partial_\mu \alpha]. \quad (65)$$

From the point of view of the diagram technique, det M can be represented in the form of a sum of closed loops, over which a scalar particle of zero mass propagates:

$$\det M = \exp(Tr \ln \Box^{-1} M) = \exp\left[-\sum_n \frac{g^n}{n} Tr \int A_{\mu_1}(x_1) \ldots \times \ldots A_{\mu n}(x_n)\, \partial_{\mu_1} D^0(x_1 - x_2) \ldots \times \ldots \partial_{\mu n} D^0(x_n - x_1)\, dx_1 \ldots dx_n \right]. \quad (66)$$

Here D^0 is the Feynman Green's function of the D'Alembertian operator. Each loop is multiplied here additionally by -1. Expression (66) can be written in compact form by introducing the auxiliary Fermi field c (ref. 5):

$$\det M \int \exp\left\{ Tr \frac{1}{2} \int [\bar{c} \Box c + g[\bar{c}, \partial_\mu c] A_\mu]\, dx \right\} dc\, d\bar{c}. \quad (67)$$

The field c must be regarded as a fermion in order to ensure a minus sign in front of each cycle.

The free Green's function corresponding to (56) is

$$D^c_{\mu\nu} = \frac{1}{(2\pi)^4} \cdot \frac{g^{\mu\nu} - k^\mu k^\nu (1-\alpha)(k^2 - m^2\alpha)^{-1}}{k^2 - m^2}. \quad (68)$$

As $k \to \infty$ we have $D^c \sim k^{-2}$. Since the interaction Lagrangian (37) contains only triple vertices with one derivative and quadrupole vertices without derivatives, the theory is renormalizable. The Lagrangian (37) in a gauge of general form leads to a mixing of the fields A_μ and B. Before we construct a perturbation theory, it is convenient to diagonalize this Lagrangian by changing over to the variables

$$B' = B - M \Box^{-1} \partial_\mu A_\mu. \quad (69)$$

The parameter α, which fixes the longitudinal part of the Green's function, can be chosen to be arbitrary. In particular, $\alpha = 0$ corresponds to the Landau gauge $\partial_\mu A_\mu = 0$. The renormalized matrix elements, generally speaking, depend on the parameter α, inasmuch as we have seen that the transition from one gauge to another is accompanied by an additional renormalization of the external lines. However, the dependence on α drops out completely from the renormalized matrix elements. (A detailed discussion of this question can be found in ref. 30.)

The proof of the equivalence of the S matrix in the unitary and renormalized gauges was presented so far under the assumption that an invariant regularization has been effected. We now must demonstrate that such a regularization is indeed feasible and that an improper limiting procedure that permits the lifting of the intermediate regularization can be defined without violating gauge invariance. The violation of the gauge invariance would destroy the equivalence of the unitary and renormalizable gauges, and consequently violate one of these properties.

4. INVARIANT REGULARIZATION AND THE WARD IDENTITIES

The invariant-regularization method is applicable to an arbitrary gauge-invariant theory, both symmetrical and with spontaneously broken symmetry.[11] It is also applicable to a rather extensive class of theories with a nontrivial internal-symmetry group, and in particular to nonlinear chiral models.[31]

By way of the first example, we consider the Yang–Mills Lagrangian. We introduce in this Lagrangian an invariant structure containing higher derivatives:

$$L = \frac{1}{2} Tr\left\{ \frac{1}{2} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\Lambda^2} D_\alpha F_{\mu\nu} D_\alpha F_{\mu\nu} - D_\alpha\varphi D_\alpha\varphi - \frac{1}{\Lambda^2} D^2\varphi D^2\varphi + V(\varphi) \right\}, \quad (70)$$

where $D_\alpha F_{\mu\nu}$ is the covariant derivative of the tensor $F_{\mu\nu}$:

$$D_\alpha F_{\mu\nu} = \partial_\alpha F_{\mu\nu} + g[F_{\mu\nu}, A_\alpha]. \quad (71)$$

It is easy to verify that under gauge transformations we have

$$D_\alpha F_{\mu\nu} \to \Omega^{-1} D_\alpha F_{\mu\nu} \Omega, \quad (72)$$

and consequently the Lagrangian (70) is invariant to the transformations (31).

The free propagators defined by the Lagrangian (70) take the form

$$D^{ab}=\frac{\delta ab}{(2\pi)^4}\,(k^2+k^4\Lambda^{-2}-m^2)^{-1};$$
$$D^{ab}_{\mu\nu}=\frac{\delta ab}{(2\pi)^4}\,(g^{\mu\nu}-k^\mu k^\nu k^{-2})\,(k^2-\Lambda^{-2}k^4)^{-1}. \qquad (73)$$

(for the sake of argument we have written out here the Green's function of the vector field in a transverse gauge). The regularized propagators decrease as k^{-4} as $k \to \infty$. However, in order to retain the invariance of the regularized theory, it would be necessary to introduce new vertices into the Lagrangian. From the explicit form of the Lagrangian (70) it follows that, in comparison with the initial Lagrangian, it contains additional triple vertices with a maximum of three derivatives, quadruple vertices with two derivatives, quintuple vertices with one derivative, and sextuple vertices without derivatives. Let us calculate the divergence index of an arbitrary diagram. Each internal line corresponds to a factor d^4k in the numerator, and to a factor k^{-4} as the result of the propagation function. Therefore the divergence index does not depend on the number of internal lines. Each triple, quadruple, or quintuple vertex makes a maximum contribution k^3, k^2, or k respectively. To each vertex there corresponds a factor $\delta^4(k)$, which is equivalent to k^{-4}. One of the δ functions expresses the law of conservation of the total four-momentum. Summing all these factors, we find that the maximum index of a diagram with n triple, m quadruple, l quintuple, and S sextuple vertices is equal to

$$3n+2m+l-4\,(n+m+l+s)+4=4-n-2m-3l-4s. \qquad (74)$$

It follows from (74) that only a finite number of diagrams diverge in the regularized model, namely, the single-loop diagrams of second, third, and fourth order. This conclusion can be extended automatically to any gauge-invariant theory. (In some models it may be useful to use for the regularization covariant derivatives of higher degree, e.g., $D^2F\mu\nu D^2F\mu\nu$.)

