Levinson's theorem

P. Beregi, B. N. Zakhar'ev, and S. A. Niyazgulov

Joint Institute for Nuclear Research, Dubna

Fiz. ÉL Chast. Atom. Yad., 4, 512-545 (1973)

Papers in which relationships are established between the phase shifts at zero and infinite energy and the number of bound states of a system are reviewed. The cases when the two-body potential is local, nonlocal, velocity-dependent, and energy-dependent, and the scattering of charged particles are discussed. Levinson's theorem is also formulated for many-channel scattering, and for systems of three of more particles, etc.

INTRODUCTION

During the development of any branch of science, when a large body of material is accumulated, certain results are crystallized out from the mass of useful information that have the character of general laws, reflecting, in a very concentrated form, especially deep relationships between the objects studied. In the quantum theory of scattering. Levinson's theorem is a fundamental result of this kind.

In its original form,[1] this theorem establishes[1)] a relationship between the difference of the partial-wave phase shifts $\delta_l(E)$ (l is the orbital angular momentum) at zero and infinite energy E of the relative motion of two colliding particles and the number m of bound states of these particles with the given l:

$$\delta_l(0)-\delta_l(\infty)=m_l\pi. \tag{1}$$

It is also assumed that the particles do not have spin and that they interact through a local spherically symmetric potential that also satisfies the restrictions

$$\int_0^\infty r\,|V(r)|\,dr<\infty;\quad \int_0^\infty r^2|V(r)|\,dr<\infty. \tag{2}$$

The first of the conditions (2) means that the absolute value of the potential must not increase as $r\to 0$ faster than $1/r^{2-\varepsilon}$ (where $\varepsilon > 0$) and the second that $|V(r)|$ decreases as $r\to\infty$ faster than $1/r^3$.

Later, it was found that the theorem can be generalized to more complicated problems and that there exist various modifications of it in the cases when it cannot be applied directly. Up to the present time, several dozen investigations have been made in this field and Levinson's idea has been appreciably developed.

New and in some cases more effective methods of proof have been proposed. They have made it possible to extend the theorem to a larger class of interactions: Nonlocal potentials, the interaction of charged particles, other forces having a singular behavior, and to potentials that depend on the velocity and the energy. It has been possible to relate $\delta_l(0)-\delta_l(\infty)$ to the number of Regge poles of a definite type and extend the theorem to many-channel processes. Levinson's theorem has been formulated for a three-body system. A whole series of investigations has been undertaken to establish its form in field theory.

Thus, the group of phenomena for which Levinson's theorem holds in some form or another is fairly large and it would clearly be useful to summarize these results in a single review. This would enable one rapidly and fairly thoroughly to acquaint oneself with the problem and to obtain a systematic list of original papers, which are distributed in numerous journals.

The theorem was first proved in connection with the study of the uniqueness of the inverse scattering problem, i.e., the problem of determining the potential from the phase shifts when bound states are present. But it can also be useful in the phenomenological choice of interaction parameters on the basis of experimental data. For example, a potential well must be chosen sufficiently deep to guarantee the existence of the number of levels that the energy behavior of the phase shifts indicates in accordance with Levinson's theorem. Very frequently it is found that the information obtained in this manner is indispensable.

A discussion of the questions associated with the theorem leads one naturally to consider various facts that pertain to scattering theory and are closely related to the theorem although not directly covered by it.

The reader of this review will still have to examine the original sources if he wishes to acquaint himself more fully with the different proofs and other details. Our aim has been to give a general picture of the state of the art and to acquaint the reader with the most interesting aspects of the problem and, in certain cases, to put forward additional explanations and illustrations of aspects that have not hitherto been presented sufficiently clearly.

The methods of proof of relation (1) at present available in the literature have a rather formal character. It is therefore advisable to begin with a simple example, discussing in more detail the relationship between the phase shifts $\delta(0)$ and $\delta(\infty)$ and the number of bound states, the aim being to render the relation more obvious.

1. SIMPLE CASE OF POTENTIAL SCATTERING

Let us consider the Schrödinger equation for a wave function $\Psi(k, r)$ that depends for each fixed value of the energy on one variable r; this may be the radial equation for the partial S wave ($l = 0$) of a particle in a spherically symmetric potential (here and throughout the review, $\hbar = 1$, $2M = 1$, where M is the reduced mass):

$$-\frac{d^2}{dr^2}\Psi(k,r)+[V(r)-k^2]\,\Psi(k,r)=0. \tag{3}$$

We shall assume V(r) satisfies the relations (2). We specify the following boundary conditions at r = 0 and as $r\to\infty$:

$$\Psi(k,0)=0;\quad \Psi(k,r)\underset{r\to\infty}{\longrightarrow}\sin(kr+\delta(k))/k. \tag{4}$$

The expression of the asymptotic behavior of Ψ in the form (4) emphasizes the fact that δ is the phase shift of the function $\Psi(k, r)$ at large r relative to the function $\Psi_{free}(k, r) = \sin(kr)/k$ for the free motion of the particle $[V(r) \equiv 0]$, for which the Schrödinger equation has the form

$$-\frac{d^2}{dr^2}\Psi_{free}(k, r) - k^2\Psi_{free}(k, r) = 0. \quad (5)$$

We begin by considering the behavior of $\delta(k, r)$ at large energies $E = k^2$.

Intuitively, it is clear that at an energy much greater than $|V(r)|$ (if the potential is a bounded function) the influence of the field V(r) on the motion of the particle can be ignored, i.e., $\Psi(k, r) \xrightarrow[k\to\infty]{} \Psi_{free}(k, r)$ and therefore the phase shift of Ψ relative to Ψ_{free} must vanish with increasing k: $\delta(\infty) = 0$.

This can be seen quite clearly for the example of a rectangular well of radius a and depth V_0. With increasing energy, the difference in the effective kinetic energies of a particle in the region $r < a$ for free motion and motion in the potential will remain constant and equal to the depth V_0 of the well:

$$k'^2 - k^2 = V_0. \quad (6)$$

The difference in the frequencies of the oscillations of the two solutions for $r < a$ will decrease; for in accordance with (6)

$$k'^2 - k^2 \equiv (k' - k)(k' + k) = V_0, \quad (6a)$$

and therefore

$$k' - k = \frac{V_0}{k' + k} = \frac{V_0}{\sqrt{E + V_0} + \sqrt{E}} \xrightarrow[E,k\to\infty]{} 0, \quad (7)$$

and this means that with increasing energy the switching on of the potential has less and less influence on the change of the frequency of the solution in the interaction region. The solution $\Psi(k, r)$ will tend to the free solution and $\delta(k) \xrightarrow[k\to\infty]{} 0$.

We shall now show that in this case at zero energy the phase shift $\delta(0)$ is equal to $m\pi$, as we require in accordance with (1). It is easy to see that $\delta(0)$ takes only values that are multiples of π [apart from the exceptional case when the potential well V(r) contains a "level" with vanishing binding energy].

Outside the interaction region (any one!) the wave function $\Psi(k = 0, r)$ has the form of a straight line:

$$\Psi(k=0, r) = \lim_{k\to 0}\frac{1}{k}\sin(kr + \delta(k)) = r + b, \quad (8)$$

where $b = \lim_{k\to 0} [\sin\delta(k)/k]$.

Essentially, the straight line (8), like the straight line

$$\Psi_{free}(0, r) = \lim_{k\to 0}\frac{1}{k}\sin kr = r, \quad (9)$$

is a function of the degenerate oscillations with infinitely large period (with oscillation frequency equal to zero). And since both these functions, $\Psi(0, r)$ and $\Psi_{free}(0, r)$, have intersections with the r axis ("nodes") at a finite distance $\Delta r = b$ from each other, this distance being negligibly small compared with the infinite period of their "oscillations," the phase shift $\delta(0)$ must be assumed a multiple of π:

$$\delta(0) = n\pi. \quad (10)$$

It only remains to show that n is equal to the number of levels m in the well V(r).

For simplicity, let us consider once more V(r) in the form of a rectangular well with depth V_0 and radius a [one can go over to a more general case of V(r) by representing V(r) approximately as a step function and then going to the limit of an infinite number of steps].

The solution $\Psi(k, r)$ has in the two regions $r \geq a$ and $r \leq a$ a sinusoidal form with different frequencies:

$$\Psi(k, r \leqslant a) = A\sin\{\sqrt{V_0 + k^2}\, r \equiv \varphi_1(k, r)\}; \quad (11a)$$

$$\Psi(k, r \geqslant a) = \frac{1}{k}\sin\{kr + \delta(k) \equiv \varphi_2(k, r)\}. \quad (11b)$$

Let us find the relation between the phases $\varphi_2(k, r)$ and $\varphi_1(k, r)$, in which we are interested since $\varphi_2(0, a)$ is equal to the desired phase shift $\delta(0)$, $\varphi_1(0, \delta)$ is related to the number of levels in the potential V(r).

Since Ψ is continuous, the functions (11a) and (11b) must be equal at $r = a$, i.e., they can vanish at $r = a$ only together and their phases, $\varphi_1(k, a)$ and $\varphi_2(k, a)$, must both become multiples of π. Moreover, since $d\Psi/dr$ is also continuous, the first derivatives of (11a) and (11b) must also vanish together at $r = a$ when $\varphi_1(k, a)$ and $\varphi_2(k, a)$ become multiples of an odd number of $\pi/2$. Obviously, $\varphi_1(k, a) = \varphi_2(k, a)$ when $V_0 = 0$ and when V_0 increases they both pass through the values $n\pi/2$ (n = 1, 2, ...) and thus

$$\text{if } \varphi_1(k, a) = n\pi, \text{ then also } \varphi_2(k, a) = n\pi; \quad (12a)$$

$$\text{if } \varphi_1(k, a) = (n + 1/2)\pi, \text{ then also } \varphi_2(k, a) = (n + 1/2)\pi; \quad (12b)$$

$$\text{if } n\pi < \varphi_1 < (n + 1/2)\pi, \text{ then also } n\pi < \varphi_2 < (n + 1/2)\pi. \quad (12c)$$

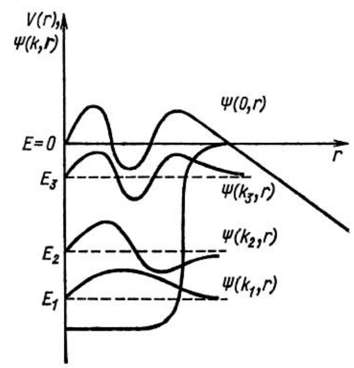

Fig. 1. Wave functions of bound states, $\psi(k_i, r)$, with energies E_i ($E_1 < E_2 < E_3 < 0$) and states of the continuous spectrum $\Psi(E, r)$ in the limiting case $E \to 0$ for a particle in a potential well.

