Configuration mixing effects in single-nucleon transfer reactions

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The present status of the theory of single-nucleon transfer reactions to bound and unbound states of the final nucleus is considered. Various methods are described for solving the equations for the form factors of single-nucleon transfer reactions within the framework of various model representations of the nuclear structure. It is shown that configuration mixing effects and allowance for the continuum can play a significant role.

INTRODUCTION

Reactions with direct nucleon transfer (stripping and pickup) occupy a special place among nuclear reactions. The characteristic features of their angular distribution and of their energy dependence made it possible long ago to segregate them into a separate class in accordance with purely experimental attributes. The relative simplicity of the mechanism made it possible to develop a sufficiently complete theory of these reactions as long as 20 years ago.[1] It became clear at that time that stripping and pickup reactions make it possible to extract in very simple fashion spectroscopic information on definite nuclear levels. Even at that stage, much valuable information was obtained and has made more precise our ideas concerning the shell model, in enabling us to proceed to a quantitative description. The development of the idea of the optical potential has led to the development of the distorted-wave method and has made it possible to refine the then existing theory of these reactions. From then on, the direct-transfer reaction has become the only process in which the nucleons of the colliding particles become redistributed and for which it is possible to calculate the absolute values of the cross sections with an accuracy almost equal to that of the experimental data. In turn, this has made it possible to verify, with the aid of the direct-transfer reactions, the quantitative predictions of more complicated spectroscopic models. It can be stated that the study of the stripping and pickup reactions has contributed during the last 20 years more to nuclear spectroscopy than the study of all other redistribution reactions taken together.

Unfortunately, the great simplicity of the theory has always entailed a large number of approximations. Therefore the theory of direct-transfer reactions is in a position similar to that of the shell model in the recent past, namely, the number of purely theoretical arguments against it exceeds the number of arguments in its favor. One can hope, however, that the presently developing methods of reaction theory will serve as the basis equally as well as the theory of the Fermi liquid served for the shell model. For the time being, the existing and practically employed methods can be regarded as semiphenomenological. Nuclear spectroscopy has recently progressed into the region of highly excited nuclear states.[2] Practical interest attaches therefore to the question of the extent and manner in which the existing methods of calculating the transfer reactions must be modified to be able to use the information they yield for description of the structure of highly excited states of nuclei. In the present review we attempt to answer this question by using the most modern methods of the theory of reactions and nuclear spectroscopy.

We formulate first a few fundamental premises that can yield numerical information on the experimental data concerning the direct nucleon transfer reactions. For greater clarity we shall henceforth refer to the deuteron stripping reaction A(d, p)B. The description of the pickup reaction is then obtained in a trivial manner by using the detailed balance principal. A formal generalization to the case of more complicated incident particles with transfer of a larger number of nucleons likewise entails no difficulty, although the picture of the deuteron stripping can become much less simple and less realistic in such a formal transition, inasmuch as these cases contain specific difficulties connected with the complicated structure of the incident particles themselves as well as their interactions with the nuclei A and B. On the other hand, the problems that are raised in the theory of deuteron stripping must be solved also in the case of more complicated particles.

Thus, let us consider the amplitude of the process A(d, p)B. The Lippman–Schwinger equation permits us to write it in the form

$$T=\langle\Phi_f\,|\,V_f\,|\,\Psi_i\rangle. \tag{1}$$

The total wave function Ψ_i, the final-state wave function Φ_f, and the final-state interaction are determined here by the equations

$$\left.\begin{aligned}H\Psi_i&=(T_n+T_p+V_{np}+V_{nA}+V_{pA}+H_A)\,\Psi_i=E\Psi_i;\\ H_f\Phi_f&=(T_n+T_p+V_{pA}+H_A+V_{nA})\,\Phi_f\\ &=(T_p+V_{pA}+H_B)\,\Phi_f=E\Phi_f;\\ V_f&=H-H_f=V_{np}.\end{aligned}\right\} \tag{2}$$

The approximations used in practical stripping calculations are divided into two groups, one connected with the mechanism of the reaction and the other with the structure of the initial and final states of the nuclei A and B. The first group of approximations consists of replacing the wave functions Φ_f and Ψ_i, defined by Eqs. (2), with model wave functions: Ψ_i is replaced by the function Φ_{dA}, which describes the scattering of the deuteron by the target nucleus, and Φ_f is replaced by the function Φ_{pB}, which describes the scattering of the protons by the nucleus B. In the simplest variant we have

$$\left.\begin{aligned}\Phi_{dA}&=\Psi^{(+)}(\mathbf{k}_d\mathbf{r}_d)\,\Psi_A(\xi)\,\varphi_d(|\mathbf{r}_n-\mathbf{r}_p|);\\ \Phi_{pB}&=\Psi^{(-)}(\mathbf{k}_p\mathbf{r}_p)\,\Psi_B(\mathbf{r}_n,\,\xi).\end{aligned}\right\} \tag{3}$$

Here $\Psi_A(\xi)$, $\Psi_B(\mathbf{r}_n, \xi)$ and $\varphi_d(|\mathbf{r}_n - \mathbf{r}_p|)$ are the internal wave functions of the target nucleus, of the final nucleus, and of the deuteron, respectively. For the relative-motion wave functions $\Psi^{(+)}(\mathbf{k}_d\mathbf{r}_d)$ and $\Psi^{(-)}(\mathbf{k}_p\mathbf{r}_p)$ one chooses the wave functions of the elastic-scattering problem in the optical potential. The complex optical potential parameters are adjusted to obtain the best description of the experimental data on elastic scattering. The amplitude (1) with this choice of the wave functions is very similar to the amplitude of the process in the first Born approximation. This is why the approximation (2) is called the distorted-wave Born approximation (DWBA) in the foreign literature and the "distorted-wave method" in the Soviet literature. The physical meaning of this approximation is clear: The deuteron arrives at the nucleus without exciting the latter, leaving a neutron in the field of the nucleus A, while the outgoing proton does not act on the produced nucleus B. The Born character of the reaction amplitude indicates that the connection between the deuteron and proton channels is considered only in the first order of perturbation theory. We neglect here the reaction of the proton channel on the deuteron channel. This approach can be qualitatively justified if it is recognized that the direct process occurs in the peripheral region of the nucleus, where the deuteron only "brushes" slightly against the target. In this region, the only source of "pumping over" from the proton channel to the deuteron channel can be the pickup reaction, the probability of which is comparable with the stripping probability. Since the stripping cross section usually does not exceed 10% of the total reaction cross section for the deuteron, perturbation theory works well in this case. Of course, the coupling between the channels can be much stronger in the internal region of the nucleus, where the compound system A + d is produced. However, other competing reaction channels also become coupled in the internal region, and this leads to a suppression of the deuteron and proton wave functions. The presence of such channels is taken into account phenomenologically in the amplitude of the distorted-wave method by introducing absorptive imaginary parts in the deuteron and proton optical potentials. This causes the contribution to the approximate stripping amplitude to come from the peripheral region of the nucleus, as follows indeed from physical considerations. To take into account the contribution of the internal region to the (d, p) process that proceeds via a compound nucleus, one can use the Hauser–Feshbach theory. This process leads to a more or less isotropic angular distribution of the protons, and can compete with the direct mechanism only at large proton-scattering angles. In the region of the characteristic mechanism of the angular distribution that is typical of direct processes, the contribution from the process via the compound nucleus is usually negligible. Of course, if the amplitude of the direct process turns out to be suppressed for some reason, the compound process becomes predominant. But this always manifests itself in the experiments (by the smoothing of the angular distributions and by the fluctuations of the cross sections as functions of the energy of the incident deuterons). Recently, attempts were made to estimate the validity of the distorted-wave method. Among these attempts are the rigorous solutions[3] of the three-body problem (n and p in a fixed potential) with a separable interaction potential V_{np}, and also the less accurate but more practical methods of taking into account the strong coupling of the deuteron and proton channels.[4] The results of such comparisons confirm the qualitative conclusions reached above, namely, that the weak-coupling approximation works well in the peripheral region (for large impact parameters ρ_d and ρ_p). In the language of angular distributions, this means that at small proton-scattering angles, the differential cross section at the maximum, from which the quantitative information on the structure is extracted, is described with good accuracy by the distorted-wave method. Another method of evaluating the distorted-wave method is to compare the theoretical estimates with the experimental ones in those cases when the structures of the nuclei A and B are extremely simple and known beforehand (for example, in the case of stripping by the doubly magic target ^{208}Pb). It follows from this criterion that the distorted-wave method can extract spectroscopic information[5] with an accuracy 20-30%.

Recently, refinements have been introduced into the distorted-wave approximation by taking into account explicitly the coupling between elastic and inelastic scattering channels for the deuterons and protons.[6] In this case the wave functions Φ_{dA} and Φ_{pB} take the form

$$\left.\begin{aligned}\Phi_{dA} &= \varphi_d \sum_i \Psi_i^{(+)}(\mathbf{k}_d\mathbf{r}_d)\,\Psi_{A_i}(\xi);\\ \Phi_{pB} &= \sum_j \Psi_j^{(-)}(\mathbf{k}_p\mathbf{r}_p)\,\Psi_{Bj}(\mathbf{r}_n,\ \xi).\end{aligned}\right\} \tag{4}$$

Here Ψ_{A_i} and Ψ_{B_j} are the wave functions of all the possible states of the target nucleus and of the final nucleus. The use of the approximation (4) for the wave functions in the amplitude (1) is called in the Soviet literature the coupled-channel method, and in the foreign literature the coupled-channel Born approximation (CCBA). The substitution of the functions (4) in the amplitude makes it possible to take into account multistep stripping processes with prior excitation of the nucleus A by the deuteron, followed by transitions of the nucleus B from the state Ψ_{B_j} to the state $\Psi_{B_{j'}}$ under the influence of the field of the outgoing proton. Such processes are important in stripping from deformed nuclei having low-lying collective levels. The theory of these processes will be considered in greater detail later on, and at present we proceed to an analysis of another group of approximations used in the ordinary methods of calculating direct reactions, namely approximations connected with the structures of the nuclei A and B. Returning to the distorted-wave approximation (3), we express the reaction amplitude in the form

$$\begin{aligned}T = \int d\mathbf{r}_n\, d\mathbf{r}_p \Psi^{(-)}(\mathbf{k}_p\mathbf{r}_p)\, V_{np}(|\mathbf{r}_n - \mathbf{r}_p|)\\ \times \varphi_d(|\mathbf{r}_n - \mathbf{r}_p|)\, \Psi^{(+)}(\mathbf{k}_d\mathbf{r}_d)\, f_{AB}^*(\mathbf{r}_n).\end{aligned} \tag{5}$$

It is seen from this form that the entire information on the structures of the nuclei A and B is contained in the amplitude of the process via the so-called stripping form factor $f_{AB}(\mathbf{r}_n)$ (we omit the spin multipliers for simplicity):

$$f_{AB}(\mathbf{r}_n) = \int \Psi_A^*(\xi)\, \Psi_B(\mathbf{r}_n,\ \xi)\, d\xi. \tag{6}$$

It is clear that in the simplest case of a doubly magic target A, the states of B can be adequately described by the single-particle shell model:

$$\Psi_B(\mathbf{r}_n, \xi) = \Psi_{nlj}(\mathbf{r}_n)\,\Psi_A(\xi), \tag{7}$$

and the form factor is simply the single-particle wave function of the transferred nucleon:

$$f_{AB}(\mathbf{r}_n) = \Psi_{nlj}(\mathbf{r}_n). \tag{8}$$

Expression (8) makes it easy to calculate the amplitude and the cross sections of stripping to a single-particle state $(d\sigma/d\Omega)_{s.p}$.

In the case of more complicated nuclei A and B, expressions (7) and (8) no longer hold. Were we to know the structures of the functions Ψ_A and Ψ_B, we could again use expressions (6) and (5) to calculate the reaction cross section $(d\sigma/d\Omega)_{\text{theor}}$. This cross section could in turn be compared with the experimental value $(d\sigma/d\Omega)_{\text{exp}}$, and the structure assumptions could thus be verified. Until recently, however, the reasoning used in the analysis of the stripping reactions was different, namely, expressions of the type (8) were used to calculate the model cross section $(d\sigma/d\Omega)_{s.p}$. This was solved by determining the ratio

$$S_{AB} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{exp}} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{\text{s. p}}. \tag{9}$$

This expression, called the spectroscopic factor, served as a numerical criterion for the deviation of the structure of the nucleus B from the single-particle structure. The quantities S_{AB} were identified with the squares of the coefficients of the expansion of the wave function Ψ_B in terms of the product $\varphi_{nlj}(\mathbf{r})\Psi_A(\xi)$, calculated in one spectroscopic model or another. Questions connected with the uniqueness of such an interpretation in the case of the shell model, with allowance for configuration mixing, will be considered later on. This will be followed by a study of analogous problems that arise in the analysis of stripping on different states. This analysis is becoming more and more pressing in connection with the already mentioned shift of the interests of nuclear spectroscopy into the region of highly excited states of nuclei. In the analysis of highly excited states it is undoubtedly necessary to take into account also multistep stripping processes. A consideration of such processes also calls for a correct treatment of the form factors. This problem will also be considered in the present review. We discuss here problems connected with the use of wave functions of a semimicroscopic approach that differs from the shell model.

