Continuous analog of Newton's method in nonlinear problems of physics

E. P. Zhidkov, G. I. Makarenko, and I. V. Puzynin

Joint Institute for Nuclear Research, Dubna

Fiz. Él. Chast. Atom. Yad., 4, 127-166 (January-March 1973)

A new method of numerical investigation of nonlinear problems of physics is described. The method has made it possible to investigate a large class of nonlinear problems of physics: beam extraction from an accelerator, plasma physics, reconstruction of the potential from the limiting phase shift in the inverse problem of scattering theory, and the calculation of particle-like solutions of an equation encountered in nonlinear field theory.

INTRODUCTION

Many problems of modern physics — for example, the analysis of the motion of charged particles and the behavior of bunches of charged particles in electromagnetic fields, plasma problems, problems of scattering theory, dispersion relations, nonlinear field theory, to mention only some — lead to the solution of nonlinear equations.

In the investigation of these problems great importance attaches not only to analytic and qualitative methods but also to effective algorithms for solving the problem numerically with a computer. Not infrequently, a numerical solution is the only practical possibility because of the complexity of the mathematical problem. And in a certain sense the creation of a well-founded and effective algorithm for the numerical solution of the problem and its realization in a computer program are equivalent to a complete solution.

Great attention has been devoted in the Soviet Union and other countries to the development of methods of numerical solution of nonlinear problems. This problem has become especially acute at the Joint Institute for Nuclear Research with the development of new methods of collective acceleration and also with the investigation of the parameters of beam extraction systems in new accelerators of charged particles.

The present paper is an attempt to give a systematic exposition of a continuous analog of Newton's method developed to solve certain physical problems by a group of workers in the computational mathematics branch of the Laboratory of Computational Technology and Automation at the Joint Institute for Nuclear Research. Methods of introducing a continuous parameter, which gives the name to the method, reduce an originally stationary problem to an evolution problem. In many cases, methods of solution of evolution problems can be more conveniently realized on a computer. Alongside the mathematical foundation and its application to the solution of concrete mathematical problems, we consider some definite physical problems for whose numerical solution the method can be used. Some of the physical problems considered in the present review were solved for the first time by this method. In other problems the method has proved to be much more effective than the previously used methods.

1. APPROXIMATE SOLUTION OF NONLINEAR EQUATIONS IN A BANACH SPACE

A great many mathematical formulations of physical problems lead one to consider an abstract nonlinear equation in order to develop a unified approach to the solution of these problems. Consider the nonlinear equation

$$\varphi(x)=0, \quad (1)$$

where the nonlinear operator $y = \varphi(x)$ maps the Branch space X with elements x into the Banach space Y ($y \in Y$). Solving Eq. (1) is tantamount to finding elements $x^* \in X$ that are carried by φ to the null element of the space Y.

Before we describe the continuous analog of Newton's method for solving this problem, let us illustrate the idea behind the method for a simple example.

Consider the equation

$$f(x)=0, \quad (2)$$

where $y = f(x)$ is a sufficiently smooth function of a single real variable (Fig. 1). The classical method of Newton (the method of tangents) gives the approximate value of the root x_n of the equation in accordance with the equation

$$x_n=x_{n-1}-f(x_{n-1})/f'(x_{n-1}), \quad n=1,\ 2,\ \ldots$$

Geometrically, it reduces to finding successively the points of intersection x_n of the tangents to the curve $y = f(x)$ through the points $(x_{n-1}, f(x_{n-1}))$ with the Ox axis. Usually, Newton's method is employed in cases when the initial approximation x_0 of the desired solution x^* is chosen in such a way that $f'(x) \neq 0$ and $f''(x)$ does not change its sign in a certain neighborhood of x^* that contains x_0. In such a case the successive approximations should not carry one out of the neighborhood. If the above conditions are not satisfied, Newton's process need not converge. For example, if the initial approximation is taken to be $x_0^{(1)}$ (see Fig. 1), then Newton's sequence is $x_0^{(1)}$, $x_1^{(1)}$, $x_1^{(1)}$, ... Thus, the process diverges in this case because $f''(x)$ changes sign on the interval $[x_0^{(1)}, x_1^{(1)}]$. If the initial approximation is taken to be $x_0^{(2)}$, then Newton's pro-

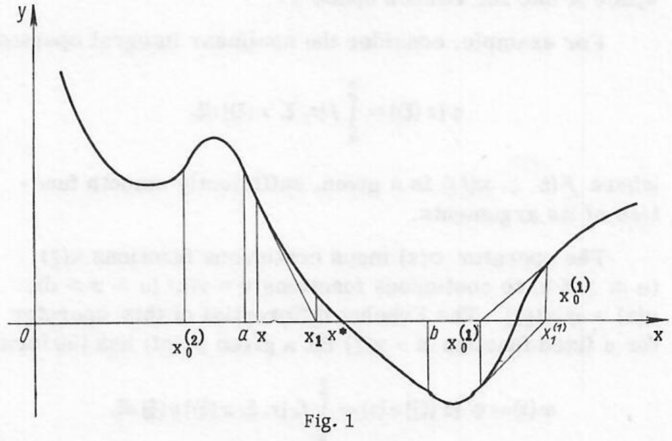

Fig. 1

cess obviously diverges in this case too. If the initial approximation is chosen in the neighborhood (a, b) of the root x^*, convergence is obtained. The continuous analog of Newton's method is a generalization of this method.

We shall assume that x is a function of a continuous parameter $t\ (0 \leq t < +\infty)$: $x = x(t)$, and this dependence is such that for all t we have

$$df[x(t)]/dt = -f[x(t)].$$

Assuming that $f'(x) \neq 0$, we rewrite the last relation in the form

$$dx(t)/dt = -f[x(t)]/f'[x(t)]. \tag{3}$$

Equation (3) is a differential equation of the first order for the unknown function $x = x(t)$. If this differential equation is augmented with the initial condition

$$x|_{t=0} = x_0 \tag{4}$$

(initial approximation to the desired solution), then, solving the Cauchy problem (3)-(4), we obtain the function $x = x(t)$, and we can expect that

$$\lim_{t\to+\infty} x(t) = x^*,$$

where x^* is the desired solution of Eq. (2). The solution of the Cauchy problem (3)-(4) instead of Eq. (2) is the continuous analog of Newton's method. The classical method of Newton is obtained from the continuous analog if the Cauchy problem (3)-(4) is solved approximately by Euler's method with an integration step $\tau = 1$. Newton's method has been generalized to the abstract equation (1) by Kantorovich.[1] In this case, the method has come to be known as the Newton-Kantorovich method.

In ref. 2, differential equations like (3) are obtained as continuous analogs of single-step iteration processes. In particular, the Newton-Kantorovich process is associated with a differential equation in a Banach space:

$$dx/dt = -[\varphi'(x)]^{-1}\varphi(x) \tag{5}$$

with initial condition

$$x(0) = x_0. \tag{6}$$

Here $\varphi'(x)$ is the Fréchet derivative[3] of the operator $\varphi(x)$, a linear (for fixed x) operator that carries the Banach space X into the Banach space Y.

For example, consider the nonlinear integral operator

$$\varphi[x(\xi)] = \int_a^b f[z, \xi, x(\xi)]\, d\xi,$$

where $f(z, \xi, x(\xi))$ is a given, sufficiently smooth function of its arguments.

The operator $\varphi(x)$ maps continuous functions $x(\xi)$ $(a \leq \xi \leq b)$ to continuous functions $y = y(z)$ $(c \leq z \leq d)$; $y(z) = \varphi[x(\xi)]$. The Fréchet differential of this operator for a fixed function $x = x(\xi)$ (at a given point) has the form

$$w(z) = \varphi'[x(\xi)]\, v(z) = \int_a^b f'_x[z, \xi, x(\xi)]\, v(\xi)\, d\xi.$$

Here $\varphi'(x)^{-1}$ is in the inverse operator of the operator $\varphi'(x)$. Given an element w of the space Y, the operation $\varphi'(x)^{-1}$ amounts to finding an element $v \in X$ from the equation $\varphi'(x)v = w$, i.e., $v = \varphi'(x)^{-1}w$. In the example considered, the operation $\varphi'(x)^{-1}$ amounts to finding the solution of the linear integral equation $w(z) = \int_a^b f'_x[z, \xi, x(\xi)]\, v(\xi)d\xi$ for $v(\xi)$ for a given function $w(z)$ and given kernel $f'[z, \xi, x(\xi)]$.

In ref. 2 a study is made of the convergence of the continuous analog of Newton's method for the nonlinear equation (1) to the solution of this equation. M. K. Gavurin proves a theorem whose conditions guarantee the existence of a solution of the original equation (1) and the convergence in the limit $t \to +\infty$ of the solution x(t) of the problem (5)-(6) to this solution.

<u>Theorem 1.</u> Suppose that in the sphere

$$\|x - x_0\| < B\|\varphi(x_0)\| \qquad (B > 0,\ \text{const}) \tag{7}$$

there exist the Fréchet derivative $\varphi'(x)$ and the linear Gateaux derivative $\varphi''(x)$, the linear operator $\varphi'(x)$ having an inverse $\varphi'(x)^{-1}$ for which

$$\|\varphi'(x)^{-1}\| \leqslant B, \tag{8}$$

and $\varphi''(x)$ being bounded in the neighborhood of each point of the sphere (7).

Then, first, Eq. (5) with the initial condition (6) has a solution $x = x(t)$ for values of t in the interval $0 \leq t < \infty$, which is a root of Eq. (1). In ref. 2, some considerations concerning the numerical solution of the problems (5) and (6) are put forward.

The ideas of ref. 2 were further developed in ref. 4, in which a study is made of continuous analogs of many-step iteration methods as well as ways of their approximate representation, but only for linear functional equations.

In ref. 5, Davidenko develops a method of variation of a parameter, an iteration method of variation of a parameter, and a modification of it for solving functional equations. A study is made of methods of numerical solution of different problems in analysis on the basis of these methods.

Analyzing the existing ways of solving nonlinear functional equations, one can readily see that the construction of many methods of solution of these problems is related to a greater or a lesser extent with the question of the existence of a unique solution of the problem. This circumstance predetermines the restriction, often severe, on the classes of problems to which the results that are obtained can be applied. The problem of the existence and uniqueness of a solution of a nonlinear problem must be studied separately and in the majority of cases is very complicated.

When one considers many problems of modern physics, one can see that there is frequently considerable a priori information about the existence and the qualitative behavior of the solution of the problem. This information can be obtained from an analytic and qualitative investigation of the problem, from various physical arguments, from the study of simplified model problems, and also in many other ways. Therefore, by separating the problems of the existence of a solution of a nonlinear problem and its approximate solution, one can greatly extend the class of problems solvable by means of such methods. On the other hand, in the solution of physical problems with a nonunique solution it is necessary to have sufficiently flexible computational schemes, which, in particular, can be constructed on the basis of the evolution equations (5). Of particular promise are methods of introducing a continuous parameter for which the convergence with respect to this parameter to the desired solution and, possibly, the stability are governed by the manner in which it is introduced.

These considerations suggest that the continuous analog of Newton's method is one of the most effective means of constructing algorithms for the numerical solution of nonlinear physical problems. Here it is natural to investigate the possibility of applying this method for the approximate solution of nonlinear problems under the assumption that solutions to the problem exist.

In ref. 6, a theorem is proved on the basis of the method developed by Gavurin;[2] it concerns the convergence of the continuous analog of Newton's method under the assumption that there exists an isolated solution of the nonlinear equation (1).

Theorem 2. Suppose Eq. (1) has a unique solution x^* in an open domain D of a Banach space (B space) X. Suppose that in D there exist the continuous Fréchet derivatives $\varphi'(x)$ and $\varphi''(x)$. Suppose, further, that in D there exists an inverse operator $\varphi'(x)^{-1}$ for which the inequality (8) holds. Then there exists a sphere S: $\|x - x^*\| \leq \varepsilon$ belonging to D such that for any $x_0 \in S$ the differential equation (5) with the initial condition (6) has a solution x(t) for $0 \leq t < \infty$ and $\lim_{t\to\infty} \|x(t) - x^*\| = 0$.