It is well known that $\Psi(k_n, r)$, the function of the n-th bound state, has n − 1 nodes, not counting the point r = 0 (Fig. 1). At the point of the last node, the phase of the function $\Psi(k_n, r)$ is $(n-1)\pi$. After this point, $\Psi(k_n, r)$ also passes through one maximum (or minimum), as a result of which the phase of $\pi/2$ in the well increases further by at least $\pi/2$. Clearly, the phase of the solution at E = 0 must be ahead of the phase of the uppermost, m-th, level since the oscillation frequency of the solution with E = 0 must be greater than for a bound state with E < 0. It is also clear that $\varphi_1(0, a)$ cannot exceed the value $(m + 1/2)\pi$; for otherwise we should have more than m levels in the well:

$$(m - 1/2)\pi \leqslant \varphi_1(0, a) \leqslant (m + 1/2)\pi. \qquad (13)$$

And it follows from (12) and (13) that

$$(m - 1/2)\pi \leqslant \varphi_2(0, a) = \delta(0) \leqslant (m + 1/2)\pi, \qquad (14)$$

and this, in conjunction with (10) and $\delta(\infty) = 0$, gives (1). This is what we wanted to prove. In the special case when there is an (m + 1)-th level with vanishing binding energy, the derivative of $\Psi(k, r)$ vanishes at $r = a$ and $\varphi_1(0, a) = (m + 1/2)\pi$ since the solution $\Psi(0, r)$ must be fitted at the edge of the well to a constant, into which its exponentially decreasing tail goes over in the limit $E_{m+1} \to 0$: $\Psi(k, r \geqslant a) \sim \exp(-\varkappa_{m+1} r) \xrightarrow[\varkappa_{m+1}\to 0]{} \text{const}$ (Fig. 2).

In accordance with (12b) and the equation $\varphi_2(0, a) = \delta(0)$, we have

$$\delta(0) = (m + 1/2)\pi. \qquad (15)$$

This corresponds to the fact that the function (8), which "oscillates" (with zero frequency), has reached its extremum and that its phase shift relative to (9) is, instead of (10), a multiple of an odd number of $\pi/2$.

Thus, we can say that a level at E = 0 gives only half the contribution of a normal level to Levinson's relation; the point E = 0 corresponds half to a bound state and half to a scattering state.[2)]

It is helpful to augment the above with two interesting illustrations.

1. So far, we have been concerned with the value of the phase shift at fixed values of the energy. We have shown that as the potential becomes deeper, $\delta(0)$ changes abruptly by π when the new level arises. Let us consider how this is manifested in the energy dependence of $\delta(k)$. In Figs. 3a and 3b we show a number of curves[3,4] of $\delta(k)$ corresponding to potentials V(r) of different depths. The curves are numbered in the order corresponding to stronger attraction of V(r). It should be noted that for $l \neq 0$ the appearance of a new level is preceded by the formation of a resonance (a quasistationary state for E > 0). This resonance gradually approaches E = 0. Figure 3b shows the discontinuity of $\delta(0)$ when the quasistationary level is transformed into a stationary level. A similar picture is obtained when the well is made wider instead of deeper.

2. Let us now consider the behavior of the phase function[5,6] $\delta(k, r)$ for some potential well V(r). Let us first establish the meaning of the function $\delta(k, r)$ (see the equation for it in Sec. 5). For each fixed value of $r = r_0$ it corresponds to the ordinary phase shift at energy k^2 in the "amputated" potential $V_{r_0}(r)$, which is equal to the original V(r) for $r < r_0$ but vanishes for $r \geq r_0$:

$$V_{r_0}(r) = \begin{cases} V(r) \text{ for } & r < r_0, \\ 0 \text{ for } & r \geqslant r_0. \end{cases}$$

Thus, the entire function $\delta(k, r)$ for given k describes the dependence of the phase shift of scattering on the potential amputated at $r = r_0$, i.e., with "tail" cutoff, on the position of the point of amputation.

The phase function varies in a very characteristic manner (Fig. 4), which demonstrates the change of the phase shifts when there is a continuous change in the scattering potential, that is, a gradual growth of its tail.[5] At high energies, the entire function $\delta(k, r)$ is nearly zero. With decreasing $k = \sqrt{E}$ [the curves of $\delta(k, r)$ in Fig. 4 are numbered in this sequence], the dependence of the phase function on r becomes more and more step-like. At k = 0, it is a strictly step-like function. Each step is of height π: The step occurs at the values of r at which a new level occurs in the amputated potential. For any $r = r_0$, we have the equation $\delta_{r_0}(0) - \delta_{r_0}(\infty) = n_{r_0}\pi$, where n_{r_0} is the number of bound states in the potential amputated at r = 0.

Let us prove this. We have pointed out above that several methods of proving Levinson's theorem have already been proposed. And the advantage of this is not only that they are convenient in different situations but that each of these variants reveals certain new aspects of the laws that determine the behavior of quantum systems.

In 1964, Wellner[7] proposed a comparatively simple

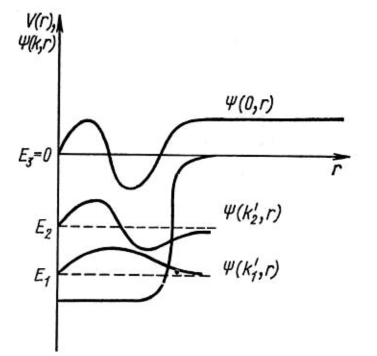

Fig. 2. Well with two bound states and one "level" with energy E = 0.

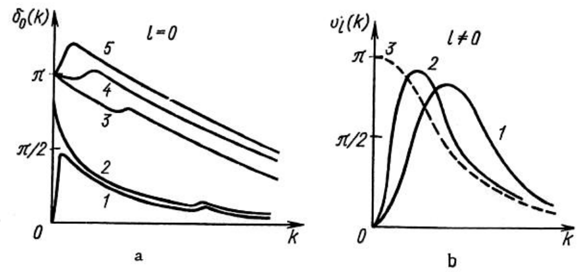

Fig. 3. Energy dependence of the phase shift for potentials of different dephs.

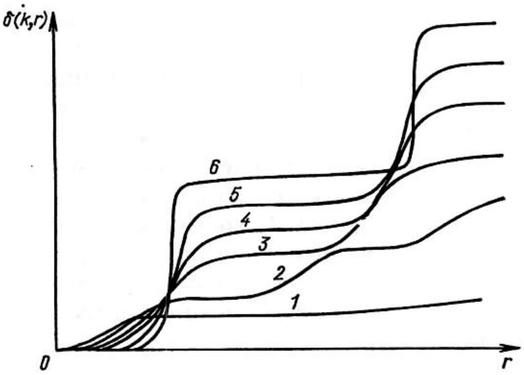

Fig. 4. Phase functions δ(k, r) for potential scattering at different energies ($E_1 > E_2 > \ldots > E_6 \approx 0$).

proof of the theorem. The fact that he chose for publication the Journal of the American Association of Physics Teachers shows that he wished to make his proof accessible to the largest possible circle of readers interested in physics.

We shall here reproduce the main parts of Wellner's paper. We differentiate the Schrödinger equation

$$-\frac{d^2}{dr^2}\Psi + V(r)\Psi - k^2\Psi = 0$$

with respect to k (we denote the operation of this differentiation by a dot above the function):

$$-\frac{d^2}{dr^2}\dot{\Psi} + V(r)\dot{\Psi} - k^2\dot{\Psi} - 2k\Psi = 0. \quad (16)$$

We now multiply (3) by $\dot{\Psi}$ and (16) by Ψ and subtract one from the other:

$$-\frac{d}{dr}\left(\dot{\Psi}\frac{d}{dr}\Psi - \Psi\frac{d}{dr}\dot{\Psi}\right) + 2k\Psi^2 = 0. \quad (17)$$

We integrate (17) with respect to r from 0 to r and, using the boundary condition $\Psi(0) = 0$, we obtain

$$\dot{\Psi}\frac{d}{dr}\Psi - \Psi\frac{d}{dr}\dot{\Psi} = 2k\int_0^r \Psi^2(r')\,dr'. \quad (18)$$

In the region of large values of r, where we can use for the wave function Ψ the asymptotic form $\Psi = \sin(kr + \delta)$, we obtain [we assume that the potential V(r) does not contain a zero-energy level]

$$\dot{\delta} = -r + \sin 2(kr+\delta)/2k + 2\int_0^r \Psi^2(r')\,dr'. \quad (19)$$

The relation (19) can be transformed [see Appendix 1] to

$$\dot{\delta} = \int_0^\infty (2\Psi^2 - 1)\,dr. \quad (20)$$

In conjunction with the completeness condition for the solutions of the Schrödinger equation, Eq. (20) enables us to obtain (1).

