Method of generalized hyperspherical functions

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A method of generalized hyperspherical functions is presented which makes possible the study of structure in nuclear bound states. A basis of generalized hyperspherical functions in an (A − 1)-dimensional space is used to isolate collective motions in a system of A nucleons and to construct the wave functions for the collective motions.

INTRODUCTION

Among the nuclear models currently most widely used are the shell model and the Bohr–Mottelson collective model.[1,2] Even a cursory comparison of these two models reveals that they are founded on very different concepts. Whereas the shell model begins with the premise that the nucleons move independently in a self-consistent field, the collective model asserts that, because of a strong correlation of nucleon motion within the nucleus, well-defined collective degrees of freedom are present. In order to reconcile these conflicting assumptions, one usually resorts to the notion of independent nucleon motion in a nonspherical self-consistent field[3] which then enables one to relate the collective degrees of freedom to the parameters which characterize the deformed field. Unfortunately, while this approach is both simple and physically very graphic, it leads to the appearance of extraneous variables.

The extraneous variables can, in principle, be eliminated either by introducing some coupling between the nucleon coordinates or by projecting the wave function onto a state with a definite value of angular momentum. However, in actual practice this approach forces one to deal with unjustifiably complicated calculations so that nothing remains of the initial simplicity of the method. What does remain is the current search for a way to combine the shell model and the collective model in a manner which permits practical calculations to be performed. In addition to the need for some rational approximation which harmoniously combines the collective and single-particle aspects of nucleon motion, the problem also demands a means for the gradual transition from this approximation to an exact description.

In discussing this problem we shall first try to determine which results from the shell model and the collective model must be included in the combined model.

The collective model's primary success has been in the interpretation of phenomena which are due to the nonspherical shape of nuclei.[4,5] This interpretation is possible on the basis of the dynamical equations for small fluctuations in the shape of the nuclear surface, as discussed by Bohr.[1] These equations themselves were taken from hydrodynamics and applied to the atomic nucleus because of the analogy (noticed somewhat earlier by Bohr and Wheeler) between the motion of nucleons in a nucleus and the motion of molecules in a drop of an incompressible fluid. At first the kinematic parameters in the equations were selected as required by the hydrodynamic results, and the force constants were deduced in accord with the surface energies in the Weizsäcker equation. However, it quickly became clear that this method of selecting the parameters did not produce agreement with experiment, and phenomenological model parameters were chosen on the basis of experiments with the lowest collective excitations.[5] However, this method (like the first) of selecting the parameters in the model was not totally unambiguous. What is required is a more careful derivation of the collective model's dynamical equation based on a many-particle Schrödinger equation, especially since the dynamical equation is valid only for small deformations, and the experimental deformations are not always small. Finally, a microscopic interpretation is needed for the dynamical variables; their relationship to the individual nucleon coordinates must be established.

Turning to the shell model, we find that the assumption of independent nucleon motion allows one to formulate a simple and effective method of constructing antisymmetric fermion wave functions, within the confines of this model. This is the most constructive factor in the shell model, and it would be difficult to overestimate its value for solving practical problems in nuclear physics. We now ask if it is possible to retain this constructive feature of the method while relaxing the assumption concerning the independence of the nucleon motions. The answer appears to be positive.

The shell-model wave functions are appropriate to the limiting system of independent particles and take into account only those correlations in the nucleons' motion which are derived from the Pauli principle and the requirement that the system wave function correspond to a state with definite angular momentum and definite parity. If we use single-particle wave functions for nucleons in a oscillator field, the shell model then also includes the correlations which result from extracting the center of mass motion for the system of nucleons. All these correlations are kinematic in nature and, unlike the dynamical correlations, are not directly related to the forces acting between the nucleons. The dynamical correlations have been partially taken into account with the help of K harmonics.[6]

If the shell-model wave functions are constructed on an oscillator basis, then each of them can be written as the product of a hyperspherical function and a function of a "global radius"; any single hyperspherical function will correspond to an infinite number of shell-model functions which differ only in their dependence on the global radius factor.

The main thrust of the K-harmonic method is to provide a more precise determination of this global radius factor, and to exchange the functional dependence on the global radius which is suggested by the shell model for a dependence which is better fitted to the internucleon forces.

Whereas, in systematic application of the shell model the nuclear wave function is written in the form of an expansion in terms of a complete set of antisymmetrized single-particle functions and the expansion coefficients are found by diagonalizing the Hamiltonian for the nucleon system, in the K-harmonic method the expansion is in terms of hyperspherical functions and the problem reduces to finding the solutions to an infinite system of ordinary differential equations for functions of the global radius.

A better description of the functional dependence on the global radius will enable us to detect and describe the dynamical correlations connected with the volume oscillations of the nucleus. As for the dynamical correlations which cause vibrations in the nuclear surface shape, one finds that when they are taken into account using K harmonics a very large number of basis functions is required, just as when a spherically symmetric oscillator basis is used in the shell model.

An effective description of the surface collective motions in a system of A nucleons can be obtained by using a basis of generalized hyperspherical functions in an n-dimensional space where $n = A - 1$.

Recall that the hyperspherical functions for a system of A nucleons are generated by the rotation operator in a 3n-dimensional space and are classified in terms of the system's total orbital angular momentum L, the representations of the permutation group (the Young tableaux), and the momentum in the 3n-dimensional space (the number K). In a rotation in the 3n-dimensional space the motion takes place on the surface of a 3n-dimensional sphere whose radius is ρ (the global radius). One can, however, consider a more restricted class of motions which take place on the surface of a 3n-dimensional inertia ellipsoid whose principal axes a, b, and c retain their orientations and magnitudes. These motions can be represented by the rotations of three mutually perpendicular vectors in an n-dimensional space:

$$\vec{A}_\xi = \{a_{1\xi},\ a_{2\xi},\ \ldots,\ a_{n\xi}\};$$

$$\vec{A}_\eta = \{a_{1\eta},\ a_{2\eta},\ \ldots,\ a_{n\eta}\};\quad \vec{A}_\zeta = \{a_{1\zeta},\ a_{2\zeta},\ \ldots,\ a_{n\zeta}\}.$$

The components of the first vector are the ξ-e projections of the Jacobi vectors in a coordinate system whose axes ξ, η, and ζ coincide with the principal axes of the inertia ellipsoid; the components of the second vector are the η-e projections of the Jacobi vectors in this same system, and so on. The n-dimensional vectors are orthogonal because of the definition of the coordinate system which is related to the principal axes of the inertia ellipsoid:

$$\sum_{k=1}^{n} a_{k\xi} a_{k\eta} = 0;\quad \sum_{k=1}^{n} a_{k\xi} a_{k\zeta} = 0;\quad \sum_{k=1}^{n} a_{k\eta} a_{k\zeta} = 0.$$

It is useful to note that when particles are permuted the triplet of n-dimensional vectors undergoes a rotation in the n-dimensional space without any change in their lengths. For certain permutations the rotation is accompanied by a reflection. The orientation of this triplet of vectors in the n-dimensional space is given by $3(n-2)$ angles which can be used as the dynamical variables in the A-body problem, together with the three principal semiaxes of the inertia ellipsoid and the three Euler angles (φ, θ, ψ) which define the orientation of the inertia ellipsoid's principal axes in the usual three-dimensional space.

Of special importance for the work to follow will be the reference frame in the n-dimensional space (which we shall call the privileged frame) which has one unit vector directed along the vector $\vec{A}_\xi$, its second unit vector along the vector $\vec{A}_\eta$, and its third along the vector $\vec{A}_\zeta$. The advantage of this system of unit vectors is due to the possibility of very simple expressions for the rate of change of any of the vectors $\vec{A}_\xi$, $\vec{A}_\eta$, $\vec{A}_\zeta$ in terms of the angular rates Ω_{sl} at which they rotate in the planes of the reference system:

$$\dot{\vec{A}}_\xi = \{a\Omega_{A-3,1},\ a\Omega_{A-3,2},\ \ldots,\ a\Omega_{A-3,A-4},$$

$$\dot{a},\ a\Omega_{A-3,A-2},\ a\Omega_{A-3,A-1}\};$$

$$\dot{\vec{A}}_\eta = \{b\Omega_{A-2,1},\ b\Omega_{A-2,2},\ \ldots,\ b\Omega_{A-2,A-4},$$

$$b\Omega_{A-2,A-3},\ \dot{b},\ b\Omega_{A-2,A-1}\};$$

$$\dot{\vec{A}}_\zeta = \{c\Omega_{A-1,1},\ c\Omega_{A-1,2},\ \ldots,\ c\Omega_{A-1,A-4},$$

$$c\Omega_{A-1,A-3},\ c\Omega_{A-1,A-2},\ \dot{c}\},$$

where a, b, and c are the lengths of the vectors $\vec{A}_\xi$, $\vec{A}_\eta$ and $\vec{A}_\zeta$; $\Omega_{sl} = -\Omega_{ls}$ is the angular rate of rotation in the plane (sl).

In addition to the angular rotation rates in the $A-1$ dimensional space it is useful to introduce the angular momentum J in this space, together with its projections J_{sl} on the planes (sl). By definition the projection J_{sl} of the angular momentum is the quantity which is canonically conjugate to the rotation angle in the plane (sl). Like the Ω_{sl}, the components of J_{sl} can be expressed in terms of the angles which give the orientation of the privileged reference frame in the $A-1$ dimensional space, along with the time derivatives of these angles.

Even in three-dimensional space there is some difference between the angular momentum of a point (a one-dimensional system) and the angular momentum of a gyroscope. The rotation of a point is completely specified by two angles, and the projection of the angular momentum on the axis passing through the point and the origin is zero. However, the rotation of a gyro is characterized by three angles and all three projections of the angular momentum on the intrinsic gyro axes are nonzero. We can then analogously speak of the angular momentum of a point (a one-dimensional system), of a two-dimensional gyro, a three-dimensional gyro, and so on in an n-dimensional space. Thus the angular momentum of a three-dimensional system (the three orthogonal n-dimensional vectors) has 3n nonzero projections on the planes of the privileged reference system, and in this respect it differs from the angular momentum of an n-dimensional gyro.

