Static models in the dispersion approach

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Results obtained for static models by the dispersion approach are reviewed. The emphasis is on the methods used to obtain equations for the models and on the exact solutions of these equations. Use of these models in various approximation schemes and in analyzing low-energy πN scattering is discussed.

INTRODUCTION

Since its introduction, the static model[1] has occupied an important position in the theory of elementary particles. It would be difficult to find a textbook or monograph in which it is not mentioned. The exact meaning of the static-model concept itself has changed over the course of time. Histories and bibliographies on the use of the static model in classical and quantum field theory have been published by Henley and Thirring[3] and by Goebel.[73]

The review below is limited primarily to the static limit of the dispersion relations. Figure 1 illustrates the logical structure of the contents. We first consider which aspects of the static-model concept have been explained on the basis of the Hamiltonian approach, the dispersion relations, and the Mandelstam representation. In 1955, Bogolyubov[6] gave a rigorous proof of the dispersion relations for nonvanishing momentum transfer and thereby made it possible to distinguish the importance of the principle of causality (or analyticity) and crossing symmetry in deriving dispersion relations for the static model.[5] As one of the basic symmetry principles in nature,[4] crossing symmetry is an integral part of the static model. It is incorporated only approximately in this model, but the severity of the resulting restrictions can be adjusted. This latter assertion can also be applied to the condition of two-particle unitarity, which is used in the static model. It is the simultaneous incorporation of both these principles which is responsible for the advantages and difficulties of this model. Going beyond the scope of the approximations in the unitarity condition and crossing symmetry, Serebryakov and Shirkov arrived at the concept of short-range repulsion. The equations which arise in this approach generalize the equations of the static model and permit effective use of the results of current algebra and chiral symmetries.[28,95] Despite the relative simplicity of the static model, its equations have not yet been completely analyzed because of their nonlinearity. However, even approximate solutions of these equations have played an important role in interpreting experimental data, for example, on p-wave πN scattering. We will be concerned primarily with exact solutions of the equations of the static model. There are many cases in which explicit equations cannot be found for these solutions, so we are confronted with the problem of proving the existence and uniqueness theorems as basic tools for studying the solutions as well as for constructing them (approximately). Here it is extremely useful to use the methods of functional analysis. The difficulties which arise are not unique to the static models, so it can be hoped that the overcoming of these difficulties in the simplest cases will stimulate progress on more realistic models. Apparently the most suitable analytic approach to the Chew–Low equations is to use them in their dynamical form, which arises in a study of the amplitudes for scattering by their Riemann surfaces, which are of fundamental importance in the entire approach.

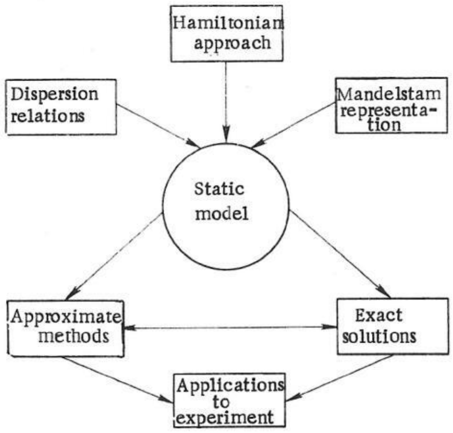

Fig. 1

In terms of a uniformizing variable, the method of continuing amplitudes from one sheet to another can be thought of as an equation. Then systems of nonlinear difference equations – the dynamical form of the Chew–Low equations – arise in the static models; analysis of these equations can yield certain interesting new results. No concrete interaction mechanism is specified in the Chew–Low equations. In the language of the N/D method this assertion is equivalent to that of arbitrariness in the choice of the function N: the generalized potential. However, exact inclusion of the crossing-symmetry condition imposes restrictions on N and thereby singles out a definite class of permissible forms of the generalized potential. The dynamical form of the Chew–Low equations permits significant progress toward the study of solutions of physical interest, which have simultaneously the appropriate cutoff behavior and a Born pole. Important difficulties arise in the analysis of such solutions by the methods of functional analysis.

An interesting application of the Chew–Low equations was the attempt by Low and Huang to formulate a criterion for choosing solutions of the bootstrap type. For several soluble models they established relations between these solutions and the form of the cutoff function and the number of subtractions in the dispersion relations. These results were subsequently extended to models for which exact solutions are not known.

The next important question which must be resolved on the basis of the static models is that of whether it is possible to predict internal symmetry groups. We can now answer this question in the negative on the basis of certain unsuccessful attempts to determine whether the SU(2) group can be singled out in connection with a criterion for choosing bootstrap solutions. Nevertheless, an extremely interesting attempt to solve this problem was undertaken on the basis of the inclusion of inelastic channels in the static models.[70]

Finally, comparison of the approximate and exact solutions with experimental data shows that the static model describes the experimental results on low-energy scattering within 20%. The physical picture which arises here is consistent with the conclusions of other approaches (effective Lagrangians, as a consequence of chiral symmetrices, etc.) but differs from them in the simultaneous inclusion of the principles of crossing symmetry, analyticity, and unitarity.

I. THE CHEW–LOW EQUATIONS

1. Derivation of the Chew – Low equations and origin of the cutoff function. In describing the scattering of π mesons by nucleons at low energies, it is a good first approximation to neglect the nucleon motion in comparison with that of the π mesons. Then the nucleon field can be treated as a classical field in which π mesons are scattered and created. The fixed nucleon can emit and absorb π mesons only in the $l = 1$ state. The Hamiltonian for the interaction of the quantized meson field $\varphi_i(\mathbf{x})$ with the classical nucleon field $u(\mathbf{x})$ is

$$H = \frac{f}{\mu} \sum_{i=1}^{3} \int u(\mathbf{x}) (\sigma\nabla) \tau_i \varphi_i(\mathbf{x})\, dx, \qquad (1)$$

where $f^2 \approx 0.08$ is the pion–nucleon coupling constant and μ is the mass of the π meson.

The Hamiltonian (1) is invariant with respect to rotations in ordinary space and in isotopic space. The function $u(\mathbf{x})$ incorporates the spatial distribution of the fixed nucleon source; its density function is conveniently assumed normalized: $\int u(\mathbf{x})d\mathbf{x} = 1$. The Fourier transform of the source function is $u(\mathbf{q}) = \int \exp(i\mathbf{q}\mathbf{x})\, u(\mathbf{x})d\mathbf{x}$.

If we assume that the source function is spherically symmetric and has a finite radius R, we find that for $q \geq 1/R$ the function $u(q^2)$ is small. The quantity $q_{max} = 1/R$ represents the maximum momentum of π mesons, which effectively participate in interaction (1).

If inelastic processes are neglected, it can be shown that the Chew–Low equations[1,2] hold for the model with interaction (1):

$$\operatorname{Re} h_i(\omega) = \frac{\lambda_i}{\omega} + \frac{1}{\pi} \int_{\mu}^{\infty} \left[\frac{\operatorname{Im} h_i(\omega')}{\omega' - \omega} + \frac{\sum_{j=1}^{3} A_{ij} \operatorname{Im} h_j(\omega')}{\omega' + \omega} \right] d\omega', \qquad (2)$$

where

$$h_i(\omega) = \frac{\exp[i\delta_i(\omega)] \sin \delta_i(\omega)}{q^3 u^2(q^2)};$$

$i = \{2T, 2J\}$; T is the total isopin, J is the total angular momentum, δ_i is the scattering phase shift in a state with definite T and J values, $\omega = (q^2 + \mu^2)^{1/2}$ is the total energy of the meson, and the crossing matrix A_{ij} and the numbers λ_i are given by

$$A = \frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}; \quad \lambda = \frac{2}{3}\begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}\left(\frac{f}{\mu}\right)^2. \qquad (3)$$

Since the spin and isopin variables appear symmetrically in the Hamiltonian (1), the interactions in states {1.3} and {3.1} are the same. The model with interaction (1) is not the only model which leads to the Chew–Low equations; models based on the following Hamiltonians also lead to Eqs. (2):

a) that for the interaction of neutral scalar mesons with a source,

$$H_{in} = \sqrt{4\pi}\, g \int u(\mathbf{x}) \varphi(\mathbf{x})\, dx;$$

b) that for the interaction of charged scalar mesons with a source,

$$H_{in} = \sqrt{4\pi}\, g \int u(\mathbf{x}) (\tau_1\varphi_1(\mathbf{x}) + \tau_2\varphi_2(\mathbf{x}))\, dx;$$

c) that for the interaction of neutral pseudoscalar mesons with a source,

$$H_{in} = \sqrt{4\pi}\, \frac{g}{\mu} \int u(\mathbf{x}) (\sigma\nabla) \varphi(\mathbf{x})\, dx;$$

d) that for the symmetric interaction of charged scalar mesons with a source,

$$H_{in} = \sqrt{4\pi} g \int u(\mathbf{x}) \tau_i \varphi_i(\mathbf{x})\, dx.$$

In these models the mesons interact with a source in states with $l = 0, 1$, and in all these cases the scattering problem leads to the Chew–Low equations.

Equations (2) have turned out to be quite important for studying low-energy pion–nucleon scattering. This model has received widespread acceptance, so it is quite natural to pose the following question: Can we analyze the Chew–Low equations on a more general basis than the specific form of the interaction Hamiltonian? The first step in the exploration of this possibility was taken by Logunov and Tavkhelidze,[5] who derived Eqs. (2) from the causality condition in the Bogolyubov formulation.[6] The next step was taken by Oehme[7] and Chew et al.,[8] who obtained the Chew–Low equations as the static limits of the rigorously proved relativistic dispersion relations.[9] Equations for s waves analogous to those for p waves were derived in ref. 8; the same method can be used to find equations for higher (d and f) waves for πN scattering.

For simplicity we describe the CGLN (Chew–Goldberger–Low–Nambu) method[8] for the example of the scattering of neutral π mesons by spinless nucleons N. The incorporation of spin and isospin does not significantly affect the method. The transition amplitude for this process is expressed in the following manner in terms of the S matrix:

$$\langle q_2, p_2 | S - 1 | q_1, p_1 \rangle = i(2\pi)^4 \delta(p_1 + q_1 - p_2 - q_2) \times \frac{1}{(2\pi)^6} \cdot \frac{M}{(4q_1^0 q_2^0 p_1^0 p_2^0)^{1/2}} T(p_1 q_1;\, p_2 q_2). \qquad (4)$$

Here q_i (p_i) is the 4-momentum of the pion (nucleon), and M is the nucleon mass. The Lorentz-invariant amplitude $T(p_1q_1;\, p_2q_2)$ depends on two variables, which we can choose to be any two of the Mandelstam variables s, u, t:

$$s=(p_1+q_1)^2=M^2+\mu^2+2ME;$$
$$u=(p_1-q_2)^2;\ t=-2q^2(1-\cos\theta),$$

where q is the momentum, θ is the c.m. scattering angle, and E is the meson energy in the laboratory system.

For the scattering amplitude we have the expansion

$$T(s,t)=4\pi\frac{W}{M}\sum_{l\geqslant 0}(2l+1)f_l(s)P_l\left(1+\frac{t}{2q^2}\right), \tag{5}$$

where W is the total energy. The condition of two-particle unitarity is

$$\operatorname{Im} f_l(s)=q(s)\,|f_l(s)|^2. \tag{6}$$

We write the dispersion relations in terms of s for a fixed value of t without subtractions:

$$T(s,t)=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\operatorname{Im}T(s',t)\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'. \tag{7}$$

The Chew–Low equations follow from Eq. (7) if we make several assumptions: First, we neglect all inelastic processes in (7). Second, we restrict expansion (5) to the lowest-order partial waves, setting $f_l(s)=0$ for $l>1$. Third, we go to the static limit in the equations for the partial waves ($\mu/M\to 0$, $\omega/M\to 0$). After some straightfoward calculation we find

$$\left.\begin{aligned} f_0(\omega)&=\frac{2f^2}{\mu^2}+\frac{3}{\pi}\int_{\mu^2}^{\infty}\operatorname{Im}\frac{f_1(\omega')}{q'^2}d\omega'^2\\ &+\frac{1}{\pi}\int_{\mu^2}^{\infty}\frac{\operatorname{Im}f_0(\omega')}{\omega'^2-\omega^2}d\omega'^2;\\ f_1(\omega)&=-\frac{2}{3}\cdot\frac{f^2}{\mu^2}\cdot\frac{q^2}{\omega}+\frac{q^2}{\pi}\int_{\mu^2}^{\infty}\frac{\operatorname{Im}f_1(\omega')}{q'^2(\omega'^2-\omega^2)}d\omega'^2,\end{aligned}\right\} \tag{8}$$

where

$$\omega=\lim_{M\to\infty}E \text{ and } f=g(\mu/2M).$$

Equations (8) incorporate all the features of the πN scattering problem in the static limit which were established by Chew.[8] First, the crossing-symmetry condition T(s, u, t) = T(u, s, t), which relates different partial waves, reduces to a series of uncoupled equations of the type $f_l(-\omega)=f_l(\omega)$. Second, the different partial waves are coupled because of the pole term. For each concrete problem we must specify: a) the quantum numbers of the meson and source as well as the group with respect to which the interaction is invariant; b) the assumption regarding the mass spectrum of the meson + source system; c) the assumption regarding the growth of the functions $f_i(\omega)$ at infinity. Condition a can be used to determine the form of the crossing matrix A_{ij}, while conditions b and c fix the pole and the growth rate of the functions $f_i(\omega)$.

In one aspect, Eqs. (2) differ significantly from the equations derived in ref. 8. They contain, instead of the functions $h(\omega)$, the partial-wave amplitudes $f_i(\omega)=h_i(\omega)\cdot u^2(q^2)$, i.e., different functions have the same analytic properties in the Chew–Low equations (2) and the CGLN equations. Very little is known about the source function,[3] so it is clear that the partial waves in the Chew–Low equations can have analytic properties quite different from those in the CGLN equations.