Corresponding to the fact that the states $\Psi(k_n, r)$ and $\Psi(k, r)$ of the discrete and the continuous spectrum of Eq. (3) form a complete system, we have the equation (see ref. 4, p. 345):

$$\sum_{n=1}^m \Psi(k_n, r)\Psi(k_n, r') + \frac{2}{\pi}\int_0^\infty \Psi(k, r)\Psi(k, r')\,dk = \delta(r - r'). \quad (21)$$

The solutions of the equations of free motion also form a complete system:

$$\frac{2}{\pi}\int_0^\infty \sin kr \sin kr'\,dk = \delta(r - r'). \quad (22)$$

We subtract (22) from (21) and set $r = r'$, after which we integrate the equation with respect to r [using the fact that the functions $\Psi(k_n, r)$ of the bound states are normalized]:

$$m + \frac{2}{\pi}\int_0^\infty dk\int_0^\infty dr\,[\Psi^2(k, r) - \sin^2 kr] = 0. \quad (23)$$

We rewrite (23) in the form

$$m = \frac{2}{\pi}\int_0^\infty dk\left\{\int_0^\infty [\Psi^2(k, r) - 1/2]\,dr - \int_0^\infty [\sin^2 kr - 1/2]\,dr\right\} = 0. \quad (23')$$

The second integral in the braces in Eq. (23') vanishes, and the first can be replaced in accordance with (20) by $\dot{\delta}/2$. As a result[3)] we obtain (1).

2. CHARGED PARTICLES AND OTHER EXAMPLES OF SINGULAR INTERACTIONS

An important feature of the Coulomb interaction is, for example, the fact that the phase shift for the scattering of charged particles is unbounded.[4)] Moreover, in the case of Coulomb attraction there are infinitely many bound states. Therefore, Levinson's theorem in its usual form becomes meaningless for such long-range forces. Nevertheless, a certain analog of the theorem can be formulated,[8,9] but not for the phase shifts of purely Coulomb scattering compared with the case of free motion.

Suppose that, in addition to the Coulomb potential, there is a further short-range potential V(r). The Schrödinger equation for the l-th partial wave then takes the form ($\hbar = 1$; $2M = 1$)

$$[d^2/dr^2 + k^2 - l(l+1)/r^2 - V(r) - \beta/r]\,\Psi_l(k, r) = 0, \quad (24)$$

where $\beta = Z_1Z_2e^2$; Z_1 and Z_2 are the charges of the colliding particles in units of the elementary charge e.

It can be shown that a relation of the type (1) holds for the phase shifts $\eta_l(k)$ that characterize the deviation of the scattering by the combined field $V + \beta/r$ from purely Coulomb scattering.

The partial wave $\Psi_l(k, r)$ has the asymptotic form

$$\Psi_l(k, r) \underset{r\to\infty}{\longrightarrow} \exp[-i(kr - l\pi/2 - \beta(\ln 2kr)/2k + \sigma_l)] - \exp[i(kr - l\pi/2 - \beta(\ln 2kr)/2k + \sigma_l)]\,S_l(k). \quad (25)$$

Here $\sigma_l = \arg\Gamma(i\beta/2k + l + 1)$ and

$$S_l(k) = \exp[2i\eta_l(k)] = f_l(k)/f_l(-k). \quad (26)$$

For a purely Coulomb field, $S_l = 1$, and by definition $\eta_l = 0$.

We shall consider separately the cases of repulsive ($\beta > 0$) and attractive ($\beta < 0$) Coulomb fields.

1. Suppose $\beta > 0$. Then bound states can exist only because of the additional short-range potential V(r). Let us denote by n_l the number of levels in the potential $V(r) + \beta/r$ with given l.

For $\beta > 0$, the modified Levinson theorem has the form

$$\left.\begin{aligned}&\eta_l(0) - \eta_l(\infty) = n_l\pi, \quad \text{if} \quad l \geqslant 1 \ \text{or} \ f_l(0) \neq 0 \quad \text{for } l = 0,\\ &\eta_0(0) - \eta_0(\infty) = (n_0 + 1/2)\pi, \quad \text{if} \quad f_0(0) = 0 \ (l = 0).\end{aligned}\right\} \quad (27)$$

The last relation corresponds to a level at E = 0. Assuming that V(r) satisfies the conditions [cf. (2)]

$$\lim_{r\to 0} r^2 V(r) = 0 \ \text{and} \ \lim_{r\to\infty} r^3 V(r) = 0, \quad (28)$$

one can show[8] that $\eta_l(\infty) = 0$.

The procedure used here to go over from the ordinary phase shift δ to η is helpful when one wishes to generalize Levinson's theorem to other potentials (besides the Coulomb) that do not satisfy the conditions (2).

2. Suppose $\beta < 0$. In the potential $V(r) + \beta/r$ there are infinitely many bound states. They are all shifted in energy relative to the levels of the purely Coulomb case, for which we have the simple equation

$$E_{nl} = -\beta^2/4n^2, \quad (29)$$

where n is the principal quantum number.

For the position of mixed levels (for $V + \beta/r$) one can also write down a relation of the type (29), except that in this case one must replace the integers n on the right-hand side of (29) by certain quantities n'_l specially chosen in such a way as to describe the discrete spectrum of the combined field:

$$E'_{nl} = -\beta^2/4n_l'^2. \quad (30)$$

As a characteristic of the departure of the Coulomb levels from the levels in the potential $V + \beta/r$ one introduces the concept of the quantum defect of the n_l-th bound state:

$$\mu_l(n) = n - n'_l. \quad (31)$$

Obviously, $\mu_l(n) = 0$ for the purely Coulomb interaction.

We denote by μ_l the limiting value of $\mu_l(n)$ as $n \to \infty$:

$$\mu_l = \lim_{n\to\infty} \mu_l(n). \quad (32)$$

For $\beta < 0$, the analog of Levinson's theorem can be formulated as follows:

$$\eta_l(0) - \eta_l(\infty) = \mu_l\pi, \quad (33)$$

where $\eta_l(\infty) = 0$ if the conditions (28) hold.

The limiting quantum defect μ_l can be determined approximately from spectroscopic measurements for large n or by a numerical solution of the Schrödinger equation for k = 0 (see ref. 8).

Let us consider potentials that have singular behavior as $r \to 0$.

Apart from the Coulomb interaction, the singular potentials $V(r) = \beta/r^2$ have been studied (see ref. 10). Clearly, they can be regarded simply as an "addition" to the centrifugal barrier $l(l+1)/r^2$:

$$l(l+1)/r^2 + \beta/r^2 = \nu(\nu+1)/r^2, \quad (34)$$

where

$$\nu = [-1 + (1 + 4l + 4l^2 + 4\beta)^{1/2}]/2. \quad (35)$$

As in the case of the ordinary centrifugal barrier, one obtains an energy-independent phase shift:

$$\delta_l(k) = \pi(l - \nu)/2. \quad (36)$$

Potentials of the type $\beta\varphi(r)/r^2$, where $\varphi(0) = 1$, and $\lim_{r\to\infty}\varphi(r) = 0$ give the same phase shift at infinite energy: $\delta_l(\infty) = \pi(l - \nu)/2$.

Levinson's theorem can be formulated for $V = \beta/r^2$ and $V = \beta\varphi(r)/r^2$ in the usual manner. Note, however, that when $\beta < -l(l+1)$ the wave function becomes unbounded as $r \to 0$, and for $\beta < -1/4 - l(l+1)$ the particle "collapses" onto the center and one cannot introduce the concept of a phase shift.

Potentials that increase faster than $1/r^2$ as $r \to 0$ give an unbounded phase shift as $k \to \infty$. Thus, for potentials that behave for $r \to 0$ as $1/r^{2+\varepsilon}$, where $0 < \varepsilon < \infty$, we have

$$\delta(k \to \infty) = -k^{\varepsilon/(2+\varepsilon)}. \quad (37)$$

This result is obtained in ref. 10, but it is not rigorously proved, and potentials with a hard core [$V(r) = \infty$ for $r < a$] give

$$\delta(k \to \infty) = -ka. \quad (38)$$

Although a relation of the type (1) becomes meaningless in the last two cases, there still remains a relationship between the phase shift as $k \to 0$ and the number of levels m (see ref. 5):

$$\delta(0) = (m + \tau/2)\pi, \quad (39)$$

where τ is the multiplicity of the level for k = 0.

3. NONLOCAL POTENTIALS

Potentials $V(\mathbf{r})$ that depend on the vector $\mathbf{r}$, the relative distance between the particles, are said to be local. They are a special case of an interaction of a more general type that can be specified in the Schrödinger equation by means of the kernel $V(\mathbf{r}, \mathbf{r}')$ of an integral operator:

$$[-\Delta_{\mathbf{r}} - k^2]\Psi(\mathbf{r}) + \int V(\mathbf{r}, \mathbf{r}')\Psi(\mathbf{r}')\,d\mathbf{r}' = 0. \quad (40)$$

The nonlocal potential $V(\mathbf{r}, \mathbf{r})$ makes Eq. (40) an integrodifferential equation. It goes over into an ordinary differential Schrödinger equation with local $V(\mathbf{r})$ if $V(\mathbf{r},$

r') has a special form:

$$V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}'). \tag{41}$$

There are certain indications that fundamental nucleon–nucleon forces have a nonlocal nature. One can reduce the interaction of complicated systems to a nonlocal potential (for example, in the unified theory of nuclear reactions[11]). Recently, nonlocal potentials have been widely used in nuclear physics to solve different three-particle problems.

Forces of this kind have long attracted interest, but they have still not yet been adequately studied. And one could imagine that Levinson's theorem would be a helpful auxiliary means to a more complete mastery of these interactions. But if one wishes to use relation (1) for nonlocal potentials, one must take care.

The first paper in which it is shown that Levinson's theorem in the form (1) does not necessarily apply in the case of nonlocal potentials was published[12] in 1957. In this paper, the so-called separable potentials are considered. Let us dwell briefly on their definition.