1. EQUATIONS FOR THE STRIPPING FORM FACTOR (WITH CONFIGURATION MIXING IN THE SHELL MODEL AS AN EXAMPLE)

Let us examine expression (6) for the form factor. It is clear that if the wave functions of the target nucleus and of the residual nucleus were known with good accuracy, the determination of the form factor would entail no difficulty. It might seem that the results of arbitrary spectroscopic models for Ψ_A and Ψ_B can be used in expression (6). We shall demonstrate, however, using the shell model as an example, that this is not always the case. In the shell model, the Hamiltonian of the nucleus B can be expressed in the form

$$H_B(\mathbf{r}_n, \xi') = H_A(\xi') + T_n + V_0(r_n) + V_{\text{res}}(\mathbf{r}_n, \xi') = H^0 + V_{\text{res}}(\mathbf{r}_n, \xi'). \tag{10}$$

We call attention to the fact that, in the spirit of the shell model, we are considering only the interaction of the "valence" particles above the magic core. The core particles serve only as the source of the self-consistent field $V_0(r_n)$. The valence nucleons interact with the core particles likewise via such a field, and interact with one another with the aid of residual pair interactions $V_{\text{res}}(\mathbf{r}_n, \xi')$. The contribution of these particles to the self-consistent case is neglected here. Therefore the prime of the coordinate ξ' in expression (10) means that this is the coordinate of the valence nucleon. Using the shell-model basis, we can construct a complete set of eigenfunctions of the operator $H^0 = H_A(\xi) + T + V_0(r_n)$ and expand the wave function of the state B in terms of this set. Then expression (6) allows us to write for the form factor

$$f_{AB}(\mathbf{r}) = \int_n C^B_{Anlj}\Phi_{nlj}(\mathbf{r}). \tag{11}$$

For simplicity, we consider the case of a target A with zero spin (otherwise it would be necessary to sum also over the quantum numbers l and j). The symbol $\int_n$ denotes summation over the discrete single-particle states Φ_{nlj} of the shell model and integration over the states $\varphi_{lj}(\mathbf{k}\cdot\mathbf{r})$ of the continuum. On the other hand, using (10) and (6), we can set up equations for the determination of the form factor:[7,8]

$$(\mathscr{E}_{BA} - T - V_0)\, f_{AB}(\mathbf{r}) = \int d\xi' \Psi_A(\xi')\, V_{\text{res}}(\mathbf{r}, \xi')\, \Psi_B(\mathbf{r}, \xi') = P_{AB}(\mathbf{r}). \tag{12}$$

As already mentioned in the Introduction, the main contribution to the amplitude (5) of the direct process is made by the periphery of the nucleus. Although formally the integration in (5) with respect to r_n and r_p extends to infinity, the integrals converge rapidly owing to the exponential decrease of the form factor. Therefore the calculated stripping cross sections are very sensitive to the asymptotic behavior of the form factors. An analysis of the right-hand side of (12) with allowance for the relation between the binding energies E_A and E_B for the systems A and B shows that in the case of short-range residual interactions the asymptotic form of $f_{AB}(\mathbf{r})$ is determined by the binding energy of the transferred nucleon in the state B, i.e., by the quantity $\mathscr{E}_{BA} = E_B - E_A$. On the other hand, the asymptotic form of the single-particle functions Φ_{nlj} in the expansion (11) is determined by the binding energies $\mathscr{E}_{nlj}$ in the shell-model potential. To take correct account of the asymptotic behavior of the form factor it is therefore necessary to make allowances also for the states of the continuum. This constitutes the fundamental difference between the spectroscopic accuracy criteria and the accuracy criteria of the wave functions used in stripping theory. In the configuration-mixing method, the basis set Φ_{nlj} is cut off from above and from below, i.e., only one term in the sum over n is taken. The contribution from the terms with large n (and from the continuum) is neglected, since it is assumed that the corresponding coefficients in the expansion of (11) will turn out to be small. As will be shown below, this is indeed the case. For large values of r, however, the contribution from the dominant expansion component will turn out sooner or later to be less than the contribution of the higher orbitals, since the smallness of the expansion coefficients will be offset by the weaker damping of the considered component. In most cases one calculates in nuclear spectroscopy the matrix elements in which the principal value is assumed by the internal region of the nucleus, and the correctness of the asymptotic behavior of the wave functions is of so little importance that the basis wave functions in the Woods–Saxon potential are frequently replaced by the harmonic-oscillator functions. In the calculation of direct reactions, the asymptotic behavior of the functions becomes more important. Knowledge of the wave functions in the internal region is in this case also necessary for determination of the correct normalization factor. This raises the problem of obtaining a sufficiently accurate calculation of the wave functions in both the internal and external regions of the nucleus. Let us consider different possibilities of solving this problem.

The most obvious method is to extend the basis in the ordinary spectroscopic calculations, i.e., to include the continuum in the set of functions Φ_{nlj} that make up the basis. This raises many technical difficulties, connected with the fact that standard matrix-diagonalization methods cannot be directly used for states of the continuum. To reduce the problem to the usual form, an approximate breakup of the integrals into sums over finite energy intervals is used.[9] It becomes necessary then to monitor not only the convergence of the solution for different upper cutoff limits of the set, but also the convergence over the energy intervals. On the other hand, the choice of smaller intervals, which increases formally the description accuracy, leads to a simultaneous decrease of the values of the admixture coefficients corresponding to this interval, and work with small coefficients is a complicated practical problem even for modern computers. This method was used in ref. 9 to calculate the form factors of the reaction $^{42}Ca(p, d)^{41}Ca$. To calculate the wave functions of the O^+ states of the ^{42}Ca nucleus, the matrices of the Hamiltonian H_B were analyzed with the continuum taken into account. Integration over the continuum was carried out up to the energy E_{max} = 300 MeV. The energy interval could be increased in this case to 10 MeV. Ibarra and Bayman[9] were successful to a certain degree in that there were no single-particle resonances in the nuclei selected by them. Otherwise it would be necessary to decrease the energy interval to a value on the order of the width of the resonance (which is not known beforehand) or to introduce more complicated criteria for selecting this interval.

A rather natural modification of this method is the use of the basis set of functions of the Sturm–Liouville problem, as proposed in refs. 10 and 11. The wave function of the system of the interacting particles (we consider for simplicity the case of two nucleons above the core with a symmetrical spatial part of the wave function) can be sought for the bound state in the form

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \sum_{ij} b_{ij} G_i(\mathbf{r}_1) G_j(\mathbf{r}_2). \tag{13}$$

The functions G_n are here the eigenfunctions of the Sturm–Liouville problem:

$$\left.\begin{aligned} [T + \alpha_i^{(1)} V_0(r_1)]\, G_i(\mathbf{r}_1) &= E_1 G_i(\mathbf{r}_1); \\ [T + \alpha_j^{(2)} V_0(r_2)]\, G_j(\mathbf{r}_2) &= E_2 G_j(\mathbf{r}_2) \end{aligned}\right\} \tag{14}$$

with the orthonormalization condition

$$\int G_n(\mathbf{r}) V_0(r) G_{n'}(\mathbf{r})\, d\mathbf{r} = -\delta_{nn'}. \tag{14a}$$

In this case, the problem of diagonalizing the total Hamiltonian reduces to a solution of a system of equations for the coefficient b_{ij}, in the form

$$\begin{aligned} (E_1 + E_2 - E) b_{nm} &+ \sum_i (\alpha_i^{(1)} - 1) b_{im} \langle n | V_0^2 | i \rangle \\ &+ \sum_j (\alpha_j^{(2)} - 1) b_{nj} \langle m | V_0^2 | j \rangle \\ &+ \sum_{ij} b_{ij} \langle nm | V_0(1) V_0(2) V_{\text{res}}(1, 2) | ij \rangle = 0. \end{aligned} \tag{15}$$

By solving this system we obtain the spectrum of the eigenvalues $E^{(k)}$ of our two-nucleon problem and the sets of coefficients $b_{nm}^{(k)}$ that determine the corresponding $\Psi_k(\mathbf{r}_1, \mathbf{r}_2)$. Since the system of discrete functions G_n is complete for the bound states, the question of including the continuum, with all the ensuing difficulties, does not arise in this problem. Another advantage of the expansion (13) is the freedom of choosing the parameters E_1 and E_2 in (14), so that the rate of convergence of the series (13) can be regulated. If we choose $E_1 = E_A$ and $E_2 = \mathscr{E}_{BA}$, then the description of the state Ψ_{k_0} of interest to us, with energy $E_{k_0} = E_B$, requires the minimal number of coefficients $B_{nm}^{(k_0)}$. The single-particle asymptotic expressions will then be automatically correct. To obtain an equally accurate description of the other levels of our two-nucleon system it is necessary to use a larger set of coefficients. A minor shortcoming in this approach is the use of the orthogonality condition with a weight [see (14a)]. Owing to this condition, it is necessary to evaluate integrals of the type $\int \Psi_A^+(\mathbf{r}_1) G_n(\mathbf{r}_1) d\mathbf{r}_1$ when calculating the form factor f_{AB} from Eqs. (6) and (13).

As already indicated, another method of finding the form factor is to solve Eq. (12). Since the right-hand side of the equation contains the unknown function Ψ_B, the problem does not differ in principle in any way from the problem of diagonalizing the Hamiltonian matrix. We note that if we introduce expansions in the complete set of the states of the system A

$$\Psi_B(\mathbf{r}, \xi) = \sum_i [\Psi_{A_i}(\xi) f_{AB}^i(\mathbf{r})]^{I_B}, \tag{16}$$

then the problem (12) reduces to a system of equations for coupled channels, a system widely used at present in the theory of nuclear reactions. In the terminology of reaction theory, the form factor is the wave function of the relative motion in the given channel, so that the system of integrodifferential equations for f^1_{AB} can be solved also in the manner used in reaction theory, i.e., without resorting to expansion in the set of basis functions. We shall return to this problem later on, and dwell for the time being on different approximate methods connected with the solution of (12). All are based on subdividing the system of equations by making various asumptions concerning the function $P_{AB}(\mathbf{r})$.

The simplest is the well-depth prescription (WDP) procedure, based on the substitution

$$V_0(r) f_{AB}(\mathbf{r}) + P_{AB}(\mathbf{r}) \approx \alpha V_0(r) f_{AB}(\mathbf{r}). \tag{17}$$

The parameter α is fitted in such a way that the equation

$$(T + \alpha V_0 - \mathscr{E}_{BA}) f^0_{nlj}(\mathbf{r}) = 0 \tag{18}$$

has a solution at a fixed value of $\mathscr{E}_{BA}$. The quantum number n is chosen to be the number of the dominating component in the expansion of f_{AB} in the shell-model basis (11). The form factor f^0_{nlj} obtained in this manner is normalized to unity and is used to calculate the model stripping cross section $d\sigma/d\Omega_{s.p}$, which enters in expression (9) for the spectroscopic factor. This approximation is universally used to this day when spectroscopic information is extracted from stripping reactions. Its main advantage is that, as seen from (18), the asymptotic behavior of f^0_{nlj} is accounted for correctly. This leaves two questions unclear: 1) To what extent does the approximation (17) distort the form factor in the peripheral and external regions? 2) What is the relation between the spectroscopic factors calculated with the aid of f^0_{nlj} and the expansion coefficients C_{nlj} used in the ordinary spectroscopic calculations? We shall attempt to answer these questions at the end of the present section, but for the time being we turn to other approximate methods of solving the integrodifferential equation (12).

