Straight-line path approximation in quantum field theory

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The aim of this review is to acquaint the reader with the basic ideas and elements of the mathematical apparatus on which the straight-line path approximation in quantum field theory is based. A number of applications of the approximation to the interaction of high-energy particles are considered and its connection with the quasipotential description in quantum field theory is considered.

INTRODUCTION

The study of strong interaction processes at high energies is one of the central problems of elementary particle physics at the present time. The general principles of quantum field theory provide the theoretical foundations for the interpretation of the empirical laws established for these processes. In the study of the interaction of high-energy hadrons an important role is played by the idea introduced by Bogolyubov in the theory of dispersion relations[8,9] that the scattering amplitude is a single analytic function of the physical variables.

It is this concept, which expresses the important requirement that the different physical processes be interrelated, that is the basis of the majority of theoretical and phenomenological approaches to the description of strong interactions at high energies. The following approaches have been developed furthest: dispersion relations and equations, dispersion sum rules, the asymptotic approach, phenomenological Regge and eikonal approaches and the quasipotential approach. In recent years, the fruitfulness of the quasipotential method for the study of high-energy processes has been demonstrated.

The quasipotential method[11,14,34] was proposed in 1963 and was significantly developed in the subsequent years and widely used for very different applications in quantum field theory. The success of this method is based on the fact that it combines the rigor of the fundamental principles of quantum field theory with a lucid physical interpretation, which enables one to exploit both empirical and heuristic arguments pertaining to the interaction of particles at high energies.

In some recent investigations pertaining to the description of various processes at high energies attention was drawn to the significance of the heuristic idea of smoothness of a local quasipotential; this idea is due to Blokhintsev[26] and Logunov and his collaborators.[22] The assumption of a smooth local quasipotential enables one, on the one hand, to reproduce the basic features of high-energy hadron scattering and, on the other, it leads to a very simple qualitative picture of particle interaction at asymptotically high energies. As a result of this, the hadrons during a scattering process at high energy retain in each scattering event, which is decribed by a smooth complex quasipotential, large fractions of their longitudinal (in the center of mass system) momenta, undergoing small transverse momentum transfers. In a certain sense, this behavior resembles the classical picture of the scattering of fast particles that move along approximately straight paths and suffer only small angular deflections. An important difference, however, is the absorptive, i.e., essentially inelastic, nature of the high-energy scattering of hadrons and the approximate constancy of the total cross section and the diffraction behavior at small momentum transfer that are inherent to high-energy scattering.

The qualitative picture of approximately rectilinear motion of the interacting particles that we have described above can also be extended to inelastic processes at high energies, since one of the most important empirical features of these processes is the boundedness of the transverse momenta and the dominance of the longitudinal (in the direction of the collision) components of the secondary-particle momenta. It is therefore of great importance to develop methods based on the idea of rectilinear paths, i.e., on the assumption that there are small momentum transfers in processes of elastic and inelastic interaction of particles at asymptotically high energies. We shall refer to such methods as straight-line path approximations. They are the subject of the present review.

The straight-line path approximation was formulated relativistically for the first time in, for example, refs. 4 and 23 using functional integration in quantum field theory. The choice of this method was not fortuitous. First, functional integration, as was shown in the pioneering papers of Feynman[27] and Bogolyubov,[7] is very convenient for finding closed expressions for the total Green's functions. Secondly, in a functional integral representation of the amplitudes of different processes as sums over the paths of colliding particles the notion of straight-line paths can be simply and lucidly realized. In this approach, the approximation consists of allowing for paths of particles that most nearly approximate the classical paths, which in the case of high-energy scattering through small angles coincide with rectilinear trajectories. The approximate methods used to calculate the functional integrals in this case are similar to the approximation procedures proposed by Fradkin[21] and Barbashov[1] in an investigation of the infrared asymptotic behavior of Green's functions in quantum electrodynamics.

In the present review we set forth the general method of constructing two-particle Green's functions and scattering amplitudes as functional integrals. The subjects of our investigations are standard field-theory models: the model of scalar nucleons that exchange scalar mesons, and the model of scalar nucleons and vector mesons. Closed expressions are obtained for the two-particle Green's functions that contain contributions of all possible graphs taking into account radiative corrections, closed nucleon loops, etc. An important stage in the construction of the scattering amplitude is the development of a method for the correct transition to the mass shell and an analysis of the renormalization problem. A feature of the general expression for the scattering amplitude expressed as a functional integral is associated with the specific factorization in the amplitude of the contributions that describe the effects of nucleon self-interaction, exchange effects, and vacuum polarization. We also discuss the physical interpretation of the scattering amplitude as a sum over the deviations of the paths from the classical trajectories. Further, on the basis of the expression for the two-particle Green's function, we construct the amplitude of a process with the production of a certain number of meson quanta and we consider the general properties of the amplitudes of inelastic processes.

Taking as an example the expressions for the amplitudes of elastic and inelastic scattering expressed as functional integrals, we give a relativistic formulation of the straight-line approximation. This approximation leads to a modification of the nucleon propagators in which one ignores combination of momenta of virtual mesons. One can also make an even simpler approximation, which leads to the vanishing of the quadratic terms in the propagators. However, in some cases such a method is inapplicable because of difficulties that then arise, in particular, with divergences of the Feynman integrals.

As an example of the use of the straight-line path approximation we consider the scattering amplitude of two particles in the limit of high energies at fixed momentum transfers. An important feature is the factorization of the radiative corrections in the scattering amplitude, which in the given approximation has the form of an eikonal representation with Yukawa interaction potential. There are no retardation effects in the final result. It is interesting to note that the sum of ladder graphs with crossings in the straight-line path approximation reduces effectively to a sum of ladder graphs of quasipotential type. The diffraction behavior of the scattering amplitude is due to our allowance for radiative corrections, which lead naturally to a smooth complex effective quasipotential.

As a second example of the use of the approximation, we consider multiple production of particles. Among the various approaches that have been developed in this direction the one that is closest to the straight-line path method is the one in which meson production in strong interaction is interpreted by analogy with bremsstrahlung emission of soft particles in electrodynamics. A common assumption of such approaches is that one can partly or completely ignore recoils when secondary particles are emitted. Among the results set forth in this part of the review we may mention the following: prediction of a Poisson nature of the multiplicity distribution for fixed t; the discovery of a region of scaling behavior of the cross sections summed over the number of secondary particles; approximate linearity of the mean multiplicity in t.

The straight-line path approximation has been developed further in recent papers. In particular, it has been used to study other field-theory models, a larger class of diagrams has been taken into account, and different asymptotic regions has been considered. In the present review we have not set ourselves the task of making an extensive review of all problems associated with this approach and the reader can turn to the review in ref. 2, which contains an interesting analysis of the eikonal approximation and its connection to the quasiclassical approximation and optical models of particle scattering.

Further, we formulate some mathematical realizations of the concept of straight-line paths that use the methods of functional integration and enable one to allow consistently for the deviations of the particle paths from rectilinear trajectories. It is important to study this question because it is only in this way that one can arrive at a consistent resolution of the problem of justifying the approximation and extending its applicability. We show that allowance for correction terms leads to the appearance of retardation effects. We also note the singular character of the correction terms at short distances, which can lead ultimately to the appearance of noneikonal contributions to the scattering amplitude.

We formulate an operator method of solving quasipotential equations and establish its connection to the approximate methods of functional integration considered earlier in the review. We show that if the local quasipotential is smooth the operator method provides a consistent justification of the eikonal representation of the scattering amplitude and enables one to find the corrections to it.

We investigate the structure of the noneikonal contributions to the two-nucleon scattering amplitude. In particular, we show that the sum of all ladder diagrams of eighth order in the scalar model contains terms that are absent in the orthodox eikonal equation and vanish in the limit $\mu/m \to 0$, where μ and m are the meson and nucleon masses. The terms we obtain correspond to contributions to the effective quasipotential resulting from the exchange of nucleon–antinucleon pairs.

1. REPRESENTATION OF SCATTERING AMPLITUDES AS PATH INTEGRALS

Construction of the Two-Particle Green's Function

For simplicity, we shall first consider the model of scalar nucleons interacting with a scalar meson field; the model is described by the interaction Lagrangian $L_{int} = g{:}\psi^+\psi\varphi$. The results will be generalized to the case of scalar nucleons interacting with a neutral vector field later.

The one-particle nucleon Green's function in a given external scalar field $\varphi(x)$ satisfies the equation

$$[\Box + m^2 - g\varphi(x)]\, G(x, y \mid \varphi) = \delta(x-y). \tag{1}$$

A formal solution of Eq. (1) can be represented as the functional integral[1]

$$G(x, y \mid \varphi) = \mathrm{i} \int_0^\infty d\tau \exp(-\mathrm{i}\tau m^2) \int [\delta\nu]_0^\tau$$
$$\times \exp\left\{\mathrm{i}g \int_0^\tau d\xi \varphi\left[x + 2\int_0^\xi \nu(\eta)\, d\eta\right]\right\} \delta\left[x - y + 2\int_0^\tau \nu(\eta)\, d\eta\right], \tag{2}$$

where

$$[\delta\nu]_{\tau_1}^{\tau_2} = \left\{\delta\nu \exp\left[-\mathrm{i}\int_{\tau_1}^{\tau_2} \nu^2(\eta)\, d\eta\right]\right\} \Big/ \left\{\int \delta\nu \exp\left[-\mathrm{i}\int_{\tau_1}^{\tau_2} \nu^2(\eta)\, d\eta\right]\right\}; \tag{3}$$

$\delta\nu$ is the volume element of the function space of four-dimensional real functions $\nu(\eta)$ defined on the interval $\tau_1 \le \eta \le \tau_2$.

The Fourier transform of the Green's function is

$$G(p,\ q\,|\,\varphi)=\int d^4x\, d^4y \exp(ipx-iqy)\, G(x,\ y\,|\,\varphi)$$
$$=\mathrm{i}\int_0^\infty d\tau \exp[i\tau(p^2-m^2)]\int d^4x \exp[ix(p-q)]\int[\delta\nu]_0^\tau$$
$$\times\exp\left\{ig\int_0^\tau d\xi\varphi\left[x+2p\xi+2\int_0^\xi \nu(\eta)\,d\eta\right]\right\}. \qquad (4)$$

Using the expression (4), one can find the two-nucleon Green's function:

$$G(p_1,\ p_2;\ q_1,\ q_2)=\left[\exp\frac{i}{2}\int D\frac{\delta^2}{\delta\varphi^2}\right]$$
$$\times G(p_1,\ q_1\,|\,\varphi)\,G(p_2,\ q_2\,|\,\varphi)\,S_0(\varphi)\,|_{\varphi=0}, \qquad (5)$$

where

$$\left[\exp\frac{i}{2}\int D\frac{\delta^2}{\delta\varphi^2}\right]$$
$$\equiv\exp\left[\frac{i}{2}\int d^4x_1\, d^4x_2 D(x_1-x_2)\frac{\delta^2}{\delta\varphi(x_1)\,\delta\varphi(x_2)}\right]; \qquad (6)$$

$S_0(\varphi)$ is the mean value of the S matrix over the fluctuations of the nucleon vacuum in the presence of the external scalar field φ. It is well known that $S_0(\varphi)$ can be represented in the form

$$S_0(\varphi)=\exp[i\pi(\varphi)], \qquad (7)$$

where the functional $\pi(\varphi)$ in the framework of the models considered here corresponds to a sum of connected diagrams containing one closed nucleon loop with an arbitrary number of ends of the external meson field.

We introduce the notation

$$\int j_i\varphi\equiv\int dz\varphi(z)\, j(x_i-z;\ p_i;\ \tau_i\,|\,\nu_i),\quad i=1,\ 2, \qquad (8)$$

where

$$j(x_i-z;\ p_i;\ \tau_i\,|\,\nu_i)$$
$$=\int_0^{\tau_i} d\xi\delta\left[x_i-z+2p_i\xi+2\int_0^\xi \nu_i(\eta)\,d\eta\right]. \qquad (9)$$

Using this notation, we represent (5) in the form

$$G(p_1,\ p_2;\ q_1,\ q_2)$$
$$=\mathrm{i}^2\int_0^\infty d\tau_1\, d\tau_2 \exp[i\tau_1(p_1^2-m^2)+i\tau_2(p_2^2-m^2)]$$
$$\times\int d^4x_1\, d^4x_2 \exp[ix_1(p_1-q_1)+ix_2(p_2-q_2)]$$
$$\times\int[\delta\nu_1]_0^{\tau_1}[\delta\nu_2]_0^{\tau_2}\mathscr{E}(x_{1,2};\ p_{1,2};\ \tau_{1,2}\,|\,\nu_{1,2}), \qquad (10)$$

where

$$\mathscr{E}=\left[\exp\frac{i}{2}\int D\frac{\delta^2}{\delta\varphi^2}\right]\exp\left[ig\int\varphi(j_1+j_2)\right]S_0(\varphi)\,|_{\varphi=0}. \qquad (11)$$

Let us consider in more detail the structure of $\mathscr{E}$. We define for each functional $A(\varphi)$

$$\bar{A}(\varphi)=\left[\exp\frac{i}{2}\int D\frac{\delta^2}{\delta\varphi^2}\right]A(\varphi), \qquad (12)$$

so that the mean value of $A(\varphi)$ over the vacuum fluctuations of the meson field is $\bar{A}=\bar{A}(\varphi)(\varphi)|_{\varphi=0}$.

We now consider the mean value over the meson vacuum of the product of two functionals:

$$\overline{A\cdot B}=\left[\exp\frac{i}{2}\int D\frac{\delta^2}{\delta\varphi^2}\right]A(\varphi)\,B(\varphi)\,|_{\varphi=0}. \qquad (13)$$

It is not difficult to prove the identity

$$\overline{A\cdot B}=\left[\exp\frac{i}{2}\int D\left(\frac{\delta}{\delta\varphi_1}+\frac{\delta}{\delta\varphi_2}\right)^2\right]A(\varphi_1)\,B(\varphi_1)\,|_{\varphi_1=\varphi_2=0}$$
$$=\left[\exp i\int D\frac{\delta^2}{\delta\varphi_1\delta\varphi_2}\right]\bar{A}(\varphi_1)\,\bar{B}(\varphi_2)\,|_{\varphi_1=\varphi_2=0}$$
$$=\bar{A}\left(i\int D\frac{\delta}{\delta\varphi_2}\right)\cdot\bar{B}(\varphi_2)\,|_{\varphi_2=0}. \qquad (14)$$

Choosing as the two functionals

$$A(\varphi)=\exp\left[ig\int\varphi(j_1+j_2)\right]\text{and}B(\varphi)=S_0(\varphi), \qquad (15)$$

we obtain

$$\bar{A}(\varphi)=\exp\left[ig\int\varphi(j_1+j_2)-\frac{ig^2}{2}\int D(j_1+j_2)^2\right]; \qquad (16)$$

$$\bar{S}_0(\varphi)=\exp[i\Pi(\varphi)], \qquad (17)$$

where the functional $\Pi(\varphi)$ corresponds to the sum of all connected diagrams with an arbitrary number of closed nucleon loops and external ends with allowance for all possible internal meson pairings.

Using the identity (14) and Eqs. (15)-(17), we find for $\mathscr{E}$ the expression

$$\mathscr{E}=\exp\left[-\frac{ig^2}{2}\int D(j_1+j_2)^2\right.$$
$$\left.-g\int D(j_1+j_2)\frac{\delta}{\delta\varphi}\right]\exp\left[i\Pi(\varphi)\right]_{\varphi=0}$$
$$=\exp\left[-\frac{ig^2}{2}\int D(j_1+j_2)^2+i\Pi\left(-g\int D(j_1+j_2)\right)\right]. \qquad (18)$$

Expanding (18) in a series in the coupling constant, substituting it into (10), and making some simple functional integrations with respect to ν_i, we obtain the usual unrenormalized perturbation series for the two-particle Green's function.

Let us here emphasize an important fact that will be used later. The expression (18) enables one to separate in a general form the effects of interaction between the two nucleons (exchange effects) from the effects of nucleon self-interaction (radiative corrections) and vacuum renormalization.

Indeed, the first term in the exponential (18) can be rewritten trivially in the form

$$\frac{ig^2}{2}\int D(j_1+j_2)^2 = ig^2\int Dj_1j_2 + \frac{ig^2}{2}\int Dj_1^2 + \frac{ig^2}{2}\int Dj_2^2, \quad (19)$$

where the first term on the right-hand side corresponds to one-meson exchange between the two nucleons, and the remainder lead to radiative corrections.

Accordingly, we represent the second term in the exponential in the form

$$\Pi = \Pi_{12} + \Pi_1 + \Pi_2 + \Pi(0), \quad (20)$$

where

$$\Pi_{12} = \Pi\left[-g\int D(j_1+j_2)\right] - \Pi\left(-g\int Dj_1\right) - \Pi\left(-g\int Dj_2\right) + \Pi(0) \quad (21)$$

and

$$\Pi_i = \Pi\left(-g\int Dj_i\right) - \Pi(0), \; i=1,2. \quad (22)$$

We show that the expressions (21) and (22) can be represented in terms of the polarization operator of the meson field:

$$P(x_1, x_2|\varphi) = -\frac{\delta^2}{\delta\varphi(x_1)\,\delta\varphi(x_2)}\Pi(\varphi) \quad (23)$$

or the total Green's function of the meson field:

$$D(x_1, x_2|\varphi) = D(x_1-x_2) + \int d^4y_1\,d^4y_2 D(x_1-y_1)\,P(y_1, y_2|\varphi)\,D(y_2-x_2) \quad (24)$$

in the presence of external sources. As a result, we obtain for $\mathscr{E}$ the expression[35]

$$\mathscr{E} = \exp[i\Pi(0)]\,\mathscr{E}^{(1)}\mathscr{E}^{(2)}\mathscr{E}^{(12)}, \quad (25)$$

where

$$\mathscr{E}^{(i)} = \exp\left[-\frac{ig^2}{2}\int D_i^* j_i^2\right], \quad i=1, 2 \quad (26)$$

and

$$\mathscr{E}^{(12)} = \exp\left[-ig^2\int D_{12}^* j_1 j_2\right]. \quad (27)$$

Here we have used the notation

$$D_i^* = 2\int_0^1 d\sigma \int_0^\sigma d\lambda D\left(x_1, x_2|-g\lambda\int Dj_i\right), \quad i=1, 2; \quad (28)$$

$$D_{12}^* = \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 D\left(x_1, x_2|-g\lambda_1\int Dj_1 - g\lambda_2\int Dj_2\right). \quad (29)$$

We note that D_i^* is the Green's function of the meson field interacting with the external source j_i corresponding to the i-th nucleon, while D_{12}^* is the Green's function of the meson field interacting with both the nucleons at once.

Thus, $\mathscr{E}$, which determines the two-nucleon Green's function, factorizes into factors that represent, respectively, the interaction between the two nucleons, the radiative corrections, and the vacuum renormalization.