For this, it is convenient to rewrite Eq. (40) in the momentum representation. Then the kinetic energy operator, $-\Delta_r$, becomes simply a number p^2 and (40) takes an integral form (rather than integrodifferential):

$$(p^2-k^2)\,\Psi(\mathbf{k})+\int\langle\mathbf{k}\,|\,V\,|\,\mathbf{k}'\rangle\,\Psi(\mathbf{k}')\,d\mathbf{k}'=0. \tag{42}$$

Assuming that the two-particle interaction is central, we can write the nonlocal potential in the momentum representation in the form

$$\langle\mathbf{k}\,|\,V\,|\,\mathbf{k}'\rangle=4\pi\sum_{lm}V_l(k,k')\,Y^*_{lm}(\Omega_{\mathbf{k}})\,Y_{lm}(\Omega_{\mathbf{k}'}), \tag{43}$$

where each term describes interaction between particles in a state with definite orbital angular momentum l, and k and k' are the momenta of the relative motion of the particles before and after the interaction.

The use of nonlocal separable potentials makes it possible to obtain a solution of Schrödinger's equation and the Lippmann–Schwinger equation in analytic form.[13] For such potentials, the partial-wave terms $V_l(k, k')$ can be written in the form (for the remainder of this section we shall omit the subscript l)

$$V(k,k')=\sum_{i=1}^{n}C_i g_i(k)\,g_i(k') \tag{44}$$

and then the T matrix, the solution of the Lippmann–Schwinger equation, can be represented by means of the functions g_i in the form

$$T(k,k',z)=\sum_{i,\,j=1}^{n}C_i g_i(k)\,g_j(k')\,M^{-1}_{ij}(z), \tag{45}$$

where

$$M_{ij}(z)=\delta_{ij}-C_j\frac{1}{2\pi^2}\int_0^\infty\frac{g_i(p)\,g_j(p)\,p^2\,dp}{z-p^2}, \tag{46}$$

and $z = E + i\varepsilon$.

The wide use of potentials of the type (44) is explained by the fact that for them one obtains an appreciable simplification not only of the solution of two-particle problems but also of Faddeev's equations, which describe the motion of a system of three bodies. Using (44), one can reduce the multidimensional Faddeev integral equations to an equivalent system of one-dimensional integral equations that can be solved relatively easily on a computer.

However, simplicity alone would not be a sufficient ground for using the potentials (44). It can be shown that, using them, one can reproduce the majority of the characteristics of three-particle systems. This question has been considered in great detail in ref. 13, which contains many references to original papers.

In the study of the inverse problem for scattering in a nonlocal separable potential, Gourdin and Martin[12] pointed out that, notwithstanding (1), there are cases when

$$\delta(0)-\delta(\infty)>m\pi,$$

where m is the number of bound states of the two-particle system. They took the potential

$$V(k,k')=\varepsilon g(k)\,g(k');\qquad \varepsilon=\pm1, \tag{47}$$

for which the phase shift is given by

$$k\,\mathrm{tg}\,\delta(k)=-\frac{\pi}{2}\cdot\frac{f(k)}{R(k)}, \tag{48}$$

where

$$f(k)=(\varepsilon/2\pi^2)\,k^2g^2(k); \tag{49}$$

$$R(k)=1-P\int_0^\infty f(p)\,dp/(k^2-p^2). \tag{50}$$

For Levinson's theorem in its ordinary form to be violated it is necessary and sufficient for $f(k)$ and $R(k)$ to vanish simultaneously.

One of the features of nonlocal potentials is the fact that for them one can have "bound" states with positive energy (in the continuum). The above condition for Levinson's theorem to be violated corresponds, as it happens, to a case in which there is a bound state at positive energy. For local potentials (in the one-channel case) this is impossible.

Martin[14] showed that for a real nonlocal interaction, $V(r, r') = V(r', r)$, with the property $V(r, r') = 0$ for $r > r_0$ or $r' > r_0$, one can formulate an equivalent of Levinson's theorem:

$$\delta(0)-\delta(\infty)=\pi(m+m'), \tag{51}$$

where m is the number of bound states; m' is the number of "spurious" bound states with $E > 0$ that have wave functions and decrease for large r. Such states arise because under certain conditions the inhomogeneous integral equation obtained from the Schrödinger equation and the corresponding homogeneous equation can simultaneously have a solution. In this case, the solution of the Schrödinger equation becomes nonunique and, in addition to the ordinary solution, which has the asymptotic behavior

$$\Psi(k, r) \underset{r\to\infty}{\sim} \sin(kr + \delta(k)),$$

there also exists a solution for which

$$\Psi(k, r) \xrightarrow[r\to\infty]{} 0.$$

In refs. 14 and 15, some examples are given in which one requires a modification of the original form of Levinson's theorem.

In ref. 16 a more general class of interactions is considered than in ref. 14. True, here too the potentials are only those with $l = 0$, but it is only assumed that V(r, r') is real and symmetric:

$$V(r, r') = V^*(r, r') = V(r', r) \quad (52)$$

and that there exists a real number $\alpha > 0$ for which

$$\int \exp(\alpha r)\, dr \int r' \exp(\alpha r') \,|V(r, r')|\, dr' < \infty. \quad (53)$$

In this case we have

$$\delta(0) - \delta(\infty) = \pi(m + m' + q/2), \quad (54)$$

where m is the number of bound states; m' is the number of spurious bound states; q is the number of bound states with zero energy.

A result analogous to (54) is obtained in refs. 17 and 18, in which a study is made of the inverse scattering problem in the case of separable potentials, and it is assumed that the form factors g(k) in V(k, k') satisfy the condition[17]

$$k^{\nu}\,|g(k)| \xrightarrow[k\to\infty]{} 0, \quad (55)$$

where $\nu > 3/2$ and g(k) are continuous and differentiable everywhere.

Bound states with positive energy are unstable under small changes in the nonlocal potential, and they disappear for small variations of the interaction parameters. If such states are present, one can have two different formulations of Levinson's theorem.[19] If one defines the phase shift as a continuous function of the energy, one must include among the bound states the spurious ones, as is done in (51) and (54) (Fig. 5). Curve 1 corresponds to a level with $E > 0$; curve 2 corresponds to a slightly modified potential in which this level disappears;[20] curve 3 corresponds to a choice of the phase shift in which it changes discontinuously by π at the energy of the bound state in the continuum. However, one can also use the ordinary form of Levinson's theorem (without m' on the right-hand side), though in this case the phase shift is defined in such a way that it has discontinuities of magnitude π at each bound state in the continuum (see curve 3 in Fig. 5). At one time, this ambiguity in the definition of the phase shifts led to the erronous opinion that the theorem is violated.[20]

4. MOMENTUM-DEPENDENT POTENTIALS

For the phenomenological description of the low-energy nucleon–nucleon interaction one introduces potentials of the type

$$V(r, p) = V_0(r) + [p^2 V_1(r) + V_1(r)\, p^2]/2; \quad r = |\mathbf{r}|; \quad p = |\mathbf{p}|, \quad (56)$$

where p is the differential momentum operator. Such potentials are said to depend on the momentum (velocity). The Schrödinger equation for them has the form

$$-\Delta\Psi(\mathbf{r}) + V_0(r)\Psi(\mathbf{r}) - \tfrac{1}{2}[\Delta V_1(r) + V_1(r)\Delta]\,\Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \quad (57)$$

For $V_1(r) > 0$, the momentum-dependent part of the potential can be replaced by the necessary repulsion of nucleons at short distances (see the references in ref. 21). The use of such V(r, p) instead of the ordinary potentials V(r), which have infinitely repulsive cores, enables one to avoid the difficulties that arise in the solution of many-particle problems because of the unbounded growth of V(r).

In ref. 21, Levinson's theorem is established for velocity-dependent potentials (56). Omitting the derivation, we here give only the final result. In ref. 21, one considers superpositions of Yukawa potentials:

$$V_i(r) = \frac{1}{r}\int_m^{\infty} \sigma_i(\mu)\exp(-\mu r)\, d\mu; \quad i = 0, 1, \quad (58)$$

where

$$\int_m^{\infty} \sigma_0(\mu)\, d\mu < \infty; \quad \int_m^{\infty} \mu^j \sigma_1(\mu)\, d\mu < \infty; \quad j = 0, 1, 2.$$

Levinson's theorem is expressed in the form

$$\delta_l(0) = \pi(n_l + \tau/2), \quad (59)$$

where τ, as usual, is nonvanishing only for $l = 0$ and is equal to the number of "bound" states[5)] at E = 0.

5. ENERGY-DEPENDENT POTENTIALS

Investigation of energy-dependent potentials provides us with a good opportunity for giving another method of proving Levinson's theorem.

Potentials V(E, r) have been studied for which the phase shift at infinite energy, $\delta(\infty)$, equals zero.[22] For this, it is sufficient to make V(E, r) satisfy the condition

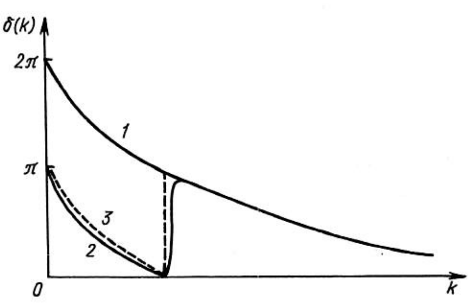

Fig. 5. Energy dependence of the phase shift for a nonlocal potential: Curves 1 and 3 correspond to different ways of defining the phase shift when there is a "bound state" in the continuum; curve 2 shows the phase shift for the case when this "bound state" is transformed into a narrow resonance. As one can see, curve 3 is a limiting case of curve 2 for a vanishing width of the resonance.

$$|V(E, r)| < M(E) r^{a-2}, \quad a > 0, \tag{60}$$

where $\lim_{E\to\infty} [M(E)E^{-b}] = 0$ for $a \neq 1$; $b = 1/2$ for $a > 1$; $b = a/2$ for $a = 1$; $\lim_{E\to\infty} [M(E)E^{1/2} \ln E] = 0$ for $a = 1$.

To determine the phase shift $\delta(k = 0)$ one can use the method of phase functions (or variable phase method).[5,6] [The concept of phase function $\delta(k, r)$ has already been used in Sec. 1.]