For single-loop diagrams there is no general method of invariant regularization. From each specific model it is necessary to use a special method. In some cases, e.g., for the γ_5 gauge group, there is no invariant regularization of the single-loop diagrams at all, and this makes it impossible to satisfy the normal Ward identities in these theories.

This leads to the following important theorem:[11] Anomalous Ward identities can be generated only by single-loop diagrams.

The method described can be extended to the case of a model with spontaneously broken symmetry. The corresponding Lagrangian can be obtained from (70) by making the formal substitution

$$m^2\to -m^2;\quad \varphi\to\varphi+\xi. \qquad (75)$$

In the theory obtained in this manner, the three propagators can take a form analogous to (73):

$$D^{ab}_{\mu\nu}=\delta^{ab}\,(g^{\mu\nu}-k^\mu k^\nu k^{-2})\,(k^2+k^4\Lambda^{-2}-M^2)^{-1};$$
$$D^{ab}_i=\delta^{ab}\,(k^2+k^4\Lambda^{-2}-m_i^2)^{-1}. \qquad (76)$$

The maximum number of derivatives in the additional vertices remains the same as in the symmetrical model. Consequently, the foregoing conclusions extend automatically to this case, too.

As to the single-loop diagrams, the fact that they have no anomalies in this model can be simply verified by direct calculation. It is also possible to use the method proposed in ref. 28 for the invariant regularization of single-loop diagrams. The idea of this method is the following. We consider a five-dimensional generalization of the Yang–Mills Lagrangian. The momentum k_μ now has five components, while the fields A_μ have 15 components. Since this theory remains gauge-invariant as before, the single-loop diagrams satisfy formally the usual Ward identities. We set five components of all the external momenta equal to zero. Then, by virtue of the conservation law, the fifth component of the momentum in all the internal lines will have the same value. We can, in particular, fix the fifth component of the momentum in the internal lines of the cycle, putting $k_5^2 = \mu^2$. We thus obtain an aggregate of diagrams coinciding with the usual diagrams in the Yang–Mills theory, but in which all the internal lines acquire an additional mass μ. For example, $k^{-2} \to (k^2 - \mu^2)^{-1}$. In addition, diagrams appear in which five components of the fields A_μ participate. The corresponding additional vertices are shown in Fig. 2. The diagrams generated by the vertices (c) and (d) satisfy the Ward identities by themselves, since they describe the gauge-invariant interaction of scalar particle. They can therefore be omitted. The remaining aggregate of diagrams coincides for $\mu = 0$ with the diagrams of the nonregularized theory. If we take the sum of the diagrams with $\mu = 0$ and μ_i different from zero, and choose the coefficients c_i and μ_i such that

$$\sum c_i=0;\quad c_0=1;$$
$$\sum c_i\mu_i^2=0;\quad \mu_0=0, \qquad (77)$$

then we obtain a final result that satisfies the normal Ward identities.[1)]

The invariant-regularization method described makes it possible to go over correctly from a unitary (Coulomb) gauge to an explicitly relativistic invariant gauge in the

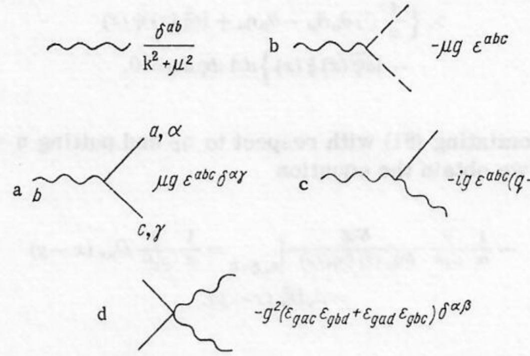

Fig. 2 The solid line denotes the propagator of the Yang–Mills field, and the dashed line the propagator of a fictitious C particle.

case of the usual Yang–Mills theory, and from a unitary gauge to a renormalized gauge in models with spontaneously broken symmetry. However, in order to determine the improper limiting transition $\Lambda \to \infty$, it is necessary to introduce counterterms that depend on Λ. If one succeeds in choosing these counterterms in a gauge-invariant manner, then the entire reasoning of the preceding section can be applied to the renormalized theory, and this proves by the same token the unitarity of the renormalized S matrix.

The relations between the counterterms follow from equations similar to the Ward identities. We describe below a general method of obtaining relations of the type of Ward identities in gauge-invariant theories. These relations are useful also from the practical point of view, since they make it possible to use in the calculations any convenient regularization (not necessarily invariant), mainly taking care that the finite parts of the diagrams satisfy Ward identities.

Let us illustrate the idea of the method using electrodynamics as the example. The generating functional for the Green's functions is defined by the equation

$$Z(\eta_\mu, \xi, \bar{\xi}) = z^{-1}(0) \int \exp i \left\{ \int \left[L(x) + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \eta_\mu A_\mu + \bar{\xi}\psi + \bar{\psi}\xi \right] dx \right\} dA\, d\bar{\psi}\, d\psi, \quad (78)$$

where L(x) is the gauge-invariant Lagrangian of the electromagnetic and the electron–positron fields. The Ward identities are the direct consequence of the invariance of L(x). To obtain them we make in the integral of (78) a gauge change of variables:

$$A_\mu \to A_\mu + \partial_\mu \varphi; \quad \psi \to \exp(ie\varphi)\psi; \quad \bar{\psi} \to \exp(-ie\varphi)\bar{\psi}. \quad (79)$$