To determine the meaning of the cutoff function $u(q^2)$ we turn to the analytic properties of the scattering amplitude T(s, u, t) as a function of any two at the Mandelstam variables s, u, t (ref. 10). For scattering of neutral pions by spinless nucleons the Mandelstam representation is

$$\begin{aligned}T(s,u,t)&=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)\\&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}ds'\int_{(M+\mu)^2}^{\infty}du'\frac{\rho(s',u')}{(s'-s)(u'-u)}\\&+\frac{1}{\pi^2}\int_{(M+\mu)^2}^{\infty}dx\int_{4\mu^2}^{\infty}dt'\rho_1(x,t')\left[\frac{1}{(x-s)(t'-t)}+\frac{1}{(x-u)(t'-t)}\right].\end{aligned} \tag{9}$$

The double spectral representation in (9) simultaneously describes three processes:

$$\left.\begin{aligned}&\text{I. } \pi+N\to\pi'+N' && s\\ &\text{II. } \bar{\pi}'+N\to\bar{\pi}+N' && u\\ &\text{III. } \pi+\bar{\pi}'\to\bar{N}+N' && t\end{aligned}\right\}\ \text{are energy variables.}$$

From (9) we can easily find (7). As a function of the variable t, the quantity Im T(s,t) has cuts outside the physical region for process I. To determine the consequences of these cuts, we use the approach of Cini and Fubini,[11] writing the imaginary part of the scattering amplitude as

$$\operatorname{Im}T(s,t)=\{\operatorname{Im}T(s,t)\}_{\text{ela}}+\{\operatorname{Im}T(s,t)\}_{\text{inela}}. \tag{10}$$

It follows from spectral representation (9) that the first term in (10) has a cut as a function of t beginning at $t=16\mu^2$, while the second, which is nonvanishing only for $s>(M+2\mu)^2$, has a cut beginning at $t=4\mu^2$. We substitute (10) into (7) and neglect the s and u dependences in the integrals of the second term [this approximation is valid for $|s|\sim|u|<(M+2\mu)^2$]:

$$\begin{aligned}T(s,u,t)&=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)\\&+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\{\operatorname{Im}T(s't)\}_{\text{ela}}\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'+\frac{1}{\pi}\int_{4\mu^2}^{\infty}\frac{c(t')}{t'-t}dt'.\end{aligned} \tag{11}$$

The cut as a function of t, beginning at $t=4\mu^2$, is related to the circumstance that the amplitude T also describes process III. We approximate the imaginary part of this process by a series of δ functions,

$$c(t)=\pi\sum_i c_i\delta(t-t_i).$$

We now convert (10) to

$$\begin{aligned}T(s,u,t)&=g^2\left(\frac{1}{M^2-s}+\frac{1}{M^2-u}\right)+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\{\operatorname{Im}T(s',t)\}_{\text{ela}}\\&\times\left[\frac{1}{s'-s}+\frac{1}{s'-u}\right]ds'+\sum_i\frac{c_i}{t-t_i}.\end{aligned} \tag{12}$$

We single out the partial waves from (12) by the differen-

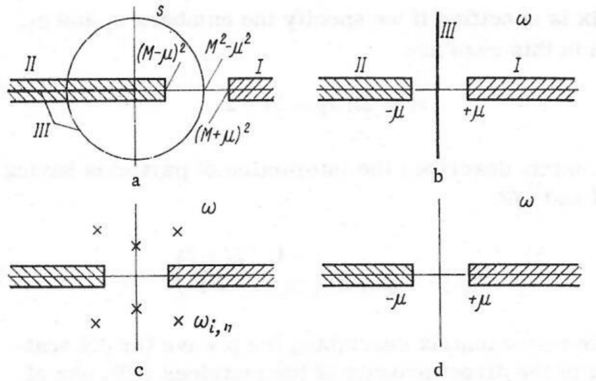

Fig. 2

tial procedure of ref. 12, i.e., by combining the spectral representation (12) for forward scattering with that for back-scattering:

$$\left.\begin{aligned}4\pi(W/M)f_0(s)=[T(s,0)+T(s,-4q^2)]/2;\\4\pi(W/M)f_1(s)=[T(s,0)-T(s,-4q^2)]/2.\end{aligned}\right\}\quad(13)$$

It follows from these equations that $f_0(s)$ and $f_1(s)$ have the same system of poles t_i. Now taking the static limit in (13), we find that the functions $f_0(\omega)$ and $f_1(\omega)$ satisfy equations whose right sides contain the same system of ω poles. This system of poles is symmetric with respect to the line $\mathrm{Re}\,\omega = 0$.

The last step in the derivation of the Chew−Low equations from the representations (9) is to construct an auxiliary function having the same poles as the t-channel contribution to representation (12):

$$V(q^2)=\prod_i[1/(\omega-\omega_i)];\quad u(q^2)=V(q^2)/V(-1).$$

Instead of dealing with the partial wave $f_l(\omega)$ we can now deal with the function $f_l(\omega)/u(q^2)=h_l(\omega)$, whose analytic properties are the same as those of the corresponding function in the Chew−Low equation (2).

These arguments are generalized to the case of meson−nucleon scattering. The amplitude T has the spin−isospin structure described by

$$T=A+\frac{1}{2}(\hat{q}_1+\hat{q}_2)B;\quad A_{\substack{\alpha\beta\\t_1t_2}}=A^{(+)}\delta_{\alpha\beta}\delta_{t_1t_2}+A^{(-)}(1/2)[\tau_\alpha\tau_\beta]_{t_1t_2}\quad(14)$$

and the analogous equation for B, where $\alpha\beta$ (t_1, t_2) are the isospin indices for the pions (nucleons). From the Mandelstam representation we find the analytic properties of the partial waves $f_i(s)$ (refs. 13, 14). Figure 2a shows the arrangement of cuts; the same cuts are shown in the ω plane in Fig. 2b. The cut $-\infty$, i due to process III can be represented by a system of poles (Fig. 2c). We construct the function

$$u(q^2)=\frac{V(q^2)}{V(-1)},\quad V=\prod_n\frac{1}{\omega-\omega_n}.$$

The functions $h_i(\omega)=f_i(\omega)/u(q^2)$ obviously have the analytic properties shown in Fig. 2d. Assuming also that $f_{13}=f_{31}$, we find the Chew−Low equations (2).

2. Properties of the crossing matrix for elastic processes. The amplitude $T_{\substack{\alpha\beta\\t_1t_2}}(s,u,t)$ for elastic πN scattering satisfies the crossing-symmetry property

$$T_{\substack{\alpha\beta\\t_1t_2}}(s,u,t)=T_{\substack{\beta\alpha\\t_1t_2}}(u,s,t).\quad(15)$$

This symmetry property can be easily reformulated for partial waves; the resulting relations are conveniently thought of as a consequence of the static limit of (15):

$$T_{\substack{\alpha\beta\\t_1t_2}}(\mathbf{p},-\mathbf{p},\omega)=T_{\substack{\beta\alpha\\t_1t_2}}(-\mathbf{p},\mathbf{p},-\omega);\quad(16)$$

where the indices α, β give the isospin states of the pions but can also be thought of as parameters characterizing the irreducible representation of same symmetry group with respect to which the interaction of the particle and the source is invariant.

To expand the T matrix in a sum over the representation into which the direct product of representations α and t_1 decomposes, we introduce a system of projection operators which project onto these representations. This system has the properties of completeness and orthogonality:

$$\sum P_i=1;\quad P_iP_j=\delta_{ij}P_j.\quad(17)$$

Using the projection operators we can write (16) as

$$\sum_i T_i(\omega)(P_i)_{\alpha\beta}=\sum_i T_i(-\omega)(P_i)_{\beta\alpha}.\quad(18)$$

The indices giving the state of the source are the same on both sides of Eq. (18), so we omit them. Since system P_i is complete, we can write the following expansion for any matrix element $(P_i)_{\beta\alpha}$:

$$(P_i)_{\beta\alpha}=\sum_j(A^T)_{ij}(P_j)_{\alpha\beta}.\quad(19)$$

Equation (19) serves as a definition of the crossing matrix. Substituting (19) into (18), we find

$$T_i(\omega)=\sum_j A_{ij}T_j(-\omega).\quad(20)$$

Using condition (20) twice, and exploiting the orthogonality of the operators P_i, we easily find the property

$$A^2=E.\quad(21)$$

Carrying out the summation over i in (19) and using the completeness condition (17), we find

$$\sum_j A_{ij}=1\quad\text{for any}\quad i.\quad(22)$$

Taking the trace of both sides of (19), we find

$$c_iA_{ij}=c_j;\quad c_i=\mathrm{Tr}\,P_i>0.\quad(23)$$

Equations (21)−(23) do not depend on which transformation group the T matrix is invariant with respect to. If we supplement these equations with the symmetry properties of the projection operators P_i with respect to the interchange $\alpha \rightleftharpoons \beta$, we can construct the crossing matrix on this basis.[15] These additional symmetry properties are

$$c_i^{(m)} A_{ij} = \pm c_j^{(m)}, \tag{24}$$

where $c_j^{(m)}$ are known numbers.

From the basic properties [(21)-(23)] of matrix A we can draw certain general conclusions about this matrix. It is easy to prove the following lemmas:

Lemma 1. If A and B are two crossing matrices for elastic processes of orders n and m, respectively, their direct product

$$A \times B$$

is also a crossing matrix, of order n · m.

Lemma 2. If A and B are two crossing matrices of the same order, the matrices

$$ABA, \quad BAB$$

are also crossing matrices of the same order.[16]

From (20) we find the eigenvalues of matrix A to be ± 1, while (22) shows that the eigenvalue + 1 always belongs to the eigenvector $\{1, 1, \ldots, 1\}$.

A matrix satisfying condition (21) can be written as

$$A = u \begin{pmatrix} \pm 1, & 0, \ldots, & 0 \\ 0, \pm & 1, \ldots, & 0 \\ \cdot & \cdot \cdot \cdot \cdot \cdot & \cdot \\ 0, & 0, \ldots, & \pm 1 \end{pmatrix} u^{-1}, \tag{25}$$

where u is an arbitrary nonsingular matrix. The choice of signs in (25) determines the trace of matrix A. Let us consider several possibilities.

We first assume n = 2 and trace A = 0. Then from (25) we find an equation with two parameters for A:

$$A = \begin{pmatrix} a & \frac{1-a^2}{b} \\ b & -a \end{pmatrix}.$$

Using (22), we can simplify this equation to

$$A = \begin{pmatrix} a & 1-a \\ 1+a & -a \end{pmatrix}. \tag{26}$$

Equation (26) constitutes the most general form of the two-row crossing matrix. The single parameter is given by condition (23):

$$a = \frac{c_1 - c_2}{c_1 + c_2}. \tag{27}$$

We now assume n = 3 and trace A = 1. From (25) and (22) we find an equation with three parameters:

$$A = E - 2 \begin{pmatrix} 1, & a, & -(1+a) \\ b, & ab, & -(1+a)\,b \\ c, & ac, & -(1+a)\,c \end{pmatrix} \frac{1}{1+ab-(1+a)\,c}. \tag{28}$$

We turn now to some examples of constructing a crossing matrix for the group SU(2). A two-row crossing matrix is specified if we specify the numbers c_1 and c_2, which in this case are

$$c_1 = 2l; \; c_2 = 2l + 2.$$

This matrix describes the interaction of particles having spin l and 1/2:

$$A_{l,\,1/2} = \frac{1}{(2l+1)} \begin{pmatrix} -1, & 2l+2 \\ 2l, & +1 \end{pmatrix}. \tag{29}$$

The four-row matrix describing the p wave for πN scattering is the direct product of the matrices (29), one of which refers to the isospin variables and the other to the spin variables.

The basic conditions are insufficient to construct a three-row crossing matrix and must be supplemented with Eqs. (24). After constructing the projection operators for this case, we can easily find eigenvectors (24) with eigenvalues ± 1. The matrix $A_{l,1}$ has the form (28), where

$$a = \frac{2l+1}{2l-1} \cdot \frac{1}{l+1}; \quad b = \frac{1}{l+1}; \quad c = -\frac{l}{l+1}. \tag{30}$$

We can also use general theorems to construct the crossing matrix in the group SU(2). For example, Rose and Yang[17] have shown that

$$\mathrm{Tr}\, A = 1, \, 0$$

for matrices of odd and even order, respectively. A column-summation rule was proved in ref. 16 for a crossing matrix of order n and for the group SU(2):

$$\sum_i (-1)^i A_{ij} = -(-1)^{j+(n-1)}.$$

Analogous assertions have been made for SU(n) (refs. 18-20).

3. Formulation of the boundary-value problem.

Equations (2) determine analytic functions $h_i(z)$ with the following properties:

1) $h_i(z)$ are analytic in the complex z plane with the cuts[1)] $(-\infty, -1]$, $[+1, +\infty)$ (Fig. 2d);

2) $h_i^*(z) = h_i(z^*)$;

3) $\mathrm{Im}\, h_i(\omega + i0) = q^{2l+1} u^2(q^2)\, |h_i(\omega + i0)|^2$, where

$$h_i(\omega + i0) = \lim_{\varepsilon \to +0} h_i(\omega + i\varepsilon), \quad \omega > 1$$

is the unitarity condition;

4) $h_i(-z) = \sum_j A_{ij} h_j(z)$ are the crossing-symmetry conditions;

5) $h_i(z)$ has a first-order pole at the origin, with $\mathrm{Res}\, h_i(\omega) = \lambda_i$;

6) $h_i(z) \to 0$ for $|z| \to \infty$ and $\mathrm{Im}\, z \neq 0$.

Using the Cauchy theorem and properties 1-6, we can derive Eqs. (2).

We have already mentioned that physically different scattering problems lead to the Chew–Low equations (2).

We show below that the solutions of even highly simple equations like (2) with specified numbers λ_i are ambiguous. Functions arise in the course of the solution which can lead to the presence of a pole in ω and a certain behavior at infinity. These properties are specific to each specific problem. The general properties of the functions satisfying various equations like (2) are properties 1-4.

For convenience below we analyze, instead of the partial amplitudes, the matrix elements of the S matrix,

$$S_i(\omega) = 1 + 2iq^{2l+1}u^2(q^2)\,h_i(\omega), \tag{31}$$

where $q = (\omega^2 - 1)^{1/2}$ is defined such that $[(\omega + i0)^2 - 1]^{1/2} > 0$ at $\omega > 1$. Then $q^*(\omega) = -q(\omega^*)$. We assume that $u(q^2)$ are meromorphic functions and that $u^*(\omega^2 - 1) = u(\omega^{*2} - 1)$. The functions $S_i(\omega)$ contain, in addition to the poles of the partial amplitudes $h_i(\omega)$, poles of $u(q^2)$. If we consider general properties 1-4 only, we find no stipulation regarding poles in $h_i(\omega)$. It is therefore not necessary to consider the $u(q^2)$ poles, so the first property of $h_i(\omega)$ can be transferred completely to the function $S_i(\omega)$.