A very successful approximation that greatly simplifies the problem of finding f_{AB} was proposed in ref. 8. It consisted of replacing the true wave function Ψ_B in the right-hand side of (12) by the ordinary function obtained in spectroscopic calculations with a limited basis. This approximation makes it possible to change over directly from the system of coupled equations to a single equation for the form factor. This approach can be justified by the following physical consderations. It is known that the nucleon density in the nucleus has a very sharp boundary (the nuclear surface). Naturally, the outer nucleon can sense the residual interactions only at a distance on the order of the radius of these interactions (1-2 F) from the nuclear boundary. In the more remote region, the influence of the residual interactions, i.e., of the term P_{AB} in (12), should be vanishingly small. These considerations are corroborated by the results of the already mentioned exact calculations[9] of the behavior of the form factors. The solid line of Fig. 1 denotes the form factor for the state $1f_{5/2}$ in the nucleus ^{41}Ca, calculated by the diagonalization method with allowance for the continuum. The radius of the nucleus is $R_{nuc} = 4.3$ F. We see that the form factor begins its asymptotic behavior already in the region $r \approx 5$-6 F. The results of the calculation of the form factors $1f_{7/2}$, $2p_{3/2}$, and $2p_{1/2}$ are similar (Figs. 2-4). Let us return now to Eq. (12). There is every reason for believing that in the internal region $r \leq R_{nuc}$ the wave functions are adequately described by the wave functions Ψ^0_B obtained in the configuration-mixing method of the shell model as a result of the residual interactions V_{res}. We can therefore express (12) in the internal region, with good accuracy, in the form

$$\left.\begin{aligned} (\mathscr{E}_{BA} - T - V_0) f^0_{AB}(\mathbf{r}) &= P^0_{AB}(\mathbf{r}), \quad r \leqslant R; \\ P^0_{AB}(\mathbf{r}) &= \int d\xi' \Psi^*_A(\xi') V_{res} \Psi^0_B(\mathbf{r}, \xi'). \end{aligned}\right\} \tag{12a}$$

On the other hand, in the region outside the nucleus, good results will be obtained with any approximate method that leads to the cutoff of $P_{AB}(\mathbf{r})$ at a distance on the order of $R_{nuc} + \sigma$ (σ is the radius of the residual pairing interactions). We can propose several approximations for the function Ψ^0_B: matching the wave functions of the shell model at the point R_{nuc} to the functions that have a correct asymptotic behavior, cutoff of $P^0_{AB}(\mathbf{r})$ at the point R_{nuc}, and finally replacement of the true Ψ_B by the shell-model function Ψ^0_B. The last method should be particularly effective if the shell-model expression consists of one dominant component and several small ones. Then, as will be shown below, the true asymptotic form of the dominant component will be very close to the shell-model asymptotic form, and the small admixtures are still unable to distort significantly the behavior of Ψ_B in the re-

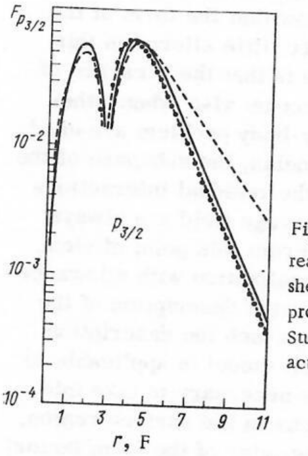

Fig. 1. The form factors $F_{p_{3/2}}$ for the reaction ^{42}Ca(p, d)^{41}Ca; dashed line – shell model; dash–dot line – WDP approximation; solid line – using the Sturm–Liouville method and the exact calculation.[9]

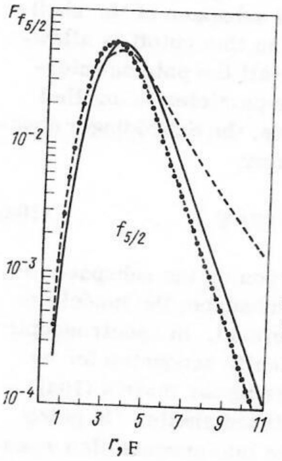

Fig. 2. The form factors $F_{f_{5/2}}$ for the reaction ^{42}Ca(p, d)^{41}Ca (the notation is the same as in Fig. 1).

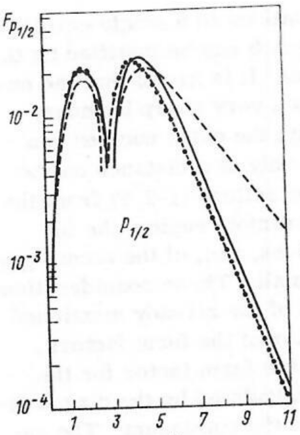

Fig. 3. The form factors $F_{p_{1/2}}$ for the reaction ^{42}Ca(p, d)^{41}Ca (the notation is the same as in Fig. 1).

gion $R_{nuc} \le r \le R_{nuc} + \sigma$. It is precisely for this case that exact calculations of the form factors are available[9] and admit of comparison with the approximate methods. It should be noted that in the case when there is no dominant component in Ψ_B^0, the behavior of P_{AB}^0 even in the surface region will be determined by the admixtures with the minimal binding energy, and the form factor f_{AB}^0 can differ strongly from the exact form factor. On the other hand, if we are interested in the behavior of f_{AB}^0 at large distances, then it is easy to verify that its asymptotic form can be determined not by the experimental binding energy $\mathscr{E}_{BA}$, but by a combination of the asymptotic forms of the model functions contained in P_{AB}^0. Indeed, the asymptotic behavior of the solution of the differential equation (12a) is determined by $\mathscr{E}_{BA}$ only if the right-hand side of the equation decreases more rapidly than $\exp\{-k_{BA} r\}$, where $k_{BA} = \sqrt{2M/\hbar^2 | \mathscr{E}_{BA}|}$. Caution is therefore always advisable when the model function Ψ_B^0 is used.

We now pause to discuss one detail, which is usually lost sight of in the approximate calculations, and which is connected with the form in which expressions (12) and (12a) are written out. As already indicated, the consistent form of the Hamiltonian (10) in the shell model presupposes the presence of an inert core, which serves as the source of the external field $V_0(r)$ for the valence nucleons. It is therefore clear immediately that in the shell model one can claim to describe the excited states that do not affect the core. In many cases, this approach is quite realistic. The residual interactions V_{res} are determined in this approach only in the cut-off subspace of the shell-model states. The most essential in this cutoff is allowance for the Pauli principle, since all the pairing interactions take place only for valence particles in unfilled shells. Strictly speaking, therefore, the Schrödinger equation in the shell model takes the form

$$(H^0 + PV_{\text{res}}P)\,P\Psi = EP\Psi, \tag{10a}$$

where P is the operator of projection on the subspace of the unfilled shells. Outside this subspace, the model residual interaction is simply not defined. In spectroscopic calculations, this fact is automatically accounted for by using as the basis, when diagonalizing the matrix (10a) only, the wave functions of the unfilled shells. On going to Eq. (12), this fact must be taken into account also when

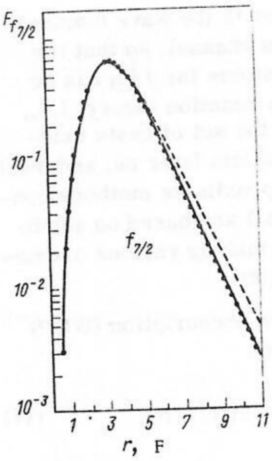

Fig. 4. The form factors $F_{f_{7/2}}$ for the reaction ^{42}Ca(p, d)^{41}Ca (the notation is the same as in Fig. 1).

$$\begin{aligned}(\mathscr{E}_{BA} - T - V_0) f_{AB} &= \langle \Psi_A P \,|\, PV_{\text{res}}P \,|\, P\Psi_B \rangle \\ = \langle \Psi_A P \,|\, (1 - Q) V_{\text{res}} P \,|\, P\Psi_B \rangle &= \langle \overline{\Psi}_A (\xi) \,|\, V_{\text{res}} (\mathbf{r}, \xi) \,|\, \overline{\Psi}_B (\mathbf{r}, \xi) \rangle \\ &- \sum_{n_0} |\varphi_{n_0}(\mathbf{r})\rangle \langle \varphi_{n_0}(\mathbf{r}) \overline{\Psi}_A (\xi) \,|\, V_{\text{res}} \,|\, \overline{\Psi}_B (\mathbf{r}, \xi) \rangle.\end{aligned} \tag{12b}$$

The operator Q is defined here as Q = 1 − P, and the property $P^2 = P$ is used. In addition, we have introduced the definition $\bar{\Psi} = P\Psi$, and the summation is carried out over the states of the filled core of the shell model. It is easy to verify that in this notation the expansion (11) for the form factor consists automatically of only the wave functions of the unfilled states, since it follows from (12b) that

$$(\mathscr{E}_{BA} - \mathscr{E}_{n_0}) \langle \varphi_{n_0} \,|\, f_{AB} \rangle = 0 \tag{19}$$

and f_{AB} is orthogonal to the functions of all the filled states of the core. The orthogonality requirement (19) was indeed the reason why the authors of ref. 9 introduced a projection term in Eq. (12). No such projection was carried out in any of the papers devoted to the calculation of the form factors with the aid of the approximate solution of Eq. (12). This, as we shall show, can appreciably alter the form of the form factor in the case of filling of a shell with n ≠ 1.

Notice should be taken here of two other circumstances. First, strictly speaking, V_{res} is determined for the shell model without allowance for the continuum, and this fact must be taken into account when the shell-model basis is expanded. The results of diagonalization with allowance for the continuum,[9] however, show that the form of the residual interactions is only very little altered in this case. The second circumstance is that the presence of projection must be taken into account also when other model Hamiltonians of the many-body problem are used. Of course, in any model Hamiltonian, the subspace of the basis functions is bounded and the residual interactions which are not included in the average field are always defined only in this subspace. From this point of view, even the exact problem of diagonalization with allowance for the continuum can give a correct description of the form factor only to the extent to which the description within the framework of the shell model is applicable to our state. If, for example, it is necessary to take into account the clustering of nucleons in the surface region, which can greatly distort the behavior of the form factor, then we are faced with two possibilities. The clustering of nucleons can be described adequately on the basis of the shell and residual-interaction model. In this case, the effect of nucleon clustering is already accounted for in the right-hand side of (12b). It many turn out that this basis is insufficient for description of the clustering phenomena. We then expand the subspace, adding to the shell-model basis also the space of closed cluster channels and suitably renormalizing the residual interactions. There are grounds for hoping that effects of this type can exert an influence on the form factor only when the investigated states of the nuclei A and B have large reduced widths in the cluster channels. An experimental indication of this fact may be the selectivity of the excitation of such states in reactions of direct cluster transfer. Unfortunately, effects of this type are strongest in light nuclei, where the applicability of the shell model is subject to considerable doubt and there are practically no other models of the nuclear structure.

We turn again to the approximate equation (12b). There are different methods of solving it. Most of them are based on the expansion of the form factor in the complete set of basis functions. In ref. 8 the employed set were the harmonic-oscillator functions $\Phi_n(\mathbf{r})$:

$$f_{AB}=\sum_n C_n\Phi_n(\mathbf{r});\ (T+V_{\text{osc}}-E_n)\,\Phi_n(\mathbf{r})=0.$$

This leads to the necessity of solving a system of equations of the type

$$\sum_n C_n\{\mathscr{E}_{BA}-E_n\delta_{nn'}-\langle n'\,|\,V_0-V_{\text{osc}}\,|\,n\rangle\}$$
$$=\int\Phi^*_{n'}(\mathbf{r})\,P_{AB}(\mathbf{r})\,d\mathbf{r}\equiv I_{n'}. \tag{20}$$

To facilitate the calculation of the matrix elements $I_{n'}$, the authors of ref. 8 used the functions Ψ_A and Ψ_B obtained with the oscillator basis in the shell model. As indicated above, all the approximate methods in which the functions Ψ_A and Ψ_B are chosen on the basis of a model from structure calculations lead to a correct asymptotic behavior of the form factors, provided that $P_{AB}(r)$, which is constructed from such functions, decreases sufficiently rapidly with increasing r. Since the oscillator functions satisfy this requirement, this choice of P_{AB} leads to good results. The form factors of the (p, d) pickup reaction from the ground state of ^{58}Ni, calculated with the aid of Eqs. (20), and the exact form factor obtained in ref. 9, are shown in Fig. 5. It is possible that the good agreement between the approximate and exact results was helped by the large value of the binding energy ($\mathscr{E}_{BA}$ = 12.98 MeV) and by the presence of a dominant component $2p_{3/2}$ (with amplitude 0.922) in the wave function of the investigated state of ^{58}Ni. A major shortcoming in such an approach is the use of the oscillator basis not only in the right-hand side of the equation, but also for the expansion of the form factor itself. Since the asymptotic behavior of f_{AB} differs significantly from the asymptotic behavior of the oscillator functions, it is necessary to mix 7-10 oscillator functions to account for the correct asymptotic behavior of f_{AB} even in the limited region up to $2R_2$. The number of terms would increase very greatly if the experimental binding energies were lower, or in the case of subbarrier stripping, where the form factor must be known up to $(4\text{-}6)R_{\text{nuc}}$.