By virtue of the invariance of L(x), this change alters only the terms with the sources and the term that fixes the gauge. Since the change of variables does not alter the integral, we can put

$$dz/d\varphi \,|_{\varphi=0} = 0. \quad (80)$$

This relation represents in compact form the infinite system of Ward identities. Its explicit form is

$$\int \exp i \left\{ \int \left[L(x) + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \eta_\mu A_\mu + \bar{\xi}\psi + \bar{\psi}\xi \right] dx \right\} \times \left\{ \frac{1}{\alpha} \Box \partial_\mu A_\mu - \partial_\mu \eta_\mu + ie\bar{\xi}(x)\psi(x) - ie\bar{\psi}(x)\xi(x) \right\} dA\, d\bar{\psi}\, d\psi = 0. \quad (81)$$

Differentiating (81) with respect to η_ν and putting $\eta = \xi = \bar{\xi} = 0$, we obtain the equation

$$-\frac{1}{\alpha}\frac{\partial}{\partial x^\mu}\frac{\delta^2 Z}{\delta\eta_\mu(x)\,\delta\eta_\nu(y)}\Big|_{\eta,\,\xi=0} = \frac{1}{\alpha}\frac{\partial}{\partial x^\mu} D_{\mu\nu}(x-y) = \partial_\nu D_0^c(x-y), \quad (82)$$

which denotes the absence of renormalization of the longitudinal part of the Green's function. Differentiating (81) with respect to ξ and $\bar{\xi}$, we obtain the relation

$$\frac{1}{\alpha}\Box_x \frac{\partial}{\partial x^\mu}\langle T A_\mu(x)\bar{\psi}(y)\psi(z)\rangle = e\{\delta(x-y) - \delta(x-z)\}\langle T\bar{\psi}(y)\psi(z)\rangle, \quad (83)$$

which is the well known Ward identity for the vertex part.

In principle, the same procedure can be used also for the Yang–Mills theory. In this case the generating functional for the Green's functions is

$$Z(\eta_\mu, \eta) = z^{-1}(0) \int \exp i \left\{ \int \left[L(x) + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \eta_\mu A_\mu + \eta B \right] dx \right\} \Delta(A)\, dA\, dB. \quad (84)$$

Making the gauge change of variables (31) in this integral and putting $(dZ/d\varphi)|_{\varphi=0} = 0$, we obtain a system of Ward identities. The relations obtained in this manner, however, are meaningless. Under the transformations (81), the terms $(\partial_\mu A_\mu)^2$ and $\Delta(A)$ vary in a complicated manner, and the relations $dz/d\varphi = 0$ include in this case matrix elements of the type

$$\langle T\partial_\nu\partial_\mu A_\mu(x) A_\nu(x) A_\rho(y)\rangle,$$

which are not encountered in the expansion of the S matrix. These terms depend on the product of operators of like arguments, and call for a special additional definition.

To obtain the Ward identities relating only the physical matrix elements, we make a gauge change of variables of special type, with a function φ satisfying the equation

$$M^{ab}\varphi^b = \chi^a, \quad (85)$$

where χ is an arbitrary function, and the operator M is defined in (65). This transformation leaves invariant the integration measure $\Delta(A)\, dA$. To verify this, it suffices to calculate the Jacobian of the transformation and the variation of $\Delta(A)$:

$$\delta\Delta(A) = \delta Tr \ln M = \int \delta M^{ab} M^{-1ba}_{xy}\big|_{y=x} dx, \quad (86)$$

where the inverse operator M_{xy}^{-1} satisfies the equation

$$\Box M^{-1ab}_{xy} + g\varepsilon^{acd}\partial_\mu\{A^c_\mu(x) M^{-1db}_{xy}\} = \delta^{ab}\delta(x-y). \quad (87)$$

The operator M_{xy}^{-1ab} is the Green's function of the fictitious c particles in the external field A_μ.

Equating the coefficient of $\chi(x)$ to zero, we obtain the relation

$$\int \exp i \left\{ \int \left[L(x) + \eta_\mu A_\mu + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2 + \eta B \right] dx \right\} \times \left\{ \frac{1}{\alpha}\partial_\mu A^d_\mu(y) + \int [\eta^a_\mu(z)\,\partial^z_\mu M^{-1ad}_{zy} + g\varepsilon^{abc}\eta^a_\mu(z) A^b_\mu(z) M^{-1cd}_{zy} + g\varepsilon^{abc}\eta^a(z)\varphi^b(z) M^{-1cd}_{xy}]\, dz \right\} \times \Delta(A)\, dA\, dB = 0, \quad (88)$$

which is a system of generalized Ward identities. Differentiating this equation with respect to $\eta^b_\nu(x)$ and putting η_ν, $\eta = 0$, we obtain

$$-\frac{1}{\alpha}\left\langle T\frac{\partial A^a_\mu}{\partial x^\mu}A^b_\nu(y)\right\rangle = \langle T\partial^y_\nu M^{-1ba}_{yx}\rangle + g\varepsilon^{bcd}\langle TA^c_\nu(y)M^{-1da}_{yx}\rangle. \quad (89)$$

Differentiating (89) with respect to y and using Eq. (87), we obtain

$$(1/\alpha)\langle T(\partial A^a_\mu/\partial x^\mu)(\partial A^b_\nu/\partial y^\nu)\rangle = \delta^{ab}\delta(x-y). \quad (90)$$

This leads to the equation

$$(1/\alpha)\,\partial^x_\mu D^{ab}_{\mu\nu}(x, y) = \delta^{ab}\partial_\nu D^c_0(x-y), \quad (91)$$

which is the exact analog of the condition (82) in electrodynamics, meaning the absence of renormalization of the longitudinal part of the Green's function. The condition (91) leads to the vanishing of the counterterm of the mass renormalization.