Representation for the Two-Nucleon Scattering Amplitude

The two-nucleon scattering amplitude is expressed in terms of the two-particle Green's function (5) by

$$(2\pi)^4\delta(p_1+p_2-q_1-q_2)\,iF(p_1, p_2; q_1, q_2) = \lim_{p_i^2, q_i^2\to m^2} \prod_{i=1,2}(p_i^2-m^2)(q_i^2-m^2)\,G(p_1, p_2; q_1, q_2). \quad (30)$$

Here we ignore the renormalization problem, which is discussed in the next section, and we omit on the right-hand side of (30) the factor $\exp[i\Pi(0)]$, which does not contribute to the scattering.

As we have mentioned in the introduction, the development of a correct procedure for the transition to the mass shell in the construction of the scattering amplitude by means of the definition (30) in a general form is an important problem. Many approximate methods that are reasonable from the physical point of view when used before the transition to the mass shell shift the positions of the poles of the Green's function and render the procedure for finding the scattering amplitude mathematically incorrect.

We shall here use a method for separating the poles of the Green's functions that generalizes the method introduced in refs. 3, 5, 20, and 23 in the determination of the scattering amplitude in the model of scalar nucleons interacting with a scalar meson field, the contributions of closed nucleon loops being ignored.

Using the expression for the two-particle function (5) and Eqs. (9) and (25)-(29), we represent the definition (30) of the scattering amplitude in the form

$$\begin{aligned}
&(2\pi)^4\delta(p_1+p_2-q_1-q_2)\cdot F(p_1, p_2; q_1, q_2) \\
&= \lim_{p_i^2, q_i^2\to m^2}\prod_{i=1,2}(p_i^2-m^2)(q_i^2-m^2) \\
&\times \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^{\tau_1} d\xi_1 \int_0^{\tau_2} d\xi_2 \exp[i\tau_1(p_1^2-m^2) + i\tau_2(p_2^2-m^2)] \\
&\times \int d^4x_1\,d^4x_2 \int d^4z_1\,d^4z_2 \exp[ix_1(p_1-q_1) + ix_2(p_2-q_2)] \\
&\times \int [\delta\nu_1]_0^{\tau_1}[\delta\nu_2]_0^{\tau_2}\, i\mathscr{F}(z_1, z_2|j_1, j_2) \\
&\times \delta\left(x_1 - z_1 + 2p_1\xi_1 + 2\int_0^{\xi_1}\nu_1\,d\eta\right) \\
&\times \delta\left(x_2 - z_2 + 2p_2\xi_2 + 2\int_0^{\xi_2}\nu_2\,d\eta\right),
\end{aligned} \quad (31)$$

where

$$\mathscr{F}(z_1, z_2|j_1, j_2) = g^2\mathscr{E}^{(1)}\mathscr{E}^{(2)}D_{12}^*\int_0^1 d\gamma \exp\left[-i\gamma g^2\int D_{12}^* j_1 j_2\right]. \quad (32)$$

In deriving (31) and (32) we have used the fact that the free part of the Green's function, which is not associated with the interaction of the two nucleons, can be subtracted in accordance with the equation

$$\mathscr{E}^{(12)} \to \mathscr{E}^{(12)} - 1 = ig^2 \int D^*_{12} j_1 j_2 \int_0^1 d\gamma \exp\left[-i\gamma g^2 \int D^*_{12} j_1 j_2\right]. \quad (33)$$

Taking into account the identity

$$\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^{\tau_1} d\xi_1 \int_0^{\tau_2} d\xi_2 \ldots \equiv \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \int_{\xi_1}^\infty d\tau_1 \int_{\xi_2}^\infty d\tau_2 \ldots \quad (34)$$

and making the following substitution of the ordinary and the functional variables:

$$\left.\begin{aligned} &\tau_i \to \tau_i + \xi_i; \\ &x_i \to x_i - 2p_i\xi_i - 2\int_0^{\xi_i} \nu_i d\eta; \\ &\nu_i(\eta) \to \nu_i(\eta - \xi_i) - (p_i - q_i)\,\vartheta(\eta - \tau_i), \end{aligned}\right\} \quad (35)$$

we obtain

$$\begin{aligned} &(2\pi)^4 \delta(p_1 + p_2 - q_1 - q_2) F(p_1, p_2; q_1, q_2) \\ &= \lim_{p_i^2, q_i^2 \to m^2} \prod_{i=1,2} (p_i^2 - m^2)(q_i^2 - m^2) \\ &\times \int_0^\infty d\tau_1 d\tau_2 d\xi_1 d\xi_2 \exp[i\tau_1(p_1^2 - m^2) + i\tau_2(p_2^2 - m^2) \\ &+ i\xi_1(q_1^2 - m^2) + i\xi_2(q_2^2 - m^2)] \int d^4x_1 d^4x_2 \\ &\times \exp[ix_1(p_1 - q_1) + ix_2(p_2 - q_2)] \\ &\times \int [\delta\nu_1]^{\tau_1}_{-\xi_1} [\delta\nu_2]^{\tau_2}_{-\xi_2} i\mathscr{F}(x_1, x_2 \mid j_1, j_2). \end{aligned} \quad (36)$$

Taking into account the translational symmetry[1] of the functional $\mathscr{F}$, and going to the limit in (36), we find the final expression for the two-nucleon scattering amplitude:

$$\begin{aligned} F(p_1, p_2; q_1, q_2) &= \int [\delta\nu_1]^\infty_{-\infty} \mathscr{E}^{(1)}(p_1, q_1 \mid \nu_1) \\ &\times \int [\delta\nu_2]^\infty_{-\infty} \mathscr{E}^{(2)}(p_2, q_2 \mid \nu_2)\, ig^2 \int d^4x \exp(ix\Delta) \\ &\times D^*_{12}(x; p_i; q_i \mid \nu_i) \int_0^1 d\gamma \exp\left(-i\gamma g^2 \int D^*_{12} j_1 j_2\right), \end{aligned} \quad (38)$$

where

$$\Delta = (p_1 - q_1) = -(p_2 - q_2), \quad x = x_1 - x_2.$$

All the quantities in (38) are functionals of the sources:

$$j_i \equiv \int_{-\infty}^\infty d\zeta \delta\left[x_i - z + 2p_i\zeta\vartheta(\zeta) + 2q_i\zeta\vartheta(-\zeta) + 2\int_0^\zeta \nu_i d\eta\right]. \quad (39)$$

Note that the expression (39) defines the scalar density of a classical point particle that moves along the curvilinear path $x_i(s)$, which depends on the proper time $s = 2m\zeta$ and satisfies the equation

$$m\, dx_i(s)/ds = p_i\vartheta(\zeta) + q_i\vartheta(-\zeta) + \nu_i(\zeta) \quad (40)$$

subject to the condition $x_i(0) = x_i$, $i = 1, 2$.

For this reason, the representation (38) of the scattering amplitude can be regarded as a functional sum over all possible nucleon paths in the scattering process. It is not difficult to generalize the results obtained here to the model of scalar nucleons interacting with a neutral vector field with interaction Lagrangian

$$L_{int} = ig\!: A_\mu \psi^+ \overleftrightarrow{\partial_\mu} \psi\!: + \frac{g^2}{2}\!: A_\mu^2 \psi^+\psi\!:. \quad (41)$$

For the two-nucleon scattering amplitude in this model we obtain

$$\begin{aligned} F(p_1, p_2; q_1, q_2) &= \int [\delta\nu_1]^\infty_{-\infty} \mathscr{E}^{(1)}(p_1, q_1 \mid \nu_1) \\ &\times \int [\delta\nu_2]^\infty_{-\infty} \mathscr{E}^{(2)}(p_2, q_2 \mid \nu_2)\, l^{(1)}_\alpha l^{(2)}_\beta ig^2 \int d^4x \exp[ix\Delta] \\ &\times D^{\alpha\beta *}_{12}(x; p_i, q_i \mid \nu_i) \int_0^1 d\gamma \exp\left[-i\gamma g^2 \int D^{\alpha\beta *}_{12} j^{(1)}_\alpha j^{(2)}_\beta\right], \end{aligned} \quad (42)$$

where

$$l^{(i)}_\alpha = [p_i + q_i + 2\nu_i(0)]_\alpha, \quad i = 1, 2. \quad (43)$$

As in the foregoing case, all the quantities in the representation (42) can be expressed in terms of the Green's functions of vector fields interacting with the sources:

$$\begin{aligned} j^{(i)}_\alpha &= \int_{-\infty}^\infty d\zeta\, [2p_i\vartheta(\zeta) + 2q_i\vartheta(-\zeta) + 2\nu_i(\zeta)]_\alpha \\ &\times \delta\left[x_i - z + 2p_i\zeta\vartheta(\zeta) + 2q_i\zeta\vartheta(-\zeta) + 2\int_0^\zeta \nu_i(\eta)\, d\eta\right]. \end{aligned} \quad (44)$$

It is easy to see that the expression (44) determines the current density of a point particle moving along the curvilinear path (40), and that the current density satisfies the continuity equation

$$\partial_\alpha j^{(i)}_\alpha = 0, \quad i = 1, 2. \quad (45)$$

Discussion of the Renormalization Problem

It is obvious that the one-particle Green's function G(p) of interacting nucleons defined by

$$\begin{aligned} &(2\pi)^4 \delta(p - q) G(p) \\ &= \left[\exp \frac{i}{2} \int D \frac{\delta^2}{\delta\varphi^2}\right] G(p, q \mid \varphi) S_0(\varphi)\big|_{\varphi=0} \end{aligned} \quad (46)$$

does not, in the general case, have the same position of its pole and value of its residue as the Green's function of free nucleons, i.e.,

$$G(p) = \frac{1}{m^2 - p^2 + \Sigma(p^2)}\bigg|_{p^2 \to m^2_{ph}} \sim \frac{z^{-1}}{m^2_{ph} - p^2}, \quad (47)$$

where

$$\left.\begin{aligned} &m^2_{ph} = m^2 + \Sigma(m^2_{ph}) = m^2 + \delta m^2; \\ &z = 1 - \frac{\partial \Sigma}{\partial p^2}(m^2_{ph}). \end{aligned}\right\} \quad (48)$$

For this reason, when defining the scattering amplitude (30) as the residue of the two-particle Green's function, we must write $z(p_i^2 - m_{ph}^2)$ instead of $(p_i^2 - m^2)$. Note however that because of the renormalizations of the mass and field the functional integrals in (36) diverge, i.e.,

$$\int [\delta\nu_1]^{\tau_1}_{-\xi_1} [\delta\nu_2]^{\tau_2}_{-\xi_2} \mathscr{F} \xrightarrow[\tau_i,\, \xi_i \to \infty]{} \exp\left\{-i \sum_{k=1,\,2} \tau_k \times [\delta m^2 + (1-z)(p_k^2 - m_{ph}^2)]\right\} \quad (49)$$

$$\times \exp\left[-i \sum_{k=1,\,2} \xi_k [\delta m^2 + (1-z)(q_k^2 - m_{ph}^2)]\right] \int_R [\delta\nu_1]^{\infty}_{-\infty} [\delta\nu_2]^{\infty}_{-\infty} \mathscr{F},$$

where the symbol $\int_R$ denotes the regularized value of the functional integral obtained after the separation of the divergent exponential factors. It follows from this that the scattering amplitude defined as the residue of the two-particle Green's function at the physical poles (with allowance for the renormalization of the masses and fields) is given by Eqs. (38) and (42), in which the regularized values of the functional integrals are used.

The procedure for regularizing the functional integrals in a general investigation of the structure of the scattering amplitude can be appreciably simplified if one assumes the existence of the limits

$$\frac{\mathscr{E}^{(i)}(p,\, q;\, \tau,\, \xi \mid \nu)}{\int [\delta\nu]^{\tau}_{-\xi} \mathscr{E}^{(i)}(p,\, q;\, \tau,\, \xi \mid \nu)} \xrightarrow[\tau,\, \xi \to \infty]{} e^{(i)}(p,\, q \mid \nu), \quad (50)$$

where all the momenta p and q lie on the mass shell and $\mathscr{E}^{(i)}(p, q; \tau, \xi \mid \nu)$ are defined by Eqs. (26), in which the nucleon currents are given by the expression (37), which depends on τ and ξ. The meaning of the limit (50) reduces to the existence of the following improper functional integrals:

$$\int [\delta\nu]^{\infty}_{-\infty} e^{(i)}(p,\, q \mid \nu) = 1; \quad (51)$$

$$\int [\delta\nu]^{\infty}_{-\infty} e^{(i)}(p,\, q \mid \nu)\, A(\nu) = [A]_{e^{(i)}} \quad (52)$$

for a suitable class of functionals $A(\nu)$.

Using the definition (50) and Eqs. (51) and (52), we can represent the two-nucleon scattering amplitude (38) in the form

$$F(p_1,\, p_2;\, q_1,\, q_2) = r^{(1)}(t)\, r^{(2)}(t)\, f(p_1,\, p_2;\, q_1,\, q_2), \quad (53)$$

where

$$f(p_1,\, p_2;\, q_1,\, q_2) = \int [\delta\nu_1]^{\infty}_{-\infty} e^{(1)}(p_1,\, q_1 \mid \nu_1) \times \int [\delta\nu_2]^{\infty}_{-\infty} e^{(2)}(p_2,\, q_2 \mid \nu_2)\, ig^2 \int d^4x$$

$$\times \exp[ix\Delta]\, D^{*}_{12}(x,\, p_i,\, q_i \mid \nu_i) \int_0^1 d\gamma \exp\left[-i\gamma g^2 \int D^{*}_{12} j_1 j_2\right]; \quad (54)$$

$$r^{(i)}(t) = \int_R [\delta\nu_i]^{\infty}_{-\infty} \mathscr{E}^{(i)}(p_i,\, q_i \mid \nu_i), \quad t = (p_i - q_i)^2. \quad (55)$$

Thus, in accordance with (53), some of the radiative corrections to two-nucleon scattering are factorized in the scattering amplitude in the form of factors that depend only on the square of the momentum transfer.

These radiation factors have a simple physical meaning: They describe the interaction of asymptotically free nucleons in the initial and the final state with the fluctuations of the meson vacuum.

The representation (53) may be particularly convenient for a study of the asymptotic behavior of amplitude at high energies, since it separates out from the scattering amplitude in a closed form the factors that do not depend on the energy.

Construction of Amplitudes of Inelastic Processes

We consider here a generalization of the above method to the construction of inelastic amplitudes. To be specific, we shall consider meson production in inelastic collisions of two nucleons. These processes can be described by means of the two-particle Green's functions of two nucleons in the presence of an external meson field φ^{ext}:

$$G(p_1,\, p_2;\, q_1,\, q_2 \mid \varphi^{ext}) = i^2 \int_0^{\infty} d\tau_1\, d\tau_2$$

$$\times \exp[i\tau_1(p_1^2 - m^2) + i\tau_2(p_2^2 - m^2)]$$

$$\times \int d^4x_1\, d^4x_2 \exp[ix_1(p_1 - q_1) + ix_2(p_2 - q_2)]$$

$$\times \int [\delta\nu]^{\tau_1}_0 [\delta\nu_2]^{\tau_2}_0 \mathscr{E}(\varphi^{ext}), \quad (56)$$

where

$$\mathscr{E}(\varphi^{ext}) = \left[\exp \frac{i}{2} \int D \frac{\delta^2}{\delta\varphi^2}\right] \exp\left[ig \int \varphi(j_1 + j_2)\right] S_0(\varphi)\big|_{\varphi = \varphi^{ext}}$$

$$= \exp\left\{-\frac{ig^2}{2} \int D(j_1 + j_2)^2 + ig \int \varphi^{ext}(j_1 + j_2) + i\Pi\left[\varphi^{ext} - g \int D(j_1 + j_2)\right]\right\}. \quad (57)$$

It is convenient to rewrite the expression (57) in the form

$$\mathscr{E}(\varphi^{ext}) = \exp[i\Pi(\varphi^{ext})]\, \mathscr{E}R(\varphi^{ext}), \quad (58)$$

where

$$R(\varphi^{ext}) = \exp\left\{ig \int \varphi^{ext}(j_1 + j_2) + i\Pi\left[\varphi^{ext} - g \int D(j_1 + j_2)\right] - i\Pi(\varphi^{ext}) - i\Pi\left[-g \int D(j_1 + j_2)\right]\right\}; \quad (59)$$

$\mathscr{E} = \mathscr{E}(\varphi^{ext} = 0)$ is defined by Eq. (18) and corresponds to purely elastic scattering of nucleons.

We now introduce

$$\Gamma_i(z \mid \varphi^{ext}) = j_i + \int_0^1 d\sigma \int_0^1 d\lambda \int dy\, (Dj_i)_y\, P\left(z,\, y \mid \sigma\varphi^{ext} - \lambda g \int Dj_i\right), \quad i = 1,\, 2; \quad (60)$$

$$\Gamma_{12}(z \mid \varphi^{ext}) = \int_0^1 d\sigma \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 \int dy_1\, dy_2\, (Dj_1)_{y_1}$$

$$\times (Dj_2)_{y_2}\, \Gamma\left[z,\, y_1,\, y_2 \mid \sigma\varphi^{ext} - g \int D(\lambda_1 j_1 + \lambda_2 j_2)\right], \quad (61)$$

where

$$\Gamma(z, y_1, y_2 | \varphi) = \frac{\delta^3}{\delta\varphi(z)\,\delta\varphi(y_1)\,\delta\varphi(y_2)} \Pi(\varphi) \quad (62)$$

is the generalized vertex operator of the coupling of free mesons in the presence of the external field and the polarization operator is defined by Eq. (23).