The components J_{sl} of the angular momentum can be put into correspondence with the operators $\hat{J}_{sl}$ which coincide with the generators of the rotation group for the n-dimensional space. We shall use these operators to write the kinetic energy operator for a system of A identical particles[7-9]

$$\hat{T} = -\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial a^2} + \frac{A-4}{a}\cdot\frac{\partial}{\partial a} + \left(\frac{1}{a^2-b^2} - \frac{1}{c^2-a^2}\right) 2a\frac{\partial}{\partial a}\right.$$
$$+\frac{\partial^2}{\partial b^2} + \frac{A-4}{b}\cdot\frac{\partial}{\partial b} + \left(\frac{1}{b^2-c^2} - \frac{1}{a^2-b^2}\right) 2b\frac{\partial}{\partial b}$$
$$+\frac{\partial^2}{\partial c^2} + \frac{A-4}{c}\cdot\frac{\partial}{\partial c} + \left(\frac{1}{c^2-a^2} - \frac{1}{b^2-c^2}\right) 2c\frac{\partial}{\partial c}$$
$$-\sum_{s=1}^{A-4}\left(\frac{1}{a^2}\hat{J}^2_{s,n-2} + \frac{1}{b^2}\hat{J}^2_{s,n-1} + \frac{1}{c^2}\hat{J}^2_{s,n}\right)$$
$$-\frac{b^2+c^2}{(b^2-c^2)^2}(\hat{I}^2_\xi + \hat{I}^2_{n-1,n}) - \frac{c^2+a^2}{(c^2-a^2)^2}(\hat{I}^2_\eta + \hat{I}^2_{n,n-2})$$
$$-\frac{a^2+b^2}{(a^2-b^2)^2}(\hat{I}^2_\zeta + \hat{J}^2_{n-2,n-1}) - \frac{4ab}{(b^2-c^2)^2}\hat{I}_\xi\hat{J}_{n-1,n}$$
$$\left.-\frac{4ca}{(c^2-a^2)^2}\hat{I}_\eta\hat{J}_{n,n-2} - \frac{4ab}{(a^2-b^2)^2}\hat{I}_\zeta\hat{J}_{n-2,n-1}\right\}.$$

The variables a, b, and c vary over the regions defined by these inequalities: $0 < a < \infty$, $0 < b < \infty$, $0 < c < \infty$. In this region the volume element takes the form $|a^2 - b^2|\,|b^2 - c^2|\,|c^2 - a^2|\,(abc)^{A-4}\,da\,db\,dc$. The quantities a, b, and c, which define the instantaneous values of the inertia ellipsoid's principal semiaxes, characterize the mass distribution of the nucleons in space. These variables are suitable for describing the volume and surface oscillations of the inertia ellipsoid. The surface vibrations of the inertia ellipsoid can be associated with the quadrupole vibrations of the surface of the nucleus in the Bohr–Mottelson collective model. This association allows one to relate many of the collective-model concepts to the surface of the inertia ellipsoid rather than to the nuclear surface itself; we can therefore extend the collective model to the light nuclei, which, unlike the heavier nuclei, do not have well-defined surfaces, but which do have an inertia ellipsoid (like the heavier nuclei). Later we shall present equations which formally relate the variables a, b, and c to the variables β, γ of the collective model.

In analogy with the generalized spherical functions of Wigner, the eigenfunctions of the angular-momentum operator $\hat{J}^2 = \sum_{s<l}\hat{J}^2_{sl}$, $1 \le s$, $l \le n$, in n-dimensional space will be aptly named generalized hyperspherical functions. Of these eigenfunctions we shall be interested only in those which correspond to zero eigenvalues for all the operators $\hat{J}_{sl}$, $1 \le s$, $l \le A-4$, which do not appear explicitly in the expression for the kinetic energy operator, because it is these generalized hyperspherical functions which are, like the wave functions of an A-particle system, invariant under rotations in the planes (sl), where $1 \le s$, $l \le A-4$.

The wave function for an A-particle system can be written as a series in terms of the generalized hyperspherical functions. In particular, turning to the wave functions for the shell model constructed with an oscillator basis, it is not difficult to see that each of them is a linear combination of a certain number of eigenfunctions for the operator $\hat{J}^2$, with the coefficients of this linear combination being dependent on the variables a, b, and c and on the Euler angles φ, θ, ψ. The improvement in the functional dependence of these coefficients on a, b, and c, when compared with the dependence given by the shell model, allows one to approximately transfer both the volume and the surface dynamical correlations in the motion of the nucleons. The problem of describing the volume and surface correlations then reduces to finding functions dependent on a, b, and c, and this is achieved by solving equations of the Bohr–Mottelson type.

1. THE GENERALIZED EULER ANGLES AND THE CARTESIAN COMPONENTS OF THE JACOBI VECTORS IN THE INERTIA ELLIPSOID SYSTEM

The components of the three mutually perpendicular vectors $\vec{A}_\xi = \vec{A}^{(1)}$, $\vec{A}_\eta = \vec{A}^{(2)}$, $\vec{A}_\zeta = \vec{A}^{(3)}$ in the n-dimensional space can be expressed in terms of the generalized Euler angles. Following Vilenkin,[10] we introduce these angles as parameters of rotation in the n-dimensional Euclidean space.

Assume that the set of vectors $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ forms an orthonormal basis. Then the vector $A^{(i)}$ can be expanded in the form

$$\vec{A}^{(i)} = \sum_{k=1}^{n} a_{k,i}\vec{e}_k. \tag{1}$$

When the coordinate system is rotated, we go from the basis vectors $\vec{e}_m$ to the new vectors $\vec{e}'_m$:

$$\vec{e}'_m = g\vec{e}_m = \sum_k g_{km}\vec{e}_m. \tag{2}$$

Let us assume that the coordinate system is rotated by an angle θ in the (l,k) plane in a direction going from the vector $\vec{e}_l$ to the vector $\vec{e}_k$. The unit vectors $\vec{e}_l$ and $\vec{e}_k$ then transform into new vectors:

$$g^{l\to k}(\theta)\,\vec{e}_l = \cos\theta\,\vec{e}_l + \sin\theta\,\vec{e}_k;\quad g^{l\to k}(\theta)\,\vec{e}_k = -\sin\theta\,\vec{e} + \cos\theta\,\vec{e}_k,$$

while the remaining unit vectors are unchanged:

$$g^{l\to k}(\theta)\,\vec{e}_m = \vec{e}_m,\quad m \neq l,\ k.$$

An arbitrary rotation of the coordinate system in an n-dimensional Euclidian space can be represented by

$$g = G^{(n-1)}G^{(n-2)}\ldots G^{(1)}, \tag{3}$$

where

$$G^{(l)} = g^{1\to 2}(\theta_1^{n-l})\,g^{2\to 3}(\theta_2^{n-l+1})\ldots g^{l\to l+1}(\theta_l^{n-1}), \tag{4}$$

and the generalized Euler angles θ_j^k vary between the limits

$$0 \leqslant \theta_j^k \leqslant (1+\delta_{j,1})\,\pi. \tag{5}$$

In the general case the transformation of Eq. (3) contains $n(n-1)/2$ generalized Euler angles. However, let us assume that in this n-dimensional Euclidean space we have three mutually orthogonal vectors which are oriented in an arbitrary fashion in some original coordinate system. Then the rotation of the coordinate system which brings the axes n, n − 1, and n − 2 of the new coordinate system (which was earlier termed the privileged system) into coincidence with the aforementioned vectors can be written in the form of a transformation:

$$G=G^{(n-1)}(\theta_1, \theta_2, \ldots, \theta_{n-1})\,G^{(n-2)}(\theta_1', \ldots, \theta_{n-2}')\times G^{(n-3)}(\theta_1'', \ldots, \theta_{n-3}''), \tag{6}$$

which contains $3n-6$ parameters, the Euler angles. The rotations $G^{(i)}$, where $i < n-3$, do not affect the unit vectors $\vec{e}_n$, $\vec{e}_{n-1}$, $\vec{e}_{n-2}$; therefore we shall not concern ourselves with them.

We now put

$$\vec{A}^{(i)}=a^{(i)}G\vec{e}_{n-3+i},\ i=1,\,2,\,3,\quad a^{(1)}=a,\ a^{(2)}=b,\ a^{(3)}=c. \tag{7}$$

It follows from Eq. (1) that the $a_{k,i}$ of the three mutually perpendicular n-dimensional vectors can be expressed in terms of the elements of the G matrix:

$$a_{k,\,i}=a^{(i)}G_{k,\,n-3+i}(\theta_1, \ldots, \theta_{n-1}, \theta_1', \ldots, \theta_{n-2}', \theta_1'', \ldots, \theta_{n-3}''), \tag{8}$$

that is, in terms of the $3A-9$ generalized Euler angles; this then solves the problem posed at the outset.