For the numerical realization of the continuous analog of Newton's method, one can apply different methods of approximate solution of differential equations. The convergence of one of these – Euler's – on a finite interval of the parameter t, $0 \leq t \leq T$, is studied and proved in ref. 6. In these papers one studies the equation

$$dx/dt = \psi(x) \tag{9}$$

with the initial condition

$$x(0) = x_0. \tag{10}$$

Here, x is an element of the B space X; $\psi(x)$ is a function that maps X into itself. It is assumed that in a certain open domain $D \subset X$

$$\|\psi(x)\| \leqslant M \quad (M > 0, \text{ const}) \tag{11}$$

and $\psi(x)$ satisfies a Lipschitz condition in D, i.e., for all $x_1, x_2 \in D$:

$$\|\psi(x_1) - \psi(x_2)\| \leqslant L\|x_1 - x_2\| \qquad (L > 0, \text{ const}). \tag{12}$$

These conditions ensure the existence of a unique solution for the problem (9)–(10), which belongs to D, on a certain interval of variation of the real parameter t, $0 \leq t \leq T$; this follows from the existence theorem (see ref. 3, pp. 422–423). The computational scheme of Euler's method consists of finding elements $\tilde{x}_k$ by means of the relations

$$\tilde{x}_0 = x_0; \; \tilde{x}_k = \tilde{x}_{k-1} + \tau_{k-1}\psi(\tilde{x}_{k-1}), \quad r = 1, 2, \ldots, n \tag{13}$$

for a nonuniform mesh that divides the interval $0 \leq t \leq T$ by points t_k: $t_k = t_{k-1} + \tau_{k-1}$, $t_0 = 0$, $t_n = T$, and $\tau = \max \tau_k$. Under the conditions formulated above, the convergence of the approximate solution (13) of the problem (9)–(10) to the exact solution is proven if $\tau \to 0$ and if for any n the inequality $\tau \leq KTn^{-1}$ holds (K > 0 is a constant).

As shown in ref. 7, for Eq. (5) with the initial condition (6) and provided the conditions of Theorem 2 hold, the conditions that ensure convergence of Euler's method are also satisfied. When the continuous analog of Newton's method is made discrete in this way, then at every step k one must solve the linear problem

$$\varphi'(x_k)\,\Delta x_k = -\varphi(x_k) \tag{14}$$

for Δx_k and find the new value x_{k+1} by means of the relation

$$x_{k+1} = x_k + \tau_k \Delta x_k. \tag{15}$$

Theorem 2 quoted here on the convergence of the solution of the problem (5)–(6) to one of the solutions of Eq. (1) is formulated under the assumption that there exists a localized (in the case of nonuniqueness) solution. This distinguishes it from Theorem 1. The assumption that there exists a solution of the problem eliminates the relation between the size of the domain considered and the estimate in it of the norm of the operator $\varphi'(x)^{-1}$ in Theorem 1. In practice, the fulfillment of this condition in certain concrete problems cannot be established. Like the well known theorems on the convergence of Newton's method,[1] Theorem 2 presupposes the existence of a good initial approximation to the solution of the original problem. However, in Theorem 2 there are no actual restrictions on the norms of the operators φ', φ'', and $(\varphi')^{-1}$ in the neighborhood of the desired solution. Such restrictions do occur in the theorems on the convergence of Newton's method in ref. 8, in which it is also assumed that there exists a solution of the original functional equation.

Numerical methods of solution of a number of mathematical and physical problems that will be described below have been developed by means of the above approach.

2. BOUNDARY-VALUE PROBLEMS FOR ORDINARY NONLINEAR DIFFERENTIAL EQUATIONS

In this section we describe methods of numerical solution of boundary-value problems for nonlinear ordinary differential equations. The direct study of individual physical effects and certain mathematical problems, for example, the method of straight lines in the solution of boundary-value problems for partial differential equations, lead to the approximate solution of these problems.

As examples of physical problems we may mention problems of the motion and extraction of accelerated particles,[9] nonlinear field theory,[10] and problems in the investigation of plasmas.[11] We should here mention that completely new approaches to the investigation of a large number of already traditional physical problems have been developed on the basis of the method of introducing a parameter for the approximate solution of nonlinear problems.

Much literature is devoted to the approximate solution of boundary-value problems for nonlinear ordinary differential equations. A great many investigations (see, for example, ref. 12, pp. 395-406, and also refs. 13-18) are united by the idea of reducing a continuous boundary-value problem to a discrete analog by means of a finite-difference approximation.

The iteration method of solving a system of nonlinear difference equations obtained by making a boundary-value problem discrete:

$$y''=f(x, y, y'); \tag{16}$$

$$y(a)=\beta_1;\ y(b)=\beta_2, \tag{17}$$

is investigated in ref. 12 by the application of the method of finite differences. The convergence of the iteration process of solving the nonlinear algebraic system is found in ref. 12 under restrictions on the nonlinear part of the equation that ensure the applicability of the fixed point principle. Sapagovas[13] used the same method to investigate a more complicated equation. Lasota[14] considered a difference method of solving the boundary-value problem (16)-(17) subject to the condition

$$(b-a)^2 L_1/\pi^2+(b-a) L_2/4<1,$$

where L_1 and L_2 are Lipschitz constants for $f(x, y, y')$ with respect to the variables y and y' in the region $-\infty < y, y' < +\infty$. Apart from the convergence of the method, these conditions ensure the existence and uniqueness of a solution of the problem (16)-(17). Vashakmadze[15] makes boundary-value problems discrete by means of quadrature formulas that ensure a higher order of convergence of the solution of the discrete problem to the desired solution. In some investigations, for example, refs. 16-18, Newton's method or a modification of it is used to solve the nonlinear system of difference equations.

Henrici (ref. 16, pp. 347-388) considers the question of approximating the solutions of the boundary-value problem

$$\varphi(y)\equiv y''+f(x, y)=0; \tag{18}$$

$$y(0)=y(1)=0 \tag{19}$$

subject to the condition that f and f'_y are continuous in the strip S(x, y):

$$0\leqslant x\leqslant 1,\ -\infty<y<\infty \text{ and } f'_y(x, y)\leqslant 0 \text{ for } (x, y)\in S.$$

This ensures the existence and uniqueness of a solution of the boundary-value problem. Lees,[17] generalizing and developing ref. 16, imposes on f'_y the condition

$$\sup_S f'_y=\eta<\pi^2. \tag{20}$$

This condition is used to prove the existence of a solution of the difference boundary-value problem and prove the applicability of Newton's method for finding the solution. Tamme[18] finds the solution of the discrete problem by means of the generalized method of Newton. The restrictions that are obtained on f'_y, which have an integral nature, are basically similar to the conditions (20).

The method of Newton developed by Kantorovich[1] to solve nonlinear functional equations has been successfully used in various investigations (see, for example, refs. 19-23) and to construct algorithms for the approximate solution of boundary-value problems for nonlinear differential equations. This approach is frequently justified under conditions that, apart from the convergence of the method, ensure the existence and uniqueness of a solution of the problem. This leads to restrictions on the nonlinearity similar to those already considered above. For example, Glinskaya and Mysovskikh[19] prove the convergence of Newton's method for the approximate solution of the boundary-value problem (18)-(19) under restrictions on f'_y that have the form $|f'_y|\leq M<8$ and $f'_y\leq 0$. Vinokurov and Ivanov[20] introduce a discrete dynamical parameter – a descent parameter – in the traditional Newton scheme of approximate solution of the boundary-value problem (16)-(17). They assume that a rational choice of this parameter accelerates the process and, possibly, guarantees its convergence even in cases when the classical method does not.

Bellman and Kalaba[24] investigate the method of quasilinearization, which reduces the solution of a nonlinear boundary-value problem to the solution of an infinite sequence of linear problems and is a development of Newton's functional method. Therefore, the restrictions inherent in Newton's method are to a certain extent present in their method. For example, if the method is used to solve the problem (16)-(17) approximately, one uses the property of "convexity" of the function f (see ref. 24, pp. 36-37), which ultimately imposes restrictions on the signs of the first and the second partial derivatives of the function f. The convergence of various variants of Newton's method for nonlinear boundary-value problems is also investigated by Shamanskii in ref. 21. Bakhvalov[22] proposes a modification of Newton's method suitable for computer solution of boundary-value problems for systems with a large number of equations. Budak and Gol'dman[23] study some modifications of Newton's method and also their difference analogs under the assumption that the nonlinear parts of the systems of differential equations are smooth in the neighborhood of the desired solution.

Finally, let us consider some methods of approximate solution of nonlinear boundary-value problems whose basic idea consists of reducing the original stationary problem to an evolution process by introducing an additional continuous parameter into the problem in a certain manner. These include, for example, the "stabilization" method, which is based on the convergence of the solution of the problem for a parabolic equation to the solution of a boundary-value problem for a corresponding ordinary differential equation.

The idea of using an "accompanying" equation to construct and investigate methods of approximate solution of stationary boundary-value problems can be found, for example, in ref. 24 (pp. 61-63 and 105-106). The conditions under which the stabilization of the solution of nonlinear parabolic equations is investigated are very important and restrict the class of problems that can be solved by this method. For example, Friedman[25] establishes stabilization for nonlinearity of the form $f(x, u) = c(x)u + k(x, u)$, where $c(x) \leq 0$, $|k(x, u)| \leq \varepsilon |u|$, and the constant $\varepsilon > 0$ is sufficiently small.

We also mention the method of variation of a parameter proposed by Davidenko to solve nonlinear problems. This method has been applied by Shidlovskaya[26] to solve the boundary-value problem (16)-(17). In addition, Shamanskii (ref. 21, pp. 94-106, 137-149) applies this method to construct algorithms for the numerical solution of nonlinear systems with nonlinear boundary conditions.

The method of approximate solution of boundary-value problems for ordinary differential equations based on the continuous analog of Newton's method for functional equations is to a certain extent free of the shortcomings that we have mentioned above. First, the presence of the parameter τ in the scheme that is discrete with respect to the parameter t makes it more flexible than the traditional method of Newton. The parameter τ, which becomes the value of the step in the approximate integration of the differential equation describing the continuous process of Newton can be chosen optimally on the basis of certain considerations. Secondly, the convergence of the method has been investigated under the assumption that there exists (and is localized in the case of nonuniqueness) a solution of the original boundary-value problem. These assumptions lead to natural requirements of smoothness of the nonlinear part f of the equation in the neighborhood of the desired solution but do not impose any actual restrictions on its magnitude or the signs of it or its derivatives.

In ref. 27, the continuous analog of Newton's method is justified for the following boundary-value problem:

$$y^{(n)}+f(x, y, y', \ldots, y^{(n-1)})=0; \quad (21)$$

$$y^{(i)}(a)=y^{(j)}(c)=0, \quad i=0, 1, \ldots, k-1; \quad (22)$$

$$j=0, 1, \ldots, n-k-1.$$

The nonlinear problem (21)-(22) is replaced by the system

$$v^{(n)}+f'_{y^{(n-1)}}(x, y, y', \ldots, y^{(n-1)})\,v^{(n-1)}+\ldots+f'_y(x, y, y', \ldots, y^{(n-1)})\,v = -[y^{(n)}+f(x, y, y', \ldots, y^{(n-1)})]; \quad (23)$$

$$\partial y(x, t)/\partial t=v(x, t), \quad (24)$$

which is solved in the half-strip $a \leq x \leq b$, $0 \leq t < \infty$ under the conditions

$$v^{(i)}(a, t)=v^{j}(b, t)=0, \quad i=0, 1, \ldots, k-1;$$

$$j=0, 1, \ldots, n-k-1;$$

$$y(x, 0)=y_0(x). \quad (25)$$

Here $y_0(x)$ is the initial approximation to the desired solution of the problem.

Under certain assumptions pertaining to the smoothness of the function f, the existence of solutions of the problem (21)-(22), and the choice of the function $y_0(x)$, it is proved that the problem (23)-(25) can be solved for all $0 \leq t < \infty$ and $\lim_{t\to\infty} y(x, t) = y^*(x)$, where $y^*(x)$ is the solution of the boundary-value problem (21)-(22). Similar results are obtained in ref. 6 for certain special forms of the boundary-value problem (21)-(22) that are most frequently encountered in different physical problems.

By way of an example let us consider the conditions of convergence of the continuous analog of Newton's method for the boundary-value problem (18)-(19).

<u>Theorem 3.</u> Suppose a solution of the boundary-value problem (18)-(19) exists and in the case of nonuniqueness that it can be localized. This means: 1) One can construct functions u(x) and U(x) that are twice continuously differentiable on [0, 1], satisfy the boundary conditions (19), do not have common tangents at the points $x = 0$ and $x = 1$, and for $0 < x < 1$ satisfy the inequality $u(x) < U(x)$; 2) in the closed domain $D[0 \leq x \leq 1,\ u(x) \leq y \leq U(x)]$ of the plane there is only one solution, $y^*(x)$, of the boundary-value problem (18)-(19), and $y^*(x)$ does not have common tangents at the point $x = 0$ and $x = 1$ with the functions u(x) and U(x) and for $0 < x < 1$ we have $u(x) < y^*(x) < U(x)$.