To obtain the analog of the identity (83), we differentiate (88) twice with respect to η_ν and put η_ν, $\eta = 0$. We have

$$\begin{aligned}(1/\alpha)\langle TA^a_\mu(x)A^b_\nu(y)\partial_\rho A^c_\rho(z)\rangle &= \langle T\partial_\mu M^{-1ac}_{xz}A^b_\nu(y) + \langle TA^a_\mu(x)\partial_\nu M^{-1bc}_{yz}\rangle \\ &+ g\varepsilon^{ade}\langle TA^d_\mu(x)M^{-1ec}_{xz}A^b_\nu(y)\rangle \\ &+ g\varepsilon^{bde}\langle TA^a_\mu(x)A^d_\nu(y)M^{-1ec}_{yz}\rangle. \end{aligned} \quad (92)$$

In contrast to electrodynamics, the divergence of the vertex function is no longer expressed only in terms of a two-point Green's function. To obtain from (92) a relation between the renormalization constants, we differentiate this equation with respect to y^ν and separate in it the structure that is transverse with respect to the momentum p conjugate to the coordinate x. Changing over to Fourier transforms, we have.

$$D^{tr}_{\mu\nu}(p)\frac{k^\nu}{k^2}\cdot\frac{(p+k)^\rho}{(p+k)^2}\Gamma^{abc}_{\lambda\nu\rho}(p, k) = \frac{k^\nu}{k^2}G(p+k)\gamma^{abc}_{\mu\nu}(p, k). \quad (93)$$

Here $\Gamma^{abc}_{\lambda\nu\rho}$ denotes the proper vertex part

$$\begin{aligned}\langle TA^a_\mu(x)A^b_\nu(y)A^c_\rho(z)\rangle &= \int D^{aa'}_{\mu\mu'}(x-x')D^{bb'}_{\nu\nu'}(y-y') \\ &\times D^{cc'}_{\rho\rho'}(z-z')\Gamma^{a'b'c'}_{\mu'\nu'\rho'}(x', y', z')\,dx'\,dy'\,dz', \end{aligned} \quad (94)$$

and we have used (87) and (91). G(q) is the Green's function of the c particle,

$$\delta^{ab}(2\pi)^{-4}\int\exp[ik(x-y)]G(k)\,d^4k = \langle TM^{-1ab}_{xy}\rangle, \quad (95)$$

and the function $\gamma^{abc}_{\mu\nu}$ is connected with the proper vertex by the function γ^{abc}_ν, which corresponds to a transition of two c particles into a single vector particle, by the relation

$$\gamma^{abc}_\nu = ip_\mu\gamma^{abc}_{\mu\nu}. \quad (96)$$

The symbol tr denotes that we have separated in Eq. (93) the part transverse with respect to the variable p. The vertex part $\Gamma^{abc}_{\lambda\nu\rho}$ can be represented in the form

$$\begin{aligned}\Gamma^{abc}_{\lambda\nu\rho}(p, k, q) &= i\varepsilon^{abc}\{z_1^{-1}[g^{\nu\lambda}(p-k)^\rho \\ &+ g^{\nu\rho}(p-q)^\lambda + g^{\lambda\rho}(q-\rho)^\nu] + 0_{\lambda\nu\rho}, \end{aligned} \quad (97)$$

where $0_{\lambda\nu\rho}$ is a finite part that vanishes for $p^2 = k^2 = q^2$. We can represent in similar form the other functions that enter in (93):

$$D^{tr}_{\mu\nu}(p) = (g^{\mu\nu} - p^\mu p^\nu p^{-2})\,p^{-2}\{z_2 + 0(p^2)\}; \quad (98)$$

$$G(q) = \{\tilde{z}_2 + 0(q^2)\}\,q^{-2}; \quad (99)$$

$$\gamma^{abc}_{\mu\nu}(p, q) = \varepsilon^{abc}\{g_{\mu\nu}\tilde{z}_1^{-1} + 0_{\mu\nu}(p, q)\}. \quad (100)$$

The constant z_1 is, by virtue of (96), the renormalization constant of the vertex function γ^{abc}_ν. Substituting these expressions in (93) and putting $p^2 = k^2 = q^2 = 0$, we obtain

$$z_1 z_2^{-1} = \tilde{z}_1\tilde{z}_2^{-1} \quad (101)$$

This relation replaces the equality $z_1 = z_2$ in electrodynamics. By direct calculation in the lower orders we can demonstrate that $z_1 \neq z_2$.

We can obtain analogously relations for the renormalization constants of the quadrupole vector vertex z_4, and also for the constants z_{1B}, z_{2B}, z_{4B}, which renormalize the Green's function and the vertex function of the material field:

$$z_4 = z_1^2 z_2^{-1}; \quad z^{-1}_{2B}z_{1B} = \tilde{z}_2^{-1}\tilde{z}_1; \quad z_{4B} = z^2_{1B}z_2^{-1}. \quad (102)$$

These relations ensure gauge invariance of the renormalized Lagrangian

$$\begin{aligned}L &= -\frac{z_2}{4}f^R_{\mu\nu}f^R_{\mu\nu} - \frac{z_1 g_R}{2}f^R_{\mu\nu}[\mathbf{A}^R_\mu\mathbf{A}^R_\nu] - z_4\frac{g^2_R}{4}[\mathbf{A}^R_\mu\mathbf{A}^R_\nu][\mathbf{A}^R_\mu\mathbf{A}^R_\nu] \\ &- \tilde{z}_2\bar{\mathbf{c}}^R\,\Box\,\mathbf{c}^R + \tilde{z}_1 g_R A^R_\mu[\bar{\mathbf{c}}^R\partial_\mu\mathbf{c}^R] + \frac{1}{2}z_{2B}\partial_\mu\mathbf{B}^R\partial_\mu\mathbf{B}^R - z_{1B}g_R\mathbf{A}^R_\mu[\mathbf{B}^R\partial_\mu\mathbf{B}^R] \\ &+ \frac{z_{4B}}{2}g^2_R[\mathbf{A}^R_\mu\mathbf{B}^R][\mathbf{A}^R_\mu\mathbf{B}^R] + V(\mathbf{B}^R). \end{aligned} \quad (103)$$

Changing over to nonrenormalized fields

$$\begin{aligned}A^R_\mu &= z_2^{-1/2}A_\mu; \\ c_R &= \tilde{z}_2^{-1/2}c; \\ B^R &= z_{2B}^{-1/2}B \end{aligned} \quad (104)$$

and taking (102) into account, we can rewrite this expression in the form

$$\begin{aligned}L = &-\frac{1}{4}F_{\mu\nu}(g)F_{\mu\nu}(g) + \frac{1}{2}D_\mu B(g)D_\mu B(g) \\ &- \bar{c}\,\Box\,c + gA_\mu[\bar{c}\partial_\mu c] + V_R(B), \end{aligned} \quad (105)$$

where the "bare" charge g is connected with the physical charge by the relation

$$g = z_1 z_2^{-3/2}g_R. \quad (106)$$

We see that the renormalization does not violate the gauge invariance. (The parameters of the gauge transformation, however, are changed in this case.) Thus, we have succeeded in constructing a Lagrangian that is gauge-invariant for any finite Λ and admits of an improper transition to the limit as $\Lambda \to \infty$. By the same token, we prove the gauge invariance and the unitarity of the renormalized S matrix.