Property 2 of the partial amplitudes $h_i(\omega)$ also holds for the functions $S_i(\omega)$ if we take into account the choice of the $q(\omega)$ branch. The unitarity condition 3 leads to

$$S_i(\omega + i0) = \exp[2i\delta_i(\omega)] \text{ for } \omega > 1.$$

We can also transfer crossing-symmetry condition 4 to the functions $S_i(\omega)$ if we note that $q(-\omega) = q(\omega)$ and use property (22). We finally find:

I. $S_i(z)$ are analytic in the complex z plane with cuts along the real axis in the intervals $(-\infty, -1]$, $[+1, +\infty)$;

II. $S_i^*(z) = S_i(z^*)$;

III. $|S_i(\omega + i0)|^2 = 1$, $\omega > 1$, $S_i(\omega + i0) = \lim_{\varepsilon \to +0} S_i(\omega + i\varepsilon)$;

IV. $S_i(-z) = \sum_j A_{ij} S_j(z)$.

II. EXACT SOLUTIONS IN THE THEORY OF CHEW–LOW EQUATIONS

4. Algebraic formulation of the Chew–Low equations. The Chew–Low equations are usually solved by the method worked out by Salzman and Salzman[21] or by the N/D method.[22] Each of these methods involves a regularization of the singular equations (2). This approach frequently leads to spurious or unreliable solutions; an example of a spurious approximate solution of the problem by the N/D method can be found in ref. 61. These disadvantages can be avoided by using an algebraic formulation of the equations.[23] In this approach, a system of algebraic equations equivalent to the Chew–Low equations is derived. In certain cases the algebraic system is more convenient for theoretical studies, and it is definitely preferable for numerical calculations.

To derive an algebraic system from the boundary-value problem formulated in Sec. 3 we consider the function

$$z = \frac{2z'}{1 + z'^2}. \tag{32}$$

This function conformally maps the z plane into the z' plane; it maps the unit circle $|z'| < 1$ into a plane with cuts $(-\infty, -1]$ and $[+1, +\infty)$, and it maps points ± 1 of the z' plane into points ± 1 of the z plane. In terms of this new variable the basic properties of the boundary-value problem are:

1) $h_\alpha(z')$ is analytic in the unit circle $|z'| < 1$, except at the point $z' = 0$, where it has a pole with a residue $\lambda_\alpha/2$; the function $h_\alpha(z')$ for $|z'| = 1$ is piecewise continuous. The conformal mapping (32) maps cuts in the z plane into the unit circle $|z'| = 1$ according to the equation $\cos\varphi = 1/x$;

2) $\operatorname{Im} h_\alpha(\varphi) = F(\varphi)|h_\alpha(\varphi)|^2$, $-\pi/2 < \varphi < \pi/2$, the unitarity condition. The function $F(\varphi)$ is related to the source function. We assume that $F(\varphi)$ is piecewise continuous;

3) $h_\alpha(\varphi + \pi) = \Sigma A_{\alpha\beta} h_\beta(\varphi)$ is the crossing-symmetry condition. The problem thus reduces to one of finding the functions $h_\alpha(\varphi)$. It follows from condition 1 that $h_\alpha(z')$ can be written as

$$h_\alpha(z') = \lambda_\alpha/(2z') + \sum_{n\geq 0} a_n^{(\alpha)} z'^n. \tag{33}$$

It follows from the condition $h_\alpha^*(z') = h_\alpha(z'^*)$ that all the coefficients in (33) are real. On the unit circle $|z'| = 1$ we find two Fourier series from (33):

$$\left.\begin{aligned} \operatorname{Re} h_\alpha(\varphi) &= a_0^{(\alpha)} + (\lambda_\alpha/2 + a_1^{(\alpha)})\cos\varphi + \sum_{n=2}^{\infty} a_n^{(\alpha)}\cos n\varphi; \\ \operatorname{Im} h_\alpha(\varphi) &= (-\lambda_\alpha/2 + a_1^{(\alpha)})\sin\varphi + \sum_{n=2}^{\infty} a_n^{(\alpha)}\sin n\varphi. \end{aligned}\right\} \tag{34}$$

The function $F(\varphi)$ has the properties $F(\varphi) = -F(-\varphi)$; $F(\varphi) = F(\varphi + \pi)$. We can thus write

$$F(\varphi) = \sum_{n=1}^{\infty} F_{2n}\sin 2n\varphi, \tag{35}$$

where F_{2n} are known numbers.

The function

$$D_\alpha(\varphi) = \operatorname{Im} h_\alpha(\varphi) - F(\varphi)|h_\alpha(\varphi)|^2 \tag{36}$$

is periodic with a period of 2π and can be written as the Fourier series

$$D_\alpha(\varphi) = \sum D_n^{(\alpha)}\sin n\varphi. \tag{37}$$

A necessary and sufficient condition for the vanishing of series (37) on the interval $-\pi/2 < \varphi < \pi/2$ is that the coefficients D_n^α satisfy

$$(\pi/2)D_{2n}^{(\alpha)} - 4n(-1)^n \sum D_{2p+1}^{(\alpha)} \frac{(-1)^p}{(2n)^2 - (2p+1)^2} = 0. \tag{38}$$

Writing $D_n^{(\alpha)}$ in terms of $a_n^{(\alpha)}$, we find a system of algebraic equations,

$$\sum_{m=0}^{\infty} A_{km}^{(\alpha)} a_m^{(\alpha)} + \sum B_{kmn}^{(\alpha)} a_m^{(\alpha)} a_n^{(\alpha)} = 0, \tag{39}$$

which is equivalent to the unitarity relation. The coefficients $A^{(\alpha)}$ and $B^{(\alpha)}$ can be easily expressed in terms of the known numbers F_{2n}. Substituting (33) into property 3, we find a system of algebraic equations corresponding to the crossing relations:

$$\sum_{\beta=1}^{N}[A_{\alpha\beta}-(-1)^n\delta_{\alpha\beta}]\,a_n^{(\beta)}=0,\quad \begin{matrix}\alpha=1,\,2,\,\ldots,N,\\ n=0,\,1,\,2,\,\ldots .\end{matrix} \tag{40}$$

The problem has thus been reduced to systems of algebraic equations (39) and (40).

We assume that the solution depends on s parameters, i.e., that, under certain assumptions, each curve $\operatorname{Im} h_\alpha(\varphi)$ passes through s points with abscissas $\varphi_1, \varphi_2, \ldots, \varphi_s$ and with specified ordinates $H_1^{(\alpha)}, H_2^{(\alpha)}, \ldots, H_s^{(\alpha)}$. We thus have

$$\operatorname{Im} h_\alpha(\varphi_p)=(-\lambda_\alpha/2+a_1^{(\alpha)})\sin\varphi_p + \sum_{n=2}^{\infty}a_n^{(\alpha)}\sin n\varphi_p=H_p^{(\alpha)}$$
$$(p=1,\,2,\,\ldots,s;\ \alpha=1,\,2,\,\ldots,N). \tag{41}$$

Systems (39)-(41) can be solved jointly by Newton's method.

An important advantage of this approach is its independence of the number of partial waves. The conformal mapping (32) does not incorporate the circumstance that the Riemann surface of the functions $h_i(\omega)$ can have an infinite number of sheets. As we will see below, this is in fact the case. Accordingly, expansion (33) cannot have a convergence radius larger than 1 for all problems. The convergence of expansion on a unit circle is analyzed, and this method is developed further, in ref. 24. The method of reducing the problem to a system of nonlinear algebraic equations was used independently by Amatuni[25] in solving the equations for $\pi\pi$ scattering.

5. Case of one partial wave. Solution of Castillejo, Dalitz, and Dyson. The form of the functions having the basic properties depends on the number of these functions ($i = 1, 2, \ldots, N$) and is governed by the matrix A for a given N. The simplest case, that of $N = 1$ and $A = 1$, was analyzed in detail by Castillejo et al.[26]

The CDD (Castillejo–Dalitz–Dyson) method[26] is based on Wigner's[27] R function. One of the governing properties of the R function is that Im R(z) has the same sign as Im z. Furthermore, the function R(z) is required to be a meromorphic function over the entire z plane. The functions $h_i(z)$ satisfy all properties of the R function, but they are meromorphic in a plane with cuts (Fig. 2d). Such functions have been called[26] "generalized R functions." It is then easy to show that the function $H_i(z) = -1/h_i(z)$ is also a generalized R function. Then the Herglotz theorem[26] can be used to establish the most general form of the H function. For a neutral scalar field, we have

$$H(\omega)=A\omega-\frac{c}{\omega}+\frac{2\omega}{\pi}\int_1^{\infty}\frac{q\,|u(q)|^2\,d\omega'}{\omega'(\omega'^2-\omega^2)}+S(\omega), \tag{42}$$

where $A \geq 0$, $c \geq 0$, and

$$S(\omega)=\sum R_n\,[1/(\omega_n-\omega)-1/(\omega_n+\omega)] \tag{43}$$

is an odd R function. The sequence $\omega_1, \omega_2, \ldots$ can be either finite or infinite. The solution is thus not single-valued and contains a countable number of parameters. The most important feature of this solution method is the transformation from the function $h_i(z)$ to $H_i(z)$, which linearizes the boundary-value problem on the right-hand cut. This method has found many applications to dispersion relations, e.g., in ref. 28, where exact solutions were found for several low-energy models for $\pi\pi$ scattering, and the physical consequences of these solutions were analyzed.

There is a simpler method for solving the problem for the case $N = 1$, $A = 1$. We start from the boundary-value problem specified by conditions I-IV in Sec. 3. We consider the function which is the inverse of function (32):

$$z'=\eta(\omega)=(1+iq)/\omega. \tag{44}$$

Obviously, the function $\eta(\omega)$ itself satisfies basic conditions I-IV and is a highly simple solution. However, it is not the most general function which satisfies condition III. This would be the Blaschke function, which obeys the inequality $|b[\eta(\omega)]| \leq 1$ within the unit circle and has only zeroes:

$$b(\eta)=\eta^{\lambda}\prod_{k=1}^{n}\frac{|\eta_k|}{\eta_k}\cdot\frac{\eta_k-\eta}{1-\eta_k^*\eta}. \tag{45}$$

It follows from conditions III and II that the zeroes of η_k must be symmetric with respect to the axis $\operatorname{Im}\eta = 0$. The crossing-symmetry condition shows that these zeroes are symmetric with respect to the $\operatorname{Re}\eta = 0$ axis, so that n is always even:

$$b(\eta)=\eta^{\lambda}\prod_{n_1}\frac{\eta_{n_1}-\eta^2}{1-\eta_{n_1}^2\eta^2}\prod_{n_2}\frac{-\eta_{n_2}^2+\eta^2}{1+\eta_{n_2}^2\eta^2}\times\prod_{n_3}\frac{(|\eta_{n_3}|^2+\eta^2)+(2\operatorname{Re}\eta_{n_3}^*\eta)^2}{(1+|\eta_{n_3}|^2\eta^2)^2+(2\operatorname{Re}\eta_{n_3}\eta)^2}, \tag{46}$$

where n_1, n_2, n_3 are the numbers of poles on the positive real and imaginary half-lines and in the first quadrant of the η plane; λ is the order of the singularity at the zero. Equation (46) gives a different form of solution (42) and is convenient in that the independent parameters in it are related in a simple manner to such physical characteristics as the positions and widths of the resonance and the masses of the bound states. The number of singularities can also be infinite; the limitations which arise on η_{n_i} in this case have been analyzed thoroughly.[29] If we assume the functions $b(\eta)$ to be bounded functions within the unit circle, we can replace (46) by a more general representation, which turns out to be important in the solution of several problems[68] (see Sec. 12 below).

6. Solution of the Chew – Low equations with a two-row crossing matrix. As the simplest example we consider a two-row matrix:

$$A=\frac{1}{3}\begin{pmatrix}-1 & 4\\ 2 & 1\end{pmatrix}. \tag{47}$$

This matrix corresponds to spins 1, 1/2 of the interacting particles and is a matrix of type (29). The solution for this case has been given by many investigators.[30–34] We will consider this solution here in more detail.

We first determine the limitations which are imposed on $S_i(z)$ by the crossing-symmetry conditions. We write S_1 and S_3 as[2])

$$S(z)=\begin{pmatrix} S_1(z) \\ S_3(z) \end{pmatrix}. \quad (48)$$

Then condition IV from Sec. 3 can be written in matrix form:

$$AS(z)=S(-z). \quad (49)$$

We break up S(z) into parts which are even and odd functions of z. Then the most general form of the column matrix S(z) satisfying (49) is

$$S(z)=\begin{pmatrix} s(z)-2a(z) \\ s(z)+a(z) \end{pmatrix}, \quad (50)$$

where $s(z) = s(-z)$, $a(z) = -a(-z)$, and, according to condition II, $s^*(z) = s(z^*)$, $a^*(z) = a(z^*)$. In particular, if we set $a(z) = 0$, we find

$$S(z)=\begin{pmatrix} 1 \\ 1 \end{pmatrix} s(z). \quad (51)$$

The symmetry function s(z) satisfies conditions I-III, i.e., is the Blaschke function (46); Eq. (51) gives the even solution of the problem defined by conditions I-IV for matrix (43). To find the most general solution of the problem, we note that unitarity condition III yields

$$|s(\omega)-2a(\omega)|^2=|s(\omega)+a(\omega)|^2=1 \text{ for } \omega>1. \quad (52)$$

Equations (52) give two unit circles centered at the points $2a(\omega)$, $-a(\omega)$. Obviously, these circles do not intersect at any value of the ratio $s(\omega)/a(\omega)$, i.e., Eqs. (52) are compatible. These equations can be easily rewritten as

$$|s/a-2|^2=|s/a+1|^2; \quad (53)$$

$$|s+a|^2=1. \quad (54)$$

The points of the circles are defined by

$$s/a+(s/a)^*=1. \quad (55)$$

This condition is a consequence of (52) and gives the ratio s/a for $\omega > 1$. Since the functions s/a are odd, the right side of (55) changes sign at $\omega < 1$. We define $s(\omega)/a(\omega) = \Phi(\omega)$. Then it follows from the definition of $S_i(\omega)$ that (55) includes $\Phi^{(+)}(\omega) = \Phi(\omega + i0)$, and from condition II we find $(s/a)^* = \Phi^{(-)}(\omega) = \Phi(\omega - i0)$. We finally find an inhomogeneous linear Riemann boundary-value problem:

$$\Phi^{(+)}(\omega)+\Phi^{(-)}(\omega)=\pm 1; \quad \begin{matrix}\omega>1, \\ \omega<-1,\end{matrix} \quad (56)$$

whose solution is given, e.g., in ref. 35. The solution is

$$\Phi(\omega)=\frac{1}{\pi}\arcsin\omega+i\sqrt{\omega^2-1}\,\beta(\omega);$$

$$\beta(\omega)=-\beta(-\omega). \quad (57)$$

The arbitrariness in the solution is governed by the requirements that $\Phi(\omega)$ be bounded and integrable on the cut; these requirements lead to a special form for $\beta(\omega)$. In our problem, $S_i(\omega)$ are meromorphic functions in a plane with cuts, so $\beta(\omega)$ is an arbitrary odd meromorphic function of ω. In the analysis below a particular solution of (57) is quite important:

$$w=\frac{1}{\pi}\arcsin\omega; \quad w+w^*=\pm 1, \quad \omega \gtrless \pm 1. \quad (58)$$

In the complex w plane this solution is the equation of two straight lines, Re w = ± 1/2.