The functional $R(\varphi^{ext})$ in terms of the expressions (60) and (61) has the form

$$R(\varphi^{ext}) = \exp[-i\Pi(0)] \exp\left[ig \int \varphi^{ext}(\Gamma_1 + \Gamma_2) + ig^2 \int \varphi^{ext}\Gamma_{12}\right]. \quad (63)$$

The generating function of the inelastic amplitudes is defined as the residue of the two-particle Green's function of two nucleons in the presence of the external field (56) after the transition to the nucleon mass shell:

$$iF(p_1, p_2; q_1, q_2 | \varphi^{ext}) = \lim_{p_i^2, q_i^2 \to m^2} \prod_{i=1,2} (p_i^2 - m^2)(q_i^2 - m^2) \times G(p_1, p_2; q_1, q_2 | \varphi^{ext}). \quad (64)$$

Omitting the factor $\exp[i\Pi(\varphi^{ext})]$ on the right-hand side of (64) – it does not contribute to the interaction of the particles – and making the change of variables (35), we obtain the following representation for the generating functional (64):

$$F(p_1, p_2; q_1, q_2 | \varphi^{ext}) = \int d^4x_1\, d^4x_2 \times \exp[ix_1(p_1 - q_1) + ix_2(p_2 - q_2)] \int [\delta\nu_1]_{-\infty}^{\infty} \times \mathscr{E}^{(1)}(p_1, q_1 | \nu_1) \exp\left[ig \int \varphi^{ext}\Gamma_1\right] \int [\delta\nu_2]_{-\infty}^{\infty} \times \mathscr{E}^{(2)}(p_2, q_2 | \nu_2) \exp\left[ig \int \varphi^{ext}\Gamma_2\right] i\tilde{\mathscr{F}}(x_1, x_2 | j_1, j_2; \varphi^{ext}), \quad (65)$$

where

$$\tilde{\mathscr{F}}(x_1, x_2 | j_1, j_1; \varphi^{ext}) = g^2 \tilde{D}_{12}^* \int_0^1 d\gamma \exp\left[-i\gamma g^2 \int \tilde{D}_{12}^* j_1 j_2\right]. \quad (66)$$

Here

$$\tilde{D}_{12}^* = D_{12}^*(x_1, x_2; p_i, q_i | \nu_i) - \int_0^1 d\sigma \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 \times \int dz\, dy_1\, dy_2 \varphi^{ext}(z) D(x_1 - y_1) \times \Gamma_{12}\left[z, y_1, y_2 | \sigma\varphi^{ext} - g \int D(\lambda_1 j_1 + \lambda_2 j_2)\right] D(y_2 - x_2) \quad (67)$$

and this quantity corresponds to the total Green's function of the meson field interacting simultaneously with the external field φ^{ext} and the sources j_1 and j_2.

Note that the functional dependence of (67) on φ^{ext} is determined by the interaction of the external field with the exchange currents in the two-nucleon system.

The amplitude for the production of N secondary mesons in a two-nucleon collision can be calculated by means of the functional derivatives of the generating functional (65), i.e.,

$$(2\pi)^4 \delta\left(q_1 + q_2 - p_1 - p_2 - \sum_{i=1}^N k_i\right) \times F(p_1, p_2; q_1, q_2; k_1, k_2, \ldots, k_N) = \prod_{i=1}^N \int dy_i \exp(iy_i k_i) \frac{\delta}{\delta\varphi^{ext}(y_i)} F(p_1, p_2; q_1, q_2 | \varphi^{ext})|_{\varphi^{ext}=0}. \quad (68)$$

For example, the amplitude for the production of one meson with momentum k is

$$F(p_1, p_2; q_1, q_2; k) = ig \int [\delta\nu_1]_{-\infty}^{\infty} \mathscr{E}^{(1)}(p_1, q_1 | \nu_1) \times \int [\delta\nu_2]_{-\infty}^{\infty} \mathscr{E}^{(2)}(p_2, q_2 | \nu_2) \int d^4x \exp(ix\Delta) \times \Big\{[\Gamma_1(p_1, q_1; k | \nu_1) + \Gamma_2(p_2, q_2; k | \nu_2)] g^2 D_{12}^* \times \int_0^1 d\gamma \exp\left[-i\gamma g^2 \int D_{12}^* j_1 j_2\right] + \tilde{\Gamma}_{12}(x; p_i, q_i; k | \nu_i) \times \exp\left(-ig^2 \int D_{12}^* j_1 j_2\right)\Big\}. \quad (69)$$

Here we have used the notation

$$\Gamma_i(p_i, q_i; k | \nu_i) = \int dz \exp(izk) \Gamma_i(z | \varphi^{ext})|_{x_i=0,\ \varphi^{ext}=0}, \quad i = 1, 2; \quad (70)$$

$$\tilde{\Gamma}_{12}(x; p_i, q_i; k | \nu_i) = -g \int_0^1 d\lambda_1 \int_0^1 d\lambda_2 \int dz\, dy_1\, dy_2 \exp(izk) \times D(y_1 - x/2) D(y_2 + x/2) \Gamma\left[y_1, y_2, z | -g \int D(\lambda_1 j_1 + \lambda_2 j_2)\right]. \quad (71)$$

The amplitude for the production for two or more mesons can be constructed similarly.

2. STRAIGHT-LINE PATH APPROXIMATION AND ASYMPTOTIC BEHAVIOR OF SCATTERING AMPLITUDES AT HIGH ENERGIES

Elastic Scattering

If vacuum polarization effects are ignored, $\pi = 0$, the two-nucleon elastic scattering amplitude is given by the expression (42), in which

$$D_{12}^{\alpha\beta *} = D_i^{\alpha\beta *} = D^{\alpha\beta} = \frac{1}{(2\pi)^4} \int d^4k \frac{\exp(ikx)}{\mu^2 - k^2} (g^{\alpha\beta} - k^\alpha k^\beta / k^2). \quad (72)$$

As a result, we obtain the following closed expression for the two-particle scattering amplitude:[4]

$$F(p_1, p_2; q_1, q_2) = \frac{ig^2}{(2\pi)^4} \int d^4x \exp(ix\Delta) D^{\alpha\beta}(x) \times \int [\delta\nu_1]_{-\infty}^{\infty} [\delta\nu_2]_{-\infty}^{\infty} [2\nu_1(0) + p_1 + q_1]_\alpha \times [2\nu_2(0) + p_2 + q_2]_\beta \int_0^1 d\gamma \exp - \Big\{\frac{ig^2}{2} \int d^4k D^{\alpha\beta}(k) \times \Big[\sum_{i=1,2} j_\alpha^{(i)}(k; p_i, q_i | \nu_i) j_\beta^{(i)}(-k; p_i, q_i | \nu_i) + \gamma \exp(ikx) j_\alpha^{(1)}(k; p_1, q_1 | \nu_1) j_\beta^{(2)}(-k; p_2, q_2 | \nu_2) + (k \to -k)\Big]\Big\}, \quad (73)$$

where

$$j_\alpha^{(i)}(k;\ p_i,\ q_i\,|\,\nu_i) = 2i\int_{-\infty}^{\infty} d\zeta\,[\nu_i(\zeta) + p_i\vartheta(\zeta) + q_i\vartheta(-\zeta)]_\alpha$$
$$\times\exp\left\{2ik\left[p_i\zeta\vartheta(\zeta) + q_i\zeta\vartheta(-\zeta) + \int_0^\zeta \nu_i\,d\eta\right]\right\} \quad (74)$$

is the transition current, which satisfies the continuity equation

$$k_\alpha j_\alpha^{(i)}(k;\ p_i,\ q_i\,|\,\nu_i) = 0. \quad (75)$$

Note that the terms $j^{(i)}j^{(i)}$, i = 1, 2, in (73) describe radiative corrections in each of the nucleon lines, while the terms $j^{(1)}j^{(2)}$ describe the interaction between the two nucleons.

Let us dwell briefly on the physical meaning of the functional variables ν_1 and ν_2. Introduced formally in the derivation of the solution for the Green's function, these variables describe the deviation of the particle path from rectilinearity; for if we set $\nu = 0$ in the equation (74) for the transition current, we obtain

$$j_\alpha(k;\ p,\ q\,|\,0) = -[2p_\alpha/(2pk + i0) - 2q_\alpha/(2qk - i0)], \quad (76)$$

which corresponds to the classical current of a nucleon that moves with momentum p for $\zeta > 0$ and momentum q for $\zeta < 0$.

Note, however, that the approximation $\nu = 0$ is certainly false for proper times s of the particle near zero, when the classical trajectory of the particle changes direction. In the language of Feynman diagrams, this is equivalent to ignoring the quadratic dependence on the momentum k in the nucleon propagators, i.e.,

$$\frac{1}{m^2-(p+k)^2} \to -\frac{1}{2pk},$$

which can lead to the appearance of divergences of the integrals with respect to d^4k at the upper limit.

A better approximation for the classical nucleon current, which takes into account recoil, is given by the mean value of the current (74) with respect to the functional variable ν, i.e.,

$$\overline{j_\alpha}(k;\ p,\ q\,|\,\nu) = \int[\delta\nu]_{-\infty}^{\infty}\, j_\alpha(k;\ p,\ q\,|\,\nu)$$
$$= i\int_{-\infty}^{\infty} d\zeta\,[k\varepsilon(\zeta) + 2p\vartheta(\zeta) + 2q\vartheta(-\zeta)]_\alpha$$
$$\times\exp\{2ik\,[p\zeta\vartheta(\zeta) + q\zeta\vartheta(-\zeta)] + ik^2|\zeta|\}$$
$$= -[(2p_\alpha + k_\alpha)/(2pk + k^2 + i0) - (2q_\alpha - k_\alpha)/(2qk - k^2 - i0)]. \quad (77)$$

The straight-line path approximation when used to find the elastic scattering amplitude consists in substituting into the argument of the exponential in (73) the products of the currents averaged over the functional variables ν_1 and ν_2:

$$\overline{j_\alpha^{(1)}(k;\ p_1,\ q_1)\,j_\beta^{(2)}(-k;\ p_2,\ q_2)}$$
$$= \left(\frac{2p_{1\alpha}+k_\alpha}{2p_1k+k^2+i0} - \frac{2q_{1\alpha}-k_\alpha}{2q_1k-k^2-i0}\right)\left(\frac{2p_{2\beta}-k_\beta}{2p_2k-k^2+i0} - \frac{2q_{2\beta}+k_\beta}{2q_2k+k^2-i0}\right); \quad (78)$$

$$\overline{j_\alpha^{(i)}(k;\ p_i,\ q_i)\,j_\beta^{(i)}(k;\ p_i,\ q_i)}$$
$$= \left(\frac{2p_{i\alpha}+k_\alpha}{2p_ik+k^2+i0} - \frac{2q_{i\alpha}+k_\alpha}{2q_ik+k^2-i0}\right)$$
$$\times\left(\frac{2p_{i\beta}+k_\beta}{2p_ik+k^2+i0} - \frac{2q_{i\beta}+k_\beta}{2q_ik+k^2-i0}\right),\quad i = 1,\ 2. \quad (79)$$

Therefore, the expression for the elastic scattering amplitude ignoring terms corresponding to the replacement of the momenta of the outgoing particles takes the following form in the straight-line path approximation:

$$f_{el}(p_1,\ p_2;\ q_1,\ q_2) = \frac{ig^2}{(2\pi)^4}\, r^{(1)}(t)\, r^{(2)}(t)\,(p_1+q_1)_\alpha\,(p_2+q_2)_\beta$$
$$\times\int d^4x\exp[ix(p_1-q_1)]\,D^{\alpha\beta}(x)\int_0^1 d\gamma\exp[-i\gamma\chi^{(0)}(x;\ p_i,\ q_i)], \quad (80)$$

where

$$\chi^{(0)}(x;\ p_i,\ q_i) = \frac{g^2}{(2\pi)^4}\int d^4k\exp(ikx)\,D^{\alpha\beta}(k)$$
$$\times\left(\frac{2p_1+k}{2p_1k+k^2+i0} - \frac{2q_1-k}{2q_1k-k^2-i0}\right)_\alpha\left(\frac{2p_2-k}{2p_2k-k^2+i0} - \frac{2q_2+k}{2q_2k+k^2-i0}\right)_\beta; \quad (81)$$

$$r^{(1)}(t) = \exp\left[\frac{g^2}{2i}\int\frac{d^4k}{(2\pi)^4}D(k)\right.$$
$$\left.\times\left(\frac{2p_1+k}{2p_1k+k^2} - \frac{2q_1+k}{2q_1k+k^2}\right)_\alpha^2\right]; \quad (82)$$

$$r^{(2)}(t) = \exp\left[\frac{g^2}{2i}\int\frac{d^4k}{(2\pi)^4}D(k)\left(\frac{2p_2+k}{2p_2k+k^2} - \frac{2q_2+k}{2q_2k+k^2}\right)_\alpha^2\right]. \quad (83)$$

It is interesting to note that the contribution of the radiative corrections to ladder diagrams in this approximation is factorized in the form of the factor $r^{(1)}r^{(2)}$, which depends only on the square of the momentum transfer: $t = (p_1 - q_1)^2$. A similar factorization of the contributions of the radiative corrections in the case of quantum electrodynamics was found in ref. 30.

In the high-energy limit $s \to \infty$ at fixed momentum transfers t bounded by the condition $|t/m^2| \ll g^2$, the quantities $\chi^{(0)}$ and $r^{(i)}$ take the form

$$\chi^{(0)} = \frac{g^2}{8\pi}\int\frac{d^2\mathbf{k}_\perp}{\mathbf{k}_\perp^2+\mu^2}\exp(-i\mathbf{k}_\perp\mathbf{x}_\perp) = \frac{g^2}{4\pi}K_0(\mu|\mathbf{x}_\perp|); \quad (84)$$

$$r^{(1)}r^{(2)} = \exp(at), \quad (85)$$

where K_0 is a MacDonald function of zeroth order, and

$$a = \frac{g^2}{3(2\pi)^2m^2}\,[\ln(m^2/\mu^2) + 1/2 + O(\mu^2/m^2)]. \quad (86)$$

Thus, in the given asymptotic limit the expression for the amplitude of elastic scattering of two scalar nucleons interacting with a vector field has the form[2)]

$$f_{el}(s,\ t) = f^{(0)}(s,\ t)\exp(at), \quad (87)$$

where

$$f^{(0)}(s,t) = \frac{i(s-u)}{2}\int d^2\mathbf{x}_\perp\exp(i\mathbf{x}_\perp\boldsymbol{\Delta}_\perp)$$
$$\times\left\{\exp\left[-(ig^2/4\pi)K_0(\mu|\mathbf{x}_\perp|)\right] - 1\right\}, \quad (88)$$
$$t = -\Delta_\perp^2. \quad (89)$$

As can be seen from (87), allowance for radiative effects leads to a diffraction behavior of the amplitude of scattering of high-energy particles through small angles, which corresponds to a Gaussian form of the local quasipotential of elastic scattering with range of order $g\hbar/(mc)$. The forces due to the exchange of mesons between nucleons obviously have range $\hbar/(\mu c)$, it being assumed that $g\hbar/(mc) \ll \hbar/(\mu c)$. Thus, in the region of momentum transfer $\mu^2 \leqslant |t| < g^2m^2$ it is important to take into account multiple meson exchange, which leads to an eikonal structure of $f^{(0)}(s, t)$.

As was shown in ref. 24, allowance for the interaction of the nucleons with the zero-point fluctuations of the meson vacuum enables one to give a qualitative explanation of the smoothness property of the local quasipotential. Representing the amplitude (87) in the eikonal form, we find the expression for the eikonal function corresponding to this representation:

$$\exp[2i\chi(\mathbf{x}_\perp)] = \int \frac{d^2\rho}{4\pi a} \exp(-\rho^2/4a) \exp[2i\chi^{(0)}(\mathbf{x}_\perp + \rho)]. \quad (90)$$

From this it is not difficult to show that $\chi(\mathbf{x}_\perp)$ is a complex quantity with positive-definite imaginary part $|\exp(2i\chi)| < 1$ in accordance with the unitarity condition.

Expanding the exponential in the integrand of (90) in powers of $\chi^{(0)}$, we can find χ as a series:

$$\chi(\rho) = \frac{g^2}{8\pi} \int \frac{d^2\mathbf{k}_\perp \exp(-i\mathbf{k}\mathbf{x}_\perp)}{\mathbf{k}_\perp^2 + \mu^2} \exp(-a\mathbf{k}_\perp^2)$$
$$+ i\left(\frac{g^2}{8\pi}\right)^2 \int \frac{d^2\mathbf{k}_\perp\, d^2\mathbf{k}'_\perp \exp[-i\mathbf{x}_\perp(\mathbf{k}_\perp + \mathbf{k}'_\perp)]}{(\mathbf{k}_\perp^2 + \mu^2)(\mathbf{k}'^2_\perp + \mu^2)}$$
$$\times \{\exp[-a(\mathbf{k}_\perp + \mathbf{k}'_\perp)^2] \exp[-a(\mathbf{k}_\perp^2 - \mathbf{k}'^2_\perp)]\} + \ldots \quad (91)$$

The first term is purely real and corresponds to scattering on a Yukawa potential whose force center is distributed randomly in accordance with a Gaussian law. The second term in the equation gives a contribution to the imaginary part of the quasipotential.

Thus, allowance for radiative effects in two-particle scattering leads naturally to a smooth complex quasipotential whose imaginary part is positive-definite, in accordance with unitarity.

Inelastic Processes

The inelastic amplitudes describing the production of a certain number of particles of the vector field in a collision of two high-energy scalar nucleons can be found by means of the generating function $f(p_1, p_2; q_1, q_2 | A^{ext})$.

In the framework of the straight-line path approximation, $f(p_1, p_2; q_1, q_2 | A^{ext})$ takes the form

$$f(p_1, p_2; q_1, q_2 | A^{ext}) = g^2 \int d^4x_1\, d^4x_2$$
$$\times \exp[ix_1(p_1 - q_1) + ix_2(p_2 - q_2)] (p_1 + q_1)_\alpha (p_2 + q_2)_\beta$$
$$\times D^{\alpha\beta}(x_1 - x_2) \exp\left\{ig \int d^4k A_\gamma^{ext}\right.$$
$$\times [\overline{j_\gamma^{(1)}}(k; p_1, q_1) \exp(ikx) + \overline{j_\gamma^{(2)}}(k; p_2, q_2) \exp(-ikx)]\Big\}$$
$$\times \int_0^1 d\lambda \exp\left\{\frac{ig^2\lambda}{2} \int d^4k D^{\alpha\beta}(k)\right.$$
$$\times \Big[\sum_{i=1,2} \overline{j_\alpha^{(i)}(k; p_i, q_i) j_\beta^{(i)}(-k; p_i, q_i)}$$
$$+ \lambda \exp[ik(x_1 - x_2)]$$
$$\times \overline{j_\alpha^{(1)}(k; p_1, q_1) j_\beta^{(2)}(-k; p_2, q_2)} + (k \to -k)\Big]\Big\}, \quad (92)$$

where the functional means of the current and its bilinear combinations are defined by (77)-(79), respectively.

The amplitude for the production of N particles of the vector field is defined by means of the variational derivatives with respect to the field A^{ext}:

$$(2\pi)^4 \delta(p_1 + p_2 - q_1 - q_2 - \sum_{i=1}^N k_i)$$
$$\times f(p_1, p_2; q_1, q_2; k_1, k_2, \ldots, k_N)$$
$$= \prod_{i=1}^N e_\alpha^*(k_i) \frac{\delta}{\delta A_\alpha^{ext}(k_i)} f(p_1, p_2; q_1, q_2 | A^{ext})|_{A^{ext}=0}$$
$$= g^2 \int d^4x_1 d^4x_2 \exp[ix_1(p_1 - q_1) + ix_2(p_2 - q_2)]$$
$$\times \prod_{i=1}^N e_\alpha^*(k_i) [\overline{j_\alpha^{(1)}}(k_i; p_1, q_1) \exp(ik_i x_1)$$
$$+ \overline{j_\alpha^{(2)}}(k_i; p_2, q_2) \exp(ik_i x_2)] (p_1 + q_1)_\sigma (p_2 + q_2)_\rho$$
$$\times D^{\sigma\rho}(x_1 - x_2) \int_0^1 d\lambda \exp\left\{\frac{ig^2\lambda}{2} \int d^4k D^{\mu\delta}(k)\right.$$
$$\times \Big[\sum_{i=1,2} \overline{j_\mu^{(i)}(-k; p_1, q_1) j_\delta^{(i)}(k; p_2, q_2)}$$
$$+ \lambda \overline{j_\mu^{(1)}(k; p_1, q_1) j_\delta^{(2)}(-k; p_2, q_2)}$$
$$\times \exp[-ik(x_1 - x_2)] + (k \to -k)\Big]\Big\}, \quad (93)$$

where $e_\alpha(k)$ is a polarized vector meson with momentum k.

