Two-photon production of particles and the equivalent-photon approximation

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A covariant derivation is offered for the equivalent-photon method, and the accuracy of this method is estimated. Differential distributions are calculated for two-photon production with inclusion of the polarization of the virtual photons. A method is described for estimating lower limits on the cross sections. There is a discussion of two-photon production of e^+e^- pairs, the energy loss of nuons in matter, the contribution to the measured σ^{tot}_{pA} values, F_π, F_K, the possible calibration of cross sections in colliding pp beams, measurement of the polarization of high-energy photons, production of massive muon pairs in hadron collisions, and some processes of high order in α. There is a detailed discussion of what information about the $\gamma\gamma \to h$ transition can be extracted from experiment and how this information can be extracted.

INTRODUCTION

In a discussion of particle production in electromagnetic interactions, attention is usually focused on processes which are of low order in the electromagnetic coupling constant $\alpha = 1/137$, e.g., particle production during one-photon annihilation. Until recently two-photon production of particles had not received much attention, since the corresponding cross sections are smaller by four orders of magnitude (α^2).

There are a number of cases, however, when this rule does not hold and two-photon production (Figs. 1 and 2) becomes predominant over one-photon production. As accelerator energies (E) increase, these situations become more frequent, since the cross sections for one-photon (annihilation) production fall off no more slowly than E^{-2}, while the cross sections for two-photon production increase logarithmically with the energy. Furthermore, two-photon events frequently make possible objective studies not possible in the one-photon channel, e.g., $\gamma\gamma \to$ hadrons transitions. In colliding e^+e^- beams, two-photon production of hadrons should become predominant at an energy as low as $\sqrt{s} > 4$ GeV. Two-photon production of e^+e^- pairs was recently observed in πp scattering[1] and in accelerators with colliding e^+e^- beams at Novosibirsk[2] and Frascati.[3]

Interest in two-photon reactions began with Anderson's discovery of the positron (1932) and his experimental studies of the interaction of fast particles with matter.[4] Among the simplest electromagnetic processes in which positrons are produced (Figs. 1 and 3a), the two-photon mechanism in Fig. 1, $AB \to ABe^+e^-$, predominates at high energies. Anderson's studies[4] were soon followed by studies[5−9] of the processes in Figs. 1 and 2. At present the two-photon production of lepton pairs is primarily of interest as a research tool rather than of intrinsic interest. These production processes are discussed from this point of view in Sec. III.

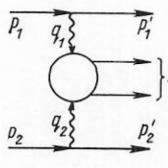


Fig. 1

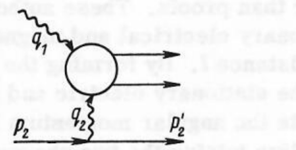


Fig. 2

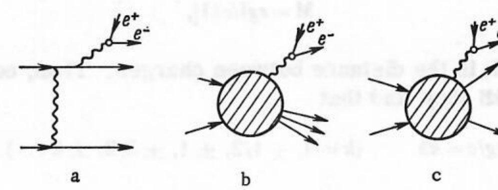


Fig. 3

Two-photon processes have again attracted considerable interest,[10−16] because they open up a fundamentally new research field in high-energy physics: study of the reaction $\gamma\gamma \to$ hadrons as a function of the photon "mass" and energy. These topics are discussed in Sec. IV.

Exact calculations of the cross sections for two-photon production are either very laborious or altogether impossible, because some of the quantities are simply not known (e.g., the cross section for $\gamma\gamma \to$ hadrons). For this reason frequent use has been made of the approximate equivalent-photon method, which has frequently been called the "Weizsäcker—Williams" method.[6,17] The preliminary estimates of total cross sections and the information on the nature of the differential distributions which are necessary in certain cases can also be found conveniently by this method. On the other hand, the literature reveals no estimates of the accuracy of the equivalent-photon method, nor is there a complete understanding of the nature of the method or its applicability in the case of two-photon production.[18] Equations related to the method have frequently contained explicit errors (see, e.g., ref. 15). We will therefore begin this review with an exposition of the method.

I. THE EQUIVALENT-PHOTON METHOD

The idea behind the method was that of Fermi, who pointed out that the field of a fast charged particle is very similar to radiation, which can be thought of as a flux of photons distributed with a certain density $n(\omega)$ over the frequency spectrum. The electromagnetic interaction of this particle with, e.g., a nucleus, can thus be reduced to the interaction of these photons with the nucleus. This

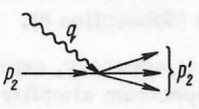
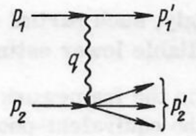


Fig. 4



$p_1^2 = p_1'^2 = m^2,\ p_2^2 = M^2.$

Fig. 5

fact was used and developed by Williams and Weizsäcker[6] in analyzing the interaction of a relativistic charged particle with matter. Initially, a semiclassical approach was used to find the equivalent-photon spectrum $n(\omega)d\omega$; when this approach is used, it is difficult to estimate the accuracy of the approximation. The development of Feynman-diagram techniques opened up the possibility of a simple and rigorous justification of the method. This problem was in fact solved by Pomeranchuk and Shmushkevich.[21] Gribov et al.[22] offered a covariant form of this derivation.

The essence of the equivalent-photon method can be described in the following manner. The cross section for the electromagnetic interaction of charged particles 1 and 2 (Fig. 5) is expressed in terms of the known cross section $\sigma_{\gamma 2}$ for the absorption of a photon by particle 2 (Fig. 4):

$$d\sigma_{12} = n_1(\omega)\, d\omega \sigma_{\gamma 2}(\omega). \quad (1)$$

An important simplification results from the fact that it is sufficient to use in the calculations simply the amplitudes for photoproduction on the mass shell. (2)

Furthermore, in the extension of (1) to the differential distributions, another postulate is frequently incorporated in the formulation of the method:

The photon polarization is unimportant, so the result incorporates a photoproduction cross section which is averaged over the photon polarizations. (3)

However, the virtual photons are polarized (as in the case of bremsstrahlung), so this postulate is wrong.[14,18] Accordingly, instead of (1) and the similar expression for the two-photon case within the framework of the equivalent-photon method, (2), we find slightly more complicated expressions, some of which are given in subsections 3, 5, and 6. However, for a broad class of distributions of practical interest, postulate (3) turns out to be justified (see subsections 3, 5, and 6).

1. One-Photon Exchange. Relation to Photoabsorption Cross Sections

For definiteness, we consider the inelastic scattering of an electron by a proton resulting in the production of some system of hadrons f: $ep \to e + f$. The diagram in Fig. 5, where the momentum of the virtual photon is $q = p_1 - p_1'$ ($q^2 < 0$), represents this process in the lowest order in terms of the electromagnetic interaction. The cross section is expressed in terms of the amplitude M^μ of the process of Fig. 4 — the absorption of a virtual photon by a proton involving the production of a final hadron system f. Averaging over the initial spin states and summing over the final spin states, we find

$$d\sigma = \frac{4\pi\alpha}{q^2} M^\mu M^{*\nu} \rho^{\mu\nu} \frac{(2\pi)^4 \delta(p_1 + p_2 - p_1' - p_2')\, d\Gamma}{4\sqrt{(p_1p_2)^2 - m^2M^2}} \frac{d^3p_1'}{2E'(2\pi)^3}. \quad (4)$$

Here Γ is the phase volume of the product particles f, which have a total momentum p_2', and the tensor $\rho^{\mu\nu}$ comes from the electron:

$$\rho^{\mu\nu} = \mathrm{Sp}\,[(\hat{p}_1 + m)\gamma^\mu(\hat{p}_1' + m)\gamma^\nu]/2q^2 = g^{\mu\nu} - q^\mu q^\nu/q^2 + [(2p_1 - q)^\mu (2p_1 - q)^\nu]/q^2. \quad (5)$$

(The factor q^{-2} is introduced for convenience, since as $q^2 \to 0$ we have, by virtue of current conservation, $p_1^\mu p_1^\nu M^\mu M^{*\nu} \sim q^2$.) The quantity $\rho^{\mu\nu}$ represents the density matrix of the virtual photon produced by the electron. For a virtual photon produced by other particles, this matrix is written below [Eq. (24)]. Density matrix (5) is not diagonal, i.e., the virtual photons are polarized.

Equation (4) is written in a form for which it is natural to introduce the terminology adopted in the equivalent-photon method: Instead of speaking of an ep collision, we can speak of a collison of virtual photons with a proton; the number of these photons having a given polarization μ, ν in a given phase-space element $d^3p_1' = d^3q$ is proportional to the quantity $\rho^{\mu\nu} d^3q/q^2$.

The helicity of the virtual photon (with $q^2 < 0$) can take on the values $\pm 1, 0$. The photons having a helicity ± 1 are "transverse" (T), and those having a helicity 0 are "scalar" or time-like (S) [real photons ($q^2 = 0$) can be only transverse]. We denote by σ_T the cross section for the absorption of a photon, and we denote by σ_S that for the absorption of a scalar photon. Near the mass shell (as $q^2 \to 0$) the scattering amplitudes for transverse photons convert into the corresponding amplitudes for the real photoprocess, while the amplitudes for reactions involving scalar photons vanish by virtue of gradient invariance. In particular, we have

$$\sigma_T(\omega, q^2) \to \sigma_\gamma(\omega); \quad \sigma_S(\omega, q^2) \backsim q^2 \text{ as } q^2 \to 0. \quad (6)$$

Here σ_γ is the cross section for the absorption of unpolarized real photons. After integrating over the phase space Γ of the particles produced, we can write cross section (4) in terms of σ_T and σ_S in the standard manner[23] (Appendix 2):

$$d\sigma = (\alpha/4\pi^2 q^2)\,[(\omega^2 - q^2)/(E^2 - m^2)]^{1/2} \times [2\rho(1,1)\,\sigma_T + \rho(0,0)\,\sigma_S]\,(d^3p_1'/E_1'), \quad (7)$$

where

$$\left.\begin{aligned} -\rho(1,1) &= \frac{E^2 + (E-\omega)^2 - q^2/2}{\omega^2 + q^2} + \frac{2m^2}{q^2}; \\ \rho(0,0) &= 1 - 4(E - \omega/2)^2(\omega^2 - q^2)^{-1}; \end{aligned}\right\} \quad (8)$$

$$\omega = qp_2/M = (p_1 - p_1')\,p_2/M; \quad E = p_1p_2/M;$$

$$d^3p_1'/E' = d(-q^2)\,d\omega\, d\varphi/(2\sqrt{E^2 - m^2}). \quad (9)$$

The coefficients $\rho(a, b)$ are the elements of density matrix (5) in the helicity basis. In the laboratory system ($p_2 = 0$) the invariant E is the electron energy, ω is the photon energy, and φ is the azimuthal position of the scattered electron; in this case the integration over φ is trivial.

The coefficients of σ_T and σ_S in Eq. (7) can be interpreted as the number of transverse and scalar photons, respectively.

2. One-Photon Exchange. The Equivalent-Photon Method. Accuracy of the Approximation

In the transition to approximate equations in the equivalent-photon method, the scalar-photon contribution σ_S is discarded, and $\sigma_T(\omega, q^2)$ is evaluated on the mass shell, so we have

$$\left.\begin{aligned} d\sigma &= \sigma_\gamma(\omega)\, dn; \\ dn &= \{\alpha\omega/[2\pi(E^2-m^2)]\}\,\rho(1,1)\,[d(-q^2)\,d\omega/q^2]. \end{aligned}\right\} \quad (10)$$

For fixed ω, the integration over $-q^2$ is carried out from a q^2_{min} value which is determined purely kinematically,[1])

$$q^2_{min} = m^2\omega^2\,[1 + O(m^2/(E-\omega)^2)]\,[E(E-\omega)]^{-1}, \quad (11)$$

up to some q^2_{max}. This upper limit is governed either by the characteristic cutoff parameter of the integral over q^2 or by the experimental conditions (e.g., only those events in which the electron is scattered through a small angle might be considered). The validity of the conversion from (7) to (10) requires a special analysis in each particular case.

For photoabsorption by a proton the cross section σ_T is known to fall off with increasing $-q^2$, while the cross section σ_S in (6) initially increases, but never exceeds σ_T. We denote the characteristic scale of the σ_T increase by Λ_γ^2; in this particular case we have $\Lambda_\gamma \sim m_\rho$. If we assume that the characteristic scale for the change in σ_S is no smaller than Λ_γ^2, we find

$$\begin{aligned} \sigma_S(\omega, q^2) &\leqslant (|q^2|/\Lambda_\gamma^2)\,\sigma_\gamma(\omega) \quad \text{for } |q^2| < \Lambda_\gamma^2, \\ \sigma_T(\omega, q^2) &= \sigma_\gamma(\omega)\,[1 + O(q^2/\Lambda_\gamma^2)], \end{aligned} \quad (12)$$

and for $|q^2| > \Lambda_\gamma^2$ cross sections σ_T and σ_S fall off according to a power law.

In our example the quantities $\rho(1,1)$ and $\rho(0,0)$ are on the same order of magnitude, and over a broad range $|q^2| < \omega^2$ they are essentially independent of q^2. Accordingly, for $|q^2| < \Lambda_\gamma^2\omega^2$ the dependence of $d\sigma$ in (7) on q^2 is basically logarithmic (dq^2/q^2). At larger $|q^2|$ values the cross section $d\sigma/dq^2$ falls off much rapidly. The integral contribution of the terms taken into account in the equivalent-photon method is on the order of $\int dq^2/q^2 \sim \ln(q^2_{max}/q^2_{min})$, while the terms discarded in the transition from (7) to (10) give a contribution $\sim \int dq^2/\Lambda_\gamma^2 \sim q^2_{max}/\Lambda_\gamma^2$. Accordingly, the error of the approximation can be described by

$$\begin{aligned} \eta &\sim (q^2_{max}/\Lambda_\gamma^2)\,[\ln q^2_{max}/q^2_{min}]^{-1} \\ &\sim (q^2_{max}/\Lambda_\gamma^2)\,[\ln q^2_{max}E^2/m^2\omega^2]^{-1}. \end{aligned} \quad (13)$$

In an evaluation of the total cross section or of $d\sigma/d\omega$, we have $q^2_{max} \sim \Lambda_\gamma^2$, and the error of the equivalent-photon method is logarithmic. If, on the other hand, q^2_{max} is limited by the experimental conditions, so that we have $q^2_{max} \ll \Lambda_\gamma^2$, the error is a power-law error: $\eta \sim q^2_{max}/\Lambda_\gamma^2$. Accordingly, such partial cross sections can be used to obtain reliable lower estimates (Subsection 8).

Within the framework of this accuracy, expressions (10) for the equivalent-photon spectrum simplify; the corresponding equations are collected in Subsection 4.

For many cases of practical interest the dependences of σ_T and σ_S on q^2 are of the same nature as in this example, and the equivalent-photon approximation is valid. For a photon produced, not by an electron, but by, e.g., a pion, the quantities $\rho(a, b)$ are cut off because of the decrease in the form factors at some $|q^2| = \Lambda^2$. If we have $\Lambda \ll \Lambda_\gamma$, and the form factors are well known up to $|q^2| \sim \Lambda_\gamma^2$ or up to those values $|q^2| < \Lambda_\gamma^2$ at which they become negligibly small, the error of the equivalent-photon method is a power-law error, $\eta \sim (\Lambda/\Lambda_\gamma^2)$.

If the increase in $-q^2$ is accompanied by a slower decrease in σ_T, e.g., a logarithmic decrease, the equivalent-photon approximation cannot be used. A corresponding example is discussed in Subsection 7.

In the conversion to differential distributions it is customary to use a natural generalization of (2) which is based on the adoption statement (3):

$$d\sigma_{12} = d\sigma_{\gamma 2}\, dn. \quad (14)$$

This condition does not always hold, however, since the virtual photon is polarized. The photon polarization is inconsequential if its target is unpolarized and if an integration is carried out over the momenta of the product particles (so no direction is singled out). It is precisely for this reason that (7) contains only the diagonal elements of the density matrix $\rho(1,1)$ and $\rho(0,0)$ and thus (10) contains only the cross section σ_γ for the absorption of unpolarized light.

If one particle is singled out in the final state, however, a direction is singled out also. Then (7) and (14) are replaced by a more complicated expression which contains, even on the mass shell, and additional term proportional to $\mathrm{Re}\,\rho(1,-1)$, the off-diagonal element of the density matrix. Here we have $\mathrm{Re}\,\rho(1,-1) \sim \cos 2\varphi$, where φ is the azimuthal position of this particle with respect to the electron-scattering direction (see, e.g., ref. 24).

Precisely this situation arises in two-photon production if two scattered particles are singled out in the final state (see, e.g., ref. 23a).