Suppose that in D the function $f(x, y)$ has continuous partial derivatives to second order inclusive. Suppose, in addition, 3) the boundary-value problem $v'' + f'_y(x, y)\,v = 0$; with $v(0) = v(1) = 0$ has only the trivial solution for any continuously differentiable function $y(x) \in D$; 4) $\|y_0'' + f(x, y_0)\| \leq \varepsilon$, where $\varepsilon > 0$ is sufficiently small and $y_0(x)$, a twice continuously differentiable function in D, is the initial approximation to the desired solution $y^*(x)$.

Then the system for the functions y(x, t) and v(x, t),

$$\begin{cases} v''_{xx}+f_y(x, y)\,v=-[y''_{xx}+f(x, y)]; \\ y'_t=v, \end{cases} \quad (26)$$

has in the half-strip s ($0 \leq x \leq 1$, $0 \leq t < \infty$) a unique solution that satisfies the conditions

$$v(0, t)=v(1, t)=0; \ y(x, 0)=y_0(x); \quad \text{and} \lim_{t\to\infty}\sum_{k=0}^{2}\max_{0\leq x\leq 1}|y^{(k)}_{xk}(x, t)-y^{*(k)}(x)|=0. \quad (27)$$

The proof of this theorem consists of verifying the conditions of Theorem 2 for the nonlinear operator $\varphi(y)$ of the problem (18)-(19). Note that the first proof of this theorem using an analog of Euler's broken lines for the problem (26)-(27) was given in ref. 28.

To implement the method numerically, it is necessary to have a discrete representation of the method. The method can be made discrete with respect to the parameter t on the basis of Euler's method[13] for the approximate solution of the Cauchy problem (9)-(10), where $\psi = -(\varphi')^{-1}\varphi$; $\varphi(y) = y'' + f(x, y)$. The semidiscrete scheme obtained in this manner can be interpreted as an analog of the method of straight lines for the approximate solution of the problem (26)-(27). In it, the half-strip s ($0 \leq x \leq 1$, $0 \leq t < \infty$) is divided by straight lines parallel to the Ox axis: $t = t_i$, $t_{i+1} - t_i = \tau_i$, and the second equation of the system (26) is replaced by the difference analog

$$\widetilde{y}(x, t_{i+1})=\widetilde{y}(x, t_i)+\tau_i\widetilde{v}(x, t_i). \quad (28)$$

We introduce the notation $\widetilde{y}_i(x) = \widetilde{y}(x, t_i)$, $\widetilde{v}_i(x) = \widetilde{v}(x, t_i)$. The course of the calculations can be described as follows. We assume that on the layer $t = t_i$ the function $\tilde{y}_i(x)$ is already known. Then the function $\tilde{v}_i(x)$ is determined as a solution of the boundary-value problem for the linear equation

$$\tilde{v}_i'' + f_y'(x, \tilde{y}_i(x))\,\tilde{v}_i = -[f(x, \tilde{y}_i(x)) + \tilde{y}_i''(x)]; \qquad (29)$$

$$\tilde{v}_i(0) = \tilde{v}_i(1) = 0. \qquad (30)$$

Further, using Eq. (28), we can calculate the value of $y_{i+1}(x)$ on the following layer. The method converges as $\tau \to 0$ if the problem (29)-(30) is solved exactly on each layer.

To make the method completely discrete, as is needed if the boundary-value problem (18)-(19) is to be solved numerically on a computer, the solution of the linear boundary-value problem (29)-(30) for fixed $t = t_0$ must also be found numerically. For this one can use the method of finite differences. The complete discrete scheme will then include the solution for fixed parameter t of the discrete linear boundary-value problem that approximates the corresponding continuous problem and the advance to the next value of t by means of the difference relation (28). This scheme is justified in the case of a three-point difference approximation of second order of accuracy in ref. 29. The computational schemes of approximate solution of boundary-value problems obtained on the basis of the continuous analog of Newton's method can be readily realized in the form of computer programs.

A very convenient algorithm for the numerical solution of linear difference boundary-value problems is the well known sweep method (see ref. 30, pp. 176-186). The integration step τ can be chosen in accordance with certain considerations of optimality. For example, analyzing the discrepancy of the approximations at each step,[31] one can increase or decrease the integration step τ depending on the change in the discrepancy. This greatly accelerates the process of convergence of the approximations to the solution of the difference problem that approximates the original problem.

3. DIRICHLET PROBLEM FOR A QUASILINEAR EQUATION OF SECOND ORDER OF ELLIPTIC TYPE

This section is devoted to an exposition of an approximate method of solving a Dirichlet problem: to find a function z(x,y) that satisfies the equation

$$\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = f(x, y, z) \text{ in the domain G}$$

and $z = \varphi(s)$ on the boundary Γ of G, where $\varphi(s)$ and $f(x, y, z)$ are given functions and the domain G of the XOY plane is bounded by a smooth curve Γ. Problems of this kind arise in different branches of physics, for example, in the theory of elasticity and plasticity,[32] in the study of spatial motions of a gas at velocities near that of sound,[33] in magnetohydrodynamics,[24,34] in the stationary theory of a thermal explosion,[35] and so forth. Alongside analytic and qualitative investigations of boundary-value problems for nonlinear elliptic equations, there is great importance in the development of methods of numerical solution of these problems.

Some investigations have been devoted to the convergence of the method of finite differences. For example, Parter[36] considers the conditions of convergence of the finite-difference problem that approximates the Dirichlet problem. Restrictions of Hölder type (with respect to the variables x and y) and of the Lipschitz type with respect to the variable z are imposed on the function $f(x, y, z)$. Under the additional conditions of "weak nonlinearity" on the function f the original Dirichlet problem reduces to a simplified one and the convergence of the difference method is proved for it on the basis of the maximum principle.

For the solution of certain nonlinear problems of the theory of elasticity and plasticity described by quasilinear elliptic equations with divergent right-hand sides, Sapagovas[37] has developed an iteration method. Note that the Dirichlet problem for a quasielliptic equation of second order has been studied in ref. 38. The method of straight lines was used to find an approximate solution of this problem. The solvability and convergence of the method have been proved. A large number of investigations have been devoted to the application of the generalized method of Newton developed by Kantorovich.[1] This method has made it possible to prove the solvability of the Dirichlet problem for a quasilinear elliptic equation in the case of "slight" nonlinearity in the coefficients and for certain nonlinear equations of a special form.

Mysovskikh[39] investigates the Dirichlet problem for the equation

$$\Delta z = k(x, y)\, z^2.$$

The existence of a solution of this problem and the convergence of Newton's method to it is proved.

In the construction of methods of approximate solution of the Dirichlet problem, it is convenient to use evolution processes, introducing an additional continuous parameter into the problem. For example, Shidlovskaya[26] points out the possibility of applying the method of variation of a parameter to solve the following boundary-value problem:

$$\Delta z + f(x, y, z, z_x, z_y) = 0;$$
$$z|_\Gamma = \varphi(s).$$

The continuous analog of Newton's method is one of the possible methods of introducing a continuous parameter. In its discrete interpretation on the basis of Euler's method (Sec. 1), the nonlinear problem reduces to the solution of a sequence of linear elliptic boundary-value problems. This makes it possible to use the well developed methods of numerical solutions of boundary-value problems for linear elliptic equations. Here we must above all mention the results of Tikhonov and Samarskii,[40] Yanenko,[41] and Forsythe and Wasaw.[42] The most recent work in this field is Samarskii's monograph.[43] Below, on the basis of refs. 44 and 45, we shall show how the method of stabilization with respect to the parameter can be applied to the solution of the problem we have formulated.

Consider the quasilinear differential equation

$$\Delta z(x, y) + f(x, y, z) = 0, \qquad (31)$$

where Δ is the Laplacian; $f(x, y, z)$ is a twice continuously differentiable function of its arguments. We seek a solution of Eq. (31) defined in a bounded domain G that satisfies on the boundary Γ the condition

$$z(x, y)|_{\Gamma}=0. \tag{32}$$

We assume that the boundary Γ of G is sufficiently smooth.

We consider the complete linear normed space X(G, H) of functions z(x, y) that are twice continuously differentiable in the closed domain $\bar{G} = G + \Gamma$, vanish on the boundary, and are such that the second derivatives of z(x, y) satisfy a Hölder condition in $\bar{G}$ with exponents λ ($0 < \lambda < 1$). The norm of z(x,y) in the space X(G, H) is defined by

$$\|z(x, y)\|_X=\sum_{l=0}^{2}\sum_{m=0}^{l}\max_{\bar{G}}\left|\frac{\partial^l z(x, y)}{\partial x^m\,\partial y^{l-m}}\right|+H_{z''_{x^2}}+H_{z''_{xy}}+H_{z''_{y^2}}, \tag{33}$$

where $H_{z''_{x^2}}$, $H_{z''_{xy}}$, $H_{z''_{y^2}}$ are lower bounds of the Hölder constants with the exponent λ for the functions $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\,\partial y$, $\partial^2 z/\partial y^2$, respectively, in $\bar{G}$. We introduce the Banach space Y(G, H) of functions w(x, y) that are continuous in $\bar{G}$ and satisfy a Hölder condition with exponent λ. We define the norm in Y(G, H) by the equation

$$\|w(x, y)\|_Y=\max_{\bar{G}}|w(x, y)|+H_w, \tag{34}$$

where H_W is the lower bound of the Hölder constants for the function w(x, y) in $\bar{G}$.

Theorem 4. Suppose that in the domain D,

$$\|z-z^*\|_X<M, \tag{35}$$

of the space X(G, H) the boundary-value problem (31)-(32) has a solution z*(x, y), which is moreover unique. Suppose, further, that $f(x, y, z)$ is a twice continuously differentiable function of its arguments. Suppose that for any z(x, y) ∈ D and any w(x, y) ∈ Y(G, H) the boundary-value problem for the function v(x, y):

$$\Delta v(x, y)+f'_z(x, y, z)\,v(x, y)=w(x, y); \tag{36}$$

$$v(x, y)|_{\Gamma}=0 \tag{37}$$

has a unique solution v(x, y) ∈ X(G, H). Then there exists a sphere S,

$$\|z-z^*\|_X\leqslant\varepsilon\quad(\varepsilon<M), \tag{38}$$

such that for any $z_0(x, y)$ ∈ S in the cylinder

$$\Omega=G\times[0\leqslant t<\infty)$$

there exists a unique solution of the system of equations

$$\begin{cases}\Delta v(x, y, t)+f'_z(x, y, z)\,v(x, y, t)=-[\Delta z+f(x, y, z)];\\ \partial z(x, y, t)/\partial t=v(x, y, t),\end{cases} \tag{39}$$

that satisfies the conditions

$$\begin{cases}v(x, y, t)|_B=0;\\ z(x, y, t)|_{t=0}=z_0(x, y);\ z_0(x, y)_{\Gamma}=0,\end{cases} \tag{40}$$

where B is the lateral surface of the cylinder Ω and

$$\lim_{t\to\infty}\|z(x, y, t)=z^*(x, y)\|_X=0. \tag{41}$$

Note that $z_0(x, y)$ is the initial approximation of the desired solution of the Dirichlet problem. The more accurately $z_0(x, y)$ is chosen, the faster the solution is obtained with the desired accuracy "in the time t."

A detailed proof of the theorem is given in ref. 44. Here, we note only that for its proof one must verify that all the conditions of Theorem 2, formulated for a Banach space, are satisfied. In doing so it is important to use the theorem on the boundedness of the inverse operator (see, for example, ref. 3, p. 157) and also the theorem on the Hölder continuity of the solution of an elliptic equation and uniformity of the estimate of $H_{v''}$ for the solution of the Dirichlet problem (36)-(37).[46-47]

Scheme for the numerical solution of the problem (39)-(40). We choose a step for the motion with respect to the parameter t, denoting it by τ. We split the domain $\Omega_T = G \times [0 \le t \le T]$ into n parts by planes parallel to XOY: $t_0 = 0$; $t_1 = \tau$; $t_2 = 2\tau, \ldots, t_n = n\tau = T$. We replace the second equation of the system (39) by the difference relation

$$[z_{k+1}(x, y)-z_k(x, y)]/\tau=v_k(x, y),$$

where $z_k(x, y)$ and $v_k(x, y)$ are the approximate values at the point $t = t_k$ of the functions $z(x, y, t_k)$ and $v(x, y, t_k)$, respectively.