Since we know the ratio $s(w)/a(w)$ [Eq. (57)], the problem of solving Eq. (54) reduces to one of finding, for example, the function $a(w)$. We seek the solution as a function of w; then $i\sqrt{\omega^2-1}\,\beta(\omega) = \cos\pi w\beta(\sin\pi w) = \beta_0(w)$. It is easy to see that $\beta_0(w) = \beta_0(w + 1)$, $\beta_0(w) = -\beta_0(-w)$, and

$$s(w)/a(w)=w+\beta_0(w). \quad (59)$$

For the function $\varphi(w) = s(w) + a(w)$ we find from (54) and (58) the functional equation

$$\varphi(w)\varphi(1-w)=1. \quad (60)$$

Since the function $a(w)$ is odd, we find a second functional equation for $\varphi(w)$:

$$\varphi(w)/[w+\beta_0(w)+1]=-\varphi(-w)/[-w-\beta_0(-w)+1]. \quad (61)$$

The general solution of functional equations (60) and (61) is

$$\varphi(w)=\frac{w+\beta_0(w)}{w+\beta_0(w)-1}\exp[h(w-1/2)],$$

where $h(w) + h(-w) = 0$; $h(w) + h(w + 1) = 0$. We finally find the general solution of the problem with the matrix (47):

$$S(w)=\begin{pmatrix} w+\beta_0(w)+2 \\ w+\beta_0(w)+1 \end{pmatrix}\frac{w+\beta_0(w)}{[w+\beta_0(w)]^2-1}\exp[h(w-1/2]. \quad (62)$$

This method can be extended immediately to the case of the matrix $A_{l,1/2}$ in (29), where l is an arbitrary integer. The form of the function $S_l(w)$ for this case is[34]

$$S_l(w)=\begin{pmatrix} \frac{w+l\beta_0(w)-(l+1)}{w+l\beta_0(w)+l} \\ 1 \end{pmatrix}\varphi_l(w), \quad (63)$$

where

$$\varphi_l(w)=\prod_{n=1}^{l}\frac{w+l\beta_0(w)-1/2+(-1)^{n-1}[1/2+(l-n)]}{w+l\beta_0(w)-1/2-(-1)^{n-1}[1/2+(l-n)]} \times\exp[h(w-1/2)]. \quad (64)$$

Solutions can also be found for nonintegral l. In this case the form of the matrix A is not governed by the specific transformation group with respect to which the interaction is assumed invariant. Accordingly, nonintegral values of l can be used for parametrization in Eq. (29); the solution is again of the form (63), but now, instead of (64), we have

$$\varphi_l(w)=\mathrm{ctg}\frac{\pi}{2}\times(w-1/4)\frac{\Gamma(-(w+l)/2+1)\,\Gamma(-(-w+l)/2)}{\Gamma((w-1/4-l)/2+1)\,\Gamma(-(w-1/4+l)/2)}. \quad (65)$$

Let us consider the properties of the solution (62). We define

$$D(\omega)=\exp[h(w-1/2)]. \tag{66}$$

The function $D(\omega)$ satisfies

$$|D(\omega)|^2=1,\quad \omega\geqslant 1,\quad D(-\omega)=D(\omega). \tag{67}$$

This condition gives the function $D(\omega)$ as the Blaschke function (45) of variable η [Eq. (44)]. The position of each zero of this function is specified by two parameters, corresponding to the two parameters of the CDD pole.[26] It follows that an ambiguity of the CDD-pole type also arises in more complicated cases and is the same in both partial waves.

The remaining part of S(w) depends on the arbitrary meromorphic function $\beta_0(w)$. The poles and zeros of S(w) are given by

$$w+\beta_0(w)+n=0,\quad n=0,\ \pm 1,\ -2. \tag{68}$$

It follows from the periodicity of $\beta_0(w)$ that the roots of Eq. (68) can be determined simply by solving the one equation with n = 0:

$$w+\beta_0(w)=0. \tag{69}$$

Let us consider the ratio $[w+\beta_0(\pi)]/\{[w+\beta_0(w)]-1\}$, which appears in S(w). The zeros of the denominator obviously lie one unit away from those of the numerator, i.e., the poles and zeros of this ratio are symmetric with respect to the line Re w = 1/2. The equation for $S_1(w)$ contains the additional factor $\{[w+\beta_0(w)]-2\}/\{[w+\beta_0(w)]+1\}$, whose zeros and poles are also symmetric with respect to the line Re w = 1/2. This symmetry is a consequence of the unitary condition on the cut $\omega > +1$.

From Eq. (69) we find the following properties for the set of roots W of this equation:

a) Set W is symmetric with respect to the origin.

b) Set W is symmetric with respect to the Im axis. Accordingly, to determine set W it is sufficient to find all roots of Eq. (69) which lie in the first quadrant of the W plane, including its boundary.

We can specify the properties of set W in more detail only by making certain assumptions about $\beta_0(w)$.

7. Riemann surface of the functions $S_i(z)$ and some of their properties. We conclude from the two-particle unitarity condition and the Schwarz reflection principle[3)] that the functions $S_i(z)$ have first-order algebraic branch points at z = + 1. The crossing-symmetric condition allows us to extend this conclusion to the point z = −1.

To determine the nature of the branching of these functions at infinity we must circumvent the points ± 1 simultaneously. If unitarity condition III holds at the right-hand cut, it does not at the left-hand cut, and the functions $S_i(-z)$ are determined with the help of the crossing-symmetry condition. Accordingly, the two points ± 1 can be circumvented simultaneously through the use of conditions III and IV. Here Ningu Hu[36] was apparently the first to show that the point $z\to\infty$ is a branch point of the logarithmic type or, more precisely, that all sheets of the Riemann surface join at infinity. He treated the matrix elements of the S matrix as functions of the energy and momentum, working from unitarity condition III and the odd nature of the phase as a function of the momentum $S_i(\omega, q) = 1/[S_i(\omega, -q)]$. He showed that for the neutral and charged models the Riemann surface of functions $S_i(\omega, q)$ has two sheets. This result agrees with those of ref. 26. An analogous treatment for s-wave and p-wave πN scattering leads to the conclusion that the Riemann surface of these functions is not double-sheeted.

Jones[37] used the concept of a universal covering surface to analyze the Riemann surface of the partial amplitudes according to a potential-scattering model. He also pointed out the importance of correctly taking the form of the Riemann surface into account in physical applications.

The explicit solution of the problem with a two-row crossing-symmetry matrix shows that an infinitely remote point is a logarithmic branching point. Working on the basis of this result and the conclusions of Ning Hu, we assume that the Riemann surface of the functions $S_i(z)$ has an infinite number of sheets because of the branch point at infinity. After determining the form of the Riemann surface of the functions $S_i(z)$, we can solve the problem of their analytic continuation from the first sheet to any other. The continuation to the second sheet follows from conditions II and III: $S_i^{(2)}(z) = 1/S_i(z)$.

For the continuation of the functions $S_i(z)$ to all sheets of the Riemann surface it is more convenient to transform to the variable w in (58) and write the properties of the functions $S_i(w)$ in matrix form.[15] We introduce the column matrix

$$S(w)=\begin{pmatrix} S_1(w)\\ S_2(w)\\ \vdots\\ S_n(w)\end{pmatrix}. \tag{70}$$

We denote by I the nonlinear operation

$$IS(w)=\begin{pmatrix} 1/S_1(w)\\ 1/S_2(w)\\ \vdots\\ 1/S_n(w)\end{pmatrix}. \tag{71}$$

The conformal transformation (58) leads to uniformization of the functions $S_i(w)$. Then conditions I-IV in Sec. 3 take the following form in terms of the variable w:

I'. S(w) is a column matrix of meromorphic functions in the w plane;

II'. $S^*(w) = S(w^*)$;

III'. $IS(w) = S(1-w)$ – the unitarity condition;

IV'. $S(-w) = AS(w)$ – the crossing-symmetry condition.

The functional form of unitarity condition III' follows from (58). Combining properties III' and IV', we find $AIS(w) = S(w-1)$; $IAS(w) = S(w+1)$. Repeating this opera-

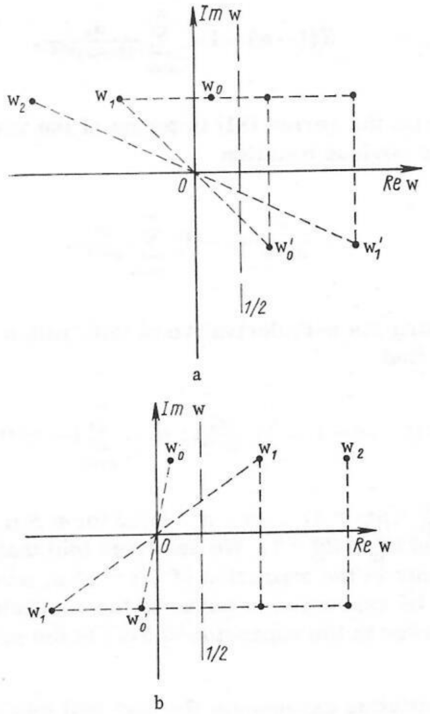

Fig. 3

tion n times, we draw the following conclusions:[4)]

$$(AI)^n S(w) = S(w-n); \quad\quad (72)$$
$$(IA)^n S(w) = S(w+n). \quad n=1,\ 2,\ \ldots \quad\quad (73)$$

Accordingly, knowing the values of the function S(w) at the pole $|\mathrm{Re}\ w\ | \leq 1/2$, i.e., on the physical sheet, we can use (72) and (73) to continue this function to any plane. The operator $(AI)^n$ is the operator for the continuation to the left to the strip $|\mathrm{Re}\,(w+n)\,| \leq 1/2$, while the operator $(IA)^n$ represents the continuation to the strip $|\mathrm{Re}\,(w-n)\,| \leq 1/2$. If we initially assume that conditions I'-IV' hold only for the first sheet of the Riemann surface, we can easily show that they hold for the entire w plane. Figure 3 illustrates the geometric method for analytic continuation along straight lines parallel to Im w by means of the operators $(AI)^n$ and $(IA)^n$.

From unitarity condition III we can determine the form of the column matrix S(w). We introduce the new function $g_i(w) = \ln S_i(w)$, where the logarithm is understood to represent that branch of the multivalued function for which $[\ln S_i(w)]^* = [\ln S_i(w^*)]$. The functions $g_i(w)$ satisfy the equation

$$g_i(w) + g_i(1-w) = 0,$$

whose general solution is

$$g_i(w) = \Delta_i(w-1/2), \quad \Delta_i(w) = -\Delta_i(-w).$$

We thus have

$$S(w) = \begin{pmatrix} \exp \Delta_1(w-1/2) \\ \exp \Delta_2(w-1/2) \\ \cdots\cdots\cdots \\ \exp \Delta_n(w-1/2) \end{pmatrix}. \quad\quad (74)$$

where each of the functions $\Delta(w)$ is antisymmetric with respect to its argument. On the line Re w = 1/2 all the functions $\Delta_i(w-1/2)$ are purely imaginary, and the unitarity condition holds. Over the remainder of the plane, on the other hand, the $\Delta_i(w-1/2)$ values are complex. The unitarity of the column matrix S(w) does not couple the functions Δ_i. This coupling is established by the requirement of crossing symmetry, IV'.

Finally, we note that the ambiguity in the solution of problem I-IV found in the case of a two-row crossing matrix is again found in the general case of a matrix of arbitrary order n × n. Accordingly, if we find a column of functions S(w) satisfies conditions I'-IV', then the following column also satisfies these conditions:

$$\begin{pmatrix} S_1(w+\beta(w)) \\ S_2(w+\beta(w)) \\ \cdots\cdots \\ S_n(w+\beta(w)) \end{pmatrix} D(w); \quad\quad (75)$$

$D(w)\,D(1-w) = 1$; $D(w) = D(-w)$; $D^*(w) = D(w^*)$; $\beta(w) = \beta(w+1)$; $\beta(w) = -\beta(-w)$; $\beta^*(w) = \beta(w^*)$.