We note further that

$$G_{k,\,s}=\sum_{lm}G_{kl}^{(n-1)}G_{lm}^{(n-2)}G_{ms}^{(n-3)} \tag{9}$$

But the following expressions are valid for all the matrix elements $G^{(l)}$:

$$G_{k,\,m}^{(l)}(\theta_1, \ldots, \theta_l)=\delta_{km},\quad \text{if}\quad m>l+1;$$

$$G_{k,\,l+1}^{(l)}(\theta_1, \ldots, \theta_l)=\begin{cases}(-1)^{l+1-k}\,\varepsilon_{k,\,l+1}(\theta_1, \ldots, \theta_l), & m=l+1,\ k\leqslant l+1;\\ 0, & m=l+1,\ k>l+1;\end{cases}$$

$$G_{k,\,m}^{(l)}(\theta_1, \ldots, \theta_l)=\begin{cases}(-1)^{m+1-k}\varepsilon_{k,\,m+1}(\theta_1, \ldots, \theta_{m-1}, \theta_m-\pi/2), & m\leqslant l,\ k\leqslant m+1;\\ 0, & m\leqslant l,\ k>m+1,\end{cases}$$

where the function $\varepsilon_{km}(\theta_1, \ldots, \theta_{m-1})$ coincides with the k-th Cartesian component of the unit vector in an m-dimensional space ($m > 1$), expressed in terms of hyperspherical coordinates:

$$\varepsilon_{1,\,m}(\theta_1, \ldots, \theta_{m-1})=\sin\theta_{m-1}\sin\theta_{m-2}\ldots\sin\theta_2\sin\theta_1;$$

$$\varepsilon_{k,\,m}(\theta_1, \ldots, \theta_{m-1})=\sin\theta_{m-1}\sin\theta_{m-2}\ldots\times\sin\theta_k\cos\theta_{k-1},\ m-1\geqslant k\geqslant 2;$$

$$\varepsilon_{m,\,m}(\theta_1, \ldots, \theta_{m-1})=\cos\theta_{m-1}.$$

2. WAVE FUNCTIONS FOR COLLECTIVE EXCITATIONS OF MAGIC NUCLEI IN THE OSCILLATOR SHELL-MODEL APPROXIMATION

Let a, b, and c be the semiaxes of the inertia ellipsoid for a system of A nucleons, φ, θ, ψ be the Euler angles defining the orientation of the inertia ellipsoid's principal axes, and α_i be the generalized Euler angles in a space of $A-1$ dimensions which define in this space the directions of the three mutually perpendicular unit vectors. Then, in the oscillator approximation for the shell model the ground-state wave function for a magic nucleus can be written as[11]

$$\Psi_{0K}(a,\,b,\,c;\,\alpha_i,\,\sigma_i,\,\tau_i)=C_K(abc)^K\times\exp\left(-\frac{a^2+b^2+c^2}{2}\right)\chi_K(\alpha_i,\,\sigma_i,\,\tau_i), \tag{10}$$

where C_K is the normalization constant

$$C_K^2=\frac{64\sqrt{\pi}}{\Gamma(A+2K-1/2)\,\Gamma((A+2K-2)/2)\,\Gamma(A+2K-3)/2)}, \tag{11}$$

K is a quantum number which is related to the number A of nucleons in the nucleus and the number of the last filled oscillator shell m:

$$K=mA/4=[m(m+1)(m+2)(m+3)]/6, \tag{12}$$

$\chi_K(\alpha_i, \sigma_i, \tau_i)$ is a function of the spin–isospin variables (σ_i, τ_i) and the generalized Euler angles (α_i) in an $A-1$ dimensional space; this function is antisymmetric relative to interchanges of the coordinates of any pair of nucleons and is normalized to unity.

Among the excited states of the magic nucleus one can isolate those whose wave functions, like Eq. (10), can be written as products:

$$\Psi_{\Lambda LMK}(a,\,b,\,c,\,\varphi,\,\theta,\,\psi;\quad \alpha_i,\,\sigma_i,\,\tau_i)=\Phi_{\Lambda LMK}(a,\,b,\,c,\,\varphi,\,\theta,\,\psi)\,\chi_K(\alpha_i,\,\sigma_i,\,\tau_i). \tag{13}$$

These functions differ from the ground-state wave function only in the factor containing the collective coordinates a, b, c, φ, θ, ψ. The quantum numbers L and M define the value of the total system angular momentum and its projection onto some specified axis, while the quantum number Λ characterizes the excitations connected with the variables a, b, c.

The functions $\Phi(a, b, c, \varphi, \theta, \psi)$ satisfy the equation

$$-\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial a^2}+\frac{A-4}{a}\cdot\frac{\partial}{\partial a}+\left(\frac{1}{a^2-b^2}-\frac{1}{c^2-a^2}\right)2a\frac{\partial}{\partial a}-\frac{K(A+K-5)}{a^2}\right.$$

$$+\frac{\partial^2}{\partial b^2}+\frac{A-4}{b}\cdot\frac{\partial}{\partial b}+\left(\frac{1}{b^2-c^2}-\frac{1}{a^2-b^2}\right)2b\frac{\partial}{\partial b}-\frac{K(A+K-5)}{b^2},$$

$$+\frac{\partial^2}{\partial c^2}+\frac{A-4}{c}\cdot\frac{\partial}{\partial c}+\left(\frac{1}{c^2-a^2}-\frac{1}{b^2-c^2}\right)2c\frac{\partial}{\partial c}-\frac{K(A+K-5)}{c^2}-$$

$$-\frac{b^2+c^2}{(b^2-c^2)^2}I_\xi^2-\frac{c^2+a^2}{(c^2-a^2)^2}I_\eta^2-\frac{a^2+b^2}{(a^2-b^2)^2}I_\zeta^2\Big\}\Phi$$

$$+\frac{1}{2}m\omega^2(a^2+b^2+c^2)\Phi=E\Phi,$$

where $\omega^2=\hbar^2/m^2$, which is a result of our choice of units for measuring a, b, c. For the particular case in which A = 16 and K = 4, Zickendraht[9] has obtained Eq. (14) and its eigenfunctions for the lowest states. After putting

$$\Phi(a,\,b,\,c,\,\varphi,\,\theta,\,\psi)$$
$$=\tilde{\Phi}(a,\,b,\,c,\,\varphi,\,\theta,\,\psi)(abc)^K\exp[-(a^2+b^2+c^2)/2], \qquad (15)$$

we obtain an equation for $\tilde{\Phi}(a$, b, c, φ, θ, $\psi)$ which is somewhat simpler than Eq. (14):

$$\Big\{\frac{\partial^2}{\partial a^2}+\frac{A+2K-4}{a}\cdot\frac{\partial}{\partial a}+\Big(\frac{1}{a^2-b^2}-\frac{1}{c^2-a^2}-1\Big)2a\frac{\partial}{\partial a}$$
$$-\frac{b^2+c^2}{(b^2-c^2)^2}I_\xi^2+\frac{\partial^2}{\partial b^2}+\frac{A+2K-4}{b}\cdot\frac{\partial}{\partial b}$$
$$+\Big(\frac{1}{b^2-c^2}-\frac{1}{a^2-b^2}-1\Big)2b\frac{\partial}{\partial b}$$
$$-\frac{c^2+a^2}{(c^2-a^2)^2}I_\eta^2+\frac{\partial^2}{\partial c^2}+\frac{A+2K-4}{c}\cdot\frac{\partial}{\partial c}$$
$$+\Big(\frac{1}{c^2-a^2}-\frac{1}{b^2-c^2}-1\Big)2c\frac{\partial}{\partial c}$$
$$-\frac{a^2+b^2}{(a^2-b^2)^2}I_\zeta^2\Big\}\tilde{\Phi}=[\varepsilon-3(A+2K-1)]\tilde{\Phi},$$

in which $\varepsilon=2mE/\hbar^2$. The functions $\tilde{\Phi}$ take the form of polynomials in a, b, c, and the degree of the polynomial is higher, the higher the excitation.

If the number of particles is large (A ≫ 1), Eq. (14) can be rather simply transformed into an expansion in terms of inverse powers of $A^{1/3}$; keeping just the main terms we have

$$-\frac{\hbar^2}{2m}\Big\{\frac{1}{\rho^{3A-4}}\cdot\frac{\partial}{\partial\rho}\rho^{3A-4}\frac{\partial}{\partial\rho}$$
$$+\frac{6K}{\rho^2}\Big[\frac{1}{\beta^4}\cdot\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2}\cdot\frac{1}{\sin 3\gamma}\cdot\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}$$
$$-\frac{I_\xi^2}{4\beta^2\sin^2(\gamma-2\pi/3)}-\frac{I_\eta^2}{4\beta^2\sin^2(\gamma+2\pi/3)}$$
$$-\frac{I_\zeta^2}{4\beta^2\sin^2\gamma}-\beta^2\Big]-\frac{3K(3K+3A-15)}{\rho^2}-\rho^2\Big\}\Phi=E\Phi. \qquad (16)$$

The variables ρ, β, γ are related to a, b, and c as follows:

$$\rho^2=a^2+b^2+c^2;\quad [\rho^2/\sqrt{3(K+A-5)}]\beta\sin\gamma$$
$$=(\sqrt{3}/2)(a^2-b^2);$$

$$[\rho^2/\sqrt{3(K+A-5)}]\beta\cos\gamma=c^2-(a^2+b^2)/2.$$

The operator found in square brackets in Eq. (16) coincides with the Bohr–Mottelson model Hamiltonian for small surface oscillations relative to the spherically symmetric equilibrium shape of the nucleus. Thus, if we identify the shape of the nucleus with the shape of its inertia ellipsoid, then for collective excitations of magic nuclei with a large number of particles, the oscillator shell model leads to a wave function having the same dependence on the parameters β and γ and the Euler angles φ, θ, ψ as found with the collective model.