The equation for $\delta(k, r)$ in the case $l = 0$ has the form

$$\frac{d}{dr}\delta(k, r) = -\frac{V(E, r)}{k}\sin^2[kr + \delta(k, r)] \tag{61}$$

with the boundary condition

$$\delta(k, 0) = 0. \tag{62}$$

At large r, where V(r) vanishes, $\delta(k, r)$ goes over into the phase shift $\delta(k)$. Equation (61) and the condition (62) uniquely determine a bounded and a continuous function $\delta(k, r)$.

Because of the fact that k occurs in the denominator on the right-hand side of Eq. (61), this equation is not suitable for determining $\delta(k, r)$ in the limit $k \to 0$. But one can go over to a different function, $f(k, r)$, which is related to $\delta(k, r)$ as follows (R is any positive value of r):

$$\operatorname{tg}\delta(k, r) = kR \operatorname{tg} f(k, r); \quad f(k, 0) = 0. \tag{63}$$

For $f(k, r)$ we have the equation

$$\frac{d}{dr} f(k, r) = -\frac{V(E, r)}{R}\left[\frac{\sin kr}{k}\cos f(k, r) + R\cos kr \sin f(k, r)\right]^2. \tag{64}$$

Taking into account the continuity of the functions $f(k, r)$ and $\delta(k, r)$ for $k \neq 0$ and their being equal at $r = 0$, we can assert that

$$(2n - 1)\pi/2 < \delta(k, r) < (2n + 1)\pi/2, \tag{65}$$

if

$$(2n - 1)\pi/2 < f(k, r) < (2n + 1)\pi/2, \tag{66}$$

where n is an integer.

In accordance with (63), $\tan\delta(0, r) = 0$ if f satisfies (66) and, therefore,

$$\delta(0, r) = n\pi, \quad \text{if} \quad (2n - 1)\pi/2 < f(0, r) < (2n + 1)\pi/2. \tag{67}$$

At large r, the phase function $\delta(0, r)$ goes over into $\delta(0)$ and (67) gives

$$\delta(0) = n\pi, \quad \text{if} \quad (2n - 1)\pi/2 < f(0, r \to \infty) < (2n + 1)\pi/2. \tag{68}$$

To prove Levinson's theorem, it only remains to show that n in (68) is equal to the number of bound states, m, in the potential V(E, r). This can be done if one requires that the rate of change of V(E, r) with the energy for negative values of E be bounded in such a way that the number m in V(E, r) does not differ from a number of bound states for the potential V(r) = V(0, r), which does not depend on the energy:

$$\partial V(E, r)/\partial E \leqslant 1 \quad \text{for} \quad E \leqslant 0. \tag{69}$$

The number of zeros (nodes) of the radial part of the wave function, not counting the point r = 0, at E = 0 is equal to the number m of bound states; this is the largest integer for which $(2m - 1)\pi/2 < f(0, r \to \infty)$ (see ref. 22), i.e.,

$$(2m - 1)\pi/2 < f(0, r \to \infty) < (2m + 1)\pi/2. \tag{70}$$

As a result, we obtain the ordinary Levinson relation

$$\delta(0) - \{\delta(\infty) = 0\} = m\pi. \tag{71}$$

6. RELATIONSHIP BETWEEN REGGE POLES AND PHASE SHIFTS

We consider the radial Schrödinger equation for the l-th partial wave:

$$-\frac{d^2}{dr^2}\Psi_l(r) + \frac{l(l+1)}{r^2}\Psi_l(r) + V(r)\Psi_l(r) = E\Psi_l(r). \tag{72}$$

We denote by $N_l(E = 0)$ the number of Regge poles[6] situated at E = 0 on the real axis Re l to the right of the given fixed value of the orbital angular momentum l. It has been shown[23] that $N_l(E = 0)$ is equal to the number of bound states m_l and that Levinson's theorem can be rewritten in the form

$$\delta_l(0) - \delta_l(\infty) = \pi N_l(E = 0). \tag{73}$$

Moreover, the relation (73) is also true for all real $l > -1/2$, and not only for integral l. Thus, (73) is a further generalization of Levinson's theorem.

Let us explain the fact that the number of levels m_l in a state with definite orbital angular momentum l is equal to $N_l(E = 0)$. To do this, let us follow the behavior of any eigenvalue E_{nl} when l in Eq. (72) increases continuously and monotonically. It is clear that when the centrifugal barrier $l(l + 1)/r^2$ increases, the depth of the effective potential well $l(l + 1)/r^2 + V(r)$ becomes less and less and, at the same time, the level E_{nl} will rise until at some $l = l_n^{(0)}$ its binding energy becomes equal to zero. To the bound state E_{nl} there corresponds a Regge pole on Re l that moves to the right along Re l as E_{nl} increases, i.e., the part of the corresponding Regge trajectory[7] for $E \leq 0$ is situated on the real l axis. When the energy is increased further, the trajectory leaves the axis Re l (Fig. 6; the arrows on the trajectories indi-

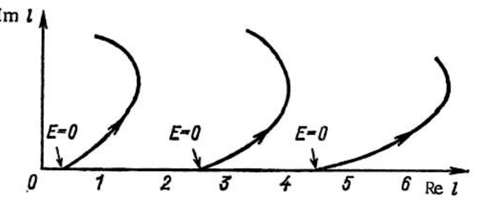

Fig. 6. Regge trajectories for potential scattering.

cate the direction of increasing energy). These arguments make it obvious that for each of the m_l levels there is a corresponding Regge trajectory which for E = 0 gives one of the N_l (E = 0) poles. And, conversely, to each of the N_l (E = 0) poles there corresponds a Regge trajectory that, with decreasing energy, must necessarily pass through all values of l that lie to the left on Re l (Re $l > -1/2$), where poles are associated with the levels E_{nl}.

A further relation of the type of Levinson's theorem for Regge poles was obtained in ref. 24. It can be shown that the number n(k) of Regge poles lying in the right-hand part of the $\lambda = l + 1/2$ plane at a given energy $E = k^2$ is proportional to the magnitude of the phase shift $\omega(\lambda, k)$ at r = 0 of the irregular Jost solution $f(\lambda, k, r)$ of Eq. (72):

$$\lim_{r\to\infty} f(\lambda, k, r) \exp(ikr) = 1$$

for $\lambda = 0$. In ref. 24 it is shown that

$$\omega(0, k) = -\pi n(k). \tag{74}$$

An analogous theorem for the case of the Coulomb interaction is considered in ref. 25.

7. MANY-CHANNEL PROBLEMS

So far, we have restricted ourselves to single-channel problems. But in the great majority of cases one must solve many-channel systems of equations in order to describe quantum effects. In the form in which it was formulated in the foregoing sections, Levinson's theorem cannot be applied to systems with several channels, if only because instead of the single phase shift $\delta(k)$ the result of the collision is determined by the entire matrix S(k). And, moreover, when the energy changes, the dimensionality of S(k) can also change. When the energy E passes through a threshold E_α at which n new channels are opened, the rank of S changes by n. In connection with what we have said, Levinson's theorem must be significantly modified for such problems.[26–29]

We write down first a typical system of many-channel equations. Then, on the basis of this system, we introduce the concept of eigenphase shifts, for which Levinson's theorem is formulated in this case.

In a number of problems, Schrödinger's equation

$$(H - E)\Psi = 0 \tag{75}$$

for particles A and B reduces to coupled (!) equations for the functions ψ_α of the individual partial channels (ref. 26). These functions $\psi_\alpha(r)$ are radial wave functions of the relative motion of A and B for a given choice α of the quantum numbers that determine the state (channel) of the system A + B for all its remaining degrees of freedom. One can regard the $\psi_\alpha(r)$ as the coefficients of the expansion of Ψ in the functions Φ_α that characterize A and B (for example, their spins, isospins, and state of internal motion if A and B have an internal structure) and they also include the angular dependence $Y_{lm}(\Omega_r)$ of the relative motion of A and B with definite orbital angular momentum l and projection m, and so forth:

$$\Psi = \sum_\alpha \psi_\alpha(r)\Phi_\alpha. \tag{76}$$

If Φ_α is an orthonormalized set, then, substituting (76) into (75), multiplying the resulting equation from the left by Φ_α^*, and integrating with respect to the variables τ on which the Φ_α depend, we obtain a system of many-channel equations:

$$-\frac{d^2}{dr^2}\psi_\alpha(r) + \sum_\beta V_{\alpha\beta}(r)\psi_\beta(r) - (E - \varepsilon_\alpha)\psi_\alpha(r) = 0. \tag{77}$$

Here

$$V_{\alpha\beta}(r) = \int \Phi_\alpha^* V_{AB}\Phi_\beta\, d\tau$$

is the interaction matrix, which couples the individual equations (the channels) in (77). This relationship arises because of the mixing of states of the system corresponding to different channels because of the interaction V_{AB} of the particles A and B. The constants ε_α are the energies of the internal state of particles A and B. They determine the threshold values of the energy at which the channels α are opened. If $E > \varepsilon_\alpha$ the channel is open, but if $E < \varepsilon_\alpha$ it is closed. In the case of short-range forces, the interaction matrix vanishes for large r and the system (77) splits up into individual Schrödinger equations for the free motion of particles A and B in the channels:

$$-\frac{d^2}{dr^2}\psi_\alpha(r) - (E - \varepsilon_\alpha)\psi_\alpha = 0. \tag{78}$$

The solution of these equations is well known and determines the asymptotic behavior of the system's wave function. For open channels, $\lim_{r\to\infty}\psi_\alpha$ is a combination of the ingoing and the outgoing wave:

$$\psi_\alpha(r) \underset{r\to\infty}{\longrightarrow} A_\alpha \exp(-ik_\alpha r) + B_\alpha \exp(ik_\alpha r); \quad k_\alpha = \sqrt{E - \varepsilon_\alpha}, \tag{79}$$

and for closed channels it is an exponential damped function. The amplitudes A_α of the ingoing waves in the open channels are fixed in accordance with the conditions of the given physical problem and the amplitudes of the outgoing waves are then determined by the scattering matrix $S_{\alpha\beta}$:

$$B_\alpha = S_{\alpha\beta}A_\beta. \tag{80}$$

In its turn, the S matrix is determined from the solution of the system (77).