In what follows, we shall be interested in the case when the produced mesons are "soft:"

$$\frac{1}{\sqrt{s}} \sum_{i=1}^N k_{0i} \ll 1, \quad \left|\sum_{i=1}^N \mathbf{k}_{i\perp}\right| \ll |\mathbf{p}_{1\perp} - \mathbf{q}_{1\perp}| \approx |\mathbf{p}_{2\perp} - \mathbf{q}_{2\perp}|, \quad (94)$$

where the components of the particle momenta are specified in the center of mass system $\mathbf{p}_1 + \mathbf{p}_2 = 0$, and the momenta of the initial nucleons are taken along the z axis. Under these conditions, the amplitude for the production of N mesons factorizes and it can be written in the form

$$f_{inel}(N) \equiv f(p_1, p_2; q_1, q_2; k_1, k_2, \ldots, k_N)$$
$$= f_{el} \prod_{i=1}^{n_1} g e_{i\alpha}^*(k_i) \overline{j_\alpha^{(1)}(k_i; p_1, q_1)}$$
$$\times \prod_{l=1}^{n_2} g e_{l\beta}^*(k'_l) \overline{j_\beta^{(2)}(k'_l; p_2, q_2)}, \quad (95)$$

where

$$\overline{j_\alpha^{(i)}(k; p_i, q_i)} = \left(\frac{2p_i + k}{2p_i k + k^2} - \frac{2q_i - k}{2q_i k - k^2}\right)_\alpha, \quad i = 1, 2;$$
$$t = \Delta^2 = \left(q_1 - p_1 + \sum_{i=1}^{n_1} k_i\right)^2 = \left(q_2 - p_2 + \sum_{l=1}^{n_2} k'_l\right)^2, \quad n_1 + n_2 = N. \quad (96)$$

The differential cross section for the production of N mesons in a two-nucleon collision is given by

$$d\sigma_N = \frac{1}{2\sqrt{s(s-4m^2)}} |f_{\text{inel}}(N)|^2 (2\pi)^4 \times \delta\left(p_1 + p_2 - q_1 - q_2 - \sum_{i=1}^{N} k_i\right) \times \frac{1}{(2\pi)^6} \cdot \frac{d\mathbf{q}_1 d\mathbf{q}_2}{2q_{10} \cdot 2q_{20}} \cdot \frac{1}{N!} \prod_{i=1}^{N} \frac{d\mathbf{k}_i}{2k_{0i}} \cdot \frac{1}{(2\pi)^3}, \quad (97)$$

where $s = (p_1 + p_2)^2$.

Using (97) and making the transformation

$$\delta\left(p_1 + p_2 - q_1 - q_2 - \sum_{i=1}^{n_1} k_i - \sum_{l=1}^{n_2} k_l'\right) = \int d^4\Delta\,\delta\left(p_1 - q_1 - \sum_{i=1}^{n_1} k_i + \Delta\right)\delta\left(p_2 - q_2 - \sum_{l=1}^{n_2} k_l' - \Delta\right) \quad (98)$$

we can represent the differential cross section of meson production in the form[4]

$$(d\sigma)_{n_1, n_2} \xrightarrow[\text{fixed } \Delta]{s\to\infty} \frac{1}{2s} \cdot \frac{d^4\Delta}{(2\pi)^4} |f_{\text{el}}(s, t)|^2 W_{n_1}(p_1, \Delta) W_{n_2}(p_2, -\Delta), \quad (99)$$

where

$$W_{n_1}(p_1, \Delta) = \frac{2\pi}{n_1!} \int \frac{d^4 q_1}{2q_{10}} \delta\left(p_1 - q_1 - \sum_{i=1}^{n_1} k_i + \Delta\right) \times \prod_{i=1}^{n_1} \frac{d\mathbf{k}_i}{2k_{0i}} \cdot \frac{(-g^2)}{(2\pi)^3} |\overline{j_\alpha^{(1)}(k; p_1, q_1)}|^2 \quad (100)$$

and the expression for $W_{n_2}(p_2, -\Delta)$ is similar.

The quantities $W_{n_1}(p_1, \Delta)$ and $W_{n_2}(p_2, -\Delta)$ depend on the variables

$$t = \Delta^2, \quad \nu_1 = p_1\Delta, \text{ and } t = \Delta^2, \quad \nu_2 = -p_2\Delta, \quad (101)$$

respectively.

Using the variables (101), we transform the volume element $d^4\Delta$ to

$$d^4\Delta = \frac{4\pi}{\sqrt{s(s-4m^2)}} dt\, d\nu_1\, d\nu_2 \frac{d\Phi}{2\pi}, \quad (102)$$

where Φ is the azimuthal angle, and the physical range of the variables of integration is determined by

$$-t \leqslant 2\nu_i \leqslant s, \quad i = 1, 2; \qquad -s \leqslant t \leqslant 0, \quad m^2 \ll s. \quad (103)$$

In what follows we shall be interested in the differential cross section $(d\sigma/dt)_{n_1, n_2}$ in the limit $s \to \infty$ for fixed t. Integrating the expression (100) with respect to $d\nu_1$ and $d\nu_2$ and using (87), we obtain for $|t/m^2| \ll g^2$

$$(d\sigma/dt)_{n_1, n_2} \xrightarrow[\text{fixed } t]{s\to\infty} (1/4\pi)\, v^2(t)\, w_{n_1}(s, t)\, w_{n_2}(s, t), \quad (104)$$

where

$$w_n(s, t) = \frac{\exp(at)}{\pi} \int d\nu W_n(t, \nu) = \exp(at) \frac{1}{n!} \int_{\Omega_p} \prod_{i=1}^{n} \frac{d\mathbf{k}_i}{2k_{0i}} \cdot \frac{(-g^2)}{(2\pi)^3} |\overline{j_\alpha^{(l)}}(k_i, p_l, q_l)|^2, \; l = 1, 2. \quad (105)$$

The region of integration Ω_p with respect to the momenta of the secondary mesons is determined by the condition

$$-t \leqslant 2p \sum_{i=1}^{n} k_i - \left(\Delta - \sum_{i=1}^{n} k_i\right)^2 \leqslant s \quad (106)$$

or, since in our case $(\Delta - \sum_{i=1}^{n} k_i)^2 \approx \Delta^2$, by the condition

$$0 \leqslant 2p \sum_{i=1}^{n} k_i \leqslant s + t. \quad (107)$$

We now consider the approximation in which one can ignore the total momentum of the emitted mesons in accordance with the condition of softness (94). In this approximation, the expression (105) becomes a Poisson distribution:[25]

$$w_n(s, t) = \frac{1}{n!} \exp(at) [\bar{n}(s, t)]^n, \quad (108)$$

where[3)]

$$\bar{n}(s, t) = -\frac{g^2}{(2\pi)^3} \int \frac{d\mathbf{k}}{2k_0} |\overline{j_\alpha^{(l)}}(k; p, q)|^2, \quad l = 1, 2, \quad (109)$$

is the mean number of particles produced in a two-nucleon collision at high energies $s \to \infty$ for fixed t.

Using (96) for $\bar{j}_\alpha$, we find that for $|t| \ll g^2m^2$ we have

$$\bar{n}(s, t) = -bt. \quad (110)$$

Here b depends on the chosen method of cutting off the integrals at the upper limit with respect to the momenta of the emitted mesons. In particular, under the conditions $R_\perp^2 \sim m^2$ and $1 \gg \alpha^2 \gg \mu^2/m^2$, where $\alpha = R_z/p_0$, we obtain

$$b = [2g^2/(3(2\pi)^2 m^2)] [\ln(m^2/\mu^2) + 1/2 + O(\mu/m)], \quad (111)$$

which is equal to twice the slope parameter of the diffraction exponential (86). Note also that the equation $2a = b$ holds in the infrared asymptotic limit $\mu \to 0$. In this case, when we sum in the expression (104) over the number of all emitted mesons, the dependence on the variable t cancels, which leads to the vanishing of the diffraction peak in the differential cross section. A similar feature has been noted in ref. 19 and is analogous to the scaling behavior of deep inelastic processes of hadron interaction at high energies.[17,18]

3. SOME MATHEMATICAL REALIZATIONS OF THE CONCEPT OF STRAIGHT-LINE PATHS IN THE METHOD OF FUNCTIONAL INTEGRATION

Above we have studied some applications of the straight-line path approximation in its simplest form. Putting it crudely, we have used the approximation $\int [\delta\nu] \exp F \approx \exp(\int [\delta\nu] F)$ or even $\exp(F_{\nu=0})$. In other words, we have assumed that when particles are scattered in the asymptotic region of high energies and bounded momentum transfer only the paths that differ least from the classical trajectories of particles contribute to the Feynman path integral.

Below, we consider a number of approximation systems that are different mathematical realizations of the physical concept of straight-line paths in the method of functional integration. We may mention that methods of measure theory and integration in function spaces have been widely used in recent years in quantum field theory investigations. This approach is based on a representation of the solutions of the exact equation of the theory in the form of functional integrals. However, because of the absence of a developed technique of calculation of general quadratures the functional integrals are "a law unto themselves" in the sense that necessary information can be extracted only by some approximation procedure.

The simplest and best known approximation procedures are those in which one deals with only Gaussian quadratures at each stage of the calculations. It is in this class of approximations that the $k_ik_j = 0$ approximation and the straight-line path approximation discussed above belong. The approximation procedured developed below go back conceptually to the idea of rectilinear paths and enable one, in particular, to estimate consistently the effects of deviation of the particle paths from rectilinear trajectories in high-energy scattering processes.

Formulation of Approximations

We consider a functional integral in Gaussian measure:

$$\int \frac{\delta\nu}{\text{const}} \exp\left[-i \int d\xi \nu^2(\xi)\right] \exp(g\pi[\nu]), \tag{112}$$

where $\pi[\nu]$ is functional and the constant is the normalization constant. It is well known that the calculation of (112) can be reduced to the finding of functional derivatives in accordance with the equation

$$\int [\delta\nu] \exp(g\pi[\nu]) = \exp\left\{\frac{1}{4i} \int d\xi \frac{\delta^2}{\delta\nu^2(\xi)}\right\} \exp(g\pi[\nu])\Big|_{\nu=0}. \tag{113}$$

In addition, in some quantum field theory problems (see, for example, ref. 31) one needs to determine the action of the differential operator

$$\exp\left\{\frac{i}{2} \int D \frac{\delta^2}{\delta\nu^2}\right\},$$

where

$$\int D \frac{\delta^2}{\delta\nu^2} = \int d\xi_1 d\xi_2 D(\xi_1, \xi_2) \frac{\delta^2}{\delta\nu(\xi_1)\delta\nu(\xi_2)};$$

$D(\xi_1, \xi_2)$ is a function of propagator type. Having in mind later applications, we unite the two problems as follows. It is required to find the functional $\Pi[\nu]$ from the relation

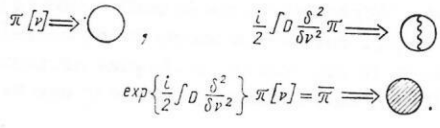
$$\exp(\Pi[\nu]) = \exp\left\{\frac{i}{2} \int D \frac{\delta^2}{\delta\nu^2}\right\} \exp(g\pi[\nu]) \equiv \overline{\exp(g\pi[\nu])}, \tag{114}$$

where $\pi[\nu]$ is a given functional; D is a function of two variables. If

$$D = -\delta(\eta_1 - \eta_2)/2, \tag{115}$$

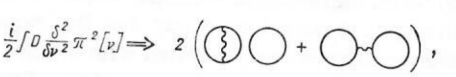
the value of the functional $\Pi[\nu]$ for $\nu = 0$ determines the functional integral in accordance with (113). To simplify the equations, we shall denote the action of the differential operator in some cases by the averaging sign, as in (114).

For greater clarity, we introduce the graphical notation



(116)

In this notation, for example, we obtain

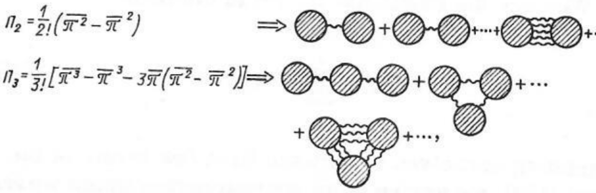


where the first two terms will be called unconnected graphs in the usual terminology. We emphasize that although the graphs (116) obviously have an analogy with Feynman graphs, in many cases their superficial appearance will have nothing in common with ordinary Feynman graphs.

We now assume that the structure of the functional $\pi[\nu]$ is such that there is a smallness parameter associated with loops. In this case, there exists an approximation procedure, which we shall call the correlation procedure and in accordance with which we seek the functional $\Pi[\nu]$ in the form of a series:

$$\Pi = \sum_{n=1}^{\infty} g^n \Pi_n. \tag{117}$$

Substituting this expression in (109), we readily obtain



(118)

Considering the graphs (118), we readily see that the correlation method does indeed correspond to an expansion in the number of loops, and that only the connected part of the sum of all graphs with n loops contributes to Π_n.

Terminating the series (117), we obtain an approximate expression for the functional Π. This approximation is valid if for all $n \geq 2$

$$\overline{\pi^n}\big|_{\substack{\text{connected}\\ \text{part}}} \ll \overline{\pi^n}\big|_{\substack{\text{unconnected}\\ \text{part}}} \tag{119}$$

In this case, when $\exp\Pi$ is expanded in a series in powers of g, allowance for only Π_1 gives the principal terms in each order, while allowance for Π_2 gives the corrections to it, etc.

The correlation procedure is intimately related to an expansion of the form

$$\overline{\exp(g\pi)} = \exp(g\bar{\pi})\left[1 + \sum_{n=2}^{\infty} \frac{g^n}{n!}\overline{(\pi - \bar{\pi})^n}\right]. \tag{120}$$

Such an expansion was encountered in refs. 1 and 5. In general, it has the same region of applicability as the correlation approximation, and differs from it in that it gives fewer correction terms in each order in g. However, the higher correction terms in the correlation expansion have, in our opinion, a simpler geometrical meaning [see (118)], which makes it easier to use to a certain extent.

As we have pointed out above, these approximations are good when there exists a smallness parameter associated with loops. It may also occur that the theory contains a smallness parameter associated with a line, i.e., that arises when the functional $\pi[\nu]$ is varied. Then one can make an expansion in the number of lines joining different loops. Representing π in the form

$$\pi[\nu] = \int d\eta \tilde{\pi}[\eta] \exp\left[-i\int \eta(\xi)\,\nu(\xi)\,d\xi\right] \tag{121}$$

and substituting (121) into (120), we obtain

$$\exp(\Pi_\varepsilon[\nu]) = 1 + \sum_{n=1}^{\infty} \frac{g^n}{n!} \int \prod_{j=1}^{n} \{\delta\eta_j \tilde{\pi}[\eta_j]\}$$
$$\times \exp\left[-i\int \nu\left(\sum_{j=1}^{n}\eta_j\right) - \frac{i}{2}\int D\left(\sum_{j=1}^{n}\eta_j^2\right)\right]$$
$$\times \exp\left[-i\varepsilon\int D\left(\sum_{i<j}\eta_i\eta_j\right)\right], \tag{122}$$

where the smallness parameter ε is assigned to the terms with different η and $\Pi[\nu] = \Pi_\varepsilon[\nu]$ for $\varepsilon = 1$.

We seek the functional $\Pi_\varepsilon[\nu]$ in the form

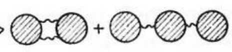
$$\Pi_\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n \Pi_{n+1}. \tag{123}$$

Restricting ourselves to only the first few terms of the series (123), we arrive at an approximation which we shall call the $\eta_i\eta_j$ approximation.

The calculations lead to the following expressions for the first terms:

$$\Pi_1 = g\bar{\pi} \Longrightarrow$$  ,

$$\Pi_2 = \frac{ig^2}{2}\int D\left(\frac{\delta\bar{\pi}}{\delta\nu}\right)^2 \Longrightarrow$$  ,

$$\Pi_3 = \frac{g^2}{2i}\int D_{13}D_{24}\frac{\delta^2\bar{\pi}}{\delta\nu_1\delta\nu_2}\left(\frac{1}{2i}\cdot\frac{\delta^2\bar{\pi}}{\delta\nu_3\delta\nu_4} + g\frac{\delta\bar{\pi}}{\delta\nu_3}\cdot\frac{\delta\bar{\pi}}{\delta\nu_4}\right) \Longrightarrow$$  , (124)

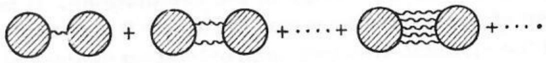
where the numbers indicate the order of a convolution, i.e.,

$$\int D_{13}D_{24}\frac{\delta^2\bar{\pi}}{\delta\nu_1\delta\nu_2}\cdot\frac{\delta^2\bar{\pi}}{\delta\nu_3\delta\nu_4} \equiv \int d\xi_1\,d\xi_2\,d\xi_3\,d\xi_4$$
$$\times D(\xi_1, \xi_2)\,D(\xi_2, \xi_4)$$
$$\times \frac{\delta^2\bar{\pi}}{\delta\nu(\xi_1)\,\delta\nu(\xi_2)}\cdot\frac{\delta^2\bar{\pi}}{\delta\nu(\xi_3)\,\delta\nu(\xi_4)} \quad \text{etc.}$$

Thus, we actually obtain an expansion in the number of lines connecting different loops. Since we are dealing with connected graphs, k and n, the numbers of such lines and loops, satisfy the inequality

$$k \geqslant n - 1. \tag{125}$$

This means that the sum of the first n terms of the $\eta_i\eta_j$ approximation is contained in the analogous sum of the correlation approximation, i.e., the region of applicability of the former is larger than that of the latter. Its use, however, may appreciably simplify calculations, since in the case of the former there is no need to consider the sum



The first terms in all these approximations are equal, and differences appear only when the corrections are calculated. This reflects the fact that when these methods are used to calculate high-energy scattering amplitudes they are different variants of the straight-line path approximation.