A perfectly analogous procedure can be used in the theory with spontaneously broken symmetry. The Ward identities for this case are given as before by Eq. (88), in which it is only necessary to make the substitution $B \to B + \xi$. In the same manner we can also obtain relations between the renormalization constants. However, since the Lagrangians (30) and (37) contain very many vertices, the corresponding manipulations are quite cumbersome. Some of them are given in ref. 33.

Another way of proving the existence of an improper transition to the limit as $\Lambda \to \infty$ was proposed in ref. 34. The authors used the fact that the matrix elements in the model with spontaneously broken symmetry can be obtained from the matrix elements of the symmetrical theory by analytic continuation in the mass. As we have shown, in the symmetrical theory all the divergences are eliminated by a finite number of gauge-invariant counterterms. The result of ref. 34 is that the same counterterms eliminate the divergences also in the model with broken symmetry.

The foregoing proof of the renormalizability and unitarity of the S matrix extends also to the case when a gauge-invariant interaction with other fields is included. Serious complications arise, however, if the gauge group includes γ_5 transformations, corresponding to the axial-vector interaction between a Yang–Mills field and fermions. At the same time, it is just models of this kind which are of greatest interest for the description of weak and electromagnetic interactions.

5. RENORMALIZATION OF MODELS OF WEAK AND ELECTROMAGNETIC INTERACTIONS

The spontaneous symmetry-breaking mechanism described above makes it possible to construct gauge-invariant models that combine weak and electromagnetic interactions. The first and most popular model of this kind was proposed by Weinberg.[4]

Weinberg's model is based on the gauge group U(2). The initial Lagrangian is

$$L = \bar{R}\gamma^\mu (\partial_\mu + ig_1 B_\mu) R + \bar{L}\left(\partial_\mu + ig\boldsymbol{\tau}\mathbf{A}_\mu + \frac{ig_1}{2} B_\mu\right) L - G\{(\bar{L}\varphi) R + \bar{R}(\varphi^+ L)\} - \frac{1}{4} f_{\mu\nu} f_{\mu\nu} - \frac{1}{4}\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu} + \frac{1}{2}\left(\partial_\mu\varphi + ig\boldsymbol{\tau}\mathbf{A}_\mu + \frac{ig_1}{2} B_\mu\right)^2 + \frac{m^2}{2}\varphi^2 - h^2\varphi^4. \quad (107)$$

L and R denote respectively a lepton doublet with left-hand helicity and a singlet with right-hand helicity:

$$L = \frac{1}{2}(1 - \gamma_5)\begin{pmatrix} \nu_e \\ e \end{pmatrix}; \quad R = \frac{1}{2}(1 - \gamma_5)\, e. \quad (108)$$

In addition to the already considered isovector field A_μ, the Weinberg model employs a neutral field B_μ connected with the abelian gauge group U(1).

The Lagrangian (107) is invariant to the gauge transformations

$$\begin{aligned} \mathbf{A}_\mu &\to \mathbf{A}_\mu + g[\mathbf{A}_\mu \boldsymbol{\xi}] + \partial_\mu \boldsymbol{\xi}; \\ B_\mu &\to B_\mu + \partial_\mu \eta; \\ L &\to L + ig\boldsymbol{\xi\tau} L + \frac{ig_1}{2}\eta L; \\ R &\to R + ig_1 \eta R; \\ \varphi &\to \varphi + ig\boldsymbol{\xi\tau}\varphi + \frac{ig_1}{2}\eta\varphi. \end{aligned} \quad (109)$$

The transformations (109), when expressed in terms of the canonical fields e and ν_e, include both vector and axial-vector gauge transformations.

Before we construct a perturbation theory for the Lagrangian (107), we must, as before, perform the canonical transformation

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \to \begin{pmatrix} \varphi^+ \\ \varphi^0 + \xi \end{pmatrix}. \quad (110)$$

As a result of this transformation, three components of the scalar field φ are transformed into massless Goldstone bosons, while one component acquires a real mass. Three of the four vector fields ($A_\mu B_\mu$) acquire a finite mass, and one remains massless, corresponding to invariance of the transformed Lagrangian with respect to charge gauge transformations. The canonical transformation gives rise also to a mass term for the leptons

$$-G\left\{\left[\bar{L}\begin{pmatrix} 0 \\ \xi \end{pmatrix}\right] R + \bar{R}[(0, \xi) L]\right\} = -G\xi\bar{e}e. \quad (111)$$

The Lagrangian (107) leads to a mixing of the fields A_μ and B_μ. Diagonalization results in two neutral fields,

$$Z_\mu = (g^2 + g_1^2)^{-1/2}(gA_\mu^3 + g_1 B_\mu) \quad (112)$$

and

$$A_\mu = (g^2 + g_1^2)^{-1/2}(-g_1 A_\mu^3 + gB_\mu) \quad (113)$$

with masses $(\xi/2)(g^2 + g_1^2)^{1/2}$ and 0, respectively. The component A_μ interacts with a conserved electric current and is identified with the electromagnetic field; the field Z_μ describes a neutral intermediate boson that induces the interaction of neutral leptonic currents.