3. MATRIX ELEMENTS OF THE POTENTIAL-ENERGY OPERATOR IN TERMS OF THE FUNCTIONS $\chi_K(\alpha_i,\,\sigma_i,\,\tau_i)$

By using the generalized hyperspherical functions and the spin–isospin functions for states with definite values of total spin and isopin, one can construct a basis of the functions $\chi_S(\alpha_i,\,\sigma_i,\,\tau_i)$ which is antisymmetric with respect to particle interchange. This basis can then be used to expand the wave function for a system of interacting nucleons. The expansion coefficients will depend on the collective coordinates a, b, c, φ, θ, ψ and must satisfy an infinite set of linked equations which is equivalent to the many-particle Schrödinger equation for the original wave function. An approximate solution to the problem, which satisfies the Ritz variational principle, can be obtained by breaking off the expansion after some number of terms. For magic nuclei the very simplest approximation is the one in which, of all the functions, only the function $\chi_K(\alpha_i,\,\sigma_i,\,\tau_i)$ discussed above[1)] is retained. Then the equation for the single collective-coordinate-dependent coefficient is that given by Eq. (14) if the oscillator potential $m\,\omega^2(a^2+b^2+c^2)/2$ is replaced by U(a, b, c), which is the average value of the system's potential-energy operator in the state with the wave function $\chi_K(\alpha_i,\,\sigma_i,\,\tau_i)$. The algorithm for averaging the potential-energy operator of a system of nucleons over a state with the wave function χ_K has already been established.[11]

First we recall that in the oscillator shell model the matrix elements of the potential-energy operator for the two-particle interaction

$$\hat{U}=\sum_{l>K}V(|\vec{r}_l-\vec{r}_K|) \qquad (17)$$

take the following simple form for wave functions of the ground state of magic nuclei:

$$\langle\Psi_{0K}|\hat{U}|\Psi_{0K}\rangle=\int_0^\infty\exp(-q^2)P_K(q^2)V(\sqrt{2}r_0q)q^2\,dq; \qquad (18)$$

r_0 is the oscillator radius; the $P_K(q^2)$ are well-known polynomials in q^2, the degree and polynomial coefficients of which depend on the magic nuclei to which they correspond. These polynomials are given in ref. 12 for a number of magic nuclei.

The matrix elements of the operator $\hat{U}$ in the functions $\chi_K(\alpha_i,\,\sigma_i,\,\tau_i)$,

$$U(a,\,b,\,c)=\int\chi_K(\alpha_i,\,\sigma_i,\,\tau_i)$$
$$\times\sum_{l>k}V(|\vec{r}_k-\vec{r}_l|)\chi_K(\alpha_i,\,\sigma_i,\,\tau_i)\,d\vec{\alpha}_{A-1}, \qquad (19)$$

can be represented by threefold integrals over the two-particle potential

$$U(a,\,b,\,c)=P_K\Big(-\frac{1}{2}\cdot\frac{\partial}{\partial\lambda}\Big)I(\lambda;\,a,\,b,\,c)|_{\lambda=1/2}, \qquad (20)$$

where

$$I(\lambda;\ a,\ b,\ c)=\frac{\sqrt{\pi}}{4(2\pi\lambda)^{3/2}}\cdot\frac{\Gamma((A+2K-1)/2)}{\Gamma((A+2K-4)/2)}\int_0^{\pi}d\alpha_1\int_0^{\pi}d\alpha_2\int_0^{\pi}d\alpha_3$$

$$\times(\sin\alpha_1)^{A+2K-3}(\sin\alpha_2)^{A+2K-4}(\sin\alpha_3)^{A+2K-5}V\left(\sqrt{(a^2/\lambda)\sin^2\alpha_1\sin^2\alpha_2\cos^2\alpha_3+(b^2/\lambda)\sin^2\alpha_1\cos^2\alpha_2+(c^2/\lambda)\cos^2\alpha_1}\right);\quad(21)$$

the $P_K(x)$ are the polynomials defined above.

In deriving Eqs. (20) and (21) we shall follow the method suggested by Surkov[13] for calculating matrix elements of the potential-energy operator with hyperspherical functions. First consider the integral

$$j=\int G(\alpha_i)\,d\vec{\alpha}_{A-1},\quad(22)$$

in which $\vec{\alpha}_{A-1}$ indicates a set of generalized Euler angles in an $A-1$ dimensional space, and $G(\alpha_i)$ is some function which depends on them.

Let q_{kx}, q_{ky}, and q_{kz} be the Cartesian components of the Jacobi vectors in an arbitrary coordinate system; $a_{K\xi}$, $a_{K\eta}$, $a_{K\zeta}$ are the Cartesian components of the same vectors in a coordinate system whose axes are directed along the principal axes of the inertia ellipsoid, and φ, θ, ψ are the Euler angles which specify the orientation of the inertia ellipsoid's axes in space. We further assume that

$$\sum_{k=1}^{A-1}a_{k\xi}^2=a_0^2;\quad\sum_{k=1}^{A-1}a_{k\eta}^2=b_0^2;\quad\sum_{k=1}^{A-1}a_{k\zeta}^2=c_0^2.\quad(23)$$

By using the two identities

$$8\int\delta(a^2-a_0^2)\,\delta(b^2-b_0^2)\,\delta(c^2-c_0^2)\,a_0b_0c_0\,da_0\,db_0\,dc_0=1;\quad(24)$$

$$|a_0^2-b_0^2|\,|b_0^2-c_0^2|\,|c_0^2-a_0^2|$$

$$\times\int\delta\left(\sum_{l=1}^{A-1}q_{lx}q_{ly}\right)\delta\left(\sum_{l=1}^{A-1}q_{ly}q_{lz}\right)\delta\left(\sum_{l=1}^{A-1}q_{lz}q_{lx}\right)d\varphi\sin\theta\,d\theta\,d\psi=1,\quad(25)$$

we can transform j into the following form:

$$j=\frac{8}{(abc)^{A-5}}\int\delta\left(\sum_l q_{lx}q_{ly}\right)\delta\left(\sum_l q_{ly}q_{lz}\right)\delta\left(\sum_l q_{lz}q_{lx}\right)$$

$$\times\delta\left(a^2-\sum_{l=1}^{A-1}q_{lx}^2\right)\delta\left(b^2-\sum_{l=1}^{A-1}q_{ly}^2\right)\delta\left(c^2-\sum_{l=1}^{A-1}q_{lz}^2\right)G(\alpha_i)\,d\tau_{3A-3},\quad(26)$$

where

$$d\tau_{3A-3}\equiv d\vec{q}_1\,d\vec{q}_2\,\ldots\,d\vec{q}_{A-1}$$

$$=(a_0b_0c_0)^{A-4}\,|a_0^2-b_0^2|\,|b_0^2-c_0^2|\,|c_0^2-a_0^2|$$

$$\times\,da_0\,db_0\,dc_0\,d\vec{\alpha}\,d\varphi\sin\theta\,d\theta\,d\psi.$$

Let us now introduce the integral representation for the δ function

$$\delta(a^2-a_0^2)\,\delta(b^2-b_0^2)\,\delta(c^2-c_0^2)$$

$$=\frac{1}{(2\pi^3)}\iiint_{-\infty}^{\infty}dk_1\,dk_2\,dk_3\exp\left[ik_1(a^2-a_0^2)\right.$$

$$\left.+ik_2(b^2-b_0^2)+ik_3(c^2-c_0^2)\right];\quad(27)$$

$$\delta\left(\sum_{l=1}^{A-1}q_{lx}q_{ly}\right)\delta\left(\sum_{l=1}^{A-1}q_{ly}q_{lz}\right)\left(\sum_{l=1}^{A-1}q_{lz}q_{lx}\right)$$

$$=\frac{1}{(2\pi)^3}\iiint_{-\infty}^{\infty}dp_1\,dp_2\,dp_3$$

$$\times\exp\left[-i\sum_{l=1}^{A-1}(p_1q_{ly}q_{lz}+p_2q_{lz}q_{lx}+p_3q_{lx}q_{ly})\right].\quad(28)$$

The next step is to reduce to diagonal form the expressions in the exponent of the product of the integral representation of the δ functions; these expressions are forms which are quadratic in the Cartesian components of the Jacobi vectors. This is accomplished by rotating the coordinate axes of the three-dimensional space in which the Jacobi vectors are given, and the angles of rotation are determined from the values of the parameters.

Thus, we reduce the quadratic form $L(x,y,z)=k_1x^2+k_2y^2+k_3z^2+p_1yz+p_2zx+p_3xy$ to diagonal form. This is easily accomplished if we temporarily introduce spherical coordinates in place of the Cartesian coordinates: $x=r\sin\theta_1\cos\varphi_1$; $y=r\sin\theta_1\sin\varphi_1$; $z=r\cos\theta_1$, for then

$$L(x,\ y,\ z)=\frac{2}{\sqrt{3}}tr^2+\sqrt{\frac{4\pi}{15}}\,r^2\sum_{\mu=-2}^{2}\tilde{\alpha}_\mu Y_{2\mu}(\theta_1,\ \varphi_1);$$

$$t=\frac{k_1+k_2+k_3}{2\sqrt{3}};$$

$$\tilde{\alpha}_{\pm2}=\frac{k_1-k_2\mp ip_3}{\sqrt{2}};\quad\tilde{\alpha}_{\pm1}=\frac{p_2\mp ip_1}{\sqrt{2}};\quad\tilde{\alpha}_0=\frac{2}{\sqrt{3}}\left(k_3-\frac{k_1+k_2}{2}\right).\quad(29)$$

The following transformation of the sum on the right side of Eq. (29) is well known:[1]

$$\sum_\mu{}'\tilde{\alpha}_\mu Y_{2\mu}(\theta_1,\ \varphi_1)=\tilde{\lambda}\cos\delta\,Y_{20}(\theta_1',\ \varphi_1')$$

$$+\tilde{\lambda}\sin\delta\left[Y_{22}(\theta_1',\ \varphi_1')+Y_{2,-2}(\theta_1',\ \varphi_1')\right]/\sqrt{2}.\quad(30)$$

This transformation, a rotation of the coordinate system in three-dimensional space, solves our problem, for after reintroducing the Cartesian components the quadratic form is found to be diagonalized:

$$L(x',\ y',\ z')=\frac{2}{\sqrt{3}}\left[t+\frac{\tilde{\lambda}}{2}\cos\left(\delta-\frac{2\pi}{3}\right)\right]x'^2$$

$$+\frac{2}{\sqrt{3}}\left[t+\frac{\tilde{\lambda}}{2}\cos\left(\delta+\frac{2\pi}{3}\right)\right]y'^2+\frac{2}{\sqrt{3}}\left[t+\frac{\tilde{\lambda}}{2}\cos\delta\right]z'^2.\quad(31)$$

