Theory of nuclear rotational bands

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The effects which arise due to the rotation of deformed nuclei are briefly summarized. Methods for theoretically describing these effects based on quantum many-body theory are outlined.

INTRODUCTION

Study of nuclear rotation was spurred by the discovery of the anomalously intense E2 transitions in rare-earth nuclei ($150 \leq A \leq 190$) and in transuranium nuclei ($228 \leq A$). The intensity of these transitions was explained by Rainwater[1] and later by Bohr and Mottelson,[2,3] who hypothesized a static quadrupole deformation of the self-consistent field of the nucleons involved. The deformation of this field (such that the nuclear matter fills a volume whose surface is approximately an ellipsoid of revolution) explains the large static nuclear quadrupole moment. The arbitrariness in the orientation of the symmetry axis of the field leads to the existence of a series of states having identical nucleon distributions with respect to field levels but differing in the weight factors corresponding to different orientations. These "rotational states" predicted theoretically by Bohr and Mottelson have been identified experimentally in many cases. Several regularities have led to the grouping of the states of deformed nuclei into states of various "rotational bands." The interpretation of the rotational states offered in refs. 1-3 has found much support. On the other hand, it became clear that this interpretation required refinement in connection with the structure of the internal part of the deformed-state wave function and in the description of the differences among the wave functions of the states of a rotational band. It was also necessary to develop a mathematical apparatus for the theory of rotation in a quantum system of many nucleons. Below we focus on this latter point.

1. PROPERTIES OF THE ROTATIONAL STATES

The states of deformed nuclei have been combined into rotational bands on the basis of several considerations, in particular, the dependence of the state energy on the angular momentum quantum number L. Members of a rotational band have the same spatial parity π. A rotational band can begin with any integral (in even nuclei) or half-integral (in odd nuclei) value of I (I_{min} is customarily denoted by the index "K"). The angular-momentum composition of the bands implies nuclear symmetry with respect to discrete rotations of the internal axes (as in a symmetric diatomic molecule). In the even–even nuclei the lowest-lying excited states belong to the K = 0 band ($\pi = +1$) and have angular momenta I = 0, 2, 4, ... The $K \neq 0$ bands contain states having angular momenta I = K, K + 1, K + 2, ... and have been observed in both even and odd nuclei. Among odd–odd nuclei there are states which can be classified as rotational states having K = 0; I = 1, 3,... . This part of the K = 0 band turns out to be displaced toward higher energies[4] from the states having I = 0, 2, 3, ...

The excitation energy of a rotational level can be written in the form

$$E_I - E_0 = \frac{I(I+1)}{2J}, \tag{1.1}$$

characteristic of a rotator having a moment of inertia J. In contrast with a rigid rotator, the nuclear moment of inertia changes smoothly along the band, increasing at the beginning of the band nearly linearized with excitation energy:

$$J = b + c(E_I - E_0)/2. \tag{1.2}$$

Equations (1.1) and (1.2) lead to a simple expression for the energy of a "nonrigid" rotator:[5]

$$E_I - E_0 = (b/c)\left(\sqrt{1 + cI(I+1)/b^2} - 1\right). \tag{1.3}$$

This expression gives the energies of the first six or seven levels of the ground rotational bands of even–even nuclei within an error comparable to the experimental error.[5,6] The values of parameters b and c for deformed nuclei typically are such that we can speak of an adiabatically slow rotation only for states having

$$I \ll I_0 = \sqrt{b^2/c} \approx 14 - 18.$$

There are other ways to parametrize E_I, e.g., by using the Harris equation:[7]

$$\left.\begin{aligned} E_I - E_0 = E(I) &= [J_0 + (3/4c)\,\omega^2]\,\omega^2/2; \\ I(I+1) &= [J_0 + (1/2c)\,\omega^2]^2\,\omega^2. \end{aligned}\right\} \tag{1.4}$$

Interest in these equations results from, first, the possible use of these equations to find a systematic classification of the experimental data over a very broad range of nuclei, extending to nuclei in the transition region and to spherical nuclei.[8] Second, Eqs. (1.4) correspond to a simple physical picture showing how a state dependence arises in the moment of inertia, drawn on the basis of the model with a variable moment of inertia.[9,10] In this model the energy of a state having a fixed value of I is set equal to the minimum energy E(J, I). This energy depends on some collective variable whose value also determines J. In the case

$$E(J, I) = c(J - J_0)^2/2 + I(I+1)/2J, \tag{1.5}$$

the equations giving the equilibrium value of J and the corresponding energy reduce to form (1.4).

Another regularity of rotational states is found in the probabilities for transitions under the influence of an electromagnetic field[11] (and for α and β transitions). In the K = 0 bands of even–even nuclei, e.g., we find a ratio $B(E2.4 \to 2)/B(E2.2 \to 0) \approx 10/7$, and the static quadrupole moment is proportional to $I/(2I + 3)$. These and other regularities can be explained by describing the reduced matrix element for the electric λ-pole moment of the nucleus with an equation[12] based on the familiar "adiabatic

wave function" (see, e.g., the reviews in ref. 13):

$$\langle K_f I_f \| \mathscr{M}(\lambda) \| I_i K_i \rangle = \left[\frac{(2I_f+1)(2I_i+1)}{(1+\delta_{K_f})(1+\delta_{K_i})} \right]^{1/2} - (1)^{I_f - K_f} \times \left\{ \begin{pmatrix} I_f & \lambda & I_i \\ -K_f & K_f - K_i & K_i \end{pmatrix} \langle K_f | \mathscr{M}'(\lambda, K_f - K_i) | K_i \rangle + (-1)^{I_i + K_i} \begin{pmatrix} I_f & \lambda & I_i \\ -K_f & K_f + K_i & -K_i \end{pmatrix} \times \langle K_f | \mathscr{M}'(\lambda, K_f + K_i) | \overline{K}_i \rangle \right\}. \quad (1.6)$$

The consequences of Eq. (1.6) have been tested thoroughly in the first few rotational excitations for the operators corresponding to the electric dipole, quadrupole, and octupole moments. The analogous expression available for the magnetic dipole moment describes the magnetic moments and M(1) transitions of deformed nuclei.

The quantities $\langle K_f | \mathscr{M}' | K_i \rangle$ in Eq. (1.6) represent the matrix elements of operator $\mathscr{M}$, referred to the internal axes. The wave functions $|K\rangle$ are approximated by relatively simple combinations of the wave functions of independent fermions in a field of axial symmetry (see, e.g., ref. 14, which describes in detail the structure of nonrotational states of deformed nuclei).

We might expect that Eq. (1.6) would reproduce the experimental situation for transitions in roughly the same manner that rigid-rotator equation (1.1), with a constant moment of inertia, reproduces the energies of the rotational states. However, measurements of the lifetimes of high-I rotational levels[15,16] show that important corrections must be made to (1.6) even for the ground band of even–even nuclei.

The importance of the terms neglected in (1.6) increases, becoming appreciable even at small I, for transitions forbidden on the basis of approximate quantum numbers associated with the structure of the deformed $|K\rangle$ states. The necessary corrections can be made by taking into account phenomenologically the mixing of various $|K\rangle$ in the stationary-state wave function which is caused by Coriolis forces.[17,18] These forces arise because of the coupling of the collective angular momentum **R** and the internal angular momentum **j**; they are described by the rotary-model Hamiltonian

$$H = H_{\text{intr}} + \sum_{k=x,\,y} \frac{R_k^2}{2J}. \quad (1.7)$$

The matrix elements of $\mathbf{I} = \mathbf{R} + \mathbf{j}$ in the internal coordinate system are[19]

$$\langle K \pm 1 I | I_\pm | IK \rangle = \sqrt{(I \pm K)(I \mp K + 1)} \quad (1.8)$$
$$(I_\pm = I_x \pm i I_y).$$

The matrix elements of **j** can be either treated as adjustable parameters or approximated by the matrix elements of the total angular momentum between various deformed wave functions, where the latter are assumed internal wave functions.

In some cases the mixing effects can be analyzed on the basis of perturbation theory. Perturbation-theory equations giving the corrections to (1.6) on the basis of more general assumptions regarding the Hamiltonian than those reflected in (1.7) are given in ref. 20.

Coriolis mixing also affects the energies of rotational states and turns out to be extremely important in the K = 1/2 bands.[13]

For certain nuclei single-particle levels having a large $j \gg 1$ admixture turn out to lie close to the Fermi surface (e.g., the $i_{13/2}$ level in Yb, Er, Dy, etc., isotopes; or the $j_{15/2}$ level in the actinide region). The matrix elements for the Coriolis interaction in such nuclei may reach values comparable to the excitation energy of nonrotational states. In these cases coupling of states differing in K becomes of decisive importance for the structure of the lowest-lying states.[21]

In summary, nuclear rotation has the following features:

1) There is usually no serious difficulty in experimentally distinguishing rotational states against the background of other states because the rotational states have characteristic energy and transition-probability properties.

2) Rotation in nuclei is greatly distorted by the coupling with internal motion. Only a few levels can be said to be in adiabatically slow rotation. This distortion is responsible for the ambiguity involved in a theoretical discrimination between rotation and other modes, i.e., in an optimal determination of the collective angles[13] of the unified model.

3) The rotation cannot be treated as a process in which the nuclear shape undergoes slight changes, i.e., it cannot be treated by analogy with vibrations. The wave packets describing the nuclear rotation around some axis (e.g., the 0y axis),

$$|t, \omega\rangle = \exp(-i\omega I_y t) |\omega\rangle, \quad (1.9)$$

must include many components of the rotational band, so the information obtained from these packets must reflect a certain averaging over a large part of the band.

The last two circumstances are responsible for the limited accuracy of the properties predicted for rotational states on the basis of semiclassical arguments.

2. MICROSCOPIC MODELS FOR NUCLEAR ROTATIONAL BANDS

This term covers a rather large number of nuclear models developed to describe the basic properties of rotational states, primarily the energies, on the basis of data on the internal structure of deformed nuclei. These models have yielded important relations among the characteristics of the simplest (quasiparticle) excitations of deformed nuclei and the structure of the rotational bands.

Cranking model. The first category of methods for describing rotational bands consists of refinements of the cranking model.[22] This category includes a variational method[23,24] for states whose time dependence is governed by Eq. (1.9). There is an equivalent formulation based on the equation for the single-particle Green's function of a system of fermions, written in a rotating coordinate system.[25]

The form of the multifermion wave functions used in the variational approach is chosen to best approximate the internal nuclear state. Good results can be found by taking into account only extremely specific correlations between nucleons – correlations of the superconducting type – and using mathematical methods developed for the theory of superconductivity.[14,23,26-28]

In the Hartree–Fock–Bogolyubov theory with a time dependence, we seek states $|t, \omega\rangle$ in which the expectation value of $H' - i\hbar\partial/\partial t$ is stationary with respect to small variations $|t, \omega\rangle = |t, \omega\rangle + |\delta\rangle$. Here $H' = H - \lambda N$ is the Hamiltonian minus the part showing the contribution to the energy which is proportional to the number of nucleons. The stationarity condition consists of the equations

$$\langle \omega, t | [Q, (H' - i\hbar\partial/\partial t)] | t, \omega\rangle = 0. \quad (2.1)$$

Here $[A, B] \equiv AB - BA$ and Q is an operator of the type

$$Q = \sum_{\mu\nu} (p_{\mu\nu}\alpha_\mu^+\alpha_\nu^+ + q_{\mu\nu}\alpha_\mu\alpha_\nu + s_{\mu\nu}\alpha_\mu^+\alpha_\nu^+). \quad (2.2)$$

Without any loss of generality we can assume that the quasiparticle operators α_μ, α_ν^+ are such that for any ω we have

$$\alpha_\mu | \omega, t\rangle = 0 \quad (2.3)$$

(i.e., the state $|\omega, t\rangle$ is the vacuum state with respect to α_μ).

We consider two states satisfying the stationarity conditions and corresponding to two infinitesimally close values of ω; we write

$$|\omega + \delta\omega\rangle = (1 + i\delta\omega\hat{J\theta}) | \omega\rangle, \quad (2.4)$$

where $\tilde{\theta}$ is a Hermitian operator of the type

$$\hat{\theta} = \sum_{\mu\nu} (\theta_{\mu\nu}\alpha_\mu^+\alpha_\nu^+ + \theta_{\mu\nu}^*\alpha_\nu\alpha_\mu), \quad (2.5)$$

and the numerical factor J in (2.4) is introduced to normalize $\hat{\theta}$ according to the condition

$$\langle \omega | [I_y, \hat{\theta}] | \omega\rangle = 1/i. \quad (2.6)$$

Replacing Q in Eq. (2.1) by $i\delta\omega J\hat{\theta}$, we can relate the variation in the expectation value of Hamiltonian H' ($E = \langle \omega | H' | \omega\rangle$) to that of the projection of the angular momentum ($M = \langle \omega | I_y | \rangle$):

$$\delta E = \omega\delta M, \quad (2.7)$$

where

$$\delta M = J\delta\omega. \quad (2.8)$$

In classical mechanics Eqs. (2.7) and (2.8) give the energy and moment of inertia of a rotating body as functions of the angular rotation frequency. To use these equations to analyze the energies of nuclear rotational bands, we introduce a seemingly natural relation between the expectation value of the projection of the angular momentum in the rotating state, M, and quantum number I of the state:

$$M^2 = I(I + 1). \quad (2.9)$$

The information contained in Eqs. (2.7) and (2.8) allows us to expand the energy $E = E_I$ in powers of $I(I + 1)$ and to obtain the phenomenological equations discussed in the preceding section. Finding the coefficient in the power-law expansion

$$M = \sum_{k=0, 1, \ldots} \xi_k \omega^{2k+1}, \quad (2.10)$$

we can thus integrate Eq. (2.7):

$$E - E_0 = \sum_{k=0, 1, \ldots} \frac{2k+1}{2k+2} \xi_k \omega^{2k+2}. \quad (2.11)$$

The correspondence between Eqs. (1.4) and Eqs. (2.10) and (2.11) is obvious.[29]

Description of the rotational bands in the variational approach thus requires solution of the equations for $|\omega\rangle$ which arise from conditions (2.1). This very complicated problem can be avoided by introducing approximations for the function $|\omega\rangle$ (and, correspondingly, for the nuclear Hamiltonian H'). Information can be obtained about moment of inertia J by solving the system of linear algebraic equations for the coefficients $\theta_{\mu\nu}$ of operator $\hat{\theta}$ in Eqs. (2.4) and (2.5). We can write the equations for $\hat{\theta}$ in a compact form:

$$\langle \omega | [Q, [(H' - I_y^2/2J), \hat{\theta}]] | \omega\rangle = 0. \quad (2.12)$$

This equation follows from Eqs. (2.1) and (2.4) under a condition which is a strengthening of condition (2.6). According to (2.6), $[I_y^2, \theta]$ is equal to $2I_y/i$ when an averaging is carried out over state $|\omega\rangle$. To find the correct result we must assume $\langle \omega | [Q, [I_y^2/2J, \theta]] | \omega\rangle = \langle \omega | [Q, 2I_y/i] | \omega\rangle$. Evaluation of the expression in (2.12) can be simplified by writing Hamiltonian H' in second-quantization form:

$$H' = H_0' + \sum_\mu \varepsilon_\mu\alpha_\mu^+\alpha_\mu + \frac{1}{4}\sum_{ijkl} V_{ij,kl} : a_i^+a_j^+a_la_k : \quad (2.13)$$

In (2.13) the $a_i^+(a_j)$ are the creation (absorption) operators for the nucleons in arbitrary single-particle states i (j); α_μ^+, α_ν are quasiparticle operators[14] with respect to which $|\omega = 0\rangle$ is the vacuum state; and ":...:" denotes conversion to normal form,[30] achieved by writing all operators in terms of α_μ^+, α_ν and then permuting with α^+ the operators on the left of the operators α and multiplying by $(-1)^P$ (where P is the parity of the permutation).