If the function $z_k(x, y)$ is known on the layer $t = t_k$, then the function $v_k(x, y)$ on this layer is determined by the solution of the linear boundary-value problem for $v_k(x, y)$ as a function of x and y:

$$\Delta v_k(x, y)+f'_z(x, y, z_k(x, y))\,v_k(x, y)$$
$$=-[\Delta z_k(x, y)+f(x, y, z_k(x, y))]; \tag{42}$$

$$v_k(x, y)|_B=0. \tag{43}$$

Finding $v_k(x, y)$ from this in any known method, for example, by the method of meshes,[12] we determine the function $z_{k+1}(x, y)$ on the next layer $t = t_{k+1}$:

$$z_{k+1}(x, y)=z_k(x, y)+\tau v_k(x, y).$$

Since the function $z(x, y, 0) = z_0(x, y)$ is specified on the layer $t = t_0$, the process of calculating the functions $z_k(x, y)$ and $v_k(x, y)$ is well defined. If it is assumed that the boundary-value problem (42)-(43) is solved exactly, then in the limit $\tau \to 0$ we obtain convergence of the approximate solution to the exact solution [6]. The complete discrete scheme of numerical solution of the original problem can be obtained by a method similar to that described in Sec. 2.

4. NONLINEAR INTEGRAL EQUATIONS

In this section we consider the application of the continuous analog of Newton's method to the numerical solution of nonlinear integral equations. In elementary-particle physics, the solution of certain problems reduces to nonlinear integral equations and systems. These include the problem of the passage of light through matter, the frequency dependence of the permittivity, scattering of elementary particles of the type[48] $a+b \to a+b$, and determination of the form factors of elementary particles.[49]

Important directions in the investigation of these problems are the proof of their solvability by means of analytic methods[50,51] and the selection of solutions with physical meaning. It is also necessary to solve nonlinear integral equations in plasma physics,[52] nonlinear mechanics,[53] and other branches of physics.

Guseinov[54] and his pupils have studied nonlinear singular integral equations and the application of these equations to boundary-value problems of the theory of analytic functions.

Let us mention some investigations devoted to the numerical solution of nonlinear integral equations. Panov[55] applies Chaplygin's method; Vetchinkin[56] combines Newton's method with the method of successive approximations; Zagadskii[57] uses a modification of Newton's method. Gekht[58] proves a theorem on the convergence of the method of successive approximations. Sokolov[59] introduces a method of averaging of functional corrections and applies it to solve nonlinear integral equations and systems of such equations. Kurpel'[60] considers a generalized algorithm of Sokolov's method and gives new sufficient conditions for the convergence of the successive approximations and corresponding error estimates. Using Kantorovich's theorem[1] on the convergence of Newton's method, Mysovskikh[61] derives an estimate for the error that arises when a nonlinear integral equation is solved by the method of mechanical quadrature. Mamedov[62] transfers the number-theory methods of Korobov developed for the approximate solution of linear equations to the nonlinear case. Bel'tyukov and Shil'ko[63] apply a method of approximate solution of Hammerstein's equation,

$$\varphi(s)=\int_a^b \mathscr{K}(s,\ t,\ \varphi(t))\,dt \quad (b>0),$$

in the form of a series in powers of μ:

$$\varphi(t)=\sum_{n=0}^{\infty}\mu^{n+1}\varphi^{(1,\ n)}(t).$$

The undetermined positive parameter μ is introduced into the equation as an upper limit of the integral.

Davidenko[5] applies his previously proposed method of variation of a parameter; he derives iteration formulas of improved accuracy for the determination (improvement) of numerical solutions of nonlinear integral equations of the form

$$\varphi(x)=\int_a^b F(x,\ t,\ \varphi(t))\,dt+f(x).$$

Iteration formulas are constructed on the basis of an arbitrary quadrature formula.

In what follows we shall explain the use of the method of introduction of a continuous parameter along the lines of refs. 7 and 64. Here, we give a difference scheme analogous to the difference scheme in Davidenko's paper.[5] In Theorems 5 and 6 we study the question of the solvability of difference equations and the convergence of the solution of a difference equation to the solution of the original equation.

Consider the nonlinear integral equation

$$u(x)=\int_a^b f[x,\ \xi,\ u(\xi)]\,d\xi,\quad a\leqslant x\leqslant b, \tag{44}$$

where $f(x,\ \xi,\ u)$ is a given twice continuously differentiable function of its arguments, and u(x) is the desired function.

We seek a solution in the space of functions $C^L_{[a,b]}$, in which the norm of a function is defined by

$$\|u\|=\max_{a\leqslant x\leqslant b}|u(x)|+L_u, \tag{45}$$

where L_u is the greatest lower bound of the Lipschitz constants of the function u(x) on [a, b]. We assume that Eq. (44) in $C^L_{[a,\,b]}$ has at least one solution; if there are several solutions, they are assumed to be isolated. One can show that the nonlinear operator

$$\Phi[u(x)]\equiv u(x)-\int_a^b f[x,\ \xi,\ u(\xi)]\,d\xi \tag{46}$$

maps elements of space $C^L_{[a,b]}$ into elements of the same space.

The first and the second Fréchet derivatives of the operator $\Phi[u]$ are defined by the equations

$$\Phi'[u]\,v=v(x)-\int_a^b f'_u[x,\ \xi,\ u(\xi)]\,v(\xi)\,d\xi;$$

$$\Phi''[u]vw=-\int_a^b f''_{u^2}[x,\ \xi,\ u(\xi)]\,v(\xi)\,w(\xi)\,d\xi,$$

where u, v, w $\in C^L_{[a,b]}$. By virtue of the assumption that the second derivatives of $f(x,\ \xi,\ u)$ are continuous we obtain continuity of $\Phi'[u]$ and $\Phi''[u]$ at each point u $\in C^L_{[a,b]}$. Using the operator Φ, we formulate the problem of solving Eq. (44) as follows: to find a solution of the equation $\Phi[u(x)] = 0$ in the function space $C^L_{[a,b]}$ under the assumption that such solutions (one or several) exist. The following theorem holds.

Theorem 5. Let $f(x,\ \xi,\ u)$ be a given twice continuously differentiable function of its arguments. Further, suppose that in an open bounded domain G of the space $C^L_{[a,\,b]}$ there exists a unique solution $u^*(x)$ of Eq. (44). Suppose that for any fixed function $\bar{u}(x) \in$ G the linear equation

$$v(x)-\int_a^b f'_u[x,\ \xi,\ \bar{u}(\xi)]\,v(\xi)\,d\xi=0 \tag{47}$$

has only the trivial solution $v(x) \equiv 0$. Then there exists a sphere S: $\|u-u^*\| \leq R$, which belongs to G, that is such that for any function $u_0(x) \in$ S in the half-strip $\Omega = [a,\,b] \times [0 \leq t < \infty)$ there exists a unique solution of the system of equations

$$\begin{cases} v(x,\ t)-\int_a^b f'_u[x,\ \xi,\ u(\xi,\ t)]\,v(\xi,\ t)\,d\xi \\ =\int_a^b f[x,\ \xi,\ u(\xi,\ t)]\,d\xi-u(x,\ t); & (48) \\ \partial u(x,\ t)/\partial t=v(x,\ t) & (49) \end{cases}$$

with the initial condition

$$u(x, 0) = u_0(x), \tag{50}$$

where

$$\lim_{t\to\infty} u(x, t) = u^*(x). \tag{51}$$

The convergence in this relation is understood in the sense of the metric (45) of the space $C^L_{[a,b]}$. Here $u_0(x)$ is the initial approximation to the desired solution of Eq. (44).

The proof of Theorem 5 reduces to verifying that the conditions of Theorem 2, formulated for a Banach space, are satisfied.

The difference scheme is based on the continuous analog of Newton's method, i.e., the difference scheme is constructed for the problem (48)-(50). This problem is solved by Euler's method in two stages:

1) approximate solution of the linear Fredholm integral equation of second kind (48) for the unknown function v(x, t) for fixed t. For given t, the function u(x, t) in (48) is assumed known;

2) completion of one integration step with respect to the variable t in accordance with the equation

$$u(x, t+\tau) = u(x, t) + \tau v(x, t).$$

Further, it is assumed that all the conditions of Theorem 5 are satisfied. We split [a, b] into N equal parts by means of the division points

$$a = x_0;\ x_1 = x_0 + h;\ x_2 + x_1 + h,\ \ldots;\ x_N = x_{N-1} + h = b; \tag{52}$$
$$h = (b-a)/N.$$

With each function u(x) in $C^L_{[a,b]}$ we associate N + 1 numbers: $u(x_0)$, $u(x_1)$, ..., $u(x_N0)$, the coordinates of the vector $u_h = (u_0, u_1, \ldots, u_N)$, where $u_i = u(x_i)$, $0 \le i \le N$, which we call the mesh image of the function u(x). In its turn, we associate the mesh image u_h with a polygon $u_h(x)$ constructed from the lattice points. The function $u_h(x)$ is continuous and satisfies a Lipschitz condition. By the norm of the vector u_h we shall understand the norm of the function $u_h(x)$ in the metric of $C^L_{[a,b]}$. Using some quadrature formula, for example, the trapezium or Simpson formula, we replace the integral in (44) by the sum corresponding to the division points (52). We obtain a system of nonlinear equations for u_0, u_1, ..., u_N:

$$u_i = \sum_{h=0}^{N} A_h^{(N)} f[x_i, \xi_k, u_k], \quad i = 0, 1, \ldots, N. \tag{53}$$

If we take N sufficiently large, i.e., h sufficiently small, and the initial value to be $u_0 = (u_0^*, u_1^*, \ldots, u_N^*)$, which is the mesh image of the function $u^*(x)$, and a solution of equation (44), and apply to the system (53) the continuous analog of Newton's method and Theorem 1 of [2], we can show that this system is solvable. Consider the operator

$$\varphi(u) \equiv u_i - \sum_{h=0}^{N} A_h^{(N)} f[x_i, \xi_k, u_h] = z_i, \quad i = 0, 1, \ldots, N. \tag{54}$$

It is shown in ref. 7 that $\varphi(u)$ has a continuous Fréchet derivative $\varphi'(u)$ and that the operator $\varphi'(u)$ has a bounded inverse, i.e., $\| [\varphi'(u)]^{-1} \| \le B$, if N is sufficiently large and if the initial value is taken to be the mesh image of the function u(x) in the sphere S: $\|u - u^*\| \le R$, S G. It follows from the smoothness of the function $f(x, \xi, u)$ that there exists a bounded second Fréchet derivative $\varphi''(u)$. Consider the sphere S_1:

$$\| u - u^* \| \leqslant B \| \varphi(u^*) \|. \tag{55}$$

Clearly, $\| \varphi(u^*) \| \le Mh^p$, where M = const > 0, p > 1. For sufficiently small h, the sphere (55) belongs to S. Hence, Theorem 1 applies to the system (53). This system is uniquely solvable in the sphere (55) and its solution can be obtained as a solution of the Cauchy problem

$$\begin{cases} du/dt = -[\varphi'(u)]^{-1} \varphi(u); \\ u(x, 0) = u^*(x). \end{cases}$$

Knowing that a solution of the system (53) exists and is unique, and also using the results of ref. 7, we can formulate the following theorem.

<u>Theorem 6.</u> Consider the system of equations

$$\left. \begin{aligned} & w_{ij} - \sum_{h=0}^{N} A_h^{(N)} f'_u [x_i, \xi_h, u_{hj}] w_{kj} \\ & = \sum_{h=0}^{N} A_h^{(N)} f [x_i, \xi_h, u_{hj}] - u_{ij}; \\ & u_{i,\, j+1} - u_{ij} = w_{ij}\tau;\ i = 0, 1, \ldots, N;\ j = 0, 1, \ldots \end{aligned} \right\} \tag{56}$$

where τ is sufficiently small and N is sufficiently large, with the given initial condition

$$u_0 = (u_{00}, u_{10}, \ldots, u_{N0}). \tag{57}$$

Then there exists a number $\varepsilon > 0$ such that if $\|u_0 - u^{**}\| \le \varepsilon$, where u^{**} is the solution of the system (53), then the system (56) with the initial condition (57) is uniquely solvable for all i = 0, 1, 2, ..., N and j = 0, 1, 2, ... The solution $w_j = (u_{0j}, u_{1j}, \ldots, u_{Nj})$ of the problem (56)-(57) has the property that

$$\lim_{j\to\infty} | u_j - u^{**} | = 0.$$

5. CALCULATION OF PARTICLE-LIKE SOLUTIONS OF AN EQUATION OF NONLINEAR FIELD THEORY

In certain problems that arise in nonlinear field theory and the statistical theory of nuclei one considers the following boundary-value problem for ordinary differential equations of second order.

1. To find a solution y = y(x) of the equation

$$y'' + 2y'/x - y + y^n = 0, \quad n > 1, \quad x \geqslant 0, \tag{58}$$

satisfying the conditions

$$y(0) = y_0 < \infty,\ y'(0) = 0;\ y(\infty) = 0, \tag{59}$$

where y_0 is an unknown positive parameter.