3. Two-Photon Production of Particles

In the case of two-photon production of particles (Fig. 1) the colliding particles emit "bremsstrahlung" photons having momenta and $q_i = p_i - p_i'$ ($i = 1, 2$). During the collision these photons convert into a particle system f having a total momentum $k = q_1 + q_2$ and a total mass $W^2 = \sqrt{k^2}$. The cross section for this reaction can be expressed in terms of the $M^{\mu\nu}$ amplitudes for the $\gamma\gamma \to f$ transition:

$$d\sigma = \frac{(4\pi\alpha)^2}{q_1^2 q_2^2}\,\rho_1^{\mu\mu'}\rho_2^{\nu\nu'} M^{\mu\nu} M^{*\mu'\nu'} \times \frac{(2\pi)^4\,\delta(q_1+q_2-k)\,d\Gamma}{4\sqrt{(p_1p_2)^2 - m_1^2m_2^2}}\,\frac{d^3p_1'\,d^3p_2'}{2E_1'2E_2'(2\pi)^6}. \quad (15)$$

Here Γ is the phase space of the product system f, and ρ_i is the density matrix of the i-th virtual photon, given by (5) and (24).

After integration over the phase space of the product particles, the cross section is expressed in terms of six invariant functions, instead of the two invariant functions σ_T and σ_S as in (7). Four of these functions are the cross sections for the $\gamma\gamma \to f$ transition away from the mass shell for the corresponding photons: σ_{TT}, σ_{TS}, σ_{ST}, σ_{SS}. For example, σ_{TS} is the cross section for the $\gamma\gamma \to f$ transition in the collision of the transverse photon with momentum q_1 and a scalar photon with momentum q_2. In contrast to the one-photon case, off-diagonal τ_{TT} and τ_{TS} terms arise which correspond to scattering of photons involving a change in the individual helicities of the two photons, but not in their total helicity. For example, we have

$$\tau_{TT} = \int M_{11} M^*_{-1-1} (2\pi)^4 \delta(q_1 + q_2 - k)\, d\Gamma / 4\sqrt{(q_1 q_2)^2 - q_1^2 q_2^2}, \quad (16)$$

where M_{ab} are the helicity amplitudes for the $\gamma\gamma \to f$ transition. The quantities σ_{ab} and τ_{ab} depend only on $W^2 = (q_1 + q_2)^2$, and the photon mass is $q_i^2 < 0$.

The two-photon analog of Eq. (7) is (Appendix 2)[14,25]

$$d\sigma = \frac{\alpha^2}{16\pi^4 q_1^2 q_2^2} \left[\frac{(q_1 q_2)^2 - q_1^2 q_2^2}{(p_1 p_2)^2 - m_1^2 m_2^2}\right]^{1/2} [4\rho_1(1, 1)\rho_2(1, 1)\sigma_{TT} + 2|\rho_1(1, -1)\rho_2(1, -1)|\tau_{TT}\cos 2\tilde{\varphi} + 2\rho_1(1, 1)\rho_2(0, 0)\sigma_{TS} + 2\rho_1(0, 0)\rho_2(1, 1)\sigma_{ST} + \rho_1(0, 0)\rho_2(0, 0)\sigma_{SS} - 8|\rho_1(1, 0)\rho_2(1, 0)|\tau_{TS}\cos\tilde{\varphi}]\, \frac{d^3p_1' d^3p_2'}{E_1' E_2'}. \quad (17)$$

The coefficients $\rho_i(a, b)$ are the elements of the density matrix ρ_i in the helicity basis in the center-of-mass system of the photons, while $\tilde{\varphi}$ is the azimuthal angle between the vectors $\mathbf{p}_1'$ and $\mathbf{p}_2'$ in the same system. Exact expressions for these two quantities are derived in Appendix 2. A significant point for our purposes is that with $|q^2| < W^2$ all the $|\rho_1(a, b)|$ [as well as the $|\rho_2(a, b)|$] are on the same order of magnitude and are essentially constant over a broad range.

Near the mass shell (as $q_i^2 \to 0$) the cross sections for processes involving scalar photons vanish, like (6), and σ_{TT} and τ_{TT} convert into the corresponding quantities for the real photoprocess; in particular $\sigma_{TT}(q_i^2 = 0)$ is equal to the cross section $\sigma_{\gamma\gamma}$ for the $\gamma\gamma \to f$ transition for real unpolarized light (Appendix 2):

$$\sigma_{TT}(W^2, q_i^2) \to \sigma_{\gamma\gamma}(W^2); \tau_{TT}(W^2, q_i^2) \to \tau_{TT}(W^2); \sigma_{TS} \sim q_2^2; \sigma_{ST} \sim q_1^2; \quad \sigma_{SS} \sim q_1^2 q_2^2; \ \tau_{TS} \sim \sqrt{q_1^2 q_2^2} \ \text{for } q_i^2 \to 0. \quad (18)$$

In certain cases of physical interest the dependences of σ_{ab} and τ_{ab} on q_i^2 are similar to the q^2 dependences of the photoabsorption cross sections, i.e., there is a characteristic scale Λ_γ^2 such that we have

$$\sigma_{TS}, \ \sigma_{ST}, \ \sigma_{SS}, \ \tau_{TS} \leqslant |q_i^2|\sigma_{\gamma\gamma}/\Lambda_\gamma^2; \quad \sigma_{TT} = \sigma_{\gamma\gamma}[1 + O(q_i^2/\Delta_\gamma^2)] \ \text{for } |q_i^2| < \Lambda_\gamma^2, \quad (19a)$$

while for $|q_i^2| > \Lambda_\gamma^2$ the cross sections σ_{ab} and the τ_{ab} fall off according to a power law.

The quantities of σ_{ab} and τ_{ab} are given in Appendix 3 for the production of a lepton pair ($l \equiv e, \mu$). We see from these expressions that for small $W \sim m_l$ we have $\Lambda_\gamma \sim W$. (For $W \gg m_l$, the quantity σ_{TT} falls off only logarithmically at first; this situation is discussed in Subsection 7.)

As yet we have no experimental information for $\gamma\gamma \to$ h (hadrons) transitions. It has been established in studies of the proton and neutron form factors and of the cross sections for γ scattering that the characteristic scale for changes in these quantities as functions of q^2 is on the order of m_ρ^2. It is natural to expect that for $\gamma\gamma \to$ h transitions also we would have $\Lambda_\gamma \sim m_\rho$ (at least for $W \geqslant m_\rho$). As a result we find

$$\Lambda_\gamma \sim W\,(\gamma\gamma \to l^+l^-, \ W \sim m_l); \ \Lambda_\gamma \sim m_\rho\,(\gamma\gamma \to h, \ W \geqslant m_\rho). \quad (19b)$$

For the transition to the equivalent-photon method in the two-photon case it is convenient to use the invariants ω_i and E and to expand the vectors q_i in terms of the vectors p_1, p_2 and in the plane orthogonal to p_1, p_2:

$$\left.\begin{array}{c} \omega_i = q_i(p_1 + p_2)/2E; \ s = (p_1 + p_2)^2 = 4E^2; \\ q_i = a_i p_2 + b_i p_1 + q_{i\perp}; \ q_{i\perp} p_{1,2} = 0; \\ d^3p_1' d^3p_2'/E_1' E_2' \equiv \\ \equiv \{(\pi s/2)[(p_1 p_2)^2 - m_1^2 m_2^2]\} \\ \times d(-q_1^2)\, d(-q_2^2)\, d\omega_1\, d\omega_2\, d\varphi; \\ \cos\varphi = -(q_{1\perp} q_{2\perp})/\sqrt{q_{1\perp}^2 q_{2\perp}^2}; \ (q_{i\perp}^2 < 0). \end{array}\right\} \quad (20)$$

In the c.m. system of the colliding particles (in the case of colliding beams), E is the beam energy, ω_i is the energy of the i-th photon, and φ is the angle between $q_{1\perp}$ and $q_{2\perp}$, i.e., between the scattering planes for particles 1 and 2. If we direct the z axis of the system along $\mathbf{p}_1 = -\mathbf{p}_2$, we have $q_{i\perp} = (0, q_{ix}, q_{iy}, 0)$. In the important region

$$m_i^2/(E - \omega_i)^2, \ |q_i^2|/W^2, \ |q_i^2|/\omega_i E, \ |q_i^2|\, m_i^2/(\omega_i E)^2 \ll 1, \quad (21)$$

we can use the approximations

$$\left.\begin{array}{c} \cos\varphi = \cos\tilde{\varphi}; \ q_i^2 = q_{i\perp}^2 (1 - \omega_i/E)^{-1} - q_{i\,\min}^2; \\ q_{i\,\min}^2 = m_i^2 \omega_i^2 / E(E - \omega_i); \\ W^2 = 4\omega_1\omega_2; \ (q_1 q_2)^2 - q_1^2 q_2^2 = (2\omega_1\omega_2)^2. \end{array}\right\} \quad (22)$$

From the nature of these dependences of σ_{ab} and $\rho_i(a, b)$ on q_i^2 we can conclude that the basic contribution to the cross section comes from the region $|q_i^2| < \Lambda_\gamma^2$, W^2. For q_i^2 values we can discard the contributions of scalar photons in (17), retaining only the terms containing σ_{TT} and τ_{TT} on the mass surface. As a result, the equivalent-photon approximation two-photon production becomes

$$\left.\begin{array}{c} d\sigma = [\sigma_{\gamma\gamma}\, dn_1\, dn_2 + \tau_{TT}(\cos 2\varphi)\, dn_1^\tau\, dn_2^\tau/2]\, d\varphi/2\pi; \\ dn_i = (\alpha/\pi)\sqrt{s/2(p_1 p_2)^3}\,\rho_i(1, 1)\, d(-q_i^2)\,\omega_i\, d\omega_i/q_i^2; \\ dn_i^\tau = (\alpha/\pi)\sqrt{s/2(p_1 p_2)^3}\,|\rho_i(1, -1)/q_i^2| \times \\ \times d(-q_i^2)\,\omega_i\, d\omega_i. \end{array}\right\} \quad (23a)$$

The accuracy of approximation (23) can be evaluated in the same manner as for the one-photon case, (13), from which this approximation differs in that it contains, in addition to the cross sections $\sigma_{\gamma\gamma}$ for the scattering of unpolarized photons, an interference term τ_{TT}. This term results from the polarization of the virtual photons and vanishes in an azimuthal averaging, which gives, in agreement with (3),

$$d\sigma = \sigma_{\gamma\gamma}\, dn_1\, dn_2. \quad (23b)$$

TABLE 1

Particle	C	D
Point spinless	0	1
$l = e, \mu$	1	1
π	0	$F_\pi(q^2)$
p	G_M^2	$[4m_p^2 G_E^2 - q^2 G_M^2](4m_p^2 - q^2)^{-1}$

4. Equivalent-Photon Spectra

For processes in which photons can be produced by various particles, we generalize density matrix (5) for the virtual photon, determining its structure on the basis of the gradient-invariance condition:

$$\rho^{\mu\nu} = (g^{\mu\nu} - q^\mu q^\nu / q^2)\, C(q^2) + [(2p-q)^\mu (2p-q)^\nu / q^2]\, D(q^2). \quad (24)$$

The values of C and D for various particles are shown in Table 1. Here G_E, G_M, and F_π are the form factors for the proton and pion. For a nucleus having a charge Ze we have $D(0) = Z^2$. We can now construct the equivalent-photon spectra for approximation (21). Using Eqs. (A.9) and (A.4) from Appendices 1 and 2, we find the following for (23):

$$dn_i = \frac{\alpha}{\pi}\left(1 - \frac{\omega_i}{E}\right)\frac{d\omega_i}{\omega_i}\cdot\frac{d(-q_i^2)}{|q_i^2|}\left[\left(1 - \left|\frac{q_{i\,\min}^2}{q_i^2}\right|\right) D_i + \frac{q_{i\,\min}^2}{2m_i^2} C_i\right] = \frac{\alpha}{\pi}\cdot\frac{d\omega_i}{\omega_i}\cdot\frac{d(-q_i^2)}{|q_i^2|}\left[\left|\frac{q_{i\perp}^2}{q_i^2}\right| D_i + \frac{q_{i\,\min}^2}{2m_i^2} C_i\right]. \quad (25)$$

The quantity dn_i^T is found from dn_i at $C_i = 0$.

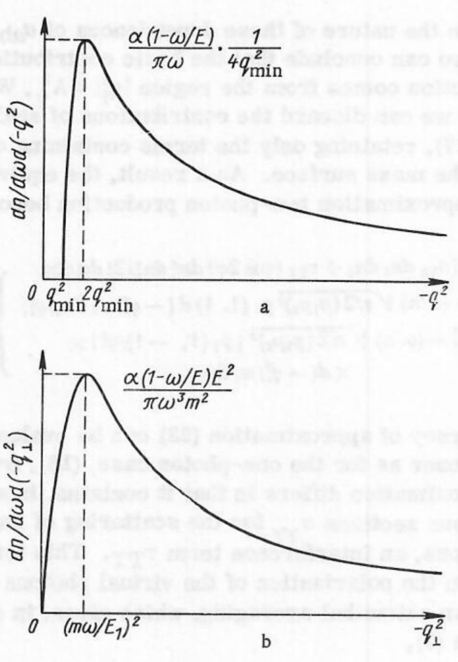


Fig. 6

The coefficient $(\omega_i^2/2E)(E - \omega_i)$ of C_i in Eq. (25) frequently allows us to discard this term in dn_i and thus neglect the magnetic moment (the spin). The we have $dn_i^T = dn_i$. Fig. 6 shows dn as a function of q^2 and $q_\perp^2$ (for the case C = 0).

In certain cases the integral over $-q^2$ can be evaluated easily;[2)] e.g., for point spinless particles we have C = 0, D = 1, and

$$dn_i = n(\omega_i)\, d\omega_i = (\alpha/\pi)(d\omega_i/\omega_i)(1 - \omega_i/E) \times [\ln q_{i\,\max}^2 / q_{i\,\min}^2 - 1 + q_{i\,\min}^2 / q_{i\,\max}^2]. \quad (26a)$$

For leptons we have C = D = 1 and

$$n_l(\omega_i)\, d\omega_i = (\alpha/\pi)(d\omega_i/\omega_i)[(1 - \omega_i/E + \omega_i^2/2E^2) \times \ln q_{i\,\max}^2 / q_{i\,\min}^2 - (1 - \omega_i/E)(1 - q_{i\,\min}^2/q_{i\,\max}^2)]. \quad (26b)$$

If we can neglect the q^2 dependence of the form factors, we conclude that Eq. (26a) holds for pions and kaons while for proton $[D(0) = 1;\ C(0) = G_M^2(0) = \mu_p^2]$, we have

$$n_p(\omega_i)\, d\omega_i = (\alpha/\pi)(d\omega_i/\omega_i)[(1 - \omega_i/E + \mu_p^2\omega_i^2/2E^2) \times \ln q_{i\,\max}^2 / q_{i\,\min}^2 - (1 - \omega_i/E)(1 - q_{i\,\min}^2/q_{i\,\max}^2)]. \quad (26c)$$

Finally, we write the spectrum of equivalent photons produced by a proton, taking account of the q^2 dependence of C and D. In the usual dipole approximation, with $G_E = G_M/\mu_p = (1 - q^2/q_0^2)^{-2}$ and $q_0^2 \approx 0.71\ \mathrm{GeV}^2$, we have

$$\left.\begin{aligned} n_p(\omega_i)\, d\omega_i &= \frac{\alpha}{\pi}\frac{d\omega_i}{\omega_i}\left[1 - \frac{\omega_i}{E}\right] \times \left[\varphi_i\left(\frac{q_{i\,\max}}{q_0^2}\right) - \varphi_i\left(\frac{q_{i\,\min}^2}{q_0^2}\right)\right]; \\ \varphi_i(x) &= \left[1 + y_i\left(\frac{\mu_p^2 + 1}{4} + 4\frac{m^2}{q_0^2}\right)\right] \times \left[-\ln\left(1 + \frac{1}{x}\right) + \sum_{k=1}^{3}\frac{1}{k(1+x)^k}\right] \\ &\quad + \frac{m^2 y_i}{q_0^2 x (1+x)^3} + (\mu_p^2 - 1)\left(1 + \frac{y_i}{4}\right) \times \left[\left(1 - \frac{4m_i^2}{q_0^2}\right)^{-4}\ln\frac{x + 4m^2/q_0^2}{1+x} + \sum_{k=1}^{3}\frac{1}{k(1+x)^k\left(1 - \frac{4m^2}{q_0^2}\right)^{4-k}}\right]; \\ y_i &= \frac{q_{i\,\min}^2}{m^2} = \frac{\omega_i^2}{E(E - \omega_i)}. \end{aligned}\right\} \quad (26d)$$

The accuracy of the description given by the spectra (26) depends on the formulation of the experiment; we consider two possible cases.