The form of the equations for $\theta_{\mu\nu}$ in lowest order in ω and the equation for the moment of inertia J ($\omega = 0$) are given in refs. 23-25 taking account of the residual interaction (see the text below and Appendix 2). If the operator $\hat{\theta}$ corresponding to the "collective angular variable" commutes with the interaction operator, we find simple equations for $\theta_{\mu\nu}$ and J:

$$\theta_{\mu\nu} = \frac{i}{J} \cdot \frac{\langle \mu | I_y | \nu\rangle}{\varepsilon_\mu + \varepsilon_\nu} (u_\mu v_\nu - u_\nu v_\mu); \quad (2.14)$$

$$J = \sum_{\mu\nu} \frac{|\langle \mu | I_y | \nu\rangle|^2}{\varepsilon_\mu + \varepsilon_\nu} (u_\mu v_\nu - u_\nu v_\mu)^2. \quad (2.15)$$

Inglis's equation (2.15), corrected for pairing, has been used to calculate the moments of inertia of deformed nu-

clei[31] and allows us to find a quite good agreement between the experimental and theoretical moments of inertia with a pairing corresponding to an even–odd mass difference and a good description of the gap in even nuclei.

The residual interactions are taken into account in an analysis of the nonadiabatic corrections to the energies of the rotational states. These corrections were calculated in refs. 32 and 35 using the theory of Green's functions and using a semiclassical approximation to calculate the matrix elements $\langle \mu \mid I_y \mid \nu \rangle$. It was shown that the coupling between rotation and pairing predominates in band formation at large I. Estimates were found for the critical moments at which pairing correlation vanishes. This effect – the vanishing of pairing correlations in rapid nuclear rotation – has an analog in the physics of extended superconductors and was predicted for nuclear physics by Mottelson and Velatin.[34]

At present several models related to the variational approach are available which allow a simpler extraction of information about the coupling of various types of excitations of the internal state and rotation.[35] These models use the Inglis equation with pairing, (2.15), which gives the moment of inertia as a function of characteristics of the internal state, and they use ideas from the rotary model with a variable moment of inertia, discussed above [before Eq. (1.5)].

The projection method. The possibilities of the Hartree–Fock–Bogolyubov and equivalent methods have scarcely been exhausted, but they do suffer from several serious shortcomings. The use of semiclassical ideas to analyze the spectra of nuclei in which the moments of inertia may differ appreciably in neighboring quantum states renders theoretical predictions unreliable. This theory is inadequate for calculating the non-energy-dependent characteristics of the rotational states, and unresolved problems remain in this area.

One of the earliest methods for describing rotation which is not based on the semiclassical approximation is the method[36] of projecting the deformed wave functions onto states with fixed quantum numbers IM. The wave functions are written as

$$| IM \rangle = \sum_k P^I_{MK} \xi_K \mid K \rangle, \qquad (2.16)$$

where

$$P^I_{MK} = \sum_\nu | \nu IM \rangle \langle KI\nu | = \frac{2I+1}{8\pi^2} \int d\omega D^{I*}_{MK}(\omega) \hat{R}(\omega). \qquad (2.17)$$

In Eqs. (2.16) and (2.17) P^I_{MK} is the projection operator, $| K \rangle$ is a state whose angular-momentum projection on the 0z axis of the laboratory coordinate system is equal to K, the ξ_K are numerical coefficients, $D^I_{MK}(\omega) = \langle MI\nu \mid \hat{R}(\omega) \mid \nu IK \rangle$ is a generalized spherical harmonic, $\hat{R}$ is the rotation operator, and ν denotes states of the complete set for the given nucleus. Projection allows us to determine moments of inertia and matrix elements which are physically observable on the basis of information about the nuclear Hamiltonian H when these approximations regarding $| K \rangle$ and ξ_K are used. The states $| K \rangle$ (ξ_K) are usually treated as unknown. A variational procedure is used to find them; it is better to carry out the variation after the projection to take into account coupling with internal modes. This method has been applied to heavy nuclei in, e.g., ref. 37. In an interesting approach,[38] the projection method was combined with the Green's function method to obtain, in the lowest approximations, the results of the cranking model for the moments of inertia.

The projection method (involving projection onto states with fixed angular momenta) turns out to be unsuitable for solving a simple problem – that of determining the nuclear mass – if the projected function is not separated into two cofactors. In this problem a model analogous to the cranking model gives the correct result for any approximations of the internal state.[1)] Because of this circumstance and the technical difficulties involved in projection, there is interest in seeking a formulation of the quantum theory of rotational bands which is less sensitive to the model of the many-fermion wave function describing the internal structure.

3. EFFECT OF THE SPIN OF THE NUCLEAR STATE ON THE MOTION OF THE INDIVIDUAL NUCLEONS

In the preceding section we mentioned that it is possible to study the spin dependence of collective parameters (the gap and deformation parameters) within the framework of the cranking model. Several nuclear properties can be naturally explained if we assume that the nuclear rotation severely distorts the motion of one or several of the least bound nucleons. The model of a quasiparticle coupled with a rotator by Coriolis forces,[18] e.g., allows us to describe the effects of the mixing of quasiparticle states having different K of an odd nucleus. The analytic formulation of the model incorporates the assumption that the internal part of the wave function of the deformed nucleus can be approximated by a relatively simple multinucleon wave function constructed on the basis of configurations of the deformed potential.[14] We will show in this section that this assumption can lead to serious errors under certain conditions, and we will give more accurate equations describing the effect of spin on the distribution of quasiparticles with respect to levels of the average field.

In Appendix 1 we derive, in a method free of these errors, a model taking into account excitation of deformed-field configurations coupled by conservation of nuclear spin.[35] We write the wave function of the nucleus as

$$| \nu IM \rangle = \sum_p c_p(\nu I) \mid pIM \rangle; \qquad (3.1)$$

$$| pIM \rangle = P^I_{MKp} \Omega_p \mid - \rangle. \qquad (3.2)$$

where

$$| - \rangle = \sum_I \eta_I \mid I,\ M = 0 \rangle \qquad (3.3)$$

is a superposition of states of the ground band of the nearest even–even nucleus. Projection operator P^I_{MK} is given by Eq. (2.17).

Here Ω_p denotes the creation (absorption) operator for the quasiparticle or a combination of such operators, which may include phonons of collective nonrotational

states.[14] We assume state $\Omega_p|-\rangle$ to be an eigenstate of operator I_z (in the laboratory system), and we denote the corresponding eigenvalue K_p.

We assume the nuclear Hamiltonian H and the state $|-\rangle$ in (3.2) to be unknown, and we assume the function $h[I(I+1)] = E_I$ to be unknown; this function gives the spin dependence of the energy of states in $|-\rangle$. The elements determining H, h, and $|-\rangle$ can either be treated as phenomenological parameters which can be adjusted independently or sought by the methods discussed above. Examples of empirical equations with two parameters approximating $h[I(I+1)]$ can be found in Sec. 1. In the simple case of a rigid rotator we have $h = I(I+1)/2J_0$, where J_0 is the moment of inertia.

The equations for the column vector $c(\nu I)$, whose coefficients are the amplitudes $C_p(\nu I)$ in Eq. (3.1), can be written as

$$\left.\begin{aligned}\rho^I(\mathcal{H}^I - E_{\nu I})\,C(\nu I) = 0;\\ (\mathcal{H}^I - E_{\nu I})\,\rho^I C(\nu I) = 0.\end{aligned}\right\} \tag{3.4}$$

Here we are using matrix notation and we have used

$$\mathcal{H}^I = \mathcal{H}^{(0)} + X^I \tag{3.5}$$

The diagonal matrix elements of $\mathcal{H}^{(0)}$ represent the adiabatic limit of the excitation energies of states $\Omega_p|-\rangle$; the residual interactions also contribute to this matrix [see Eq. (A.2)]. Matrix X^I is defined by

$$f(X^I) = \mathring{X}^I/2J_0;\ f(E_2) = I(I+1)/2, \tag{3.6}$$

$$\begin{aligned}\mathring{X}^I_{pp'} = [I(I+1) - 2K_{p'}^2]\,\delta_{pp'} + (j^2)_{pp'}\\ -\sqrt{(I+K_{p'})(I-K_{p'}+1)}\,(j_-)_{pp'}\\ -\sqrt{(I-K_{p'})(I+K_{p'}+1)}\,(j_+)_{pp'}.\end{aligned} \tag{3.7}$$

Appendix 1 gives equations for matrices $\mathring{X}^I$, X^I; the quantities $j_\pm$, j^2 appearing in them, and the overlap-integral matrix ρ^I.

Equation (3.4) converts into the equation for the model of the quasiparticle coupled with a rotator[18] under the following conditions:

1) The state $|-\rangle$ is assumed to be a solution of the Hartree–Fock–Bogolyubov problem for the nucleus.

2) Expansion (3.1) for $|\nu IM\rangle$ is limited to the series of single-quasiparticle states, i.e., Ω_p are the Bogolyubov quasiparticle operators α_p^+ ($\alpha_p|-\rangle = 0$), and in the matrix $\mathcal{H}^0_{pp'}$ we take into account only that part which does not change the number of quasiparticles ($\mathcal{H}^0_{pp'} = \varepsilon_p\delta_{pp'}$).

3) A simple expression is used for $h(I^2)$:

$$[h(I^2) = I^2/2J_0,\quad X^I_{pp'} = \mathring{X}^I_{pp'}/2J_0].$$

4) Overlap matrix ρ^I is assumed to have the form $\rho^I_{pp'} = \text{const}\,\delta_{pp'}$.

The derivation in Appendix 2 allows us to refine each of these approximations. This derivation also shows that all effects associated with the presence of a spin in state $|\nu IM\rangle$, including coupling of excitations Ω_p by Coriolis forces, arise from the last term in Hamiltonian $h(I^2)$ in Eqs. (A.1). This term contains powers of the two-particle operator I^2 and can be thought of as the quasiparticle interaction operator. Writing $P^I_{K_{p_2}K_{p_1}}\Omega_{p_1}h(I^2)$ as in Eq. (A.8), with matrix $X^I_{p_1p_2}$ given by Eqs. (A.7), (3.6), and (3.7), we single out from operator $h(I^2)$ the rotational energy of the core, represented by state $|-\rangle$, and the term showing the coupling of the core with the excitation, described by operator Ω_p. As a result in matrices $\mathring{X}^I_{pp'}$, $X^I_{pp'}$ those matrix elements which couple states having a number of quasiparticles $n_p > n_{p'}$ vanish. The projections of angular momentum $I_\varkappa$ are single-particle operators, and their commutators with Ω_p in Eqs. (A.3) and (A.4) contain the same number of quasiparticle operators α^+, α as do the operators Ω_p themselves. Nevertheless, there is a mixing of wave-function components having a large number of quasiparticles with the single-quasiparticle component in state (3.1), (3.2). This mixing results from the residual-interaction operator in the Hamiltonian [the last term of the Hamiltonian in Eq. (2.13)].

In the calculations the number of basis functions in (3.1) is always limited by the small number of terms, and Eqs. (3.4), which follow from (A.1), may not be satisfied. We find a better approximation to the solution by using a variational principle; for this purpose we turn to the condition for stationarity of the energy with respect to small variations $C^+(\nu I)$ and $C(\nu I)$. We write the expectation value of the Hamiltonian in state (3.1) as

$$\overline{H}^{\nu I} = \frac{C^+(\nu I)\,\{\rho^I\mathcal{H}^I + \mathcal{H}^I\rho^I\}\,C(\nu I)}{2C^+(\nu I)\,\rho^I C(\nu I)} \tag{3.8}$$

(of the several possible forms in which we could write $\overline{H}^{\nu I}$ we have chosen that corresponding to the situation in which the matrix contracted with C^+ and C is Hermitian). The stationarity condition leads to

$$\left.\begin{aligned}\widetilde{\mathcal{H}}^{\nu I}C(\nu I) = E_{\nu I}C(\nu I);\\ \widetilde{\mathcal{H}}^{\nu I} = \tfrac{1}{2}(\rho^I\mathcal{H}^I + \mathcal{H}^I\rho^I) + (1-\rho^I)\,E_{\nu I}.\end{aligned}\right\} \tag{3.9}$$

The matrix elements of matrix ρ^I are obtained in Appendix 1, where the role of overlap matrix ρ^I in Eqs. (3.4) and (3.9) is also demonstrated. The difference between $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ can be interpreted as a renormalization of the inertial parameters of the core. The renormalized moments of inertia turn out to differ in different matrix elements of $\widetilde{\mathcal{H}}^I$. The moments of inertia $\tilde{J}$ in the diagonal matrix elements of $\widetilde{\mathcal{H}}^I$ [Eqs. (A.17) and (A.18)] slightly modify the moments of inertia of the even core. The matrix elements of $\widetilde{\mathcal{H}}^{\nu I}_{pp'}$ with $K_p = K_{p'} \pm 1$ (the Coriolis-interaction terms) also differ from the corresponding elements $\mathcal{H}^{\nu I}_{pp'}$. A reflection of the fact that the condition for orthogonality of states is more complicated than in the formulation of ref. 18 is the dependence of the matrix elements on energy $E_{\nu I}$ [see Eqs. (A.19) and (A.20)]. Renormalization has no effect in the calculation of the decoupling parameters in states with $K = \pm 1/2$ in the lowest order in the interaction, since the difference between the energies in the numerator of (A.20) vanishes in this case. Large changes occur in the values of matrix elements of this type when the single-quasiparticle energies $\varepsilon_p +$

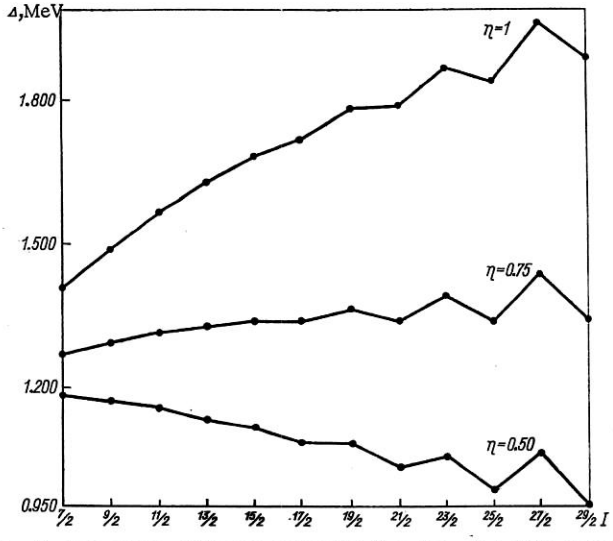

Fig. 1. Dependence of the gap parameter Δ on the spin I of the ground rotational band of ^{169}Yg for various attenuation factors $\eta = \eta^{I}_{pp'}$ of the Coriolis forces for an accurate reproduction of energies.