The alternate approach consists of choosing as the basis the functions of the Sturm–Liouville problem:

$$f_{AB}(\mathbf{r})=\sum_n a_n g_n(\mathbf{r}); \tag{21}$$

$$(T+\alpha_n V_0(r)-\mathscr{E}_{BA})\,g_n(\mathbf{r})=0. \tag{22}$$

Substitution of this basis set in equations such as (12) leads to a diagonal matrix for the expansion coefficients

$$a_n=I^0_n/(1-\alpha_n), \tag{23}$$

where

$$I^0_n=\int g_n(\mathbf{r})\,P_{AB}(\mathbf{r})\,d\mathbf{r}. \tag{24}$$

The choice of this basis guarantees a correct asymptotic behavior of f_{AB}, just as in the already mentioned well-depth fitting (WDP) procedure. The WDP procedure itself consists of retaining in the expansion (21) only one dominant term with $\alpha_{n_0}\approx 1$ (this is in fact the parameter α in Eq. (18)]. In the case of the basis (21), it is quite easy to consider also the convergence question. Quasi-classical estimates show that in the case of large n we have $\alpha_n\approx n^2$. Thus, the behavior of the denominator in (23) guarantees that the main contribution to the expansion is made by the first few terms of the series, and particularly by the dominant term n_0. The behavior of the matrix element (24) can likewise be easily understood. For the first few terms of the expansion, contribution to the integral of (24) is made mainly by the region inside the nucleus. The behavior of the form factor determined by the predominant term n_0 is thereby corrected in the interior of the nucleus. The form factor f^0_{nlj} of the WDP procedure, which has the correct asymptotic behavior, acquires in the internal region of the nucleus the more correct form determined by the usual calculations within the framework of the shell basis. This intense restructuring of the form factor terminates when the contribution made to the matrix elements I^0_n by the internal region becomes negligibly small owing to the increasing number of oscillations of the functions $g_n(\mathbf{r})$. This occurs usually already at n ≈3-4. For large values of n, the value of I^0_n is determined only by the integral of the external region, where the functions $g_n(\mathbf{r})$ and $P_{AB}(\mathbf{r})$ are exponentially small. It is now easy to understand the character of the limitations imposed on the accuracy of the approximate methods based on the use of the model functions Ψ^0_B. The point is that at large n the integrals I^0_n become practically constant and the entire convergence of the nucleus is determined only by the denominator of (23). However, if P_{AB} attenuates in the external region very rapidly, then

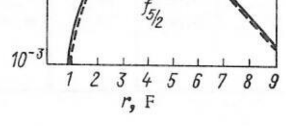

Fig. 5. Form factor $F_{f_{5/2}}$ for the reaction ^{58}Ni(p, d)^{57}Ni: solid curve – exact form factor obtained in ref. 9; dashed curve – form factor calculated by expanding $F_{f_{5/2}}$ in the oscillator wave functions.[8]

the common denominator of all the discarded terms of the series is very small, and this leads to a very small contribution from the unaccounted-for parts of the series. On the other hand, if P_{AB} attenuates slowly and the contribution made to I_n^0 by the external region is appreciable, the very idea of substituting the model function in P_{AB} becomes certainly incorrect. None of the methods of solving the equation with an approximately known right-hand side gives a good result in this case.

There are also other methods of solving the inhomogeneous equation (12). One of them[12] is based on the fact that it is possible to transfer the term with the potential $V_0(r)$ from the left to the right-hand side of the equation. The solution of the equation with the right-hand side can then be expressed in terms of the Green's function $G(\mathbf{r}, \mathbf{r}') = 1/(T - \mathscr{E}_{BA})$ of the homogeneous equation, i.e., in terms of a product of functions that are regular at zero and at infinity. But then numerical integration of products of exponentially increasing functions by exponentially decreasing functions becomes inevitable, and this makes the proposed method technically very complicated. In another direct method[13] of numerically integrating Eq. (12), difficulties of the same type make it necessary to invert a 150 × 150 matrix and to perform numerous iterations.

Summarizing, we can state that if $P_{AB}(\mathbf{r})$ can be replaced by an approximate function, the most convenient method of finding the form factor is the use of the basis of the Sturm–Liouville problem (21). Naturally, in the case of a pure shell state we have for the transferred nucleon $V_{res} = 0$ and $\alpha_{n_0} = 1$, and the form factor obtained in this manner coincides with the WDP form factor and with the single-particle wave function of the shell model. If $V_{res} \neq 0$, a small deviation from the wave function of the shell model will appear in the form factor connected with the weakest component of the expansion of Ψ_B in terms of the states of Ψ_A. The coefficients of the form-factor expansion for the reaction $^{42}Ca(p, d)^{41}Ca$ in terms of Sturm–Liouville functions are given in Table 1. We see that in the worst cases (the form factors of $2p_{1/2}$ and $1f_{5/2}$) the convergence is so strong that the series can be terminated with the third or fourth term. The form factors obtained thereby coincide fully with the form factors from the exact calculations.[9] The differential cross sections of the (p, d) reaction, obtained with the aid of form factors containing an ever-increasing number of terms n in the expansion (21), are shown in Fig. 6. We see that the absolute value of the cross section also saturates quite rapidly, and the angular distributions vary quite little, as noted in ref. 9. The form factors of $2p_{1/2}$ and $2p_{3/2}$, calculated with and without allowance for the projection term in the right-hand side of (12b), are shown in Fig. 7, where Fig. 8 shows the corresponding angular distributions of the (p, d) reaction. We see that inclusion of the projection term affects strongly the behavior of the form factor and the absolute value of the reaction cross section.

We proceed now to the question of the value of the spectroscopic factor determined from the single-nucleon transfer reactions. It is usually determined in the following manner: The form factor obtained in the WDP procedure [see (17) and (18)] is normalized to unity. This form factor can be used to calculate the single-particle reaction cross section $(d\sigma/d\Omega)_{s.p.}$. This expression enables us to find the spectroscopic factor S_{AB} from Eq. (9). It is usually assumed that the spectroscopic factor obtained in this manner is connected with the coefficient C_{nlj} obtained from the usual spectroscopic calculations:

$$S_{AB} = |C_{nlj}|^2. \tag{25}$$

We see now, however, that Eq. (25) is meaningless. The point is that when we use the exact expression for the form factor f_{AB}, obtained by solving (12), the question of the spectroscopic factor does not arise at all, since Eq. (12) defines also a non-unity normalization of f_{AB} (all that are normalized to unity are the wave functions Ψ_A and Ψ_B in the right-hand side of the equation). In analogy with the WDP form factor, we can obtain the normalization multiplier for the exact form factor

$$\int_0^\infty |f_{AB}|^2 \, d\mathbf{r} = N_{AB} \tag{26}$$

and introduce the concept of the spectroscopic factor $\overline{S}_{AB}$ with the aid of the definition

$$\overline{S}_{AB} = N_{AB}, \tag{27}$$

which enables us to connect the exact value of the cross section $(d\sigma/d\Omega)_{exp}$ with the value $d\bar{\sigma}/d\Omega$ calculated with the aid of the form factor $f_{AB} = f_{AB}/\sqrt{N_{AB}}$, which now is already normalized to unity:

$$\overline{S}_{AB} = \left(\frac{d\sigma}{d\Omega}\right)_{exp} \Big/ \frac{d\bar{\sigma}}{d\Omega}\,.$$

TABLE 1. Coefficients a_{nlj} of the Expansion of the Form Factors $f_{AB}(r)$ in Sturm–Liouville Functions for the Reaction $^{42}Ca(p, d)^{41}Ca$

$1f_{7/2}$		$1f_{5/2}$		$2p_{3/2}$		$2p_{1/2}$	
basis states	a_{nlj}	basis states	a_{nlj}	basis states	a_{nlj}	basis states	a_{nlj}
$1f_{7/2}$	0.9922	$1f_{5/2}$	0.9300	$1p_{3/2}$	−0.0851	$1p_{1/2}$	−0.1121
$2f_{7/2}$	0.0222	$2f_{5/2}$	0.1590	$2p_{3/2}$	0.9575	$2p_{1/2}$	0.9416
$3f_{7/2}$	0.0032	$3f_{5/2}$	0.0427	$3p_{3/2}$	0.1129	$3p_{1/2}$	0.1285
$4f_{7/2}$	0.0015	$4f_{5/2}$	0.0166	$4p_{3/2}$	0.0271	$4p_{1/2}$	0.0491
$5f_{7/2}$	0.0007	$5f_{5/2}$	0.0083	$5p_{3/2}$	0.0106	$5p_{1/2}$	0.0210
				$6p_{3/2}$	0.0063	$6p_{1/2}$	0.0117

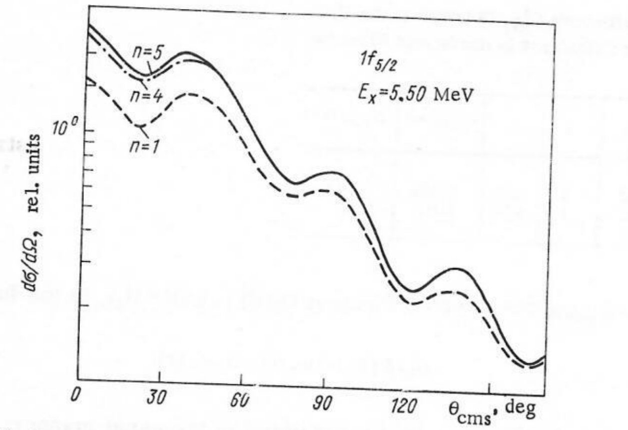

Fig. 6. Cross sections of the reaction ^{42}Ca(p, d)^{41}Ca($f_{5/2}$), obtained with the aid of form factors containing different numbers of terms, n, in the expansion (21).

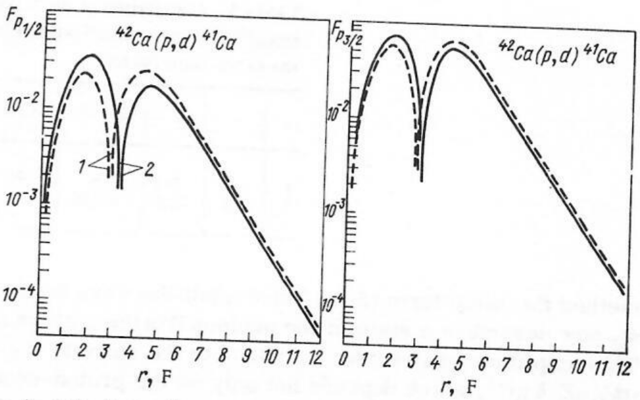

Fig. 7. Form factors $F_{p_{1/2}}$ and $F_{p_{3/2}}$ for the reaction ^{42}Ca(p, d)^{41}Ca, calculated with (1) and without (2) allowance for the projection term in the right-hand side of (12b).

It should be noted that by virtue of the difference between the exact f_{AB} and the value f^0_{AB} obtained in the WDP procedure, the WDP spectroscopic factor S_{AB} obtained in the usual manner may differ from the exact factor, i.e., $S_{AB} \neq \bar{S}_{AB}$. In the investigated case of ^{42}Ca, the differences can reach 50%. This fact is indeed usually emphasized in recent articles devoted to the determination of the exact form factor. It should be added that in principle the exact spectroscopic factors $\bar{S}_{AB} = N_{AB}$ also differ from the squared coefficients C_{nlj}, since the C_{nlj} are connected with an expansion in terms of the shell-model functions, while N_{AB} are connected with expansions in terms of functions of the channel basis [see (16)]. The results[11] for the case of the nucleus ^{42}Ca, listed in Table 2, show that the difference between these coefficients can be very small (~ 10-20%). However, more serious discrepancies can be expected for weakly bound states of complicated structure. It is therefore necessary also to approach with greater caution the sum rules for the spectroscopic factors,[14] since they were all obtained for the basis of the functions φ_{nlj}, and not f_{A_iB}. The normalization conditions for the functions Ψ_{A_i} and Ψ_B yield directly the sum rule $\sum_i N_{A_iB} = 1$, which is the equivalent of the usual relation $\sum_{lj} |C_{nlj}|^2 = 1$. However, the fact that the form factors f_{A_iB} are not orthogonal does not enable us to obtain the equivalent of the other sum rule which, disregarding the spin dependence and the standard fractional-parentage coefficients,[14] is of the form

$$\sum_B |C^B_{nlj}|^2 = 1. \tag{27a}$$

It should be noted that in none of the experimentally investigated cases was this rule satisfied with accuracy better than 30-40%.

It is thus not obvious whether the wave functions of the shell model can be used in approximate methods of solving Eq. (12) for the form factors. These methods work well in the only case for which an exact solution is available, but it is not certain how well they work in other cases. We therefore consider it timely to search for technical and economical means of solving the system (12b) for closed coupled channels. One such method, using as the zeroth iteration the shell-model wave functions, was proposed by Rost.[15] However, the convergence of Rost's iteration procedure has not yet been rigorously proved. The problem of solving the system (12) in the case of a deformed potential was solved by P. É. Nemirovskii and V. A. Chepurnov.[16] In our case, however, we are faced with an additional difficulty connected with the fact that it is necessary not only to find the eigenvalues of the system (12), but also to obtain agreement (possibly by varying the parameters V_{res}) with the experimental values of the binding energy $\mathscr{E}_{BA}$.

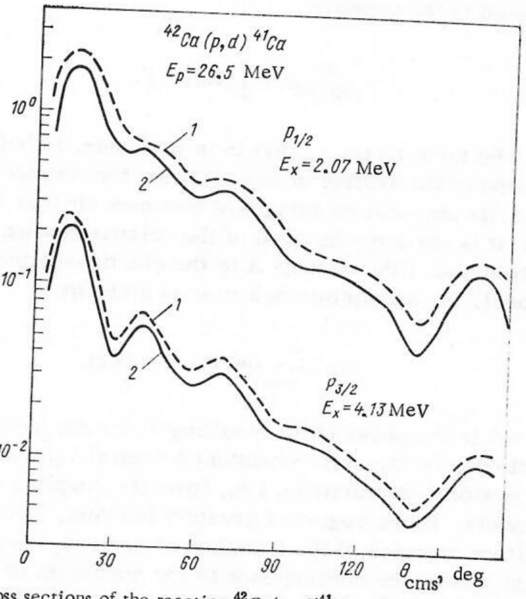

Fig. 8. Cross sections of the reaction ^{42}Ca(p, d)^{41}Ca, obtained with the aid of form factors calculated (1) with and (2) without allowance for the projection term in the right-hand side of (12b).