Corrections to the Eikonal Equation

Let us consider the application of the above methods for the actual examples of the scattering amplitude of two scalar nucleons in the model $L_{int} = g{:}\psi^+\psi\varphi$, which can be represented in the following form[5] if radiative corrections and the vacuum polarization contributions are ignored:

$$f_{el}(p_1, p_2; q_1, q_2) = \frac{ig^2}{(2\pi)^4}\int d^4x D(x)\exp[-ix(p_1 - q_1)]\int_0^1 d\lambda$$
$$\times S_\lambda(x; p_1, p_2; q_1, q_2) + (q_1 \leftrightarrow q_2), \tag{126}$$

where

$$S_\lambda = \int [\delta\nu_1]_{-\infty}^{\infty}[\delta\nu_2]_{-\infty}^{\infty}\exp\left\{ig^2\lambda\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\tau D\left[-x + 2\xi a_1(\xi)\right.\right.$$
$$- 2\tau a_2(\tau) - 2\int_{-\xi}^{0}d\eta\nu_1(\eta)$$
$$\left.\left. + 2\int_{-\tau}^{0}d\eta\nu_2(\eta)\right]\right\} \overset{\text{def}}{=} \int [\delta\nu_1]_{-\infty}^{\infty}[\delta\nu_2]_{-\infty}^{\infty}\exp(ig^2\lambda\pi); \tag{127}$$

$$a_{1,2}(\xi) = p_{1,2}\vartheta(\xi) + q_{1,2}\vartheta(-\xi). \tag{128}$$

We shall seek the asymptotic behavior of the functional integral S_λ at large $s = (p_1 + p_2)^2$ and fixed momentum transfers $t = (p_1 - q_1)^2$. The calculations show that in this case the parameter $1/s$ is associated with the loops and $1/\sqrt{s}$ is associated with lines[4]) and therefore with allowance for the relation (125) in the n-th order in g^2 for S_λ for fixed $x \neq 0$ the principal term has the asymptotic behavior $1/s^n$ and the next correction is $1/s^n\sqrt{s}$. Thus, if it necessary to calculate only the first two asymptotic terms in each order in g^2, it is convenient to use the $\eta_i\eta_j$ approximation generalized for the two functional variables ν_1 and ν_2, and it is sufficient to use the approximation $\exp \Pi_2 = 1 + \Pi_2 + \ldots$ with respect to the type of the expansion (120). Thus, the approximate expression for S_λ is

$$S_\lambda \approx \exp(ig^2\lambda\bar{\pi})\left[1+\frac{i\lambda^2 g^4}{4}\int d\xi \sum_{i=1,2}\left(\frac{\delta\bar{\pi}}{\delta\nu_i(\eta)}\right)^2\right]\Big|_{\nu=0}. \quad (129)$$

Proceeding from Eq. (127) for S_λ, we obtain the expression

$$\bar{\pi}|_{\nu=0} = \frac{1}{(2\pi)^4}\int d^4kD(k)\exp(-ikx) \times \int_{-\infty}^{\infty} d\xi\, d\tau \exp\{2ik[\xi a_1(\xi) - \tau a_2(\tau)]\}$$
$$\times \exp[ik^2(|\xi|+|\tau|)] = \frac{1}{(2\pi)^4 s}\int d^4kD(k)\exp(-ikx) \times \int_{-\infty}^{\infty} d\xi\, d\tau \exp\{2ik[\xi a_1(\xi)/\sqrt{s} - \tau a_2(\tau)/\sqrt{s}]\}$$
$$\times \exp\left[i\frac{k^2}{\sqrt{s}}(|\xi|+|\tau|)\right]. \quad (130)$$

In this equation, we have made the change of variables $\xi, \tau \to \xi/\sqrt{s}, \tau/\sqrt{s}$.

Similarly,

$$\frac{i\lambda^2 g^4}{4}\int d\eta\left[\left(\frac{\delta\bar{\pi}}{\delta\nu_1(\eta)}\right)^2+\left(\frac{\delta\bar{\pi}}{\delta\nu_2(\eta)}\right)^2\right]\Big|_{\nu=0}$$
$$= -\frac{i\lambda^2 g^4}{(2\pi)^8 s^2}\int d^4k_1 d^4k_2 D(k_1)D(k_2)\exp[-ix(k_1+k_2)](k_1k_2)$$
$$\times \int_{-\infty}^{\infty} d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2 \exp\left\{2ik_1\left[\xi_1\frac{a_1(\xi_1)}{\sqrt{s}} - \tau_1\frac{a_2(\tau_1)}{\sqrt{s}}\right]\right.$$
$$\left.+ i\frac{k_1^2}{\sqrt{s}}(|\xi_1|+|\tau_1|)\right\}\exp\left\{2ik_2\left[\xi_2\frac{a_1(\xi_2)}{\sqrt{s}} - \tau_2\frac{a_2(\tau_2)}{\sqrt{s}}\right]\right.$$
$$\left.+ i\frac{k_2^2}{\sqrt{s}}(|\xi_2|+|\tau_2|)\right\}\frac{1}{\sqrt{s}}[\Phi(\xi_1,\xi_2)+\Phi(\tau_1,\tau_2)], \quad (131)$$

where

$$\Phi(\xi_1,\xi_2) = \vartheta(\xi_1,\xi_2)[|\xi_1|\vartheta(|\xi_2|-|\xi_1|)+|\xi_2|\vartheta(|\xi_1|-|\xi_2|)]. \quad (132)$$

We now find the asymptotic behavior of the expressions (130) and (131) as $s \to \infty$ for fixed t. The expression (130) must then be calculated to terms $O(1/s^2)$ and (131) to $O(1/s^3)$. For this, we go over to the center-of-mass system and take the z axis along the momenta of the incident particles. Then

$$\left.\begin{aligned} p_{1,2} &= \left\{\frac{\sqrt{s}}{2},\ 0,\ 0,\ \pm\frac{\sqrt{s-4m^2}}{2}\right\};\\ q_{1,2} &= \left\{\frac{\sqrt{s}}{2},\ \pm\Delta_\perp\sqrt{1+\frac{t}{s-4m^2}},\right.\\ &\left.\pm\frac{\sqrt{s-4m^2}}{2}\left(1+\frac{2t}{s-4m^2}\right)\right\};\\ \Delta_\perp^2 &= -t, \end{aligned}\right\} \quad (133)$$

and, substituting (133) into (128), we obtain in the limit $s \to \infty$ for fixed t

$$\left.\begin{aligned} \frac{a_1(\xi)}{\sqrt{s}} &\approx \frac{1}{2}n^+ + \frac{\Delta_\perp}{\sqrt{s}}\vartheta(-\xi)+O\left(\frac{1}{s}\right);\\ \frac{a_2(\xi)}{\sqrt{s}} &\approx \frac{1}{2}n^- - \frac{\Delta_\perp}{\sqrt{s}}\vartheta(-\xi)+O\left(\frac{1}{s}\right);\\ n^\pm &= (1,\ 0,\ 0,\ \pm 1). \end{aligned}\right\} \quad (134)$$

Using (134), we obtain asymptotic expressions for (130) and (131). Namely

$$\bar{\pi} = \frac{1}{(2\pi)^4 s}\int d^4kD(k)\exp(-ikx) \times \int_{-\infty}^{\infty} d\xi\, d\tau \exp[i\xi(k_0-k_z) - i\tau(k_0+k_z)]$$
$$\times\left\{1 - 2i\frac{\mathbf{k}_\perp\Delta_\perp}{\sqrt{s}}[\xi\vartheta(-\xi)+\tau\vartheta(-\tau)]\right.$$
$$\left.+\frac{ik^2}{\sqrt{s}}(|\xi|+|\tau|)\right\}+O\left(\frac{1}{s^2}\right) \approx -\frac{1}{8\pi^2 s}\int\frac{d^2\mathbf{k}_\perp}{\mathbf{k}_\perp^2+\mu^2}\exp(i\mathbf{k}_\perp\mathbf{x}_\perp)$$
$$+\frac{i\Delta_\perp}{s\sqrt{s}\,8\pi^2}[(x_0+x_z)\vartheta(-x_0-x_z)+(x_z-x_0)\vartheta(x_0-x_z)]$$
$$\times\int d^2\mathbf{k}_\perp\exp(i\mathbf{k}_\perp\mathbf{x}_\perp)\frac{\mathbf{k}_\perp}{\mathbf{k}_\perp^2+\mu^2}+\frac{i}{16\pi^2 s\sqrt{s}}(|x_0+x_z|+|x_0-x_z|)$$
$$\times\int d^2\mathbf{k}_\perp\exp(i\mathbf{k}_\perp\mathbf{x}_\perp)\frac{\mathbf{k}_\perp}{\mathbf{k}_\perp^2+\mu^2}+O\left(\frac{1}{s^2}\right)$$
$$= -\frac{1}{4\pi s}K_0(\mu|\mathbf{x}_\perp|)-\frac{\mu}{4\pi s\sqrt{s}}\frac{\Delta_\perp\mathbf{x}_\perp}{|\mathbf{x}_\perp|}$$
$$\times[(x_0+x_z)\vartheta(-x_0-x_z)+(x_z-x_0)\vartheta(x_0-x_z)]$$
$$\times K_1(\mu|\mathbf{x}_\perp|)-\frac{i\mu^2}{8\pi s\sqrt{s}}(|x_0+x_z|+|x_0-x_z|)$$
$$\times K_0(\mu|\mathbf{x}_\perp|)+O\left(\frac{1}{s^2}\right); \quad (135)$$

$$\frac{i\lambda^2 g^4}{4}\int d\eta\left[\left(\frac{\delta\bar{\pi}}{\delta\nu_1(\eta)}\right)^2+\left(\frac{\delta\bar{\pi}}{\delta\nu_2(\eta)}\right)^2\right]\Big|_{\nu=0}$$
$$\approx -\frac{i\lambda^2 g^4}{(2\pi)^8 s^2\sqrt{s}}\int d^4k_1 d^4k_2 D(k_1)D(k_2)\exp[-ix(k_1+k_2)](k_1k_2)$$
$$\times\int_{-\infty}^{\infty} d\xi_1\, d\tau_1\, d\xi_2\, d\tau_2\exp[i\xi_1(k_{10}-k_{1z})-i\tau(k_{10}+k_{1z})]$$
$$\times\exp[i\xi_2(k_{20}-k_{2z})-i\tau_2(k_{20}+k_{2z})][\Phi(\xi_1,\xi_2)$$
$$+\Phi(\tau_1,\tau_2)]+O\left(\frac{1}{s^3}\right)$$
$$= -\frac{i\lambda^2 g^4\mu^2}{32\pi^2 s^2\sqrt{s}}(|x_0+x_z|+|x_0-x_z|)K_1^2(\mu|\mathbf{x}_\perp|). \quad (136)$$

Here we assume $|\mathbf{x}_\perp| \neq 0$, which ensures that all the integrals converge. The functions K_0 and K_1 are MacDonald functions of zeroth and first order and are determined by the expressions

$$K_0(\mu|\mathbf{x}_\perp|) = \frac{1}{2\pi}\int d^2\mathbf{k}_\perp \frac{\exp(i\mathbf{k}_\perp\mathbf{x}_\perp)}{\mathbf{k}_\perp^2+\mu^2};$$

$$K_1(\mu|\mathbf{x}_\perp|) = -\frac{\partial K_0(\mu|x_\perp|)}{\partial(\mu|x_\perp|)}.$$

We now substitute (135) and (136) into (129) and obtain for S_λ the desired expression[15]

$$S_\lambda \approx \exp\left[-\frac{ig^2\lambda}{4\pi s}K_0(\mu|\mathbf{x}_\perp|)\right]\left\{1-\frac{ig^2\lambda\mu}{4\pi s\sqrt{s}}\cdot\frac{\Delta_\perp\mathbf{x}_\perp}{|\mathbf{x}_\perp|}\right.$$
$$\times[(x_0+x_z)\vartheta(-x_0-x_z)+(x_z-x_0)\vartheta(x_0-x_z)]K_1(\mu|\mathbf{x}_\perp|)$$
$$+\frac{g^2\lambda\mu^2}{8\pi s\sqrt{s}}(|x_0+x_z|+|x_0-x_z|)K_0(\mu|\mathbf{x}_\perp|)$$
$$\left.-\frac{ig^4\lambda^2\mu^2}{32\pi^2 s^2\sqrt{s}}(|x_0+x_z|+|x_0-x_z|)K_1^2(\mu|\mathbf{x}_\perp|)\right\}. \quad (137)$$

In this expression, the factor in front of the braces corresponds to the eikonal behavior of the scattering amplitude, while the terms in the braces determine the correction of relative magnitude $1/\sqrt{s}$.

As is well known from the investigation of the scattering amplitude in the Feynman diagrammatic technique, the high-energy asymptotic behavior can contain only logarithms and integral powers of s. A similar effect is observed here, since integration of the expression (137) for S_λ in accordance with (126) leads to the vanishing of the coefficients of the half-integral powers of s. Nevertheless, allowance for the terms that contain half-integral powers of s is needed for the calculations of the next corrections in the scattering amplitude. It is interesting to note the appearance in the correction terms of a dependence on x_0 and x_z, i.e., the appearance of the so-called retardation effects, which are absent in the principal asymptotic term.

Making similar calculations, we can show that all the following terms decrease sufficiently rapidly compared with those we have written down. However, it must be emphasized that this by no means proves the validity of the eikonal representation for the scattering amplitude in the given framework, for the coefficient functions in the asymptotic expansion, which are expressed in terms of MacDonald functions, are singular at short distances and this singularity becomes stronger with increasing rate of decrease of the corresponding term at large s. Therefore, integration of S_λ in accordance with (126) in the determination of the scattering amplitude may lead to the appearance of terms that violate the eikonal series in the higher order in g^2. The possible appearance of such terms in individual orders of perturbation theory in models of the type φ^3 was pointed out in refs. 6 and 36.

We may mention that in the framework of the quasipotential approach in quantum field theory there is a rigorous justification of the eikonal representation on the basis of the assumption of a smooth local quasipotential. In the example just considered we have a singular interaction, which when radiative effects are ignored leads to a quasipotential of the Yukawa type, which requires special care.

4. OPERATOR METHOD AND THE STRAIGHT-LINE PATH APPROXIMATION

Formulation of the Operator Method

We consider a quasipotential equation with local quasipotential for the scattering amplitude of scalar particles:

$$T(\mathbf{p},\mathbf{p}';s) = gV(\mathbf{p}-\mathbf{p}';s) + g\int d\mathbf{q}K(\mathbf{q}^2;s)V(\mathbf{p}-\mathbf{q};s)T(\mathbf{q},\mathbf{p}';s), \quad (138)$$

where p and p' are the relative momenta of the particles in the center-of-mass system in the initial and the final state: $s = 4(p^2+m^2) = 4(p'^2+m^2)$.

To solve Eq. (138) we make a Fourier transformation:

$$V(\mathbf{p}-\mathbf{p}';s) = \frac{1}{(2\pi)^3}\int d\mathbf{r}\exp[i(\mathbf{p}-\mathbf{p}')\mathbf{r}]V(\mathbf{r};s); \quad (139)$$

$$T(\mathbf{p},\mathbf{p}';s) = \int d\mathbf{r}\,d\mathbf{r}'\exp(i\mathbf{p}\mathbf{r}-i\mathbf{p}'\mathbf{r}')T(\mathbf{r},\mathbf{r}';s). \quad (140)$$

Substituting (139) and (140) into (138), we obtain

$$T(\mathbf{r},\mathbf{r}';s) = \frac{g}{(2\pi)^3}V(\mathbf{r};s)\delta^{(3)}(\mathbf{r}-\mathbf{r}') + \frac{g}{(2\pi)^3}\int d\mathbf{q}K(\mathbf{q}^2;s)V(\mathbf{r};s)\exp(-i\mathbf{q}\mathbf{r}) \times\int d\mathbf{r}''\exp(i\mathbf{q}\mathbf{r}'')T(\mathbf{r}',\mathbf{r}'';s). \quad (141)$$

Introducing the representation

$$T(\mathbf{r},\mathbf{r}';s) = \frac{g}{(2\pi)^3}V(\mathbf{r};s)F(\mathbf{r},\mathbf{r}';s), \quad (142)$$

we have

$$F(\mathbf{r},\mathbf{r}';s) = \delta^{(3)}(\mathbf{r}-\mathbf{r}') + \frac{g}{(2\pi)^3}\int d\mathbf{q}K(\mathbf{q}^2;s)\exp(-i\mathbf{q}\mathbf{r}) \times\int d\mathbf{r}''\exp(i\mathbf{q}\mathbf{r}'')V(\mathbf{r}'';s)F(\mathbf{r}',\mathbf{r}'';s). \quad (143)$$

We define the pseudodifferential operator

$$\hat{L}_\mathbf{r} = K(-\nabla_\mathbf{r}^2;s), \quad (144)$$

and then

$$K(\mathbf{r};s) = \int d\mathbf{q}\exp(-i\mathbf{q}\mathbf{r})K(\mathbf{q}^2;s) = \hat{L}_\mathbf{r}(2\pi)^3\delta^3(\mathbf{r}). \quad (145)$$

With allowance for (145), Eq. (143) can be rewritten in the symbolic form

$$F(\mathbf{r},\mathbf{r}';s) = \delta^{(3)}(\mathbf{r}-\mathbf{r}') + g\hat{L}_\mathbf{r}[V(\mathbf{r};s)F(\mathbf{r},\mathbf{r}';s)]. \quad (146)$$

We shall seek the solution of this equation in the form

$$F(\mathbf{r},\mathbf{r}';s) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\exp[W(\mathbf{r};\mathbf{k};s)]\exp[-i\mathbf{k}(\mathbf{r}-\mathbf{r}')]. \quad (147)$$

Substituting (147) into (146), we obtain an equation for the function

$$\exp[W(\mathbf{r};\mathbf{k};s)] = 1 + g\hat{L}_{\mathbf{r}}\{V(\mathbf{r};s)\exp[W(\mathbf{r};\mathbf{k};s) - i\mathbf{kr}]\}\exp(i\mathbf{kr}). \quad (148)$$

Using the idea of modified perturbation theory in the exponential,[21] we write the function W (r; k; s) as an expansion in the coupling constant g:

$$W(\mathbf{r};\mathbf{k};s) = \sum_{n=1}^{\infty} g^n W_n(\mathbf{r};\mathbf{k};s). \quad (149)$$

Then from Eq. (148) we obtain expressions for the functions

$$W_1(\mathbf{r};\mathbf{k};s) = \int d\mathbf{q}V(\mathbf{q};s)K[(\mathbf{k}+\mathbf{q})^2;s]\exp(i\mathbf{qr}); \quad (150)$$

$$W_2(\mathbf{r};\mathbf{k};s) = -\frac{W_1^2(\mathbf{r};\mathbf{k};s)}{2} + \frac{1}{2}\int d\mathbf{q}_1 d\mathbf{q}_2 \exp(-i\mathbf{q}_1\mathbf{r} - i\mathbf{q}_2\mathbf{r}) \times V(\mathbf{q}_1;s)V(\mathbf{q}_2;s)K[(\mathbf{q}_1+\mathbf{q}_2+\mathbf{k})^2;s] \times \{K[(\mathbf{q}_1+\mathbf{k})^2;s] + K[(\mathbf{q}_2+\mathbf{k})^2;s]\}, \quad (151)$$

etc.