$[I(I+1) - 2K_p^2]/2J$ of the coupled states are greatly different. The eigenenergies $E_{\nu I}$ are approximately equal to the energies of the quasiparticle states, so the factor η governing the renormalization is given approximately by

$$\eta^{I}_{pp'} \approx 1 \pm \frac{J\Delta E}{\rho^{I}_{pp}\langle - | I^2 | - \rangle} = 1 \pm \frac{\Delta E}{\rho^{I}_{pp}\Delta\tilde{E}}. \qquad (3.10)$$

Here ΔE is the energy spacing between bands coupled by Coriolis forces, and the combination having the dimensionality of energy is $\Delta\tilde{E} = \langle - | I^2 | - \rangle/2J$. The sign of the correction is negative for the lower of the two coupled levels and positive for the upper. Finally, Eq. (A.21) shows that in this approximation we also find $\mathcal{H}$ terms coupling states with $K_{p'} = K_p \pm 2$.

The need to introduce a renormalization in the Hamiltonian for the model of a deformed nucleus with Coriolis mixing of nonrotational states has been mentioned frequently in the literature. It was shown in ref. 40, e.g., that the energies of states of the ground rotational band of ^{169}Yb can be reproduced most accurately within the framework of the rotary model by reducing the Coriolis force to 0.8 of the renormalized value. Figure 1 shows the gap parameter Δ whose substitution into the rotary-model equations for this nucleus leads to an accurate reproduction of the energies of the $K^{\pi} = (7/2)^+$ ground band and of the position of the first levels of the $K^{\pi} = (5/2)^+$, $(3/2)^+$ bands for various values of the normalization parameters for the Coriolis forces.[39] This figure shows that the even–even core of the nucleus ^{169}Yb can be thought of as constant for all states of the rotational band ($\Delta \approx$ const) if we simply introduce an attenuation factor $\eta \sim 0.75$ for matrix elements $\mathcal{H}^{I}_{K,\,K\pm1}$. The analysis in ref. 21 gives us basis to assume that taking into account the residual interactions which mix known collective internal excitations with the single-quasiparticle phonon states does not significantly change these matrix elements.[2)] On the other hand, inclusion of the overlap matrix ρ^{I} on the basis of approximate equations (A.18) and (A.20) gives the correct order of magnitude for the Coriolis-force attenuation coefficients (Table 1).

4. MICROSCOPIC QUANTUM THEORY OF NUCLEAR ROTATIONAL BANDS

There are several versions of the theory based on the equations of motion for operators acting in the space of states of the rotational bands. Equations were written in ref. 41 for the fractional parentage coefficients relating states of nuclei having approximately the same A. This theory is based on the assumption that only those coefficients coupling states of rotational bands based on single-quasiparticle (phononless) states are significantly different from zero. A similar approach was used in refs. 42 and 43, where the matrices of simple operators (in particular, the quadrupole moment), written in the eigenbasis of the nucleus, were limited to the space of states of a single rotational band.

This procedure unambiguously fixes the matrix structure of all the electromagnetic multipoles $m(\lambda)$ in the space of rotational-band states. The dependences of the reduced matrix elements $\langle I_i \| m(\lambda) \| I_f \rangle$ on I_i and I_f turn out to be precisely the same as those in Eq. (1.6) with $K_i = K_f = 0$ or $2K_i > \lambda$. Accordingly, the formalism of ref. 43 seems inadequate for describing deviations from the adiabatic limit of the unified model.

The rotational band ($I = K, K+1, \ldots; -I \le M \le I$) was described in refs. 44 and 45 by introducing roton operators, whose matrices in the eigenbasis $|IM\rangle$ coincide with the matrices of spherical harmonics in the basis of normalized generalized spherical harmonics:

$$R^{+}_{lm} = \sum_{\substack{I_1M_1\\ I_2M_2}} [(2I_1+1)(2I_2+1)]^{1/2} \times \begin{pmatrix} l & I_1 & I_2 \\ 0 & K & -K \end{pmatrix} \begin{pmatrix} l & I_1 & I_2 \\ m & M_1 & -M_2 \end{pmatrix} (-1)^{M_2-K} |I_2M_2\rangle\langle M_1I_1|. \qquad (4.1)$$

Equation (4.1) yields relations for Hermitian conjugation, commutation with the angular-momentum operator I, and multiplication of operators R^{+}_{lm}. It also shows that the R^{+}_{lm} commute with each other. These properties are identical to the corresponding properties of matrices of spherical harmonics (Appendix 2). These relations can be used to construct all possible operators R^{+}_{lm} if a single operator $R^{+}_{l_1m_1}$ (with odd l_1) is known. If operator $R^{+}_{l_1m_1}$ is known, but l is even, these rules allow us to determine all the other R^{+}_{lm} with even l. The transformation of R^{+}_{lm} with spatial (P) and time (τ) inversions is governed by properties of the states in (4.1) discussed in the preceding

TABLE 1. Renormalization of the Parameter Corresponding to the Moment of Inertia, J_0/J, and the Parameter Corresponding to the Coriolis Forces, $\eta^{I}_{pp'}$, for the Ground States of the Rotational Bands of ^{169}Yb According to Eqs. (A.18) and (A.20)

IK	J_0/J	$K_pK_{p'}(I)$	$\eta^{I}_{pp'}$
7/2, 7/2	0.895	7/2, 5/2 (7/2)	0.556
5/2, 5/2	0.995	5/2, 3/2 (5/2)	0.840
3/2, 3/2	1.000	3/2, 1/2 (3/2)	0.895
1/2, 1/2	1.000	1/2, 1/2 (1/2)	1.000

section:

$$PR^{+}_{lm}P^{-1}=R^{+}_{lm};\quad \tau R^{+}_{lm}\tau^{-1}=(-1)^{l}R_{lm}. \tag{4.2}$$

The effect of Hamiltonian H on the states of one rotational band ($|-\rangle$) is equivalent to the effect of some operator $h(\mathbf{I}^2)$ which depends on only the operator representing the square of the angular momentum, such that we have $h[I(I+1)] = E_I$, where E_I are the energies of the band states:[45]

$$[H-h(\mathbf{I}^2)]\,|-\rangle=0. \tag{4.3}$$

The rotons of Eq. (4.1) transform the states $|\,\mathrm{IM}\,\rangle$, leaving them in the space of band states, and they contain the projection operator onto this space. We can therefore write

$$[H-h(\mathbf{I}^2),\ R^{+}_{lm}]=0. \tag{4.4}$$

We treat both R^{+}_{lm} and the "model Hamiltonian" $h(\mathbf{I}^2)$ in (4.4) as unknowns. Taking the matrix elements of (4.4) between eigenstates of the nucleus, we find a system of linear homogeneous equations for $\langle M_2 I_2 \mid R^{+}_{lm} \mid I_1 M_1\rangle$, which has a nontrivial solution under the condition

$$h(I_2(I_2+1))-h(I_1(I_1+1))=E_{I_2}-E_{I_1} \tag{4.5}$$

(E_I is the eigenenergy on one of the nuclear states with angular momentum I). The same system of equations, written in the basis of an arbitrarily chosen "simple" operator $H_0 = H - V$ leads to

$$(R^{+}_{l})_{kp}=\frac{(\vec{J}_l\cdot\mathbf{R}^{+}_{l})_{kp}}{E_k-E_p};$$

$$(\vec{J}_l\cdot\mathbf{R}^{+}_{l})_{kp}=[h(\mathbf{I}^2),\ \mathbf{R}^{+}_{l}]_{kp}$$

$$-\sum_{\lambda}{}'\{V_{k\lambda}(E_\lambda-E_p)^{-1}(\vec{J}_l\cdot\mathbf{R}^{+}_{l})_{\lambda p}-(\vec{J}_l\cdot\mathbf{R}^{+}_{l})_{k\lambda}(E_k-E_\lambda)^{-1}V_{\lambda p}\}. \tag{4.6}$$

Here $\mathbf{R}^{+}_{l}$ denotes the column with components R^{+}_{lm}, E_k are the eigenenergies of H_0, the prime on the summation symbol shows that terms with $\lambda = p(\lambda = k)$ are neglected in the first (or second) term in braces, and $V_{k\lambda}$ is the matrix element of the "interaction" $V = H - H_0$.

An iterative solution of system (4.6) for the commuting tensor operators R^{+}_{lm} and the parameters of the function $h(\mathbf{I}^2)$ can be found if among the states $|k\rangle$ there is a state such that

$$R^{+}_{lm}|0\rangle=\left\{\begin{array}{l} O(r_x^2,\ r_y^2) \\ -\frac{1}{2}\sqrt{l(l+1)}\,(r_x+ir_y) \\ 1+O(r_x^2,\ r_y^2) \\ \frac{1}{2}\sqrt{l(l+1)}\,(r_x-ir_y) \\ O(r_x^2,\ r_y^2) \end{array}\right\}|0\rangle\quad\begin{array}{l} m>1 \\ m=1 \\ m=0 \\ m=-1 \\ m<-1 \end{array} \tag{4.7}$$

where $\sum_k |r_x|^2_{k0} \ll 1$; $\sum_k |r_y|^2_{k0} \ll 1$.

There exist states $|\hat{\theta}=0\rangle$ and $|\hat{\theta}=\pi\rangle$ such that $R^{+}_{lm}|\hat{\theta}=0\rangle=\delta_{m0}|\hat{\theta}=0\rangle$ and $\hat{R}_{lm}|\hat{\theta}=\pi\rangle=(-1)^{l}\delta_{m0}|\hat{\theta}=\pi\rangle$; they are

$$|\hat{\theta}=0\rangle=\sum_{I=K,\ K+1,\ \ldots}(2I+1)^{1/2}|I,\ M=K\rangle;$$

$$|\hat{\theta}=\pi\rangle=\sum_{I=K,\ K+1}(2I+1)^{1/2}(-1)^{I+K}|I,\ M=-K\rangle.$$

The coefficients of the expansion of the many-fermion configurations in the deformed potential in terms of states with a fixed angular momentum are proportional to $(2I+1)^{1/2}$ for $I(I+1) < \langle|\,\mathbf{I}^2\,|\rangle$. It is therefore reasonable to begin under the assumption that relation (4.7) holds when applied to states $|0\rangle$ including a small number of configurations of the deformed potential.

To describe the ideal rotational band $h = a\mathbf{I}^2$ we use in (4.6) an approximate expression which follows from (4.7):

$$[h,\ R^{+}_{l,\ m=\pm1}]_{k0}\approx\mp a\sqrt{2l(l+1)}\,(I_{\pm1})_{k0}, \tag{4.8}$$

where $I_{\pm1}=\mp(1/\sqrt{2})\,I_{\pm}=\mp(1/\sqrt{2})\,(I_x\pm iI_y)$. We find moment of inertia $J = (2a)^{-1}$ from one of the commutation relations for $I_{m=\pm1}$ and $R^{+}_{l,m=\pm1}$ for the operators used to derive Eq. (4.8) [see Eq. (A.25)]:

$$[I_{+1},\ R^{+}_{l,\ m=-1}]_{00}=-\sqrt{l(l+1)/2}. \tag{4.9}$$

These expressions are reminiscent of the equations used in the cranking method and will lead to similar arguments about the Hamiltonian for Eq. (2.15) for the moment of inertia. There are differences: 1) in the improved definition of state $|0\rangle$; and 2) in the absence of restrictions on the matrix elements of operators R^{+}_{lm} in contrast with the cranking model, in which we are restricted to rotations in a fixed plane. Use of roton operators allows us to take into account the kinematics of three-dimensional rotations. That refinement of the cranking model taking account of the residual interactions[23-25] which is associated with the first of these differences was pursued in ref. 46 (see also Appendix 2), where it was shown that choosing state $|0\rangle$ and the quasiparticle-state energies according to Eq. (3) can significantly increase the predicted moments of inertia, bringing them closer to the experimental values. Inclusion of the more complicated dependence of $h(\mathbf{I}^2)$ on $\mathbf{I}^2$ has the opposite effect, slightly reducing the value of the first term in the expansion of $h(\mathbf{I}^2)$ in powers of $\mathbf{I}^2$ (see the second paper cited in ref. 5).

The operators $R^{+}_{1m} \equiv r_m$ and I can be thought of as generators of a transformation group in the space of states of one rotational band [group IO (3); ref. 47]. One of the two normalized states in the space of band states can be found from the other by combining spatial rotations and transformations:

$$|2\rangle=\exp(i\mathbf{a}\cdot\mathbf{r})\,|1\rangle. \tag{4.10}$$

It is easy to find a relation between the expectation values of the angular momentum ($\mathbf{M} = \langle|\,\mathbf{I}\,|\rangle$) and the energy ($E = \langle|\,H\,|\rangle$) in states $|1\rangle$ and $|2\rangle$:

$$\mathbf{M}_2=\mathbf{M}_1+\langle 1\,|\,\mathbf{r}\,|\,1\rangle\times\mathbf{a}; \tag{4.11}$$

$$E_2=E_1+i\mathbf{a}\langle 1\,|\,[h,\ \mathbf{r}]\,|\,1\rangle + \sum_{i,\ j=x,\ y,\ z}\frac{a_ia_j}{2}\langle 1\,|\,[r_i,\ [h,\ r_j]]\,|\,1\rangle+O(a^3). \tag{4.12}$$

Operator r thus serves as a "rotation generator," governing the change in the angular momentum in wave-packet transformations (4.10). If $h = \mathbf{I}^2/2J$, terms $O(a^3)$ vanish identically, and we find the energy to be

$$E_2=E_1+\frac{1}{2J}\mathbf{a}\langle 1\,|\,(\mathbf{I}\times\mathbf{r}-\mathbf{r}\times\mathbf{I})\,|\,1\rangle+$$

$$+\frac{1}{2J}\langle 1|(\mathbf{a}^2-(\mathbf{a},\mathbf{r})^2)|1\rangle. \qquad (4.13)$$

These arguments can be used to evaluate the importance of the second difference mentioned above between the roton formulation and cranking model. We assume state $|1\rangle$ is such that we have $\mathbf{I}_z|1\rangle = K|1\rangle$; substituting into (4.11) and (4.13) the vector $\mathbf{a} = (a, 0, 0)$, we find the energy of a nucleus rotating around the 0y axis with a moment of $\mathbf{M} = (0, M_y = a\langle 1|r_x|1\rangle, K)$:

$$E_2 = E_1 + \frac{M_y^2}{2J}\cdot\frac{\langle 1|(r_y^2+r_z^2)|1\rangle}{\langle 1|r_x|1\rangle^2}. \qquad (4.14)$$

The quantity $\frac{\langle 1|r_x|1\rangle^2}{\langle 1|(r_y^2+r_z^2)|1\rangle}J = J'$ is the moment of inertia determined from the recipe of the cranking model. It is easy to see that $\frac{\langle 1|r_x|i\rangle^2}{\langle 1|(r_y^2+z_z^2)|1\rangle} < 1$ (if $\langle 1|\mathbf{I}^2|1\rangle < \infty$) i.e., the cranking model leads to some decrease in the moment of inertia because of kinematic features of three-dimensional rotation. Calculations neglecting the role of the residual interactions show that this decrease is only a few percent.