2. To find a solution $\eta = \eta(x)$ of the equation

$$\eta'' = \eta - \eta^n / x^{n-1}, \quad n > 1, \quad x \geqslant 0, \tag{60}$$

satisfying the conditions

$$\eta(0)=0, \quad \eta'(0)=\alpha<\infty, \quad \eta(\infty)=0, \tag{61}$$

where α is an unknown positive parameter.

The solution of these problems is said to be particle-like.[10] Equation (60) can be obtained from Eq. (58) by the substitution $\eta(x) = xy(x)$. The solutions of Eq. (58) with the initial conditions

$$y(0)=y_0, \quad y'(0)=0 \tag{62}$$

correspond to the solutions of Eq. (60) with the initial conditions

$$\eta(0)=0; \quad \eta'(0)=\alpha=y_0. \tag{63}$$

The problems (58)-(59) and (60)-(61) with n = 2 and 3 arise in nonlinear field theory when one studies the interaction of elementary particles. For n = 3/2, Eq. (58) is an equation of the Thomas–Fermi type, and the corresponding problem (58)-(59) arises in the statistical theory of nuclei. In the same field one encounters the more complicated problem

$$\eta_i''=m_i\eta_i-k_i\frac{(\eta_2-\eta_1)^n}{x^{n-1}}, \quad n>1, \; x\geqslant 0 \; (i=1, 2);$$

$$\eta_i(0)=0, \quad \eta_i'(0)=\alpha_i>0, \quad \eta_i(\infty)=0.$$

The problem (60)-(61) for n = 2 and 3 has been considered by several authors, for example, in refs. 10, 65-68. In these papers some arguments are put forward relating to the existence and the properties of the solutions of this boundary-value problem and the results of computer calculations are given. In ref. 10 a numerical method is used to find five values of the initial derivative α corresponding to five different solutions of the problem.

In ref. 69, the problem (60)-(61) is reduced to a boundary-value problem for the linear equation

$$y''=y-\varepsilon^{n-1}y/x^{n-1}, \quad \varepsilon>0$$

by the method of linearization of nonlinear equations. However, the method of linearization is not completely justified.

In refs. 70-72 the existence of a solution of the problem (58)-(59) for n satisfying the condition $1 < n \leq 4$ is established. In ref. 70 the existence of a solution for $1 < n \leq 3$ is proved. Reference 70 also includes a qualitative investigation of the behavior of the solutions; this makes it possible to replace the boundary-value problem on the straight line approximately by a boundary-value problem for a finite interval. This circumstance is exploited in the numerical determination of the solution.

In ref. 71 the existence of solutions of the boundary-value problem is established by an investigation of the minimum of a certain functional. Such a method is effective for the investigation of the existence of solutions, but it does not enable one to study their qualitative behavior, which is important when one wishes to choose initial approximations in a numerical solution. In ref. 73 it is established that for $n \geq 5$ there are no solutions of the boundary-value problem. Thus, the problem (58)-(59) does not have solutions for $0 < n \leq 1$ and $n \geq 5$. For $1 < n \leq 4$ positive solutions exist. For $4 < n < 5$ the question of the existence of solutions of the boundary-value problem has not been investigated.

Apart from their independent physical interest, these problems are of great mathematical interest. In these problems questions of numerical determination of solutions are intimately related to investigations of the qualitative behavior. For it is on the basis of such an investigation that one can apply a numerical method developed on the basis of the continuous process of Newton most effectively. The numerical solution of these problems brings out especially clearly the basic aspects of a given numerical method: The need to make a thorough preliminary analysis of the problem in order to construct effective initial approximations and the great scope of the method, which enables one to solve problems with a countable set of solutions.

We begin by listing the principal results of the qualitative analysis of the problem made in refs. 70, 73, and 74. Basically, they refer to the boundary-value problem (60)-(61), but, as we have already noted, they can be transferred to the problem (58)-(59).

Since the solutions of the problems (58)-(59) and (60)-(61) belong to the set of solutions of the Cauchy problems (58), (62), and (60), (63), respectively, a number of facts pertaining to the solution of these problems have been established. In ref. 73 it is shown that solutions of the problems (58), (62), and (60), (63) exist, are unique, and continuable for all $x \geq 0$ and rational n [for $n = (2p + 1)/2q$ we assume $\eta'' = 0$ for $\eta < 0$]. The solutions together with their first derivative depend continuously on the initial values y_0 and α. As in ref. 74, it is proved that in the neighborhood of the singular point x = 0 the solution of the Cauchy problem, and therefore of the boundary-value problem can be represented by convergent power series. This last fact is important when one constructs algorithms for the numerical solution of these problems. The method by which these assertions are proved enables one to establish a number of additional useful facts about the behavior of the solutions of the problem (60), (63). These include estimates of the first maximum of the solution as a function of the initial derivative α, and also about the nature of the further behavior of the solution. These facts are set forth in more detail in ref. 70.

With regard to the solution of the boundary-value problems (58)-(59) and (60)-(61), the following existence theorem holds.[75]

Theorem 7. For any integer $i \geq 0$ and any $n = (2p + 1)/(2q + 1)$, where p and q are positive integers, $1 < n < 4$, there exist solutions of the problems (58)-(59) and (60)-(61) that have exactly i zeros on the interval $0 < x < \infty$.

It is shown that any such solution is unstable in the sense of Lyapunov. Note that when $n \neq (2p + 1)/(2q + 1)$ the problems (58)-(59) and (60)-(61) have only positive solutions.

In a numerical determination of particle-like solutions, the boundary-value problem on a half-axis can be replaced approximately on the basis of the results of ref. 70 by the problem on the finite interval [0, b], where $b > 0$ is chosen sufficiently large.

In the case n = 3, Eq. (60) is investigated numerically by means of the continuous analog of Newton's method in ref. 3. Particle-like solutions are calculated and the dependence of the computational process on the choice of the initial conditions and the integration step τ_k in Euler's scheme is investigated. For the construction of initial approximations, the estimates given in ref. 70 for the extrema of the solutions are used. Experience gained from calculations by an optimal algorithm for the choice of τ_k that ensures convergence of the process when one is calculating particle-like solutions with several zeros shows that the change in the step is proportional to the change in the discrepancy between the difference equations in the foregoing steps.

We are interested in studying the process of convergence of solutions of boundary-value problems on the finite interval [0, b], y(0) = y(b) = 0 (b → ∞) to a particle-like solution. The developed algorithm was used to solve a sequence of boundary-value problems for Eq. (60) (n = 3) for b = 2, 4, 6, 8, 10, 12. Positive solutions were calculated. On the interval [0, 2] the function $\eta_0(x) = 1.3 \cdot \sin \pi x/2$ was taken as an initial approximation. On each following interval the initial function was constructed from the solution obtained for the boundary-value problem on the foregoing interval with continuation by the same zero on the remaining part of the interval. The results of the numerical experiment are given in[31] Table 1. Boldface incates an agreement in the given decimal place with the result for the next value of b.

This table shows that for the calculation of positive particle-like solutions the half-axis $0 \le x < \infty$ can be replaced with sufficient accuracy by the finite interval [0, 10].

6. EXTRACTION OF A BEAM OF CHARGED PARTICLES FROM AN ACCELERATOR

An important problem in the design of an accelerator is the choice of its parameters in such a way as to ensure the required properties of the beam extracted from the accelerator.

In refs. 76-84 the method of calculating the beam extraction is usually reduced to a solution of a Cauchy problem for a certain differential equation that describes the motion of a particle in the magnetic field of the accelerator. One specifies certain parameters of the deflecting magnets and calculates the trajectory of particles corresponding to the chosen values of the parameters. If the trajectories that are obtained do not satisfy the requirements imposed upon them, other parameters of the deflecting magnets are specified and the Cauchy problem is solved once more. The process of choosing the parameters is continued until the particle trajectories satisfy the posed requirements. It is clear that such a (intuitive) choice of the parameters entails the calculation of many trajectories and is therefore wasteful of computer time. In addition, the accuracy of the desired parameters is not high.

An interesting mathematical formulation of the problem of optimizing the parameters of focusing systems of charged particles can be found in ref. 85. In this paper an algorithm is developed choosing the parameters of a magnetic system to ensure optimal conditions (in a certain sense) of the focusing of charged particles. The problem is solved in the single-particle approximation. The algorithm is based on the construction of a certain variational functional, whose extremal values are determined by numerical methods. Here we consider a new numerical method of solving the problem of the extraction of a beam of charged particles from an accelerator. For the first time the mathematical formulation of the problem of beam extraction from an accelerator is reduced to the solution of boundary-value problems for a nonlinear ordinary differential equation of second order. Using this method, which is based on the continuous analog of Newton's method, one can find the necessary values of the parameters automatically. Man's interference becomes superfluous. In conjunction with a fast computer program, this economizes time and raises the accuracy of the end results.

We give a simplified formulation of our problem. The detailed formulation and results of numerical calculations are given in ref. 9. The motion of a charged particle in the median plane of the accelerator is described by the equation

$$\rho'' = 2\rho'^2/(R_s+\rho) + R_s + \rho - (R_s+\rho)^2 F(\rho)/R_1 = -f(\varphi, \rho, \rho'). \quad (64)$$

Here, $\rho = \rho(\varphi)$ is the desired function; φ is the argument; $R = R_S$ is the equilibrium orbit; $\rho = \rho(\varphi)$ is the deviation of the particle from the equilibrium orbit; R_1 is a given constant; $F = F(\rho)$ is a given function, which describes the magnetic field of the accelerator.

One has to find a solution of Eq. (64) satisfying the boundary conditions

$$\rho(\varphi_1) = \rho_1; \quad \rho(\varphi_2) = \rho_2, \quad (65)$$

where ρ_1 and ρ_2 are given constants. We shall seek the solution in the interval $\varphi_1 \le \varphi \le \varphi_2$. The boundary conditions (65) mean that one has to transfer a particle from a given point (φ_1, ρ_1) of the accelerator to another point (φ_2, ρ_2). This can be achieved by means of an initial slope $\rho'|_{\varphi=\varphi_1} = \rho_0'$ of the curve. The desired slope ρ'_0 at $\varphi = \varphi_1$ can be ensured by the arrangement of the deflecting magnet in the region $\varphi < \varphi_1$. After solving the problem (64)-(65), we find $d\rho/d\varphi$ at the point $\varphi = \varphi_1$. This enables us to determine the parameters of the deflecting magnet set up in the region $\varphi < \varphi_1$.

In accordance with the method of Newton we have described for ordinary nonlinear differential equations (see Sec. 2), the problem (64)-(65) reduces to the following.

To solve the system of equations

TABLE 1

x	$b=2$	$b=4$	$b=8$	$b=10$	$b=12$
0.3	1.083713	1.025938	1.024517	1.024516	1.024516
0.6	1.156584	1.207219	1.208096	1.208096	1.208096
0.9	0.887350	1.018838	1.021596	1.021597	1.021597
1.2	0.602958	0.786233	0.790178	0.790179	0.790179
1.5	0.355318	0.589509	0.594513	0.594515	0.594515
1.8	0.137357	0.436771	0.443066	0.443068	0.443068
2.1	0.0	0.320983	0.329033	0.329035	0.329035
2.4	0.0	0.233528	0.244007	0.244011	0.244011
2.7	0.0	0.167019	0.180849	0.180853	0.180853
3.0	0.0	0.115862	0.134004	0.134010	0.134010
3.3	0.0	0.074614	0.099281	0.099289	0.099289

$$\begin{cases} v''_{\varphi^2} + \mathcal{P} v'_\varphi + Qv = R; & (66) \\ \rho'_t = v, & (67) \end{cases}$$

where $v = v(\varphi, t)$ and $\rho = \rho(\varphi, t)$ are the desired functions of φ and t, with $\varphi_1 \leq \varphi \leq \varphi_2$, $0 \leq t < +\infty$; $\mathcal{P} = f'_{\rho'}$; $Q = f'_\rho$; and $R = -[\rho''_{\varphi^2} + f(\varphi, \rho, \rho')]$ are known functions of φ, ρ, and ρ'.

The system (66)-(67) is solved with the initial condition

$$\rho(\varphi, 0) = \rho_0(\varphi), \ \rho_0(\varphi)_1 = \rho_1, \ \rho_0(\varphi_2) = \rho_2 \quad (68)$$

and the boundary conditions

$$v(\varphi_1, t) = v(\varphi_2, t) = 0. \quad (69)$$

This problem can be readily solved on a computer.