1. Particles scattered through small angles $\theta_i \leq \theta_{\max} \ll 1$ and having momenta p_1', p_2' are detected (these particles can be extracted from the beam by means of a magnetic field because their momenta are reduced[10,14]). Here, for $q_{i\,\max}^2 = E(E - \omega_i)\theta_{\max}^2 + q_{\min}^2 < \Lambda_\gamma^2$, the accuracy of approximation (13) is a power-law accuracy, increasing with decreasing $\theta_{\max}$.

2. Only particles formed in $\gamma\gamma$ collisions (e.g., the production of l^+l^- pairs in pp collisions) are detected; here we have $q_{\max}^2 \sim \Lambda_\gamma^2$, and the error is logarithmic.

Using the crude replacement of q^2_{min} by m^2W^2/s, we find from (13) that the error of the approximation is $\eta \sim [\ln(\Lambda_\gamma^2 s/m^2 W^2)]^{-1}$. Accordingly, for the production of hadrons in colliding ee beams we have $\eta^{-1} \sim \ln(m_\rho^2 s/m_e^2 W^2)$. In evaluating the asymptotic form ($E \to \infty$) we can significantly simplify Eqs. (26), finding the basic logarithmic approximation:

$$n(\omega_i)\, d\omega_i = (\alpha/\pi)(d\omega_i/\omega_i) \ln (q^2_{i\,max}/q^2_{i\,min});\quad q^2_{i\,min} = (m_i\omega_i/E)^2. \tag{27}$$

Equations (25)-(27), in which the index i should be assigned a specific value, hold for the problem of one-photon exchange.

II. USE OF THE EQUIVALENT-PHOTON METHOD FOR TWO-PHOTON PRODUCTION OF PARTICLES

We have seen under which conditions the equivalent-photon method can be used and how accurate it is. We will now use these results to write convenient approximations for cases of practical interest.

5. Distribution of the Total Momentum $k = q_1 + q_2$ of the Product System Distribution with Respect to $W^2 = k^2$

Since we have $W^2 = 4\omega_1\omega_2$ (for $|q_i^2| \ll W^2$), we find from Eqs. (23)-(27)

$$d\sigma/dW^2 = \sigma_{\gamma\gamma}(W^2) \int\limits_{\omega_{min}}^{\omega_{max}} n(\omega_1)\, n(\omega_2 = W^2/4\omega_1)\, d\omega_1/4\omega_1. \tag{28}$$

The region of the integration over ω is governed by the conditions $\omega_i < E$ or $q^2_{min} < q^2_{max}$, i.e., we have

$$\omega_{i\,max} = \min\{E,\ E\sqrt{q^2_{i\,max}}/m_i\};\quad \omega_{1\,min} = W^2/4\omega_{2\,max}. \tag{29}$$

For electron beams, for which generally $q^2_{max} > m_e^2$, we find $\omega_{i\,max} = E$.

For the integration over this region we have, in the first logarithmic approximation,[26,27]

$$\begin{gathered} d\sigma/dW^2 = (2/3)(\alpha/\pi)^2 (\sigma_{\gamma\gamma}/W^2) \\ \times \left\{ \begin{array}{ll} L[L^2 + 3L(l_1 + l_2) + 6l_1 l_2] & \text{for } l_i > 0; \\ (L + l_2)^2 (L + l_2 + 3l_1) & \text{for } l_1 > 0,\ l_2 < 0; \\ (L + l_1 + l_2)^3 & \text{for } l_i < 0; \end{array} \right\} \\ L = \ln s/W^2;\quad 2l_i = \ln(q^2_{i\,max}/m_i^2). \end{gathered} \tag{30}$$

An unusual situation arises in the production of hadrons in colliding electron beams. Over a large energy range in this case $l_1 = l_2 = \ln m_\rho/m_e \gg L/2 = (\ln s/W^2)/2$, so we can improve the accuracy of the result by accurately calculating the coefficient of l_1^2 in (30); we find [ref. 25][3)]

$$\begin{gathered} d\sigma/dW^2 = (2\alpha/\pi)^2 (\sigma_{\gamma\gamma}/W^2) \{ l_1^2 [(1 + W^2/2s)^2 L \\ - (3 + W^2/s)(1 - W^2/s)/2] + l_1 L^2 + L^3/6 \} \end{gathered} \tag{31}$$

[we recall that here the accuracy of η is $\sim (2l_1 + L)^{-1}$]. To illustrate the importance of this refinement, we note that, even for $s/W^2 \sim 30$, Eq. (30) gives a value which is too large, roughly twice that given by (31). At the energies available today, $l_1 \gg L/2$, and it is a good approximation to retain in (31) only the term proportional to l_1^2.

Distribution with respect to $\varepsilon = \omega_1 + \omega_2$ and k_z. In the c.m. system of p_1 and p_2, the quantity ε is the total energy of the product particles. For $|k_\perp^2| \ll W^2$, the longitudinal momentum of the system of product particles is $k_z \approx \sqrt{\varepsilon^2 - W^2} \approx \omega_1 - \omega_2$. Taking this circumstance into account, we find from (28)

$$\left. \begin{gathered} d\sigma = (d\varepsilon\, dW^2/4\sqrt{\varepsilon^2 - W^2})\, n(\omega_1)\, n(\omega_2)\, \sigma_{\gamma\gamma}(W^2) \\ = (dk_z\, dW^2/4\sqrt{k_z^2 + W^2})\, n(\omega_1)\, n(\omega_2)\, \sigma_{\gamma\gamma}(W^2); \\ k_z = \omega_1 - \omega_2 = \omega_1 - W^2/4\omega_1;\quad \omega_{1,2} = (\varepsilon \pm k_z)/2. \end{gathered} \right\} \tag{32}$$

In the first logarithmic approximation, we have

$$\begin{gathered} d\sigma = (\alpha/\pi)^2 (dk_z\, dW^2/W^2 \sqrt{k_z^2 + W^2})\, \sigma_{\gamma\gamma} \ln [E^2 q^2_{1\,max}/m_1^2\omega_1^2] \\ \times \ln [E^2 q^2_{2\,max}/m_2^2\omega_2^2]. \end{gathered} \tag{33}$$

Distribution with respect to $k_\perp = q_{1\perp} + q_{2\perp}$, ω_1, ω_2. This distribibution is an important particular case of distributions for which account must be taken of the polarization of the virtual photons, i.e., of the contribution of the term τ_{TT} in (23).[27,28] We can rewrite original equation (23) as follows (for simplicity, we assume C = 0, D, D = 1):

$$\left. \begin{gathered} d\sigma = \left(\frac{\alpha}{\pi^2}\right)^2 \left(1 - \frac{\omega_1}{E}\right)\left(1 - \frac{\omega_2}{E}\right) \frac{d\omega_1\, d\omega_2\, d^2k_\perp}{\omega_1\omega_2} \\ \times \int \frac{q^2_{1\perp} (k - q_1)^2_\perp [\sigma_{\gamma\gamma} + \tau_{TT} \cos 2\varphi/2]}{(q^2_{1\perp} - m_1^2\omega_1^2/E)^2 [(k - q_1)^2_\perp - m_2^2\omega_2^2/E^2]^2}\, d^2 q_{1\perp}; \\ \cos\varphi = -q_{1\perp}(k - q_1)_\perp / \sqrt{q^2_{1\perp}(k - q_1)^2_\perp}. \end{gathered} \right\} \tag{34}$$

In the region $|k_\perp^2| \leqslant (m_i\omega_i/E)^2$ either we have $|q^2_{i\perp}| \sim (m_i\omega_i/E)^2$ for both quantities or we have $q_{1\perp} \approx -q_{2\perp}$; in either case there is no averaging over φ which will cause the τ_{TT} contribution to vanish. In particular, for $k_\perp = 0$, we have $\cos 2\varphi = 1$, and the cross section[4)] is proportional to $(\sigma_{\gamma\gamma} + \tau_{TT}/2)$. Cheng and Wu[18] pointed out that for small $k_\perp$, the answer cannot be expressed in terms of $\sigma_{\gamma\gamma}$ alone. We emphasize that this result by no means implies that the equivalent-photon method of (2) is "inapplicable" for two-photon production of particles; it simply corresponds to a violation of additional assumption (3).

However, the basic contribution to the cross section comes from large $|k_\perp^2| \gg (m_i\omega_i/E)^2$. In the integration over $q_{1\perp}$ (for fixed $k_\perp$) the basic contribution comes from two symmetric regions: $(m_1\omega_1/E)^2 \ll |q^2_{1\perp}| \ll |k_\perp^2|$ and $(m_2\omega_2/E)^2 \ll |(k - q_1)^2_\perp| = |q^2_{2\perp}| \ll |k_\perp^2|$. In the first of these we have $k_\perp - q_{1\perp} \approx k_\perp$, and the only part of the integrand in (34) which depends on φ is $\cos 2\varphi$. In this region integration over $q_{1\perp}$ in the first logarithmic approximation thus causes the τ_{TT} contribution to vanish. The integration over $q^2_{1\perp}$ gives the large logarithm $\ln|(E^2k_\perp^2) \cdot (m_1^2\omega_1^2)^{-1}|$. Taking both regions into account, we find

$$\begin{gathered} d\sigma = 2(\alpha/\pi)^2 [d\omega_1\, d\omega_2\, d(-k_\perp^2)/(\omega_1\omega_2 |k_\perp^2|)]\, \sigma_{\gamma\gamma} \\ \times \ln [k_\perp^2 (p_1p_2)/(W^2 m_1 m_2)];\quad |k_\perp^2| \gg (m_i\omega_i/E)^2. \end{gathered} \tag{35}$$

6. Differential Distribution with Respect to the Momenta of the Product Particles

To construct the equivalent-photon method in this case in accordance with (2), we discard the contribution of scalar photons and evaluate the other amplitudes on the mass shell. In this case the differential cross section (15) is expressed in terms of five helicity amplitudes M_{ab} for the $\gamma\gamma \to f$ transition (a, $b = \pm 1$ are the photon helicities) (cf. ref. 29). To obtain a direction from which to reckon the azimuthal angles, we select one of the product particles, having a momentum k_1. Then in approximation (21) we have

$$\left.\begin{aligned} d\sigma = {} & (\alpha^2/q_1^2q_2^2)\{2\rho_1(1,1)\rho_2(1,1)[|M_{11}|^2+|M_{1-1}|^2] \\ & +2|\rho_1(1,-1)\rho_2(1,-1)|[M_{11}M^*_{-1-1} \\ & \times\cos 2(\varphi_1-\varphi_2)+M_{1-1}M^*_{-11}\cos 2(\varphi_1+\varphi_2)] \\ & +4\rho_1(1,1)|\rho_2(1,-1)|\operatorname{Re}(M_{11}M^*_{1-1})\cos 2\varphi_2 \\ & +4|\rho_1(1,-1)|\rho_2(1,1)\operatorname{Re}(M_{11}M^*_{-11})\cos 2\varphi_1\} \\ & \times[\delta(q_1+q_2-k)\,d\Gamma/4\sqrt{(p_1p_2)^2-m_1^2m_2^2}] \\ & \times[d^3p_1'd^3p_2'/E_1'E_2']; \\ & -\rho_i(1,1)=C_i+2D_iE^2q_{i\perp}^2/\omega_i^2q_i^2; \\ & |\rho_i(1,-1)|=2D_iE^2q_{i\perp}^2/\omega_i^2q_i^2; \\ & \cos\varphi_i=-(q_{i\perp}k_{1\perp})/\sqrt{q_{i\perp}^2k_{1\perp}^2}; \\ & \omega_{1,2}=q_{1,2}(p_1+p_2)/2E\approx kp_{2,1}/\sqrt{2p_1p_2}, \end{aligned}\right\} \quad (36)$$

where φ_i is the angle between vectors $q_{i\perp}$ and $k_{1\perp}$. Amplitudes M_{ab} depend only on invariants such as q_ik_j, k_jk_l, W^2, and q_i^2. On the mass shell we set $q_i^2 = 0$ and $q_{i\perp} = 0$ in these invariants.

The differential cross section with respect to the momenta of the product particles is found through an integration over p_i', which can be carried out conveniently through the use of

$$\begin{aligned} \delta(q_1+q_2-k)\,d^3p_1'\,d^3p_2'/E_1'E_2' &= d^2q_{1\perp}/[2(E-\omega_1)(E-\omega_2)] \\ &= d^2q_{1\perp}/[2E^2(1-\varepsilon/E+W^2/4E^2)]. \end{aligned} \quad (37)$$

The characteristic scale μ^2 for the change in amplitudes M_{ab} as functions of $q_{i\perp}^2$ is at any rate no smaller than the characteristic distance to the nearest singularity in the plane of the variables $(q_i - k_j)^2$, i.e., we have $\mu \sim m_e$ for lepton production and $\mu \gtrsim m_\pi$ for hadron production. The integrals over $q_{i\perp}^2$ in (36), on the other hand, are cut off at $|q_{i\perp}^2| \sim |k_\perp^2| \gtrsim (m_i\omega_i/E)^2$, so for $|k_\perp^2| \ll \mu^2$ we can also neglect the dependences of form factors C and D on q^2. Then the remaining integration over $q_{1\perp}$ can be carried out in terms of elementary functions.[18,30] The result looks formidable, but it has a power-law error: $\sim |k_\perp|/\mu$.

We restrict the evaluation of the integral of (36) to the first logarithmic region, $(m_i\omega_i/E)^2 \ll |k_\perp^2| \ll \mu^2$.[27,28] In the integration over $q_{1\perp}$ the basic contribution comes, as in the derivation of (35), from two symmetric regions: $(m_1\omega_1/E)^2 \ll |q_{1\perp}^2| \ll |k_\perp^2|$ and $(m_2\omega_2/E)^2 \ll |(k - q_1)_\perp^2| = |q_{2\perp}^2| \ll |k_\perp^2|$. In the first of these regions the angle φ_1 is the same as the variable integration angle between $q_{1\perp}$ and $k_{1\perp}$, while φ_2 is the same as the fixed angle φ_0 between $k_\perp$ and $k_{1\perp}$ ($\cos\varphi_0 = -(k_\perp k_{1\perp})/(k_\perp^2 k_{1\perp}^2)^{1/2}$. Accordingly, after integrating over $q_{1\perp}$, taking both these regions into account, we find

$$\left.\begin{aligned} d\sigma &= [(8\pi)^3\alpha^4/W^4|k_\perp^2|]\{A_1A_2R_{\gamma\gamma}+B_1B_2R^a_{\gamma\gamma}\cos 2\varphi_0\} \\ &\quad\times\ln[k_\perp^2(p_1p_2)/W^2m_1m_2]\,d\Gamma; \\ A_i &= 1-\omega_i/E+C_i(0)\,\omega_i^2/2E^2;\quad B_i=1-\omega_i/E; \\ (4\pi\alpha)^2R_{\gamma\gamma} &= (|M_{11}|^2+|M_{1-1}|^2)/2 \\ &= g^{\mu\mu'}g^{\nu\nu'}M^{\mu\nu}M^{*\mu'\nu'}/4; \\ (4\pi\alpha)^2R^a_{\gamma\gamma}\cos 2\varphi_0 &= -\tfrac{1}{2}\operatorname{Re}[M^*_{11}(M_{-1-1}+M_{-11}]\cos 2\varphi_0 \\ &= g^{\mu\mu'}(2k_\perp^\nu k_\perp^{\nu'}/k_\perp^2-g^{\nu\nu'})M^{\mu\nu}M^{*\mu'\nu'}/4. \end{aligned}\right\} \quad (38)$$

In particular, for the $\gamma\gamma \to l^+l^-$ transition, we have

$$\begin{aligned} R_{\gamma\gamma} &= 2W^2/(m_l^2-k_{1\perp}^2)-4(m_l^4+k_{1\perp}^4)/(m_l^2-k_{1\perp}^2)^2; \\ R^a_{\gamma\gamma} &= -8m^2k_{1\perp}^2/(m_l^2-k_{1\perp}^2)^2. \end{aligned} \quad (39)$$

In addition to the term $R_{\gamma\gamma}$, which corresponds to the scattering of unpolarized photons, the results contain the term $R^a_{\gamma\gamma}$ corresponding to the polarization of the virtual photons. Accordingly, supplementary assumption (3) is not valid here.[5)]

However, for many distributions of practical interest which are found from (38) through a partial integration over some parameters of the product particles, the quantity $R^a_{\gamma\gamma}$ drops out. In particular, it is clear that $R^a_{\gamma\gamma}$ vanishes in an integration over the angular coordinates of the vector $k_\perp$, since $R^a_{\gamma\gamma}$ and $R_{\gamma\gamma}$ do not depend on $k_\perp$ on the mass shell.