2. REACTIONS OF NUCLEON TRANSFER TO UNBOUND (RESONANT) STATES

It has become clear of late that to refine our concepts concerning nuclear structure it is necessary to consider the properties of not only the low-lying excited states, but also of the resonant levels in the continuum. In particular, one can investigate transfer reactions of the type A + d → (A + n) + p → A + n + p with excitation of such resonant states. The amplitude of this process has in the distorted-wave

TABLE 2. Comparison of the Mixing Coefficients C^2_{nlj} Obtained in the Shell Model[9] with the Normalized Coefficients Calculated in the Present Paper for the Exact Form Factor

n	l	j	C^2_{nlj}, 10^{-2}	N_{AB}, 10^{-2}	n	l	j	C^2_{nlj}, 10^{-2}	N_{AB}, 10^{-2}
1	3	7/2	97.8	99.7	2	1	3/2	0.165	0.183
1	3	5/2	0.686	0.709	2	1	1/2	1.05	1.22

method the same form (5) as before, but the wave function Ψ_B now describes a state of the nucleus B in the continuum. This amplitude can be used to calculate the quantity $d^3\sigma \cdot (d\Omega_p\, dE\, d\Omega_n)^{-1}$, which depends not only on the proton-scattering angle θ_p but also on the energy E of the transferred neutron and on the direction of its emission from the resonant state Ω_n. Since the neutron-emission direction is not determined in most experiments, we are usually interested in the quantity

$$\frac{d^2\sigma}{d\Omega_p\, dE} \sim \int d\Omega_n \,|\mathcal{T}\,|^2. \tag{28}$$

The form factor $f_{AB}(r)$ is in this case, as before, a solution of the system of Eqs. (12) for the coupled channels. Its physical meaning now becomes clearer [as in (8)]: It is the wave function of the relative motion of the neutron and of the nucleus A in the elastic-scattering channel. Its asymptotic behavior is given by

$$f_{AB}(r) \underset{r\to\infty}{\sim} \sin(kr + \delta - l\pi/2). \tag{29}$$

Here δ is the phase shift resulting from the potential scattering by the self-consistent potential $V_0(r)$ and from the residual interactions, i.e., from the coupling of the channels. In the region of greatest interest, that of small positive energies of the transferred neutron, the position of the resonance corresponds to the maximum of the elastic-scattering cross section, i.e., $\delta_{res} \approx \pi/2$. The form factor has therefore at the resonance maximum the asymptotic form

$$f_{AB}(r) \sim \cos(kr). \tag{29a}$$

Just as in the case of bound states, the form factor at the resonance maximum can be obtained in several ways. The most exact method is to solve the system (12), which describes the elastic scattering of the neutron by the nucleus A, for the case of coupled channels. This yields the form factor. Technically, however, this is a rather complicated problem, and its solution can be obtained at present only in a few simplest cases. By analogy with stripping into bound states, we can therefore attempt to use approximate procedures of finding the form factors, particularly the WDP procedure. Inasmuch as for bound states the form factor f^0_{AB} of the WDP procedure is the dominant term in the expansion of the exact form factor f_{AB} in the functions of the Sturm–Liouville problem, it is desirable to define an analog of the Sturm–Liouville problem for the states of the continuum.[17,18] In this case it is more convenient to introduce the eigenvalues λ_n of the Sturm–Liouville problem, which are connected with the eigenvalues α_n of Eq. (22) by the relation $\lambda_n = 1/\alpha_n$. Equation (22) itself can be rewritten, introducing the Green's function $G_0(E) = 1/(E - H_0)$, in the form

$$G_0(E)\, V_0(r)\, \varphi_n(r) = \lambda_n \varphi_n(r). \tag{30}$$

The functions φ_n are normalized in the usual manner:

$$\langle \varphi_n | V_0 | \varphi_{n'} \rangle = + \delta_{nn'}. \tag{31}$$

The condition at zero takes likewise the form

$$\varphi_n(r) \underset{r\to 0}{\sim} r^{l+1}. \tag{32}$$

The condition at infinity can be chosen in the form (29a):

$$\varphi_n(r) \underset{r\to\infty}{\sim} \cos(kr). \tag{33}$$

Since the definition of the Green's function contains also the boundary conditions, the asymptotic form (33) corresponds to Eq. (30) rewritten in the form

$$P\,\frac{1}{E - H_0}\, V_0(r)\, \varphi_n(r) = \lambda_n \varphi_n(r). \tag{30a}$$

Here P is the symbol of the principal value of the integral whose kernel is the Green's function. With this choice of the boundary conditions, the functions $\varphi_n(r)$ form, in accordance with Mercer's theorem, a complete set. In principle, it is possible also to choose other boundary conditions, for example in the form of waves that diverge at infinity.[19] The eigenvalues λ_n are then complex. If we let the energies E in (30) assume complex values, the condition of diverging waves at infinity can be satisfied also with real values[20] of λ_n. In this case the $\varphi_n(r)$ can be connected with the Gamow functions of the quasistationary states. The solution of (12) for the form factor can now be sought in the form

$$f_{AB}(r) = c\, j_l(kr) + \sum_n a_n \varphi_n(r), \tag{34}$$

where $j_l(kr)$ is a spherical Bessel function describing the incident wave; the coefficient c is connected in simple fashion[17] with the amplitude for the scattering by the potential $V_0(r)$, and the expansion coefficients a_n take the form

$$a_n = \frac{\lambda_a}{1 - \lambda_n}\, [c\, \langle \varphi_n | V_0 | j \rangle + \langle \varphi_n | P_{AB} \rangle]. \tag{35}$$

In complete analogy with the case of coupled states, the case $\lambda_n = 1$ corresponds to single-particle resonance in the solution of (30). The coefficient c tends to zero in this case. If we confine ourselves to the dominant term

φ_{n_0} with $\lambda_{n_0} \approx 1$ in the expansion (34), then the form becomes

$$f^0_{AB}(r) \approx \frac{\langle \varphi_{n_0} | P_{AB} \rangle}{1-\alpha_{n_0}} \varphi_{n_0}(r). \tag{36}$$

We thus arrive again at the WDP procedure in the case of resonant states. It should be noted that for very narrow resonances the wave functions φ_{n_0} in the internal region of the nucleus depend very weakly on the choice of the boundary conditions, and in practice we can choose any of the forms listed above for the basis φ_n. The boundary conditions (33), however, have the advantage that, first, they are the only ones for which the completeness of the set φ_n can be demonstrated and, second, the stripping amplitudes contain no additional divergences that result when the Gamow functions are used.

It is clear that f^0_{AB} will coincide with the exact form factor in the case when $P_{AB}(r)$ in (12) is equal to zero, and one of the values of λ_n is unity. In this case the resonance in the elastic-scattering cross section will be connected only with the conditions of the reflection of the neutron wave in the single-particle potential $V_0(r)$. This is the so-called single-particle potential resonance. The strongest differences between f^0_{AB} and f_{AB} can be then expected when the experimentally observed resonance, resulting from the coupling of the reaction channels, lies somewhere in between the positions of the successive single-particle states (resonances) with n and n+1 nodes of the internal wave function. This is precisely the case shown in Fig. 9.[21]

Solution of the system of equations for the coupled channels in the model problem of $^{15}N+n$ scattering leads to the appearance of a narrow resonance in the $d_{5/2}$ elastic-scattering channel. The wave function at the maximum of this resonance (the exact $d_{5/2}$ form factor) is shown in Fig. 9 by the solid line, while the dashed line shows the WDP procedure form factor for the $1d_{5/2}$ state. Although this single-particle state in the self-consistent potential is 6 MeV lower in energy, we see that the WDP form factor duplicates quite well the behavior of the exact form factor in the surface region. The curves in the figure are normalized to the same asymptotic value [$f^0_{AB} = f_{AB} = \cos(kr)$].

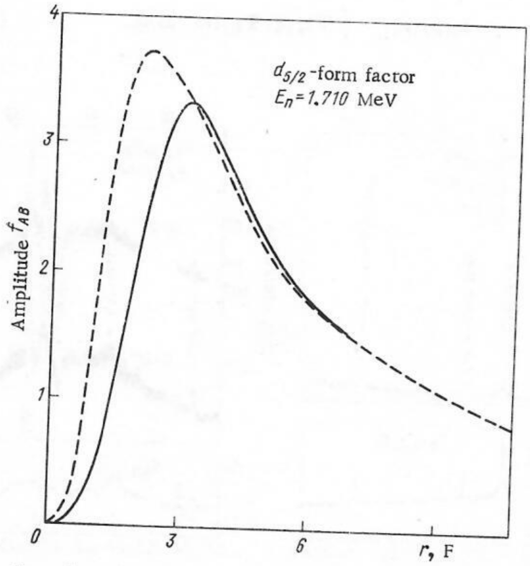

Fig. 9. Exact form factor $F_{d_{5/2}}$ for the $n+{}^{15}N$ scattering and the form factor in the WDP approximation.

If, in analogy with stripping to bound states, we normalize the two form factors to unity in the internal region (r < 7.5 F), then the cross sections calculated with these form factors will have the same angular dependence and will differ in magnitude by 25-30% (as against the 50% difference between the exact cross section and the WDP cross section in the case of the bound $1f_{5/2}$ state in ^{42}Ca). This demonstrates that the convergence of the expansion (34) in the case of the continuum is not worse than the convergence of the expansion in the Sturm–Liouville functions in the case of the discrete spectrum. A comparison of the spectroscopic factors with the coefficients C_{nlj} is impossible in the present case simply because the use of the basis of the channel functions is obvious in the case of the continuum. In the case of nucleon transfer to the resonant state, however, it is possible to trace the analogy with bound states, and also to introduce the concept of the spectroscopic factors as measures of the single-particle behavior of the investigated resonance.

We consider first the case $V_{res} = 0$. The form factor $f_{AB}(E, r)$ is then simply the wave function of the elastic scattering of the neutron by the potential $V_0(r)$. As already indicated, in this case only single-particle potential resonances are possible. It is known[22] that the wave function for the internal region of the potentials can be represented in the vicinities of such resonances in the factorized form

$$f_{AB}(E, r) = C(E)\, u(r). \tag{37}$$

Here u(r) is a solution of the single-particle Schrödinger equation that is regular at zero. It can be normalized, like the wave function of the discrete spectrum, to unity inside the effective radius of the potential. The coefficient C(E) then takes the form

$$C(E) = \sqrt{\frac{1}{2\pi} \cdot \frac{\Gamma_0}{(E-E_0)^2 + \Gamma_0^2/4}}. \tag{38}$$

Substitution of this form factor into the amplitude (5) leads[23] to a resonant form of the stripping cross section $d^2\sigma/d\Omega dE$. Integrating this expression with respect to the energy in the vicinity of the resonance, we obtain

$$\frac{d\sigma}{d\Omega} = \int dE \frac{d^2\sigma}{d\Omega\, dE} = \left(\frac{d\sigma}{d\Omega}\right)_{s.p} \times \int \frac{1}{2\pi} \cdot \frac{\Gamma_0}{(E-E_0)^2 + \Gamma_0/4} dE \approx \left(\frac{d\sigma}{d\Omega}\right)_{s.p.} \tag{39}$$

The cross section $(d\sigma/d\Omega)_{s.p}$ is calculated here with the function u(r) and is a direct analog of the single-particle cross section for stripping to bound states. The role of the spectroscopic factor S is taken by the expression $S = \int dEC(E)^2$. In our case of single-particle resonance, it is equal to unity, in complete analogy with the single-particle bound states.