7. INVERSE PROBLEM OF SCATTERING THEORY

We consider the Schrödinger equation for the radial wave function (the case $l = 0$)

$$u'' + [k^2 - v(x)] = 0 \ (k^2 \text{ is a parameter}) \quad (70)$$

with the initial conditions

$$u(0, k) = 0, \quad u'(0, k) = k. \quad (71)$$

The real function v(x) ($0 < x < \infty$), which is called the potential, is assumed to be sufficiently smooth and to satisfy the condition

$$\int_0^\infty x\,|v(x)|\,dx < \infty. \quad (72)$$

The solution u(x, k) of the problem (70)-(71) when the condition (72) holds for large values of x has the asymptotic behavior

$$u(x, k) = A(k)\sin[kx + \delta(k)] + 0(1), \quad (73)$$

where A(k) and δ(k) are continuous functions of the parameter k; A(k) is known as the asymptotic amplitude and δ(k) as the limiting phase shift.

The inverse problem of scattering theory consists of finding the potential v(x) from given limiting phase shift $\delta(k)$ ($0 \leq k < \infty$). The first important results in the investigation of the inverse problem are due to Levinson.[86] In what follows we shall, in addition to this formulation of the inverse problem, use the following as well. Instead of Eq. (70) we consider the phase equation

$$y'(x, k) = -v(x)\sin^2[kx + y(x, k)]/k \ (k^2 \text{ is a parameter}), \quad (74)$$

which was first obtained by Drukarev.[87] The derivation of Eq. (74) from Eq. (70) can also be found in ref. 88.

The initial conditions (71) then go over into the initial condition

$$y(0, k) = 0 \quad (75)$$

for Eq. (74).

If the condition (72) is satisfied, the solution of the problem (75)-(76) has the property

$$\lim_{x\to\infty} y(x, k) = \delta(k), \quad (76)$$

where δ(k) is the limiting phase shift. The inverse problem is now posed as follows: given the function δ(k) ($0 \leq k < \infty$), to find a function v(x) ($0 < x < \infty$) such that the solution y(x, k) of the Cauchy problem (74)-(75) with the potential v(x) gives, by virtue of (76), the given δ(k).

We consider questions of the solvability and the number of solutions of the inverse problem of scattering theory. A great many investigations have already been devoted to the problem of reconstructing the potential v(x) in the Schrödinger equation (70) from certain given asymptotic properties of its solutions (for example, from the limiting phase shift or from the spectral function). We mention foremost the well known papers of Levinson,[86] Gel'fand and Levitan,[89] Krein,[90] and Marchenko.[91] A detailed review of the results on the inverse problem can be found in the monograph of Agranovich and Marchenko,[92] in Faddeev's paper,[93] and also in the books of de Alfaro and Regge,[94] Wu and Ohmura,[95] Newton,[96] and Calogero.[97]

For example, Marchenko[91] derives the basic integral equation of the inverse theory, formulates the well known conditions for the existence and uniqueness of a solution of this equation, and gives a method for reconstructing the potential from given scattering data. Here, we shall not dwell in detail on the well known results that apply to the inverse problem, but merely mention that on the basis of these results one can formulate conditions of unique solvability of the inverse problem.

For this it is required that the function δ(k) given on the interval $[0, \infty)$ be sufficiently smooth and satisfy the condition

$$\delta(0) = \delta(\infty) = 0. \quad (77)$$

Then the potential in Eq. (74) can be uniquely reconstructed. Thus, to reconstruct the potential, δ(k) must be known for all positive values of k.

In practice, δ(k) can be determined only on a finite interval $[k_1, k_2]$, since the scattering is always observed in a finite energy range. In addition, at sufficiently high energies, the scattering process can no longer be described by the Schrödinger equation.

One is led to this question: Can the potential v(x) be found approximately if δ(k) is given only in the finite energy range $[k_1, k_2]^2$? This question has been studied by Mel'nikov,[98] Marchenko,[91] and Lundina and Marchenko.[99]

In Marchenko's papers it is proved, in particular, that it is in principle possible to recover the potential (with a given accuracy) if δ(k) is given on a finite interval $[k_1, k_2]$. Lyantse[100] has studied scattering problems in the case of a complex potential. A number of results on scattering by highly singular potentials have been established by Limic,[101] and by Rofe-Beketov, and Khristov.[102]

Let us consider an approximate method of solving the inverse problem and the results of computer calculations.

Among the approximate methods used to recover the potential from the limiting phase shift, we mention the methods based on the use of Marchenko's integral equations (see, for example, ref. 103) or the Gel'fand–Levitan integral transformation (see, for example, ref. 104). In other papers (for example, ref. 105) the analytic form of the potential is specified, in particular, the Yukawa potential:

$$v(x) = \sum_{k=1}^{m} A_k \exp(-kx)/x. \quad (78)$$

In this case, the coefficients A_k are determined from a given experimental phase shift.

Let us dwell in more detail on the approximate solution of the inverse problem by the introduction of a continuous parameter.[106] Equation (74) with the initial condition (75) and the relation (76) defines a certain transformation P such that with every potential in a certain class there is associated a limiting phase shift: $\delta(k) = P[v(x)]$.

Our problem is to find a potential $v^*(x)$ such that the operator P maps to a given experimental phase shift $\delta_e(k)$ $(0 \leq k < \infty)$ such that $\delta_e(0) = \delta_e(\infty) = 0$. In other words, we must solve the nonlinear operator equation

$$\Phi(v) \equiv P(v) - \delta_e(k) = 0 \tag{79}$$

for $v(x)$ $(0 < x < \infty)$.

To solve Eq. (79) we use the method of introducing a continuous parameter. Consider the equation

$$d\Phi[v(x, t)]/dt = -\Phi[v(x, t)] \tag{80}$$

with the initial condition $v(x, 0) = v_0(x)$, where $v_0(x)$ is the initial value of the potential.

Assuming that the potential $v(x, t)$ corresponds to a calculated limiting phase shift $\delta_c(k, t) = P[v(x, t)]$ and using (79) and (80), we obtain

$$d\delta_c(k, t)/dt = -[\delta_c(k, t) - \delta_e(k)]. \tag{81}$$

Let us find an expression for the left-hand side of Eq. (81). To do this, we differentiate both sides of Eq. (74) with respect to t under the assumption that y and v depend on x and on t. Writing $y'_t(x, t) = w(x, t)$ and $v_t(x, t) = z(x, t)$, we find

$$\begin{aligned} w'_x(x, t) = & -z(x, t)\sin^2[kx + y(x, t)]/k \\ & - v(x, t)\sin 2[kx + y(x, t)]\, w(x, t)/k. \end{aligned} \tag{82}$$

Solving this linear differential equation for $w(x, t)$ with the obvious boundary condition $w(0, t) = y'_t(0, t) = 0$, we obtain

$$\begin{aligned} w(x, t) = & (-1/k)\int_0^x z(s, t)\sin^2[ks + y(s, t)] \\ & \times \exp\left\{-(1/k)\int_s^\infty v(\xi, t)\sin 2[k\xi + y(\xi, t)]\, d\xi\right\} ds, \end{aligned} \tag{83}$$

from which, going to the limit $x \to \infty$, we obtain

$$\begin{aligned} w(\infty, t) = & -(1/k)\int_0^\infty z(s, t)\sin^2[ks + y(s, t)] \\ & \times \exp\left\{-(1/k)\int_s^\infty v(\xi, t)\sin 2[k\xi + y(\xi, t)]\, d\xi\right\} ds. \end{aligned} \tag{84}$$

Taking into account (76) and that the fact that $w(x, k, t) = y'_t(x, k, t)$, and also assuming that we can reverse the order of the operations of going to the limit (with respect to x) and taking the derivative (with respect to t), we find

$$w(\infty, k, t) = \lim_{x\to\infty} y'_t(x, k, t) = d\delta(k, t)/dt,$$

i.e., (81) takes the form

$$\begin{aligned} & \int_0^\infty z(x, t)\sin^2[kx + y(x, t)] \\ & \times \exp\left\{-(1/k)\int_x^\infty v(\xi, t)\sin 2[k\xi + y(\xi, t)]\, d\xi\right\} dx \\ & = k[\delta_c(k, t) - \delta_e(k)], \end{aligned} \tag{85}$$

where

$$z(x, t) = dv(x, t)/dt. \tag{86}$$

The system (85)–(86) is solved with the initial condition

$$v(x, 0) = v_0(x). \tag{87}$$

For a suitable choice of $v_0(x)$, we obtain a solution of Eq. (79) as the limit of $v(x, t)$ as $t \to \infty$:

$$\lim_{t\to\infty} v(x, t) = v^*(x). \tag{88}$$

We now explain the scheme of the numerical solution of the problem (85)–(87). Substituting the initial value of the potential, $v_0(x)$, into Eq. (74), and solving the problem (74)–(75), we obtain $y(x, k, 0)$ and $\delta_c(k, 0)$. Then $v_0(x)$, $y(x, k, 0)$, and $\delta_c(k, 0)$ are substituted into the Fredholm equation of the first kind (85), from which the unknown function $z(x, 0)$ is determined.

Of course this problem is not well posed. For its solution one can apply the well known method of regularization developed by Tikhonov.[107] We replace Eq. (86) with $t = 0$ by the approximate difference relation

$$[v(x, \tau) - v(x, 0)]/\tau = z(x, 0),$$

from which we obtain

$$v(x, \tau) = v(x, 0) + \tau z(x, 0), \tag{89}$$

where τ is the step with respect to the variable t.

This cycle of calculations is repeated with $v(x, \tau)$, obtained from (89) as the new initial value of the potential. The computational process is continued until the calculated potential is stabilized. As stabilization criterion one takes, for example, the condition

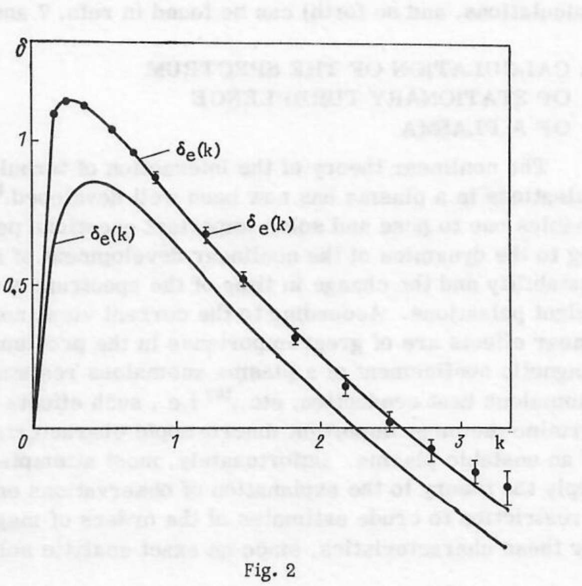

Fig. 2

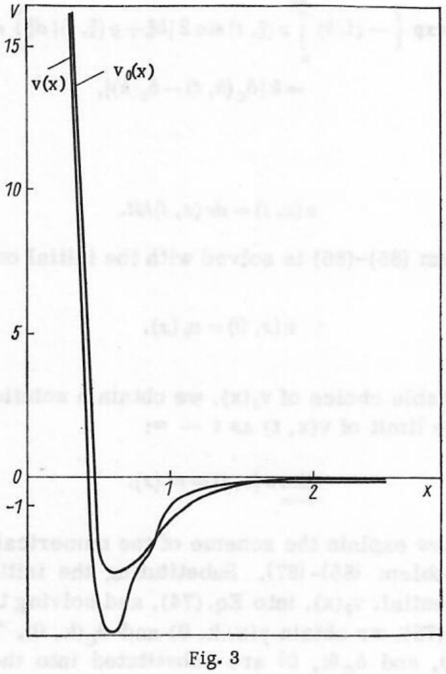

Fig. 3

$$\sigma_c=\sqrt{\sum_{i=1}^{m}[\delta_c(k_i)-\delta_e(k_i)]^2/m}\leqslant\sigma_e,$$

where

$$\sigma_e=C\sqrt{\sum_{i=1}^{m}(\Delta\sigma_i)^2/m};$$

$\Delta\sigma_i$ is the error of the phase shift $\delta_e(k)$ at the point $k = k_i$; $C \geq 1$. The quantity C depends on the interval $[k_1, k_2]$ on which the limiting phase shift is given and certain other factors. We give the results of numerical calculations with a computer. Figure 2 shows the dependence of the experimental phase shift[108] $\delta_e(k)$, the phase shift $\delta_0(k)$ corresponding to the initial value of the potential $v_0(x)$, and the theoretical limiting phase shift $\delta_c(k)$ calculated from the recovered potential v(x). The dependences of the initial value of the potential $v_0(x)$ and of the potential v(x) are shown in Fig. 3. A more detailed description of the numerical calculations (tables of results, accuracy of the calculations, and so forth) can be found in refs. 7 and 106.