The description of the nonenergetic characteristics of rotational states was also studied in refs. 44 and 45. For an arbitrary tensor operator $\mathbf{F}_{LM}$ we can define a function f_{LM} of $\mathbf{R}^+_{lm}$ and $\mathbf{I}$ such that we have

$$\langle B|(F_{LM} - f_{LM}(R^+_{lm}, \mathbf{I}))|A\rangle = 0, \qquad (4.15)$$

if $|A\rangle$ and $|B\rangle$ are superpositions of states of one band [i.e., $\mathbf{F}_{LM}$ is equivalent to $(\doteq) f_{LM}$]. If there are only a few nonvanishing commutators of $\mathbf{F}_{LM}$ and $\mathbf{R}^+_{lm}$, we can write f_{LM} in the compact form

$$F_{LM} \doteq f_{LM} = \sum_{l,\, n_1,\, n_2} f(l, n_1, n_2)\,(\mathbf{I}^2)^{n_1}\{R^+_{L-l}, T_l\}_{LM}\,(\mathbf{I}^2)^{n_2}, \qquad (4.16)$$

where $f(l, n_1, n_2)$ are constants; tensor operator $\mathbf{T}_{lm}$ is defined by its component, $T_{ll} = (I_+)^l (T_1 = \mathbf{I})$; and $\{\mathbf{R}^+_{L-l}, \mathbf{T}_l\}_{LM}$ denotes vector coupling of the operators $\mathbf{R}^+_{L-l, m_1}$ and $\mathbf{T}_{lm_2}$:

$$\frac{1}{2}\{\mathbf{R}^+_{L-l}, \mathbf{T}_l\}_{LM} = \sum_{m_1 m_2}(L-l,\ l;\ m_1 m_2|LM)\,R^+_{L-l,\,m_1}T_{lm_2} = \sum_{m_1 m_2}(L-l,\ l;\ m_1 m_2|LM)\,T_{lm_2}R^+_{L-l,\,m_1}. \qquad (4.17)$$

In (4.16) we can now restrict the treatment to $n_1 = 0, 1$ or $n_2 = 0, 1$. This procedure corresponds to an alternative asymmetric form of Eq. (4.16) in which, e.g., all the $\mathbf{I}$ are the components on the right of $\mathbf{R}^+$, i.e., $n_1 = 0$, but in addition to

$$\{\mathbf{R}^+_{L-l}, \mathbf{T}_l\}_{LM} = \{\mathbf{R}^+_{L-l}, \mathbf{T}_l\}$$

we have the term $\mathbf{R}^+_{L-l+1}\mathbf{T}_l$, where $\underset{LM}{AB}$ denotes the coupling of the two tensor operators A and B up to an angular momentum L. This quantity is determined from an equation corresponding to the first equation in (4.17). Finally, we are using $\{A, B\} \equiv AB + BA$.

The matrix elements of $\mathbf{F}_{LM}$ between rotational-band states can be evaluated from the known coefficients $f(ln_1n_2)$ in (4.16) by the Racah-coefficient technique:[48]

$$\langle M_f I_f|F_{LM}|I_i M_i\rangle = \sum_{l,\, n_1,\, n_2} f(ln_1n_2)\,[I_f(I_f+1)]^{n_1}\,[I_i(I_i+1)]^{n_2} \times (-1)^{I_f-M_f}\begin{pmatrix} I_f & L & I_i \\ -M_f & M & M_i\end{pmatrix}\langle I_f\|\{\mathbf{R}^+_{L-l}, \mathbf{T}_l\}\|I_i\rangle; \qquad (4.18)$$

$$\langle M'I'|T_{lm}|IM\rangle = \delta_{I'I}(-1)^{I-M'}\,l!\,2^{-l/2}\left[\frac{(2I+l+1)!}{(2l)!\,(2I-l)!}\right]^{1/2}\begin{pmatrix} I & l & I \\ -M' & m & M\end{pmatrix}; \qquad (4.19)$$

$$\langle I_f\|\{\mathbf{R}^+_{L-l}, \mathbf{T}_l\}\|I_i\rangle = (-1)^{I_f-K}[(2I_f+1)(2I_i+1)]^{1/2}\,l! \times 2^{-l/2}(-1)^{I_f+I_i+L}\begin{pmatrix} I_f & L-l & I_i \\ -K & 0 & K\end{pmatrix}\left[\frac{(2L+1)!\,(2I_i+l+1)!}{(2l)!\,(2I_i-l)!}\right]^{1/2} \times \begin{Bmatrix} I_f & L & I_i \\ l & I_i & L-l\end{Bmatrix}. \qquad (4.20)$$

The first term in Eq. (4.18) ($l = n_1 = n_2 = 0$) is the unified-model expression for the matrix element of an electric-multipole operator[13] (for $K = 0$ or $L < 2K$), renormalized because of the value of the coefficient $f(000)$ (see the discussion below).

These equations are of practical interest when there are no more than two commutators of $\mathbf{F}_{LM}$ and $\mathbf{R}^+_{lm}$ which are significantly different from zero. The Hermitian operators of this type are equivalent $(\doteq)$ to one of the following expressions:

$$F^{(1)}_{LM} \doteq c_0 R^+_{LM} + \frac{c_2}{2}\{R^+_{LM}, \mathbf{I}^2\} + \frac{c_3}{2}\{\mathbf{R}^+_{L-2}, \mathbf{T}_2\}_{LM} + \frac{c_4}{2}\{\mathbf{R}^+_{L-1}, \mathbf{T}_1\}_{LM}; \qquad (4.21)$$

$$F^{(2)}_{LM} \doteq \frac{c_1}{2i}[R^+_{LM}, \mathbf{I}^2]. \qquad (4.22)$$

Here $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ behave differently with respect to time inversion.

Equation (4.21) (L = 1) can be used to analyze the matrix elements of the magnetic dipole moment:

$$\mu \doteq \sqrt{\frac{3}{4\pi}}\cdot\frac{e\hbar}{2Mc}(g_r\mathbf{I} + g_{tr}\mathbf{r}),\ (r_m = R^+_{1m}). \qquad (4.23)$$

The coefficients $f(ln_1n_2)$ in the general equation (4.19) and the parameters g_r and g_{tr} in our example, (4.23), can be evaluated by exploiting the commutation and normalization properties of the roton operators and exploiting the explicit form of their matrix elements in an arbitrary basis. From (4.23) we easily find the relations

$$\left.\begin{aligned}&\langle -|[\mu_x, r_y]|-\rangle = i\sqrt{\frac{3}{4\pi}}\cdot\frac{e\hbar}{2Mc}g_r\langle -|r_z|-\rangle;\\ &\left\langle -\left|\left(\mu_z - \sqrt{\frac{3}{4\pi}}\cdot\frac{e\hbar}{2Mc}g_r I_z\right)\right|-\right\rangle = g_{tr}\langle -|r_z|-\rangle.\end{aligned}\right\} \qquad (4.24)$$

The simplest approximations for the matrix elements of the roton operators, discussed above and used in (4.24), lead to the familiar equations of the unified model for g_r and g_{tr} [see Eq. (6.7) in ref. 13].

Let us consider in slightly more detail the structure of the matrix elements of an operator having transformation properties like those of the electric multipole moment in the space of states of the K = 0 band (I = 0, 2, 4, ...). In this case we use Eq. (4.21) (L = 2, 4, ...), assuming $c_4 = 0$,

since the operator for c_4 couples states of quantum number 1 differing in parity. We can determine c_2 and c_3 from

$$\left.\begin{aligned}\langle -|[[F_{l_1,0}, R^+_{l_2,1}], R^+_{l_3,-1}]|-\rangle &\approx [l_2(l_2+1)\,l_3(l_3+1)]^{1/2}\\ &\times\left\{c_2+c_3(-1)^{l_1+1}\left[\frac{(2l_1-4)!}{(2l_1)!}\right]^{1/2} l_1(l_1-1)\right\};\\ \langle -|[[F_{l_1-2}, R^+_{21}], R^+_{l31}]|-\rangle &\approx [l_2(l_2+1)\,l_3(l_3+1)]^{1/2}\\ &\times c_3(-1)^{l_1}\frac{1}{2}\left[\frac{(l_1+1)(l_1+2)}{(2l_1-1)(2l_1-3)}\right]^{1/2},\end{aligned}\right\} \quad (4.25)$$

which hold for $R^+_{lm}|-\rangle \approx \delta_{m0}|-\rangle$ and which permit a numerical analysis based on very simple approximations regarding the structure of the roton operators. Knowing c_2 and c_3, we can determine c_0 from, e.g.,

$$c_0=\frac{1}{2}\left\langle -\left|\sum_m (-1)^m\{F_{Lm}, R^+_{L,-m}\}\right|-\right\rangle - c_2\langle -|\mathbf{I}^2|-\rangle + c_3\sqrt{\frac{2L+1}{6}}\begin{pmatrix} L-2 & 2 & L\\ 0 & 0 & 0\end{pmatrix}\langle -|\mathbf{I}^2|-\rangle. \quad (4.26)$$

Equation (4.26) replaces, in the roton scheme, the unified-model expression for the matrix element of an operator in the internal coordinate system:

$$c_0 \quad \text{(unif. model)} \quad = \langle K=0 \mid F_{L0} \mid K=0\rangle. \quad (4.27)$$

From (4.7) we see that the discrepancy between (4.26) and (4.27) must increase with increasing multipole order of the operator L.

The occurrence of coefficients c_2 and c_3 in the expansion of the operator corresponding to a physical observable in series (4.16) can be interpreted in the unified model as a result of the mixing of states differing in K (refs. 13, 14). The differences between these values of c_0, c_2, and c_3, on the one hand, and those for the unified model in the microscopic approach, on the other, are due to the refinement of the internal state.

For large angular-momentum quantum numbers of the states between which the transition occurs, the corrections to the rules of the adiabatic theory can be found in a model combining ideas of the projection method and the cranking method.[38] Coefficients c_2, c_3, etc., in Eqs. (4.21) and (4.22) can be interpreted as derivatives of the matrix elements of the operators, referred to the internal axes, with respect to the collective angular momentum M. We find a correspondence between this formalism and the results of ref. 38 by using definition (4.10) for the state in which an average over the angular momentum can take on arbitrary values, in correspondence with Eq. (4.11). For an operator with quadrupole symmetry we find

$$\left.\begin{aligned}c_2 &= (1/\sqrt{6})\,X\,(\partial^2/\partial M_y^2\langle M_y \mid F_{22}\mid M_y\rangle)_{M_y=0}\\ &+ (1/2)\,Y\,(\partial^2/\partial M_y^2\langle M_y\mid F_{20}\mid M_y\rangle)_{M_y=0};\\ c_3 &= -Z\,(\partial^2/\partial M_y^2\langle M\mid F_{22}\mid M\rangle)_{M_y=0}\\ &+ W\,(\partial^2/\partial M_y^2\langle M\mid F_{20}\mid M\rangle)_{M_y=0},\end{aligned}\right\} \quad (4.28)$$

where $|M_y\rangle = \exp[-M_y(\hat{r}_x/\langle 0|r_0|0\rangle)]$, and the coefficients X, Y, Z, W, which have the same structure for all operators F_{2m}, are

$$X=\frac{\langle 0|r_0|0\rangle^2(2\zeta-1)}{\xi\zeta+(2\zeta-1)\varepsilon/2\sqrt{6}};\quad Y=\frac{\langle 0|r_0|0\rangle^2 J}{\xi\zeta+(2\zeta-1)\varepsilon/2\sqrt{6}};$$
$$Z=\frac{\langle 0|r_0|0\rangle^2}{\xi\zeta+(2\zeta-1)\varepsilon/2\sqrt{6}};\quad W=\frac{1}{2}\cdot\frac{\langle 0|r_0|0\rangle^2\varepsilon}{\xi\zeta+\varepsilon(2\zeta-1)/2\sqrt{6}};$$
$$\xi=\left\langle 0\left|R^+_{20}\frac{1+r_0^2}{2}\right|0\right\rangle;\quad \zeta=\langle 0|r_0^2|0\rangle;\quad \varepsilon=\langle 0|R^+_{2-2}r_+^2|0\rangle.$$

Retaining only the large components of the roton operators in these equations, we find $\xi \approx \zeta \approx 1$, $\varepsilon \approx 0$ and $X \approx Y \approx Z \approx 1$, $W \approx 0$, respectively. Equation (4.28) converts into expressions describing effects of second order in M (ref. 38).

For small angular-momentum quantum numbers of the states between which the transition occurs (I_i, I_f) the dependence of the transition matrix elements on I_i and I_f found in ref. 38 differs significantly from the predictions of the adiabatic theory. The rotary formalism allows us to find both the results of the adiabatic theory for small I and the results of ref. 38, which hold for $I \gg 1$. The high coefficients (X, Y, Z, W) in Eqs. (4.28) show the limited accuracy of the cranking model results, even at large I.

5. MICROSCOPIC THEORY OF THE COUPLING OF NUCLEAR ROTATIONAL BANDS

Such a theory can serve as an alternative method for describing the coupling of rotational motion with single-particle motion (see, e.g., Sec. 3) or the coupling of vibrational motion with rotational motion (see, e.g., ref. 19). On the other hand, this theory is required for deriving a microscopic theory of transition nuclei, for which coupling effects are particularly important.

In the last year or two, efforts to construct microscopic models for the transition range have been reported. Marshalek and Weneser[50] used a combination of the cranking model with boson expansions to describe quasirotational bands of nearly spherical nuclei. They tested this method for the case of a simple model describing the transition from a spherical nucleus to a deformed one. Holzwarth[51] combined the projection method with a boson expansion, obtaining a model for collective states in the transition range, also working from the spherical limit.

The theory of Sec. 4 has been generalized[45] especially to describe the coupling of rotational bands with respect to effects related to the transition probabilities between rotational bands, as well as with respect to energetic effects. The operators used here are more complicated:

$$B^+_{\alpha IM}=\sum_{\substack{I_1M_1\\ I_2M_2\\ n_in_\alpha}}[(n_\alpha+1)(2I_1+1)(2I_2+1)]^{1/2}\begin{pmatrix} I & I_1 & I_2\\ K & K_1 & -K_2\end{pmatrix} \times\begin{pmatrix} I & I_1 & I_2\\ M & M_1 & -M_2\end{pmatrix}(-1)^{M_2-K_2} \times|n_i,\ n_\alpha+1,\ I_2M_2\rangle\langle M_1I_1,\ n_\alpha,\ n_i|. \quad (5.1)$$

These operators couple states of two neighboring nuclei ($\Delta A = 1$, where I is a half-integer; or $\Delta A = 2$, where I is an integer) or of any single nucleus ($\Delta A = 0$, where I is an integer). They may be called "quasiparticle–roton" or "phonon–roton" operators. Here n_i (n_α) denotes the number of phonons[3)] of type i (α); and K_1, K_2 are the mini-

mum values of I_1, I_2 (the K's are numbers) of the rotational bands having n_i, n_α and n_i, $n_\alpha + 1$ phonons, respectively.

The most interesting property of operators (5.1) is that they are not independent for a given α, so that their commutator is

$$[B_{\alpha I_1 M_1},\ B^+_{\alpha I_2 M_2}] = \sum_L (2L+1) \times \begin{pmatrix} I_1 & I_2 & L \\ K & -K & 0 \end{pmatrix} \times \begin{pmatrix} I_1 & I_2 & L \\ M_1 & -M_2 & M_2 - M_1 \end{pmatrix} (-1)^{M_2 - K} R^+_{L,\, M_2 - M_1}. \quad (5.2)$$

Here definition (4.1) of rotons $R^+_{L,\, M_2 - M_1}$ is supplemented by a summation over the various rotational bands on the right side. The quantities R^+, B^+_α, B_α (for different α) commute. The commutators with I are the same as for any tensor operator (see Appendix 2).