Of course both the quantities $\tilde{\lambda}$, δ, and $\tilde{\varphi}$, $\tilde{\theta}$, $\tilde{\psi}$, the Euler angles of the rotation in three-dimensional space, are functions of the variables $\tilde{\alpha}_\mu$ or, equivalently, k_i and p_i. We note further that the volume elements in the space of the variables t, $\tilde{\lambda}$, δ, $\tilde{\varphi}$, $\tilde{\theta}$, $\tilde{\psi}$ and the space of the variables k_i, p_i are related as follows:

$$dk_1\,dk_2\,dk_3\,dp_1\,dp_2\,dp_3=(1/2)\,dt\,\tilde{\lambda}^4\,d\tilde{\lambda}\,|\sin3\delta|\,d\delta\,d\tilde{\varphi}\sin\tilde{\theta}\,d\tilde{\theta}\,d\tilde{\psi}.\quad(32)$$

The result just obtained enables us to establish the following equality which will be required in subsequent transformations:

$$\int_{-\infty}^{\infty} \cdots \int dk_1\, dk_2\, dk_3\, dp_1\, dp_2\, dp_3 \exp \sum_{l=1}^{A-1} \mathrm{i} \{ -k_1 q_{lx}^2 - k_2 q_{ly}^2 - k_3 q_{lz}^2 - p_1 q_{ly} q_{lz} - p_2 q_{lz} q_{lx} - p_3 q_{lx} q_{ly} \}$$
$$= \frac{1}{2} \int_{-\infty}^{\infty} dt \int_0^{\infty} \tilde{\lambda}^4 d\tilde{\lambda} \int_0^{\pi/3} \sin 3\delta\, d\delta \int_0^{2\pi} d\tilde{\varphi} \int_0^{\pi} \sin \tilde{\theta}\, d\tilde{\theta} \int_0^{2\pi} d\tilde{\psi} \times \exp \sum_{l=1}^{A-1} \mathrm{i} \{ -s_1 q_{lx'}^2 - s_2 q_{ly'}^2 - s_3 q_{lz'}^2 \}, \tag{33}$$

where

$$s_1 = (2/\sqrt{3})\, [t + (1/2)\, \tilde{\lambda} \cos(\delta - 2\pi/3)];$$
$$s_2 = (2/\sqrt{3})\, [t + (1/2)\, \tilde{\lambda} \cos(\delta + 2\pi/3)];$$
$$s_3 = (2/\sqrt{3})\, [t + (1/2)\, \tilde{\lambda} \cos \delta].$$

We now put $G(\alpha_i) = \hat{U}\chi_K^2(\alpha_i, \sigma_i, \tau_i)$ and understand that the integral for j requires both an integration over α_j and a summation over the spin–isospin variables σ_i, τ_i.

If we now insert in the integral over $d\tau_{3A-3}$ in Eq. (26) the factor $C_K^2(abc)^{2K}$, which ensures the transformation of the expression $C_K(abc)^K \chi_K^2(\alpha_i)$ into the preexponential part of the (normalized to unity) ground-state wave function of the appropriate magic nucleus in the oscillator shell model (of course, the factor inserted must be compensated for later), and also the factor[13]

$$\int \sqrt{\frac{\Delta_1 \Delta_2 \Delta_3}{\pi^3}} \exp(-\Delta_1 R_{x'}^2 - \Delta_2 R_{y'}^2 - \Delta_3 R_{z'}^2)\, dR_{x'}\, dR_{y'}\, dR_{z'} = 1,$$

$$\Delta_1 = A(\varepsilon + \mathrm{i}s_1), \quad \Delta_2 = A(\varepsilon + \mathrm{i}s_2), \quad \Delta_3 = A(\varepsilon + \mathrm{i}s_3),$$

expressed in terms of the Cartesian components of the center of inertia ($R_{x'}$, $R_{y'}$, $R_{z'}$), we finally arrive at the following expression for the integral in Eq. (26):

$$j = \frac{1}{2} \int_{-\infty}^{\infty} dt \int_0^{\infty} \tilde{\lambda}^4 d\tilde{\lambda} \int_0^{\pi/3} \sin 3\delta\, d\delta \int_0^{2\pi} d\tilde{\varphi} \int_0^{\pi} \sin \tilde{\theta}\, d\tilde{\theta} \int_0^{2\pi} d\tilde{\psi} \times D(t, \tilde{\lambda}, \delta, \tilde{\varphi}, \tilde{\theta}, \tilde{\psi}; a, b, c)$$
$$\times \int d\tau_{3A} \hat{U} \Psi_{0б}^2 (r_{ix'} \sqrt{\varepsilon + \mathrm{i}s_1},\ r_{iy'} \sqrt{\varepsilon + \mathrm{i}s_2},\ r_{iz'} \sqrt{\varepsilon + \mathrm{i}s_3}). \tag{34}$$

Here Ψ_{0b} is the ground-state wave function of the magic nucleus as constructed from single-particle nucleon functions for an anisotropic oscillator field with three different frequencies:

$$\omega_{x'} = (\hbar/m)(\varepsilon + \mathrm{i}s_1); \quad \omega_{y'} = (\hbar/m)(\varepsilon + \mathrm{i}s_2); \quad \omega_{z'} = (\hbar/m)(\varepsilon + \mathrm{i}s_3);$$
$$D(t, \tilde{\lambda}, \delta, \tilde{\varphi}, \tilde{\theta}, \tilde{\psi}; a, b, c) = \frac{\exp(\mathrm{i}k_1 a^2 + \mathrm{i}k_2 b^2 + \mathrm{i}k_3 c^2)}{64\pi^{9/2} (abc)^{A+2K-5}} \times \frac{\Gamma((A+2K-1)/2)\, \Gamma((A+2K-2)/2)\, \Gamma((A+2K-3)/2)}{[(\varepsilon + \mathrm{i}s_1)(\varepsilon + \mathrm{i}s_2)(\varepsilon + \mathrm{i}s_3)]^{[(A+2K-3)/2]}}. \tag{35}$$

By generalizing Eq. (18) to the case of an anisotropic oscillator, we find the integral over $d\tau_{3A}$ on the right side of Eq. (34) to be

$$\int d\tau_{3A} \hat{U} \Psi_{0б}^2 (r_{ix'} \sqrt{\varepsilon + \mathrm{i}s_1},\ r_{iy'} \sqrt{\varepsilon + \mathrm{i}s_2},\ r_{iz'} \sqrt{\varepsilon + \mathrm{i}s_3})$$
$$= \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} [(\varepsilon + \mathrm{i}s_1) x^2 + (\varepsilon + \mathrm{i}s_2) y^2 + (\varepsilon + \mathrm{i}s_3) z^2] \right\} \times P_k \left\{ \frac{1}{2} [(\varepsilon + \mathrm{i}s_1) x^2 + (\varepsilon + \mathrm{i}s_2) y^2 + (\varepsilon + \mathrm{i}s_3) z^2] \right\}$$
$$\times V(\sqrt{x^2 + y^2 + z^2}) \sqrt{(\varepsilon + \mathrm{i}s_1)(\varepsilon + \mathrm{i}s_2)(\varepsilon + \mathrm{i}s_3)}\, dx\, dy\, dz$$
$$= P_k \left(-\frac{1}{2} \cdot \frac{\partial}{\partial \lambda} \right) \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \{ -\lambda [(\varepsilon + \mathrm{i}s_1) x^2 + (\varepsilon + \mathrm{i}s_2) y^2 + (\varepsilon + \mathrm{i}s_3) z^2] \} V(\sqrt{x^2 + y^2 + z^2})$$
$$\times \sqrt{(\varepsilon + \mathrm{i}s_1)(\varepsilon + \mathrm{i}s_2)(\varepsilon + \mathrm{i}s_3)}\, dx\, dy\, dz. \tag{36}$$

We still must evaluate the integral

$$j_1 = \frac{1}{2} \int_{-\infty}^{\infty} dt \int_0^{\infty} \tilde{\lambda}^4 d\tilde{\lambda} \int_0^{\pi/3} \sin 3\delta\, d\delta \int_0^{2\pi} d\tilde{\varphi} \int_0^{\pi} \sin \tilde{\theta}\, d\tilde{\theta} \int_0^{2\pi} d\tilde{\psi} \times D(t, \tilde{\lambda}, \delta, \tilde{\varphi}, \tilde{\theta}, \tilde{\psi}; a, b, c)$$
$$\times \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \exp\{ -\lambda [(\varepsilon + \mathrm{i}s_1) x^2 + (\varepsilon + \mathrm{i}s_2) y^2 + (\varepsilon + \mathrm{i}s_3) z^2] \}$$
$$\times V(\sqrt{x^2 + y^2 + z^2}) \sqrt{(\varepsilon + \mathrm{i}s_1)(\varepsilon + \mathrm{i}s_2)(\varepsilon + \mathrm{i}s_3)}\, dx\, dy\, dz. \tag{37}$$

First we note that

$$\left(\frac{\lambda}{\pi} \right)^{3/2} \iiint_{-\infty}^{\infty} \exp[-\lambda (\varepsilon + \mathrm{i}s_1) x^2 - \lambda (\varepsilon + \mathrm{i}s_2) y^2 - \lambda (\varepsilon + \mathrm{i}s_3) z^2] \times \sqrt{(\varepsilon + \mathrm{i}s_1)(\varepsilon + \mathrm{i}s_2)(\varepsilon + \mathrm{i}s_3)}\, dx\, dy\, dz = 1. \tag{38}$$