That part of the Lagrangian (107) which describes the interaction of the vector mesons with the fermions assumes after diagonalization the form

$$L_I = \frac{g}{2\sqrt{2}}\bar{e}\gamma^\mu(1 + \gamma_5)\nu W_\mu + \frac{gg_1}{(g^2 + g_1^2)^{1/2}}\bar{e}\gamma^\mu e A_\mu + \frac{(g^2 + g_1^2)^{1/2}}{4}\left[\frac{(3g_1^2 - g^2)}{g^2 + g_1^2}\bar{e}\gamma^\mu e - \bar{e}\gamma^\mu\gamma^5 e + \bar{\nu}\gamma^\mu(1 + \gamma^5)\nu\right] Z_\mu, \quad (114)$$

where W_μ are the charged components of the vector triplet (only the electronic part of the Lagrangian has been written out; the muonic part is similar in form).

Weinberg's Lagrangian includes, in addition to the usual V–A and electromagnetic interactions, a supplementary term describing weak neutral lepton current interaction. The presence of this term leads, even in the lowest order, to a difference between the predictions of the Weinberg model for the $e\nu$ or $\mu\nu$ scattering processes and the predictions of the V−A theory. We shall not dwell in greater detail on the physical consequences of Weinberg's model. We note only that in this model relations arise between the masses of the intermediate bosons and the weak and electromagnetic constants:

$$e=\frac{gg_1}{(g^2+g_1^2)^{1/2}};\quad \frac{G_W}{\sqrt{2}}=\frac{g^2}{8M_W^2}=\frac{1}{2\xi^2};$$
$$M_W \geqslant 37.3\,\text{GeV};\quad M_z > 76.4\,\text{GeV}. \tag{115}$$

The Weinberg Lagrangian is gauge-invariant. Therefore, on the basis of the reasoning of the preceding sections, it would be expected to correspond to a renormalizable theory. This conclusion, however, is in error. There is no invariant regularization of single-loop diagrams in this model, so that the lower-order single-loop diagrams satisfy not the normal Ward identities but the so called anomalous ones.[35,36] In other words, gauge invariance of the initial classical Lagrangian does not result in gauge invariance of the corresponding quantum theory. In Weinberg's model it is impossible to construct a gauge-invariant renormalized S matrix. This leads to nonequivalence of the unitary and renormalizable gauges, and consequently to incompatibility of these two requirement.

Let us illustrate the foregoing, using as an example a simple model described by the Lagrangian

$$L=\overline{\psi}\gamma^\mu\partial_\mu\psi-(1/4)\,f_{\mu\nu}f_{\mu\nu}-ig\overline{\psi}\gamma^\mu\gamma^5\psi A_\mu. \tag{116}$$

This Lagrangian is invariant with respect to the gauge transformations

$$\psi\to\exp(ig\gamma^5\varphi)\,\psi;\quad \overline{\psi}\to\exp(ig\gamma^5\varphi)\,\overline{\psi};\quad A_\mu\to A_\mu+\partial_\mu\varphi. \tag{117}$$

One might expect, just as in electrodynamics, the longitudinal part of the vector-field propagator to make no contribution to the physical matrix elements, so that the theory would be renormalizable. Actually, the renormalization procedure violates gauge invariance in this case. It is impossible to determine a third-order vertex function corresponding to the diagram shown in Fig. 3 as an integrable generalized function satisfying the normal Ward identities.

If we denote the Fourier transform of the vertex function by $T^{\mu\nu\alpha}$ (p, q), then gauge invariance stipulates that

$$p_\mu T_{\mu\nu\alpha}(p,q)=q_\nu T_{\mu\nu\alpha}(p,q)=(p+q)_\alpha T_{\mu\nu\alpha}(p,q)=0. \tag{118}$$

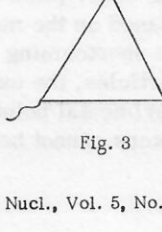

Fig. 3

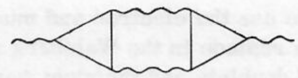

Fig. 4

At the same time, an explicit calculation of the matrix element shown in Fig. 3, with allowance for the symmetry properties, yields

$$i\,(p+q)_\alpha T_{\mu\nu\alpha}=(g^3/16\pi^2)\,\varepsilon^{\mu\nu\alpha\beta}p_\alpha q_\beta.$$

It must be emphasized that no choice of counterterms (assuming a minimal subtraction procedure[37]) can satisfy the conditions (118).

It follows from (119) [sic] that the longitudinal part of the propagator makes a nonzero contribution, e.g., to the diagram shown in Fig. 4, and thus leads in the final analysis to nonrenormalizability of the theory. Obviously, the same effect will be observed in the Weinberg model, since that part of the Lagrangian which describes the interaction with the field B_μ coincides in fact with the model considered by us. A way out of this difficulty is described in refs. 38, 39, and 40. We shall illustrate it, using the same model as an example. Assume that in addition to the field ψ there is an analogous field ψ' that differs from ψ by some internal quantum number:

$$L=\overline{\psi}\gamma^\mu\partial_\mu\psi+\overline{\psi}'\gamma^\mu\partial_\mu\psi'-\frac{1}{4}f_{\mu\nu}f_{\mu\nu}-g\overline{\psi}\gamma^\mu\gamma^5\psi A_\mu+g\overline{\psi}'\gamma^\mu\gamma^5\psi' A_\mu. \tag{119}$$

The interaction (119) can be expressed in pure vector form (not containing the matrices γ_5) by introducing new canonical variables

$$\psi_1=\frac{1}{2}\{(1-\gamma_5)\,\psi+(1+\gamma_5)\,\psi'\};\quad \psi_2=\frac{1}{2}\{(1+\gamma_5)\,\psi+(1-\gamma_5)\,\psi'\}. \tag{120}$$

The interaction Lagrangian expressed with the aid of ψ_1 and ψ_2 takes the form

$$L_I=(g\overline{\psi}_1\gamma^\mu\psi_1-g\overline{\psi}_2\gamma^\mu\psi_2)\,A_\mu \tag{121}$$

and is the analog of the Lagrangian of the electromagnetic interaction of two massless spinors. This Lagrangian is invariant with respect to the pure vector gauge group and does not lead to anomalous Ward identities. In the case of the pure vector gauge group it is always possible to satisfy the normal Ward identities, for in this case the single-loop diagrams that describe the material field can be invariantly regularized with the aid of the usual Pauli–Villars regularization. The absence of anomalies in multiloop diagrams was proved by us earlier.