Operators (5.1) can be expressed in terms of simpler operators, e.g., by using

$$B^+_{\alpha I_1 M_1} R^+_{L_2 M_2} = \sum_{IM} (2I+1) \begin{pmatrix} I_1 & L_2 & I \\ K & 0 & -K \end{pmatrix} \times \begin{pmatrix} I_1 & L_2 & I \\ M_1 & M_2 & -M \end{pmatrix} (-1)^{M-K} B^+_{\alpha IM}, \quad (5.3)$$

which, along with Eq. (A.23), allows us to reduce these operators to polynomials B^+_α, $I = |K|$, M and R^+_{1M} (or R^+_{2M}). We could also use factorization:

$$B^+_{\alpha IM} = \begin{cases} b^+_\alpha R^+_{I;\, K, M}, & |K_2| > |K_1|; \\ (-1)^{M-K} R_{I;\, -K, -M} b^+_\alpha, & |K_2| < |K_1|, \end{cases} \quad (5.4)$$

where

$$b^+_\alpha = \sum_{\substack{I'M' \\ n_i n_\alpha}} (n_\alpha + 1)^{1/2} |n_i,\ n_\alpha + 1,\ I'M'\rangle \langle M'I',\ n_\alpha,\ n_i| \quad (5.5)$$

are operators invariant with respect to rotations of the laboratory system and which represent phonon operators of the internal excitations; the "generalized-roton" matrices

$$R^+_{I;\, KM} = \sum_{\substack{I_1 M_1 \\ I_2 M_2 \\ \gamma}} [(2I_1+1)(2I_2+1)]^{1/2} \begin{pmatrix} I, & I_1 & I_2 \\ K & K_1 & -K_2 \end{pmatrix} \times \begin{pmatrix} I & I_1 & I_2 \\ M & M_1 & -M_2 \end{pmatrix} (-1)^{M_2 - K_2} |\gamma I_2 M_2\rangle \langle M_1 I_1 \gamma| \quad (5.6)$$

represent tensor operators having the same structure as the matrices D^I_{KM} of the generalized spherical harmonics. Accordingly, the operators $\frac{1}{2}(R^+_{I;\, K, M} + (-1)^{M-K} R^+_{I;\, -K, -M})$, which have suitable Hermitian-conjugation properties, may be thought of as the D functions of collective angles, giving a microscopic definition of these angles.

Any physical observable can be expressed in terms of tensor operators F_{LM}. For any operator F_{LM} there is a series expansion in powers of B^+_α, B_α, R^+, I. If we restrict the discussion to, e.g., spatial states of a single-phonon excitation, the operator F_{LM} is equivalent to the following expression (see ref. 45 for the more general case):

$$F_{LM} \doteq \sum_{\substack{\alpha\alpha' \\ l_1 l_2 n}} f(\alpha\alpha',\ l_1 l_2 n)\, B^+_{\alpha|K|M} B_{\alpha'|K'|M'} R^+_{l_1} I_{l_2} (I^2)^n. \quad (5.7)$$

$\varkappa = |K - K'|$; $\varkappa + l_1$; LM

The summation over l_1, l_2, and L is such that with $l_1 = 0$ the coupling is with any angular momentum L, while with $l_1 \neq 0$ the coupling is with the maximum (for the given l_1 and l_2) angular momentum $L_{max} = \varkappa + l_1 + l_2$ and with the angular momentum $L = L_{max} - 1 = \varkappa + l_1 + l_2 - 1$.

A different basis of operators may be more convenient for describing the transition to a spherical nucleus.[4] If we group the single-phonon states of α LM with different K, e.g., by requring that all convert into states of a given vibrational band of a spherical nucleus, and if we set $\alpha = \lambda K$ (where λ are the quantum numbers other than KIM), we can determine the operators

$$B^+_{\lambda IM} = \sum_{K=-I}^{I} B^+_{\alpha = \lambda K,\, IM}, \quad (5.8)$$

for which we can easily check the usual boson commutation relations, using (5.2). The new basis of operators is thus constructed from B^+_α, B_α, R^+, I, in terms of which we can expand any tensor operator F_{LM} as in (5.7). We see that we can also express the old operators $B^+_{\alpha IM}$ in terms of the new operators $B^+_{\lambda IM}$ and in terms of R^+_{lm} and I_μ, arranging coupling up to various angular momenta and using (5.3).

Since the actual Hamiltonian H given in (2.13) for the system is a zeroth-rank tensor operator, we can use for it an expansion in terms of B^+_α, B_α, R^+, I, as for F_{00}. In the particular case of a space of single-boson excitations we can use Eq. (5.7) with L = 0:

$$H \doteq h = \sum_{\alpha\alpha' n} h(\alpha\alpha',\ n)\, B^+_{\alpha|K|M} B_{\alpha'|K'|M'} I_\varkappa (I^2)^n, \quad (5.9)$$

$\varkappa = |K - K'|$; 00

where h can be thought of as the "model Hamiltonian" in this space of states with undetermined coefficients $h(\alpha\alpha', n)$. Considering the one rotational band $\alpha = \alpha'$, $\varkappa = 0$, we find $h \doteq h(I^2)$; we used this relation in Sec. 4. In the case of a spherical nucleus we have another particular case of (5.9):

$$h = \sum_{\lambda IM} \omega_{\lambda I} B^+_{\lambda IM} B_{\lambda IM}, \quad (5.10)$$

the harmonic-oscillator Hamiltonian. The Hamiltonian corresponding to harmonic oscillations of a spherical nucleus can be found from the general expression[45] for H = F_{00} in the space of multiphonon excitations.

The microscopic determination of any operator O^+_ν ($B_{\alpha IM}$ or R^+_{LM}) involves solving the generalized equations

TABLE 2. Deviations from the Branching Rules of the Unified Model in the Matrix Element of Operator F_{2M} between Rotational-Band States K_i, I_i, and $K_f = K$, $I_f = I$

I_i	$K_i = K$					
$I+2$	f_0	$-f_1\frac{2}{\sqrt{6}}(I+3)$	$+f_2(I+2)(I+3)$	$-$	$-$	$-$
$I+1$	f_0	$-f_1\frac{1}{\sqrt{6}}(I+4)$	$+f_2(I+1)(I+2)$	$-f_3\frac{1}{\sqrt{6}}\cdot\frac{I(I+2)}{K}$	$-f_4\frac{1}{6}\cdot\frac{I(I+2)(2I+5)}{K}$	$-$
I	f_0	$-f_1\frac{3}{\sqrt{6}}$	$+f_2 I(I+1)$	$+f_3\frac{K}{\sqrt{6}}\cdot\frac{4I(I+1)-3}{3K^2-I(I+1)}$	$-f_4\frac{K}{2}\cdot\frac{4I(I+1)-3}{3K^2-I(I+1)}$	$+f_5\frac{1}{\sqrt{6}}\times\frac{(4I(I+1)-3)I(I+1)}{3K^2-I(I+1)}$
$I-1$	f_0	$+f_1\frac{1}{\sqrt{6}}(I-3)$	$+f_2(I-1)I$	$+f_3\frac{1}{\sqrt{6}}\cdot\frac{I^2-1}{K}$	$+f_4\frac{1}{6}\cdot\frac{(I^2-1)(2I-3)}{K}$	$-$
$I-2$	f_0	$+f_1\frac{1}{\sqrt{6}}(I-2)$	$+f_2(I-2)(I-1)$	$-$	$-$	$-$

$$F_{2M}=f_0b_{00}R^+_{2M}+f_1b_{00}\underbrace{\mathbf{R}^+_2\mathbf{I}}_{2M}+f_2b_{00}R^+_{2M}\mathbf{I}^2+f_3b_{00}\underbrace{\mathbf{R}^+_1\mathbf{I}}_{2M}+f_4b_{00}\underbrace{\mathbf{R}^+_1\mathbf{I}_2}_{2M}+f_5b_{00}T_{2M}$$

$$1=f_0-\sqrt{6}f_1+6f_2,\quad a_1=\frac{1}{\sqrt{6}}f_1-f_2,\quad a_2=f_2,\quad a_3=\frac{1}{\sqrt{6}}f_3,\quad 0=f_4=f_5$$

I_i	$K_i = K\pm 1$				
$I+2$	f_0	$-f_1\frac{2}{\sqrt{6}}(I+3)$	$+f_2(I+2)(I+3)$	$-$	$-$
$I+1$	f_0	$-f_1\frac{1}{\sqrt{6}}(I+4)$	$+f_2(I+1)(I+2)$	$\mp f_3\frac{I(I+2)}{I\mp 2K}$	$\pm f_4\frac{1}{\sqrt{6}}\cdot\frac{I(I+2)(2I+5)}{I\mp 2K}$
I	f_0	$-f_1\frac{3}{\sqrt{6}}$	$+f_2 I(I+1)$	$+f_3\frac{4I(I+1)-3}{3(2K\pm 1)}$	$-f_4\frac{1}{\sqrt{6}}\cdot\frac{4I(I+1)-3}{2K\pm 1}$
$I-1$	f_0	$+f_1\frac{1}{\sqrt{6}}(I-3)$	$+f_2(I-1)I$	$\pm f_3\frac{I^2-1}{I\pm 2K+1}$	$\pm f_4\frac{1}{\sqrt{6}}\cdot\frac{(I^2-1)(2I-3)}{I\pm 2K+1}$
$I-2$	f_0	$+f_1\frac{2}{\sqrt{6}}(I-2)$	$+f_2(I-2)(I-1)$	$-$	$-$

$$F_{2M}=f_0\underbrace{\mathbf{b}_1\mathbf{R}^+_1}_{2M}+f_1\underbrace{\mathbf{b}_1\mathbf{R}^+_1\mathbf{I}}_{2M}+f_2\underbrace{\mathbf{b}_1\mathbf{R}_1\mathbf{I}^2}_{2M}+f_3\underbrace{\mathbf{b}_1\mathbf{I}}_{2M}+f_4\underbrace{\mathbf{b}_1\mathbf{I}_2}_{2M}$$

$$1=f_0+\frac{1}{\sqrt{6}}(\mp K-3)f_1\mp(\mp K+1)f_3,\quad a_1=\frac{3}{2\sqrt{6}}f_1\mp\frac{1}{2}f_3,\quad a_2=\frac{1}{2\sqrt{6}}f_1\pm\frac{1}{2}f_3,\quad 0=f_2=f_4$$

I_i	$K_i = K\pm 2$				
$I+2$	f_0	$-f_1\frac{2}{\sqrt{6}}(I+3)$	$+f_2(I+2)(I+3)$	$+f_3\frac{1}{2\sqrt{21}}(4I^2+26I+42)$	$+f'\sqrt{\frac{5}{14}}\delta_{K_f,\frac{1}{2}}(-1)^{I_f+1/2}\times\left(I_f+\frac{1}{2}\right)+f'\sqrt{\frac{5}{14}}\delta_{K_i,\frac{1}{2}}\times(-1)^{I_i+1/2}\left(I_i+\frac{1}{2}\right)$
$I+1$	f_0	$-f_1\frac{1}{\sqrt{6}}(I+4)$	$+f_2(I+1)(I+2)$	$+f_3\frac{1}{2\sqrt{21}}(-2I^2+3I+20)$	
I	f_0	$-f_1\frac{3}{\sqrt{6}}$	$+f_2 I(I+1)$	$+f_3\frac{1}{2\sqrt{21}}(-4I^2-4I+15)$	

TABLE 2 (continued)

$I-1$	f_0	$+f_1\frac{1}{\sqrt{6}}(I-3)$	$+f_2 I(I-1)$	$+f_3\frac{1}{2\sqrt{21}}(-2I^2-7I+15)$	$+f'\sqrt{\frac{5}{14}}\delta_{K_f,\frac{1}{2}}(-1)^{I_f+1/2}$ $\times\left(I_f+\frac{1}{2}\right)+f'\sqrt{\frac{5}{14}}\delta_{K_i,\frac{1}{2}}$ $\times(-1)^{I_i+1/2}\left(I_i+\frac{1}{2}\right)$
$I-2$	f_0	$+f_1\frac{2}{\sqrt{6}}(I-2)$	$+f_2(I-2)(I-1)$	$+f_3\frac{1}{2\sqrt{21}}(4I^2-18I+20)$	

$$F_{2M}=f_0 b_{2M}+f_1\underbrace{\mathbf{b}_2\mathbf{I}}_{2M}+f_2 b_{2M}\mathbf{I}^2+f_3\underbrace{\mathbf{b}_2\mathbf{I}_2}_{2M}+f'\underbrace{\vec{b}_3\mathbf{I}}_{2M}$$

$$1=f_0-\frac{1}{\sqrt{6}}(\pm 2K+5)f_1+\frac{3}{2}(2K^2\pm 10K+11)f_2;\quad a_1=\frac{1}{\sqrt{6}}f_1-\frac{3}{2}(\pm 2K+3)f_2;\quad a_2=3f_2=\frac{3}{\sqrt{21}}f_3;$$

$$a_{21}=a_{22}=\sqrt{\frac{5}{14}}f'$$

Note: Operator F_{2M} is shown by the parts of series (5.7) indicated beneath the corresponding parts of the table. Terms having powers of $\mathbf{I}$ no higher than the second and not containing[6] $\bar{b}_{1m}$, as well as the first term with $\bar{b}_{1m}=\bar{b}_{3m}$, with $K_i=K\pm 2$ are retained in (5.7). Each part of the table also shows the method for converting from our coefficients f_i in (5.7) to the parameters a_i of ref. 20. This conversion allows us to convert the equations of this table to the corrections in ref. 20 for the matrix elements of F_{2M} in the unified model.[7] The expressions in this table differ from the reduced matrix elements $\langle K_f I_f \| F_L \| I_i K_i\rangle$ $(L=2)$ [found from unified-model equation (1.6)] by a factor proportional to $[(2I_f+1)(2I_i+1)]^{1/2}(-1)^{I_f-K_f}\begin{pmatrix} I_f & L & I_i \\ K_f & K_f-K_i & K_i\end{pmatrix}$.

of motion,[45]

$$[H-h,\ O_\nu^+]=0, \tag{5.11}$$

which generalize the equations of the random-phase method[52] for oscillations [with h from (5.10) and $O_\nu^+=B_{\lambda IM}^+$] or (4.3) for rotations. The parameters h are determined from (5.11) or from auxiliary conditions giving the commutation relations, etc., for the operators O_ν^+. Equation (5.11) has been used to determine the phonon-excitation energy ω by the random-phase method for oscillations,[52] while auxiliary condition (4.9) has been used in the roton theory for rotations (Sec. 5 and Appendix 2).

The methods of this section can be used to construct branching rules for the probabilities for transitions due to multipole operator F_{LM}. The phenomenology of the rules, i.e., the procedure for evaluating matrix elements of F_{LM} between states of the rotational bands in terms of the known coefficients $f(\alpha\alpha', l_1 l_2 n)$ in (5.7), can be found by again using algebraic methods.[48] Accordingly, the equations for the corrections[20] to the transition probabilities of the unified model are reproduced, and some new equations are found. We have considered the cases $L=1$; $\Delta K=0,1,2,3,4$ and $L=2$; $\Delta K=0,1,2,3$ (ΔK is the difference between the K values of the two bands between which the transition occurs). As an example, we show in Table 2 equations[5] for $L=2$, $\Delta K=0,1,2$ and compare them with the analogous expressions from ref. 20. Accordingly, expansion of operator F_{LM} in (5.7) corresponds phenomenologically to taking account of the mixing of bands of different K which occurs in the rotary model.[20] However, Table 2 shows that the same result can be found on the basis of the same general symmetry considerations (without the use of a model) used to find expansion (5.7).