Therefore

$$\frac{64\pi^{9/2} (abc)^{A+2K-5}}{\Gamma((A+2K-1)/2)\, \Gamma((A+2K-2)/2)\, \Gamma((A+2K-3)/2)}\, j_1 \equiv j_2$$
$$= \frac{1}{2} \int \cdots \int dt \tilde{\lambda}^4 d\tilde{\lambda} \sin 3\delta\, d\delta\, d\tilde{\Omega} \exp\{ \mathrm{i}(k_1 a^2 + k_2 b^2 + k_3 c^2) \} \times \frac{\sqrt{\pi}}{4(2\pi)^{3/2}} \left(\frac{\lambda}{\pi} \right)^{3(A+2K-2)/2}$$
$$\times \iiint_{-\infty}^{\infty} \cdots \iiint_{-\infty}^{\infty} dx_1\, dy_1\, dz_1 \ldots dx_{A+2K-1}\, dy_{A+2K-1}\, dz_{A+2K-1}$$
$$\times V(\sqrt{x_1^2 + y_1^2 + z_1^2}) \times \exp\left\{ -\lambda \sum_{l=1}^{A+2K-1} [(\varepsilon + \mathrm{i}s_1) x_l^2 + (\varepsilon + \mathrm{i}s_2) y_l^2 + (\varepsilon + \mathrm{i}s_3) z_l^2] \right\}. \tag{39}$$

Thus the integrand is simplified by a transformation to the configuration space for A+2K−1 "effective particles". We must now transform from the variables t, $\tilde{\lambda}$, δ, $\tilde{\Omega}$, back to the variables k_i and p_i:

$$j_2 = \frac{\sqrt{\pi}}{4(2\pi)^{3/2}} \left(\frac{\lambda}{\pi} \right)^{3(A+2K-2)/2} \iiint_{-\infty}^{\infty} \cdots \iiint_{-\infty}^{\infty} dx_1\, dy_1\, dz_1 \ldots dx_{A+2K-1}\, dy_{A+2K-1}\, dz_{A+2K-1} \times V(\sqrt{x_1^2 + y_1^2 + z_1^2})$$
$$\times \int_{-\infty}^{\infty} \cdots \int dk_1\, dk_2\, dk_3\, dp_1\, dp_2\, dp_3 \exp[\mathrm{i}(k_1 a^2 + k_2 b^2 + k_3 c^2)]$$
$$\times \exp\left[-\mathrm{i}\lambda \sum_{l=1}^{A+2K-1} (k_1 x_l^2 + k_2 y_l^2 + k_3 z_l^2 + p_1 y_l z_l + p_2 z_l x_l + p_3 x_l y_l) \right]$$
$$= \frac{16\sqrt{\pi}}{(2\pi)^{3/2}} \left(\frac{\lambda}{\pi} \right)^{3(A+2K-4)/2} \times$$

$$\times \pi^3 \int_{-\infty}^{\infty} \ldots \int dx_1 \ldots dz_{A+2K-1} V(\sqrt{x_1^2+y_1^2+z_1^2})$$

$$\times \delta\left[a^2-\lambda \sum_{l=1}^{A+2K-1} x_l^2\right] \delta\left[b^2-\lambda \sum_{l=1}^{A+2K-1} y_l^2\right] \delta\left[c^2-\lambda \sum_{l=1}^{A+2K-1} z_l^2\right]$$

$$\times \delta\left[\sum_{l=1}^{A+2K-1} y_l z_l\right] \delta\left[\sum_{l=1}^{A+2K-1} z_l x_l\right] \delta\left[\sum_{l=1}^{A+2K-1} x_l y_l\right]. \quad (40)$$

In the configuration space of $A+2K-1$ "single-particle" vectors we introduce the inertia ellipsoid with semiaxes $\widetilde{a}_0$, $\widetilde{b}_0$, $\widetilde{c}_0$, and the single-particle variables x_i, y_i and z_i are now expressed in terms of $\widetilde{a}_0$, $\widetilde{b}_0$, $\widetilde{c}_0$, the Euler angles θ_i which define the orientation of the inertia ellipsoid's principal axes, and the generalized Euler angles α_i of the abstract $A+2K-1$ dimensional space. We then have

$$j_2 = \frac{16\sqrt{\pi}}{(2\pi)^{3/2}} \left(\frac{\lambda}{\pi}\right)^{3(A+2K-4)/2}$$

$$\times \pi^3 \int \ldots \int (\widetilde{a}_0 \widetilde{b}_0 \widetilde{c}_0)^{A+2K-4} d\widetilde{a}_0 \, d\widetilde{b}_0 \, d\widetilde{c}_0 \, d\vec{\alpha}_{A+2K-1}$$

$$\times \delta(a^2-\lambda \widetilde{a}_0^2)\, \delta(b^2-\lambda \widetilde{b}_0^2)\, \delta(c^2-\lambda \widetilde{c}_0^2)$$

$$\times V(\sqrt{\widetilde{a}_0^2 \sin^2\alpha_1 \sin^2\alpha_2 \cos^2\alpha_3 + \widetilde{b}_0^2 \sin^2\alpha_1 \cos^2\alpha_2 + \widetilde{c}_0^2 \cos^2\alpha_1})$$

$$= \frac{2\sqrt{\pi}}{(2\lambda)^{3/2}} \left(\frac{abc}{\pi^{3/2}}\right)^{A+2K-5} \int d\vec{\alpha}_{A+2K-2}$$

$$\times \int_0^{\pi} (\sin\alpha_1)^{A+2K-3} d\alpha_1 \int_0^{\pi} (\sin\alpha_2)^{A+2K-4} d\alpha_2 \int_0^{\pi} (\sin\alpha_3)^{A+2K-5} d\alpha_3$$

$$\times V\left(\sqrt{\frac{a^2}{\lambda} \sin^2\alpha_1 \sin^2\alpha_2 \cos^2\alpha_3 + \frac{b^2}{\lambda} \sin^2\alpha_1 \cos^2\alpha_2 + \frac{c^2}{\lambda} \cos^2\alpha_1}\right). \quad (41)$$

Because

$$\int d\vec{\alpha}_n = \frac{8(\pi)^{3(n-1)/2}}{\Gamma(n/2)\,\Gamma((n-1)/2)\,\Gamma((n-2)/2)},$$

it is not difficult to find that $j_1 = I(\lambda; a, b, c)$ when Eq. (40) is taken into account together with the connection between j_1 and j_2; this then demonstrates the point asserted in Eq. (20).

It is of interest to examine the limit of $I(\lambda; a, b, c)$ when $A+2K \gg 1$. This is simply carried out if we first make this change of variables under the triple integral: $\cos\alpha_1 = r\cos\theta_1$; $\sin\alpha_1 \cos\alpha_2 = r\sin\theta_1 \sin\varphi_1$; $\sin\alpha_1 \sin\alpha_2 \cdot \cos\alpha_3 = r\sin\theta_1 \cos\varphi_1$; then

$$I(\lambda; a, b, c) = \frac{\sqrt{\pi}}{4(2\pi\lambda)^{3/2}} \cdot \frac{\Gamma((A+2K-1)/2)}{\Gamma((A+2K-4)/2)}$$

$$\times \int_0^1 (1-r^2)^{(A+2K-6)/2} r^2 dr \int_0^{\pi} \sin\theta_1 \, d\theta_1 \int_0^{2\pi} d\varphi_1$$

$$\times V(\sqrt{r^2/\lambda\,(a^2 \sin^2\theta_1 \cos^2\varphi_1 + b^2 \sin^2\theta_1 \sin^2\varphi_1 + c^2\cos^2\theta_1)}). \quad (42)$$

Let us put $r^2 = [2/(A+2K-b)]q^2$. After some uncomplicated transformations, limiting ourselves to the first two terms in an expansion in terms of inverse powers of $A+2K-b$, we find

$$I(\lambda; a, b, c) \approx \frac{\sqrt{\pi}}{4(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} dx\,dy\,dz \exp(-\lambda q^2)$$

$$\times \left(1+\frac{15/4-\lambda^2 q^4}{A+2K-6}\right) V(\sqrt{a_0^2 x^2 + b_0^2 y^2 + c_0^2 z^2}), \quad (43)$$

where

$$a_0^2 = [2/(A+2K-6)]\, a^2; \quad b_0^2 = [2/(A+2K-6)]\, b^2;$$

$$c_0^2 = [2/(A+2K-6)]\, c^2.$$

After substituting Eq. (43) into Eq. (20) the principal term in the expansion gives a result which agrees with the result obtained by averaging the potential energy over a magic-nucleus state whose wave function is constructed from single-particle nucleon functions for the lowest population arrangement in an anisotropic oscillator field with the frequencies $\omega_x = \hbar/(m a_0^2)$, $\omega_y = \hbar/(m b_0^2)$, $\omega_z = \hbar/(m c_0^2)$.

4. ASYMPTOTIC SCHRÖDINGER EQUATION FOR WAVE FUNCTIONS OF COLLECTIVE EXCITATIONS IN MAGIC NUCLEI

The expressions obtained above enable us to construct the very simplest approximation for our study of the collective excitations in magic nuclei. This approximation consists of writing the trial wave function for the magic nucleus in the form

$$\Psi(a, b, c, \varphi, \theta, \psi; \alpha_i, \sigma_i, \tau_i)$$

$$= (abc)^{-(A-4)/2}\, u(a, b, c, \varphi, \theta, \psi)\, \chi_K(\alpha_i, \sigma_i, \tau_i). \quad (44)$$

The function $\chi_K(\alpha_i, \sigma_i, \tau_i)$ for the ground state of the magic nucleus is defined above, and the function $u(a, b, c, \varphi, \theta, \psi)$ must satisfy the equation

$$-\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial a^2} + \left(\frac{1}{a^2-b^2} - \frac{1}{c^2-a^2}\right) 2a\frac{\partial}{\partial a} - \frac{\Lambda(\Lambda+1)}{a^2}\right.$$

$$+ \frac{\partial^2}{\partial b^2} + \left(\frac{1}{b^2-c^2} - \frac{1}{a^2-b^2}\right) 2b\frac{\partial}{\partial b} - \frac{\Lambda(\Lambda+1)}{b^2} + \frac{\partial^2}{\partial c^2}$$

$$+ \left(\frac{1}{c^2-a^2} - \frac{1}{b^2-c^2}\right) 2c\frac{\partial}{\partial c} - \frac{\Lambda(\Lambda+1)}{c^2} - \frac{b^2+c^2}{(b^2-c^2)^2} I_\xi^2$$

$$\left. - \frac{c^2+a^2}{(c^2-a^2)^2} I_\eta^2 - \frac{a^2+b^2}{(a^2-b^2)^2} I_\zeta^2 \right\} u + U(a, b, c)\, u = Eu, \quad (45)$$

which is obtained by averaging the Schrödinger equation of the magic nucleus over the state $\chi_K(\alpha_i, \sigma_i, \tau_i)$ where $\Lambda = (A+2K-6)/2$.