For the production of a pair of particles, assumption (3) is valid for a broad class of distributions. Denoting the momentum of the second particle by $k_2 = k - k_1$, and for the case of small angles ψ showing the deviation from a planar situation (the angles between $k_{1\perp}$ and $-k_{2\perp}$), for which we have $|k_{1\perp}| \gg |k_\perp|$, we find

$$\left.\begin{aligned} \cos 2\varphi_0 &= (x^2-4\sin^2\psi/2)/(x^2+4\sin^2\psi/2); \\ k_\perp^2 &= k_{1\perp}^2(x^2+4\sin^2\psi/2); \\ x &= (|k_{1\perp}|-|k_{2\perp}|)/|k_{1\perp}|. \end{aligned}\right\} \quad (40)$$

If, in the c.m. of the initial particles, we fix $|k_1|$, ψ, and the particle-pair emission angles θ_1, θ_2, the integration over $|k_2|$ can be carried out simply by setting $k_{1\perp} = -k_{2\perp}$ (or $x = 0$) everywhere except in the rapidly changing factors $1/k_\perp^2$ and $\cos 2\varphi_0$. Noting that $|k_{2\perp}| = |k_2|\sin\theta_2$, we find[6)]

$$\left.\begin{aligned} &\int d|\mathbf{k}_2|/|k_\perp^2| = \pi/(2|\mathbf{k}_1|\sin\theta_1\sin\theta_2|\sin\psi/2|); \\ &\int(\cos 2\varphi_0/k_\perp^2)\,d|\mathbf{k}_2| = 0. \end{aligned}\right\} \quad (41)$$

Accordingly, the distribution with respect to $|k_1|$, θ_1, θ_2, ψ, or with respect to some of these parameters, is given in the first approximation by simply the first term in (38).[28] Substituting (41) into (38), we find[7)]

$$\begin{aligned} & m_1m_2/p_1p_2 < \psi^2 < 1 \text{ и } |k_{1\perp}^2| \sim W^2\ [31, 32] \\ & d\sigma = (2\alpha^4R_{\gamma\gamma}/\pi W^4)(1-\varepsilon/E)(\sin^2\theta_1/\varepsilon_1\varepsilon_2\sin^2\theta_2) \\ & \times[|\mathbf{k}_1|^3d|\mathbf{k}_1|\,d\theta_1\,d\theta_2\,d\psi/|\sin\psi/2|\ln[(\sin^2\psi/2)\,p_1p_2/m_1m_2]. \end{aligned} \quad (42)$$

To find the distribution with respect to $|k_1|$, θ_1, θ_2, it is sufficient to integrate (38) over $k_\perp$:

$$d\sigma=\frac{2\alpha^4}{\pi}\,\frac{R_{\gamma\gamma}}{W^4}\,\frac{|\mathbf{k}_1|^3\,d|\mathbf{k}_1|\sin^2\theta_1\,d\theta_1\,d\theta_2}{\varepsilon_1\varepsilon_2\sin^2\theta_2}\left[\ln\frac{p_1p_2}{m_1m_2}\right]^2. \quad (43)$$

For the production of a lepton pair, this result was reported[8)] in ref. 5; the case in which an arbitrary pair of relativistic ($W^2 \gg k_1^2$) particles is produced was discussed in ref. 33; and Eq. (43) was obtained in general form in ref. 31. In the case of the production of a pair of relativistic particles ($W^2 \gg m^2$) at large angles ($\theta_i \sim \pi/2$ in the c.m. system) and in the case of smooth dependence of $R_{\gamma\gamma}$ on $k_{1\perp}^2$, we can simply set $\theta_i = \pi/2$ or $k_{1\perp}^2 = -W^2/4$ $R_{\gamma\gamma}$.

Then it is easy to obtain a useful estimate for the cross section in the region $W > W_{min}$, $\pi - \theta_{min} > \theta_i > \theta_{min}$, where we have $\pi/2 - \theta_{min} \ll 1$, and with an angular difference $|\theta_1 + \theta_2 - \pi| > \chi_{min}$:[27,33]

$$d\sigma=\frac{\alpha^4}{\pi}\cdot\frac{R_{\gamma\gamma}\,dW^2}{W^4}\left[\frac{\pi}{2}-\theta_{min}-\frac{\chi_{min}}{2}\right]^2\ln\frac{sq_{1\,max}^2}{m_1^2W^2}\ln\frac{sq_{2\,max}^2}{m_2^2W^2};$$
$$\pi-\theta_{min}\leqslant\theta_i\leqslant\theta_{min};\quad |\theta_1+\theta_2-\pi|\geqslant\chi_{min} \quad (44)$$

In particular, for the production of a lepton pair we have, from (39), $R_{\gamma\gamma} \approx 4$ and $\int R_{\gamma\gamma}\,dW^2/W^4 = 4W_{min}^{-2}$. The quantity $W_{min}/2$ is approximately equal to the energy threshold for the detection of the particles of the pair.

7. Calculation of the Production of Massive Lepton Pairs. Means of Inclusion Streams

In this case the equivalent-photon method is inapplicable,[9)] since the last relation in (12) is violated. With $W^2 \gg m_l^2$, $|q_i^2|$ (Appendix 3), we find

$$\left.\begin{aligned}\sigma_{TT}&=\sigma_{\gamma\gamma}\{1-[\ln(1+a)+a(1+a)^{-1}]/[\ln(W^2/m_l^2)-1]\\&\qquad+O(q_i^2/W^2)\};\\a&=(m_l^2-q_1^2)(m_l^2-q_2^2)/m_l^2W^2;\ \sigma_{TS},\ \sigma_{ST},\ \sigma_{SS}\leqslant|q_i^2|\sigma_{\gamma\gamma}/W^2;\\&\qquad\tau_{TT}\leqslant m_l^2\sigma_{\gamma\gamma}/W^2.\end{aligned}\right\} \quad (45)$$

An important (power-law) decrease in σ_{TT} occurs at $|q_i^2| \sim W^2$. If $W^2 \gg m_l^2$, the dependence of σ_{TT} on q_i^2 can no longer be neglect over a large region $(Wm_l)^2 < q_1^2q_2^2 < W^4$ (in this region the decrease is only logarithmic). Accordingly, for $\ln(W^2/m_l^2) \gtrsim \ln(p_1p_2/m_1m_2)$, we replace (23) by the slightly more complicated equation for σ_{TT} given in (45):

$$d\sigma=(\alpha/2\pi)^2\,(\omega_1\omega_2\,d\omega_1\,d\omega_2/E^4)$$
$$\times\int(dq_1^2\,dq_2^2/q_1^2q_2^2)\,\rho_1(1,1)\,\rho_2(1,1)\,\sigma_{TT}(W^2,q_1^2,q_2^2). \quad (46)$$

The analogous situation for the case of bremsstrahlung production was discussed in ref. 34.

The situation is simpler in the production of massive lepton pairs in hadron collision for $m_l W \gg m_\rho^2$. In this case the integrals are actually cut off at $|q_i^2| \sim m_\rho^2/4$ because of the form factors, and we find $\sigma_{TT}(W_1^2, 0, 0) - \sigma_{TT}(W^2, q_1^2, q_2^2) \leqslant m_\rho^2\,\sigma_{\gamma\gamma}/4Wm_l$. Again equivalent photon approximation (23b) turns out to be valid. When expressions (26d) are used for the spectra, a power-law dependence is found: $\eta \sim (m_\rho^2/4Wm_l)(\ln s/W^2)^{-1}$.

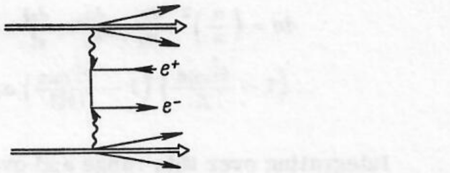


Fig. 7

In this latter case an important contribution is made to the cross section by processes involving stream production (Fig. 7), in which case the particle with p_i transforms into a stream with total momentum p_i' during the emission of the photon. At high energies, the various streams overlap slightly, so here again we can use the model in which the stream is replaced by a massive particle, and we can neglect the interference between the final states of particles 1 and 2. In this case the quantities C and D in (24) are given in terms of the familiar structure functions for deep inelastic ep scattering:

$$\left.\begin{aligned}C_i&=-(2m_i/q_i^2)\int W_1(p_i'^2,q_i^2)\,dp_i'^2;\\D_i&=(1/2m_i)\int W_2(p_i'^2,q_i^2)\,dp_i'^2;\\q_{i\,min}^2&=(p_i'^2-m_i^2)\,\omega_i/(E-\omega_i)+m_i^2\omega_i^2/E(E-\omega_i).\end{aligned}\right\} \quad (47)$$

Form factors C and D decrease with increasing $-q^2$, and the characteristic scale for this decrease is $\sim m_\rho^2$. This result allows us to use the equivalent-photon method for $W m_l \gg m_\rho^2$, as above.

8. Lower Estimates for the Cross Sections

In certain cases (particularly for evaluating possible new experiments) it is useful to have reliable lower estimates for the cross sections. This information is especially important, since in all known cases of two-photon production of particles the first corrections to the first logarithmic approximation are negative and, at the energies presently attainable, are of the same order of magnitude as the principal term.[8,35,36] On the other hand, according to (13), a sufficiently accurate calculation of the contribution of the region $|q^2| \leq q_{max}^2 \leq \Lambda_\gamma^2$ gives a reliable estimate of the cross section. This is a lower estimate since the rest of the q^2 range makes a positive contribution. As q_{max}^2 decreases, the reliability of the estimate increases, but the magnitude of this estimate also decreases. We will illustrate this situation with two examples.

The idea of obtaining a lower limit is demonstrated most simply for the case of the reaction $pp \rightarrow ppe^+e^-$.[28,37] Although the result found by Racah[8] for this case, (54), is quite accurate, this simple estimate will be useful because the region making the calculated contribution to the cross section is clearly defined in this deviation.

For this reaction, for W/m_l not too large, the characteristic parameter of the q^2 cutoff is $\Lambda_\gamma^2 \sim W^2$. Choosing a lower number η, we restrict the subsequent q_i^2 range: $-q_i^2 \leq \eta W^2$. Since in this range we have $W^2 = 4\omega_1\omega_2$ and $q_{i\,min}^2 = m_p^2\omega_i^2/E(E-\omega_i)$, the condition $q_{i\,min}^2 < \eta W^2$ also bounds the ω_i range: $Wm_p/4E\sqrt{\eta} \leq \omega_i \leq E\sqrt{\eta}/m_p$.

Using $\omega_i/E < W/m_p \ll 1$, we can write, with an error on the order of η,

$$d\sigma = \left(\frac{\alpha}{\pi}\right)^2 \frac{d\omega_1}{\omega_1}\cdot\frac{d\omega_2}{\omega_2}\cdot\frac{dq_1^2}{q_1^2}\cdot\frac{dq_2^2}{q_2^2} \times\left(1-\frac{q_{1\,min}^2}{|q_1^2|}\right)\left(1-\frac{q_{2\,min}^2}{|q_2^2|}\right)\sigma_{\gamma\gamma}(W)^2.$$

Integrating over this range and over W, for $2m_e \leq W \leq W_{max}$, we find (for $W_{max} > 3m_e$)

$$\left.\begin{aligned} &\sigma(\eta) = (28\alpha^4/27\pi m_e^2)\,[L_\eta^3 + 3L_\eta/2 - 1/2] \\ &\quad\times[1-(18m_e^2/7W_{max}^2)\ln(W_{max}/m_e)^2]; \\ &L_\eta = \ln \eta s/m_\rho^2 - 1. \end{aligned}\right\} \quad (48)$$

For $\sqrt{s} = 40$ GeV, Rach's result (54) yields $\sigma = 0.2$ mb, while (48) for $\eta = 0.1$ and $W_{max} = 20m_e$ yields $\sigma(0, 1) = 0.1$ mb, i.e., half the total cross section [the error of the $\sigma(0, 1)$ calculation is on the order[10)] of 10%].

As a second example we consider two-photon production with hadrons in colliding ee beams. Here it is useful to estimate the contribution to the cross section for the production of a hadron system having an effective mass W from the region of relatively low electron-scattering angles $\theta_i \leq \theta_{max}$ (case 1 in Subsection 4).[13,14,25] In the important case $1 \ll \gamma = 2E\sin(\theta_{max}/2)/m_e < m_\rho/m_e$ we find from Eqs. (26b) and (28) the following result, within $\sim(\gamma m_e/m_\rho)^2\,[\ln(s\gamma^2/W^2)]^{-1}$:

$$\left.\begin{aligned} &d\sigma = (2\alpha/\pi)^2\,(dW^2/W^2)\,\sigma_{\gamma\gamma}(W^2)\,J(s/W^2,\ \gamma); \\ &J(k,\gamma) = (1/4k^2)\int_1^k (dx/x)\,f(x)\,f(k/x); \\ &f(x) = (2x^2-2x+1)\ln[\gamma(x-1)] - x(x-1). \end{aligned}\right\} \quad (49)$$

Substituting into (26b) the value $q_{i\,max}^2 = |q_\perp^2|_{max}(1-\omega_i/E)^{-1}$, with fixed $|q_\perp^2|_{max} < \Lambda_\gamma^2$, we can replace (49) by the estimate found in ref. 36 for the cross section for the process $ee \to ee\mu^+\mu^-$ at not too high energies.

9. Photoproduction According to the Primakoff Scheme (Fig. 2)

This process is a one-photon exchange process, but the photoproduction cross section in the result is replaced by the cross section for the transition $\gamma\gamma \to f$. Accordingly, we can conveniently find an answer from the equations for two-photon production, (17), if we choose factors corresponding to the upper part of Fig. 1 and replace $\rho_1^{\mu\mu'}$ by the density matrix of the real incident photon. If the photon is unpolarized, we have $\rho_1^{\mu\mu'} = g^{\mu\mu'}/2$; if the photon is polarized we have $\rho_1^{\mu\mu'} = e^{\mu}e^{*\mu'}$ (e is the polarization vector). In this latter case we find

$$\left.\begin{aligned} &d\sigma = [\alpha\,(q_1q_2)/4\pi^2q_2^2\,(q_1p_2)]\,[2\rho_2(1,1)\,\sigma_{TT} + \rho_2(0,0)\,\sigma_{TS} \\ &\quad - |\rho_2(1,-1)|\,(1+2\,|e\tilde{p}_{2\perp}|^2/\tilde{p}_{2\perp}^2)\,\tau_{TT}]\,d^3p_2'/E_2'; \\ &\tilde{p}_{2\perp} \approx (s-m_2^2)\,q_{2\perp}/W^2. \end{aligned}\right\} \quad (50)$$

The exact expressions for $\rho_2(a, b)$ and $\tilde{p}_{2\perp}$ are given in Appendix 2, and the argument of σ_{TT}, σ_{TS}, and τ_{TT} is $q_1^2 = 0$. For unpolarized light, the term containing τ_{TT} in (50) drops out.

The equivalent-photon approximation is found by neglecting the σ_{TS} contribution and evaluating σ_{TT} and τ_{TT} on the mass shell. For unpolarized light we have $d\sigma = \sigma_{\gamma\gamma}dn_2$ [cf. Eq. (10)], where dn_2 is found from (25) by replacing ω_2/E by $W^2/(s-m_2^2)$. For production in the case of a nucleus having a charge Ze the only important term is that proportional to $D_2 = Z^2F^2(q_2^2)$ (F is the nuclear form factor):

$$d\sigma = \frac{Z^2\alpha}{\pi}F^2(q_2^2)\frac{dW^2}{W^2}\cdot\frac{dq_2^2}{q_2^2}\cdot\frac{q_{2\perp}^2}{q_2^2}\sigma_{\gamma\gamma}(W^2). \quad (51)$$

The principal contribution to the cross section comes from the coherent region, for which the integral in (51) is cut off by the form factor $F(q_2^2)$ at low values of $|q_2^2| = \Lambda_2^2$. If the nuclear form factor is known up to values considerably above Λ_2^2, we find, after integrating (51) over q_2^2 for $\Lambda_2 < \Lambda_\gamma$, a power-law error for the approximation: $\eta \sim (\Lambda_2/\Lambda_\gamma)^2\cdot[\ln(4\omega^2\Lambda_2^2)/W^4]^{-1}$. In the laboratory system, with $p_2 = 0$ (where ω is the frequency of the incident photon), we have

$$\left.\begin{aligned} &s - m_2^2 = 2\omega m_2;\ W^2 \approx 2\omega p_{2z}';\ -q_2^2 \approx p_2'^2; \\ &\frac{|p_{2\perp}'^2|}{p'^2} = \sin^2\theta;\ q_{2\,min}^2 = (W^2/2\omega)^2. \end{aligned}\right\} \quad (52)$$

In this case the total transverse momentum of the system of product particles is $k_\perp = q_{2\perp} = p_{2\perp}'$, and the distribution with respect to $k_\perp$ is that shown in Fig. 6.