If $V_{res} \neq 0$, resonances of more complicated nature, usually called compound resonances, can appear in the elastic-scattering channel. The physical cause of these resonances is very simple: The incident nucleon can give up part of its energy to pair collisions with target nucleons, and go over thereby to a bound state in the single-particle potential $V_0(r)$. The resultant compound-nucleus state decays only when one of the nucleons of the system succeeds in again acquiring an energy sufficient to go out to the continuum. The lifetimes of such states are much larger than the lifetime of the single-particle resonance, and their widths Γ_c are much smaller than the widths Γ_0. It is most convenient to describe such states in the Fano–Feshbach theory for a system of coupled channels. Using the results of this theory, we can show[24] that the form factor $f_{AB}(r)$ in the vicinity of a compound resonance has the factorized form (37), as before, and the coefficient $C_{AB}(E)$ is given by

$$C_{AB}(E)=\left[\frac{1}{2\pi}\cdot\frac{\Gamma_c}{(E-E_c)^2+\Gamma_c^2/4}\cdot\frac{\Gamma_c}{\Gamma_0}\right]^{1/2}. \quad (40)$$

Integrating this expression with respect to energy, we obtain the analog of the equation (39) for the compound resonance:

$$\frac{d\sigma}{d\Omega}=\left(\frac{d\sigma}{d\Omega}\right)_{s.p}\cdot\frac{\Gamma_c}{\Gamma_0}. \quad (41)$$

Comparing this expression with the definition of the spectroscopic factor [see (9)], we get

$$S_{AB}=\Gamma_c/\Gamma_0. \quad (42)$$

We recall that Γ_c is the width of the compound resonance and can be measured in elastic-scattering experiments; Γ_0 is the width that would be possessed by the single-particle resonance at the point $E = E_c$ (this width can be determined theoretically). Thus, for stripping to unbound states, the concept of spectroscopic factor can not only be introduced but also compared with values obtained from an independent elastic-scattering experiment. Factorization of (37) within the framework of the Fano–Feshbach theory enables us to obtain a more general relation between the stripping cross section $d^2\sigma/d\Omega_p\, dE_n$ and the cross section $\bar{\sigma}(E_n)$ for elastic scattering of neutrons by the same target:

$$\frac{d^2\sigma}{d\Omega_p\, dE_n}=\frac{2E_n}{\pi\Gamma_0(E_n)}\,\bar{\sigma}(E_n)\left(\frac{d\sigma(0_p)}{d\Omega_p}\right)_{s.p}. \quad (43)$$

The experimental data[25] and the theoretical calculation[21] in the coupled-channel scheme show that this relation is satisfied with good accuracy. The spectra of the protons from the stripping reaction $^{15}N(d, p)^{16}N \rightarrow {}^{15}N+n$ and of the neutrons from the reaction $^5N(n, n)^{15}N$, obtained in ref. 25, are compared in Fig. 10.

3. FORM FACTOR OF STRIPPING TO STATES WITH MORE COMPLICATED STRUCTURE

Collective Excitations

The configuration-mixing methods used by us heretofore describe fairly well the structure of the states only in the case when the number of particles above the filled shell is relatively small (or when the excited state is made up of a small number of quasiparticles). This is due not only to technical difficulties, but also to the fact that the entire shell-model approach is based on the assumption that the residual interactions are small in comparison with the self-consistent potential acting on the transferred particle. In the case of a large number of particles above the filled shell (or more complicated excited states), the situation is significantly altered. The residual interactions can now lead to coherent effects such as vibrations or rotations (including the so-called pairing vibrations). We have already attempted to emphasize that the subdivision of the theory of direct reactions into problems connected with the structure and with the mechanism of the process is quite arbitrary, since both reflect the nature of one and the same nucleus. The indicated coherent effects must therefore be taken into account when the structure of the stripping form factors is considered and when the amplitude of the entire nucleon-transfer process is calculated. In the case of the form factors, this enables us to use the simpler coordinate system of the generalized model and to write down the right-hand side of Eq. (12) for f_{AB} in relatively simple form. In the case of the amplitudes this makes it necessary to use the method of coupled channels [see Eq. (4)] instead of the distorted-wave method. Indeed, if we wish to take into account more or less accurately the excitation of the collective modes in the nucleus by the transferred nucleon, it is not consistent to neglect the excitation of these modes by the incident deuterons or by the outgoing protons. The formal substitution of the wave functions (4) of the coupled-channel method into the amplitude (1) yields an expression for the amplitude of the coupled-channel method:

$$T_{\mathrm{CCM}}=\sum_{ij}T_{ij}=\sum_{ij}\int d\mathbf{r}_n\,d\mathbf{r}_p\,\Psi_j^{(-)*}(\mathbf{k}_p\mathbf{r}_p)\,V_{np}(\mathbf{r}_n-\mathbf{r}_p)$$
$$\times\,\varphi_d(|\mathbf{r}_n-\mathbf{r}_p|)\,\Psi_i^{(+)}(\mathbf{k}_d\mathbf{r}_d)\,f_{A_iB_j}(\mathbf{r}_n), \quad (44)$$

where

$$f_{A_iB_j}(\mathbf{r}_n)=\int\Psi_{A_i}^*(\xi)\,\Psi_{B_j}(\mathbf{r}_n,\,\xi)\,d\xi. \quad (45)$$

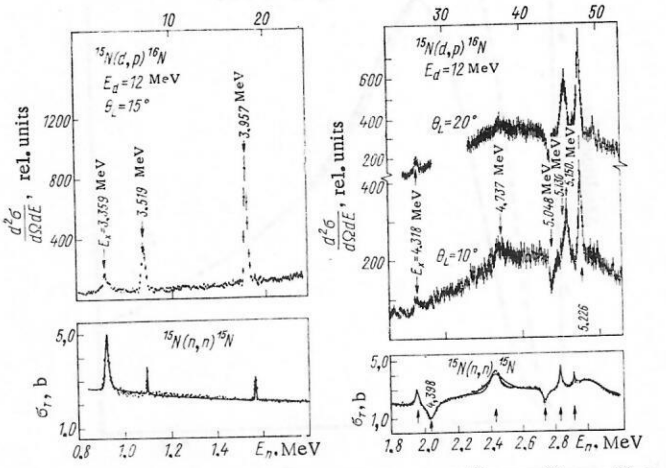

Fig. 10. Experimental cross sections of the reactions $^{15}N(d, p)^{16}N$ and $^{15}N(n, n)^{15}N$, obtained from ref. 25.

As already mentioned, the wave functions $\Psi_j^{(-)}(\mathbf{k}_p\mathbf{r}_p)$ and $\Psi_i^{(+)}(\mathbf{k}_d\mathbf{r}_d)$ are calculated with the aid of the coupled-channel method. We shall not stop here to discuss the details of these calculations, since they are the subject of a rather extensive literature,[6,26-28] and will only touch upon the qualitative aspect of stripping theory in the coupled-channel method. Using as an example the approximate approach of ref. 6 and stripping in a deformed nucleus, we can verify that the transfer of a nucleon from the state A to the state B (Fig. 11) can proceed in different ways: $A \to B$; $A \to A_1 \to B$; $A \to A_1 \to B_1 \to B$; $A \to B_1 \to B$. The amplitude of each process, which leads to the formation of the state B, is a combination of amplitudes of inelastic scattering (such as $A \to A_1$, $B \to B_1$) in the input or output channel, and amplitudes (of the type $F_{A_iB_j}$) of the stripping of a nucleon by an excited target A_i with production of the system B_j. Each of these stripping amplitudes $F_{A_iB_j}(\mathbf{r})$ is determined, in analogy with the usual stripping amplitude in the distorted-wave method, by a form factor $f_{A_iB_j}(\mathbf{r}_n)$ [see expression (45)]. It is clear that the relative contribution of each of these steps to the total amplitude of the process depends on the relative amplitudes of the intermediate excitation and stripping processes in which excited states participate. Experiment has shown[6] that in many cases the amplitude of the excitation or deexcitation of a nucleus in an inelastic-scattering process contains a small parameter. Therefore, if all the transfer amplitudes of Fig. 11 are of the same order, then the dominant process will be the so-called single-step process determined by the amplitude $F_{AB} = T_{AB}^{DWM}$. The contribution of the more complicated steps to the production of the nucleus B is large only if this amplitude is small for some reason. Such a small value, in the language of spectroscopic factors, means that the final state B is very far from the single-particle state. To assess the degree of single-particle behavior in the presence of collective excitations, let us examine in greater detail the form factors $f_{A_iB_j}$ contained in the amplitude (44).

Rotational states. We consider a transfer reaction in an even–even deformed target nucleus A. Such a nucleus can be described with a great degree of accuracy as a rigid rotator with Hamiltonian

$$H_A = \frac{\hbar^2}{2\mathscr{J}_0}\hat{I}_A^2 \tag{46}$$

and eigenfunctions

$$\Psi_A = \left(\frac{2I_A+1}{8\pi^2}\right)^{1/2}\mathscr{D}_{M_A0}^{I_A}(\omega). \tag{47}$$

Here ω is the set of Euler angles and I_A is the angular momentum with a component M_A directed along the z axis in the laboratory frame; for simplicity we consider the usual axially symmetrical case.

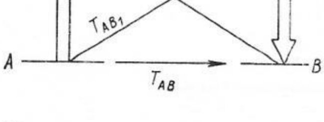

Fig. 11. Coupling of the channels that make the largest contribution to the reaction cross section.

The Hamiltonian of the odd nucleus B takes in the collective model the form

$$H_B = H_A + T + V, \tag{48}$$

where V is the deformed potential of the average field, with the aid of which we hope to take into account the greater part of the residual interactions of the nucleons above the filled core. Introducing the nucleon coordinates $\mathbf{r}'(r, \theta, \varphi)$, where θ and φ are the angle variables in the internal coordinate system connected with the rotating nucleus, we can express this potential in a Woods–Saxon form generalized to include the deformed case:

$$V(\mathbf{r}') = V_0\{1 + \exp[(r - R(\theta))/a]\}^{-1}. \tag{49}$$

The nuclear radius is here $R(\theta) = R[1 + \sum_\lambda' \beta_{\lambda 0}Y_{\lambda 0}(\theta) - n(\beta)]$; $n(\beta)$ is the correction of higher order in the degree of deformation β_λ and guarantees the conservation of the volume. Expanding the potential in the spherical functions $Y_{\lambda 0}(\theta)$ and using (46)-(48), we obtain the analog of Eq. (12) for the form factor f_{AB} in the collective model:

$$\begin{aligned}\left[T + V(r-R) - E_B + \frac{I_A(I_A+1)}{2\mathscr{J}}\right] f_{AB}(\mathbf{r}) \\ = \left\langle \Psi_A \middle| \sum_{\lambda=2} C_\lambda(r)\, Y_{\lambda 0}(\theta) \middle| \Psi_B \right\rangle.\end{aligned} \tag{50}$$

In the collective model we have the wave function

$$\Psi_B = \left(\frac{2I_B+1}{8\pi^2}\right)^{1/2} \sum_K \Psi_K(r, \theta, \varphi)\, \mathscr{D}_{M_BK}^{I_B}(\omega), \tag{51}$$

where K is the projection of the angular momentum I_B on the symmetry axis of the nucleus.

The Hamiltonian H_B can also be expressed by introducing the angular-momentum operator $\hat{j} = \hat{I}_B - \hat{I}_A$ of the transferred nucleon:

$$H_B = T + V + \frac{\hbar^2}{2\mathscr{J}_0}[\hat{I}_B^2 + (\hat{j}^2 + 2j_z^2) - (\hat{j}_+\hat{I}_{-B} + \hat{j}_-\hat{I}_{+B})], \tag{52}$$

where the subscripts "+", "−", and z of the operator $\hat{j}$ pertain to the internal coordinate system. The last term in the square brackets describes the Coriolis interaction.

In the limiting case of very large moments of inertia $\mathscr{J}_0$, both the Coriolis interaction and the entire contribution made by H_A to H_B become negligibly small. This is the case usually called the adiabatic limit. Now K becomes a good quantum number, and the wave function (51) takes the form

$$\begin{aligned}\Psi_B = \left(\frac{2I_B+1}{16\pi^2}\right)^{1/2} \{\Psi_K(r, \theta, \varphi)\, \mathscr{D}_{M_BK}^{I_B}(\omega) \\ - (-1)^{I_B+K}\mathscr{D}_{M_B-K}^{I_B}(\omega)\, \Psi_K(r, \theta, \varphi)\}.\end{aligned} \tag{51a}$$

Since the interaction $V(\mathbf{r}')$ is expressed in the compact form (49) only in the internal coordinate frame, it is much more convenient to work with an equation for the form factors in the same frame. If we expand the function Ψ_K in terms of the spin and angle functions $\mathscr{Y}_{lj}^{k}(\theta, \varphi)$

$$\Psi_K(\mathbf{r}')=\sum_{lj} R_{ljk}(r)\,\mathcal{Y}^k_{lj}(\theta,\varphi), \tag{53}$$

then Eq. (50) assumes in the internal coordinate system the simple form

$$\begin{gathered}(T+V(r-R)-E_B)\,R_{ljk}(r)\\ =\sum_{l'j'}\langle \mathcal{Y}^k_{lj}\,|\,V(r-R)-V(r-R(\theta))\,|\,\mathcal{Y}^k_{l'j'}\rangle\, R_{l'j'k}(r).\end{gathered} \tag{50a}$$

Since the form factor $f_{AB}(\mathbf{r})$ is defined in the laboratory frame, its connection with the functions R_{ljk} is given by

$$\begin{gathered}f_{AB}(\mathbf{r})=\sqrt{\frac{2(2I_A+1)}{(2I_B+1)}}\,(I_A jOK\,|\,I_B K)\\ \times(I_A jM_A m\,|\,I_B M_B)\,\frac{R_{ljk}(r)}{r}\,\mathcal{Y}^k_{lj}(\hat{r}),\end{gathered} \tag{54}$$

where $\hat{r}$ is the set of the angle variables in the laboratory frame.

We see now that Eq. (50a) is the analog of Eq. (12) for the form factors. The main differences between this equation and Eq. (12) are, first, the possibility of using the adiabatic approximation and, second, the smaller number of coordinates of the collective model. This leads to an appreciable simplification of the possible ways of solving equations (50a) in comparison with the methods described above for the solution of the system (12). From the form of (50a) we see clearly that we are again dealing with a system of coupled equations. The most accurate way of solving (50a) is, as before, the method of coupled channels, which was successfully used in ref. 16. Since we are dealing with bound states, the problem can be formulated as an eigenvalue problem. The agreement between the theoretical results and the experimental binding energies E_B is usually obtained by fitting the parameters of the potential. Since the repeated utilization of the coupled-channel method, which is essential in these approaches, greatly increases the computation time, Rost's iteration procedure[15] can likewise be used in this case.