We will assume that the internucleon forces ensure saturation and that the Coulomb interaction is absent. Then for large A the limiting energy of the system in Eq. (45) is given by the minimum of the equation

$$E_0(a, b, c) = (\hbar^2/2m)\,\Lambda(\Lambda+1)(1/a^2 + 1/b^2 + 1/c^2) + U(a, b, c). \quad (46)$$

If the minimum occurs at $a = a_0$, $b = b_0$, $c = c_0$, and $a_0 = b_0 = c_0 = \rho_0/\sqrt{3}$, it will be convenient to transform to the variables ρ, β, γ; then, after dropping terms of order $1/\Lambda^{3/4}$, Eq. (45) can be written as

$$-\frac{\hbar^2}{2m}\left\{\frac{1}{\rho^8} \cdot \frac{\partial}{\partial\rho} \rho^8 \frac{\partial}{\partial\rho} - \frac{3\Lambda(3\Lambda+3)}{\rho^2}\right.$$

$$\left. + \frac{6\Lambda}{\rho^2}(\Delta_{\beta,\gamma,\theta_i}) - \beta^2\right\} u + U(\rho, \beta, \gamma)\, u = Eu. \quad (47)$$

The normalization condition for the function $u(\rho, \beta, \gamma)$ takes the form

$$\int \cdots \int u^2 \rho^8 \, d\rho \beta^4 \, d\beta \sin 3\gamma \, d\gamma \, d\varphi \sin\theta \, d\theta \, d\psi = 1.$$

Equation (46) can also be simplified somewhat:

$$E_0(\rho) = \frac{3\hbar^2}{2m} \cdot \frac{\Lambda(\Lambda+1)}{\rho^2} + U \Big|_{\beta=0}. \tag{48}$$

For forces which ensure saturation $E_0(\rho)$ has its minimum value at some value of $\rho = \rho_0$ which is proportional to $A^{5/6}$, with the minimum value of $E_0(\rho)$ being proportional to A; furthermore, the two terms on the right side of Eq. (48), which are the system's kinetic energy (the first term) and potential energy, are also proportional to each other. We note here that for large A the dependence of Λ on A is, by definition, $\Lambda \approx (1/4)(3/2)^{4/3} A^{4/3}$.

In order to isolate the explicit dependence of ρ_0 on A for large A, we introduce r_0 and put

$$\rho_0^2 = (3/4)(3/2)^{1/3} A^{5/3} r_0^2. \tag{49}$$

Unlike ρ_0, r_0 is independent of A when A is large. It is also convenient to show the explicit A dependence in Eq. (48):

$$E_0(\rho_0) \equiv AF(r_0) = A\{(3/8)(3/2)^{1/3} \hbar^2/(mr_0^2) - V(r_0)\}, \tag{50}$$

where

$$V(r_0) = -\frac{1}{A} U \Big|_{\beta=0\,\rho=\rho_0}.$$

Equation (50) is a consequence of Eq. (47). It may be derived by retaining only the principal terms in Eq. (47) (i.e., those terms proportional to A). The terms of order $1/A^{1/3}$ give the wave function for small monopole and quadrupole oscillations. For monopole oscillations this equation takes the following form:

$$\frac{1}{A^{1/3}} \left\{ -\frac{\hbar^2}{m} \left(\frac{2}{3}\right)^{4/3} \frac{d^2}{d\xi^2} + \frac{1}{2} F''(r_0)\, \xi^2 \right\} u(\xi) = E_{1\xi} u(\xi), \tag{51}$$

where

$$\xi^2 = A^{4/3} (r - r_0)^2,$$

for quadrupole oscillations we have

$$\frac{1}{A^{1/3}} \left\{ -\frac{\hbar^2}{mr_0^2} \Delta_{\beta,\gamma,\theta_i} + \left[\frac{\hbar^2}{mr_0^2} - \frac{8}{15} \left(\frac{2}{3}\right)^{1/3} r_0^4 \right.\right.$$
$$\left.\left. \times \frac{\partial^2}{\partial (r_0^2)^2} V(r_0) \right] \beta^2 \right\} u(\beta, \gamma, \theta_i) = E_{1\beta} u(\beta, \gamma, \theta_i). \tag{52}$$

If the effective potential energy U does not depend on β, the rigidity of the β oscillations is completely determined by kinematic effects. Its nature is determined by the tendency of a system of independent particles (fermions) moving in an oscillator field and having the quantum numbers of the lowest population of oscillator shells to maintain the spherical symmetry of its inertia ellipsoid when the particle number is magic. The attractive forces between nucleons weaken this tendency, because for a given value of the global radius ρ the attractive potential energy is reduced as the nucleus becomes more deformed. As a result, when the β dependence of the potential energy is taken into account, the rigidity of the β oscillations is decreased as compared with the case where it is due solely to the kinematic effects.

By using Eqs. (51) and (52) it is not difficult to obtain expressions for the frequencies of the monopole ω_ξ oscillations and the quadrupole ω_β oscillations:

$$\omega_\xi^2 = \frac{1}{A^{2/3}} \left[3 \frac{\hbar^2}{m^2 r_0^4} - \frac{2}{m} \left(\frac{2}{3}\right)^{4/3} \frac{d^2}{dr_0^2} V(r_0) \right]; \tag{53}$$

$$\omega_\beta^2 = \frac{1}{A^{2/3}} \left[\frac{18}{5} \cdot \frac{\hbar^2}{m^2 r_0^4} - \frac{4}{5} \cdot \frac{1}{m} \left(\frac{2}{3}\right)^{4/3} \frac{d^2}{dr_0^2} V(r_0) \right]. \tag{54}$$

One important consequence of Eqs. (53) and (54) is that

$$\omega_\beta^2 = \frac{2}{5} \omega_\xi^2 + \frac{12}{5} \cdot \frac{1}{A^{2/3}} \cdot \frac{\hbar^2}{m^2 r_0^4}. \tag{55}$$

This equation shows that the frequency of the β vibrations remains finite when the frequency of the monopole oscillations goes to zero. For small values of ω_ξ the frequency of the quadrupole oscillations can be substantially greater than the frequency of the monopole oscillations.

The kinematic and dynamic anharmonic effects can be taken into account if Eq. (47) is augmented to include terms of order $1/\Lambda^{3/4}$ and $1/\Lambda^{5/4}$. To find these terms it is necessary to return to Eq. (45).

5. TRIAL WAVE FUNCTIONS FOR A VARIATIONAL CALCULATION OF THE GROUND-STATE PROPERTIES OF NONMAGIC NUCLEI

By using single-particle wave functions for a centrally symmetric field it is not difficult to construct the shell-model wave function for states with a specified angular momentum (see ref. 14, for example). However, such a function cannot convey the deformation of a nucleus. In order to describe deformed nuclei, one usually uses the single-particle functions of a nonspherical self-consistent field, but then the shell-model wave function is a superposition of states with different values of angular momentum.

Thus, the orthodox shell model forces us to choose between functions having a definite value of angular momentum, but which are not suited to the description of deformed states, and functions which are suitable for describing deformations, but which are not among the states with definite angular momentum values. For deformed nuclei the preference must be given to single-particle functions in a nonspherical field.[15-17] Nevertheless, the imprecision introduced into calculations of the energy, dimensions, and shapes of the ground states for nonspherical nuclei due to uncertainty in the angular momentum requires special study.

If the collective degrees of freedom are associated with the semiaxes of the nucleus' inertia ellipsoid rather than with the parameters of a nonspherical self-consistent field, generalized hyperspherical functions become a convenient vehicle for construction of a trial wave function for the state of definite angular momentum, and it will be suitable for describing deformed nuclei.