III. TWO-PHOTON PRODUCTION OF LEPTONS

Let us consider some of the physical effects associated with two-photon production of lepton pairs and some possible applications of this process. Here we are dealing primarily with the production of e^+e^- pairs in collisions of fast charged particles. Landau and Lifshits[5] calculated the cross section for this process in the first logarithmic approximation:[11)]

$$\begin{aligned} &\sigma = 28\,(Z_1Z_2)^2\,\alpha^4l^3/27\pi m_e^2 = 1.4\cdot10^{-30}\,(Z_1Z_2)^2\,l^3\,\mathrm{cm}^2, \\ &l = \ln 2p_1p_2/m_1m_2. \end{aligned} \quad (53)$$

We can now see two characteristic features of this process:

1) Its cross section is slightly smaller than the hadron cross section: $\alpha/m_e \approx 2/m_\pi$ and, even at moderately high energies, we have $\alpha l^3 \gg 1$, i.e., $\sigma \gg (Z_1Z_2)^2\cdot\alpha/m_\pi^2$.

2) As the energy increases, the cross section increases quite rapidly, $\sim l^3$.

Many differential distributions of the product electrons of practical interest were reported long ago[5] for the region which makes the basic contribution to the cross section. Although these distributions are written in the rest system of one of the colliding particles, a simple modification makes them suitable for the case of colliding beams.[12)] Bhabha[7] also calculated these distributions over a wider energy range.

At the energies currently obtainable, we can use (53) only for an order-of-magnitude estimate of the production cross section for e^+e^- pairs, since the first logarithmic correction, proportional to l^2, has a large negative coefficient. The coefficients of l^1 and l^0 depend strongly on the nature of the colliding particles. Racah[8] calculated the cross section for the production of e^+e^- pairs in the collision of two heavy particles A_1 and A_2:

$$\sigma_{A_1A_2\to A_1A_2e^+e^-}=\frac{(Z_1Z_2)^2\,\alpha^4}{27\pi m_e^2}\,[28l^3-178l^2+(7\pi^2+370)\,l-348-13\pi^2/2+21\xi(3)]\approx(Z_1Z_2)^2\,1{,}4\cdot10^{-30}\,[(l-2.12)^3+2.2\,(l-2.12)+0.4]\ \mathrm{cm}^2\ (\xi(3)=1.202) \tag{54}$$

[the discarded terms are of the order of m_1m_2/p_1p_2 or $(m_e/m_\pi)^2A^{2/3}$]. The coefficients of l^2 and l recently calculated in ref. 35 agree with those in (54). The cross section for the production of $\mu^+\mu^-$ pairs in e^+e^- equations was calculated by Baier and Fadin[35] and Kuraev and Lipatov:[36]

$$\left.\begin{aligned}
&\sigma_{ee\to ee\mu^+\mu^-}=\frac{28\alpha^4}{27\pi m_\mu^2}\left[l^3-\frac{178}{28}\,l^2-Bl+C\right];\\
&B=\frac{1}{28}\,[42l_\mu^2+196.4l_\mu+152+7/15+21\pi^2]\approx258;\\
&C=\frac{1}{28}\Big[14l_\mu^3+184.8l_\mu^2+l_\mu\left(1109+\frac{31}{150}-7\pi^2\right)\\
&+\left(36+\frac{7}{15}\right)\pi^2+\frac{51463}{18}-189\xi(3)-\frac{12748}{125}\Big]\approx1855;\\
&l_\mu=\ln\left(\frac{m_\mu^2}{m_e^2}\right)\approx10.67.
\end{aligned}\right\} \tag{55}$$

For $\sqrt{s}\lesssim 1$ GeV the cross section (55) is negative, so here we must take into account terms of order s^{-1}. Accordingly, at the energies currently obtainable, the lower estimates are more useful (Subsection 8).

The cross sections for the processes $ep\to epe^+e^-$ and $ee\to eee^+e^-$ were calculated in refs. 8, 35, 36 with the same accuracy as in (54), with the indistinguishability of electrons neglected. When this indistinguishability is taken into account, as in ref. 36, the coefficients of l^1 and l^0 are charged only slightly.

Various differential distributions have been calculated for the reactions $\mu p\to\mu pe^+e^-$,[8,38] $ep\to epe^+e^-$,[39] $\mu p\to\mu\mu^+\mu^-p$,[40] and $e^+e^-\to e^+e^-e^+e^-$.[32,33]

In concluding this discussion, we note that the results of early studies[5-8] unfortunately have been largely overlooked, so that studies are still appearing which are reproducing various particular results of refs. 5-8.

10. Production e^+e^- Pairs

Stopping of fast muons in matter. As fast electrons pass through matter, they lose energy primarily as a result of bremsstrahlung. For muons the bremsstrahlung cross section is $\sim\alpha^3/m_\mu^2$, much smaller than (53). However, the average energy loss per two-photon production event turns out to be smaller than that during bremsstrahlung events by a factor of m_μ/m_e. Accordingly, the energy losses due to the two mechanisms turn out to be of the same order of magnitude (see, e.g., refs. 38 and 41). The magnitude of the loss depends strongly on the exact form of the distribution of product electrons near the boundary of the region making the basic contribution to the cross section, a distribution not found in the equivalent-photon method.

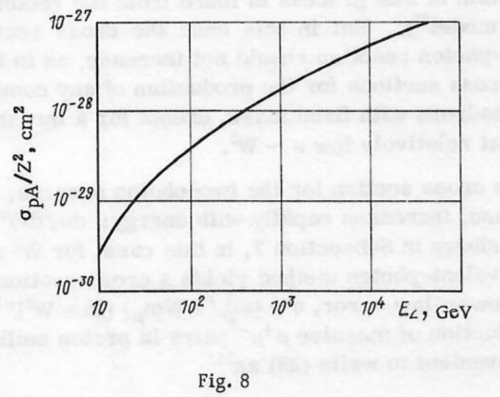


Fig. 8

Contribution to the total cross section of high-energy hadron processes. In the interaction of cosmic protons with nuclei in the atmosphere or in an emulsion, two-photon production of e^+e^- pairs makes an important contribution, which increases with the energy. Figure 8 shows the cross section calculated from (54) for this process as a function of the energy. For $E_L\sim10^4$ GeV, e.g., we have

$$\sigma_{pA\to pAe^+e^-}=0.7Z^2\ \ \mathrm{mb}. \tag{56}$$

Converting this cross section to the cross section per proton for nitrogen (Z = 7), we find 5% of σ_{pp} = 40 mb. Accordingly, this effect could turn out to be important in interpretation of the results of the experiment described in ref. 42.

Study of the pion and kaon form factors. Geshkenbein and Terent'ev[43] recently suggested that $\pi A\to\pi Al^+l^-$ and $KA\to KAl^+l^-$ reactions be used to study the π and K form factors. For $(Z\alpha)^2/|q_2^2|>R^2$ in these reactions (where R is the nuclear radius), the two-photon contribution (Fig. 1) predominates over the bremsstrahlung contribution (Fig. 3, a and c), so the π (or K) form factor can be determined from the known cross section (17). Scattering by the nucleous is coherent, and the nuclear form factor (the lower part of Fig. 1) can be neglected for $|q_2^2|<R^{-2}$. For $Z\alpha\sim1$ these conditions are equivalent.

To determine the form factor $F_\pi(q_1^2)$ in the region $-q_1^2\sim1\ \mathrm{GeV}^2$, we can carry out calculations for scattering of pions by lead. The conditions stated above are satisfied if the pion energy loss is at least 30-50 GeV; in this case cross sections of $\sim10^{-34}$ cm^2 must be measured, and the momentum of the recoil nucleus must not exceed 10-20 MeV.

Possible calibration of accelerators with colliding pp and $p\bar{p}$ beams. One of the principal parameters of a colliding-beam accelerator is the luminosity, L, defined by $\dot{N}=L\sigma$, where $\dot{N}$ is the number of events per unit time for a process with a cross section σ. If a process having a cross section σ which is not too small is understood sufficiently well, if it can be clearly distinguished from other processes, and if it is convenient for detection, experimental study of this process would constitute an independent method for determining the luminosity L of the apparatus. In this case there is no need for detailed information about the distribution of the beam density in the interaction region, which is difficult to measure.

For accelerators with e^+e^- colliding beams the problem of choosing such a calibration process is one of selecting the most suitable process from the many for which calculations have been carried out on the basis of quantum electrodynamics. For example, the basic calibration process used at the VÉPP (colliding electropositron beams) installation (at Novosibirsk) and at Orsay is the double-bremsstrahlung process. In accelerators with pp colliding beams, calculations for nearly all the reactions must use strong-interaction models, i.e., accurate calculated results cannot be expected. Such reactions are unsuitable for calibration.

It was shown in ref. 37 that calculations can be carried out on the basis of quantum electrodynamics with an error of $\sim 10^{-3}$ for the $pp \to ppe^+e^-$ reaction if the energies and emission angles of the $e^\pm$ in the c.m. system of the protons satisfy the inequalities $\varepsilon_\pm < 20 m_e E/m_p$; $|k_\perp|$, $W < 20 m_e$. In this region the cross section is described completely by the diagram in Fig. 1 and the radiation corrections to it. The contribution of this region to cross section (48) is at least 0.1 mb. This process may thus be useful for determining the luminosity in accelerators with pp colliding beams. In this range of $e^\pm$ energies the protons essentially stay within the beam, and detailed knowledge of their form factors is not necessary. The differential cross section was calculated within this error in ref. 37, and the importance of possible background processes was discussed.

Measurement of the polarization of high-energy γ's. The azimuthal asymmetry of the recoiled electron in the $\gamma e \to ee^+e^-$ reaction is governed by the photon polarization. Boldyshev and Peresun'ko suggested the use of this reaction to find the photon polarization.[44] At a photon energy $\omega_L > 100\, m_e$ the cross section is described by the diagram in Fig. 2 and is given by (50); the quantities σ_{TT}, σ_{TS}, and τ_{TT} appearing in this cross section are calculated in Appendix 3. Polarization measurements require knowledge of the azimuthal distribution of the recoil electron integrated over the momentum of this electron above some threshold value $|p_2'|_{min}$. Vinokurov and Kuraev[44] recently obtained the corresponding distributions in analytic form. The asymmetry parameter is independent of the energy and essentially independent of $|p_2'|_{min}$ over the range $0.2 m_e < |p_2'|_{min} < 10 m_e$.

11. Some Reactions of High Order in α

The increase in accelerator energies opens up the possibility of measuring reactions of extremely high order in α. The cross section for the production of two e^+e^- pairs, $ee \to eee^+e^-e^+e^-$, can be calculated in the first logarithmic approximation from (30) from the known cross section for the reaction $\gamma\gamma \to e^+e^-e^+e^-$. In contrast with $\sigma_{\gamma\gamma \to e^+e^-}$, which decreases with increasing W^2, the cross section for $\gamma\gamma \to e^+e^-e^+e^-$ is asymptotically constant, equal to 6.45 μb.[45] The cross section for the reaction[46] $ee \to eee^+e^-e^+e^-$ is thus:

$$\sigma = (\alpha^2/6\pi^2)\, \sigma_{\gamma\gamma \to e^+e^-e^+e^-}\, [\ln s/m_e^2]^4, \qquad (57)$$

equal to $6 \cdot 10^{-31}$ cm^2 for $\sqrt{s} = 7$ GeV.

This reaction can contribute an appreciable background in measurements of the total cross section for $\gamma\gamma \to$ hadrons, particularly for the case of photoproduction in nuclei, $\gamma A \to Ae^+e^-e^+e^-$.[47]

In ee colliding beams the cross section for the production of parapositronium P_s in the free state turns out

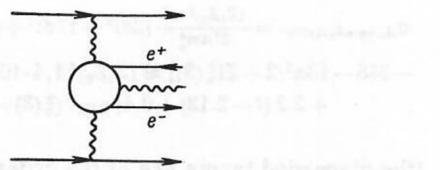


Fig. 9

not to be too small.[48] It is of the same order of magnitude as the cross section for the production of C-even hadron resonances:

$$\sigma = (\alpha^7 \xi(3)/3m_e^2)\, [\ln s/m_e^2]^3 = 6.6 \cdot 10^{-37} [\ln s/m_e^2]^3 \text{ cm}^3;$$

$$\xi(3) = 1.202. \qquad (58)$$

The P_s velocity may turn out to be so high that this particle converts into orthopositronium in the magnetic field of the accelerator, and it may even decay into an e^+e^- pair.

Fadin and Khoze[49] noted that in ee colliding beams two-photon production of an $e^+e^-\gamma$ system (Fig. 9) predominates in the production of protons having an energy of the order of 1 MeV at small angles, $\theta \sim 1$. This process is a radiation correction to the process in Fig. 1; its cross section is

$$d\sigma \sim [\alpha^5/(m_e^2 + \omega^2)] \ln^2 s/m_e^2 (d\omega/\omega) d\Omega/2\sin^2\theta. \qquad (59)$$

On the other hand, the cross sections for single and double bremsstrahlung in this region fall off as α^3/s and α^4/s. Their contributions are negligible under the condition $m^2 + \omega^2 \ll \alpha^2 s$.

12. Production of Massive $\mu^+\mu^-$ Pairs in Hadron Collisions

The Lederman group[50] carried out an experimental study of the production of massive $\mu^+\mu^-$ pairs ($W \sim 1$-6 GeV) in collisions of protons with a uranium target at an energy of $E_L \sim 20$-30 GeV. The results of this experiment attracted much interest.[51-53] Possible mechanisms for this process are being discussed, and plans are being made to continue these experiments at much higher energies (ISR and NAL). Here we will be concerned only with the role of the two-photon channel in this process.

At low energies the two-photon contribution is negligible,[26] and the one-photon channel predominates in the experiments of ref. 50. One-photon production of a $\mu^+\mu^-$ pair consists of the production of a hadron system with $J = 1$ and a subsequent transition $\to \gamma \to \mu^+\mu^-$ (the simplest description of this process is found from the vector-dominance model[51]). But in this case the cross section for the one-photon reaction should not increase, as in the case of the cross sections for the production of any combination of hadrons with fixed mass, except for a threshold increase at relatively low $s \sim W^2$.

The cross section for the two-photon process, on the other hand, increases rapidly with energy: $d\sigma/dW^2 \propto \ln^3 s$. As was shown in Subsection 7, in this case, for $W^2 \gg m_\rho^2$, the equivalent-photon method yields a cross section $d\sigma/dW^2$ with a power-law error, $\eta \sim (m_\rho^2/4\, W m_\mu)\, [\ln s/W^2]^{-1}$. For the production of massive $\mu^+\mu^-$ pairs in proton collisions it is convenient to write (28) as[27]

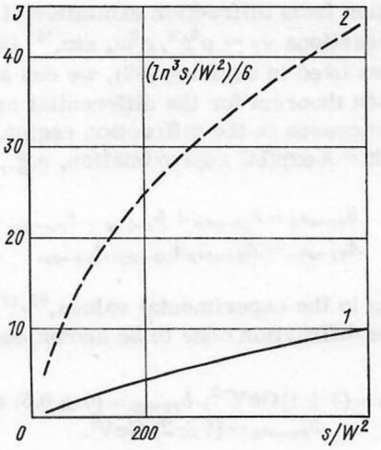


Fig. 10

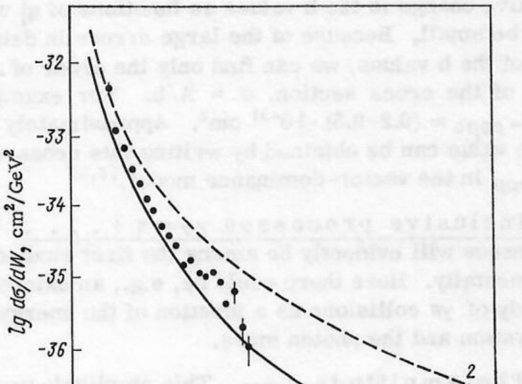


Fig. 11

$$d\sigma/dW^2 = (2\alpha/\pi)^2 (\sigma_{\gamma\gamma}/W)\, J\,(s/W^2). \qquad (60)$$

The function J(k) is calculated from the spectrum (26d). Curve 1 in Fig. 10 shows the computer-calculated J(k), while curve 2 shows the same value in the first logarithmic approximation, (30): $J(k) = (\ln^3 k)/6$. In Fig. 11 the resulting $d\sigma/dW$ values for $S = 2500\ \mathrm{GeV}^2$ are shown along with the data of ref. 50, converted to correspond to pp collisions at $s \approx 50\ \mathrm{GeV}^2$. We see that for $W > 2$ GeV these cross sections are of the same order of magnitude.

Accordingly, if the one-photon contribution does not increase rapidly with increasing s, we conclude that for $s \geqslant 1000\ \mathrm{GeV}^2$ (in the ISR experiments) the two-photon mechanism for pair production must be taken into account.[27,53] The contribution of the process involving stream formation (Fig. 7) is of the same order of magnitude as the calculated contribution (Fig. 1).