Other methods of finding solutions of the system (50a) consist, as in the case of pair interactions, in expanding the functions R_{ljk} in various sets of basis functions. The best known basis is the set of eigenfunctions of a spherical oscillator and is used in Nilsson's scheme. The advantage of this basis lies in its completeness, but its shortcoming, just as in the case of pairing interactions,[8] is that the asymptotic behavior of its components is known to be incorrect, thereby greatly complicating the calculations of the form factors. To compensate for this shortcoming, Satchler[29] proposed to use the coefficient obtained in Nilsson's scheme in combination with the Woods–Saxon spherical-potential wave functions contained in the WDP procedure. The degree to which this procedure is consistent is not yet clear. More consistent, however, is the approach of refs. 30 and 31, in which the basis of the Woods–Saxon spherical-potential eigenfunctions is used. However, in this approach the question of allowance for the continuum states arises, as always. The need for this allowance follows from the fact that in the Nilsson basis each state R_{ljk} contains a small admixture of components with large values of the principal quantum number n, corresponding to the continuum in the finite Woods–Saxon potential. A more successful choice of basis is, as before, the basis of the functions of the Sturm–Liouville problem.[28,32] The relative effectiveness of different choices of the basis can be evaluated from the data of refs. 28 and 31, which are listed in Table 3. The coefficients N_{ljk} of the coupled-channel method, given in the table, were obtained with the aid of a normalization $\int_\infty^0 R^2_{ljk}(r)dr = N^2_{ljk}$. Just as in the case described above, the coefficients of the dominant components are close to one another in the different bases. Figure 12 shows the cross section of the (d, p) reaction on ^{176}Yb, obtained in the distorted-wave

TABLE 3. Comparison of the Expansion Coefficients of Single-Particle Wave Functions of the Deformed Nucleus ^{25}Mg, Obtained by Different Methods

Bands	Basis states	Coefficients of expansion in Sturm–Liouville functions (β_{20} = 0.45, β_{40} = –0.04) for different values of N			Coefficients of expansion in Woods–Saxon basis functions (β_{20} = 0.45, β_{40} = – 0.04) for different values of N		Nilsson coefficients (β_{20} = 0.3) for different values of N	Coefficients of expansion by the coupled-channel method (β_{20} = 0.35)
		0	2	4	0	2	2	
5/2+ [202]	$d_{5/2}$		1.05	–0.073		1,0	1,0	0,993
	$g_{7/2}$			–0.083				
	$g_{9/2}$			–0.139				
1/2+ [211]	$s_{1/2}$	–0.044	0.447	–0.004	–0.035	0.408	0.37	0.548
	$d_{3/2}$		–0.742	–0.089		–0.765	–0.75	–0.711
	$d_{5/2}$		–0.490	0.023		–0.496	–0.54	–0.425
	$g_{7/2}$			–0.094				
	$g_{7/2}$			–0.086				
1/2+ [200]	$s_{1/2}$	0.148	0.724	–0.039	0.087	0.749	0.76	0.705
	$d_{3/2}$		0.638	–0.049		0.587	0.58	0.665
	$d_{5/2}$		–0.346	0.066		–0.294	–0.29	0.222
	$g_{7/2}$			0.126				
	$g_{9/2}$			–0.085				
		1	3	5	1	3	3	
1/2– [330]	$p_{1/2}$	0.038	–0.173	–0.054	–0.011	not coupled	–0.23	
	$p_{3/2}$	–0.258	0.602	0.097	–0.111	0.632	0.55	
	$f_{5/2}$		–0.075	–0.040		not coupled	–0.20	
	$f_{7/2}$		0.591	0.103		0.767	0.78	
	$h_{9/2}$			–0.015				
	$h_{11/2}$			0.083				

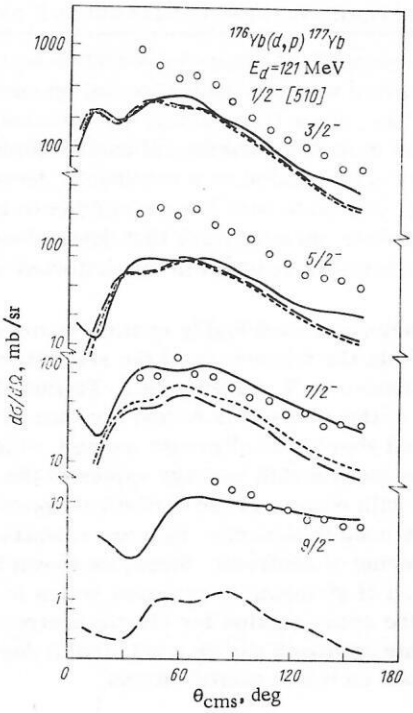

Fig. 12. Cross section for multistep stripping to the rotational states of the 1/2$^-$ [510] band of ^{177}Yb (solid curve). The single-step-transfer cross sections are shown dashed.

method and in the coupled-channel method, and also the experimental data for the ground-state rotational band of ^{177}Yb. The form factors used in the distorted-wave cross section pertained to only one (lj) set. The coefficients of the expansion of the form factor of ^{177}Yb in terms of the two bases are given in Table 4.

In all the approaches described, the role of the residual interactions was reduced only to a deformation of the self-consistent single-particle potential. The adiabaticity approximation in (52) enables us to use the concept of a pure single-particle motion, and greatly simplifies the problem. The next stage towards a more accurate allowance for the residual interactions is the inclusion of the Coriolis forces. The wave function of the nucleus B can now be expressed in the form

$$\Psi_B = \left(\frac{2I_B+1}{16\pi^2}\right)^{1/2} \sum_{K\alpha} C^{I_B}_{K\alpha} \{\mathscr{D}^{I_B}_{M_B K}(\omega)$$

$$\times \Psi^{(\alpha)}_K(\mathbf{r}') + (-1)^{I_B+K+1} \mathscr{D}^{I_B}_{M_B\,-K}(\omega)\, \Psi^{(\alpha)}_{-K}(\mathbf{r}')\}. \quad (51b)$$

The coefficients $C^{I_B}_{K\alpha}$ are the mixing coefficients resulting from the Coriolis interaction, and can be calculated in Kerman's model[33] or in the semimicroscopic approach.[34] Only one attempt has been made to date[35] to take the Coriolis interaction into account in the calculation of the stripping reactions in the coupled-channel method. The mixing coefficients were calculated in the basis of the bound states of the spherical Woods–Saxon potential, and each component of the expansion was then replaced, in analogy with Satchler's already mentioned suggestion,[29] by a WDP function so as to ensure the correct asymptotic behavior of the form factor. Calculations[35] show that while

TABLE 4. Comparison of the Expansion Coefficients of the Single-Step Wave Function 1/2$^-$[510] of ^{177}Yb, Obtained by the Sturm–Liouville Method[32] and by the Woods–Saxon Method[31]

Basis states	Coefficients of expansion in Sturm-Liouville functions (β_{20}=0.28, β_{40}=−0.008) at different values of N				Coefficients of expansion in Woods–Saxon basis functions (β_{20} = 0.28, β_{40} = 0.008) at different values of N		
	1	3	5	7	1	3	5
$p_{1/2}$	−0.021	−0.034	0.022	−0.250	−0.009	−0.022	0.177
$p_{3/2}$	0.016	0.036	0.603	−0.020	0.009	0.044	0.616
$f_{5/2}$		0.21	0.558	0.014		0.040	0.476
$f_{7/2}$		0.066	−0.418	0.074		−0.036	−0.423
$h_{9/2}$			−0.394	0.084			−0.418
$h_{11/2}$			0.137	−0.088			0.098
$i_{13/2}$				−0.099			
$i_{15/2}$				0.072			

the admixtures of Coriolis interactions in each asymptotic state are relatively small, allowance for these admixtures can alter the reaction cross section quite significantly (~30%).

Vibrational states. Two types of vibrational states can be distinguished: the so-called giant resonances with large excitation energy, and low-lying states such as shape oscillations and pairing vibrations. Not a single attempt has been made to date to include giant resonances in stripping theory; we confine ourselves therefore to low-lying vibrational states.

The energies of these states, however, are too high to use the adiabatic approximation as in the case of the rotational spectra. It is appropriate here to dwell again on the similarities and differences in the approaches of the collective model and the method of configuration mixing in the shell model. In the shell model the subspace in which the residual interactions operate is limited from the very outset to the shells above the filled core. Then additional limitations that the Pauli principle imposes on the multiparticle Schrödinger equations can be readily fitted into Eq. (12). In the case of rotational states, which was considered above, we include the greater part of the residual interactions in the self-consistent potential, with the aid of the term [V(r − R) − V(r − R(θ))]. The adiabatic approximation makes it again easy to take into account the influence of the Pauli principle: It is merely necessary to consider the unfilled single-particle states of the nucleon in the field of the deformed core. Any deviation from adiabaticity makes allowance for the Pauli principle in the collective model extremely difficult, and forces us to seek microscopic methods of describing collective excitations. Such methods are calculations in which the Tamm–Dancoff approximation is used, or else microscopic approaches[2,36] in which the random-phase approximation is used. Even in these approaches it is very difficult to take complete account of the Pauli principle, since it forces us to dispense with the quasi-Boltzmann approximation and to take exact account of blocking effects. In the calculation of spectroscopic quantities, these shortcomings are inessential and they can be offset by renormalizing the residual interaction constants. In the case of the form factor, on the other hand, as seen with (12b) as an example, such a renormalization does not suffice. Another shortcoming of semimicroscopic approaches is that they always employ the limited basis of the shell model. However, as demonstrated with the shell model as an example (see Sec. 1), the expansion coefficients agree quite well in this limited basis with the exact spectroscopic factors $\overline{S}_{AB}$. One can therefore expect stripping on an even–even target to yield information on the main components of the transition amplitude by considering only the unoccupied states for the odd nucleon above the core. This again enables us to use the collective model and write down the Hamiltonians of the nuclei A and B in the form

$$\left.\begin{aligned} H_A &= \sum_\lambda \hbar\omega_\lambda \alpha_\lambda^+ \alpha_\lambda; \\ H_B &= H_A + T + V(r) + H_c, \end{aligned}\right\} \tag{55}$$

where α_λ^+ is the operator for the production of a phonon of multiplicity λ and natural frequency ω_λ; the interaction H_c of an external particle with the vibrations of the core is given in lowest-order perturbation theory by

$$H_c = -\frac{\partial V}{\partial r} [\alpha_\lambda Y_\lambda] \sqrt{2\lambda+1}. \tag{56}$$

Using again the expansion in the channel functions

$$\Psi_B = \sum_i [f_{A_iB} \Psi_{I_{Ai}}^{Ai}]^{I_B}$$

and the equation

$$f_{AB}(\mathbf{r}) = \frac{1}{r} R_{lj}^A(r) \mathcal{Y}_{lj}^m(\Omega),$$

we can write down a system of coupled equations for the radial part of the form factor R_{lj}^A in the form

$$\begin{aligned} &[T_r + V(r) - (E_B - E_A)] R_{lj}^A \\ &= -\sum \langle [\mathcal{Y}_{lj} \Psi_{I_A}^A]^{I_B} | H_c | [\mathcal{Y}_{l_ij_i} \Psi_{I_{Ai}}^{Ai}]^{I_B} \rangle R_{l_ij_i}^{Ai}(r). \end{aligned} \tag{57}$$

The matrix element of the interaction operator (56) can be determined from experiments on inelastic scattering of nucleons with excitation of vibrational states of the target. Just as in all the previously described cases, the most accurate method of solving (57) is the coupled-channel method. The first attempt to use this method was made in ref. 37, where the cross sections of the transfer reactions ^{208}Pb(^{3}He, d)^{209}Pb and ^{208}Pb(d, p)^{209}Pb were calculated. The results of the calculations agree well with the experimental data. The problem of obtaining the form factors was greatly simplified in this case, since the experimental spectra of levels of the nucleus ^{209}Pb show that in many cases the interaction matrix elements in the right-hand side of (57) can be calculated by perturbation theory. The alternate approach is to use the basis of the Sturm–Liouville problem to calculate $R_{lj}^A(r)$. This approach was used in ref. 38 in conjunction with the coupled-channel method to determine the cross sections of the reaction ^{52}Cr(d, p)^{53}Cr. To obtain the experimental binding energies $\mathscr{E}_{BA} = E_B - E_A$, the coupling constant α_λ was varied in Eq. (57). The values obtained for this constant (0.21) agree well with the results of calculations of the inelastic-scattering reactions.

Highly Excited States (statistical approach)

The structure of the states Ψ_B becomes more and more complicated with increasing excitation energy of the final nucleus, since the residual interactions lead to a mixing of an increasing number of excited states of the core. This is accompanied by a continuous decrease of the admixture of a state with a nonexcited core to the given excited state, an admixture that determines the amplitude of the transfer reaction in the distorted-wave method.