In the considered simple model, it is easy to verify directly that there are no anomalies. The field ψ' generates a diagram analogous to that shown in Fig. 3. Since ψ' enters in the interaction Lagrangian with the constant −g, the contributions of these two diagrams cancel each other. A similar mechanism of anomaly suppression can be used also in models with broken symmetry. As a result of spontaneous symmetry breaking, the fields ψ and ψ' can acquire different nonzero masses.

The described anomaly-suppression mechanism calls necessarily for introduction of additional fields. It might seem natural to use the electron and muon as the fields ψ and ψ'. If we replace in the Weinberg model the left-hand polarized doublets and the right-hand polarized singlet by the four-component multiplets

$$D=\begin{pmatrix} \nu_{eL}+\nu_{\mu R} \\ e_L+\mu_R \end{pmatrix}; \quad S=(e_R+\mu_L) \tag{122}$$

and introduce the mass terms

$$G_1(\overline{D}_L\varphi)S+G_2(\overline{D}_R\varphi)S+\text{H.c.} \tag{123}$$

then we obtain a Lagrangian that is invariant with respect to a pure vector gauge group, and consequently is renormalizable. The electron and muon, however, enter in this case in the interaction Lagrangian with opposite helicities, in contradiction to experiment. We are left with the possibility of replacing μ and ν_μ in Eqs. (122) and (123) with some hypothetical particles θ^+ and θ^0. If we choose the mass terms such that $m_\theta \gg m_k$, then this model describes correctly the $e\nu_e$ interaction and, in addition, predicts a number of processes with participation of θ particles, which so far have remained unobservable because of their large mass. (To cancel muonic anomalies it is necessary to introduce additional heavy leptons θ^+_μ and θ^0_μ.)

There are many models that are free of anomalies and in which the mechanism described above is employed. We describe by way of example two models in which the number of hypothetical particles is minimal.

The model of Georgi and Glashow[41] is based on the gauge group SU_2. It has no neutral intermediate boson, and accordingly no interaction of neutral lepton currents. A triplet and singlet of four-component lepton fields are introduced:

$$\psi_l=\frac{1}{2}\begin{pmatrix} E^++e^- \\ \frac{1}{i}(e^--E^+) \\ \nu_{eL}\sin\beta+\theta^0_L\cos\beta+\theta^0_R \end{pmatrix};$$
$$\psi_s=(\nu_{eR}+\nu_{eL}\cos\beta-\theta^0_L\sin\beta), \tag{124}$$

where e and ν_e denote the electron and the electronic neutrino, while θ^0 and E^+ are hypothetical heavy leptons. The initial Lagrangian is

$$L=\overline{\psi}_l(\gamma^\mu\partial_\mu\psi_l+g\gamma^\mu[A_\mu\psi_l])-m\overline{\psi}_l\psi_l$$
$$+\varphi\{g_1[\overline{\psi}_l\psi_l]+g_3\overline{\psi}_l(1-\gamma_5)\psi_5\}+\text{H.c.}+L(A_\mu\varphi). \tag{125}$$

Here φ is an isovector scalar field and $L(A_\mu, \varphi)$ denotes a Lagrangian describing the gauge-invariant interaction between the field φ and the Yang–Mills field A_μ. The canonical transformation

$$\varphi^a\to\varphi^a+\delta^{a3}\xi \tag{126}$$

leads to the appearance of a mass for the two components of the vector field and to the appearance of an additional mass term for the fields ψ.

The Lagrangian (125) is invariant with respect to pure vector gauge transformations. It is therefore possible to construct for it an invariant regularization and to prove the renormalizability and unitarity of the S matrix.

The lepton-interaction Lagrangian has the standard form

$$L_I=eA_\mu(\overline{E}^+\gamma^\mu E^+-\overline{e}^-\gamma^\mu e^-)+\frac{1}{2}e\sin\beta W^+_\mu\overline{e}^-\gamma_\mu(1+\gamma_5)\nu+\ldots+\text{H.c.} \tag{127}$$

In the model of Georgi and Glashow, the following relations arise between the parameters:

$$g_W=\frac{1}{2}e\sin\beta;\quad 2m(\theta^0)\cos\beta=m(E^+)+m(e^-), \tag{128}$$

whence

$$m_W\leqslant 53\text{ GeV}.$$

The muons are introduced independently, together with their partners E^+_μ and θ^0_μ. Since the value of the angle β is not fixed in the model, no $\mu-e$ universality arises automatically, although it does not contradict the model.

The second model[42] is based on the U(2) group. In this model one introduces for the lepton pair e and ν_e only one additional neutral partner θ_e.

The leptons are unified into a four-component doublet and a singlet:

$$\psi_D=\begin{pmatrix} \nu_{eL}\sin\beta+\theta_{eL}\cos\beta+\theta_{eR} \\ e \end{pmatrix};$$
$$\psi_S=(\nu_{eR}-\nu_{eL}\cos\beta+\theta_{eL}\sin\beta). \tag{129}$$

The initial Lagrangian is

$$L=\overline{\psi}_D\gamma^\mu\left(\partial_\mu+igA_\mu\tau+\frac{ig_1}{2}B_\mu\right)\psi_D+m\overline{\psi}_D\psi_D$$
$$+G(\overline{\psi}_D\varphi)(1-\gamma_5)\psi_5+\text{H. c.}+L(B_\mu, A_\mu, \varphi). \tag{130}$$

where $L(B_\mu, A_\mu, \varphi)$ is the Lagrangian of the Yang–Mills fields B_μ and A_μ that interact with the scalar doublet φ.