8. CALCULATION OF THE SPECTRUM OF STATIONARY TURBULENCE OF A PLASMA

The nonlinear theory of the interaction of turbulent pulsations in a plasma has now been well developed.[52] It enables one to pose and solve important questions pertaining to the dynamics of the nonlinear development of an instability and the change in time of the spectrum of turbulent pulsations. According to the current view, nonlinear effects are of great importance in the problems of magnetic confinement of a plasma anomalous resistance, anomalous heat conduction, etc.,[109] i.e., such effects determine the most important macroscopic characteristics of an unstable plasma. Unfortunately, most attempts to apply the theory to the explanation of observations entail a restriction to crude estimates of the orders of magnitude for these characteristics, since an exact analytic solution of the nonlinear integral equations is frequently out of the question.

One of the important problems of the theory is to calculate the spectra of stationary turbulence of a plasma. In refs. 110 and 111 such a spectrum is calculated analytically in the asymptotic region for Langmuir plasma pulsations.

To obtain the complete turbulence spectrum (for all numbers of pulsations k) one cannot in practice use analytic methods. Here we described a method of calculation and the results of computer calculations of the complete turbulence spectrum.[112,113] The investigation is made by means of the continuous analog of Newton's method. The main equations used to calculate the spectra are the same as in ref. 110. The spectrum is assumed to be three-dimensional, isotropic, and therefore to depend only on the modulus of k. We allow for the effects of four-plasmon interaction (plasmon–plasmon scattering), induced scattering of plasmons by ions, and the absorption of plasmons in Coulomb collisions.

The problem that was solved on the computer can be stated as follows. It is required to solve the integral equation

$$\begin{aligned}y(x)=a(x)\int_0^\infty K(\xi)U[y(x),\,y(x\sqrt{\xi})]\,d\xi\\+b\Big\{\int_0^1 d\eta\int_0^\eta d\xi L(\xi,\eta)\\\times V[\xi,\eta,y(x),y(x\sqrt{\xi}),y(x\sqrt{\eta}),y(x\sqrt{1+\xi-\eta})]\\+\int_1^\infty d\eta\int_\eta^\infty d\xi M(\xi,\eta)\\\times V[\xi,\eta,y(x),y(x\sqrt{\xi}),y(x\sqrt{\eta}),y(x\sqrt{1+\xi-\eta})]\\+2\int_1^\infty d\eta\int_0^1 d\xi N(\xi,\eta)\\\times W[\xi,\eta,y(x),y(x\sqrt{\xi}),y(\sqrt{\eta}),y(x\sqrt{\xi+\eta-1})],\end{aligned}\quad(90)$$

where

$$a(x)=(25.455(x/x_0)^2;\quad b=2545.5;\quad x_0=0.6;$$

$$K(\xi)=\begin{cases}K_1(\xi) & \text{for}\quad \xi\in[1,\infty);\\ K_2(\xi) & \text{for}\quad \xi\in(0,1].\end{cases}$$

The functions $K_1(\xi)$, $K_2(\xi)$, U, V, W, L, M, N are specified by equations (see refs. 110 and 112).

Besides the trivial solution, Eq. (90) has others. A specific solution can be separated by an appropriate choice of the initial approximation $y_0(x)$.

To solve the problem (90) we used the continuous analog of Newton's method. The infinite upper limit in the integrals of Eq. (90) was replaced by a finite number R, which was sufficiently large for the behavior of the desired solution y(x) to be well described on the interval [0, R].

In accordance with the adopted method, Eq. (90) is replaced by the system of equations

$$
\left.\begin{aligned}
& a(x)\int_0^R K(\xi)\,[U'_{y(x)}U_1(x,t)+U'_{y(x\sqrt{\xi})}U_1(x\sqrt{\xi},t)]\,d\xi \\
& +b\Big\{\int_0^1 d\eta\int_0^\eta L(\xi,\eta)\,[V'_{y(x)}U_1(x,t)+V'_{y(x\sqrt{\xi})}U_1(x\sqrt{\xi},t) \\
& +V'_{y(x\sqrt{\eta})}U_1(x\sqrt{\eta},t)+V'_{y(x\sqrt{1+\xi-\eta})}U_1(x\sqrt{1+\xi-\eta},t)]\,d\xi \\
& +\int_1^R d\eta\int_\eta^R M(\xi,\eta)\,[V'_{y(x)}U_1(x,t)+V'_{y(x\sqrt{\xi})}U_1(x\sqrt{\xi},t) \\
& +V'_{y(x\sqrt{\eta})}U_1(x\sqrt{\eta},t)+V'_{y(x\sqrt{1+\xi-\eta})}U_1(x\sqrt{1+\xi-\eta},t)]\,d\xi \\
& +2\int_1^R d\eta\int_0^1 N(\xi,\eta)\,[W'_{y(x)}U_1(x,t) \\
& +W'_{y(x\sqrt{\xi})}U_1(x\sqrt{\xi},t)+W'_{y(x\sqrt{\eta})}U_1(x\sqrt{\eta},t) \\
& +W'_{y(x\sqrt{\xi+\eta-1})}U_1(x\sqrt{\xi+\eta-1},t)]\,d\xi\Big\}-U(x,t) \\
& =a(x)\int_0^R K(\xi)\,U\,d\xi+b\Big\{\int_0^1 d\eta\int_0^\eta d\xi LV \\
& +\int_1^R d\eta\int_\eta^R d\xi MV+2\int_1^R d\eta\int_0^1 d\xi NW\Big\}; \\
& \partial y(x,t)/\partial t=U_1(x,t).
\end{aligned}\right\} \quad (91)
$$

The integrodifferential system (91) was solved with the initial condition

$$y(x, 0) = f(x), \quad (92)$$

where

$$f(x)=\begin{cases}(x^2/x_0^2)\exp(1-x^2/x_0^2), & 0\leqslant x\leqslant x_0;\\ (x_0/x)^3, & x_0\leqslant x\leqslant 4.\end{cases} \quad (93)$$

Here $x_0 = 0.6$.

In the first equation of the system (91) $y(x, t)$ is replaced everywhere by its value at $t = 0$, i.e., $y(x, 0) = f(x)$. The equation for $V_1(x, 0)$ becomes a linear integral equation. If the integrals that occur in it are replaced at $x = x_i$ ($i = 1, 2, \ldots, n$) by approximate sums, we obtain a linear algebraic system for u_{i1} ($i = 1, 2, \ldots, n$), where u_{i1} is the approximate value of $U_1(x_i, 0)$. Finding the u_{i1} from this system, we substitute them into the second equation of the system (91). We replace the derivative $y'_t(x, t)$ at $t = 0$ by the approximate expression $[y(x, \tau) - y(x, 0)]/\tau = y'_t(x, 0)$ and do one step of the integration with respect to the variable t. The result is $y_i(\tau)$. Then the complete cycle of calculations is repeated from the start with the function $y_i(\tau)$ instead of $y_i(0)$.

In the limit $n \to +\infty$, the sequence $y_i(n, \tau)$ converges to y_i, the approximate solution of the problem (90). In practice, it was sufficient to perform two integration steps with respect to t in order to reach a situation in which the maximum difference between the results of the first and the second iterations does not exceed 10^{-5} with $\tau = 1$; the calculation was then terminated. That so few steps with respect to t were needed is due to the good choice of the initial approximation, given by (93).

It should be noted that the kernels of the integrals in the system (90) have singularities (becoming infinite) on certain parts of the boundary of the domain of integration. However, it is readily seen that these singularities are integrable if $y(x, t) = 0(x^2)$ near the point $x = 0$.

TABLE 2

x	y (x)	x	y (x)	x	y (x)
0,0	0,000	1,0	0,248	2,8	0,010
0,2	0,272	1,2	0,142	3,2	0,007
0,4	0,773	1,4	0,095	3,6	0,005
0,6	0,877	2,4	0,020	4,0	0,003
0,8	0,479				

As a result of the calculations we obtained an approximate solution of Eq. (90), which is given in Table 2.

The solution obtained on the interval $1.2 \leq x \leq 4$ was approximated by the method of least squares by the function $f(x) = (x_0/x)^\nu$. As a result, ν was found to be equal to 3.89.

Comparison of the solution of Eq. (90) found on the computer with the analytic solution in the asymptotic region enables one to determine the boundary of the asymptotic region.[112] In the region $x > 4x_0$, where $x_0 = 0.6$, there is good agreement between the analytic and numerical solutions. For $x < 4x_0$, the analytic solution does not give a good approximation.

1. L. V. Kantorovich, Usp. Mat. Nauk, 3, 6, 89 (1948); 11, 6, 99 (1956); L. V. Kantorovich and G. P. Alikov, Functional Analysis in Normed Space (Intern. Series of Monographs in Pure and Appl. Maths.), Vol. 46, Oxford (1964).
2. M. K. Gavurin, Izv. Vuzov. Ser. Matem., 5(6), 18 (1958).
3. L. A. Lyusternik and V. I. Sobolev, Elements of Functional Analysis [in Russian], Nauka, Moscow (1965).
4. L. Bittner, Num. Math., 6, 161 (1954).
5. D. F. Davidenko, Dokl. Akad. Nauk SSSR, 88, 601 (1953); Ukr. Matem. Zh., 5, 2, 196 (1953); 7, 1, 18 (1955); Preprints [in Russian]: IAÉ-2048 (1970); IAÉ-2081 (1971); IAÉ-2082 (1971); IAÉ-2083 (1971); IAÉ-2099 (1971); Dokl. Akad. Nauk SSSR, 162, 499 (1965).
6. E. P. Zhidkov and I. V. Puzynin, Zh. Vychis. Mat. Mat. Fiz., 7, 1086 (1967); Preprint JINR 5-3368 [in Russian], Dubna (1967); Dokl. Akad. Nauk SSSR, 180, 18 (1968).
7. E. P. Zhidkov, Some Nonlinear Problems of Modern Physics and Mathematical Methods of Their Solution, Doctoral Thesis [in Russian], Joint Institute for Nuclear Research, Dubna (1970).
8. I. P. Mysovskikh, Vestn. Leningradskogo Univ. Ser. Mat. Fiz. Khim, 11, 25 (1953).
9. E. P. Zhidkov, T. V. Ryl'tseva, and T. V. Feoktisov, Preprint JINR 5-4821 [in Russian], Dubna (1969); Zh. Vychis. Mat. Mat.Fiz., 10, 1199 (1970).
10. V. B. Glasko et al., Zh. Eksp. Teor. Fiz., 35, 452 (1958) [Sov. Phys.–JETP, 35, 312 (1959)].
11. S. G. Alikhanov, R. Z. Sagdeev, and P. Z. Chebotaev, Zh. Eksp. Teor. Fiz., 57, 1565 (1969) [Sov. Phys.–JETP, 30, 847 (1970)]; P. Z. Chebotaev, Inf. Byull. SO AN SSSR, 1, 6, 84 (1970).
12. I. S. Berezin and N. P. Zhidkov, Computing Methods, Vol. 2, Oxford (1965).
13. M. P. Sapagovas, in: Computational Mathematics, No. 1 [in Russian], Kiev (1965), p. 124.
14. A. Lasota, Ann. Polon. Math., 20, 2, 183 (1968).
15. T. S. Vashakmadze, Zh. Vychis. Mat. Mat. Fiz., 4, 623 (1964).
16. P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, Wiley, New York–London (1962).
17. M. Lees, Numerical Solution of Partial Differential Equations, Academic Press, New York (1966). (Equation 59).
18. É. É. Tamme, Zh. Vychis. Mat. Mat. Fiz., 8, 988 (1968).
19. N. N. Glinskaya and I. P. Mysovskikh, Vestn. Leningradskogo Univ. Ser. Mat. Fiz., Khim., 8, 49 (1954).
20. V. A. Vinokurov and Yu. N. Ivanov, Kosmicheskie Issledovaniya, 3, 4, 534 (1965).
21. V. E. Shamanskii, Methods of Numerical Solution of Boundary-Value