IV. TWO-PHOTON PRODUCTION OF HADRONS

Of particular interest among the two-photon production processes is hadron production. Primakoff,[54] who was the first to analyze this process, suggested measurement of the π^0 lifetime in a reaction like that in Fig. 2. In 1960 Low[11] pointed out that the π^0 lifetime could also be measured in a reaction like that of Fig. 1 in the case of e^+e^- colliding beams. At the same time Cologero and Zemach[10] analyzed the two-photon reaction $ee \to ee\pi^+\pi^-$. However, such processes could not be observed in the early accelerators with ee colliding beams, and these experimental studies have not been developed further.

Two-photon production of hadrons in ee colliding beams has attracted much attention in recent years. The production of π- and K-meson pairs has been studied.[12,13,15] In actual colliding-beam experiments one can not only measure the cross section for a reaction such as $ee \to ee\pi\pi$, but also extract information about the new reaction $\gamma\gamma \to$ h (hadrons)[14,25] (see also refs. 16, 29, 55, and 56). In this case there is the unique possibility of studying the amplitudes for $\gamma\gamma \to$ h transitions and of measuring their dependences on both W^2 and on the "masses" q_1^2 and q_2^2 of the two photons. This problem has not yet been studied experimentally, so it is worthwhile to briefly discuss from the theoretical point of view the actual object of the study: The various channels of the $\gamma\gamma \to$ h transition. This discussion is found in Subsections 13 and 14.

Significantly, the cross section for two-photon production of hadrons increases as $\ln^4 E$, and for accelerators with e^+e^- colliding beams this channel must predominate at $\sqrt{s} \geqslant 4$ GeV. Various possible experiments are discussed in Subsections 15–17.

13. Object of Study. The $\gamma\gamma \to$ h Transition on the Mass Shell ($|q_i^2| \ll W^2, m_\rho^2$)

Although the $\gamma\gamma \to$ h transition is not being studied experimentally at present, current knowledge of hadron and lepton–hadron collisions leads to several predictions, some of which are quite reliable. The major prediction refers to the nature of the W^2 dependence of the total cross section for the $\gamma\gamma \to$ h transition: It must have the form shown in Fig. 12. Near the threshold $W^2 = 4m_\pi^2$, we find $\sigma_{\gamma\gamma} \sim (\alpha/m_\pi)^2(1-4m_\pi^2 W^{-2})^{1/2}$. As W increases, in contrast with the quantum-electrodynamics result, the cross section does not decrease. The basic contribution is at first due to the S wave for $\pi\pi$ scattering (the ε meson). Then $\sigma_{\gamma\gamma}$ displays several resonance peaks, and at even larger W it gradually converts into a constant, $\sigma_{\gamma\gamma}(\infty)$, as the cross section for any hadron process does.

Pion production near the threshold. The cross section for the reaction $\gamma\gamma \to \pi^+\pi^-$ should presumably the approximately equal to that calculated in quantum electrodynamics. Current algebra and PCAC (partially conserved axial-vector current) can be used to calculate the deviation of this cross section from the quantum-electrodynamics cross section. The cross sections calculated for the reactions $\gamma\gamma \to n\pi^0$, $\gamma\gamma \to \pi^+\pi^-\pi^0$ ... on the basis of these models turn out to be small; for the reaction $\gamma\gamma \to n\pi^0$, e.g., the matrix element falls off as $(\text{momentum})^{n+1}$.[57]

Cross section for the reaction $\gamma\gamma \to \pi\pi$ for $W \leqslant 1$ GeV. This cross section can be related to the amplitude for $\pi\pi$ scattering by means of the standard methods used in low-energy physics,[58] and new information can be obtained on the S wave for $\pi\pi$ scattering with

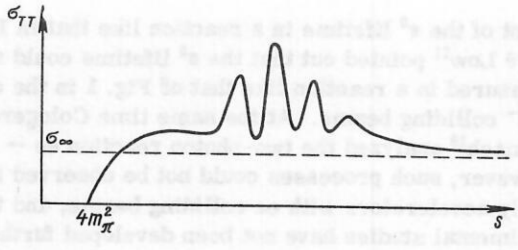


Fig. 12

T = 0, i.e., information the so-called ε meson. This problem was discussed in refs. 12, 29, and 59.

The $\gamma\gamma \to 3\pi$ reaction. This reaction is of interest in connection with the problem of the $K_L \to \mu^+\mu^-$ decay; it has been studied on the basis of the model including ρ and ε mesons and an account of the PCAC limitations at the threshold.[60]

This case presents the possibility of the most accurate study of C-even resonances[25,61] with $J \neq 1$. Except for the ε meson, all known resonances are narrow, with $\Gamma \ll M$; within $(\Gamma/M)^2$ we can write the following cross section for the production of a resonance R with mass M, spin J, and two-photon width $\Gamma^{\gamma\gamma}$.[11,61]

$$\sigma_{\gamma\gamma\to R} = 8\pi^2 (2J+1)\,\Gamma^{\gamma\gamma}\delta(W^2 - M^2)/M. \quad (61)$$

The position and width of the peak on the $d\sigma/dW^2$ curve give the mass and width of the resonance, and the corresponding cross section yields $\Gamma^{\gamma\gamma}(2J+1)$. The spin of the resonance can be determined by comparing $\sigma_{\gamma\gamma\to R}$ with $\Gamma^{\gamma\gamma}$ and by analyzing the angular distribution. The widths $\Gamma^{\gamma\gamma}$ have been calculated by means of finite sum rules[62] on the basis of tensor dominance,[63] SU3, and SU3 × SU3 models.[64]

It is interesting to study the X^0 and E mesons in connection with the problem of the pseudoscalar nonet; the spins of these mesons are not yet known sufficiently accurately. For the two possible values of the X^0 spin, J = 0 and J = 2, according to ref. 64, we have the ratio $\Gamma^{\gamma\gamma}(J=2)/\Gamma^{\gamma\gamma}(J=0) \approx 10^{-3}$. We can thus expect to obtain quite reliable conclusions regarding the spin. Detection of the E meson in these experiments would rule out the possibility $J^P(E) = 1^+$ which is currently being discussed.

Asymptotic region, $W^2 \gg 1\ \mathrm{GeV}^2$. In this region the cross section can be evaluated by using the factorization theorem for hadron processes (the theorem is valid if the Pomeranchuk singularity predominates in the photon scattering)[25]

$$\sigma_{\gamma\gamma}(\infty) = \sigma^2_{\gamma p}/\sigma_{pp} = 3\cdot 10^{-31}\,\mathrm{cm}^2 \approx (\alpha/m_\pi)^2/3. \quad (62)$$

On the basis of all the evidence available, we would conclude that the total contribution from all the resonances is large. Accordingly (by virtue of duality) we would have, for finite $W^2 \gtrsim 5\ \mathrm{GeV}^2$, $\sigma_{\gamma\gamma}(W^2) > \sigma_{\gamma\gamma}(\infty)$. This difference should fall off according to W^{-1} (the contribution of the P' and A_2 trajectories).

The reactions $\gamma\gamma \to \rho^0\rho^0, \rho^0\omega, \ldots$. For $W^2 \gg 1\ \mathrm{GeV}^2$ there should be a large contribution to the cross section from diffraction excitation of photons, i.e., from the reactions $\gamma\gamma \to \rho^0\rho^0, \rho^0\omega$, etc.[14] Using the same assumptions used in deriving (62), we can also write a factorization theorem for the differential cross sections of these processes in the diffraction region. Then the usual $d\sigma/dt = A\exp(bt)$ approximation, e.g., yields

$$\left.\begin{aligned} b_{\gamma\gamma\to\rho^0\omega} &= b_{\gamma p\to\rho^0 p} + b_{\gamma p\to\omega p} - b_{pp\to pp};\\ A_{\gamma\gamma\to\rho^0\omega} &= A_{\gamma p\to\rho^0 p}A_{\gamma p\to\omega p}/A_{pp\to pp}. \end{aligned}\right\} \quad (63)$$

Substituting in the experimental values,[65,66] we find the slope of the diffraction cone to be anomalously low:

$$b_{\gamma\gamma\to\rho^0\rho^0} = (3\pm 1)\,\mathrm{GeV}^{-2};\ b_{\gamma\gamma\to\varphi\varphi} = (0\pm 0.5)\,\mathrm{GeV}^{-2};\ b_{\gamma\gamma\to\rho^0\omega} = (4\pm 2)\,\mathrm{GeV}^2. \quad (64)$$

Effects related to the motion of the Pomeranchuk singularity $\alpha_P(t)$ can therefore be seen clearly. Hopefully, the relative change in the b values as functions of q_i^2 will not also be small. Because of the large errors in determination of the b values, we can find only the order of magnitude of the cross section, $\sigma = A/b$. For example, $\sigma_{\gamma\gamma\to\rho^0\rho^0} \approx (0.2-0.5)\cdot 10^{-31}\ \mathrm{cm}^2$. Approximately the same value can be obtained by writing this cross section $\sigma_{\rho\rho\to\rho\rho}$ in the vector-dominance model.[56,57]

Inclusive processes $\gamma\gamma \to \pi + \ldots$. These processes will evidently be among the first studied experimentally. Here there could be, e.g., an extension to a study of $\gamma\pi$ collisions as a function of the energy of the $\gamma\pi$ system and the photon mass.

The amplitude τ_{TT}. This amplitude was determined in (16). The quantity $W^2\tau_{TT}$ is the absorptive part of the amplitude for forward $\gamma\gamma$ scattering corresponding to the transformation of a pair of right-polarized photons into a pair of left-polarized photons. As $W^2 \to \infty$, this amplitude receives no contribution from Regge poles having $\alpha(0) \geq 0$. If there are no fixed singularities with Re j ≥ 0 in the amplitude, we find a superconverging sum rule, as follows from ref. 68:

$$\int \tau_{TT}(W^2)\,dW^2 = 0. \quad (65)$$

For resonances with spin J = 0, we have

$$\tau_{TT} = 2\sigma_{\gamma\gamma}. \quad (66)$$

There is no such relation for resonances with $J \neq 0$. Furthermore, the τ_{TT} value is even negative for some of these resonances, as can be seen from (65).

14. Object of Study. The $\gamma\gamma \to h$. Transition off the Mass Shell

Of particular interest are $\gamma\gamma \to h$ transitions with nonvanishing masses for both photons ($q_i^2 \neq 0$) (or with a nonvanishing mass for at least one of them). Models worked out for γp scattering which describe the dependence on the mass of one of the photons begin to yield noticeably different results for the dependence on the masses of two photons. Objects which cannot be studied in other reactions appear in this case. In particular, one can extract information about the properties of bilocal operators on the light cone. This is an important supplement to the results in a study of the product of local operators in γp scattering.

By analogy with γp scattering we would expect σ_{TT} to fall off with increasing $-q_i^2$, and we would expect σ_{ST}, σ_{TS}, and σ_{SS} to at first increase and then decrease with increasing $-q_i^2$. The characteristic scale for the changes is $\sim m_\rho^2$.

Study of the reaction $\gamma\gamma \to \pi\pi$. By studying this reaction one can determine all five invariant amplitudes and thereby carry out a detailed study of the descent from the mass shell. The values of these amplitudes are important for determining the difference between the π^+ and π^- masses.[69]

By studying the reaction $\gamma\gamma \to \pi + \ldots$ near the threshold, one can extract information on the commutator of the vector and axial currents, important for current algebra.[70]

In addition to the C-even resonances with J = 0, 2, ..., for $q_i^2 \neq 0$ resonances with $J^P = 1^+$, C = + may be produced, e.g., A1 (1070). In this case one of the photons must be scalar.

In the region $W^2 \gg q_1^2 q_i^2/m_\rho^2$, m_ρ^2, we can write a factorization theorem for the cross sections which holds if the Pomeranchuk singularity predominates in this region also [see (62) and refs. 25 and 71]:

$$\sigma_{ab}(\infty, q_1^2, q_2^2) = \sigma_a^{\gamma p}(\infty, q_1^2)\,\sigma_b^{\gamma p}(\infty, q_2^2)/\sigma_{pp}(a, b = T, S). \quad (67)$$

In particular, we have $\sigma_{ST}/\sigma_{TT} = \sigma_S^{\gamma p}/\sigma_T^{\gamma p} \approx 0.18$.[72]

One of the most interesting questions for the region $|q_i^2| > m_\rho^2$ is whether the cross section for $W^2 \gg m^2$ is dependent on essentially only one dimensional parameter $W^2/q_1^2 q_i^2$ or whether it is dependent on two dimensionless parameters, e.g., W^2/q_1^2 and W^2/q_2^2 (cf. ref. 16).

A dependence on the single parameter $W^2/q_1^2 q_2^2$ is found if it is assumed that (67) holds for the contribution of each of the Regge trajectories and that there are no other contributions,[71,73] in the parton model[74] in a study of the products of bilocal operators near the light cone,[75] and in the φ^3 model.[71] In the field model in which scale invariance is assumed at small distances, a weak dependence on the parameter $q_1^2 q_2^2$ also arises.[71]

In the resonance region, W ~ 1 GeV, the algebra for bilocal operators near the light cone predicts that the amplitude for forward $\gamma\gamma$ scattering will depend on two parameters: W^2/m_ρ^2 and q_1^2/q_2^2.[75,76] In contrast with the case of γp scattering, according to ref. 75, the amplitude for forward $\gamma\gamma$ scattering does not vanish at any q_i^2 in the resonance region.

The behavior of the multiplicity n as a function of W^2 and q_i^2, differs markedly in the different models; e.g., we have $n \sim \ln W^2$ in the ordinary multiperipheral model[77] and $n \sim \ln W^2/q_1^2 q_2^2$ in the φ^3 model.

V. e^+e^- COLLIDING BEAMS

15. Methods for Extracting Information on the Reaction $\gamma\gamma \to$ Hadrons

Accurate discrimination of the two-photon channel requires detection of the scattered electrons. From the known electron momenta one can completely determine the parameters of the hadron system as a whole. By measuring the angular and energy distributions of these electrons by means of (17), one can in principle find six quantities characterizing the reaction $\gamma\gamma \to h$ – the quantities σ_{TT}, σ_{TS}, σ_{ST}, σ_{SS}, τ_{TT}, and τ_{TS} – as functions of W^2 and q_i^2. The coefficients $\rho_i(a, b)$ of these functions in (17) depend only on the electron momenta and in this case are completely determined. This procedure for extracting information on the $\gamma\gamma \to h$ reaction is completely analogous to the "lost-mass" method used in experimental studies of deep inelastic ep scattering.

To distinguish this reaction from purely electromagnetic processes such as $ee \to ee\,\gamma\gamma$, $ee \to eee^+e^-, \ldots$, the appearance of at least one hadron should be detected along with the scattered electrons. In the important region $q^2_{i\,min} \ll |q_i^2| < W^2$ or $E_i' \gg m_e$, $[1 - (E_i'/E)]/E_i' \ll \theta_i \ll 1$ (θ_i is the scattering angle of the i-th electron), Eqs. (17) simplify considerably:[14]

$$\left.\begin{aligned} &d\sigma/dE_1'\,dE_2'\,d\Omega_1\,d\Omega_2 = (\alpha/8\pi^2)^2 \\ &\times\{(E^2+E_1'^2)(E^2+E_2'^2)/\Big[E^4(E-E_1')(E-E_2') \\ &\times \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta^2}{2}\Big] \times \sigma^{\exp}_{\gamma\gamma\to h}; \\ &\sigma^{\exp}_{\gamma\gamma\to h} = \sigma_{TT} + \varepsilon_1\sigma_{TS} + \varepsilon_2\sigma_{ST} + \varepsilon_1\varepsilon_2 \\ &\times[\sigma_{SS} + \tau_{TT}\cos(2\varphi)/2 + \delta\tau_{TS}\cos\varphi]; \end{aligned}\right\} \quad (68)$$

$$\varepsilon_i = 2EE_i'/(E^2+E_i'^2); \quad \delta = (E+E_1')(E+E_2')/E\sqrt{E_1'E},$$

where φ is the azimuthal angle between the electron scattering planes.

More detailed information can be extracted by measuring the distribution of product hadrons along with electrons. Since the virtual photons are polarized, there are also contributions to (68) in this case due to the interference of two-photon states with different helicities (cf. Subsection 6).

In a different formulation of the experiment, in which electrons are not detected (e.g., in wide-angle experiments like those of ref. 2), all the product hadrons must be detected if extensive information is to be extracted about the $\gamma\gamma \to h$ reaction.[13)] This detection is complicated by the presence of neutral hadrons and hadrons emitted at small angles. It is therefore useful to know the characteristic features of the distribution of product hadrons.