We note that in the oscillator shell model the ground-state wave function for an even−even nucleus with zero angular momentum can be written as

$$\Psi_0(a,\ b,\ c;\ \alpha_i,\ \sigma_i,\ \tau_i) = \sum_{k+l+m=n} C_{klm} a^{2k} b^{2l} c^{2m} \exp\left(-\frac{a^2+b^2+c^2}{2}\right) \chi_{lmn}(\alpha_i,\ \sigma_i,\ \tau_i).$$

The number n, the coefficients C_{klm}, and the functions χ_{klm} are determined by the oscillator configuration of the nucleus. The functions χ_{klm} are antisymmetric relative to interchanges of the spatial and spin−isospin coordinates of an arbitrary pair of nucleons. For example, the lowest oscillator configuration of a system of four neutrons (or protons) corresponds to the wave function

$$\Psi_0 = C\,[(c^2-(a^2+b^2)/2)\,\chi_0(\alpha_i,\ \sigma_i,\ \tau_i) + (\sqrt{3}/2)\,(a^2-b^2)\,\chi_2(\alpha_i,\ \sigma_i,\ \tau_i)]\exp[-(a^2+b^2+c^2)/2], \quad (56)$$

where C is the normalization constant. Ψ_0 and χ_2 are expressed in terms of the spin−isospin functions $\xi'(\sigma_i, \tau_i)$, $\xi''(\sigma_i, \tau_i)$ and spatial functions $D_0'(\alpha_i)$, $D_0''(\alpha_i)$, $D_2'(\alpha_i)$ $D_2''(\alpha_i)$ for the two-dimensional representation of the permutation group for the coordinates of four particles:

$$\chi_0 = D_0'\xi'' - D_0''\xi'; \quad \chi_2 = D_2'\xi'' - D_2''\xi',$$

in which

$$D_0' = \frac{\sqrt{5}}{4\pi}\{\sqrt{2}\sin\alpha_2\cos\alpha_2\sin\alpha_3 + \sin\alpha_2\cos\alpha_2\cos\alpha_3\};$$

$$D_0'' = \frac{\sqrt{5}}{4\pi}\left\{-\frac{1}{\sqrt{2}}\sin^2\alpha_2\sin 2\alpha_3 + \frac{1}{4}\sin^2\alpha_2\cos 2\alpha_3 + \frac{1}{4}(1-3\cos^2\alpha_2)\right\};$$

$$D_2' = \frac{1}{4\pi}\sqrt{\frac{5}{3}}\{-\sqrt{2}\cos 2\alpha_1\sin\alpha_2\cos\alpha_2\sin\alpha_3 - \sqrt{2}\sin 2\alpha_1\sin\alpha_2\cos\alpha_3 - \cos 2\alpha_1\sin\alpha_2\cos\alpha_2\cos\alpha_3 + \sin 2\alpha_1\sin\alpha_2\sin\alpha_3\};$$

$$D_2'' = \frac{1}{4\pi}\sqrt{\frac{5}{3}}\left\{-\frac{1}{\sqrt{2}}\cos 2\alpha_1(1+\cos^2\alpha_2)\sin 2\alpha_3 - \frac{1}{2}\sin 2\alpha_1\cos\alpha_2\sin 2\alpha_3 - \sqrt{2}\sin 2\alpha_1\cos\alpha_2\cos 2\alpha_2 + \frac{1}{4}\cos 2\alpha_1(1+\cos^2\alpha_2)\cos 2\alpha_3 - \frac{3}{4}\cos 2\alpha_1\sin^2\alpha_2\right\}.$$

The Euler angles α_1, α_2, α_3 are simply parametrized in terms of the Cartesian components of the three Jacobi vectors:

$$\vec{q}_1 = \vec{r}_1 - \frac{1}{3}(\vec{r}_2+\vec{r}_3+\vec{r}_4); \quad \vec{q}_2 = \frac{2\sqrt{2}}{3}\left[\vec{r}_2 - \frac{1}{2}(\vec{r}_3+\vec{r}_4)\right];$$

$$\vec{q}_3 = \sqrt{\frac{2}{3}}(\vec{r}_3-\vec{r}_4)$$

in a system of coordinates ξ, η, ζ which is rigidly connected to the principal axes of the inertia ellipsoid:

$$q_{1\xi} = (-\cos\alpha_1\cos\alpha_2\sin\alpha_3 - \sin\alpha_1\cos\alpha_3)\,a;$$

$$q_{2\xi} = (\cos\alpha_1\cos\alpha_2\cos\alpha_3 - \sin\alpha_1\sin\alpha_3)\,a; \quad q_{3\xi} = \cos\alpha_1\sin\alpha_2 a;$$

$$q_{1\eta} = (-\sin\alpha_1\cos\alpha_2\sin\alpha_3 + \cos\alpha_1\cos\alpha_3)\,b;$$

$$q_{2\eta} = (\sin\alpha_1\cos\alpha_2\cos\alpha_3 + \cos\alpha_1\sin\alpha_3)\,b; \quad q_{3\eta} = \sin\alpha_1\sin\alpha_2 b;$$

$$q_{1\zeta} = \sin\alpha_2\sin\alpha_3 c; \quad q_{2\zeta} = -\sin\alpha_2\cos\alpha_3 c; \quad q_{3\zeta} = \cos\alpha_2 c.$$

The transformation from Cartesian components for the Jacobi vectors to the variables a, b, c, φ, θ, ψ and α_i is not unique. It depends on the choice of the axes ξ, η, ζ and the choice of reference direction along each of them. However, this ambiguity does not affect the functions $\Psi(a$, b, c; α_i, σ_i, $\tau_i)$ because they are not affected by the Cartesian components of the Jacobi vectors. Therefore the wave function in Eq. (56) is invariant relative to interchanges of the principal axes of the nucleus' inertia ellipsoid.

It should be mentioned that trial wave functions constructed from single-particle functions for a nonspherical field are not invariant under permutations of the coordinate axes for the nonspherical field.

For a four-particle system the main features of the variational calculation are not masked by the technical difficulties because all the calculations can be carried to completion. Therefore we shall focus our attention on this simple example, which has only methodological interest.

The wave function of Eq. (56) is constructed of single-particle functions for a spherically symmetric oscillator field. It has just one degree of freedom, radial oscillations, and this is clearly insufficient for a variational calculation of a nonspherical nucleus. However, Eq. (56) does have a simple generalization with which we can obtain a more flexible variational description of the shape of a system of nucleons in which the angular momentum is zero:

$$\Psi = \Phi_0(a, b, c)\,\chi_0(\alpha_i, \sigma_i, \tau_i) + \Phi_2(a, b, c)\,\chi_2(\alpha_i, \sigma_i, \tau_i), \quad (57)$$

Here Φ_0 and Φ_2 are arbitrary functions of a, b, and c. The only restriction placed on Φ_0 and Φ_2 by the problem's symmetry is that in the transformation to the variables ξ, η, ζ they must transform like the functions $c^2 - (a^2 + b^2)/2$ and $\sqrt{3}(a^2 - b^2)/2$.

The system of equations for Φ_0 and Φ_2 is easily found from the requirement that the energy of the four nucleons, as computed with Eq. (57), must be a minimum:

$$\left.\begin{aligned}
&(\hat{T}_{a,b,c} + V_s - E)\,\Phi_0 + \left\{\frac{3}{2}\cdot\frac{\hbar^2}{m}\left[\frac{b^2+c^2}{(b^2-a^2)^2} + \frac{c^2+a^2}{(c^2-a^2)^2}\right] + V_0\right\}\Phi_0 + \left\{\frac{\sqrt{3}}{2}\cdot\frac{\hbar^2}{m}\left[\frac{b^2+c^2}{(b^2-c^2)^2} - \frac{c^2+a^2}{(c^2-a^2)^2}\right] - V_2\right\}\Phi_2 = 0;\\
&(\hat{T}_{a,b,c} + V_s - E)\,\Phi_2 + \left\{\frac{\sqrt{3}}{2}\cdot\frac{\hbar^2}{m}\left[\frac{b^2+c^2}{(b^2-c^2)^2} - \frac{c^2+a^2}{(c^2-a^2)^2}\right] - V_2\right\}\Phi_0\\
&\quad + \left\{\frac{1}{2}\cdot\frac{\hbar^2}{m}\left[\frac{b^2+c^2}{(b^2-c^2)^2} + \frac{c^2+a^2}{(c^2-a^2)^2} + 4\frac{a^2+b^2}{(a^2-b^2)^2}\right] - V_0\right\}\Phi_2 = 0;
\end{aligned}\right\} \quad (58)$$

where

$$\hat{T}_{b,\,b,\,c}=-\frac{\hbar^2}{2m}\left\{\frac{\partial^2}{\partial a^2}+\left(\frac{1}{a^2-b^2}-\frac{1}{c^2-a^2}\right)2a\frac{\partial}{\partial a}+\frac{\partial^2}{\partial b^2}+\left(\frac{1}{b^2-c^2}-\frac{1}{a^2-b^2}\right)2b\frac{\partial}{\partial b}+\frac{\partial^2}{\partial c^2}+\left(\frac{1}{c^2-a^2}-\frac{1}{b^2-c^2}\right)2c\frac{\partial}{\partial c}\right\};$$

$$V_s=\frac{15}{8\pi}\int d\Omega\, U\left[\sqrt{\frac{3}{2}(a^2x^2+b^2y^2+c^2z^2)}\right]\times\left\{\frac{5}{8}\lambda_{13}(x^4+y^4+z^4)+\left(\frac{47}{16}\lambda_{33}-\frac{13}{16}\lambda_{13}\right)(x^2y^2+y^2z^2+z^2x^2)\right\};$$

$$V_0=\frac{15}{8\pi}\int d\Omega U\left[\sqrt{\frac{3}{2}(a^2x^2+b^2y^2+c^2z^2}\right]\times\left\{-\frac{1}{8}\lambda_{13}\left(z^4-\frac{x^4+y^4}{2}\right)+\left(-2\lambda_{33}+\frac{5}{4}\lambda_{13}\right)\left(x^2y^2-\frac{y^2z^2+z^2x^2}{2}\right)\right\};$$

$$V_2=\frac{15}{8\pi}\int d\Omega\, U\left[\sqrt{\frac{3}{2}(a^2x^2+b^2y^2+c^2z^2)}\right]\times\left\{\frac{\sqrt{3}}{16}\lambda_{13}(x^4-y^4)+\left(-2\lambda_{33}+\frac{5}{4}\lambda_{13}\right)\frac{\sqrt{3}}{2}(x^2z^2-y^2z^2)\right\};$$

$U(r_{ij})$ is the two-particle potential for the interaction between nucleons i and j; λ_{33} and λ_{13} are coefficients which characterize the strength of the potential in various spin–isospin states, and x, y, and z are the Cartesian components of the unit vector.

Thus the variational problem reduces to solving the system of equations (58) for the best trial function among those of the type in Eq. (57).

[1)]This approximation gives a more exact value for the ground-state energy than does the direct application of the variational method to a trial function for the lowest configuration of the shell model, or than obtained from the K-harmonic method[6] in the K_{min} approximation. Moreover, with this approximation we can find both the frequency of the monopole vibrations (as in the K-harmonic method) and the frequency of the fivefold degenerate quadrupole oscillations.

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