In the microscopic approach, the stage energies can be found by diagonalizing expansion (5.9) by algebraic methods after the parameters h are determined microscopically, as mentioned above. The transition probabilities can be calculated by first finding the coefficients f in expansion (5.7) by methods analogous to those used at the end of Sec. 4, expressing these coefficients in terms of the operators O_ν^+ [by analogy with (4.25) and (4.26)]; then we use the microscopic structure found for these coefficients from (5.11). This calculation incorporates fewer assumptions than the rotary model and leads to corrections to the unified-model probabilities; these corrections are illustrated in the particular case of a rotational band in Sec. 4. The calculation is easily carried out if the coefficients f fall off rapidly with l_2 and n. In the simplest case in which we can obtain only the term $l_2=n=0$ in (5.7), we find

$$\langle f\|F_L\|i\rangle=[(2I_f+1)(2I_i+1)]^{1/2}\sum_{M_f M M_i}(-1)^{I_f-M_f}\times\begin{pmatrix} I_f & L & I_i \\ -M_f & M & M_i\end{pmatrix}\langle|[B_{\alpha_f I_f M_f},\ [F_{LM},\ B_{\alpha_i I_i M_i}^+]]|\rangle, \tag{5.12}$$

where the state $|\ \rangle$ may include an arbitrary superposition of rotational-band states coupled by operators B_f^+ and B_i^+.

The effect of operator $B_{\alpha IM}^+$ on a state containing no phonon-excitation components ($b_\alpha|\cdot\rangle=0$), and for which the direction of the internal axis is fixed in a small region near $\theta=0$ (in the laboratory coordinate system), can be found by factorization of this operator in Eq. (5.4) and by exploiting the analogy between roton operators $R_{I,KM}^+$ and the generalized spherical harmonics. Assuming that to each type of excitation (α, $K\neq 0$) there correspond two different operators $B_{\alpha IM}^+$, which we distinguish by the sign of K, we write

$$B_{\alpha IM}^+|\rangle\approx\frac{1}{\sqrt{2(1+\delta_{K0})}}(\delta_{MK}|K\rangle+(-1)^{I+K}\delta_{M,-K}|\bar{K}\rangle), \tag{5.13}$$

where

$$|K\rangle=\sqrt{\frac{2}{1+\delta_{K0}}}B_{\alpha KK}^+|\rangle,\quad |\bar{K}\rangle=\sqrt{\frac{2}{1+\delta_{K0}}}B_{\bar{\alpha}K,-K} \tag{5.14}$$

(here $\bar{\alpha}$ denotes the substitution $K\to\bar{K}=-K$).

Combining (5.12) and (5.13), we find the unified-model equation derived in Sec. 1 [Eq. (1.6)]. This equation can thus be found under the approximations stated above in the strong-deformation limit ($\theta \approx 0$).

Appendix 3 illustrates the derivation of the microscopic equations for the parameters h and the operators O_ν^+ in the case of quasirotational bands in the transition region.

APPENDIX 1

DERIVATION OF THE ROTARY MODEL AND RENORMALIZATION OF THE INITIAL PARAMETERS OF THE CORE

We begin from the Schrödinger equation for the state $|\nu IM\rangle$:

$$H|\nu IM\rangle=\sum_p c_p(\nu I)\,P^I_{MKp}\{[H,\Omega_p]+\Omega_p h(I^2)\}|-\rangle=E_{\nu I}|\nu IM\rangle \quad (A.1)$$

[here we have used Eq. (4.3), according to which H and $h(I^2)$ act identically within a rotational band]. We introduce the following notation: a) for the commutators,

$$[H,\Omega_p]=\sum_{p'}\mathcal{H}^0_{p'p}\Omega_{p'}; \quad (A.2)$$

$$[I_\pm,\Omega_p]=\sum_{p'}(j_\pm)_{p'p}\Omega_{p'},\quad ([I_z,\Omega_p]=K_p\Omega_p); \quad (A.3)$$

$$\sum_{\kappa=x,y,z}[I_\kappa,[I_\kappa,\Omega_p]]=\sum_{p'}(j^2)_{p'p}\Omega_{p'}; \quad (A.4)$$

b) for the overlap integrals of the projected many-nucleon wave functions,

$$\langle-|\Omega^+_{p_2}P^I_{K_{p_2}K_{p_1}}\Omega_{p_1}|-\rangle$$
$$=\rho^I_{p_2p_1}=\rho^{I*}_{p_1p_2}=\frac{2I+1}{2}\int_0^\pi \sin\beta\,d\beta D^{I*}_{K_{p_2}K_{p_1}}(0,\beta,0)$$
$$\times\frac{\langle-|\Omega^+_{p_2}e^{-i\beta I_y}\Omega_{p_1}|-\rangle}{\langle-|\exp(-i\beta I_y)|-\rangle}\langle-|\exp(-i\beta I_y)|-\rangle. \quad (A.5)$$

We note the following properties of the projection operators P^I_{MK}:

$$\left.\begin{aligned}&P^I_{MK}I_z=KP^I_{MK};\\&P^I_{MK}I_\pm=\sqrt{(I\pm K)(I\mp K+1)}\,P^I_{MK\mp1}.\end{aligned}\right\} \quad (A.6)$$

Using (A.6) and notation (A.3), (A.4), we find

$$P^I_{K_{p_2}K_{p_1}}\Omega_{p_1}I^2=\sum_{p'}\mathring{X}^I_{p'p_1}P^I_{K_{p_2}K_{p'}}\Omega_{p'}; \quad (A.7)$$

$$P^I_{K_{p_2}K_{p_1}}\Omega_{p_1}h(I^2)=\sum_{p'}X^I_{p'p_1}P^I_{K_{p_2}K_{p_1}}\Omega_{p'}, \quad (A.8)$$

where $\mathring{X}^I$ is given by (3.7), and X^I satisfies matrix equation (3.6) and can be determined from it by numerical methods if the energies E_I differ appreciably from the energies of the rigid rotor,[39] or by approximate analytic methods otherwise.

Using the equations given above, we can derive Eqs. (3.4) of the main text for $C(\nu I)$. A variational principle is used in the text to obtain Eqs. (3.9).

Orthogonality of the various states in (3.1) and (A.1) corresponding to different I and M is ensured by the properties of the projection operators P^I_{MK}. The orthogonality of states having identical I and M follows from (3.4) and leads to conditions on $C(\nu I)$:

$$(E_{\nu_1 I}-E_{\nu_2 I})\,C^+(\nu_1 I)\,\rho^I C(\nu_2 I)=0. \quad (A.9)$$

We also note the condition under which both equations in (3.4) have the same solution:

$$[\rho^I,\mathcal{H}^I]=0. \quad (A.10)$$

The commutativity of ρ^I and $\mathcal{H}^I$ follows from the rotational symmetry of the Hamiltonian H.

The appearance of the overlap matrix of deformed configurations ρ^I in Eqs. (3.4) and (3.9) distinguishes this model from the model having a similar physical content but based on the approximation by deformed states of internal nuclear states. The role of matrix ρ^I in Eqs. (3.4) for the vector $C(\nu I)$ with an infinite number of components reduces primarily to one of taking into account a possible linear relation among states $P^I_{MK_p}\Omega_p|-\rangle$. When the number of $|\nu IM|$ components is bounded (equal to, say, N), i.e., when we have $\sum_{p'=1}^{N}(\rho^I_{p_1p'}\mathcal{H}^I_{p'p_2}-\mathcal{H}^I_{p_1p'}\rho^I_{p'p_2})\neq0$ and when the two terms on the left side of Eq. (3.9) differ, the solution of (3.9) is affected by ρ^I. The presence of matrix ρ^I guarantees the orthogonality of different solutions of (3.9) in the sense of (A.9).

We can evaluate the matrix elements of ρ^I by approximating the integrand in (A.5) in the following manner:

$$\langle-|\exp(-i\beta I_y)|-\rangle\approx\exp[-1/4\langle-|I^2|-\rangle\beta^2]; \quad (A.11)$$

$$\frac{\langle-|\Omega^+_{p_2}\exp(-i\beta I_y)\Omega_{p_1}|-\rangle}{\langle-|\exp(-i\beta I_y)|-\rangle}\approx\delta_{p_2p_1}-i\beta\langle-|\Omega^+_{p_2}I_y\Omega_{p_1}|-\rangle+\frac{\beta^2}{2}$$
$$\times\{\delta_{p_2p_1}\langle-|I_y|-\rangle-\langle-|\Omega^+_{p_2}I^2_y\Omega_{p_1}|-\rangle\}; \quad (A.12)$$

$$D^I_{MM'}(0,\beta,0)\approx[1-\beta^2(I(I+1)/4-M^2)]\,\delta_{MM'}$$
$$+\frac{\beta}{2}\sum_{\sigma=\pm1}\sigma[(I+\sigma M')(I-\sigma M'+1)]^{1/2}\delta_{M,M'\mp\sigma}. \quad (A.13)$$

[It is assumed in Eq. (A.12) that the operators Ω_p are normalized according to the condition $\langle-|\Omega^+_{p_2}\Omega_{p_1}|-\rangle=\delta_{p_2p_1}$].

The leading terms of the matrix ρ^I with these approximations turn out to differ by a general factor

$$\rho^I_{p_2p_1}=\delta_{p_2p_1}-\frac{I(I+1)-2K^2_{p_1}+(j^2)'_{p_2p_1}}{\langle-|I^2|-\rangle}\delta_{K_{p_2}K_{p_1}}$$
$$+\frac{\sqrt{(I-K_{p_1})(I+K_{p_1}+1)}}{\langle-|I^2|-\rangle}(j_+)'_{p_2p_1}\delta_{K_{p_2},K_{p_1}+1}$$
$$+\frac{\sqrt{(I+K_{p_1})(I-K_{p_1}+1)}}{\langle-|I^2|-\rangle}(j_-)'_{p_2p_1}\delta_{K_{p_2},K_{p_1}-1}. \quad (A.14)$$

Here we have used

$$(j_\pm)'_{p_2p_1}=\langle-|\Omega^+_{p_2}I_\pm\Omega_{p_1}|-\rangle; \quad (A.15)$$

$$(j^2)_{p_2p_1}=\langle-|\Omega^+_{p_2}I^2\Omega_{p_1}|-\rangle-\delta_{p_2p_1}\langle-|I^2|-\rangle. \quad (A.16)$$

The expressions in Eqs. (A.15) and (A.16) can be related in a simple manner with the corresponding quantities in (A.3) and (A.4); the two sets of quantities are equal if $\Omega_p|-\rangle$ is a single-quasiparticle state.

We can see the role of the overlap matrix in Eq. (3.9) in the case in which $\mathscr{H}^0_{pp'}=\varepsilon_p\delta_{pp'}$ and $X^I_{pp'}=\mathring{X}^I_{pp'}/2J_0$ [see Eq. (3.7)]. (In this approximation, neither the terms of the residual interaction in the Hamiltonian nor the nonadiabatic corrections to the energies of the core rotational states are taken into account.) The matrix elements of the matrix $\mathscr{H}^{vI}$ under these assumptions are: a) for p = p',

$$\mathscr{H}^{vI}_{pp'}=\varepsilon_p+[I(I+1)-2K_p^2+(j^2)_{pp}]/2\tilde{J}, \tag{A.17}$$

where

$$\frac{1}{\tilde{J}}=\frac{1}{J_0}\left\{1-f^{(+)}_{IK_p}\frac{(j_+)_{p,\,p-1}(j_-)_{p-1,\,p}}{\rho^I_{pp}\langle-|I^2|-\rangle}-\frac{(j_-)_{p,\,p+1}(j_+)_{p+1,\,p}}{\rho^I_{pp}\langle-|I^2|-\rangle}f^{(-)}_{IK_p}\right\};$$

$$f^{(\pm)}_{IK_p}=1+\frac{\pm K_p+K_p^2-(j^2)_{pp}}{I(I+1)-2K_p^2+(j^2)_{pp}}; \tag{A.18}$$

b) for $K_p = K_{p'} \pm 1$

$$\mathscr{H}^{vI}_{pp'}=-\frac{1}{2J_0}\sqrt{(I\mp K_{p'})(I\pm K_{p'}+1)}\,(j_\pm)_{pp'}\eta^I_{pp'}, \tag{A.19}$$

where

$$\eta^I_{pp'}=1-\frac{J_0(\varepsilon_p+\varepsilon_{p'}-2E_{vI})+I(I+1)-K_p^2-K_{p'}^2+\frac{1}{2}[(j^2)_{pp}+(j^2)_{p'p'}]}{\rho^I_{pp}\langle-|I^2|-\rangle}; \tag{A.20}$$

c) for $K_p = K_{p'} \pm 2$

$$\mathscr{H}^{vI}_{pp'}=-\frac{1}{2J_0}[(I\mp K_{p'}+1)(I\mp K_{p'})(I\pm K_{p'}+1)(I\pm K_{p'}+2)]^{1/2}\times\frac{(j_\pm)_{p,\,p'\pm1}(j_\pm)_{p'\pm1,\,p}}{\rho^I_{pp}\langle-|I^2|-\rangle}. \tag{A.21}$$

APPENDIX 2

EXPRESSION FOR THE MOMENT OF INERTIA OF A DEFORMED NUCLEUS IN ROTON THEORY[8)]

A theory of nuclear rotation was formulated in ref. 44 in terms of roton operators. An expression was derived for the roton operators and for the moment of inertia of deformed nuclei, taking pairing into account.

1. Rotons and Moment of Inertia

a. Formulation of the roton theory. Rotons R^+_{lm} are defined in (4.1). They can be thought of as spherical harmonics of certain collective angles (θ, φ). The following basic properties of the rotons follow from their definition:

$$R_{l-m}=(-1)^m R^+_{lm}; \tag{A.22}$$

$$R^+_{l_1m_1}R^+_{l_2m_2}=\sum_{lm}(2l+1)\begin{pmatrix}l_1 & l_2 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}l_1 & l_2 & l\\ m_1 & m_2 & m\end{pmatrix}R_{lm}; \tag{A.23}$$

$$\sum_m R^+_{lm}R_{lm}=1,\quad R^+_{l=0,\,m=0}=1; \tag{A.24}$$

$$\left.\begin{aligned}&[I_+, R^+_{lm}]=\alpha^l_{m+1}R^+_{lm+1};\\ &[I_0, R^+_{lm}]=mR^+_{lm};\\ &[I_-, R^+_{lm}]=\alpha^l_mR^+_{lm-1}.\end{aligned}\right\} \tag{A.25}$$

here $\alpha^l_m=[(l+m)(l-m+1)]^{1/2}$; $I_0 = I_z$ and $I_\pm = I_z \pm iI_y$ are the components of the operator corresponding to the total nuclear angular momentum; and $I^2 = I_x^2 + I_y^2 + I_z^2$.

All the R^+_{lm} are combinations of powers of commuting "basis" rotons R^+_{1m} (m = 0, ± 1). To find their microscopic structure, i.e., to find their expansions in terms of creation and annihilation operators, we solve the system of equations found from (4.4):

$$\langle-|[Q,[H,R^+_{1m}]]|-\rangle=\langle-|[Q,[h,R^+_{1m}]]|-\rangle \tag{A.26}$$

for $h = E_k + I^2/2J$.