Usually, one considers cases in which there are ingoing waves only in individual (entry) channels and $A_\alpha = 0$ otherwise. There are then scattered waves in the open channels in which there are no ingoing waves. However, one can conceive of a situation in which all channels have ingoing and outgoing waves. Among such solutions of the system (77) one can distinguish the so-called *eigenchannel states*. To them there corresponds a special (s) choice of the amplitudes A_α^S, for which the B_α^S differ from the A_α^S by only a phase factor δ^S, the *eigenphase shift*, that is common to all the channels:

$$B_\alpha^s = \exp(2i\delta^s)A_\alpha^s \tag{81}$$

or

TABLE 1

System	Spin state	Number or bound states	δ(0)	Bound state
$e^- - H$	Triplet	1	π	—
	Singlet	0	π	H^-
$e^- - He$	Doublet	1	π	—
$e^- - Ne$	Doublet	2	2π	—
$e^- - A$	Doublet	3	3π	—
$n - d$, $p - d$	Quartet	1	π	—
	Doublet	0	π	^3H and ^3He
$n - t$	Triplet	1	π	—
	Singlet	1	π	—
$p - {}^3He$	Triplet	1	π	—
	Singlet	1	π	—
$p - t$	Triplet	1	π	—
	Singlet	0	π	^4He
$n - {}^3He$	Triplet	1	π	—
	Singlet	0	π	^4He
$n - \alpha$, $p - \alpha$	Doublet	1	π	—

$$S_{\alpha\beta}A^s_\beta = \exp(2i\delta^s)A^s_\alpha. \quad (82)$$

Thus, the eigenchannel states are states for which, despite the transitions between the channels in the interaction region, the flux of scattered waves is distributed over the channels in the same proportion as the ingoing waves. There are as many linearly independent eigenchannel states as there are open channels at the given energy. The corresponding sets of amplitudes A^s_α can be obtained from (82) by treating this expression as a system of homogeneous algebraic equations.

Levinson's theorem for multichannel problems is formulated as follows:

$$\sum_i \sum_s [\delta^s(\varepsilon_i) - \delta^s(\varepsilon_{i+1})] = m\pi. \quad (83)$$

Here, m is the number of bound states of the system and the summation is over all thresholds (ε_i) and all eigenphase shifts between two given thresholds. (If the last, highest, threshold is ε_N, then $\varepsilon_{N+1} = \infty$.)

The relation (83) can be given a different form[28,29] by imposing on the sum of the eigenphase shifts

$$\Delta(k) = \sum_s \delta^s(k) \quad (84)$$

the following conditions.

1. Assume $\Delta(k)$ is a continuous function of k at the threshold energies E_{thresh} if in isolated (decoupled) channels there are no bound states for E_{thresh}.

2. Assume that at points $k > 0$ where the isolated channels have bound states of multiplicity σ the function $\Delta(k)$ has discontinuities:

$$\lim_{\varepsilon\to 0}\{\Delta(k+\varepsilon) - \Delta(k-\varepsilon)\} = \pi\sigma. \quad (85)$$

For such functions $\Delta(k)$, Levinson's theorem has the form

$$\Delta(0) - \Delta(\infty) = (m + \tau/2)\pi, \quad (86)$$

where τ is the multiplicity of the system's levels at $k = 0$ and m is the number of ordinary bound states.

It would be interesting to write down Levinson's theorem for the real parts of the experimentally determined phase shifts, as is done in field theory (see Sec. 10).

8. ALLOWANCE FOR PAULI'S PRINCIPLE IN THE DESCRIPTION OF THE INTERACTION OF COMPLICATED SYSTEMS BY MEANS OF AN EFFECTIVE TWO-PARTICLE POTENTIAL

If the particles A and B in a collision are composite, a consistent description of the scattering requires one to solve the Schrödinger equation that corresponds to the interaction and motion of all the elementary particles in the system. However, in practice, to explain the experimental data at energies below the inelastic scattering threshold E_{thresh}, one frequently reduces the approximate interaction of A and B to an effective two-particle potential V_{AB}. Of course, when the problem is posed in this way the relation between the number of bound states A + B and the phase-shift difference $\delta(0) - \delta(\infty)$ obtained from Levinson's theorem for potential (v_{AB}) scattering of A by B cannot be regarded as a rigorous result. At higher energies, for example, the inelastic scattering channels are opened and the very concept of a phase shift of elastic scattering loses its original meaning. And if the system can decay into three (or more) fragments, the picture becomes even more complicated. Notwithstanding this difficulty let us discuss how Levinson's theorem can be reconciled with a choice of the phenomenological potential v_{AB} (see ref. 30). The precise statement of Levinson's theorem for three (or more) particles will be considered in Sec. 9.

It follows from the experimental data (for example, for the scattering of nucleons by light nuclei or electrons by atoms) that the correct energy dependence of the phase shift $\delta(E)$ at energies below E_{thresh} can sometimes be obtained only by choosing v_{AB} so deep that the number of levels in the potential is greater then the actual number of bound states of the system A + B. In these cases, it would seem that Levinson's theorem is violated. For example, for the scattering of an electron (e^-) by a hydrogen atom H we have for both the triplet and the singlet s phase shift[8)] the equation[9)] $\delta(0) = \pi$, although the bound state of e^- and H (the negative hydrogen ion H^-) can exist only in the singlet state. In accordance with Levinson's theorem, one should choose v_{eH} in such a way that there is a bound state H^- in both cases. Moreover, it turns out that if one calculates the binding energy of e^- and H from the scattering length and the effective radius determined from the low-energy behavior of the phase shift (in accordance with the theory of the effective radius), one obtains a negative value in the triplet state of H^- as well. In Table 1, which is taken from ref. 30, we list some other examples of such anomalies.

Swan[30] has put forward an explanation of the apparently incomprehensible behavior of the phase shift for $E < E_{thresh}$. He showed that in all such cases we obtain

$$\delta(0) - \delta(\infty) = (m + n)\pi, \tag{87}$$

where m is the number of experimentally established bound states and n corresponds to the number of states forbidden by the Pauli principle. Therefore, (87) can be regarded as a new relation of Levinson's theorem, which takes into account the identical nature of the particles that form the system.

The following arguments may serve as a certain justification of what we have said.[31,32] We shall show that the radial part, $u_0(r)$, of the wave function ψ of an electron with $l = 0$ moving in the field of a hydrogen atom (in the triplet case) must have at least one node, although the attraction between e^- and H is not sufficient for the formation of a bound state.

We denote by $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ the orbital part of the wave function of the system $e^- + H$, where $\mathbf{r}_i$ are the coordinates of the electrons (in this problem, the motion of the the heavy nucleus can be neglected with sufficient accuracy. For the function of the scattered electron in the two-particle approximation it is natural to define

$$\psi(k, \mathbf{r}_1) = \int \varphi_0^*(\mathbf{r}_2)\,\Psi(\mathbf{r}_1, \mathbf{r}_2)\,d\mathbf{r}_2, \tag{88}$$

where $\varphi_0(\mathbf{r}_2)$ is the wave function of the ground state of the hydrogen atom.

When the electrons have parallel spins (triplet state Ψ^3), the orbital function of the system must be antisymmetric under transposition of the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\Psi^3(\mathbf{r}_1, \mathbf{r}_2) = -\Psi^3(\mathbf{r}_2, \mathbf{r}_1). \tag{89}$$

We represent Ψ^3 in the form of an expansion with respect to a complete set of states φ_n of one electron in the field of the proton:

$$\Psi^3(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \psi_n(\mathbf{r}_1)\,\varphi_n(\mathbf{r}_2). \tag{90}$$

(Here, the summation sign also includes the integral over the continuum.) We now multiply (88) by φ_n and integrate both side of the resulting equation with respect to $d\mathbf{r}_1$; using also (89) and (90), we obtain

$$\int \varphi_0^*(\mathbf{r}_1)\,\psi_0(\mathbf{r}_1)\,d\mathbf{r}_1 = -\int \varphi_0^*(\mathbf{r}_2)\,\psi_0(\mathbf{r}_2)\,d\mathbf{r}_2. \tag{91}$$

One can make the change of variables $\mathbf{r}_2 \to \mathbf{r}_1$ in the integrand on the right-hand side of (91). We then find that the value of the integral in (91) does not change when its sign is reversed, i.e.,

$$\int \varphi_0^*(\mathbf{r}_1)\,\psi_0(\mathbf{r}_1)\,d\mathbf{r}_1 = 0, \tag{92a}$$

and hence

$$\int_0^\infty u_{1s}(r_1)\,u_0(r_1)\,dr_1 = 0. \tag{92b}$$

Since the radial part $u_{1s} = 2r\exp(-r)$ of the ground-state function φ_0 does not change its sign, the sign of $u_0(r)$ must change and therefore $\psi_0(\mathbf{r}) = [u_0(r)/r]Y_{00}$ if (92) is to hold. This is what we had to prove.

9. MANY-BODY SYSTEMS

The simplest of the essentially many-particle systems is the three-particle one. At the same time, the three-body problem exhibits the basic qualitative features of more complicated systems. We can therefore assume that Levinson's theorem, which we consider here for the example of three particles,[33] does not require serious modification in the transition to the general case of N bodies.

Compared with the previously considered elastic and simple inelastic scattering, one can have new types of processes for $N \geq 3$ bodies. These are *reactions with redistribution of the particles* and *reactions in which the number of fragments of the system moving apart freely in the final state is greater than two (disintegration)*. In addition, one can have collisions of more than two particles (for example, when three particles converge from different sides).