8. Construction of the function $S_i(w)$ with a finite number of singularities for a crossing matrix of arbitrary rank. Attempts to apply the method discussed in Sec. 6 to the problem with even three partial waves have been unsuccessful. A natural step would be to attempt to formulate a system of algebraic equations for the case of a Riemann surface having an infinite number of sheets, through the use of the variable w in (58), rather than z' in (32). This approach was followed by Meshcheryakov,[15] who used one other assumption: He sought those equations of problem I'-IV' for which the functions $S_i(w)$ have only a finite number of poles in the w plane. Then it follows from the unitarity condition that each function S_i can be written as

$$S_i(w) = \prod_{n=1}^{N_i} \frac{w-a_n^{(i)}}{w-(1-a_n^{(i)})} \quad\quad (76)$$

or, alternatively, as the sum of unity and an irreducible rational function:

$$S_i(w) = 1 + \frac{b_1^{(i)} w^{N_i-1} + \ldots + b_{N_i}^{(i)}}{w^{N_i} + a_1^{(i)} w^{N_i-1} + \ldots + a_{N_i}^{(i)}}. \quad\quad (77)$$

We choose from the set of poles of the functions $S_i(w)$ the pole having the maximum modulus $|1-a_n^{(i)}|$, i.e., $R = \max\{|1-a_n^{(i)}|\}$. Then in the region $|w| > R$ each of the functions is free of singularities and can be expanded in a series in inverse powers of w or $w-\alpha$. Such an expansion is known to be unique. The series

$$S_i(w) = 1 + \sum_{n=0}^{\infty} \frac{\alpha_n^{(i)}}{w^{n+1}} \quad\quad (78)$$

converges absoulutely and uniformly outside a circle of radius R. From (77) and (78) we find (for simplicity below we omit the index i)

$$b_1 w^{N-1} + b_2 w^{N-2} + \ldots + b_N$$
$$= (w^N + a_1 w^{N-1} + \ldots + a_N)$$
$$\times (\alpha_0/w + \alpha_1/w^2 + \ldots + \alpha_n/w^{n+1} + \ldots). \quad\quad (79)$$

Equating the coefficients of identical powers of w, we find an infinite system of linear equations in terms of the unknowns $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$:

$$\left.\begin{array}{l} \alpha_0=b_1; \\ \alpha_0 a_1+\alpha_1=b_2; \\ \cdots\cdots\cdots \\ \alpha_0 a_k+\alpha_1 a_{k-1}+\ldots+\alpha_k=b_{k+1}; \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_0 a_{N-1}+\alpha_1 a_{N-2}+\ldots+\alpha_{N-1}=b_N; \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_q a_N+\alpha_{q+1} a_{N-1}+\ldots+\alpha_{q+N}=0, \quad q\geqslant 0. \end{array}\right\} \tag{80}$$

The system (80) can be solved. From the first N equations we determine the sequence $\alpha_0, \alpha_1, \ldots\ldots, \alpha_{N-1}$. All subsequent α_g beginning with α_N are linear combinations of the N preceding unknowns, since the last equation in (80) can be written as

$$\alpha_g=\sum_{q=1}^{N}(-a_q)\,\alpha_{g-q}, \qquad g\geqslant N. \tag{81}$$

Accordingly, from (77) we can construct infinite series (78) on the basis of Eqs. (80). Condition (81) is written best in terms of the "infinite Hankel matrix:"

$$S=\left|\begin{array}{l} \alpha_0,\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n,\ \ldots \\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_{n+1},\ \ldots \\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \ldots,\ \alpha_{n+2},\ \ldots \\ \cdots\cdots\cdots\cdots\cdots \end{array}\right|=\|\alpha_{i+k}\|_0^\infty. \tag{82}$$

Let us consider the sequence of determinants of principal minors $D_r = |\alpha_{i+k}|_0^{r-1}$. From Eq. (81) we find

$$D_{N+1}=D_{N+2}=\ldots=D_r=0,\ r>N,\ D_N\neq 0.$$

We call the infinite Hankel matrix having this property a "matrix of finite rank N." It is clear from the construction that the rank of the Hankel matrix S is the degree of the polynomial in the denominator of the proper rational fraction (77).

Accordingly, each of the functions $S_i(w)$ can be unambiguously associated with an infinite Hankel matrix $S^{(i)}$, whose rank N_i is equal to the number of poles of the function $S_i(w)$, each counted a number of times equal to its order.[39] To construct the function $S_i(w)$ it is therefore sufficient to calculate the first $N_i + 1$ of the coefficients in expansion (78) and then use Eqs. (80) to find the numbers $b_1, b_2, \ldots, b_N, a_1, a_2, \ldots, a_N$, i.e., the functions $S_i(w)$ in the form (77).

We must now determine the limitations imposed on the expansion coefficients $\alpha_n^{(i)}$ in (78) by the crossing-symmetry IV' and the unitarity condition III'. Substituting (78) into condition IV' and equating the coefficients of identical powers of 1/w, we find

$$A_{ij}\alpha_n^{(j)}=(-1)^{n+1}\alpha_n^{(i)}. \tag{83}$$

The condition III' holds for each function $S_i(w)$ separately, so the equations for the coefficients $\alpha_n^{(i)}$ do not depend on the index i, which can thus be omitted. For the function S(1 − w) we have the expansion

$$S(1-w)=1+\sum_{n=0}^{\infty}\frac{\alpha_n}{(1-w)^{n+1}}. \tag{84}$$

We rewrite the series (84) in terms of the variable 1/w, using the obvious equation

$$\frac{1}{a-w}=(-1)\sum_{p=0}^{\infty}\frac{a^p}{w^{p+1}}. \tag{85}$$

Substituting the n-th derivative of (85), with $a = 1$, into (84), we find

$$S(1-w)=1+\sum_{n=0}^{\infty}\frac{\alpha_n'}{w^{n+1}};\ \alpha_n'=\sum_{p=0}^{n}(-1)^{p+1}C_p^n\alpha_p, \tag{86}$$

where $C_p^n = p(p-1)\ldots(p-n+1)/n!$ for $n \leq p$, $C_p^n = 0$ for $n > p$, and $C_n^0 = C_n^n = 1$. We see from (86) that each of the coefficients in the expansion of S(1 − w) in inverse powers of w can be expressed in terms of the coefficients of no higher order in the expansion of S(w) in the analogous series.

Substituting expansions (78) and (86) into the unitarity condition III', and setting the coefficients of identical powers of 1/w equal to zero, we find the following restrictions on the coefficients:

$$\alpha_n+\alpha_n'+\sum_{m=0}^{n-1}\alpha_m\alpha_{n-m-1}'=0, \qquad n=0,\ 1,\ 2,\ \ldots. \tag{87}$$

It is easy to see that the expansion coefficients in (78) with even n are independent parameters. Setting $\alpha_{-1} = 1$, we find from (87)

$$\begin{array}{c} \alpha_1=\alpha_0(\alpha_0+1)/2; \\ 2\alpha_3-2\alpha_2(\alpha_0+3/2)=-\alpha_0(\alpha_0+1)^2(\alpha_0+2)/4; \end{array} \tag{88}$$

The problem of constructing the functions $S_i(w)$ thus reduces to one of solving a system of equations of the form

$$\begin{array}{c} \alpha_{(2m+1)}^{(i)}=f\,[\alpha_{2m}^{(i)},\alpha_{2m-1}^{(i)},\ \ldots,\ \alpha_0^{(i)}];\ A_{ij}\alpha_n^{(j)}=(-1)^{n+1}\alpha_n^{(i)}; \\ \alpha_n^{(i)*}=\alpha_n^{(i)}. \end{array} \tag{89}$$

The form of the functions f does not depend on the index i and is governed by Eq. (87). The reality of the coefficients α follows from condition II'.

To determine the form of the equations which the coefficients α_0^i satisfy, we write these coefficients in a column:

$$\alpha_0=\left|\begin{array}{c} \alpha_0^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_0^{(n)} \end{array}\right|.$$

Here α_0^2 is understood to represent

$$\alpha_0^{(2)}=\left|\begin{array}{c} [\alpha_0^{(1)}]^2 \\ \cdot \\ \cdot \\ \cdot \\ [\alpha_0^{(n)}]^2 \end{array}\right|.$$

Acting on the first of Eqs. (88) with operator A, and using (83), we find

$$A\alpha_0^2 = \alpha_0^2 + 2\alpha_0; \quad A\alpha_0 = -\alpha_0. \tag{90}$$

If we have found a real solution of this system, the subsequent coefficients, $\alpha_2^{(i)}$, $\alpha_3^{(i)}$, etc., can be determined by solving the system of linear equations (88). Accordingly, the number of different sets of functions $S_i(w)$ satisfying the basic properties I'-IV' and the additional requirement (76) is no greater than the number of different real solutions of Eqs. (90).

To illustrate this method we show the results of the solution of problem I-IV for several particular cases.

1. The functions $S_i(w)$ satisfying the conditions I'-IV' for the crossing matrix

$$A_{1,1} = \begin{pmatrix} 1/3, & -1, & 5/3 \\ -1/3, & 1/2, & 5/6 \\ 1/3, & 1/2, & 1/6 \end{pmatrix} \tag{91}$$

are

$$S_1(w) = \frac{w-5/2}{w+3/2}; \quad S_2(w) = \frac{w-5/2}{w+3/2} \cdot \frac{w+1/2}{w-3/2}; \quad S_3(w) = \frac{w+1/2}{w-3/2}. \tag{92}$$

2. This method can be easily generalized to the case of the three-row crossing matrix $A_{l,1}$ in (28) (refs. 40, 41):

$$S_0(w) = \frac{w-(l+3/2)}{w+(l+1/2)}; \quad S_1(w) = \frac{w-(l+3/2)}{w+(l+1/2)} \cdot \frac{w+(l-1/2)}{w-(l+1/2)};$$
$$S_2(w) = \frac{w+(l-1/2)}{w-(l+1/2)}. \tag{93}$$

We note that these equations hold for any real l, not necessarily an integer.

3. It can be shown that the only solution (except for a trivial solution) with a finite number of poles for the three-row Chew – Low matrix (3) is

$$S_1(w) = \left[\frac{w(w-2)}{w^2-1}\right]^2;$$

$$S_2(w) = \frac{w(w-2)}{w^2-1} \cdot \frac{w}{w-1}; \quad S_3(w) = \left[\frac{w}{w-1}\right]^2. \tag{94}$$

4. For the four-row Chew – Low matrix

$$A = A_T \times A_J = \frac{1}{9} \begin{pmatrix} 1 & -4 & -4 & 16 \\ -2 & -1 & 8 & 4 \\ -2 & 8 & -1 & 4 \\ 4 & 2 & 2 & 1 \end{pmatrix} \tag{95}$$

problem I-IV has the following solutions with infinite numbers of poles:

Parameter	I	II	III
$S_{11}(w)$	$\frac{w(w-2)}{w^2-1} \cdot \frac{w(w-2)}{w^2-1}$	$\frac{w(w-2)}{w^2-1}$	$\frac{w(w-2)}{w^2-1}$
$S_{13}(w)$	$\frac{w(w-2)}{w^2-1} \cdot \frac{w}{w-1}$	$\frac{w(w-2)}{w^2-1}$	$\frac{w}{w-1}$
$S_{31}(w)$	$\frac{w}{w-1} \cdot \frac{w(w-2)}{w^2-1}$	$\frac{w}{w-1}$	$\frac{w(w-2)}{w^2-1}$
$S_{33}(w)$	$\frac{w}{w-1} \cdot \frac{w}{w-1}$	$\frac{w}{w-1}$	$\frac{w}{w-1}$

(96)

Solution I is the direct product of the two solutions corresponding to the two-row matrices A_T and A_J, discussed in Sec. 6. The structure of solutions II and III is obvious: In each of them, one of the solutions corresponding to the matrices A_T and A_J is a trivial solution (i.e., the column $\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right|$).

Working from these solutions with finite numbers of singularities, we can construct solutions containing meromorphic functions. For this purpose we must use the arbitrariness in (75), which permits functions satisfying conditions I'-IV'. Using this arbitrariness, we can attempt to satisfy the particular conditions mentioned above. However, we note that in this approach we cannot construct from solutions (94) and (95) a physical solution for the p waves of πN scattering, namely, a solution with a Born pole. The physical solution must therefore contain an infinite number of singularities in the w plane.

9. Example of functions $S_i(w)$ with an infinite number of singularities. The method described in Sec. 8 is obviously inapplicable for functions having an infinite number of singularities.[5)] The possible existence of such solutions was shown in ref. 43 for the case of the three-row crossing matrix $A_{1,1}$ in (91). A geometric interpretation of problem I'-IV' turns out to be useful in finding such solutions.[44] We introduce the ratio

$$X(w) = S_1(w)/S_2(w); \quad Y(w) = S_3(w)/S_2(w).$$

Then the problem reduces to one of solving a system of nonlinear finite-difference equations of the type

$$X(w+1) = f[X(w), Y(w)];$$
$$Y(w+1) = \varphi[X(w), Y(w)], \tag{98}$$

where f and φ are birational functions. System (98) can be interpreted as the mapping of the $\{X(w), Y(w)\}$ plane into the $\{X(w+1), Y(w+1)\}$ plane. The fixed points of this mapping are determined from a fourth-degree algebraic equation, which has a double root X = 1 (or Y = 1) and two different real roots $X_{1,2} = (7 \pm 3\sqrt{5})/2$. Since there are no general methods for solving Eqs. (98), it is natural to attempt to find solutions corresponding to an extremely simple mapping – the piecewise-linear mapping

$$X(w+1) = [\alpha X(w) - \beta]/[X(w) + \gamma]. \tag{99}$$

In the case of coincident fixed points, the transformation (99) is called "parabolic." It can be shown[43] that the solution with a finite number of poles in (92) corresponds to this case. When there are two different fixed points, the transformation (99) is called "hyperbolic" and leads to two solutions with an infinite number of poles:

$$\left.\begin{aligned} 1)\ S(w) &= \begin{pmatrix} \dfrac{\operatorname{sh}\frac{c}{2}\left(w-\frac{3}{2}\right)}{\operatorname{sh}\frac{c}{2}\left(w+\frac{1}{2}\right)} \\ 1 \\ -1 \end{pmatrix} \cdot \varphi_-(w); \\ 2)\ S(w) &= \begin{pmatrix} \dfrac{\operatorname{ch}\frac{c}{2}\left(w-\frac{3}{2}\right)}{\operatorname{ch}\frac{c}{2}\left(w+\frac{1}{2}\right)} \\ 1 \\ -1 \end{pmatrix} \cdot \varphi_+(w), \end{aligned}\right\} \tag{100}$$

where $c = \ln (1/2)(7 + 3\sqrt{5})$ and

$$\varphi_{\pm}(w) = \operatorname{tg}\frac{\pi}{2}w \frac{\prod_{k=1}^{\infty}\left\{\operatorname{ch}\frac{c}{2}\left(2k-\frac{1}{2}\right) \pm \operatorname{ch}\frac{c}{2}\left[2w-1-\frac{1}{2}(-1)^k\right]\right\}}{\prod_{k=1}^{\infty}\left\{\operatorname{ch}\frac{c}{2}\left(2k-\frac{1}{2}\right) \pm \operatorname{ch}\frac{c}{2}\left[2w-1+\frac{1}{2}(-1)^k\right]\right\}}.$$

The poles (zeros) of the functions $\varphi_{\pm}(w)$ for solutions (100) lie on the lines $\mathrm{Re}\, w = \pm k + 1/2 - (1/4)[(-1)^k \pm 1]$, $\mathrm{Re}\, w = \pm k + 1/2 + (1/4)[(-1)^k \mp 1]$ at a spacing of $i(2\pi/c)$ and have an accumulation point at infinity. The system of poles (zeros) is common to $S_1(w)$ and $S_3(w)$, and the ratio $S_1(w)/S_2(w)$ has an infinite number of poles (zeros) on the line $\mathrm{Re}\, w = -1/2$ ($\mathrm{Re}\, w = 3/2$) with a spacing $i(2\pi/c)$. The zeros and poles of the ratio $S_1(w)/S_2(w)$ do not contribute to S_1, since they are offset by the corresponding poles and zeros of the function $\varphi(w)$. Accordingly, the functions $S_i(w)$ for solutions (100) do not have poles in the strip $-1/2 \le \mathrm{Re}\, w \le 1/2$, i.e., on the physical sheet [except for the poles which can be introduced by means of the functions $\beta(w)$ and $D(w)$ in (75)]. However, there are an infinite number of zeros and poles on the nonphysical sheets. We note that the functions $S_i(w)$ are bounded by a constant on the physical sheet. For the three-row crossing matrix A^{C-L} in (3) the only solution is parabolic, i.e., the solution having a finite number of singularities, (94).