Restricting the treatment to W_1 instead of W in (147), we obtain from (147), (142), and (140) the following approximate expression for the scattering amplitude:[32]

$$T_1(\mathbf{p},\mathbf{p}';s) = \frac{g}{(2\pi)^3}\int d\mathbf{r}\exp[i(\mathbf{p}-\mathbf{p}')\mathbf{r}] \times V(\mathbf{r};s)\exp[gW_1(\mathbf{r};\mathbf{p};s)]. \quad (152)$$

To elucidate the meaning of our approximation, we expand T_1 in a series in the coupling constant g:

$$T_1^{(n+1)}(\mathbf{p},\mathbf{p}';s) = \frac{g^{n+1}}{n!}\int d\mathbf{q}_1\ldots d\mathbf{q}_n V(\mathbf{q}_1;s)\ldots V(\mathbf{q}_n;s) \times V\left(\mathbf{p}-\mathbf{p}'-\sum_{i=1}^{n}\mathbf{q}_i;s\right)\prod_{i=1}^{n}K[(\mathbf{q}_i+\mathbf{p}')^2;s] \quad (153)$$

and compare it with the (n + 1)-th iteration term of the exact equation (138):

$$T^{(n+1)}(\mathbf{p},\mathbf{p}';s) = \frac{g^{n+1}}{n!}\int d\mathbf{q}_1\ldots d\mathbf{q}_n V(\mathbf{q}_1;s)\ldots V(\mathbf{q}_n;s) \times V\left(\mathbf{p}-\mathbf{p}'-\sum_{i=1}^{n}\mathbf{q}_i;s\right)\sum_{p}K[(\mathbf{q}_1+\mathbf{p}')^2;s] \times K[(\mathbf{q}_1+\mathbf{q}_2+\mathbf{p}')^2;s]\ldots K\left[\left(\sum_{i=1}^{n}\mathbf{q}_i+\mathbf{p}'\right)^2;s\right], \quad (154)$$

where $\sum_p$ is the sum over the permutations of the momenta $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$.

It is easy to see from (153) and (154) that this approximation in the case of the Lippmann–Schwinger equation is identical to the so-called $\mathbf{q}_i\mathbf{q}_j = 0$ approximation, according to which terms of the type $\mathbf{q}_i\mathbf{q}_j$ $(i \neq j)$ are omitted in the nucleon propagators.

Operator Method and Asymptotic Behavior of the Scattering Amplitude

Taking the example of the Logunov–Tavkhelidze quasipotential equation, let us consider the case when the expressions previously obtained for the scattering amplitude can be used to find asympotic behaviors in the region $s \to \infty$ for fixed t. In the asymptotic expressions we shall take into account both the principal term and the following correction term, using the equation

$$\exp[W(\mathbf{r};\mathbf{p}';s)] = \exp[gW_1(\mathbf{r};\mathbf{p}';s)][1 + g^2W_2(\mathbf{r};\mathbf{p}';s) + \ldots], \quad (155)$$

where W_1 and W_2 are given by (150) and (151).

We take the z axis along the vector $\mathbf{p} + \mathbf{p}'$. Then

$$\mathbf{p}-\mathbf{p}' = \mathbf{\Delta}_\perp; \quad \mathbf{\Delta}_\perp\mathbf{n}_z = 0; \quad t = -\mathbf{\Delta}_\perp^2. \quad (156)$$

Bearing in mind that

$$K[(\mathbf{q}+\mathbf{p}')^2;s] = \frac{1}{\sqrt{(\mathbf{q}+\mathbf{p}')^2+m^2}}\cdot\frac{1}{(\mathbf{q}+\mathbf{p}')^2 - \frac{s}{4} + m^2 - i\varepsilon} \underset{\substack{s\to\infty \\ \text{fixed } t}}{=} \frac{2}{s(q_z - i\varepsilon)}\left[1 - \frac{3q_z + \mathbf{q}_\perp^2 - \mathbf{q}_\perp\mathbf{\Delta}_\perp}{\sqrt{s}(q_z - i\varepsilon)}\right] + O\left(\frac{1}{s^2}\right), \quad (157)$$

we obtain from (150) and (151)

$$W_1 = W_{10}/s + W_{11}/(s\sqrt{s}) + O(1/s^2); \quad (158)$$

$$W_2 = W_{20}/(s^2\sqrt{s}) + O(1/s^3), \quad (159)$$

where

$$W_{10} = 2\int d\mathbf{q}V(\mathbf{q};s)\frac{\exp(i\mathbf{qr})}{q_z - i\varepsilon} = 2i\int_z^{\infty} dz' V\left(\sqrt{\mathbf{r}_\perp^2 + z'^2};s\right); \quad (160)$$

$$W_{11} = -2\int d\mathbf{q}V(\mathbf{q};s)\exp(-i\mathbf{qr})\frac{3q_z^2 + \mathbf{q}_\perp^2 - \mathbf{q}_\perp\mathbf{\Delta}_\perp}{(q_z - i\varepsilon)^2} = -6V\left(\sqrt{\mathbf{r}_\perp^2 + z^2};s\right) + 2(\nabla_\perp^2 + i\mathbf{\Delta}_\perp\nabla_\perp) \times \int_z^{\infty} dz'(z - z')V\left(\sqrt{\mathbf{r}_\perp^2 + z'^2};s\right); \quad (161)$$

$$W_{20} = -4\int d\mathbf{q}_1 d\mathbf{q}_2 \exp[-i(\mathbf{q}_1+\mathbf{q}_2)\mathbf{r}]V(\mathbf{q}_1;s)V(\mathbf{q}_2;s) \times \frac{3q_{1z}q_{2z} + \mathbf{q}_{1\perp}\mathbf{q}_{2\perp}}{(q_{1z} - i\varepsilon)(q_{2z} - i\varepsilon)(q_{1z} + q_{2z} - i\varepsilon)} = -4i\int_z^{\infty} dz'\left\{3V^2\left(\sqrt{\mathbf{r}_\perp^2 + z'^2};s\right) + \left[\nabla_\perp\int_{z'}^{\infty} dz'' V\left(\sqrt{\mathbf{r}_\perp^2 + z''^2};s\right)\right]^2\right\}. \quad (162)$$

To determine the desired asymptotic behavior with the given accuracy, it is sufficient to write the scattering amplitude in the form

$$T(\mathbf{p},\mathbf{p}';s) = \frac{g}{(2\pi)^3}\int d\mathbf{r}_\perp dz\exp(i\mathbf{\Delta}_\perp\mathbf{r}_\perp)V\left(\sqrt{\mathbf{r}^2 + z^2};s\right) \times \exp\left(g\frac{W_{10}}{s}\right)\left(1 + g\frac{W_{11}}{s\sqrt{s}} + g^2\frac{W_{20}}{s^2\sqrt{s}} + \ldots\right). \quad (163)$$

Then, substituting (160)-(162) into (163) and making simple but fairly tedious calculations, we obtain for smooth potentials the well known expression[10]

$$T(t, s) \underset{\substack{s\to\infty \\ \text{fixed } t}}{=} \frac{s}{(2\pi)^3}\int d^2\mathbf{r}_\perp \exp(i\Delta_\perp \mathbf{r}_\perp)$$

$$\times \frac{\exp\left[\frac{2ig}{s}\int_{-\infty}^{\infty} dz V(\sqrt{\mathbf{r}_\perp^2+z^2};\, s)\right]-1}{2i}$$

$$-\frac{6g^2}{(2\pi)^3 s\sqrt{s}}\int d^2\mathbf{r}_\perp \exp(i\Delta_\perp \mathbf{r}_\perp)$$

$$\times \exp\left[\frac{2ig}{s}\int_{-\infty}^{\infty} dz' V(\sqrt{\mathbf{r}_\perp^2+z'^2};\, s)\right]\int_{-\infty}^{\infty} dz V^2(\sqrt{\mathbf{r}_\perp^2+z^2};\, s)$$

$$-\frac{ig}{(2\pi)^3\sqrt{s}}\int d^2\mathbf{r}_\perp \exp(i\Delta_\perp \mathbf{r}_\perp)\int_{-\infty}^{\infty} dz$$

$$\times\left\{\exp\left[\frac{2ig}{s}\int_{z}^{\infty} dz' V(\sqrt{\mathbf{r}_\perp^2+z'^2};\, s)\right]\right.$$

$$\left.-\exp\left[\frac{2ig}{s}\int_{-\infty}^{\infty} dz' V(\sqrt{\mathbf{r}_\perp^2+z'^2};\, s)\right]\right\}\left\{\int_{z}^{\infty} dz'\nabla_\perp^2 V(\sqrt{\mathbf{r}_\perp^2+z'^2};\, s)\right.$$

$$\left.+\frac{2ig}{s}\left[\int_{z}^{\infty} dz'\nabla_\perp V(\sqrt{\mathbf{r}_\perp^2+z'^2};\, s)\right]^2\right\}$$

$$-\frac{ig}{(2\pi)^3\sqrt{s}}\Delta_\perp^2\int d^2\mathbf{r}_\perp \exp(i\Delta_\perp \mathbf{r}_\perp)\int_{-\infty}^{\infty} z\,dz V(\sqrt{\mathbf{r}_\perp^2+z^2};\, s)$$

$$\times \exp\left[\frac{2ig}{s}\int_{z}^{\infty} dz' V(\sqrt{\mathbf{r}_\perp^2+z'^2};\, s)\right]. \qquad (164)$$

It is easy to see that the first term in this equation describes the eikonal behavior of the scattering amplitude, and all the remaining terms determine the corrections to it of relative magnitude $1/\sqrt{s}$.

Relationship between the Operator Method and the Path Integral Method

To establish what actual physical picture may correspond to our results, we establish the relationship between the operator method and the Feynman path integral method. For this, we return to Eq. (148) for the function W. The solution of this equation can be written in the symbolic form

$$\exp W = \frac{1}{1-gK[(-i\nabla-\mathbf{k})^2]V(\mathbf{r})}\times 1 \qquad (165)$$

$$= -i\int_0^\infty d\tau \exp[i\tau(1+i\varepsilon)]\exp\{-i\tau gK[(-i\nabla-\mathbf{k})^2]V(\mathbf{r})\}\times 1.$$

In accordance with the Feynman parametrization,[28] we introduce an ordering index η and rewrite (165) in the form

$$\exp W = -i\int_0^\infty d\tau \exp[i\tau(1+i\varepsilon)]$$

$$\times \exp\left\{-ig\int_0^\tau d\eta\, K[(-i\nabla_{\eta+\varepsilon}-\mathbf{k})^2]V(\mathbf{r}_\eta)\right\}\times 1. \qquad (166)$$

Using the Feynman transformation

$$\mathscr{F}[\mathbf{P}(\eta)] = \int \mathscr{D}\mathbf{p}\int_{\mathbf{x}(0)=0}\mathscr{D}\frac{\mathbf{x}}{(2\pi)^3}$$

$$\times \exp\left\{i\int_0^\tau d\eta\dot{\mathbf{x}}(\eta)[\mathbf{p}(\eta)-\mathbf{P}(\eta)]\right\}\mathscr{F}[\mathbf{p}(\eta)], \qquad (167)$$

we write the solution of Eq. (148) as the functional integral

$$\exp W = -i\int_0^\infty d\tau \exp\left[i\tau(1+i\varepsilon)\right]\int \mathscr{D}\mathbf{p}\int_{\mathbf{x}(0)=0}\mathscr{D}\frac{\mathbf{x}}{(2\pi)^3}$$

$$\times \exp\left[i\int_0^\tau d\eta\dot{\mathbf{x}}(\eta)\mathbf{p}(\eta)\right]G(\mathbf{x};\,\mathbf{p};\,\tau)\times 1. \qquad (168)$$

In this equation we have

$$G(\mathbf{x};\,\mathbf{p};\,\tau) = \exp\left[-\int_0^\tau d\eta\dot{\mathbf{x}}(\eta)\nabla_{\eta+\varepsilon}\right]$$

$$\times \exp\left\{-ig\int_0^\tau d\eta K[(\mathbf{p}(\eta)-\mathbf{k})^2]V(\mathbf{r}_\eta)\right\} \qquad (169)$$

and this function satisfies the equation

$$\left.\begin{array}{l}\frac{dG}{d\tau} = \{-igK[(\mathbf{p}(\tau)-\mathbf{k})^2]V(\mathbf{r})-\dot{\mathbf{x}}(\tau-\varepsilon)\nabla\}G;\\ G(\tau=0)=1.\end{array}\right\} \qquad (170)$$

Finding from this equation the operator function G and substituting it into (168), we find the final expression[16] for W:

$$\exp W = -i\int_0^\infty d\tau \exp[i\tau(1+i\varepsilon)]\int \mathscr{D}\mathbf{p}\int_{\mathbf{x}(0)=0}\mathscr{D}\frac{\mathbf{x}}{(2\pi)^3}$$

$$\times \exp\left[i\int_0^\tau d\eta\dot{\mathbf{x}}(\eta)\mathbf{p}(\eta)\right]\exp(g\pi), \qquad (171)$$

where

$$\pi = -i\int_0^\tau d\eta K[(\mathbf{p}(\eta)-\mathbf{k})^2]V\left[\mathbf{r}-\int_0^\tau d\xi\dot{\mathbf{x}}(\xi)\vartheta(\xi-\eta+\varepsilon)\right]. \quad (172)$$

Writing out the expansion

$$\exp W = \overline{\exp(g\pi)} = \exp(g\bar{\pi})\sum_{n=0}^{\infty}\frac{g^n}{n!}\overline{(\pi-\bar{\pi})^n}, \qquad (173)$$

in which the averaging sign denotes integration with respect to τ, $\mathbf{x}(\eta)$, and $\mathbf{p}(\eta)$ with corresponding measure [see, for example, Eq. (171)], and making the calculations, we find that

$$\bar{\pi} = W_1;\; (\overline{\pi^2}-\bar{\pi}^2)/2 = W_2\text{, etc.} \qquad (174)$$

i.e., the expansions (173) and (155) are identical.

Restricting ourselves in the expansion (173) to only the first term (n = 0), we obtain the approximate expression (152) for the scattering amplitude, which corresponds to our allowing for the particle paths that approximate most closely the classical paths and in the case of the scattering of high-energy particles through small angles coincide with rectilinear trajectories. In other words, we can say that the operator method is at high energies a realization of the concept of straight-line paths.

5. STRAIGHT-LINE PATHS AND THE EIKONAL PROBLEM

As we have shown above, the straight-line path method is based on the assumption that there is dominance of large momentum transfers in individual interactions of high-energy particles. Thus, the large momenta carried by particles in the collision process have a tendency to be conserved ("inertia" of large momenta). The kind of particle that carries the large momentum may change during the interaction process in accordance with empirical laws observed in inclusive processes. Thus, in a collision of fast nucleons one must allow for the possibility of emission of a "hard" meson, to which there is transferred an appreciable fraction of the initial-nucleon momentum.

Usually, in the derivation of the eikonal equation by summing the perturbation series, the initial particles are taken to be the leading particles that carry the large momenta. However, the existence of virtual processes in which the nature of the leading particles changes must lead to a modification of the orthodox eikonal representation. The possible appearance of extra terms in the asymptotic behavior of the sum of perturbation theory diagrams in addition to those in the ordinary eikonal formula was first noted in ref. 36.

We shall consider here the structure of the noneikonal contributions to the two-nucleon scattering amplitude, described by a sum of ladder diagrams without inclusion of radiative corrections and vacuum polarization effects in the scalar model.[33]

High-Energy Asymptotic Behavior of Feynman Graphs and Modification of the Propagators of Virtual Particles

We investigate the scattering amplitude of two scalar nucleons in the model $L_{int} = g{:}\psi^+\psi\varphi{:}$, ignoring radiative corrections and closed nucleon loops. This amplitude can be represented as a sum of diagrams of the type shown in Fig. 1, in which p_1 and p_2 are the momenta of the ingoing and q_1 and q_2 those of the outgoing particles. If there are l momenta of integration and I internal lines (in diagrams of this type, $I = 3l + 1$), then

$$F = \int dk_1 \ldots dk_l \prod_{i=1}^{I} \frac{1}{r_i^2 - m_i^2 + i\varepsilon}, \qquad (175)$$

where r_i are linear combinations of the momenta of integration k_j.

Introducing the Feynman parametrization, we obtain

$$F = (I-1)! \int_0^1 d\alpha_1 \ldots d\alpha_I \delta\left(1 - \sum_{i=1}^{I} \alpha_i\right) \int \frac{dk_1 \ldots dk_l}{[\Psi(k, \alpha, s, t)]^I}, \qquad (176)$$

where

$$\Psi = \sum_{i=1}^{I} \alpha_i (r_i^2 - m_i^2 + i\varepsilon) = \sum_{i, j=1}^{l} a_{ij} k_i k_j + 2 \sum_{i=1}^{l} b_i k_i + c, \qquad (177)$$

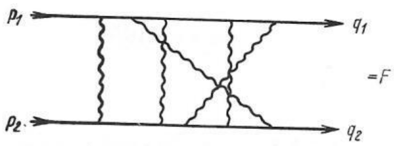


Fig. 1

after which we can integrate with respect to k_i and obtain a representation for F in Chisholm's form:[29]

$$F = (i\pi^2)^l (I - 2l - 1)! \int_0^1 d\alpha_1 \ldots d\alpha_I \delta\left(1 - \sum_{i=1}^{I} \alpha_i\right) \times \frac{[C(\alpha)]^{I-2l-2}}{[D(\alpha, s, t)]^{I-2l}}. \qquad (178)$$

In this equation, we have

$$C = \det \| a_{ij} \|; \quad D = \det \left\| \begin{array}{ccc|c} & a_{ij} & & b_1 \\ & & & \vdots \\ & & & b_l \\ \hline b_1 & \ldots & b_l & c \end{array} \right\|, \qquad (179)$$

and the Chisholm determinant D can be represented in the form

$$D(\alpha, s, t) = f(\alpha) s + g(\alpha) t + h(\alpha). \qquad (180)$$

Let us briefly review the main results of ref. 37, which we shall use to study the asymptotic behavior[5)] of the expression (178).

Definition. A t-path is a set of lines of a graph that form a continuous arc and the set is such that a) if all the lines of the set are collapsed into a point, the graph is split into two parts joined at only one vertex, the momenta p_1 and q_1 being joined to one of the parts and the momenta p_2 and q_2 to the other; b) none of the subsets of this set has the property a).