Extremely characteristic for the production of an arbitrary hadron system in the two-photon channel is a $dk_\perp^2/k_\perp^2$ distribution as in (35) with respect to the total transverse momentum $k_\perp$ in the region $|k_\perp| > m_e$. If even

TABLE 2

$R(J)$	$\varepsilon(0)$	$X(0)$	$X(2)$	$\pi_N(0)$	$\eta_0(0)$	$f(2)$	A_2	E
M, MeV	700	958	958	1016	1060	1260	1300	1420
$\Gamma^{\gamma\gamma}$, keV	20	100	0.1	1.25	5	6	30	240
$\sigma\cdot10^{33}$, cm²	2	12	0.06	0.12	0.4	1.2	5	5

one of the hadrons is not detected, the distribution with respect to total transverse momentum of the remaining hadrons becomes smoother.

Another characteristic feature of the production of a hadron pair is a $d\psi/\psi$ distribution with respect to the angle ψ giving the deviation from coplanarity of the particles of the pair, (42). The sharpness of this distribution has already been used to identify two-photon production of e^+e^- pairs in colliding beams.[2]

16. Estimates of Observable Quantities

To understand the feasibility of these experiments, we must estimate the corresponding cross sections. Crude estimates are sufficient here, and only crude estimates can be found, because of the extreme scarcity of information on two-photon amplitudes. The equations below are derived in such a manner that essentially all the uncertainty is incorporated in the quantities $\sigma_{\gamma\gamma}$.

To evaluate the possible measurement of cross sections on the mass shell it is sufficient to use the results obtained in the equivalent-photon method:

$$d\sigma=(2\alpha/\pi)^2\,(dW^2/W^2)\,\sigma_{\gamma\gamma}(W^2)\,J\,(s/W^2). \qquad (69)$$

If electrons scattered through all angles are detected, we find that J(k) is given by (31). If, on the other hand, only electrons scattered through some angles which are not too large, $\theta_i < \theta_{max}$ and $\gamma = (2E/m_e)\sin\theta_{max}/2 > m_\rho/m_e$, are detected, we find that $J(k, \gamma)$ is given by (49). Figure 13 shows plots of $J(k, \gamma)$ for this case.

At energies currently attainable the principal contribution to the cross section should come from the productionof resonances,whose cross section increases as $\ln^3 E$. This cross section can be found by substituting (61) into (69):[31]

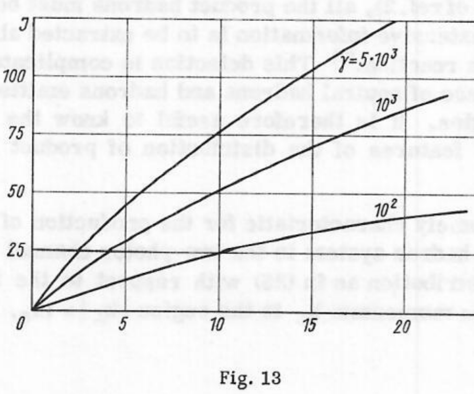


Fig. 13

$$\sigma_{ee\to eeR}=32\alpha^2\,(2J+1)\,J\,(s/M^2)\,\Gamma^{\gamma\gamma}/M^3. \qquad (70)$$

The values of this cross section calculated for[25] $\sqrt{s} = 7$ GeV are shown in Table 2. For the ε meson with a width $\Gamma \sim 400$ MeV, approximation (61) is too crude.

To evaluate the possible measurement of $\sigma_{\gamma\gamma}(W^2)$ for large W^2 we use the asymptotic value (62) and an averaging interval $\Delta W^2/W^2 \sim 1/3$. Then the values of the measurable cross sections are, according to (69) and Fig. 13, $\Delta\sigma_{ee\to eeh} \approx 10^{-34}$ cm², and we have $\Delta\sigma_{ee\to e\rho^0\rho^0} \approx 10^{-35}$ cm² for $\sqrt{s} = 7$ GeV, $W^2 = 5$ GeV², and $\theta_m = 10^0$.

Each integration over q_i^2 contributes a large factor $2\ln(m_\rho E/m_e W) \sim 15$. This integration does not occur in a measurement of the q_i^2 dependence of the cross sections, for which case this factor is replaced by $\Delta q_i^2/q_i^2$, i.e., in this case cross sections smaller by a factor of 30 must be measured (for an averaging interval of $\Delta q_i^2/q_i^2 \sim 1/2$). We thus find $\Delta\sigma \sim 3\cdot10^{-36}$ cm² if the dependences of W^2 and one of the q_i^2 are measured, and we find $\Delta\sigma \sim 10^{-37}$ cm² if dependences on W^2, q_i^2, and q_2^2 are measured.

In particular, to take into account the q_1^2 dependence in the region $q^2_{min} \ll |q_1^2| < W^2$ we should retain in (17) only the terms with σ_{TT} and σ_{ST}, integrating them over q_2^2 and ω_1 (for fixed $W^2 = 4\omega_1\omega_2$); this procedure yields,[14] by analogy with (69),

$$d\sigma=2\,(\alpha/\pi)^2\,(dq_1^2dW^2/q_1^2W^2)\,[J'_{TT}\sigma_{TT} + J'_{ST}\sigma_{ST}]. \qquad (71)$$

Graphs of the functions J'_{TT} and J'_{ST} are given in ref. 25.

To find the far asymptotic behavior of σ it is sufficient to substitute into (69) simply $\sigma_{\gamma\gamma}(\infty)$ from (62); we find

$$\sigma=8\,(\alpha^2/\pi m_\pi)^2\,(\ln E/m_e)\times(\ln E/m_e+2\ln m_\rho/m_e)\,\ln^2 E/m_\pi/9. \qquad (72)$$

Accordingly, σ increases as $\ln^4 E$. Up to $\sqrt{s} \approx m_\pi m_\rho/m_e \approx 100$ GeV, the last term, $\sim 2\ln(m_\rho/m_e)\ln^3 E$, predominates.

17. Other Hadron-Production Channels and Background

For not too high beam energies the principal hadron-production channel is the annihilation channel (Fig. 14). As $E \to \infty$, the cross section for this process falls off no more slowly than E^{-2} (see, e.g., ref. 79). The same hadron system is produced in processes similar to two-photon annihilation (Fig. 15, a and c) or similar to the bremsstrahlung emission of a hard photon (Fig. 15b). In the integration over the hadron mass W the basic contribution comes from the region $W^2 \ll s$. In the known expressions for the electromagnetic cross sections, this integration can be supplemented by ln E for each "heavy" photon, and as $E \to \infty$ we find[80,81]

$$\left.\begin{array}{l}\sigma_{15a}\leqslant(\alpha^3/E^2)\,(\ln E/m_e+C\ln^2 E/s_0);\\ \sigma_{15b}=(C_1\alpha^4/m_\rho^2)\,(\ln 16E^4/m_e m_\rho^3)\,\ln m_\rho/m_e;\\ C_1=8\pi/3\gamma_\rho^2\approx 1;\ s_0\sim 1\,\text{GeV}^2; C=\text{const.}\end{array}\right\} \qquad (73)$$

The process shown in Fig. 15c interferes with the two-photon annihilation (Fig. 15d), and we would expect to find

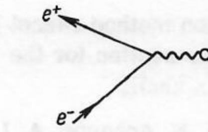


Fig. 14

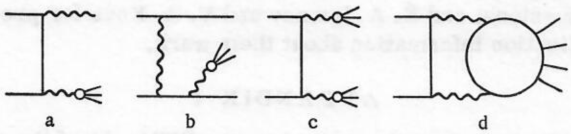


Fig. 15

$$\sigma_{15,\,c,\,d} \leqslant (\alpha^4/E^2)\ln^3 E. \qquad (74)$$

Clearly, the two-photon mechanism is the predominant mechanism for high-energy hadron production, and it must remain predominant for $\sqrt{s} \geqslant 4$ GeV.

We turn now to the background. If both scattered electrons are detected, all annihilation processes are automatically cut off. In order to cut off purely electromagnetic processes ($ee \to eee^+e^-$, $ee \to ee\gamma\gamma$), whose cross sections are very large, we must also detect at least one of the product hadrons. Then two-photon production does not differ from the bremsstrahlung production of hadrons (Fig. 15b). These processes do not interfere because of the difference in C parity.

The total cross section for bremsstrahulung production is smaller than the two-photon cross section by a factor $(m_\rho/m_\pi)^2$; the ratio of differential cross sections is approximately the same for $|q_i^2| < W^2$. Accordingly, the mass spectrum of hadrons produced in the process of Fig. 15b is $\alpha^2\, dW^2/W^4$, while for two-photon production we have $\sigma_{\gamma\gamma}(W^2, q_i^2)dW^2/W^2 \approx \alpha^2 dW^2/W^2 m_\pi^2$, i.e., $d\sigma_{15}/d\sigma_1 \sim m_\pi^2/W^2$. The discrimination of this contribution in the region $|q_i^2| \sim W^2$ was discussed in ref. 16. We note further that the bremsstrahlung-production contribution can be calculated from the familiar cross section for one-photon annihilation at low energies.

18. Other Methods for Studying the $\gamma\gamma \to$ Hadrons Reaction

Photoproduction from nuclei (the Primakoff effect) is a well-known mechanism for two-photon production of particles.. This method has been used to measure the width η_0 and to find the most accurate value of the π^0 lifetime.

In principle this mechanism can also be used to extract information about the $\gamma\gamma \to$ hadrons reaction.[54,82] The possible discrimination of the contribution of this mechanism against the background of ordinary photoproduction depends on the smallness of $k_\perp$ in the two-photon production (Subsection 9) and the characteristic distribution with respect to $k_\perp$, which has the form shown in Fig. 6: $k_\perp^2 dk_\perp^2(|k_\perp^2| + W^4/4\omega^2)^{-2}$. Observation of this dependence near the peak requires either measurement of the

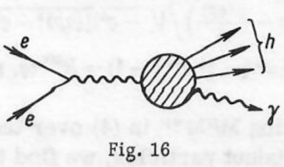


Fig. 16

momentum of the recoil nucleus or very accurate measurement of the momenta of all the product particles. Because of the smallness of $k_\perp$ (and also of the mass q^2 of the virtual photon) the cross section for the reaction $\gamma\gamma \to$ h in this case can be studied only on the mass shell. (In the case of photoproduction from $Z \gg 1$ nuclei, the smallness of the effective values of q^2 is still related to the cutoff due to the nuclear form factor.)

The cross sections for two-photon production are small; e.g., for two-photon production of the ε meson in xenon for $\omega \sim 20$-40 GeV this cross section is on the order of 10 μb.

All this discussion can be transferred essentially intact to electroproduction from nuclei.

Annihilation production of a C-even hadron system. This process (Fig. 16) in e^+e^- colliding beams was discussed in ref. 83. The cross section for this process is

$$\sigma_{16} \leqslant (\alpha^3/E^2)\ln E. \qquad (75)$$

At intermediate energies we can study radiative decay of vector mesons in this case. The corresponding estimates were made in ref. 83. Note should be taken of the difficulty in discriminating against the background of the process in Fig. 15a.

VI. CERTAIN OTHER APPLICATIONS

18. Production of Intermediate Vector Bosons W^+

Ter-Isaakyan and Khoze[84] analyzed the two-photon production of W^+W^- pairs. The cross section for the reaction $\gamma\gamma \to W^+W^-$ increases with increasing energy (in contrast with all the other reactions discussed above). In the absence of an anamalous magnetic moment we have

$$\sigma_{\gamma\gamma\to W^+W^-} = 5\pi\alpha^2 W^2/24 m_W^4 \;(W^2 \gg m_W^2). \qquad (76)$$

From this expression we find a rapidly increasing cross section for the production of W^+W^- pairs in ee collisions, using the equivalent-photon method:

$$\sigma_{ee\to eeW^+W^-} = (5\alpha^4 s/54\pi m_W^4)\,[\ln s/m_e^2]^3. \qquad (77)$$

For $m_W = 5$ GeV this cross section exceeds the cross section for annihilation production, $e^+e^- \to W^+W^-$, at $E = 160$ GeV. For the production of W^+W^- pairs in pp collisions there are no logarithmic factors $\sigma_{pp\to ppW^+W^-} \sim \alpha^4 s/m_W^4$.

19. Production of Dirac Monopoles G

In 1931 Dirac pointed out that there could be a particle carrying an elementary magnetic charge g (the magnetic monopole), where $g = h/2e$ ($n = 1, 2, 3, \ldots$). Since then various methods have been used in an experimental search for the monopole. As a result of this search it can be stated that for $m_g \leqslant 5$ GeV the cross section for monopole production is $< 10^{-43}$ cm^2.[85]

Cabibbo and Ferrari[86] recently pointed out the possibility of two-photon production of monopoles. Since the

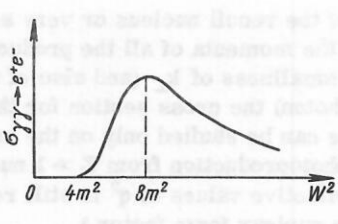


Fig. 17

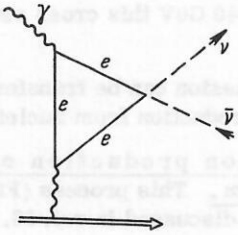


Fig. 18

interaction of monopoles with light is strong, it is clear by analogy with hadrons that the cross section for the two-photon production of n photons (through a virtual state $g\bar{g}$) is of the same order of magnitude as the cross section for the production of $g\bar{g}$. Accordingly, if Dirac monopoles in fact exist, in the case of pp scattering with $s > 4m_g^2$ we can observe a nearly isotropic background of photon having a resultant effective mass of $2m_g$.

20. Production of Heavy Leptons $L^{\pm}$

The possibility has still not been ruled out that $e^{\pm}$ and $\mu^{\pm}$ are accompanied by other charged heavy leptons $L^{\pm}$, whose production occurs by either a two-photon method (Fig. 1) or a one-photon method (Fig. 3). As Gershtein, Landsberg, and Folomeshkin[87] pointed out, if L is greater than 1/2 the cross section for the $\gamma\gamma \to L^+L^-$ reaction may increase with increasing energy, and the two-photon mechanism should become predominant at high energies.

The possible existence of leptons L with $m_e < m_L \sim (3\text{-}5)m_e$ is still being discussed.[88] The question of whether such leptons exist can be resolved in the experiments $ee \to e^+\ldots$, $eA \to eA^+\ldots$, without detection of L^+ and L^-. It is sufficient to detect simply the scattered electrons and study the lost-mass spectrum of the product system, as has been discussed in connection with the reaction $\gamma\gamma \to$ hadrons. The $\gamma\gamma \to e^+e^-$ cross section has a quasiresonance shape (Fig. 17). If a second lepton exists, a second peak should appear on the $\sigma_{\gamma\gamma}(W^2)$ curve (the double-bremsstrahlung contribution must be taken into account in the plotting of this curve, if this contribution is not eliminated experimentally).

21. Production of $\nu\bar{\nu}$ Pairs in Stars

Matinyan and Tsilosani[89] noted that two-photon production of $\nu\bar{\nu}$ (Fig. 18) could be important in astrophysics. The cross section for this process is negligibly small; for a photon frequency $\omega < m_e$,

$$\sigma = 1{,}25\ Z^2\alpha^3\ (\omega/m_e)^6 \cdot 10^{-49}\ \text{cm}^2. \qquad (78)$$

However, this process leads to an appreciable neutrino brightness for very dense "hot" stars.

The equivalent-photon method cannot be used in evaluating (78) since the cross section for the $\gamma\gamma \to \nu\bar{\nu}$ reaction vanishes on the mass shell.

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APPENDIX 1

We introduce the metric tensor $R^{\mu\nu}(a, b)$ of the subspace orthogonal to the 4-vectors a and b:

$$R^{\mu\nu}(a, b) = -g^{\mu\nu} + \frac{ab\,(a^\mu b^\nu + a^\nu b^\mu) - a^2 b^\mu b^\nu - b^2 a^\mu a^\nu}{(ab)^2 - a^2 b^2}. \qquad (A.1)$$

If the 3-momenta are directed along the z axis of the c.m. system of particles a and b, there will be only two nonvanishing components of $R^{\mu\nu}(a, b)$: $R^{xx} = R^{yy} = 1$. The tensor $R^{\mu\nu}$ evidently retains this form in any system moving along the z axis with respect to the c.m. system. The scalar product of any two vectors r_1 and r_2 transverse with respect to a and b can be written as

$$\begin{gathered}(r_{1\perp} r_{2\perp})\,|_{a,b} = -R^{\mu\nu}(a, b)\, r_1^\mu r_2^\nu = (r_{1\perp} r_2) = (r_1 r_{2\perp});\\ r_\perp^\mu\,|_{a,b} = R^{\mu\nu}(a, b)\, r^\nu.\end{gathered} \qquad (A.2)$$

To study two-photon production we use the vectors $q_{i\perp}$ — the parts of q_i orthogonal to p_1 and p_2 — and the vectors $\widetilde{p}_{i\perp}$ — the parts of p_i orthogonal to q_1 and q_2:

$$\begin{gathered}q_{i\perp}^\mu = -q_i^\nu R^{\mu\nu}(p_1, p_2);\quad \widetilde{p}_{i\perp}^\mu = -p_i^\nu R^{\mu\nu}(q_1, q_2),\\ q_{i\perp}^2 \text{ and } \widetilde{p}_{i\perp}^2 < 0.\end{gathered} \qquad (A.3a)$$

We denote the angle between the vectors $q_{i\perp}$ by φ, and we denote the angle between the vectors $\widetilde{p}_{i\perp}$ by $\widetilde{\varphi}$:

$$\cos\widetilde{\varphi} = -(\widetilde{p}_{1\perp}\widetilde{p}_{2\perp})/\sqrt{\widetilde{p}_{1\perp}^2 \widetilde{p}_{2\perp}^2};\quad \cos\varphi = -(q_{1\perp}q_{2\perp})/\sqrt{q_{1\perp}^2 q_{2\perp}^2}. \qquad (A.3b)$$

In the important region [see Eq. (21)]

$$m_i^2/(E-\omega_i)^2,\quad q_i^2/W^2,\quad q_i^2/\omega_i E,\quad q_i^2 m_i^2/(\omega_i E)^2 \ll 1,$$

these vectors are related in a simple manner:

$$\widetilde{p}_{i\perp} \approx -E q_{i\perp}/\omega_i;\quad \widetilde{\varphi} \approx \varphi. \qquad (A.4)$$

APPENDIX 2

We can rewrite (4) in terms of the helicity amplitudes for the reaction (Fig. 4) defined in the c.m. system of q and p_2.