Turning to a brief discussion of the solutions found in the last two sections, we note that their explicit forms imply the existence of functional relationships of a polynomial type. For the matrix $A_{1,1}$, solution (92) satisfies the property $S_2(w) = S_1(w)\cdot S_3(w)$, while solutions (100) satisfy the property $S_2(w) = -S_3(w)$. For the three-row matrix A^{C-L} there is a single relationship of this type:

$$S_2^2 = S_1(w)\cdot S_3(w), \qquad (101)$$

corresponding to solution (94). The functional relationships here play the same role as that played by the first integrals in the theory of differential equations, which can be used to reduce the order of an equation. Accordingly, for a system of difference equations like (98) we can attempt to find relationships which would significantly simplify the solution of these equations.

A method has been developed for constructing functional relationships of the polynomial type.[45] For the matrices $A_{1,1}$ and A^{C-L}, the only polynomial relationships have been calculated here. The mathematical literature[46] reveals a method for locally (i.e., in the neighborhood of a fixed point) constructing arbitrary functional relationships. However, these methods cannot be used effectively in our case.[47] Nevertheless, as is shown below, the analogy between differential and difference equations turns out to be useful.

10. Existence theorems for the Chew–Low equations.

All the illustrative exact solutions above have been based on a special circumstance: the linearization of Eqs. (2). When we leave this special class we are immediately confronted with a nonlinear problem in some formulation or other. For example, the system of equations (III, IV) can be solved for any crossing matrix A, but the very existence of a solution requires a special proof.

In terms of the concepts involved, the first such attempts were those by Low and Huang[48] in studies of "bootstrap solutions" (Sec. 11). Lovelace[49] subsequently analyzed the number of parameters governing a solution. Treating equations like the Chew–Low equations as equivalent to a nonlinear operator in a Banach vector space, he showed that the number of independent parameters is governed by the index of the Fréchet derivative of this operator. He did not prove the existence of the solution itself.

Warnock[50] and McDaniel and Warnock[51] proved the existence of solutions on the basis of theorems regarding fixed points or the method of contractive mappings within the framework of functional analysis. They analyzed three formulations of the Chew–Low equations: the Chew–Low equation itself (2), the N/D formulation, and the equation for the inverse amplitudes, $1/h_i(\omega)$. They assumed the cutoff function to satisfy the Hölder boundary condition

$$|\rho(t) - \rho(\tau)| < k|t-\tau|^{\mu}, \; k, \mu > 0,$$

where

$$\rho(t) = (q^3/12\pi)\, u(q^2); \; t = 1/\omega.$$

They proved the existence and uniqueness of solutions for any set of CDD poles with small residues and with an additional restriction on the magnitude of the coupling constant. These restrictions for the three specified formulations of the Chew–Low equations are $12\pi \sup_i |\lambda_i| = 0.014$, 0.11, and 0.013. For the physical value of the coupling constant, $f^2 \approx 0.087$, this value is approximately equal to 9. However, the actual construction of iterative solutions of Eqs. (2) obviously requires even smaller values of the coupling constant. In this approach the restrictions on the coupling constant are related to the analytic methods used; a possible analytic continuation to larger coupling constants was not actually carried out.

An analogous approach was used by Atkinson[52] to prove the existence of amplitudes which satisfy the Mandelstam representation and certain requirements which follow from unitarity. Although these papers were afflicted with the same difficulties involving the value of the coupling constant, they show that progress reached in one direction can be transferred to another direction. Related papers are those mentioned above, in which the Newton–Kantorovich method was used to solve the equation for low-energy $\pi\pi$ scattering[25] or used for nonlinear systems of algebraic equations equivalent to the Chew–Low equations.[23,24]

We believe that a natural approach to establishing existence theorems for the Chew–Low equations would be to start from the basic requirements I-IV and only then use the requirements fixing the physical solution. Below we use several concepts of the theory of ordinary differential equations[53] to analyze the Chew–Low equations in difference formulation I'-IV'. In this approach successful use can be made of the concepts of a fixed (stationary) point and an invariant manifold, first introduced by Poincaré.[54]

Let us consider Chew–Low equations (2) with a three-row crossing matrix A^{C-L} on an infinite-sheet Riemann surface w, given in (58). It follows from crossing-symmetry condition IV' that

$$S(w)=s_1(w)\begin{vmatrix}1\\1\\1\end{vmatrix}+s_2(w)\begin{vmatrix}4\\-2\\1\end{vmatrix}+\psi(w)\begin{vmatrix}-4\\-1\\2\end{vmatrix}, \quad (102)$$

where

$$s_1(w)=s_1(-w);\; s_2(w)=s_2(-w);\; \psi(w)=-\psi(-w).$$

We introduce the variables

$$s_1(w);\; y(w)=s_2(w)/s_1(w);\; x(w)=\psi(w)/s_1(w). \quad (103)$$

In terms of these variables we easily find the following system of equations, which is equivalent to conditions III'–IV':

$$\left.\begin{array}{l} x'=F(x, y);\; F(x, y)=\dfrac{x+2x^2-xy-2y^2}{1+3x+3y-2x^2-3xy-2y^2};\\ y'=-F(y, x);\; x(w)=-x(-w);\; y(w)=y(-w); \end{array}\right\} \quad (104)$$

$$s_1's_1(1-2y+x)(1-2y'-x')=1;\; s_1(w)=s_1(-w); \quad (105)$$

$$S(w)=s_1(w)\left\{\begin{vmatrix}1\\1\\1\end{vmatrix}+y(w)\begin{vmatrix}4\\-2\\1\end{vmatrix}+x(w)\begin{vmatrix}-4\\-1\\2\end{vmatrix}\right\}. \quad (106)$$

In Eqs. (104) and (105) the primed functions correspond to argument w + 1, while the unprimed functions correspond to argument w. Equations (104) are independent of Eqs. (105), so they can be solved successively. It is essentially this approach which led to solution (94) of the Chew–Low equations. Turning away from the functions $\beta(w)$ and D(w) of (75) in this solution, we analyze this solution on the x, y phase plane by a method similar to that used to analyze the solution of autonomous systems of differential equations in phase spaces. The solution is mapped onto the phase plane as a curve all points of which are invariant with respect to the substitution $(x, y) \to (x', y')$. This curve corresponds to functional relationship (101) and, in terms of the variables x, y, is

$$y-x^2=0. \quad (107)$$

Parabola (107) touches the y axis at y = 0. The direction of the tangent to the invariant curve at this point follows from the linear approximation of Eqs. (104): x' = x, y' = y. In other words, the only invariant lines are the coordinate axes. Solution (94) is characterized by the following function as well as by curve (107):

$$s_1(1-y^2)=1, \quad (108)$$

which is found from Eq. (106). The x, y phase plane is not adequate for a geometric mapping of function (108); it is necessary to transform to the phase space x, y, s_1 of system (104), (105). In phase space, Eqs. (107) and (108) are represented by two surfaces. Their intersection gives the line which maps solution (94). The one-dimensional space associated with this line is the simplest nontrivial invariant manifold of the Chew–Low equation.

It is natural to ask whether invariant manifolds of higher dimensionalities exist. In this case the invariant manifold can be a surface in three-dimensional space.[47] The equation of the surface is conveniently written in the form

$$s_1=\Phi(y, x^2). \quad (109)$$

The quadratic functional dependence of Φ on x automatically incorporates the invariance of the surface with respect to the crossing-symmetry transformation. The physical solution of Eq. (2) has the property $S_i(z) = 1 + O(z^2-1)^{3/2}$ as $z \to 1$. Accordingly, surface (109) must have a limiting point in phase space x, y, s_1 with coordinates (0, 0, 1). It is easy to see that surface (109) can be tangent to the y, s_1 coordinate plane at the point (0, 0, 1) or to any plane of the sheaf of planes orthogonal to the y, s_1 coordinate plane and containing the point (0, 0, 1). The function Φ obeys the equation

$$\Phi(y, x^2)\,\Phi(y', x'^2)(1-2y+x)(1-2y'-x')=1. \quad (110)$$

It can be shown that on invariant curve (107) the function

$$\Phi(y, x^2)=1/(1-y)^2 \quad (111)$$

is a solution of Eq. (110).

The solution of Eq. (110) can be sought as a double series x and y:

$$\Phi(y, x^2)=\sum_{m,\, n\geqslant 0} a_{2m,\, n}x^{2m}y^n. \quad (112)$$

In the solution of the equations for the unknown coefficients $a_{2m,n}$ a situation arises which is conceptually completely analogous to that which arose in Sec. 8 in the use of series in one variable. Technically, however, it is of course vastly more complicated. The convergence and construction of series (112) were analyzed in ref. 55. It was proved that this series converges in a certain rectangle containing the point (0, 0, 1), and it was therefore proved that a two-dimensional locally invariant manifold, invariant surface (109), exists. This apparently exhausts the invariant manifolds containing the limiting point (0, 0, 1). Actually, the arbitrariness of (75) and the two independent coordinates of the invariant surface lead to a situation in which the general solution depends on three arbitrary functions. In this manner the arbitrariness achieved is the same as that specified by crossing-symmetry condition IV.

The existence of invariant manifold (109) still does not prove the existence of the solution of the Chew–Low equations (2). This proof requires an analysis of the motion of a point over surface (109), given by Eqs. (104). Equations (104) have three different fixed points. The physical solution of the Chew – Low equation must be mapped by a line arriving at the point x = 0, y = 0. In the linear approximation the motion of any point from a small neighborhood of the origin reduces to mappings with respect to the x axis, and the initial point does not approach the origin. The position of a point with respect to the origin is governed by terms of higher order in x, y. To eliminate oscillations due to the linear terms it is convenient to analyze an iterative transformation F. Representing $F^{(n)}$ as series in increasing powers of x and y,

$$\left.\begin{aligned} x^{(k)} &= x + \sum_{m+n\geqslant 2} a_{m,n}^{(k)} x^m y^n;\\ y^{(k)} &= (-1)^k y + \sum_{m+n\geqslant 2} b_{m,n}^{(k)} x^m y^n, \end{aligned}\right\} \qquad (113)$$

we can show that the coefficients $a_{m,n}^{(k)}$ and $b_{m,n}^{(k)}$ can be calculated step by step and that the series converge.[56] It is easy to see that the set of orders of transformation F forms an infinite Abelian group G with the following multiplication law:

$$F^{(m)}F^{(n)} = F^{(m+n)}. \qquad (114)$$

In the coordinates x, y the group multiplication law becomes

$$\left.\begin{aligned} x^{(n)}(x^{(m)}, y^{(m)}) &= x^{(m)}(x^{(n)}, y^{(n)}) = x^{(m+n)}(x, y);\\ y^{(n)}(x^{(m)}, y^{(m)}) &= y^{(m)}(x^{(n)}, y^{(n)}) = y^{(m+n)}(x, y). \end{aligned}\right\} \qquad (115)$$

Subgroup G_1 of group G, corresponding to the odd powers of the initial transformation, has an important property: $(-1)^k \equiv 1$ within subgroup G_1. Accordingly, series (113) are polynomials in powers of x, y, k which identically satisfy Eqs. (115), and in subgroup G_1 these series can be discussed for all real values of k, not only the integer values. From Eqs. (115) we find the group equations

$$\left.\begin{aligned} dx^{(k)}/dk &= \delta x(x^{(k)}, y^{(k)});\ \delta x = dx^{(l)}(x^k, y^k)/dl\,|_{l=0};\\ dy^{(k)}/dk &= \delta y(x^{(k)}, y^{(k)});\\ \delta y &= dy^{(l)}(x^k, y^k)/dl\,|_{l=0}. \end{aligned}\right\} \qquad (116)$$

It follows from Eqs. (116) that the trajectory of a point in the phase space (x, y) is described by the differential equation

$$dy^{(k)}/dx^{(k)} = \delta y(x^{(k)}, y^{(k)})/\delta x(x^{(k)}, y^{(k)}). \qquad (117)$$

We can use Eq. (117) to analyze the behavior of an arbitrary point near the origin and to analyze the behavior of trajectories along which the point moves under the influence of mapping F (ref. 56). We simply state the results: All the trajectories can be classified into one of two types; the first type includes trajectories of the parabolic type, which arrive at or emerge from the origin (denoted by l in Fig. 4). Trajectories the second type are hyperbolic (L). The hyperbolic trajectories do not arrive at the origin. The motion of a point near the origin can be described as follows: If the initial point lies in the parabolic sector, there are always two trajectories l_0 and l_1 (Fig. 4) along which the initial point approaches (or leaves) the origin under the influence of transformation F, jumping from one trajectory to another. If the initial point lies in the hyperbolic sector, there are always two trajectories L_0 and L_1 along which the initial point moves under the influence of transformation F, jumping from one trajectory to another.