The structure of such highly excited states can be described within the framework of the semimicroscopic approach proposed by V. G. Solov'ev.[2] To find the stripping to each of the states it is necessary then to use the methods listed above. An alternate method, which makes it possible to operate with average values of the cross sections and with others, is the statistical approach, which is now widely used to describe neutron resonances in elastic scattering of neutrons. Since, as shown in Sec. 2, the reaction of stripping to resonant states is also connected with the cross section for elastic neutron scattering,[21,24] this approach can be used also to describe (d, p) reactions on bound excited states.

Using again the channel expansion (16) for the function Ψ_B, we introduce for convenience form factors $\bar{f}_{AB}$ normalized to unity [see Eq. (26)]. Then

$$\Psi_B = \sum_i \sqrt{N_{A_iB}} [\bar{f}_{A_iB} \Psi_A]_{I_B}. \tag{58}$$

By virtue of the finite resolution δ of the experimental setup, experiment will show only the average cross section $d\tilde{\sigma}/d\Omega$. We introduce the function $\rho(x)$ of the experimental resolution of the setup:

$$\rho(x) = \frac{1}{4\pi} \cdot \frac{\delta}{x^2 + (\delta/2)^2}. \tag{59}$$

The average experimentally observable cross section of the (d, p) reaction now takes the form (apart from spin factors)

$$d\sigma/d\Omega = \sum_B \rho(E_B - E) N_{A_0B} \frac{d\sigma}{d\Omega}(E_B)_{\mathrm{s.p}} \Delta E. \tag{60}$$

To obtain the averaged values of the spectroscopic factors N_{A_0B} we can use the following simple model.[39,40] Assume that in the absence of coupling between channels we have the single-particle state

$$\Psi_{B_0} = [f_{A_0B_0} \Psi_{A_0}]^{I_{B_0}} \tag{61}$$

with energy E_0. Turning on the residual interactions mixes this state with states of more complicated structure, corresponding to the excited core Ψ_{A_i} (for example with states of three-quasiparticle type with energy E_i). Assume that the matrix elements of these interactions V_{0i} satisfy the following conditions:

$$V_{0i} = V_{i0}; \quad V_{ik} = V_{ki} = 0; \quad k, i \neq 0. \tag{62}$$

In this case the average cross section defined by (60) assumes the Lorentz form

$$\frac{d\tilde{\sigma}}{d\Omega} \approx \left(\frac{d\sigma}{d\Omega}(E_0)\right)_{\text{s. p}} \frac{1}{2\pi}\cdot\frac{\Gamma+\delta}{(E_0+\Delta E_0-E)^2+[(\Gamma+\delta)/2]^2}, \qquad (63)$$

where the width of the Lorentz curve is

$$\Gamma=\delta\sum_i \frac{V_{0i}^2}{(E-E_i)^2+(\delta/2)^2}, \qquad (64)$$

and the shift of the maximum ΔE_0, due to the residual interactions, is

$$\Delta E_0=\sum_i \frac{V_{0i}^2\,(E-E_i)}{(E-E_i)^2+(\delta/2)^2}. \qquad (65)$$

As seen from (64), in the statistical description the width Γ is proportional to the mean square of the matrix element V_{0i} and to the number of states with energies E_i in the interval δ.

A statistical approach of this type was used in refs. 41 to describe stripping reactions to highly excited states of the ^{239}P nucleus and a number of rare-earth nuclei. The states Ψ_{B_0} were assumed to be single-particle states of the transferred nucleon in the deformed potential of the target nucleus, and were obtained by the Sturm–Liouville method described above. The stripping cross sections obtained with these form factors by the distorted-wave method were then calculated in accordance with Eq. (63), it being assumed that $\delta \ll \Gamma$. The most interesting result in this description is the function $\Gamma(E)$. The authors of the cited references have proposed that the single-particle states are connected only with doorway states of 3-quasiparticle type. The density of such states increases quadratically with the excitation energy E. If it is assumed that the matrix element V_{0i} is constant, then the function $\Gamma(E)$ should also increase as the square of the energy. It was precisely this type of $\Gamma(E)$ dependence which was used to obtain the theoretical curves (Fig. 13). If, however, we use the factorization idea described above and trace the transition from the bound states to the resonances, then (see, e.g., ref. 40) we can connect the behavior of $\Gamma(E)$ with the behavior of the imaginary part of the optical-model potential, which increases with energy much more slowly (the growth is closer to linear). One of the possible explanations of this linear dependence is that doorway states of the particle-plus-phonon type are considered. Returning to the curves of Fig. 13, we note that inasmuch as only the bound states Ψ_{B_0} were considered in the theoretical calculations, the behavior of the theoretical curves near and above the binding energy E_b of the transferred nucleon is certainly incorrectly accounted for. The falloff of the experimental curves at high excitation energies is due only to the decrease of the penetrability for the proton wave functions.

Allowance for Pairing Correlations

Residual interactions of an extremely important type are pairing correlations of the superconducting type (see, e.g., refs. 42 and 43). These phenomenological interactions make it possible to take into account in a rather simple manner the short-range part of nucleon–nucleon forces in the nucleus, and also the role of the Pauli principle and of the antisymmetrization effects. A significant advantage of this approach is the possibility of describing states of particles far from doubly magic nuclei. The most convenient method for the description of pairing correlations is second quantization. In this method, the form factor for the transfer of a nucleon to a state with single-particle quantum numbers lj is

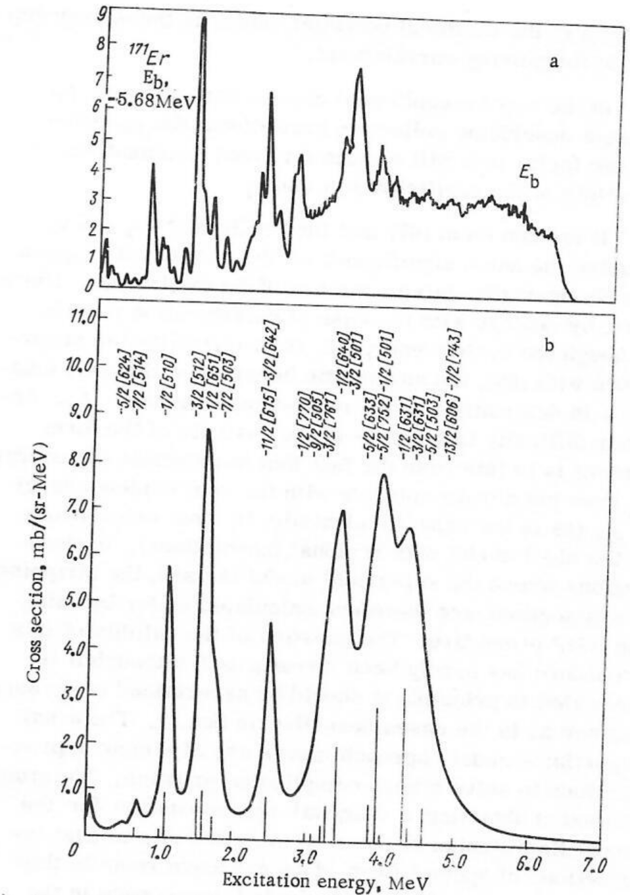

Fig. 13. Dependence of the experimental (a) and theoretical (b) cross sections of the ^{170}Er(d, p) reaction on the excitation energy of the residual nucleus ^{171}Er.

$$f_{AB}=\langle B|a_{lj}^{+}|A\rangle, \qquad (66)$$

where a_{lj}^{+} is the operator for the creation of a particle in the state lj. If we use the general form of the wave functions of nuclei A and B in the superfluid model, then it follows[43] from Eq. (66) that the role of the spectroscopic factors will be assumed by the squares of the coefficients u and v of the Bogolyubov canonical transformation:

$$S_i=\begin{cases} u_i^2 & \text{in the case of pickup;} \\ v_i^2 & \text{i n the case of stripping.} \end{cases} \qquad (67)$$

The energy of the state i of the odd nucleus B is given in this approach by

$$E_i=\sqrt{(\mathcal{E}_i-\lambda)^2+\Delta^2}, \qquad (68)$$

where $\mathcal{E}_i$ is the energy of the shell-model single-particle state, λ is the chemical potential, and Δ is the energy gap due to the pairing correlations.

In the more complicated case of semimicroscopic models describing collective excitations, the spectroscopic factor (67) will contain different combinations of products of the coefficients u_i and v_i.

It is seen from (67) that the coefficients u_i and v_i acquire the same significance as the coefficients C_{nlj} in the configuration-mixing method of the shell model. Therefore, by analogy with the case of configuration mixing, although the system energy E_i is renormalized in accordance with (68), the asymptotic behavior of the wave functions is determined by the shell-model energies $\mathcal{E}_i$. Another difficulty that arises in the analysis of the form factors is in this case the fact that in principle the energy E_i does not always coincide with the real binding energy $\mathcal{E}_{BA}$ (as is the case, incidentally, in most calculations in the shell model with residual interactions). In those regions where the superfluid model is used, the stripping cross sections are therefore calculated so far by using the WDP procedure. The question of the validity of this procedure has hardly been investigated, although it is clear that in principle it should be ascertained in the same manner as in the cases described in Sec. 1. The usual superfluid-model approach makes use of simple approximations to solve a very complicated problem. The simple method of obtaining a diagonal representation for the model Hamiltonian of the system, based on residual interactions of special form, does not leave room in this case for the introduction of a radial dependence in the right-hand side of the equation for the form factor [$P_{AB}(r)$ in Eq. (12)]. Therefore any attempt at examining more consistently the form factors in the pairing-correlation model will undoubtedly make this model much more complicated. From the practical point of view, one can hope that more accurate approaches to the problem of the form factors will not influence greatly the cross sections of the transfer reactions in this model. The arguments favoring this statement are connected, first, with the fact that the contribution from remote states (those farther than 2Δ from the Fermi surface) is very small, as can be seen from the behavior of coefficients u_i and v_i. Second, we already know that most residual interactions are taken into account in the problem by including the deformation in the single-particle potential. From this, and also from the relative smallness of Δ (~ 1 MeV), it follows that even an asymptotic expression determined by the energy $\mathcal{E}_i$ does not lead to a noticeable change in the reaction cross section.

Among the first attempts to describe more consistently the form factors in the pairing-correlation model is that of Sugawara,[44] who used a modification of the Hartree–Fock–Bogolyubov method to take into account the diagonal terms of the residual interactions and the antisymmetrization effects in the equation for the form factors. To obtain the correct values of the binding energies, Sugawara varied two parameters in the residual interactions, making the solution self-consistent (in the Hartree–Fock sense) in each variation. The ensuing calculations were much more complicated than in the ordinary approach, in spite of the fact that the coupled-channel methods were not used to their fullest extent. Sugawara obtained for the cross sections of the (d, p) and (p, d) reactions on several isotopes of Fe, Ni, and Zr values that agreed well (within 30%) with experiment, and also binding-energy values that were accurate within several percent. Unfortunately, no comparison was made of the "exact" spectroscopic factors N_{AB} with the ordinary spectroscopic factors of the WDP procedure.

CONCLUSION

In the entire review we have adhered to a unified description of the equations for the form factors and emphasized the analogies in the approach to these equations for all possible nuclear models. It is therefore clear that the applicability of different approximations in the solution of the equations for the form factors can be assessed within the framework of the simplest configuration-mixing model. The main difficulty in this case lies in obtaining for the system of equations for the coupled channels a solution that makes it possible to take accurate account of the influence of the continuum. As already indicated, in the only case for which an exact solution of the problem exists, the approximate methods make it possible to determine the spectroscopic factors with accuracy not worse than 50%. Recognizing that any attempt at a more consistent analysis of the system of equations for the form factors leads to a tremendous complication of the standard spectroscopic calculations (the best example is that of pairing correlations), a 50% value may be a perfectly satisfactory approximation. We have no guarantee, however, of not greatly exceeding this figure in the general case. It is clear that the largest difference between standard methods should be expected in the case of low binding energies and large configuration mixing.

Another question that arises in the determination of the form factors is that of the residual-interaction renormalization due to the inclusion of the continuum in the analysis. The results of an exact calculation[9] give grounds for hoping that the residual interactions used in spectroscopic calculations can serve, with a cut-off basis, as a good zeroth approximation in any iteration procedure used to find the form factors. This question, however, also calls for additional investigations.

There is also an experimental way of checking the correctness of the spectroscopic factors obtained for transfer reactions. It consists in comparing the cross sections for stripping to a resonant state with the elastic-scattering cross section. In the cases investigated heretofore[21,24,45] the relation (42) turned out to be valid with accuracy up to 30–40%. The experimental data in this region, however, are still very scanty.

It should also be noted that the question of the inclusion of the continuum in the basis of the states arises not only in the problem of the stripping form factors, but also in any spectroscopic problem in which the completeness of the basis sets is used. Investigation of the stripping form factors can therefore undoubtedly affect the results of structure calculations.

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