Thus, for three bodies the scheme of all such processes has the form (here the brackets unite pairs in a bound state and an asterisk indicates excited states of such pairs):

$1 + (23) \to 1 + (23)$ elastic scattering,

$\to 1 + (23)^*$ simple inelastic scattering,

$\to \left\{\begin{matrix} 2 + (13) \\ 3 + (12) \\ 2 + (13)^* \\ 3 + (12)^* \end{matrix}\right\}$ reactions with redistribution of particles,

$\to 1 + 2 + 3$ three-particle decay,

$1 + 2 + 3 \to \left\{\begin{matrix} \text{the same final} \\ \text{states as in a} \\ \text{binary collision} \\ 1 + (23) \end{matrix}\right\}$ triple collisions.

The three-body problem can be treated in the same way as two-particle many-channel scattering.[34] And to apply Levinson's theorem in this case in the form in which it was formulated in Sec. 4 we must first introduce the concept of *partial decay channels* and eigenphase shifts (eigenchannel states) at energies above the decay threshold. In order to make the following arguments clearer, we must emphasize the analogy between an expansion in partial waves for binary reactions ($A + B \to C + D$) and for decay reactions ($A + B \to a + b + c$).

In binary collisions, one uses an expansion in spherical functions $Y_{lm}(\Omega_r)$, where $\Omega_r \equiv \theta$ and φ are the angular variables of the vector of the relative position of the two particles. The asymptotic behavior of the wave function is decomposed into partial-wave channels with different values of the orbital angular momentum l and its projection m. Then a collection of constants — the partial-wave amplitudes of the ingoing (A_{lm}) and outgoing (B_{lm}) waves — completely characterizes the states of the system at large r.

To describe the motion of three free particles it is convenient to use the so-called hyperspherical coordinate system.[35] Instead of the ordinary Jacobi coordinates **r** (the vector of the relative distance of particles 1 and 2) and **R** (the radius vector of the third particle relative to the center of mass of the pair 1, 2) one introduces a six-dimensional vector $\boldsymbol{\rho}_6$. The modulus of this vector,

$$\rho_6 = \sqrt{R^2 + r^2}\,, \tag{93}$$

characterizes the size of this system, and five of its angular variables Ω_5 characterize the position of the triangle 1, 2, 3 (Fig. 7) in space (for this one requires three angles) and its form (a further two angles). As a generalization of the ordinary spherical functions $Y_{lm}(\Omega_r)$ to the case of six-dimensional space one introduces the hyperspherical functions $Y_K(\Omega)$, where K is the set of five quantum numbers.

An expansion with respect to $Y_K(\Omega_5)$ of the asymptotic part of the wave function Ψ of the system that corresponds to decay and a triple collision determines the partial channels of the free motion of the three particles. The partial-wave amplitudes of the ingoing (A_K) and outgoing (B_K) waves completely determine the asymptotic behavior of Ψ these channels:

$$\begin{aligned}\Psi \underset{\rho_6\to\infty}{\sim} \sum_K \{A_K \exp(-ik_6\rho_6)/\rho^{5/2} \\ + B_K \exp(ik_6\rho_6)/\rho^{5/2}\}\, Y_K(\Omega_5), \\ k_6 = \sqrt{k_r^2 + k_R^2};\end{aligned} \tag{94}$$

the discrete set of constants A_K and B_K gives the continuous angular distribution of the three particles and the continuous distribution of the total energy E between them.

The concept of partial channels for the motion of three free particles having been introduced, it is not difficult to define eigenchannel states at an energy of the system above the decay threshold. As was done for reactions in which there are only two-fragment asymptotics, one chooses a special set of the amplitudes A^S_α for the ingoing waves in all partial channels (with both two and three free fragments). The coupling of the channels must not lead to a redistribution of the flux density in the different channels as a result of scattering: The amplitudes B^S_α of the outgoing waves differ[10] only by a common phase shift $2\delta^S$ from A^S_α:

$$B^s_\alpha = \exp(2i\delta^s)\, A^s_\alpha.$$

Using the δ^S, Levinson's theorem for three particles can be formulated in exactly the same way as for ordinary many-channel scattering (see Sec. 7):

$$\sum_i \sum_s [\delta^s(\varepsilon_i) - \delta^s(\varepsilon_{i+1})] = m\pi, \tag{95}$$

where the summation $\sum_i$ is now also over the thresholds at which the particle redistribution channels are opened and includes the decay threshold.

The proof of the relation (95) for three particles was given by Wright[11)] in ref. 33; however, he evidently did not yet know the actual way to introduce partial-wave decay channels by means of the hyperspherical functions Y_K. Levinson's theorem for three-particle systems has also been considered in refs. 37-41.

10. LEVINSON'S THEOREM IN QUANTUM FIELD THEORY

The main aim of this review has been to consider Levinson's theorem in nonrelativistic quantum mechanics. However, this theorem has also been obtained in field theory on many occasions. We shall not discuss the various papers in detail but merely give references to the literature that we know on the question.

All the above forms of Levinson's relation require a Schrödinger equation in some form or other for their justification. It is well known that similar general equations of motion have not yet been formulated in quantum field theory.

In refs. 42-46 the theorem is considered in the framework of various modifications of the exactly solvable field model[47] of Lee.[12)] In this model, it is assumed that there exist but three types of particle: heavy, N and V, whose motion can be ignored (fixed extended fermion sources), and a light scalar boson θ. This last particle can undergo elastic scattering, and be absorbed and emitted in an interaction process with N and V:

$$V \rightleftarrows N + \theta.$$

The Hamiltonian H of the model can be split into two parts:

$$H = H_0 + H_1, \tag{96}$$

where H_0 is the Hamiltonian of the noninteracting particles; H_1 is the interaction operator.

Levinson's theorem can be formulated as follows if there is one θ and one N particle (single-channel case):[42,44]

$$\delta(0) - \delta(\infty) = (m - m_0)\,\pi, \tag{97}$$

where $\delta(k)$ is the phase shift of the $N-\theta$ scattering; m and m_0 are the number of states in the discrete spectrum of the Hamiltonians H and H_0, respectively. The number m_0 is equal to the number of V particles.

In refs. 43-45 the relation (97) is generalized to the case of the many-channel Lee model, in which there are several θ particles. And, in contrast to the nonrelativistic many-channel case, the theorem is sometimes written

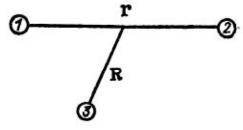

Fig. 7. Jacobi coordinates for a system of three particles.

down for the real parts of the physically measurable phase shifts $\eta_{\alpha\beta}$ of the scattering, and not for the eigenphase shifts:

$$N + \theta_\alpha \to N + \theta_\beta,$$

where the $\eta_{\alpha\beta}$ enter the scattering amplitude $T_{\alpha\beta}$ in the form

$$T_{\alpha\beta} = [\delta_{\alpha\beta} - \exp(2i\eta_{\alpha\beta})]/(2i\pi), \qquad (98)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta.

In the first papers[49,50] in which Levinson's theorem was formulated in field theory the possibility was noted of using it to distinguish one of the many solutions of Low's equation. The theorem was also obtained under the assumption of definite analytic properties of the scattering amplitude.[51–55]

The quasipotential approach in field theory[56] has much in common with the nonrelativistic formulation. However, we do not know of any papers in which Levinson's theorem has been proved by means of quasipotential equations.

CONCLUSIONS

The value of Levinson's theorem comes out clearly as an aid to choosing an interaction correctly from experimental data. But it is even more important that, in the process of becoming acquainted with the different modifications of Levinson's theorem, one must reexamine, and from a slightly different aspect, already known facts and, unexpectedly, gain a deeper insight into the important features of the theory. Each new relationship between physical quantities enables us in a more compressed and economic form to represent that which we understand by the aggregate of our concepts of quantum laws.

Levinson's relations can be used as a criterion for testing the correctness of new methods developed to solve quantum problems: The solutions obtained by means of these methods must agree with the results predicted by the theorem. Many applications of Levinson's theorem are still to be found and we shall be pleased if this review is of even slight assistance in this connection.

APPENDIX 1

The validity of (19) is shown by the following calculations:

$$\dot{\delta} = \sin 2(kr+\delta(k))/2k + 2\int_0^r \Psi^2\,dr - r = \sin 2(kr+\delta(k))/2k$$
$$+ 2\int_0^\infty \Psi^2\,dr - 2\int_r^\infty \Psi^2\,dr - \int_0^\infty dr + \int_r^\infty dr = \int_0^\infty (2\Psi^2-1)\,dr$$
$$- \int_r^\infty (2\Psi^2-1)\,dr + \sin 2(kr+\delta(k))/2k.$$

It remains to show that the last two terms cancel. Replacing the wave function by its asymptote at large r, we obtain

$$\int_r^\infty (2\Psi^2-1)\,dr = \int_r^\infty [2\sin^2(kr+\delta(k))-1]\,dr$$
$$= \int_0^\infty [2\sin^2(kr+\delta(k))-1]\,dr - \int_0^r [2\sin^2(kr+\delta(k))-1]\,dr$$
$$= \frac{1}{k}\int_{\delta(k)}^\infty [2\sin^2 x-1]\,dx + \int_0^r \cos 2(kr+\delta(k))\,dr = \left\{\frac{1}{k}\int_0^\infty (2\sin^2 x-1)\,dx = 0\right\}$$
$$+ \frac{1}{k}\int_0^{\delta(k)} [2\sin^2 x-1]\,dx + \int_0^r \cos 2(kr+\delta(k))\,dr = \sin 2(kr+\delta)/2k.$$

APPENDIX 2

We sketch the proof of Levinson's theorem by means of Jost functions, which are very frequently used in the literature (for details see ref. 4). We consider the case of s-wave scattering ($l = 0$).