Like the first model, the second is based on a pure vector gauge group, and therefore, using the rules formulated above, we can construct for it a unitary and renormalizable S matrix.

The Lagrangian (130) generates an interaction of neutral lepton currents. For $g = g_1$, however, the terms describing the interaction of the neutral electronic currents vanishes, and there remains only an interaction of neutral neutrino currents. (In the Weinberg model this cannot be obtained by any choice of g and g_1.)

For $g = g_1$, the electric charge is $e = (g/\sqrt{2})\sin\beta$, where $\cos\beta = m_e/m_\theta$. Since m should be much larger than m_e, it follows that $\sin\beta\approx 1$ and the model predicts for the intermediate meson a mass $m_W \approx 76.4$ GeV.

The muons are introduced in similar fashion. If the condition $m_\mu \ll m_{\theta_\mu}$ is satisfied, then universality arises automatically.

This concludes the description of renormalizable lepton models. All are based on the mechanism described above. Thier common shortcoming is the need for introducing hypothetical particles, the existence of which is doubtful from the experimental point of view. It should be noted that "heavy" leptons cannot be assigned an arbitrarily large mass. The mass is subject to rather stringent limitations (the specific forms of which depend on the model).

Some authors propose to construct anomaly-free models by using baryons or quarks instead of heavy leptons. We shall not dwell in detail on these models. They likewise involve difficulties connected primarily with the need for taking strong interactions into account.

Although the development of a unified theory that combines strong, electromagnetic, and weak (and possibly also gravitational) interactions is quite enticing, it seems to us that there should be a less ambitious possibility of obtaining an internally consistent description of leptons, at least in a limited energy range. The success of quantum electrodynamics is evidence favoring this point of view.

1) A general method for regularizing arbitrary diagrams by analytic continuation with respect to the number of the dimensions of space was proposed in ref. 32. According to the statement of the authors, this regularization is gauge-invariant. This fact, however, was not proved in this paper for diagrams containing more than one loop.

1. G. t'Hooft, Nucl. Phys., B35, 173 (1971).
2. P. W. Higgs, Phys. Rev. Lett., 13, 508 (1964); Phys. Rev., 145, 1156 (1966).
3. T. W. B. Kibble, Phys. Rev., 155, 1554 (1967).
4. S. Weinberg, Phys. Rev. Lett., 19, 1264 (1967).
5. L. D. Faddeev and V. N. Popov, Phys. Lett., B25, 29 (1967), Preprint ITF-67-36, Kiev (1967).
6. B. S. DeWitt, Phys. Rev., 162, 1195, 1239 (1967).
7. S. Mandelstam, Phys. Rev., 175, 1580 (1968).
8. E. S. Fradkin and I. V. Tyutin, Phys. Rev., D2, 2841 (1970).
9. A. A. Slavnov, Teor. Mat. Fiz., 10, 153 (1972); Preprint ITP-71-83E, Kiev (1971).
10. J. C. Taylor, Nucl. Phys., B33, 436 (1971).
11. A. A. Slavnov, Teor. Mat. Fiz., 13, 174 (1971); Preprint ITP-71-131E, Kiev (1971).
12. C. N. Yang and R. Mills, Phys. Rev., 96, 191 (1959).
13. R. Utiyama, Phys. Rev., 101, 1597 (1956).
14. Sh. L. Glashow and M. Gell-Mann, Ann. Phys., 15, 437 (1961).
15. J. Schwinger, Ann. Phys. (New York), 2, 407 (1957).
16. J. Sakurai, Ann. Phys. (New York), 11, 1 (1960).
17. E. S. Stuekelberg, Helv. Phys. Acta., 11, 226, 229 (1938).
18. A. A. Slavnov, Teor. Mat. Fiz., 10, 305 (1972).
19. L. D. Faddeev, Preprint LOMI No. 1, Leningrad (1972).
20. Y. Nambu and G. Iona-Lasinio, Phys. Rev., 122, 345 (1961).
21. B. A. Arbuzov, A. N. Tavkhelidze, and R. N. Faustov, Dokl. Akad. Nauk SSSR, 139, 345 (1961) [Sov. Phys.–Dokl., 6, 598 (1962)].
22. I. Goldstone, Nuovo Cimento, 19, 155 (1961).
23. N. N. Bogolyubov, Quasimean Values in Statistical-Mechanics Problems [in Russian], JINR Preprint, Dubna (1961).
24. L. D. Faddeev, Teor. Mat. Fiz., 1, 1 (1969).
25. O. I. Zav'yalov, Dissertation, JINR, Dubna (1971).
26. J. Schwinger, Phys. Rev., 115, 728 (1959).
27. E. S. Fradkin, in: Tr. FIAN SSSR, Vol. 23, Nauka, Moscow (1965).
28. G. t'Hooft, Nucl. Phys., B33, 173 (1971).
29. H. J. Borhers, Nuovo Cimento, 25, 270 (1960).
30. R. Kallosh and I. Tyutin, FIAN Preprint No. 90, Moscow (1972).
31. A. A. Slavnov, Nucl. Phys., B31, 301 (1971).
32. G. t'Hooft and M. Veltman, Nucl. Phys., B44, 189 (1972).
33. D. A. Ross and J. C. Taylor, Preprint, Oxford University 46/72.
34. B. W. Lee and J. Zinn-Justin, Phys. Rev., D6, 3121 (1972).
35. S. Adler, Phys. Rev., 177, 2426 (1969).
36. J. S. Bell and R. Jackiw, Nuovo Cimento, 60, 47 (1969).
37. A. A. Slavnov, Teor. Mat. Fiz., 7, 13 (1971).
38. C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Lett., B38, 519 (1972).
39. H. Georgi and S. Glashow, Phys. Rev., D6, 429 (1972).
40. D. Gross and R. Jackiw, Phys. Rev., D6, 477 (1972).
41. H. Georgi and S. Glashow, Phys. Rev. Lett., 28, 1494 (1972).