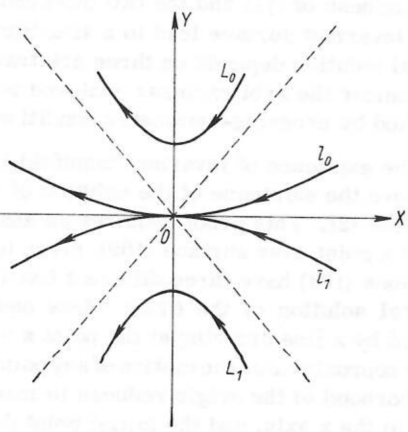

Fig. 4

We can now discuss the existence of a solution of the Chew–Low equation having the correct cutoff behavior and Born poles. This solution must pass through the point with coordinates x = 0, y = 0. This point can be approached by a finite number of mappings (104), and it can be shown the mapping of the Born term falls in a family of parabolic trajectories.

This concludes the proof of the existence of a physical solution for boundary-value problem I–IV for the function S_i without reference to the magnitude of the coupling constant f^2.

III. APPROXIMATE METHODS WITHIN THE FRAMEWORK OF THE STATIC MODEL

11. Bootstrap solutions in fixed-source models. Using static-model equation (2) we can pose questions regarding the analytic formulation and validity testing of various hypotheses of the dispersion approach; these questions are difficult to answer within the framework of the total S matrix.[57] One of these hypotheses – the hypothesis of the Chew–Mandelstam bootstrap mechanism[58] – is that hadrons are not elementary particles, but can instead be constructed from each other in such a manner that all masses and coupling constants are determined. The familiar applications of this mechanism are in ρ-meson production,[59] in the explanation of the {3,3} resonance in terms of the exchange of a nucleon in the u channel,[60] and in the explanation for other pion–nucleon resonances.[62]

Low and Huang[48] posed the question of the existence of bootstrap solutions in several models with a fixed source. As the bootstrap condition they used the Levinson theorem

$$\Delta\delta_\alpha(\omega) \equiv \delta_\alpha(\infty) - \delta_\alpha(1) = -\pi b_\alpha, \qquad (118)$$

where b_α is the number of bound states in channel α. The choice of the Levinson theorem as an additional condition for selecting bootstrap solutions is based on the analogy with potential scattering, in which case the bound states are due to the interaction and vanish when the interaction is turned off. In the N/D method for solving the dispersion relations, the functions N and D are found unambiguously by means of additional requirements on the D function: the requirement that this function have no poles and the requirement of a certain asymptotic behavior at infinity,

$$\lim_{\omega\to\infty} [\log D(\omega)/\log \omega] = 0.$$

Then the Levinson theorem is equivalent to the assertion of a one-to-one mutual correspondence between bound states and zeros of the D function. The N function with its crossing-channel poles assumes the role of a potential; this completes the establishment of the analogy mentioned above.

A test of the condition depends strongly on the asymptotic behavior of the functions $h_i(\omega)$. Equations (2) are written without subtractions. When there is one subtraction, constants c_i appear on the right side of these equations; according to the crossing-symmetry condition, these constants are related by the condition $\sum_j c_j A_{ij} = c_i$, which reduces the number of independent parameters. The number of subtractions also depends on the source function $u(q^2)$, which was chosen in ref. 48 in the form

$$u(q^2) = k^{2c}/(q^2 + k^2)^c. \quad (119)$$

Here $c = 0, 1, 2, \ldots$; and k is the cutoff momentum ($k > 1$). Accordingly, the total number of arbitrary parameters is $2 + n$, where n is the number of subtractions.

The problem is formulated analytically as follows: Given a specific static model, does a solution of the Chew–Low equations exist which satisfies the Levinson theorem (118) for any choice of conditions at infinity, including the choice of cutoff function (119) and any number of subtractions [with the restriction that a subtraction not be necessary for $u(q^2) = 1$]? Low and Huang showed that solutions of the Chew–Low equations without subtractions which satisfy bootstrap solution condition (118) do not exist in the neutral-pseudoscalar, charged-scalar, and symmetric-scalar models. When there is a single subtraction, there are several bootstrap solutions, differing in the number and positions of the bound states. To determine a unique solution we must impose two physical requirements: 1) The fixed nucleon must be a bound state in a channel with the appropriate quantum numbers; 2) there are no bound states with masses lower than the mass of a nucleon. In the special class of solutions of the symmetric-pseudoscalar theory [see solutions (96)] the bootstrap solutions are related in a simple manner to those for the neutral-pseudoscalar theory. Among these solutions, however, there are none of physical interest, i.e., which would correspond to a bound state in the $T = 1/2$, $J = 1/2$ channel.

The results of Low and Huang were generalized in ref. 61 to the case of two-row and four-row crossing matrices of a general type. The method is not related to the exact solutions and is based on estimates of the asymptotic behavior of the functions $h_i(z)$.

Generalization of the static model to the case of SU(3) symmetry leads to the prediction that resonant states arising in the scattering of two octets as a result of the exchange of a baryon in the u channel will belong to a decaplet.[63]

12. Attempts to predict internal symmetries on the basis of the static model.

According to one hypothesis, the internal-symmetry group for strong interactions is not an additional assumption but a consequence of dispersion theory.[64,65] The static model is very convenient for testing this hypothesis. The requirements that the S matrix be invariant with respect to a definite internal-symmetry group are incorporated in the concrete form of the crossing matrix in this model. The following approach can therefore be formulated for the problem of internal symmetries on the basis of the static model:[66] For which parameters of the crossing matrix satisfying only basic requirements (21) and (22) do exact solutions exist for Chew–Low equations (2)?

If it turned out that an exact solution with, say, two-row matrix (26) exists only for the parameter $a = \pm 1/(2n + 1)$, $n = 0, 1, 2, \ldots$, we could talk in terms of a prediction of SU(2) symmetry. Solutions of Chew–Low equation (2) with an arbitrary crossing matrix (26) were found above (Sec. 6). A general solution of Eqs. (2) is not known for a matrix of arbitrary order, and particular solutions, e.g., solution (93), again do not fix the internal symmetry group.

The next step in the analysis of this questions is to use bootstrap condition (118). Cunningham[67] asserted that the Chew–Low equation with a two-row crossing matrix has a solution which satisfies condition (118) only for the case of SU(2) symmetry. In the proof he used a solution with a function $\varphi_l(w)$ in the form (65). The Levinson theorem requires a finite number of poles in the function $S_i(w)$, and this situation can be arranged by means of the Blaschke function $D(w)$ in (46). The positions of its zeros are governed by equations analogous to (69) and lead to the circumstance that the infinite product does not converge. However, condition (67) allows choice of the functions $D(w)$ from a broader class. For example, the solution can be chosen in terms of bounded functions in the form[66]

$$D(z) \to A_0(z)\, D(z), \quad (120)$$

where

$$\left.\begin{aligned} A_0(x) &= \frac{[1 - i(z^2-1)^{1/2}]}{(l+1/2)\, z} \\ &\times \exp\left\{\frac{i(z^2-1)^{1/2}}{\pi} \int_1^{\infty} \frac{\ln[1+F^2(x')]\, x'\, dx'}{(x'^2-1)^{1/2}(x'^2-z^2)}\right\}; \\ F(x) &= \frac{1}{(l+1/2)}\left[\frac{1}{\pi}\ln[x+(x^2-1)^{1/2}] + \frac{x}{(x^2-1)^{1/2}}\beta(x)\right]. \end{aligned}\right\} \quad (121)$$

The function $A_0(z)$, which has a single pole at $z = 0$, was used previously[68] to construct a solution of the Chew–Low equation with a single subtraction which satisfies condition (118). Accordingly, the bootstrap condition and the Chew–Low equation again do not lead to any predictions regarding the internal-symmetry group.

In another approach[69,70] to this problem, inelastic channels are incorporated in an analysis of the static model. It is assumed that the analytic properties are the same for the elastic and inelastic amplitudes and that the latter have a definite ω parity. It turns out that in this model the inelastic amplitudes are related in such manner that the vanishing of any of them implies the vanishing of the other. Working on the basis of this result, Ehrhardt and Fairlie[70] treated the two-channel problem of finding the elements of the S matrix with a unitarity condition of the form

$$\sum_{j=1}^{2} S_{ij}(w)\, S_{jk}(1-w) = \delta_{ik},$$

where

$$S_{12}(w)=S_{21}(w).$$

A crossing-symmetry condition is not imposed on the functions $S_{11}(w)$ and $S_{22}(w)$. Assuming, first, that the inelastic amplitudes are of odd parity with respect to w and, second, that the solutions can be expanded in series in 1/w, Ehrhardt and Fairlie found that the solution they found in first order in 1/w agrees with the solution of the static problem with SU(2) symmetry [see solution (62)]. Terms of higher order in 1/w do not have this property, so we cannot agree with these investigators regarding a prediction of SU(2) symmetry. Analogous difficulties can be found in models having more channels. Despite these difficulties, these analyses are worthwhile, for they have shown that the incorporation of inelastic channels significantly extends the scope of the problem and allows a novel formulation of the question of the origin of symmetry groups.

13. Algebraic formulation of strong-coupling theory in the static limit. We turn now to yet another interesting application of the static model – in the "strong-coupling limit." It was shown on the basic of the classical theory for meson–nucleon scattering that in the strong-coupling limit there are an infinite number of isobars with quantum numbers J = T (refs. 71, 72), whose energies differ by $\Delta M \sim 1/g^2$. The existence of an infinite number of isobars and their dependence on the coupling constant g can be easily incorporated in the S matrix formulation of the static model.

We use Wick's operator form for the equations, from which the Chew – Low equations follow in the single-meson approximation:

$$f_{BA}(E)=g_B\frac{1}{\vec{M}-E}g_A+g_A\frac{1}{M-\overleftarrow{M}-\vec{M}+E}g_B, \qquad (122)$$

where f is the amplitude for the scattering of meson A by the source (whose quantum numbers are omitted), E is the total energy of the system, and $\overleftarrow{M}$ and $\vec{M}$ are the mass operators. In this formulation the cutoff function is also omitted: $u^2(q^2) = 1$. In the next step, a definite energy dependence $f_{BA}(E)$ is assumed for the amplitudes:

$$f_{BA}=-\frac{\Lambda_{BA}}{\mu+iq}, \qquad (123)$$

where $\omega = E - \vec{M}$, $\omega^2 = q^2 + \mu^2$, and Λ_{BA} are constants. The basic equations of the strong-coupling theory are found from Eqs. (123) by expanding $f_{BA}(E)$ in a series in $1/\omega$ and requiring that the first two terms of this series vanish and that the amplitudes satisfy the unitarity conditions. These equations have the form

$$[g_B, g_A]=0; \qquad (124)$$

$$\Lambda_{BA}=\left[g_B, \left[\frac{\Delta M}{2\mu}, g_A\right]\right]; \qquad (125)$$

$$\Lambda^2=\Lambda. \qquad (126)$$

We see that Eqs. (124) and (126) are of order g^2 and g^0, respectively; the coefficients of the higher-order terms in the expansion in $1/\omega$ tend toward zero in the strong-coupling limit. From a different point of view, Eqs. (124)-(126) are a consequence of the matching of initial equations (122) and hypothesis (123) with the energy dependences of the amplitudes, $f_{BA}(E)$. The strong-coupling limit allows us to restrict the treatment to the first two equations in the chain of equations which arise in the expansion in powers of $1/\omega$.

Algebraic equations (124)-(126) constitute a second-order model in the chain (general principles) → (model), which is appealed to in an analysis of the quantum-number dependence of the isobars. The crossing matrix, which plays an important role in the original Chew – Low equations, is not involved in the formulation of this model. Accordingly, the symmetry group with respect to which the interaction is assumed invariant is actually not fixed. This latter circumstance is responsible for the richness of the content of this model, which can be analyzed by purely group-theoretical method and which lies outside the scope of this review. This question is analyzed in detail by Goebel.[73]

IV. COMPARISON OF THE STATIC-MODEL PREDICTIONS FOR s AND p WAVES OF πN SCATTERING WITH EXPERIMENT

14. Quantitative s-wave theory for πN scattering. Lacking a theory for strong interactions, we are forced to resort to semiphenomenological and phenomenological equations in a quantitative analysis of processes involving hadrons. A phase-shift analysis of experimental data on elastic πN scattering, for example, presents an urgent need for an analogous technique. At present several phase-shift analyses have been carried out over broad energy ranges on the basis of special partial-wave parametrizations. Various assumptions are used in choosing the parametric equations: correspondence with the quantum-mechanical results regarding the cutoff behavior of the phase shifts of the partial-wave expansion near resonances,[74] the existence in the partial waves of cuts following from the dual spectral representations of the scattering amplitudes,[75] crossing symmetry,[76] etc. A justified parametrization of partial waves can significantly reduce the number of unknown parameters and lead to an unambiguous solution of the phase-shift analysis.

The scattering phase shifts are first subjected to the parametrization

$$\delta_l(q)\approx a_l q^{2l+1}. \qquad (127)$$

The first estimates of the s-wave scattering lengths were found by this method:[77] $a_1 = 0.16$ and $a_3 = -0.11$. A natural generalization of Eq. (127) is to incorporate higher powers of q while retaining the odd parity of the phase shift as a function of the momentum:[78]

$$\delta_l(q)=q^{2l+1}(a_l+a_l^{(1)}q^2+a_l^{(2)}q^4+\ldots). \qquad (128)$$

Successful use of Eq. (128) depends on the convergence radius of the expansion.