The concept of a Jost function is introduced by means of two auxiliary linearly independent solutions $f_\pm(k, r)$ of the Schrödinger equation (3) that for large r have the form of ingoing and outgoing waves, $\exp(\pm ikr)$:

$$\lim_{r\to\infty} \exp(\mp ikr)\, f_\pm(k, r) = 1. \qquad (A.1)$$

It is well known that any solution of an ordinary differential equation of second order can be constructed as a linear combination of two linearly independent solutions of the equation; in particular, a solution that is regular at r = 0 (at r = 0 it satisfies the conditions $\varphi(k, 0) = 0$ and for $[d\varphi(r,k)/dr]_{r=0} = 1$) is

$$\varphi(k, r) = [f_-(k)\, f_+(k, r) - f_+(k)\, f_-(k, r)]/2ik. \qquad (A.2)$$

The coefficient $f_+(k) \equiv f$ is the Jost function; it is related to $f_-(k)$ by the simple equation

$$f_-(k) = f_\ddagger^* = |f|\exp[i\delta(k)]. \qquad (A.3)$$

The ordinary physical solution differs from $\varphi(k, r)$ only by the coefficient of the ingoing wave and it is normalized to unit flux, and then the coefficient of the outgoing wave determines the S matrix:

$$S = f_-/f_+ = \exp[2i\delta(k)]. \qquad (A.4)$$

The function f is analytic in k if U(r) satisfies the conditions (2). All m zeros (poles of S) in the upper half-plane of k lie on the imaginary k axis and correspond to bound states of the system. As a result, the integral of f'/f around the contour c (Fig. 8) in the upper half-plane is

$$\int_c d\ln f = 2\pi i m. \qquad (A.5)$$

The integral around the arc of large radius vanishes because $\lim_{|k|\to\infty} f(k) = 1$. As $|k| \to 0$ we have $f(k) \sim k^\tau$, where τ is the multiplicity of the "bound" state at E = 0 and the integral around the small arc is

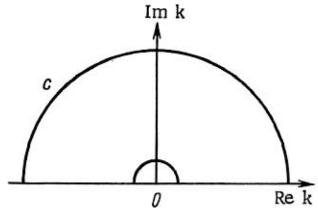

Fig. 8. Contour c around which the integral in Eq. (A.5) is taken.

$$\int d\ln f = \tau \int d\ln k = i\pi\tau. \qquad (A.6)$$

Calculating the remaining integral along the real axis and using (A.6) and the relation $\delta(-k) = -\delta(k)$, we obtain Levinson's relation instead of (A.5):

$$\delta(0) - \delta(\infty) = (m + \tau/2)\pi. \qquad (A.7)$$

---

1) The differential cross section of elastic scattering into an element of solid angle $d\Omega$ is expressed in terms of δ_l as follows:

$$d\sigma/d\Omega = |A(\theta)|^2,$$

where $A(\theta) = (1/k) \sum_{l=0}^{\infty} (2l+1) \exp(i\delta_l) \sin\delta_l P_l(\cos\theta)$. The total cross section, σ, is equal to the sum of the partial-wave cross sections σ_l:

$$\sigma = \sum_l \sigma_l = (4\pi/k^2) \sum_l (2l+1) \sin^2 \delta_l.$$

See ref. 2 for the determination of the phase shifts δ_l experimentally.

2) For a partial wave with nonvanishing orbital angular momentum ($l \neq 0$) a level at E = 0 contributes to Levinson's relation like any other state. This is due to the presence of the centrifugal barrier, which guarantees that the function $\Psi_l(0, r)$ of such a state in the region outside the well decreases with increasing r, so that $\Psi_{l\neq 0}(0, r)$ is normalizable [in contrast to $\Psi_{l=0}(0, r)$; see Figs. 1 and 2].

3) Wellner[7] notes that full rigor of the proof can be achieved by justifying the order of the passage to the limits, the use of the factors $\exp(-\varepsilon r)(\varepsilon \to +0)$, the taking of the integrals, etc.

4) The phase shift δ_l for pure Coulomb scattering compared with the case of free motion is $\delta = \lim_{r\to\infty} \left(\frac{\beta}{2k} \ln 2kr + \sigma_l\right) \to \infty$ [see Eq. (25)].

5) In ref. 21, the left-hand side of (59) also includes the quantity $\lim_{k\to\infty} [\delta_l(k) - k\alpha]$, which, however, is equal to zero.

6) Regge poles are poles of the S matrix in the complex l plane.

7) Regge trajectories are the curves described by the Regge poles in the complex l plane when the energy varies.

8) By this we mean symmetric and antisymmetric spin states of the electrons.

9) The phase shift $\delta(\infty)$ for the potential v_{eH}, chosen to describe the scattering of e^- on H below E_{thresh}, is naturally assumed equal to zero.

10) Strictly speaking, there are infinitely many partial channels and there must also be infinitely many eigenphase shifts for each value of the energy E. In practice, however, the contribution to the scattering of higher orbital momenta l and the "global" angular momentum K can be ignored at finite energies of the system.

11) Wright proved Levinson's theorem by means of the method of Moller wave operators developed by Jauch[36] for the case of single-channel scattering.

12) The nonrelativistic variant of Lee's model is analyzed in ref. 48.

---

[1] N. Levinson, Kgl. Danske Videnskab. Selskab. Mat. Fys., Medd., 25, No. 9 (1949).
[2] T. Y. Wu and T. Ohmura, Quantum Theory of Scattering, Prentice-Hall, London (1962).
[3] K. W. McVoy, Fundamentals in Nuclear Theory, Ch. 8, eds. A. de-Shalit and C. Villi, IAEA, Vienna (1967).
[4] R. G. Newton, Scattering Theory of Waves and Particles, McGraw-Hill, New York (1966).
[5] F. Calogero, Variable Phase Approach to Potential Scattering, Academic Press, New York–London (1967).
[6] V. V. Babikov, The Method of Phase Functions in Quantum Mechanics [in Russian], Nauka, Moscow (1966).
[7] M. Wellner, Amer. J. Phys., 32, 787 (1964).
[8] P. Swan, Nucl. Phys., A119, 40 (1968).
[9] D. Rein, Z. Phys., 220, 394 (1969).
[10] P. Swan, Nucl. Phys., 46, 669 (1963).
[11] H. Feshbach, Ann. Phys. (New York), 5, 357 (1958); 19, 287 (1962).
[12] M. Gourdin and A. Martin, Nuovo Cimento, 6, 757 (1957).
[13] A. G. Sitenko and V. F. Kharchenko, Preprint ITF-68-11 [in Russian], Institute for Theoretical Physics, Kiev (1968).
[14] A. Martin, Nuovo Cimento, 7, 607 (1958).
[15] Kh. Chaddan, Nuovo Cimento, 10, 892 (1958).
[16] M. Bertero, G. Talenti, and G. A. Viano, Nucl. Phys., A113, 625 (1968).
[17] R. Mills and J. F. Reading, J. Math. Phys., 10, 321 (1969).
[18] F. Tabakin, Phys. Rev., 177, 1443 (1969).
[19] M. Bolsterli, Phys. Rev., 182, 1095 (1969).
[20] J. E. Beam, Phys. Lett., 30B, 67 (1969).
[21] P. Butera and L. Girardello, Nuovo Cimento, 54A, 141 (1968).
[22] F. Calogero and G. Jogannathan, Nuovo Cimento, 47, 178 (1967).
[23] M. Ciafaloni, Nuovo Cimento, 29, 420 (1963).
[24] C. A. Lopez, Nuovo Cimento, 58A, 160 (1968).
[25] C. A. Lopez and V. Massida, Nucl. Phys., A149, 33 (1970).
[26] R. G. Newton, J. Math. Phys., 1, 319 (1960).
[27] E. Kazes, Nuovo Cimento, 13, 983 (1959).
[28] H. A. Weidenmüller, Ann. Phys. (New York), 28, 60 (1964).
[29] W. Glöckle, Z. Phys., 190, 391 (1966).
[30] P. Swan, Proc. Roy. Soc., 228, 10 (1955).
[31] A. Temkin, J. Math. Phys., 2, 336 (1961).
[32] L. Rosenberg and L. Spruch, Phys. Rev., 121, 1720 (1961).
[33] J. A. Wright, Phys. Rev., 139, 137 (1965).
[34] V. P. Zhigunov and B. N. Zakhar'ev, Fiz. El. Chast. Atom. Yad., 2, 499 (1971).
[35] L. M. Delves, Nucl. Phys., 29, 268 (1962); 29, 326 (1962).
[36] J. M. Jauch, Helv. Phys. Acta, 30, 143 (1957).
[37] S. Weiberg, Phys. Rev., 131, 440 (1963).
[38] P. E. Shanley and R. Aaron, Ann. Phys. (New York), 44, 363 (1967).
[39] I. H. Sloan, Phys. Lett., 34B, 243 (1971).
[40] I. H. Sloan, Nucl. Phys., 168A, 211 (1971).
[41] R. G. Newton, Preprint, Indiana University (1971).
[42] M. Ida, Progr. Theor. Phys., 21, 625 (1959).
[43] G. Konisi and T. Ogimoto, Progr. Theor. Phys., 22, 807 (1959).
[44] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev., 124, 1258 (1961).
[45] G. C. Ghirardi, M. Pauri, and A. Rimini, Ann. Phys. (New York), 21, 401 (1963).
[46] K. L. Nagy, Acta Phys. Hung., 24, 433 (1968).
[47] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Peterson, and Co., New York (1961).
[48] A. I. Baz', Ya. B. Zel'dovich, and A. M. Perelomov, Scattering Reactions and Decay in Nonrelativistic Quantum Mechanics, Jerusalem (1969).
[49] R. Haag, Nuovo Cimento, 5, 203 (1957).
[50] J. C. Polkinghorne, Proc. Cambr. Phil. Soc., 54, 560 (1959).
[51] R. L. Warnock, Phys. Rev., 131, 1320 (1963).
[52] J. M. Charap, Nuovo Cimento, 36, 419 (1965).
[53] M. Kato, Ann. Phys. (New York), 31, 130 (1965).
[54] D. Atkinson, K. Dietz, and D. Morgan, Ann. Phys. (New York), 37, 77 (1966).
[55] J. B. Hartle and C. E. Jones, Ann. Phys. (New York), 38, 348 (1968).
[56] V. G. Kadyshevskii, R. M. Mir-Kasimov, and N. B. Skachkov, Nuovo Cimento, 55, 233 (1967).