Transforming to the Cartesian components of the vector R^+_1:

$$\left.\begin{aligned}&R^+_1=R_1=-\frac{1}{\sqrt{2}}(R^+_{11}-R^+_{1-1});\\ &R^+_2=R_2=\frac{i}{\sqrt{2}}(R^+_{11}+R^+_{1-1});\\ &R^+_3=R_3=R^+_0;\ (x,y,z)\equiv(1,2,3),\end{aligned}\right\} \tag{A.27}$$

we can replace (A.25) by

$$[I_p, R^+_\mu]=i\varepsilon_{p\mu k}R^+_k, \tag{A.28}$$

and thus find

$$[I^2, R^+_\mu]=2(\vec{K}\cdot\mathbf{R}^+_1);\ \vec{K}=\begin{pmatrix}-1 & iI_3 & -iI_2\\ -iI_3 & -1 & iI_1\\ iI_2 & -iI_1 & -1\end{pmatrix} \tag{A.29}$$

(where k, p, μ take on the values 1, 2, 3). The quantities R_μ can be parametrized in the following manner:

$$R^+_\mu=\delta_{\mu3}+r_\mu, \tag{A.30}$$

where the r_μ make a small contribution (in comparison with unity) when the R^+_μ act on the state $|-\rangle$ in which one of the collective angles (θ) is approximately zero (corresponding to the Oz symmetry axis)[9)] (ref. 44).

b. Realization of roton operators in a quasiparticle basis. The Hartree–Fock–Bogolyubov method[14] gives us a natural basis for taking into account pairing correlations in nuclei. In this scheme the elementary (single-particle) excitations in nuclei are generated by quasiparticle operators specified by a canonical Bogolyubov transformation:

$$\alpha^+_\nu=u_\nu a^+_\nu-v_\nu a^+_{\tilde{\nu}},\ \text{etc.}, \tag{A.31}$$

where $\alpha^+_\nu(\alpha_\nu)$ are the usual creation (annihilation) operators for the particles, u_ν and u_ν ($u^2_\nu + v^2_\nu = 1$) are the coefficients in transformation (A.31), and $\tilde{\nu}$ is the state which is time-inverted with respect to ν ($u_{\tilde{\nu}}=u_\nu$, $v_{\tilde{\nu}}=v_\nu$, $a^+_{\tilde{\nu}}=-a^+_\nu$).

Vacuum operator α_ν is defined as $\alpha_\nu|\,\rangle\equiv0$, and the nuclear Hamiltonian in the Hartree–Fock–Bogolyubov basis is given by expression (2.13) ($\varepsilon_\mu \to E_\mu$).

We now express all quantities in Eqs. (A.26) in terms of the quasiparticle-creation and -annihilation operators. We assume that: a) the expansion of the quantities r_μ includes only terms which are bilinear in terms of the operators of the quasiparticles of the combination, and b) combinations of the type $\alpha^+\alpha$ can be nonvanishing, but they

can be eliminated from Eqs. (A.26) through a suitable choice of Q [see (A.33)]. Since these combinations do not appear in the expression for the moment of inertia, we will not consider them here. We note that they may turn out to be necessary for satisfying other conditions, e.g., conditions of the type[10] (A.24). For r_μ we thus find

$$\left.\begin{aligned} r_\mu &= \sum_{\lambda\nu} [(r_\mu)_{\lambda\nu}\, \alpha^+_\lambda \alpha^+_{\tilde\nu} + (r_\mu)^*_{\lambda\nu}\, \alpha_\nu \alpha_{\tilde\lambda}]; \\ (r_\mu)_{\lambda\nu} &= -(r_\mu)_{\nu\lambda},\ (r_\mu)_{\tilde\lambda\tilde\nu} = (r_\mu)^*_{\lambda\nu}, \qquad (\mu = 1, 2, 3), \end{aligned}\right\} \tag{A.32}$$

where we have taken into account the Hermitian-conjugation properties and τ parity of these operators (after singling out the odd-parity operator $\hat{r}^*$). We find the coefficients $(r_\mu)_{\lambda\nu}$ by solving Eq. (A.26). In these latter equations the vacuum state $|\ \rangle$ is used as an approximation for $|-\rangle$ (here the vacuum state is found as one of the solutions of the Hartree–Fock–Bogolyubov variational problem corresponding to the given band, although this state may not be the ground state of the nucleus). We write the operator Q as

$$Q = \alpha_i \alpha_j. \tag{A.33}$$

System (A.26) now completely determines the R operators. Since this system is inhomogeneous and has solutions for arbitrary values of the parameter J, we choose that value of J for which Eqs. (A.25), averaged over the state $|\ \rangle$, are satisfied.

c. Equations of the roton theory. Equations for the moment of inertia. Using the assumptions stated in the preceding section, we write the equations of system (A.26) in the following explicit form:

$$\begin{aligned} &(E_j + E_i)\,(r_1)_{ji} + \frac{1}{2}\sum_{\lambda\nu} [\overline{V}_{ji,\,\lambda\nu}\mathcal{U}_{ji}\mathcal{U}_{\lambda\nu} + 2\overline{V}_{j\tilde\nu,\,\tilde i\tilde\lambda}\mathcal{V}^0_{ji}\mathcal{V}^0_{\lambda\nu}]\,(r_1)_{\lambda\nu} \\ &\quad = \frac{i}{J}\left\{\frac{1}{2}(I_2)_{\tilde j\tilde i}\mathcal{V}^0_{ji} - \frac{1}{i}(r_1)_{ji} + \left(\sum_m (I_3)_{mm}\, v^2_m\right)(r_2)_{ji}\right. \\ &\qquad + \sum_\lambda [(I_3)_{j\lambda}\mathcal{U}_{j\lambda}(r_2)_{\lambda i} - (I_3)_{i\lambda}\mathcal{U}_{i\lambda}(r_2)_{\lambda j} \\ &\qquad \left. - (I_2)_{j\lambda}\mathcal{U}_{j\lambda}(r_3)_{\lambda i} + (I_2)_{i\lambda}\mathcal{U}_{i\lambda}(r_3)_{\lambda j}]\right\}. \end{aligned} \tag{A.34}$$

The analogous equations for the matrix elements of r_2 and r_3 can be found by cyclically permuting $(1 \to 2 \to 3)$ the indices of operators r and I; correspondingly, the first terms on the right sides of the expressions will be $-\frac{i}{2J}(I_1)_{\tilde j\tilde i}\mathcal{V}^0_{ji}$ and 0. Here we use the designations

$$\left.\begin{aligned} \mathcal{U}_{\alpha\beta} &= u_\alpha u_\beta + v_\alpha v_\beta; \\ \mathcal{V}^0_{\alpha\beta} &= u_\alpha v_\beta + v_\alpha u_\beta. \end{aligned}\right\} \tag{A.35}$$

As was mentioned above, the moment of inertia J is found from the requirement that Eqs. (A.25) hold on the average over the state $|\ \rangle$. Accordingly, by using, e.g., the commutator $[I_1, R_2^+]$, we find the following for the moment of inertia:

$$\sum_{\mu\nu} (I_1)_{\mu\nu}\,(r_2)_{\nu\tilde\mu}\mathcal{V}^0_{\nu\mu} = i/2. \tag{A.36}$$

This equation takes into account pairing correlations, polarization effects (the changes in the Hartree–Fock–Bogolyubov field due to rotation or due to residual interactions), and effects of the three-dimensional nature of the nuclear rotation. In the absence of pairing correlations, this equation converts into an equation found previously,[44] which takes into account only the three-dimensional effects and polarization effects. It will be shown in the next section that Eq. (A.36) is a generalization of several familiar equations, such as those found by Belyaev,[23] Thouless,[24] and of course, Inglis.[22]

2. Some Applications of the Theory

a. Moment of inertia neglecting the r_μ coupling but taking into account the residual interaction. System (A.13) is coupled by r_1, r_2, and r_3. The constituent equations are independent if we do not take into account all the terms other than the first on the right sides of the equations. Using simple transformations, we can convert these equations to the following form (e.g., for r_2):

$$\begin{aligned} &(E_j + E_i)\,(r_2)_{\tilde j i} + \sum_{\lambda\nu} [V_{\tilde j i,\,\tilde\lambda\nu}\mathcal{U}_{ji}\mathcal{U}_{\lambda\nu} + \overline{V}_{j\tilde\lambda,\,\tilde\nu i}\mathcal{V}^0_{ji}\mathcal{V}^0_{\lambda\nu}]\,(r_2)_{\lambda\nu} \\ &\qquad = \frac{i}{2J}(I_1)_{ji}\mathcal{V}^0_{ji}, \end{aligned} \tag{A.37}$$

where $\overline{V}_{ij,kl} = V_{ij,kl} - V_{ij,lk}$.

It is not difficult to see that the substitution

$$-\frac{2J}{i}(r_2)_{\tilde j i} = f_{ji}\omega^{-1} \tag{A.38}$$

converts Eq. (A.37) to one corresponding identically to the equations found by Belyaev [see Eq. (17) in ref. 23]. In (A.38) the f_{ji} are coefficients of the generalized canonical transformation, and ω is the angular velocity, all introduced by Belyaev.

b. Moment of inertia in the free-quasiparticle approximation. We can easily study the case in which the roton operators commute with terms of the residual interaction, because the terms containing $\overline{V}_{ij,kl}$ do not play a role in Eqs. (A.37). Here the solutions for the r operators are found in analytic form:

$$(r_2)_{\tilde j i} = \frac{i}{2J}\cdot\frac{(I_1)_{ji}\mathcal{V}^0_{ji}}{E_i + E_j},\ \text{etc.}; \tag{A.39}$$

Substituting them into (A.36), we find

$$J = \sum_{\mu\nu} \frac{|(I_1)_{\mu\nu}|^2}{E_\mu + E_\nu}\mathcal{V}^{02}_{\mu\nu}. \tag{A.40}$$

This equation[23] is the familiar Inglis equation for the moment of inertia of rotating deformed nuclei[22] in which pairing forces are explicitly taken into account.

The possible applications of Eq. (A.40) have been studied in detail (see, e.g., ref. 31). Having at our disposal the Woods–Saxon single-particle potential, an improvement over the Nilsson potential used earlier, we can carry out some illustrative calculations to find the effect of a change in the single-particle scheme on the results given by the Inglis equation. A more stringent criterion is used for determining the pairing-interaction constants (G_N, G_Z) than in ref. 31, namely, we seek the values of the pairing parameters which lead to better results for the pairing energy of the given nucleus. The calculations

TABLE 3. Comparison of the Calculated Moments of Inertia with the Experimental Moments and With the Data of Ref. 10 for Various Yb Isotopes

Isotope	J, MeV^{-1}			
	without correlation	with correlation	data of ref. 10	experimental
^{168}Yb	21.69	—	31.5	34.19
^{170}Yb	24.46	27.7	30.0	35.60
^{172}Yb	27.39	37.3	32.8	38.12

Note: The results found by Prior et al.[31] are satisfactorily reproduced if their procedure for choosing pairing parameters is used.

show that for nuclei far from the stability region the Inglis equation predicts moments of inertia smaller than those found by other, usually theoretical methods (see Table 3 for the nuclei ^{168}Yb, ^{170}Yb, and ^{172}Yb and the results calculated for deformation parameters $\beta_{20} = 0.300$ and $\beta_{40} = 0$).

c. Inclusion of rotational energy in the ground state. The results found above are based on the assumption that the state over which the averaging is carried out in (A.26) can be approximated by the quasiparticle vacuum state $|\ \rangle$, which minimizes the energy of the system described by the Hamiltonian H. It follows from the theory that we should have used the state corresponding to the condition

$$(H - E_k - \mathbf{I}^2/2J)\,|-\rangle = 0. \tag{A.41}$$

It is natural to approach the problem by a variational method and find the state $|0\rangle$ which minimizes the value of $\mathcal{H} = H - E_k - I^2/2J$, assuming $|-\rangle$ to be approximately identical to $|0\rangle$.

We assume that the $|0\rangle$ state is again the quasiparticle vacuum state, and we write $\mathcal{H}$ as

$$\mathcal{H} = \mathcal{H}_0 + \sum_i \varepsilon_i \alpha_i^+ \alpha_i + \mathcal{H}_{res}, \tag{A.42}$$

where $\mathcal{H}_{res}$ contains the normal product of four creation and annihilation operators. Then in the roton equations we have the Hamiltonian

$$H = \mathcal{H} + E_k + \mathbf{I}^2/2J = C + \sum_i \varepsilon_i \alpha_i^+ \alpha_i + \sum_{ij} h_{ij}^{(1,1)} \alpha_i^+ \alpha_j + \sum_{ij} h_{ij}^{(2,0)} \alpha_i^+ \alpha_j^+ + \ldots, \tag{A.43}$$

where the terms $h^{(1,1)}$, $h^{(2,0)}$, etc., have the structure

$$\left.\begin{aligned} h_{ij}^{(1,1)} &= (\langle 0 \mid \alpha_i h \alpha_j^+ \mid 0\rangle - \langle 0 \mid h \mid 0\rangle) \\ &= \frac{1}{2J} \sum_{l=1,2,3} \Big\{ \sum_\mu (I_l)_{i\mu} (I_l)_{\mu j} [2u_\mu v_\mu (u_i v_j + u_j v_i) \\ &\quad + (u_\mu^2 - v_\mu^2)(u_i u_j - v_i v_j)] + 2 \Big[\sum_\mu (I_l)_{\mu\mu} v_\mu^2 \Big] (I_l)_{ij} U_{ij} \Big\}; \\ h_{ij}^{(2,0)} &= \frac{1}{2} \langle 0 \mid \alpha_j \alpha_i h \mid 0\rangle \\ &= \frac{1}{4J} \sum_{l=1,2,3} \Big\{ \sum_\mu (I_l)_{i\mu} (I_l)_{\mu j} (u_{i\mu} \mathcal{V}^{\circ}_{j\mu} + u_{j\mu} \mathcal{V}^{\circ}_{i\mu}) \\ &\quad + 2 \Big[\sum_\mu (I_l)_{\mu\mu} v_\mu^2 \Big] (I_l)_{i\tilde{j}} \mathcal{V}^{\circ}_{ij} \Big\}. \end{aligned}\right\} \tag{A.44}$$

In solving system (A.26) with the new Hamiltonian (A.43) we must bear in mind that we cannot discard terms like $\alpha^+\alpha$ in the roton operator because of the terms $h^{(2,0)}$ in H, i.e., in this case

$$\left.\begin{aligned} r_\mu &= \sum_{\lambda\nu} [(r_\mu)_{\lambda\nu} \alpha_\lambda^+ \alpha_\nu^+ + (r_\mu)_{\lambda\nu}^* \alpha_\nu \alpha_\lambda + (r'_\mu)_{\lambda\nu} \alpha_\lambda^+ \alpha_\nu]; \\ (r'_\mu)_{\lambda\nu} &= (r'_\mu)_{\nu\lambda}^*. \end{aligned}\right\} \tag{A.45}$$

Accordingly, we replace Eqs. (A.34) with

$$\begin{aligned} &(\varepsilon_j + \varepsilon_i)(r_1)_{ji} + \sum_\lambda [h_{j\lambda}^{(1,1)}(r_1)_{\lambda i} - h_{i\lambda}^{(1,1)}(r_1)_{\lambda j}] + \sum_\lambda [h_{i\lambda}^{(2,0)}(r'_1)_{j\lambda} - h_{j\lambda}^{(2,0)}(r'_1)_{i\lambda}] \\ &\quad + \frac{1}{2} \sum_{\lambda\nu} [\overline{W}_{ji,\lambda\nu} \mathcal{U}_{ji} \mathcal{U}_{\lambda\nu} + 2\overline{W}_{\tilde{j}\nu,\tilde{i}\lambda} \mathcal{V}^{\circ}_{ij} \mathcal{V}^{\circ}_{\lambda\nu}](r_1)_{\lambda\nu} \\ &= \frac{i}{J} \Big\{ \frac{1}{2} (I_2)_{\tilde{j}\tilde{i}} \mathcal{V}^{\circ}_{ji} - \frac{1}{i} (r_t)_{ji} + \Big(\sum_m (I_3)_{mm} v_m^2 \Big) (r_2)_{ji} \\ &\quad + \sum_\lambda [(I_3)_{j\lambda} \mathcal{U}_{j\lambda} (r_2)_{\lambda i} - (I_3)_{i\lambda} \mathcal{U}_{i\lambda} (r_2)_{\lambda i} \\ &\quad - (I_2)_{j\lambda} \mathcal{U}_{j\lambda} (r_3)_{\lambda i} + (I_2)_{i\lambda} \mathcal{U}_{i\lambda} (r_3)_{j\lambda}] \Big\}, \end{aligned} \tag{A.46}$$

where $\overline{W}_{ij,kl}$ are the matrix elements of the two-particle interaction supplemented by the terms from h [see (A.43)]. The equations for r_2 (and r_3) are found by the rules in Section 2b.