Use of the effective range theory for the parametrization of the partial waves is more justified than parametrization of the phase shift itself. For a superposition of Yukawa potentials the series

$$q^{2l+1}\operatorname{ctg}\delta_l(q)=\sum_{n\geqslant 0}a_l^{(n)}q^{2n}\tag{129}$$

converges in the circle $|q| < m/2$, where m is the smallest mass in the superposition of Yukawa potentials.[78] From Eq. (129) we easily find

$$\delta_l(q)=\frac{1}{2i}\ln\frac{\sum\limits_n a_l^{(n)}q^{2n}-iq^{2l+1}}{\sum\limits_n a_l^{(n)}q^{2n}+iq^{2l+1}},\tag{130}$$

which contains logarithmic branch points, which may lie within the circle $|q| < m/2$. Accordingly, the convergence radius for expansion (128) is no larger than the convergence radius for expansion (129), for which the Mandelstam representation gives a unit value. The convergence region of expansion (129) can be expanded by choosing a different expansion center.[36] However, the practical advantage of this approach has not yet been demonstrated.

The most thorough analysis of πN scattering phases on the basis of effective range theory was that of McKinley,[79] who also pointed out certain disadvantages of this method. For s phases the following equations were found:

$$\left.\begin{aligned}\operatorname{tg}\delta_1&=q\,(0.17-0.04q^2+0.01q^4);\\ \operatorname{tg}\delta_3&=q\,(-0.10-0.036q^2+0.003q^4),\end{aligned}\right\}\tag{131}$$

which hold up to 600 MeV in the laboratory system. We note that s waves are treated independently in these equations. Actually, they are coupled by the crossing-symmetry condition. The importance of this circumstance was first pointed out by Cini et al.[80] The dispersion relations for zero-angle scattering allow us to partially take into account the coupling of πN scattering s waves. The πN scattering s wave was analyzed on the basis of dispersion relations in ref. 81, where the following equations were found:

$$\left.\begin{aligned}\frac{\sin 2\delta_1-\sin 2\delta_3}{2q}\cdot\frac{W}{M+1}&=(a_1-a_3)\,E+c^{(-)}(E)\,q^2(E);\\ \frac{\sin 2\delta_1+\sin 2\delta_3}{2q}\cdot\frac{W}{M+1}&=(a_1+2a_3)+c^{(+)}(E)\,q^2(E).\end{aligned}\right\}\tag{132}$$

The functions $c^{(\pm)}(E)$ in these equations can be assumed constant. Calculations based on Eqs. (132) yield $a_1 = -0.171 \pm 0.005$, $a_3 = -0.088 \pm 0.004$, $c^{(-)} = -0.094 \pm 0.013$, and $c^{(+)} = -0.096 \pm 0.026$. We also note that the approximate scattering lengths $a_1 + 2a_3 \approx 0$ and $a_1 - a_3 \approx 0.3$ were found on the basis of the PCAC hypotheses and current algebra.[82]

All of these approaches to a quantitative prediction of πN scattering s waves fail to incorporate one of the two fundamental requirements of two-particle unitarity and crossing symmetry. A good basis for a quantitative description of the πN scattering s waves is the static model, in which these requirements are incorporated.

In this case the problem is described by equations analogous to the Chew–Low equations (2), except that matrix (3) must be replaced by matrix (47) and we must set $\lambda_i \equiv 0$. A general solution [see Eq. (62)] is known for this problem; we write it in the form

$$\left.\begin{aligned}S_1(\omega)&=\frac{\Phi(\omega)\,[\Phi(\omega)-2]}{\Phi^2(\omega)-1}\,D(\omega);\\ S_2(\omega)&=\frac{\Phi(\omega)}{\Phi(\omega)-1}\,D(\omega).\end{aligned}\right\}\tag{133}$$

Here the functions $\Phi(\omega)$ and $D(\omega)$ are given by Eqs. (57) and (66). In solution (133) there is an arbitrariness in the form of the two functions $\beta(\omega)$ and $D(\omega)$, which obey conditions (67). Even very simple assumptions regarding these functions lead to an explanation for the strong dependence of the scattering lengths a_i on the isospin:[34] $a_1 + 2a_3 \approx 0$. This dependence remained puzzling for a long time. The approximate equality $a_1 \approx a_3$ is based solely on the crossing-symmetry property.[83] At present this dependence is related to the ρ meson.[84] In the static model this dependence finds a natural explanation on the basis of analyticity, unitarity, and crossing symmetry.

Meshcheryakov[85] analyzed the experimental s-wave data for low energies on the basis of Eqs. (133). The form of the arbitrary functions was chosen solely on the basis of a good description of the experimental data over a range as broad as possible. Several models have now furnished a good explanation for the role of ρ and σ mesons in the low-energy behavior of the phases.[86,87] This influence can be incorporated in the static model through an appropriate choice of the functions $\beta(\omega)$ and $D(\omega)$. An analysis of this type was carried out in ref. 88. Good agreement with the experimental data is found with the following (evidently

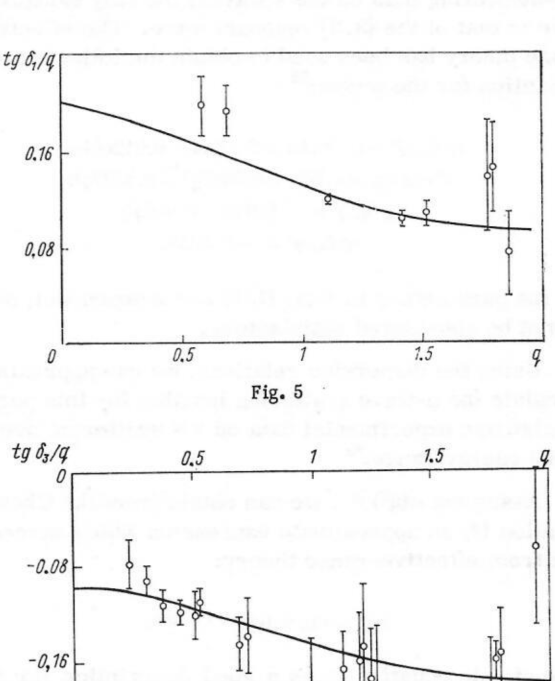

Fig. 5

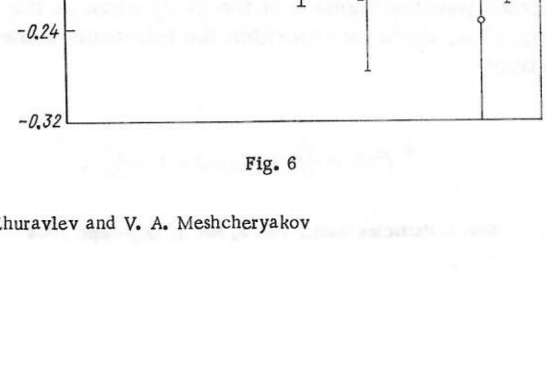

Fig. 6

not unique) choice of the functions $\beta(\omega)$ and $D(\omega)$:

$$\left.\begin{aligned} D(\omega) &= \frac{(1-iq/q_\sigma)(1-iq/q_\rho)^2(1+iq/a_0)(1+iq/a_1)^2}{(1+iq/q_\sigma)(1+iq/q_\rho)^2(1-iq/a_0)(1-iq/a_1)^2}; \\ \beta(\omega) &= \frac{\omega}{q^2}\cdot\frac{b_0+b_1q^2+b_2q^4+b_3q^6}{(1+q^2/q_\sigma^2)}, \end{aligned}\right\} \quad (134)$$

where

$$q_{\rho,\sigma} = m_{\rho,\sigma}/2,\ m_\rho = 765\ \text{MeV},\ m_\sigma = 410\ \text{MeV}.$$

The parameters in these equations were determined by the method of least squares. For the scattering lengths the values $a_1 - a_3 = 0.299 \pm 0.011$ and $a_1 + 2a_3 = 0$ were found, in excellent agreement with the results of refs. 89, 90. The theoretical curves of $\tan\delta_1/q$ and $\tan\delta_3/q$ (Figs. 5 and 6) show the good quantitative description of the energy dependence of the πN-scattering s phase shifts at energies up to 260 MeV in the laboratory system. It is interesting to compare the physical pattern of the πN-scattering s phase shifts with the approach based on effective Lagrangians. Calculating the contributions of the poles from the ρ and σ mesons and calculating the effect of remote singularities on the scattering lengths, we can find an unambiguous correspondence between them and the contributions of the corresponding Lagrangians; the contact interaction is responsible for the inclusion of the remote singularities.

15. Parametrization of the πN-scattering p waves at low energies. Much less experimental information is available on the p waves for πN-scattering than on the s waves; the only exceptional case is that of the $\{3.3\}$ resonant wave. The effective range theory has been used to obtain the following parametrization for the waves:[79]

$$\begin{aligned} \operatorname{tg}\delta_{31}/q^3 &= (-0.13+0.072\omega-0.012\omega^2)/\omega; \\ q^3\operatorname{ctg}\delta_{33} &= 4.108+0.7987q^2-0.8337q^4; \\ \operatorname{tg}\delta_{11}/q^3 &= -0.015+0.005q^2; \\ \operatorname{tg}\delta_{13}/q^3 &= -0.0035. \end{aligned} \quad (135)$$

All the parameters in Eqs. (135) are independent, so they cannot be considered satisfactory.

Using the dispersion relations, we can approximately calculate the p-wave scattering lengths; for this purpose we also use experimental data on πN scattering over a broad energy range.[90]

Assuming $u(q^2) = 1$ we can obtain from the Chew–Low equation (2) an approximate expression which agrees with that from effective-range theory:

$$\lambda_i q^3/\omega\operatorname{ctg}\delta_i(\omega) = 1-\omega r_i. \quad (136)$$

This simple equation gives a good description for the $\{3.3\}$ wave up to the energy of the $\{3.3\}$ resonance. Lyson[91] offered a parametrization of the $\{3.3\}$ wave on the basis of Eq. (136), which incorporates the influence of the source function:

$$\frac{4}{3}f^2u^2(q)\frac{q^3}{\omega}\operatorname{ctg}\delta_{33}(\omega) = 1-\frac{\omega^*}{\omega_r}, \quad (137)$$

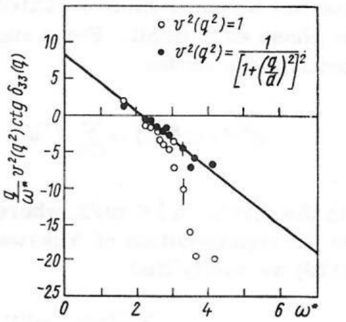

Fig. 7

where

$$f^2 = 0.087;\quad \omega_r = 2.17;$$
$$\omega^* \sim \sqrt{1+q^2}+q^2/2M;\ u(q) = [1+(q/a)^2]^{-1};\ a^{-1} = 1.8\hbar/Mc.$$

Parametrization (137) gives a qualitative description of the $\{3.3\}$ phase shift up to a laboratory energy of 600 MeV (Fig. 7).

An interesting phenomenological equation was proposed by Höhler[92] for the resonant phase shift:

$$\sin^2\delta_{33}/q^3 = A\exp[-\omega/\sigma], \quad (138)$$

where

$$A = 47.5,\quad \sigma \approx 0.397.$$

Equation (138) gives a good description of the experimental data from 189 to 525 MeV in the laboratory system. We believe it is quite significant that Eq. (138) contains a logarithmic branch point at infinity, in accordance with the structure of the Riemann surface of the static model.

The absence of an exact solution of the Chew–Low equation for p waves has led to various approximations on the basis of these equations. The most natural approximations are those in which an attempt is made to exploit the approximate equality of the interactions in the $\{1.3\}$ and $\{3.1\}$ states. However, if we set

$$h_{13}(\omega) = h_{31}(\omega) \quad (139)$$

in Eqs. (2), we immediately find a contradiction of the unitarity condition: According to approximation (139), the problem reduces to one of finding the two functions $h_{11}(\omega)$ and $h_{33}(\omega)$ which satisfy the crossing-symmetry condition with the matrix

$$A = \frac{1}{9}\begin{pmatrix} 1 & 16 \\ 4 & 1 \end{pmatrix}. \quad (140)$$

Applying (140) to the functions $h_i(\omega)$ twice, we find $h_{11} = 2h_{33}$, in contradiction of unitarity. This circumstance was first pointed out by Schwarz,[93] who suggested that the matrix in (140) be replaced by

$$A = \begin{pmatrix} 0 & 2 \\ 1/2 & 0 \end{pmatrix}. \quad (141)$$

Although this matrix does not satisfy requirements (21) and (22), it does not lead to a contradiction of the unitarity condition. Markley[94] analyzed the approximate Chew–Low equations with matrix (141); he found a good description of the resonant phase shift but he found the phase shift δ_{11} to disagree with experiment.

A possible reason for the unsuccessful application of approximation (139) to Eqs. (2) is the circumstance that these equations are physically not closed, as was pointed out by Serebyrakov and Shirkov.[95] An attempt to apply this approximation in the modified Chew–Low equations incorporating the concept of short-range repulsion was also unsuccessful.[96]

Dedushev et al.[97] found a good description of the {3.3} wave on the basis of the backward-scattering relations incorporating an approximate cancellation of the baryonic u-channel contributions. An analogous result can be obtained on the basis of the Chew–Low equation incorporating the concept of short-range repulsion. In this approach,[98] the contribution of the crossing integral for the {3.3} wave is cancelled approximately with the short-range potential, and the same equation as that of ref. 97 is found for this wave.

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1)Below we set $\mu = 1$.

2)This notation (S_1 and S_3) is used because this case corresponds to the s waves of πN scattering.

3)The Schwarz reflection principle gives a method for the analytic continuation of a function across its boundary of definition if the function is known to take on real values on part of this boundary. In our case, the functions are real on the interval (−1, + 1), so

$$S_i(z^*) = S_i^*(z).$$

4)In terms of the variable z, Eqs. (72) have the form

$$(IA)^p S(\omega) = S_p(\omega)$$

and give the analytic continuation from the physical sheet to p-th sheet (ref. 38).

5)There is one exception: The method of Sec. 8 permits a generalization if the sets of poles (zeros) of the functions $S_i(w)$ differ by only a finite number of elements.[42] In other words, these properties are formulated as

$$\lim_{|w|\to\infty} S_i(w)/S_j(w) = 1,$$

where i and j are any indices. Below we seek solutions for which at least one of the ratios $S_i(w)/S_j(w)$ has an infinite number of singularities.

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