Problems on a Computer, Vol. 2 [in Russian], Naukova Dumka, Kiev (1966).
[22]N. S. Bakhvalov, in: Computational Methods and Programming, Vol. 5 [in Russian], Izd. MGU, Moscow (1966), p. 9.
[23]B. M. Budak and N. L. Gol'dman, in: Computational Methods and Programming, Vol. 6 [in Russian], Izd. MGU, Moscow (1967), p. 19.
[24]R. E. Bellman and R. E. Kalaba, Quasilinearization and Nonlinear Boundary-Value Problems [Russian translation], Mir, Moscow (1968).
[25]A. J. Friedman, J. Math. and Mech., 8, 1, 57 (1959).
[26]N. A. Shidlovskaya, Dokl. Akad. Nauk SSSR, 107, 213 (1956).
[27]E. P. Zhidkov and Ch'oi Zai Hĕng, Preprint JINR 11-3427 [in Russian], Dubna (1967).
[28]E. P. Zhidkov and I. V. Puzynin, Preprint JINR 5-2959 [in Russian], Dubna (1966); in: Proc. Conference on Mathematical Methods of Solution of Problems of Nuclear Physics, Dubna, June 21-25 (1966) [in Russian]; Preprint JINR 5-3263, 16 [in Russian], Dubna (1967); Dokl. Akad. Nauk SSSR, 174, 271 (1967).
[29]E. P. Zhidkov and I. V. Puzynin, Zh. Vychis. Mat. Mat. Fiz., 9, 442 (1969); Preprint JINR 5-3963 [in Russian], Dubna (1968).
[30]S. K. Godunov and V. S. Ryaben'kii, Theory of Difference Schemes, Amsterdam (1964).
[31]I. V. Puzynin, Approximate Solution of Boundary-Value Problems for Nonlinear Ordinary Differential Equations of the Second Order by the Method of Introducing a Continuous Parameter. Thesis for Candidates Degree [in Russian], Joint Institute for Nuclear Research, Dubna (1969).
[32]A. I. Koshelev, Dokl. Akad. Nauk SSSR, 91, 1263 (1953); 99, 354 (1954).
[33]S. I. Pokhozhaev, Dokl. Akad. Nauk SSSR, 134, 769 (1960); b) Dokl. Akad. Nauk SSSR, 138, 305 (1961).
[34]L. Collatz, Numerical Solutions of Nonlinear Differential Equations, Wiley, New York (1966), p. 65.
[35]S. I. Khudyaev, a) Dokl. Akad. Nauk SSSR, 148, 44 (1963); b) Dokl. Nauk SSSR, 154, 787 (1964).
[36]S. V. Parter, Numerical Solution of Partial Differential Equations, Academic Press, New York (1966), p. 73.
[37]M. Sapagovas, Matem. Sb., 6, 4, 626 (1966).
[38]A. D. Lyashko and M. M. Karchevskii, Zh. Vychis. Mat. Mat. Fiz., 7, No. 3, 677 (1967).
[39]I. P. Mysovskikh, Dokl. Akad. Nauk SSSR, 94, 995 (1954).
[40]A. N. Tikhonov and A. A. Samarskii, Zh. Vychis. Mat. Mat. Fiz., 1, 5 (1961).
[41]N. N. Yanenko, a) Method of Fractional Steps of Solution of Multidimensional Problems of Mathematical Physics [in Russian], Novosibirsk (1966); b) Introduction to Difference Methods in Mathematical Physics [in Russian], Novosibirsk (1968).
[42]G. E. Forsythe and W. R. Wasaw, Finite-Difference Methods for Partial Differential Equations, New York (1960).
[43]A. A. Samarskii, Introduction to the Theory of Difference Schemes [in Russian], Nauka, Moscow (1971).
[44]E. P. Zhidkov and G. I. Makarenko, Preprint JINR R5-4128 [in Russian], Dubna (1968).
[45]E. P. Zhidkov and G. I. Makarenko, Dokl. Akad. Nauk SSSR, 187, 723 (1969).
[46]C. Miranda, Partial Differential Equations of Elliptic Type, Berlin (1970).
[47]L. Bers, F. John, and M. Schechter, Partial Differential Equations, New York (1964).
[48]D. V. Shirkov, V. V. Serebryakov, and V. A. Meshcheryakov, Dispersion Theories of Strong Interactions at Low Energy, North-Holland, Amsterdam (1969).
[49]P. S. Isaev, Fiz. Él. Chast. Atom. Yad., 2, 1971) [Particles and Nuclei, 2, No. 1, Plenum Press, New York (1972)].
[50]L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev., 101, 1 (1956).
[51]R. Denchev, Preprint JINR R-901 [in Russian], Dubna (1962).
[52]V. N. Tsytovich, Nonlinear Effects in Plasmas, New York (1970).
[53]a) N. N. Bogolyubov and N. M. Krylov, New Methods of Nonlinear Mechanics [in Russian], Gostekhizdat, Moscow (1934); b) N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York (1962).
[54]A. I. Guseinov, a) Matem. Sb., 20(62), 239 (1947); b) Izv. Akad. Nauk SSSR, Ser. Mat., 12, 193 (1948); b) Trudy Inst. Fiz. Mat Akad. Nauk AzSSR, III (1948); c) Matem. Sb., 26(68), 237 (1950).
[55]D. Yu. Panov, Izv. Akad. Nauk SSSR, OMEN, No. 6, 843 (1934).
[56]V. P. Vetchinkin Proc. Central Aerohydrodynamics Inst., No. 192 (1934).
[57]D. M.
[58]B. I. Gekht, Tr. Politekhnicheskogo Inst. Novocherkassk, 26, 436 (1955).
[59]Yu. D. Sokolov, Ukr. Matem. Zh., 9, 4, 394 (1957); 15, 1 (1963).
[60]N. S. Kurpel', in: Approximate Methods of Solution of Differential Equations [in Russian], Izd. Akad. Nauk UkrSSR Kiev, (1963), p. 47.
[61]I. P. Mysovskihk, a) Vest. Leningradskogo Univ., Ser. Mat. Mekh. Astr., 2, No. 7, 78 (1962); b) Dokl. Akad. Nauk SSSR, 153, 30 (1963).
[62]A. M. Mamedov, Izv. Akad. Nauk AzSSR, Ser. Fiz.-Tekh Mat. Nauk, No. 3, 41 (1965).
[63]B. A. Bel'tyukov and G. S. Shil'ko, in: Questions of the Accuracy and Effectiveness of Computational Algorithms, No. 3 [in Russian], Naukova Dumka, Kiev (1969), p. 17.
[64]E. P. Zhidkov and G. A. Ososkov, a) Dokl. Akad. Nauk SSSR, 180, 1279 (1968); b) Preprint JINR R5-4338 [in Russian], Dubna (1969).
[65]R. J. Finkelstein, R. Lelevier, and M. Ruderman, Phys. Rev., 83, 2, 326 (1951).
[66]N. Rosen and H. B. Rosenstock, Phys. Rev., 85, 2, 257 (1952).
[67]R. J. Finkelstein, C. Fronsdal, and P. Kaus, Phys. Rev., 103, 5, 1571 (1965).
[68]J. L. Synge, Proc. Roy. Irish Acad., 62, 3, 17 (1961).
[69]S. F. Shushurin, Investigation of Particle-Like Solutions of Nonlinear Field Equations, Thesis for Candidate's Degree [in Russian], Moscow (1961).
[70]E. P. Zhidkov and V. P. Shirikov, a) Preprint JINR R-1319 [in Russian], Dubna (1963); b) Zh. Vychis. Mat. Mat. Fiz., 4, 804 (1964).
[71]F. Nehari, Proc. Roy. Irish Acad., 62, 9, 118 (1963).
[72]V. P. Shirokov, Preprint JINR 2006 [in Russian], Dubna (1965).
[73]E. P. Zhidkov, V. P. Shirokov, and I. V. Puzynin, in: Proc. Conference on Mathematical Methods of Solution of Problems of Nuclear Physics, Dubna, November 17-20 (1964) [in Russian]; Preprint JINR 2005 [in Russian], Dubna (1965), p. 13.
[74]I. V. Puzynin, Preprint JINR R-1549 [in Russian], Dubna (1964).
[75]V. P. Shirokov, Preprint JINR R-1682 [in Russian], Dubna (1964).
[76]V. I. Kotov and V. V. Miller, Focusing and Separation by Mass of High-Energy Particles [in Russian], Atomizdat, Moscow (1969).
[77]K. G. Steffen, High Energy Beam Optics, New York (1965).
[78]B. V. Vasilishin et al., Preprint JINR 9-4223 [in Russian], Dubna (1968).
[79]V. I. Danilov, B. V. Mukhina, and É. A. Polferov, Preprint JINR R-4102 [in Russian], Dubna (1968).
[80]E. K. Tarasov, V. V. Vladimirskii, and D. G. Koshkareva, Preprint 232 [in Russian], Inst. Teor. Éksp. Fiz. (1964).
[81]V. V. Kochanova, Yu. M. Sapunov, and A. M. Frolov, Preprint OP 67-59 [in Russian], IFYÉ Serpukhov (1967).
[82]K. P. Myznikov and V. M. Tatarenko, Preprint SKU 68-16 (1968).
[83]A. P. Banford, The Transfort of Charged-Particle Beams, London (1966).
[84]V. I. Gridasov et al., Preprint JINR B2-9-4648 [in Russian], Dubna (1968).
[85]V. B. Glasko, A. G. Sveshnikov, and N. N. Semashko, Zh. Tekh. Fiz., 40, 270 (1970) [Sov. Phys.—Tech. Phys., 15, 196 (1970)].)]
[86]N. Levinson, Kgl. Danske Vid. Selskab., Mat.-Fys. Medd., 25, N. 9(1949).
[87]G. F. Drukarev, Zh. Eksp. Teor. Fiz., 19, 247 (1949).
[88]V. V. Babikov, Method of Phase Functions in Quantum Mechanics [in Russian], Nauka, Moscow (1968).
[89]I. M. Gel'fand and B. M. Levitan, a) Izv. Akad. Nauk SSSR, 15, 309 (1951); b) Dokl. Akad. Nauk SSSR, 77, 557 (1951).
[90]M. G. Krein, Dokl. Akad. Nauk SSSR, 94, 987 (1953); 97, 21 (1954); 105, 433 (1955); 113, 970 (1957).
[91]V. A. Marchenko, a) Dokl. Akad. Nauk SSSR, 72, 457 (1950); b) Tr. Moskovsk. Matematichesk. Obshch., 1, 327 (1952); c) Dokl. Akad. Nauk SSSR, 104, 695 (1955); d) Mat. Sb., 77(119), 139 (1968).
[92]Z. S. Agranovich and V. A. Marchenko, The Inverse Problem of Scattering Theory [in Russian], Izd. Khar'k. Univ., Khar'kov (1960).
[93]L. D. Faddeev, Usp. Matem. Nauk, 14, No. 4(88), 57 (1959).
[94]V. de Alfaro and T. Regge, Potential Scattering, Amsterdam (1965).
[95]Ta-You Wu and T. Ohmura, Quantum Theory of Scattering, Prentice-Hall, London (1962).
[96]R. G. Newton, Scattering Theory of Waves and Particles, McGraw-Hill, London (1966).
[97]F. Calogero, Variable Phase Approach to Potential Scattering, New York (1967).
[98]V. K. Mel'nikov, a) Preprint JINR R-265 [in Russian], Dubna (1958); b) Usp. Matem. Nauk, 14, No. 4(88), 121 (1959).
[99]D. Sh. Lundina and V. A. Marchenko, Matem. Sb., 78 (120), 475 (1969).
[100]V. É. Lyantse, Matem. Sb., 72, 537 (1967).
[101]N. Limic, Nuovo Cimento, 26, 581 (1962); 28, 1066 (1963).

[102] F. S. Rofe-Beketov and E. Kh. Khristov, Dokl. Akad. Nauk SSSR, 168, 1265 (1966); 185, 748 (1969); b) Mathematical Physics and Functional Analysis, No. 2 [in Russian] (1971), p. 122.
[103] J. Benn and G. Scharf, Nucl. Phys., 134, 481 (1969).
[104] L. A. Chudov, Preprint JINR R-175 [in Russian], Dubna (1958).
[105] a) I. Bystricky', F. Lehar, and I. Ulehla, Preprint JINR E-2440 (1965); b) P. Swan and W. A. Pearce, Nucl. Phys., 79, 77 (1966).
[106] a) Ya. Vizner, E. P. Zhidkov, and V. Lelek, Preprint JINR R5-3895 [in Russian], Dubna (1968); b) E. P. Zhidkov et al., Preprint JINR R1-5306 [in Russian], Dubna (1970).
[107] A. N. Tikhonov, Dokl. Akad. Nauk SSSR, 151, 501 (1963); 153, 49(1963); 156, 1296 (1964); 163, 591 (1965).
[108] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Preprint UCRL-70075 (Part 10), 39 (1968).
[109] B. B. Kadomstev, Plasma Turbulence, Academic Press, London (1965).
[110] S. B. Pikel'ner and V. N. Tsytovich, Zh. Eksp. Teor. Fiz., 55, 977 (1968) [Sov. Phys.—JETP, 28, 507 (1969)].
[111] V. M. Zakharov, Zh. Eksp. Teor. Fiz., 51, 688 (1966) [Sov. Phys.—JETP, 24, 455 (1967)].
[112] E. P. Zhidkov et al., a) Preprint JINR R9-4464 [in Russian], Dubna (1969); b) Plasma Phys., 12, 191 (1970).
[113] E. P. Zhidkov (Jidcov) et al., Phenomena in Ionized Gases (1969), Contributed Papers, Bucharest, Romania, September 1-6, 1969, p. 525.