Equations (A.46) were solved by assuming that all terms except the first on the right sides could be discarded, and only the contribution of the diagonal terms of $h^{(1,1)}$ and $h^{(2,0)}$ was taken into account. Here again we assumed $[W, R_\mu^+] = 0$, which eliminates the terms containing $\overline{W}_{ij,kl}$ on the left sides of the equations. Since $h^{(2,0)}$ has no diagonal part, some of the rotons $(r')_{\lambda\nu}$ drop out of the equations.

We can interpret the energies ε_i appearing in (A.42) and (A.46) in the following sense. We assume that $|0\rangle$ is a good approximation of state $|-\rangle$, which is a linear combination of states belonging to the given band of the given nucleus, and we assume that state $a_i^+|0\rangle$ is a good approximation of $|-\rangle_i$, which is a combination of states the rotational band constructed at single-quasiparticle level i in the neighboring odd nucleus, for which the moment of inertia is J_i. Then, using very simple assumptions regarding the nature of the rotational motion in the neighboring odd nucleus, we find

$$\varepsilon_i = E_i - \frac{1}{2J} K_i (K_i + 1) + \Big(\frac{1}{2J_i} - \frac{1}{2J} \Big) \langle 0 \mid \alpha_i h \alpha_i^+ \mid 0\rangle, \tag{A.47}$$

where K_i is the quantum number of angular-momentum projection on the symmetry axis in state i. The energies appearing in the roton equations and in the equation for the moment of inertia are thus related to the single-quasiparticle energies and rotational characteristics of the odd neighbor.

The consequences of the contribution of the rotational energy to the ground state have been evaluated for ^{170}Yb and ^{172}Yb. The moments of inertia found are shown in Table 3 in the column headed "with correlation."

APPENDIX 3

MICROSCOPIC MODEL FOR QUASIROTATIONAL BANDS[11]

To specify the model in which we will use the methods

described in Sec. 5 we must choose a model Hamiltonian h, i.e., cut off expansion (5.9). Here we should relax the requirements on operators O^+_{lm} (the requirements which govern the algebraic properties of these operators, found in Sec. 5), and simply assume that they are tensor operators and thus expressible in terms of B^+_α, B_α, R^+, I [see, e.g., (5.7)]. We then choose the operator h to find the phenomenological equation[53] for the energies E_I consisting of terms linear and quadratic in I and which are appropriate for the transition region. The corresponding operator satisfies the equation

$$[h,\ O^+_{lm}]=\omega_l O^+_{lm}+a\,[I^2,\ O^+_{lm}]. \tag{A.48}$$

Equation (A.48) generalizes the random-phase equation for vibrations (the first term on the right side) and the equations of roton theory for ideal rotation, (A.26) and (A.29) (the second term on the right).

The parameters of h which are to be determined are ω_l and $a = 1/2J$. For simplicity we describe the structure of O^+_{lm} in terms of particle and hole operators (pairing can be introduced by converting to Bogolyubov quasiparticles[14]):

$$O^+_{lm}=\delta_{m0}+\sum_{\underline{jK}}\left(Y^{lm}_{\underline{jk}}a^+_{\underline{j}}a_{\underline{K}}-Z^{lm}_{\underline{jK}}a^+_{\underline{K}}a_{\underline{j}}\right). \tag{A.49}$$

Here we are restricting the treatment to the same terms used in the random-phase method,[52] and we choose the free term on the basis of axial symmetry, as in (A.30). Normalization is achieved by arbitrarily fixing the value of this term for m = 0 at $\delta_{00} = 1$. (Here a bar above a state index indicates a particle state, while a bar below such an index indicates a hole state.)

We substitute (A.49) into the equations of motion found from (A.48) by replacing H by h. We assume that we have chosen a Hartree–Fock basis, not for H, but for $\mathscr{H} = H - a(I^2 - I_0^2)$. Then the energies in the single-particle term of $\mathscr{H}$ are not ε_i, but $\varepsilon_i - a\eta_i$, and the matrix elements of the residual interaction are not $V_{ij,Kl}$, but $V_{ij,Kl} - a\eta_{ij,Kl}$, where

$$\left.\begin{aligned}\eta_{il}&=\sum_K(2\delta_{K\bar{K}}-1)\,(I_{iK}I^*_{\bar{i}K}+I^*_{\bar{K}i}I_{Ki});\\ \eta_{ij,Kl}&=2\,(I_{iK}I^*_{\bar{l}j}+I^*_{\bar{K}i}I_{jl});\end{aligned}\right\} \tag{A.50}$$
$$I=I_{+1}=-\frac{1}{\sqrt{2}}(I_x+iI_y);\ \delta_{\underline{k}\bar{k}}=1;\ \delta_{\underline{k}\bar{k}}=0.$$

The equations for the coefficients Y and Z are

$$\left.\begin{aligned}&[\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}-\omega_l]\,Y^{lm}_{\underline{iK}}\\ &+\sum_{\underline{jl}}\left[(\bar{V}_{\underline{il},\,\underline{K\bar{j}}}-a\bar{\eta}_{\underline{il},\,\underline{K\bar{j}}})\,Y^{lm}_{\underline{jl}}+(\bar{V}_{\underline{i\bar{j}},\,\underline{Kl}}-a\bar{\eta}_{\underline{i\bar{j}},\,\underline{Kl}})\,Z^{lm}_{\underline{jl}}\right]=0;\\ &[\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}+\omega_l]\,Z^{lm}_{\underline{iK}}\\ &+\sum_{\underline{jl}}\left[(\bar{V}_{\underline{Kl},\,\underline{i\bar{j}}}-a\bar{\eta}_{\underline{Kl},\,\underline{i\bar{j}}})\,Y^{lm}_{\underline{jl}}+(\bar{V}_{\underline{K\bar{j}},\,\underline{i\bar{l}}}-a\bar{\eta}_{\underline{K\bar{j}},\,\underline{i\bar{l}}})\,Z^{lm}_{\underline{jl}}\right]=0.\end{aligned}\right\} \tag{A.51}$$

Here we have $\varepsilon_{iK} = \varepsilon_i - \varepsilon_K$ and $\eta^m_{iK} = \eta_i - \eta_K + m^2$, and the bar in the expressions $\bar{V}$ and $\bar{\eta}$ denotes antisymmetrized matrix elements.

To simplify the analysis of system (A.51) we adopt a model interaction of the type described by Brown,[54] and we make analogous simplifications in $\bar{\eta}_{ij,Kl}$, neglecting exchange terms:

$$\left.\begin{aligned}\bar{V}_{ij,\,Kl}&\approx-\sum_m\chi_m V^m_{iK}V^{-m}_{\bar{j}l};\\ \bar{\eta}_{ij,\,Kl}&\approx 2\,(I_{iK}I^*_{\bar{l}j}+I^*_{\bar{K}i}I_{jl}).\end{aligned}\right\} \tag{A.52}$$

We introduce the notation

$$\left.\begin{aligned}\tilde{Y}^{lm}_{\underline{\bar{j}l}}&=[l\,(l+1)]^{-1/2}\,Y^{lm}_{\underline{\bar{j}l}},\ \tilde{Z}^{lm}_{\underline{\bar{j}l}}=[l\,(l+1)]^{-1/2}\,Z^{lm}_{\underline{\bar{j}l}};\\ n^{lm}&=\sum_{\underline{\bar{j}l}}(V^{-m}_{\underline{i\bar{j}}}\tilde{Y}^{lm}_{\underline{\bar{j}l}}+V^{-m}_{\underline{\bar{j}l}}\tilde{Z}^{lm}_{\underline{\bar{j}l}}).\end{aligned}\right\} \tag{A.53}$$

Substituting (A.52) into (A.51), we find explicit expressions for the coefficients $\tilde{Y}$ and $\tilde{Z}$:

$$\left.\begin{aligned}\tilde{Y}^{lm}_{\underline{iK}}&=[\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}-\omega_l]^{-1}\\ &\times(\chi_m n^{lm}V^m_{\underline{iK}}+a\,(\delta_{m1}\,(I_+)_{\underline{iK}}+\delta_{m,-1}\,(I_-)_{\underline{iK}}));\\ \tilde{Z}^{lm}_{\underline{iK}}&=[\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}+\omega_l]^{-1}\\ &\times[\chi_m n^{lm}V^m_{\underline{K\bar{i}}}+a\,(\delta_{m1}\,(I_+)_{\underline{K\bar{i}}}+\delta_{m,-1}\,(I_-)_{\underline{K\bar{i}}})],\end{aligned}\right\} \tag{A.54}$$

where $I_\pm = I_x \pm iI_y$ and where we have used notation (A.53) and the commutation relation between O^+_{lm} and I [as for a tensor operator; see (A.25)].

To find ω_l and a, we substitute $\tilde{Y}$ and $\tilde{Z}$ from (A.54) into the expression in (A.53) for n^{lm},

$$\left.\begin{aligned}n^{lm}=\sum_{\underline{iK}}\Bigg[&\frac{V^{-m}_{\underline{K\bar{i}}}\,(\chi_m n^{lm}V^m_{\underline{iK}}+a\delta_{m,1}\,(I_+)_{\underline{iK}}+a\delta_{m,-1}\,(I_-)_{\underline{iK}})}{\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}-\omega_l}\\ &+\frac{V^{-m}_{\underline{iK}}\,(\chi_m n^{lm}V^m_{\underline{K\bar{i}}}+a\delta_{m,1}\,(I_+)_{\underline{K\bar{i}}}+a\delta_{m,-1}\,(I_-)_{\underline{K\bar{i}}})}{\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}+\omega_l}\Bigg],\end{aligned}\right\} \tag{A.55}$$

and into the commutation relation between $I_\pm$ and $O^+_{l,\pm1}$ [see (A.25)] (for $m = \pm1$):

$$1=\sum_{\underline{iK}}\Bigg[\frac{(I_\pm)_{\underline{K\bar{i}}}\,(\chi_m n^{lm}V^m_{\underline{iK}}+a\,(I_\mp)_{\underline{iK}})}{\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}-\omega_l}+\frac{(I_\pm)_{\underline{i\bar{K}}}\,(\chi_m n^{lm}V^m_{\underline{K\bar{i}}}+a\,(I_\mp)_{\underline{K\bar{i}}})}{\varepsilon_{\underline{iK}}-a\eta^m_{\underline{iK}}+\omega_l}\Bigg]. \tag{A.56}$$

We see that neglecting a in (A.55) converts this equation into the dispersion equation[54] for ω_l of the random-phase method with the Brown interaction. When ω_l is neglected in (A.56), roton-theory equation (A.46) also converts into (A.36), but with a simplified Brown interaction[54] and without inclusion of pairing. System (A.55), (A.56) thus contains both the deformed nucleus and the spherical limit. There are three unknown parameters in (A.54): ω_l, a, and n^{lm}, so this system can be written a second time for $O^+_{2l,\,m}=\underbrace{O^+_{l\mu}O^+_{l\mu'}}_{2l,\,m}$, for which we have $\omega_{2l} = 2\omega_l$. We find four equations of the type (A.55), (A.56) for l and $2l$ from which we can determine ω_l, a, n^{lm}, and $n^{2l,m}$.

In particular case of rotation ($\omega_l = 0$), Eq. (A.56) [with n^{lm} given in (A.56)] allows us to carry out a simple cal-

culation of the effect of long-range interaction V on moment of inertia $J = 1/2a$; this calculation has not been carried out previously.

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[1)] The equation for the mass of a system of A particles can be written by analogy with Eq. (2.12); here it is necessary to find the operator $\mathbf{R}^*$ and the parameter M^* (the mass) satisfying

$$\langle | [Q, [(H - \mathbf{P}^2/2M^*), \mathbf{R}^*]] | \rangle = 0. \qquad (2.18)$$

If the system is closed, certain commutation relations hold for the center-of-mass vector $\mathbf{R} = \sum_{i=1}^{A} m_i \mathbf{r}_i / \sum_{i=1}^{A} m_i$: $[H, R] = [\mathbf{P}^2/2M, R]$. The solution of Eq. (2.17) is thus $M^* = M = \sum_{i=1}^{A} m_i$, regardless of how state $|\ >$ is chosen.

[2)] In this paper we attenuate the Coriolis forces by including in matrix $j^2_{pp'}$ in (3.7) terms mixing three-quasiparticle components with the single-quasiparticle components. In light of the derivation of the model above we are forced to treat this modification of the theory as simply a formal one, devoid of a physical basis.

[3)] We are considering only the phonon case here; other cases require slight modification.

[4)] A paper on this subject is being prepared for publication.

[5)] The results shown in Table 2 were calculated by M. Kirkhbakh.

[6)] Here we have used the notation $b_{lm} = \varkappa_l \underbrace{B^+_\alpha B^+_{\alpha'}}_{lm}$ $(l = 0, 1, 2)$ and $\bar{b}_{3m} = \varkappa_3 \underbrace{B^+_\alpha B^-_{\alpha'}}_{3m}$ denotes the replacement $K' \to \bar{K}' = -K' < 0$), and the factor $\varkappa_l$ ensures that for b_{lm} the matrix elements are the same as for B^+_{lm}, i.e., the same as for the D functions.

[7)] This conversion[20] is exact (within terms of the specified power of I) if we add to the first and last rows of the table for $K_i = K(I_i = I \pm 2)$ in ref. 20 the quantity $-2f_2$ (and correct the obvious misprint; for $K_i = K$, $I_i = I + 1$ the coefficient of a_2 should be $I^2 + 2I - 2$). In general, we have more terms, so some of our terms (or some of their linear combinations) do not have corresponding terms in ref. 20 (to put our results in a form corresponding with that of ref. 20, we must set the auxiliary functions f_i in our equations equal to zero).

[8)] This is an abbreviated version of the discussion in ref. 46.

[9)] Actually, to take into account the properties of R^+_{lm} with respect to time inversion (τ), we need to introduce into Eqs. (A.30) yet another operator r, which is odd with respect to this operation: $[r, \tau]_+ = 0$. We can choose this operator such that $\hat{r}|-\rangle = |-\rangle$, so we can neglect it, except for its property of changing the τ parity of R^+_{lm}. Complications arise in the case of the K = 0 band; in this case we must either convert to R^+_{2m} rotons or develop the description of coupled (through the operator r) bands.

[10)] To satisfy this condition it is sufficient to assume that r_3 is not an independent variable governed by system (A.26) but is found from the known values of r_1 and r_2, as $r_3 = (1 - r_1^2 - r_2^2)^{1/2}$.

[11)] A paper on this subject is being prepared for publication.

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