The length of a t-path is equal to the number of lines forming it; a t-path of minimal length is called a $\bar{t}$-path.

Rule. If a graph F is such that there exists a total M of t-paths of length ρ, its asymptotic behavior is given by

$$F \approx (i\pi^2)^l \frac{(I-2l-1-\rho)!\,\rho!}{(M-1)!} \frac{(\ln s)^{M-1}}{s^\rho} \int \frac{[C_0(\alpha)]^{I-2l-2}}{(g_0 t + h_0)^{I-2l-\rho} f_0^\rho} \times \prod_{j=1}^{M} \delta\left(\sum_{\nu=1}^{\rho} \alpha_\nu^{(j)} - 1\right) \delta\left(\sum_{\nu \not\subset P} \alpha_\nu - 1\right) \{d\alpha\}. \qquad (181)$$

Here

$$g_0 t + h_0 = D(\alpha, s, t)\,|_{\alpha_\nu^{(j)}=0}; \quad C_0(\alpha) = C(\alpha)\,|_{\alpha_\nu^{(j)}=0}; \qquad (182)$$

$\alpha_\nu^{(j)}$ are the parameters of the lines that belong to the j-th $\bar{t}$-path; α_ν ($\nu \not\subset P$) are the remaining parameters; and f_0 is obtained from f [see (180)] as follows.

Make the substitution

$$\alpha_v^{(j)} \to \lambda_j \alpha_v^{(j)}; \tag{183}$$

then

$$f \to \lambda_1 \lambda_2 \ldots \lambda_M \tilde{f}(\lambda) \text{ and } \tilde{f}_0 = \tilde{f}\,|_{\lambda_j=0}. \tag{184}$$

Having written down the necessary equation, we turn to the further exposition. In the case when the momentum transfer in graphs of type F (see Fig. 1) is equal to zero, i.e., $p_1 = q_1$ and $p_2 = q_2$, the set of lines whose propagators contain the momentum p_i will be called a p-path.

Thus, in the graphs there are two p-paths, each of which forms a continuous arc. Note that each p-path is a t-path by definition. However, in contrast to t-paths, which are topological characteristics of a graph, the disposition of p-paths depends on the actual arrangement of the momenta of integration. In the graphs considered, the latter can be arranged in such a way that the p-paths coincide with any pair of t-paths that do not form a closed loop.

Assertion 1. Suppose there is given a graph such that a pair of $\bar{t}$-paths that do not have a common line contribute to its principal asymptotic behavior. If the momenta of integration are arranged in such a way that the p-paths coincide with the $\bar{t}$-paths, and one then makes the following modification in the propagators that depend upon the external momenta:

$$\frac{1}{(\sum_i k_i)^2 + 2p\sum_i k_i - m_j^2 + i\varepsilon} \to \frac{1}{2p\sum_i k_i + i\varepsilon}, \tag{185}$$

i.e., one ignores the masses and the products of the momenta of integration, then the asymptotic behavior of this graph is not affected.

Proof. The modification of the propagators (185) leads to the following changes in the determinants C and D [see (179)]. In the determinant C the parameters corresponding to the $\bar{t}$-paths vanish, i.e., C goes over into C_0. In D, there is change in c, in which the same parameters vanish. This means that in $f(\alpha)$ [see (180)] the features associated with $\bar{t}$-paths are preserved, i.e., the previous asymptotic dependence on s is preserved. The quantities C_0, $g_0t + h_0$, and $\tilde{f}_0$ calculated by (182)-(184) are also unchanged. Taking into account the expression for the asymptotic behavior of the graph (181), we therefore see that the modification of the propagators (185) does not alter anything in this expression.

Assertion 2. Suppose there is a graph such that a pair of $\bar{t}$-paths having a common line contribute to its principal asymptotic behavior. Arrange the momenta of integration in such a way that the p-paths coincide with the $\bar{t}$-paths. Then its asymptotic behavior is the same as that of the reduced graph, obtained from the original graph by collapsing the common line into a point and multiplying by the factor $\pm 1/s$. The plus sign is chosen when the external momenta in the common line have the same direction and the minus sign is chosen otherwise. Assertion 1 applies to the reduced graph.

Proof. Suppose that the common line has parameter β and its corresponding propagator is

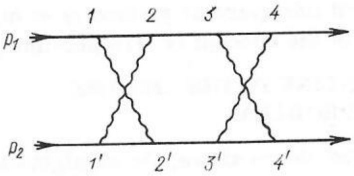


Fig. 2

$$\frac{1}{(\sum_i k_i)^2 + 2(p_1 \pm p_2)(\sum_i k_i) - M^2 \pm s + i\varepsilon}. \tag{186}$$

It is sufficient to prove that this propagator can be replaced by $\pm 1/s$; for C_0 and $g_0t + h_0$ are then unchanged, by the same arguments as we used to prove Assertion 1. The quantity f has the structure

$$f = \beta C + \begin{vmatrix} C & (\alpha^{(1)} + \beta) \\ (\alpha^{(2)} + \beta) & 0 \end{vmatrix}, \tag{187}$$

where $\alpha^{(1)}$ and $\alpha^{(2)}$ are the sets of parameters corresponding to the two $\bar{t}$-paths. It is now obvious that f can be replaced by

$$f^{(1)} = \beta C(\beta = 0) + \begin{vmatrix} C(\beta = 0) & (\alpha^{(1)}) \\ (\alpha^{(2)}) & 0 \end{vmatrix}, \tag{188}$$

which proves the assertion, since $f^{(1)}$ has the same singularities as f and leads to the same function $\tilde{f}_0$.

Eikonal and Noneikonal Contributions to the Scattering Amplitude

As is well known, the scattering amplitude of two scalar nucleons when radiative correction and vacuum polarization contributions are ignored can be represented in the form [cf. (126)-(128)]

$$f(p_1, p_2; q_1, q_2) = \frac{ig^2}{(2\pi)^4} \int d^2x D(x) \times \exp[-ix(p_1 - q_1)] \int_0^1 d\lambda S_\lambda + (q_1 \leftrightarrow q_2), \tag{189}$$

where

$$\left. \begin{aligned} S_\lambda &= \int [\delta\nu_1]_{-\infty}^{\infty} [\delta\nu_2]_{-\infty}^{\infty} \exp\Big\{ ig^2\lambda \\ &\times \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\tau D\Big[-x + 2\xi a_1(\xi) - 2\tau a_2(\tau) \\ &\quad - 2\int_{-\xi}^{0} \nu_1(\eta)\, d\eta + 2\int_{-\tau}^{0} \nu_2(\eta)\, d\eta\Big]\Big\}; \\ a_{1,2}(\xi) &= p_{1,2}\vartheta(\xi) + q_{1,2}\vartheta(-\xi). \end{aligned} \right\} \tag{190}$$

In the expressions (189) and (190) the momenta of integration of each individual diagram are arranged in such a way that the p-paths coincide with the nucleon lines. Setting ν_1 and ν_2 equal to zero, i.e., ignoring terms of the type k_ik_j in the nucleon propagators, and ignoring twisted graphs, we find by Assertion 1 the sum of the contributions

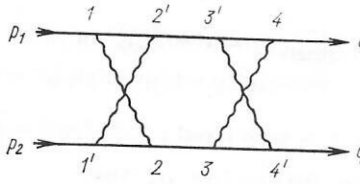


Fig. 3

to each diagram of the t-paths that coincide with nucleon lines. As a results, in the limit $s = (p_1 + p_2)^2 \to \infty$ with $t = (p_1 - q_1)^2$ fixed we obtain the well known eikonal representation of the scattering amplitude:

$$f \approx \frac{is}{(2\pi)^4} \int d^2\mathbf{x}_\perp \exp(-i\mathbf{x}_\perp \boldsymbol{\Delta}_\perp) \times \left\{ \exp\left[-\frac{ig^2}{4\pi s} K_0(\mu |\mathbf{x}_\perp|) \right] - 1 \right\}. \quad (191)$$

In this connection, the contributions from the t-paths that coincide with the nucleon lines will henceforth be called eikonal contributions.

In ref. 36 it was pointed out that in diagrams of higher orders in the coupling constant g (beginning with the eighth) one must also take into account other t-paths, that make contributions that are as great as the eikonal contributions. Let us begin our study of noneikonal contributions with the diagram of eighth order (Fig. 2).

In this diagram, which we shall call the XX diagram, there are four $\bar{t}$-paths of equal lengths of three: (1234), (1'2'3'4'), (1'234'), and (12'3'4). Formal allowance for all four paths would lead to the asymptotic behavior $(\ln^3 s)/s^3$. However, this corresponds to the vanishing of the parameters of all the lines in the diagram, which is impossible because of the factor $\delta(1 - \sum_i \alpha_i)$. Allowance for any three paths leads to the asymptotic behavior $(\ln^2 s)/s^3$, which is not the principal asymptotic behavior, since in this case its coefficient, which is proportional to C_0, will vanish because the $\bar{t}$-paths form a closed loop. Thus, one must calculate the sum of contributions from the following pairs of $\bar{t}$-paths:

$$(1234;\ 1'2'3'4'),\ (1234;\ 1'234'),$$
$$(12'3'4;\ 1'2'3'4'), \text{and } (12'3'4;\ 1'234'). \quad (192)$$

The pairs (1234; 12'3'4) and (1'2'3'4'; 1'234') do not contribute, since these $\bar{t}$-paths again form a closed loop. All pairs of $\bar{t}$-paths in (192) lead to the same asymptotic dependence on s of the form $(\ln s)/s^3$, and therefore in what follows we shall only be interested in the coefficients of this behavior.

A contribution to the XX diagram from the pair (1234; 1'2'3'4') is present in the expression (191). In this connection, we shall denote it by

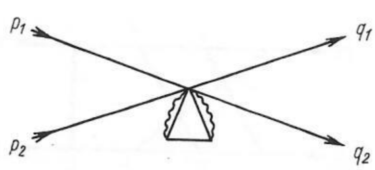


Fig. 4

$$(\ln s/s^3)\, f_{\text{eik}}^{(\text{XX})}(t). \quad (193)$$

We now find the contribution to the asymptotic behavior of XX from the $\bar{t}$-paths (12'3'4) and (1'234'). We arrange the momenta of integration in such a way that these paths coincide with p-paths. Then by Assertion 1 we can modify the propagators after the fashion of (185) in the lines in the $\bar{t}$-paths. We do this after the substitution of the momenta of integration

$$k_i \to k_i m/\mu. \quad (194)$$

Then the nucleon lines are replaced by meson lines:

$$D_m(km/\mu) = \frac{1}{k^2 m^2/\mu^2 - m^2 + i\varepsilon} = \mu^2 D_\mu(k)/m^2;$$

$$D_m(p_1 - q_1 - k) \to \mu^2 D_\mu[(p_1 - q_1)\mu/m - k]/m^2, \quad (195)$$

i.e., $t \to t\mu^2/m^2$, and the propagators corresponding to $\bar{t}$-paths merely acquire factors of the form μ/m. As a result, we can assume that all lines that occur in the $\bar{t}$-paths are modified nucleon lines. We then obtain a graph of the same form as in Fig. 2 but in which the p-paths are now directed along the nucleon lines (Fig. 3).

Thus, the desired contribution has the form

$$(\ln s/s^3)\, f_{\text{noneik}}^{(1)}(t);$$
$$f_{\text{noneik}}^{(1)}(t) = \mu^2 f_{\text{eik}}^{(\text{XX})}(t\mu^2/m^2)/m^2. \quad (196)$$

If the particle masses satisfy the condition

$$\mu^2/m^2 \ll 1, \quad t/m^2 \ll 1, \quad (197)$$

the contribution of the noneikonal $\bar{t}$-paths is small compared with the contribution of the eikonal paths.

It remains to consider only the contributions of the pairs of $\bar{t}$-paths (1'2'3'4') and (12'3'4) to the asymptotic behavior of the XX diagram. The second pair (1234) and (1'234') [see (192)], as yet uninvestigated, obviously makes the same contribution. If we collapse the $\bar{t}$-paths (1'2'3'4') and (12'3'4) into a point, we obtain a reduced graph of the form shown in Fig. 4.

It follows from this that the contribution of these $\bar{t}$-paths to the asymptotic behavior of XX does not depend on the momentum transfer, i.e., it can be represented in the form

$$\frac{\ln s}{s^3} \cdot \frac{1}{\mu^2} \varphi\left(\frac{\mu^2}{m^2}\right). \quad (198)$$

Let us find the form of the function $\varphi(\mu^2/m^2)$ when the condition (197) is satisfied. To do this, we arrange the momenta of integration in XX in such a way that the p-paths coincide with the $\bar{t}$-paths (1'2'3'4') and (12'3'4). Then, using Assertion 2, we find that the desired contribution is equal to the asymptotic behavior of the reduced graph (Fig. 5) multiplied by 1/s.

As $s \to \infty$, the asymptotic behavior of F' is, with allowance for (181),

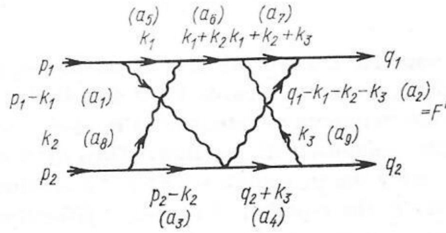


Fig. 5

$$F' \approx \frac{\ln s}{s^2}\,\mathrm{const}\int d\alpha_1 \ldots d\alpha_9 \delta(1-\alpha_1-\alpha_2)$$
$$\times\,\delta(1-\alpha_3-\alpha_4)\,\delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)\,\frac{C_0}{(g_0 t+h_0)\,\tilde{f}_0^2}\,, \quad (199)$$

where

$$g_0=0;\; h_0=-\mu^2\,[m^2(\alpha_5+\alpha_6+\alpha_7)/\mu^2+(\alpha_8+\alpha_9)]\,C_0. \quad (200)$$

From (199) and (200) we obtain an expression for the function φ defined by (198):

$$\varphi\left(\frac{\mu^2}{m^2}\right)=\mathrm{const}\int\{d\alpha\}\,\Pi\delta\left(1-\sum_i \alpha_i\right)$$
$$\times\,\frac{\delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)}{\tilde{f}_0^2\,[m^2(\alpha_5+\alpha_6+\alpha_7)/\mu^2+(\alpha_8+\alpha_9)]}\,. \quad (201)$$

For large m^2/μ^2 the main contribution comes from the region $\alpha_5+\alpha_6+\alpha_7=0$, and we can again apply Tiktopoulos' method,[37] making the substitution $\alpha_{5,6,7}\to\lambda\alpha_{5,6,7}$. Then we obtain

$$\left.\begin{array}{c} d\alpha_5\,d\alpha_6\,d\alpha_7\to\lambda^2\delta(1-\alpha_5-\alpha_6-\alpha_7)\,d\alpha_5\,d\alpha_6\,d\alpha_7\,d\lambda;\\ \delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)\to\delta(1-\alpha_8-\alpha_9);\\ \tilde{f}_0\to\lambda\tilde{\tilde{f}}_0,\end{array}\right\} \quad (202)$$

and hence

$$\varphi\left(\frac{\mu^2}{m^2}\right)=\mathrm{const}\int_0^1\frac{d\lambda}{\lambda m^2/\mu^2+1}\,,$$

i.e.,

$$\varphi(\mu^2/m^2)=\mathrm{const}\times(\mu^2/m^2)\ln(\mu^2/m^2) \quad (203)$$

if the condition (197) is satisfied. Note that the constant in (203) now includes all integrals with respect to α_i.

Remembering that $f_{\mathrm{eik}}(t=0)=\mathrm{const}/\mu^2$ and taking into account the expressions (193), (196), (198), and (203), we obtain the asymptotic expression for the XX diagram:

$$f^{(XX)}(t)\approx(\ln s/s^3)\,\{f_{\mathrm{eik}}^{(XX)}(t)+f_{\mathrm{noneik}}^{(XX)}(t)\}, \quad (204)$$

where

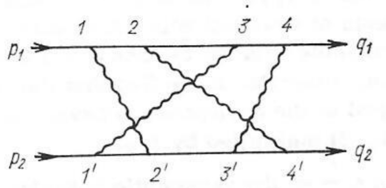


Fig. 6

$$f_{\mathrm{noneik}}^{(XX)}(t)=(\mu^2/m^2)\,f_{\mathrm{eik}}^{(XX)}(t\mu^2/m^2)$$
$$+\,\mathrm{const}\,f_{\mathrm{eik}}^{(XX)}(t=0)\,(\mu^2/m^2)\ln(\mu^2/m^2)$$

in the limit $s\to\infty$ with fixed t and $\mu^2/m^2\ll 1$.

Asymptotic Behavior of the Nucleon – Nucleon Scattering Amplitude. Eighth Order.

We have above considered one of the diagrams of eighth order. We now consider the remaining diagrams, excluding the graphs corresponding in (189) to the term $(q_1\leftrightarrow q_2)$. These include three types of noneikonal $\bar{t}$-paths that contribute to the principal asymptotic behavior.

The first type consists of the noneikonal $\bar{t}$-paths that do not have common lines. Besides the XX diagram, there is only one other diagram with such $\bar{t}$-paths (Fig. 6), and the two diagrams that are cross symmetric with respect to it.

The contribution to the asymptotic behavior of the diagram (see Fig. 6) can be written in a form analogous to (196):

$$f_{\mathrm{noneik}}^{(2)}(t)=\frac{\ln s}{s^3}\cdot\frac{\mu^2}{m^2}\,\mathrm{cross}\,f_{\mathrm{eik}}^{(2)}(t\mu^2/m^2). \quad (205)$$

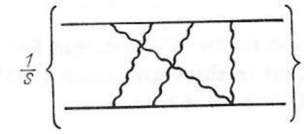
If we add the eikonal contributions of the diagrams XX and the one shown in Fig. 6 to the cross symmetric diagrams, ln s cancels, and we obtain the contribution $(1/s^3)f_{\mathrm{eik}}(t)$. Then, in accordance with (197) and (205), the contribution of the noneikonal $\bar{t}$-paths to the same sum is

$$f_{\mathrm{noneik}}(t)=(\mu^2/m^2)\,f_{\mathrm{eik}}(t\mu^2/m^2). \quad (206)$$

In the eighth order there are no other noneikonal contributions that depend on the momentum transfer.

The second type consists of noneikonal $\bar{t}$-paths that have one common nucleon line. The contribution of such $\bar{t}$-paths does not depend on the momentum transfer and was considered in the foregoing section for the diagram XX [see (198)-(204)]. However, such contributions cancel in the sum of all diagrams containing such $\bar{t}$-paths.