The photon polarization vectors $e^\mu(\pm 1)$ are orthogonal with respect to both q and p_2, while the vector $e^\mu(0)$ is orthogonal to q and $e^\mu(\pm 1)$, just as it is orthogonal to $p_2^\mu - q^\mu(qp)/q_2^2$. We can therefore write [see (A.1)]

$$\left.\begin{gathered}e(0) = k = \left(p - \frac{q\,(qp)}{q^2}\right)\Big/\sqrt{-q^2/[(qp)^2 - q^2p^2]};\\ e^\mu(1)\, e^{*\nu}(1) + e^\mu(-1)\, e^{*\nu}(-1) = R^{\mu\nu}(q, p).\end{gathered}\right\} \qquad (A.5)$$

After integrating $M^\mu M^{*\nu}$ in (4) over the entire phase space $d\Gamma$ of the product particles, we find the absorptive part of the forward amplitude for the Compton effect, $W^{\mu\nu}$. Since helicity is conserved in forward scattering, and the amplitudes for the scattering of photons having helicities +1 and −1 are the same by virtue of P invariance, we can write this part of the amplitude in terms of the cross section for absorption of a virtual photon, a transverse photon σ_T, or a scalar photon σ_S:

$$\int M^{\mu}M^{*\nu}(2\pi)^4\delta(q+p-p')\,d\Gamma/2=W^{\mu\nu}=2\sqrt{(qp)^2-q^2p^2}\times(-R^{\mu\nu}\sigma_T+k^{\mu}k^{\nu}\sigma_S). \qquad (A.6)$$

Substitution of Eqs. (A.6) and (5) into (4) immediately yields (7)-(9) with

$$\left.\begin{aligned}&\rho(1,1)=[\rho(1,1)+\rho(-1,\,-1)]/2=\\&\rho^{\mu\nu}[e^{\mu}(1)\,e^{*\nu}(1)+e^{\mu}(-1)\,e^{*\nu}(-1)]=-\rho^{\mu\nu}R^{\mu\nu}(q,\,p);\\&\rho(0,\,0)=\rho^{\mu\nu}e^{\mu}(0)\,e^{*\nu}(0)=\rho^{\mu\nu}k^{\mu}k^{\nu}.\end{aligned}\right\}\quad (A.7)$$

We also note that the tensor $W^{\mu\nu}$ is obviously regular at $q^2 = 0$, while, according to (A.5), the factor $k^{\mu}k^{\nu}$ in (A.3) behaves like $(q^2)^{-1}$ as $q^2 \to 0$. It follows that $\sigma_S \propto q^2$ as $q^2 \to 0$.

For two-photon production the quantity $M^{\mu\nu}M^{*\mu'\nu'}$ in (15) becomes, after integration over $d\Gamma$, the absorptive part of the amplitude for forward $\gamma\gamma$ scattering, $W^{\mu\nu\mu'\nu'}$. In precisely the same manner as for the derivation of (A.6), we find the following result for this case:[68]

$$\begin{aligned}&\int M^{\mu\nu}M^{*\mu'\nu'}(2\pi)^4\delta(q_1+q_2-k)\,d\Gamma/2\equiv W^{\mu\nu\mu'\nu'}\\&=2\sqrt{X}\{R^{\mu\mu'}R^{\nu\nu'}\sigma_{TT}+R^{\mu\mu'}k_2^{\nu}k_2^{\nu'}\sigma_{TS}\\&+R^{\nu\nu'}k_1^{\mu}k_1^{\mu'}\sigma_{ST}+k_1^{\mu}k_1^{\mu'}k_2^{\nu}k_2^{\nu'}\sigma_{SS}\\&+[R^{\mu\nu}R^{\mu'\nu'}+R^{\mu\nu'}R^{\mu'\nu}-R^{\mu\mu'}R^{\nu\nu'}]\,\tau_{TT}/2\\&-[R^{\mu\nu}k_1^{\mu'}k_2^{\nu'}+R^{\mu\nu'}k_1^{\mu'}k_2^{\nu}+(\mu\nu\leftrightarrow\mu'\nu')]\,\tau_{TS}\\&+[R^{\mu\nu}R^{\mu'\nu'}-R^{\mu\nu'}R^{\mu'\nu}]\,\sigma_{TT}^{a}\\&-[R^{\mu\nu}k_1^{\mu'}k_2^{\nu''}-R^{\mu\nu'}k_1^{\mu'}k_2^{\nu}+(\mu\nu\leftrightarrow\mu'\nu')]\,\tau_{TS}^{a},\end{aligned}\qquad (A.8)$$

where $R^{\mu\nu} \equiv R^{\mu\nu}(q_1, q_2)$, $k_1 = \sqrt{-q_1^2/X}[q_2 - q_1(q_1q_2/q_1^2)] = e_1(0)$, $k_2 = e_2(0) = \sqrt{-q_2^2/X}[q_1 - q_2(q_1q_2/q_2^2)]$, and $X = (q_1q_2)^2 - q_1^2q_1^2$. Substitution of Eq. (A.8) into Eq. (15) yields (17); the coefficients $\rho_i(a, b)$, the elements of the density matrix $\rho_i^{\mu\nu}$ in the helicity basis in the c.m. system of the photons, are

$$\left.\begin{aligned}&\rho_i(1,1)=-\rho^{\mu\nu}R^{\mu\nu}/2=-C_i-2D_i\frac{\widetilde{p}_{i\perp}^{\,2}}{q_i^2};\\&|\rho_i(1,-1)|=2D_i\frac{\widetilde{p}_{i\perp}^{\,2}}{q_i^2};\\&|\rho_i(1,0)|=-2\sqrt{2}D_i\frac{|\widetilde{p}_{i\perp}|(p_ik_i)}{q_i^2};\\&\rho(0,0)=C_i+4D_i\frac{(p_ik_i)^2}{q_i^2}.\end{aligned}\right\}\quad (A.9)$$

If the colliding particles are polarized, the density matrix $\rho_i^{\mu\nu}$ ceases to be symmetric, and (17) is supplemented by two terms, proportional to σ_{TT}^{a} and τ_{TS}^{a}. However, the coefficients of these terms in the measured cross sections are usually very small: $\sim (m_i/W)^2$.[25]

APPENDIX 3

Using the calculations of ref. 90 for the $\gamma\gamma \to l^+l^-$ reaction, and taking into account the projection operators defined in (A.8), we find ($m \equiv m_l$):

$$\left.\begin{aligned}&\sigma_{TT}=\frac{\pi\alpha^2L}{X}\Big\{2q_1q_2+(2m^2+q_1^2+q_2^2)\,W^2(q_1q_2)\,X^{-1}\\&\quad+4m^4(q_1q_2)^{-1}+\frac{q_1^2q_2^2W^4}{4X^2(q_1q_2)}[2X+3q_1^2q_2^2]\Big\}\\&\quad-\frac{\pi\alpha^2\Delta t}{X}\Big\{1+(m^2+q_1^2+q_2^2)\frac{W^2}{X}+q_1^2q_2^2\Big[\frac{1}{T}+\frac{3W^4}{4X^2}\Big]\Big\};\\&\sigma_{TS}=-\pi\alpha^2q_2^2\frac{W^2}{X^2}\Big\{\Delta t\Big[1+\frac{q_1^2}{T}\Big(6m^2+q_1^2+\frac{3q_1^2q_2^2W^2}{2X}\Big)\Big]\\&-\frac{L}{q_1q_2}\Big[4Xm^2/W^2+q_1^2\Big(W^2+2m^2+q_1^2+q_2^2+\frac{3}{2}q_1^2q_2^2W^2X^{-1}\Big)\Big]\Big\};\\&\sigma_{ST}=\sigma_{TS}(q_1^2\leftrightarrow q_2^2);\\&\sigma_{SS}=\pi\alpha^2q_1^2q_2^2(W^4/X^3)\{(L/q_1q_2)\\&\quad[2X-3q_1^2q_2^2]-\Delta t\,(2+q_1^2q_2^2/T)\};\\&\tau_{TT}=-\frac{\pi\alpha^2}{8X}\Big\{\frac{2\Delta t}{X}\Big[2m^2W^2+(q_1^2-q_2^2)^2\\&\quad+\frac{3}{2}q_1^2q_2^2W^4X^{-1}\Big]+\frac{L}{q_1q_2}\Big[16m^2(m^2-q_1^2-q_2^2)\\&\quad-q_1^2q_2^2\Big(8+\frac{4W^2(2m^2+q_1^2+q_2^2)}{X}+3q_1^2q_2^2W^4X^{-2}\Big)\Big]\Big\}.\end{aligned}\right\}\quad (A.10)$$

Here

$$\left.\begin{aligned}&\Delta t=t_2-t_1=\sqrt{4X(1-4m^2/W^2)};\quad T=(t_1+m^2)(t_2+m^2)\\&\quad=4Xm^2/W^2+q_1^2q_1^2;\\&L=\ln(t_2+m^2)/(t_1+m^2)\\&=\ln[q_1q_2+\sqrt{X(1-4m^2/W^2)}]/[q_1q_2-\sqrt{X(1-4m^2/W^2)}];\\&t_{1,2}=[W^2-q_1^2-q_2^2-2m^2\mp\sqrt{4X(1-4m^2/W^2)}]/2.\end{aligned}\right\}\quad (A.11)$$

On the mass shell (for $q_i^2 = 0$)

$$\left.\begin{aligned}&\sigma_{TT}(W^2,\,0,\,0)\equiv\sigma_{\gamma}(W^2)=4\pi\alpha^2/W^2\{(1+4m^2/W^2\\&\quad-8m^4/W^4)\,L-\Delta t\,(1/W^2+4m^2/W^4)\};\\&\tau_{TT}(W^2,\,0,\,0)=-8\pi\alpha^2m^2[\Delta t+2m^2L]/W^6;\\&L=2\ln[W+\sqrt{W^2-4m^2}]/[W-\sqrt{W^2-4m^2}]\\&=2\ln[W/2m+\sqrt{(W/2m)^2-1}];\ \Delta t=W^2\sqrt{1-4m^2/W^2}.\end{aligned}\right\}\quad (A.12)$$

From Eqs. (A.10) and (A.11) we see that for $W^2 \gg |q_i^2|m^2$ the main asymptotic term is

$$\sigma_{TT}=(4\pi\alpha^2/W^2)[\ln(W^2/(t_1+m^2))-1-q_1^2q_2^2/(t_1+m^2)W^2];$$
$$t_1=(m^2-q_1^2)(m^2-q_2^2)/W^2.$$

---

1)To avoid confusion, we note that there is some inconsistency in the notation: $q^2_{min} = \min(-q^2)$; $q^2_{min} \le -q^2 \le q^2_{max}$.

2)An important consideration here is that for the production of leptons or hadrons the range of the integration over $-q^2$ is bounded from above, at least because of the decrease in the cross sections for some q^2_{max}. The quantity q^2_{max} is usually much smaller than the kinematic limit of $-q^2$, which is on the order of s. In ref. 19 this circumstance was neglected, and the derivation of the spectrum incorporated an integration over $-q^2$ over the entire kinematically allowed range, so incorrect expressions were found for these spectra for the case of $s \gg \Lambda_\gamma^2$. These expressions were subsequently used in certain other studies.[15]

3)In certain studies[11,15] of processes in colliding electron beams use has been made of equations for the spectra in which the correct value $\ln(q^2_{i\max}/q^2_{i\min} = \ln[q^2_{i\max}E(E-\omega_i)/m_e^2\omega_i^2]$ was replaced by $\ln(E^2/m_e^2)$. At the energies currently obtainable, this replacement does not lead to large errors in the calculation of the cross sections for two-photon production of hadrons, since in this case $\omega_i^2 \sim W^2 \sim q^2_{\max} = \Lambda_\gamma^2$. However, this procedure does lead to an incorrect functional dependence on E. For example, the coefficient of $\ln^3 E$ is incorrect, i.e., the equations give values which are very wrong at very high energies.[11,15] In particular, an expression was found in ref. 15 for the production of e^+e^- or $\mu^+\mu^-$ pairs which is larger than the correct expression by a factor of 3/2 for $E \to \infty$.[5] As E increases, the accuracy of the equivalent-photon method improves.

4)We can estimate the order of magnitude of τ_{TT} for certain cases; for small W we have $\tau_{TT} \approx 2\sigma_{\gamma\gamma}$, and for the $\gamma\gamma \to l^+l^-$ transition with $W \gg m_l$ we have $\tau_{TT} \sim (m_l/W)^2\sigma_{\gamma\gamma}$ (Appendix 3).

5)The term $R^a_{\gamma\gamma}$ was neglected in the original equation in ref. 31.

6)It is assumed here that the x dependence of M_{ab} is weak. This is a valid assumption for the production of a narrow resonance. For fixed values of all the other momentum variables, the effective range of x near the resonance maximum is on the order of the ratio of the width of this maximum to the mass, Γ/M. For the coefficient of $R^a_{\gamma\gamma}$ to vanish, this range must be much larger than $2\sin(\psi/2)$ as we see from Eqs. (40) and (41). Accordingly, for $\psi \gtrsim \Gamma/M$ the coefficient of $R^a_{\gamma\gamma}$ does not vanish.

7)The distribution with respect to ψ was found in a numerical computer calculation in ref. 15 for the reaction $ee \to ee\pi^+\pi^-$ on the basis of quantum electrodynamics. The resulting curves were reported to have the form $d\psi/\psi$ for $\psi > (m_e/E)^{1/2} \sim 2^0$.

8)In ref. 5 calculations were carried out in the laboratory system $p_2 = 0$ on the basis of the variables $\varepsilon_{iL} = k_1p_2/m_2$ and $k_{1\perp}$. To transform to the c.m. system it is sufficient to use the trivial substitution:

$$\frac{d\varepsilon_{1L}\, d\varepsilon_{2L} d^2k_{1\perp}}{\varepsilon_{1L}\varepsilon_{2L}} \to 2\pi \frac{|\mathbf{k}_1|^3 d|\mathbf{k}_1| \sin^2\theta_1\, d\theta_1\, d\theta_2}{\varepsilon_1\varepsilon_2 \sin^2\theta_2};$$
$$W^2 \approx \frac{(m^2 - k^2_{1\perp})(\varepsilon_{1L}+\varepsilon_{2L})^2}{\varepsilon_{1L}\varepsilon_{2L}} \to 4\omega_1\omega_2. \tag{43a}$$

The subsequent integration over ε_{iL} and $k_{1\perp}$ is also reported in ref. 5 (see also ref. 17).

9)The total cross sections for lepton production are calculated correctly in the equivalent-photon method because the basic contribution to these cross sections comes from the region $W \sim m_l$, where inequalities (12) are satisfied.

10)For $L_\eta > 3$ and $\eta = 1/3$, Eqs. (48) and (54) agree to within less than 10%.

11)Only the diagram in Fig. 1 is important in this approximation. The cross section for bremsstrahlung production is $\sigma_{\rm brem} \sim (m_e/M l)^2\sigma$, where M is the mass of the lighter of the colliding particles.

12)See footnote 8.

13)In such experiments, only the cross sections integrated over some q_i^2 interval near $q^2_{i\min} \sim 0$ are measured.

14)There has been some discussion of "deep inelastic scattering of electrons by a photon target",[78] but from our point of view this formulation of the experiment is simply a particular case of the general problem corresponding $q_2^2 \approx 0$.

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