Let us consider, for example, the diagram shown in Fig. 7, in which the paths (1'2'3'4') and (13'4'4) are of the second type. Their contribution can be calculated using Assertion 2; for the asymptotic behavior of this diagram can be represented graphically in the form



(207)

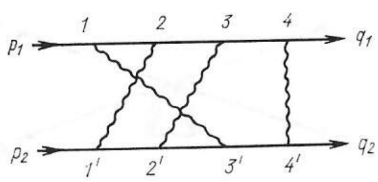


Fig. 7

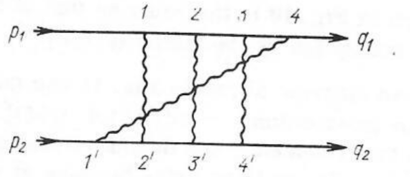


Fig. 8

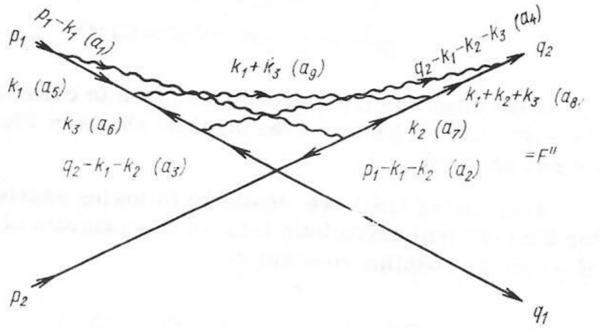
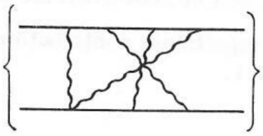
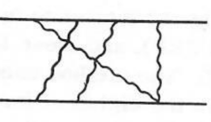
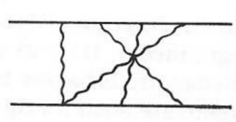


Fig. 9

The asymptotic behavior of the diagram obtained by mirror reflection of the vertices 1 and 2 with respect to 4 can be represented as follows:

$\frac{1}{s}\Bigg\{$  $\Bigg\}$. (208)

We now consider the cross symmetric graphs. By Assertion 2, the common lines must be replaced by the factor $-1/s$. We then obtain

$-\frac{1}{s}\Bigg\{$  +  $\Bigg\}$. (209)

The first term in (209) corresponds to the noneikonal contribution to the diagram that is cross symmetric with respect to the one shown in Fig. 7. Adding (207)-(209), we see that the contribution of the noneikonal $\bar{t}$-paths of the second type cancel in their sum. It is obvious that these arguments hold for the other diagrams of this type.

The third type consists of noneikonal $\bar{t}$-paths that have a common meson line. The contribution of such $\bar{t}$-paths is also independent of the momentum transfer. In the eighth order there are several diagrams that have $\bar{t}$-paths of the third type. Let us consider, for example, only one of them (Fig. 8), remembering that all the results apply equally to all such diagrams.

In this diagram the paths (1'4'34') and (12'1'4) are noneikonal $\bar{t}$-paths of the third type. Their contribution can be written in a form analogous to (198):

$$\frac{\ln s}{s^3}\cdot\frac{1}{\mu^2}\Phi\left(\frac{\mu^2}{m^2}\right). \tag{210}$$

As above, we shall seek the form of the function Φ when the condition (197) holds. We arrange the momenta of integration in such a way that the p-paths coincide with the $\bar{t}$-paths (1'434') and (12'1'4) (see Fig. 8). By Assertion 2, the desired contribution is equal to the asymptotic behavior of the reduced graph (Fig. 9) multiplied by $1/s$.

Using (181), we obtain in the limit $s \to \infty$ the asymptotic behavior

$$F'' \approx \frac{\ln s}{s^2}\,\mathrm{const}\int d\alpha_1\ldots d\alpha_9\,\delta(1-\alpha_1-\alpha_2)\,\delta(1-\alpha_3-\alpha_4) \times\delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)\,\frac{C_0}{(g_0 t+h_0)\tilde{f}_0^2}, \tag{211}$$

where

$$g_0=0;\; h_0=-\mu^2 C_0\,[m^2(\alpha_5+\alpha_6+\alpha_7+\alpha_8)/\mu^2+\alpha_9]. \tag{212}$$

From (211) and (212) we obtain an expression for the function $\Phi(\mu^2/m^2)$ defined by (210):

$$\Phi\left(\frac{\mu^2}{m^2}\right)=\mathrm{const}\int\{d\alpha\}\,\Pi\delta\Big(1-\sum_i\alpha_i\Big) \times\frac{\delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)}{\tilde{f}_0^2\,[m^2(\alpha_5+\alpha_6+\alpha_7+\alpha_8)/\mu^2+\alpha_9]}. \tag{213}$$

At large m^2/μ^2 a contribution is given by the region $\alpha_5+\alpha_6+\alpha_7+\alpha_8=0$. Making the substitution $\alpha_{5,6,7,8}\to\lambda\alpha_{5,6,7,8}$, we obtain

$$\left.\begin{array}{c} d\alpha_5\ldots d\alpha_8\to\lambda^3\delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8)\,d\alpha_5\ldots d\alpha_8\,d\lambda;\\ \delta(1-\alpha_5-\alpha_6-\alpha_7-\alpha_8-\alpha_9)\to\delta(1-\alpha_9);\\ \tilde{f}_0\to\lambda\tilde{\tilde{f}}_0, \end{array}\right\} \tag{214}$$

from which we obtain

$$\Phi(\mu^2/m^2)=\mathrm{const}\int_0^1 d\lambda\,\frac{\lambda}{\lambda m^2/\mu^2+1},$$

i.e.,

$$\Phi(\mu^2/m^2)=\mathrm{const}\times(\mu^2/m^2), \tag{215}$$

if the condition (214) is satisfied.

The results of the first and the second section can be reduced to a single equation, which allows for the cancelling of ln s when one adds the diagrams to the cross symmetric diagrams. Namely, for large s, the asymptotic expression for the nucleon–nucleon scattering amplitude in the eighth order in g is

$$f^{(8)}\approx\frac{g^8}{s^3}\left\{\frac{1}{8\cdot 4!\,(2\pi)^8}\int d^2\mathrm{x}_\perp\exp(-i\mathrm{x}_\perp\Delta_\perp) \times K_0^4(\mu\,|\,\mathrm{x}_\perp|)+f^{(8)}_{\mathrm{noneik}}(t)\right\}, \tag{216}$$

where

$$f^{(8)}_{\mathrm{noneik}}(t)=(\mu^2/m^2)\,f_{\mathrm{eik}}(t\mu^2/m^2)+(\mathrm{const}/\mu^2)\,\Phi(\mu^2/m^2).$$

In the last expression, $f_{\mathrm{eik}}(t)$ is the coefficient of the principal asymptotic behavior of the sum of diagrams shown in Figs. 2 and 6 and the cross symmetric diagrams, only the contributions of the eikonal $\bar{t}$-paths being allowed

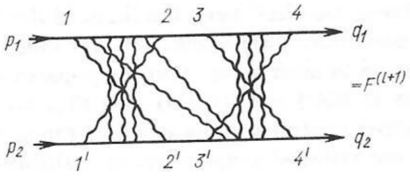


Fig. 10

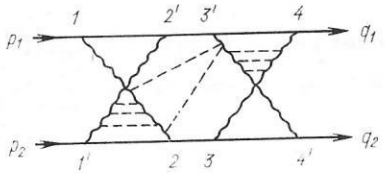


Fig. 11

for. The function $\Phi(\mu^2/m^2)$ for $\mu^2/m^2 \ll 1$ behaves as μ^2/m^2. The first term in the braces is the sum of the eikonal contributions of all eighth-order graphs [cf. (191)].

If the ratio of the meson and nucleon masses is small, one can ignore the dependence on the momentum transfer t in the correction terms to the eikonal expression if $t/m^2 \ll 1$:

$$f_{\text{eik}}(t\mu^2/m^2) \approx f_{\text{eik}}(0) = \text{const}/\mu^2, \tag{217}$$

and (216) takes the form

$$f^{(8)}\Big|_{\substack{s\to\infty \\ \text{fixed } t \\ \mu^2/m^2 \ll 1}} \approx \frac{g^8}{s^3}\left\{\frac{1}{8\cdot 4!\,(2\pi)^8}\int d^2\mathbf{x}_\perp \exp(-i\mathbf{x}_\perp\mathbf{\Delta}_\perp) \times K_0^4(\mu|\mathbf{x}_\perp|) + \text{const}/m^2\right\}. \tag{218}$$

To conclude this subsection, we emphasize that our entire exposition has been based on an investigation of the contributions from the different $\bar{t}$-paths associated with the zeros of the function $f(\alpha)$ [see (178)-(181)].

Asymptotic Behavior of the Nucleon – Nucleon Scattering Amplitude. Higher Orders.

Above, we have considered the high-energy behavior of the scattering amplitude in the eighth order in the coupling constant g. In this order we have seen that there are diagrams that make noneikonal contributions to the asymptotic behavior of the same order in s as the eikonal contributions. However, it was pointed out in ref. 36 that in the higher orders there are diagrams whose noneikonal contributions have a stronger asymptotic behavior than the eikonal contributions. A characteristic example of such diagrams with noneikonal paths of the first type is the diagram shown in Fig. 10.

In this diagram, as in the diagram XX, there are two $\bar{t}$-paths of length three: (12'3'4) and (1'234'). To investigate its asymptotic behavior we use the same method as in the first subsection, i.e., we direct the p-paths along the $\bar{t}$-paths and make a change of the momenta. If we are dealing with a diagram of order $2l+2$ in g (there are $l+1$ meson lines), then the asymptotic behavior of the graph in Fig. 10 is the same as that of the graph in Fig. 11 multiplied by the factor $(\mu^2/m^2)^{l-2}$.

In the diagram shown in Fig. 11 one must bear in mind the substitution $t \to t\mu^2/m^2$ [cf. (195)]. The dashed lines in this reduced graph denote virtual particles with mass μ^2/m. These lines arise because of meson lines (see Fig. 10) that do not belong to $\bar{t}$-paths:

$$D_\mu(k) \to D_\mu(km/\mu) = \frac{1}{k^2m^2/\mu^2 - \mu^2 + i\varepsilon} = \mu^2/m^2 D\mu^2/m\,(k)/m^2. \tag{219}$$

If the condition (197) is satisfied, then to calculate the asymptotic behavior of the diagram shown in Fig. 11 we can set t = 0.

Then, using (181), we obtain the following expression for the principal asymptotic term of this diagram of order $2l+2$ in the coupling constant g:

$$F^{(l+1)} \approx \frac{\ln s}{s^3}\cdot\frac{\text{const}}{\mu^{2(l-2)}}\int\{d\alpha\}\{d\beta\}\{d\gamma\}\,\Pi\delta\Big(1-\sum_i\gamma_i\Big) \times \delta\Big(1-\sum_i\alpha_i-\sum_i\beta_i\Big)\frac{C_0}{\tilde{f}_0^3\left(\frac{m^2}{\mu^2}\sum_i\alpha_i+\sum_i\beta_i\right)^{l-2}},\quad l \geqslant 3. \tag{220}$$

In this expression the parameters α_i correspond to the wavy meson lines, β_i to the dashed lines, and γ_i to the nucleon lines. It can be shown that the singularity $\sum_i \alpha_i = 0$ does not make a significant contribution to the integral[6] (220) for $m^2/\mu^2 \gg 1$.

But then

$$F^{(l+1)} \approx \frac{\ln s}{s^3}\cdot\frac{\text{const}}{(m^2)^{l-2}},\quad l \geqslant 3. \tag{221}$$

In the $(2l+2)$-th order in g there are diagrams with noneikonal $\bar{t}$-paths of the third type (Fig. 12).

In the diagram shown in Fig. 12 there are two $\bar{t}$-paths of length three: (12'1'3) and (1'323'), and these lead to the asymptotic behavior $\ln s/s^3$. The method used above to investigate such a diagram in the eighth order leads here to an equation similar to (221). The noneikonal $\bar{t}$-paths of the second type, whose contributions cancel when added in the eighth order, lead in the higher orders to a weaker asymptotic behavior.

All the diagrams of the given order $2l+2$ in g either belong to the type described above, and then contribute to the asymptotic behavior in accordance with (221), or have $\bar{t}$-paths of length greater than three, which leads to a weaker asymptotic behavior as $s \to \infty$. With allowance for the canceling of ln s when the graphs are added to the cross symmetric graphs, we obtain the following expression for the asymptotic behavior of the scattering amplitude $f^{(2l+2)}$ in the order $2l+2$ in g:

$$f^{(2l+2)}\Big|_{\substack{s\to\infty \\ \text{fixed } t \\ \mu^2/m^2 \ll 1}} \approx \frac{1}{s^3}\cdot\frac{\text{const}}{(m^2)^{l-2}},\quad l \geqslant 3. \tag{222}$$

Note that the eikonal equation (191) for t = 0 in the same order in g leads to the expression

$$f_{\text{eik}}^{(2l+2)}(t=0) = \text{const}/(s^l\mu^2). \tag{223}$$

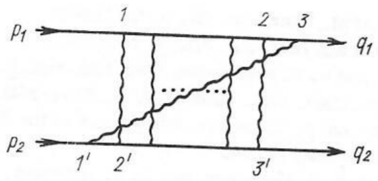


Fig. 12

Thus, the ratio of the noneikonal and eikonal amplitudes in this order in the coupling constant is

$$\left.\frac{f^{(2l+2)}_{\text{noneik}}}{f^{(2l+2)}_{\text{eik}}}\right|_{\substack{s\to\infty\\ \text{fixed } t\\ \mu^2/m^2\ll 1}} \approx \text{const}\,(\mu^2/m^2)\,(s/m^2)^{l-3},\quad l\geqslant 3. \qquad (224)$$

It can be seen from (224) that in the region

$$s\to\infty;\ \mu^2/m^2\ll 1;\ s\sim m^2;\quad t=0 \qquad (225)$$

the eikonal amplitude is much greater than the noneikonal one, i.e., (191) correctly reproduces the principal asymptotic terms in each order in g^2. But if we are in the region defined by the relations (225) but we assume $s \gg m^2$, then, as follows from (224), the contributions of the noneikonal $\bar{t}$-paths dominate over the eikonal contributions. Thus, our study of the class of ladder diagrams in the scalar model shows that the eikonal equation corresponds to allowance in the asymptotic behavior for the $\bar{t}$-paths that coincide with nucleon lines. In this case the leading particle that carries the large momentum is a nucleon and it does not change its nature in virtual processes. The noneikonal contributions to the amplitude are due to processes in which the nature of the leading particle changes, i.e., to processes in which momentum is transferred from the nucleons to the mesons and vice versa. There then arises the important question of the role of "twisted" graphs, corresponding to the original ladder graphs with replacement of the final momenta ($q_1 \leftrightarrow q_2$) [see Fig. 1 and (189)]. The possibility that a large momentum can be transferred by a meson means that the contribution to the asymptotic behavior of the scattering amplitude may dominate over the eikonal contribution in the same order in the coupling constant. For example, in the fourth order the twisted graph (Fig. 13) has the asymptotic behavior $\ln s/s$.

Note that whereas the orthodox eikonal equation corresponds to scattering by a Yukawa quasipotential due to one-meson exchange, allowance for the graph in Fig. 13 leads to the appearance of a correction to the quasipotential of non-Yukawa type. This correction corresponds to the exchange of nucleon–antinucleon pairs and has an effective range $\sim \hbar/2m$, and at short distances it behaves as $\ln r/r$.

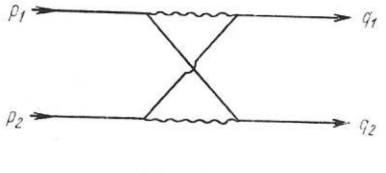


Fig. 13

The example considered here demonstrates that it is important to study the successive corrections to the effective quasipotential at high energies and argues in favor of the quasipotential interpretation of the eikonal representation in quantum field theory.

CONCLUSIONS

The aim of this review was to acquaint the reader with the basic ideas and elements of the mathematical apparatus on which the straight-line path approximation in quantum field theory is based. The individual results obtained by this method have been published in various papers and their exposition in a more complete form in one article should give an overall picture of the basic concept of straight-line paths.

This approximation has been developed as a result of the working out of a system of approximations in quantum field theory that is not based on perturbation theory and the associated assumption of a weak interaction and at the same time encompasses the main features of particle interaction at high energies. One of the principal features that characterizes collision of high-energy hadrons and is the basis of the concept of straight-line paths is the dominance of large momentum transfer in individual interaction events. In the framework of the quasipotential approach this feature for the case of elastic interactions has been reflected in the assumption of a smooth quasipotential. As we have pointed out above, smoothness of the local quasipotential enables one to a give rigorous justification of the eikonal expansion of the scattering amplitude. A consistent working out of the straight-line path conception and the assumption of a smooth interaction can be found in the operator method of solution of quasipotential equations.

For the case of purely field-theoretical problems the method of functional integration is evidently the most suitable language for expressing the straight-line path conception. The system of corrections of the straight-line path approximation is associated with successive allowance for the deviation of the particle paths from rectilinear trajectories and is not based on the diagram technique of perturbation theory.

The most important problem is that of justifying the straight-line path approximation in at least the framework of definite field-theory models. The main hindrance preventing the solution of this problem is the singular nature of the original relativistic interactions corresponding to the simplest field-theory models that have been studied. Another important question is the field-theory interpretation of the smoothness property of the local quasipotential, which guarantees validity of the eikonal expansion of the scattering amplitude of high-energy particles. We note that a formal derivation of the eikonal representation with phase shift corresponding to a singular effective interaction does not solve the problem of justifying the eikonal equation, and a painstaking analysis must be made of the omitted terms.

Finally, an important problem in which methods of functional integration are used is the need to allow for complicated effects of vacuum polarization (closed nucleon loops) and also the need to go beyond the simplest field-theory models and attempt to derive general results that do not depend on the actual details of model interactions. It is to be hoped that a further study of the straight-line path approximation will extend the number of its applications and make it one of the effective methods of quantum field theory.

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1) Note that under the translations $x_i \to x_i + h$ the functional variables of $\mathscr{F}$, that is, the currents j_i, which on account of the change of variables (35) have the form

$$j_i = \int_{-\xi_i}^{\tau_i} d\zeta\delta \left[x_i - z + 2p_i\zeta\vartheta(\zeta) + 2q_i\zeta\vartheta(-\zeta) + 2\int_0^{\zeta} v_i\, d\eta \right], \qquad (37)$$

must also be transformed.

2) Allowance for the identical nature of the nucleons leads after symmetrization of (87) to terms that vanish in the limit $s \to \infty$ with t fixed.

3) The integration in (109) is effectively bounded by the conditions $|k_z| \le R_z$ and $|\mathbf{k}_\perp| \le R_\perp$.

4) We recall that we are speaking of loops and lines defined by the rules (116).

5) Results similar to those of ref. 37 were also obtained in refs. 12 and 13.

6) This can be readily seen by calculating the power of λ that appears in the numerator when the substitution $\alpha \to \lambda\alpha$ is made [cf. (214) and (215)].

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