# Relativistic invariance in quantum theory. II

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A manifestly covariant formalism is developed for equal-time commutators of general form with an arbitrary finite number of gradient terms. In this formalism, a construction of relativistically invariant systems of equal-time commutators reduces to standard manipulations with tensor indices. General conditions of microcovariance reflecting the pseudo-Euclidean property of four-space at all points are obtained. It is shown that these conditions are more stringent than the usual requirement of invariance under transformations of the Poincaré group. On the basis of the conditions of microcovariance of the equal-time commutators, a set of assumptions is formulated for quantum field theory in which local commutativity holds automatically.

## 6. INFINITESIMAL LORENTZ TRANSFORMATIONS FOR EQUAL-TIME COMMUTATORS

1. A nontrivial example of the use of the general formulation of the conditions of relativistic invariance developed in Sec. 2 (see Vol. 3, No. 3, 1972 of this journal[1]) is the problem of constructing systems of equal-time commutators that are closed under transformations of the Poincaré group. Our aim is a general solution of this problem for equal-time commutators containing an arbitrary number of gradient terms [or, and this is the same thing, Schwinger terms[1]] under the additional independent assumption that all the local variables in the commutators transform in accordance with finite-dimensional representations of the Lorentz group.

2. The importance of investigating the relativistic properties of equal-time commutators is due primarily to the stormy development of current algebra (see, for example ref. 5). Relativistic invariance is obvious for the already classic algebra of integrated conserved charges:

$$[q_a, q_b] = iG^c_{ab}q_c, \tag{1}$$

where $a$, b, and c are the indices of the SU(3) components;

$$q_a = \int d^3x J^0_a(x); \tag{2}$$

$J^\alpha_a(x)$ is the conserved current,

$$\partial J^\alpha_a(x)/\partial x^\alpha = 0. \tag{3}$$

But already for nonconserved currents the question of the relativization of (1) is complicated, because the $q_a$ in (2) transform in the general case when (3) is not satisfied in accordance with infinite-dimensional representations of the Lorentz group.

The problem of relativization is further complicated on the transition to an algebra of unintegrated commutators containing gradient terms. In the general case, such a commutator can be written in the form

$$A(\mathbf{x}, x^0), B(\mathbf{y}, x^0)] = i\sum_{k=0}^{n} \frac{1}{k!} F^{i_1\cdots i_k}(\mathbf{x}, x^0) \times \frac{\partial}{\partial y^{i_1}}\cdots\frac{\partial}{\partial y^{i_k}}\delta^3(\mathbf{x}-\mathbf{y}), \tag{4}$$

where A(x) and B(y) are the components of local quantities (fields, currents, and their derivatives of finite orders) transforming in accordance with finite-dimensional representations of the Lorentz group. Since the commutator (4) does not have a manifestly covariant form, there is a danger of the relativistic invariance getting lost when this commutator is used. The determination of the transformation properties of the commutator (4) is complicated by the fact that, as we shall see in Sec. 8, the quantities $F^{i_1\cdots i_k}(x)$ transform in the general case in accordance with infinite-dimensional representations of the Lorentz group, even if the representations for A and B are finite-dimensional. To solve the problem of making (4) covariant, we must find all the ways of invariant truncation of the representations for $F^{i_1\cdots i_k}$ to finite-dimensional representations.

It should be emphasized that we are concerned with active and not passive invariance (see Sec. 2.1 in ref. 1), i.e., in actual and not formal "covariantization" of (4). Passive invariance is readily achieved by introducing a unit numerical four-vector $n^\alpha$ ($n^\alpha n_\alpha = 1$) oriented in an arbitrary but fixed manner. Using $n_\alpha$, we can write the commutator (4) in the form

$$\delta\{n_\alpha(x^\alpha - y^\alpha)\}[A(x), B(x)] = i\sum_{k=0}^{n}\frac{1}{k!}\mathscr{F}^{\alpha_1\cdots\alpha_k}(x)\frac{\partial}{\partial y^{\alpha_1}}\cdots\frac{\partial}{\partial y^{\alpha_k}}\delta^4(x-y), \tag{5}$$

where $\mathscr{F}^{\alpha_1\cdots\alpha_k} n_{\alpha 1} = 0$, etc. The form (5), which is introduced in ref. 6 and used in the subsequent investigations,[7,8] is not a step forward in the determination of the covariant properties of equal-time commutators and does not contain any more information than the original commutator (4).

3. Let us consider the methods used until recently to study the covariant properties of relations of the type (4).

The first investigations of the covariant properties of current algebras and equal-time commutators were made by means of Lagrangian theories or simply by analogy with the current commutators composed of free fields. The results obtained by such methods are not proofs but merely heuristic arguments. In this paper we shall not use such methods. We shall investigate properties of equal-time commutators whose derivation does not require us to multiply fields at one point.

In ref. 9 it is asserted without proof that the three-dimensional $\delta$ function $\delta^3(\mathbf{x}-\mathbf{y})$ behaves as the time component of a four-vector. Understood literally, this assertion is not true, and when applied to the right-hand sides of equal-time commutators allows a correct interpretation only in a single very special case.

For the commutator (4) one can readily obtain a Lorentz transformation; however, it is not finite but only infinitesimal (see refs. 6, 10 and some later papers). To obtain these transformations, we can use two equivalent methods: commutation of (4) with the operator $M^{i0}$, and differentiation of (5) with respect to $n^{\alpha}$. In the first method, the investigated physical system is rotated with respect to the hypersurface $x^0 = \text{const}$; in the second, this hypersurface is rotated with respect to the physical system.

Commuting the operator $M^{\alpha\beta}$ with (4), we obtain

$$[(\hat{L}^{\alpha\beta}+\hat{S}^{\alpha\beta})\,A(\mathbf{x},x^0),\,B(\mathbf{y},x^0)]$$
$$+[A(\mathbf{x},x^0),\,(\hat{L}^{\alpha\beta}+\hat{S}^{\alpha\beta})\,B(\mathbf{y},x^0)]$$
$$=\mathrm{i}\sum_{k=0}\frac{1}{k!}\,(\hat{L}^{\alpha\beta}+\hat{S}^{\alpha\beta})\,F^{i_1\ldots i_k}(\mathbf{x},x^0)\frac{\partial}{\partial y^{i_1}}\ldots\frac{\partial}{\partial y^{i_k}}\,\delta^3(\mathbf{x}-\mathbf{y}).\quad(6)$$

Here, by definition,

$$\hat{L}^{\alpha\beta}A(x)=\mathrm{i}\,(x^{\alpha}\partial/\partial x_{\beta}-x^{\beta}\partial/\partial x_{\alpha})\,A(x),\ \text{etc.}\qquad(7)$$

$\hat{S}^{\alpha\beta}$ is the matrix of an infinitesimal Lorentz transformation applied to the spin variables of the local quantity to its right. In deriving (6), we have used the Jacobi identity[2)] and the infinitesimal form of the condition (29) (ref. 1, p. 310):

$$[M^{\alpha\beta},A(x)]=(\hat{L}^{\alpha\beta}+\hat{S}^{\alpha\beta})\,(A(x).\qquad(8)$$

The orbital parts of (6) can be partly eliminated by using the commutator of $P^{\gamma}$ with (4) and the fact that

$$[P^{\alpha},A(x)]=(1/\mathrm{i})\,(\partial A(x)/\partial x_{\alpha}).\qquad(9)$$

As a result, for the infinitesimal Lorentz transformation, we obtain

$$[\hat{S}^{i0}A(\mathbf{x},x^0),\,B(\mathbf{y},x^0)]$$
$$+[A(\mathbf{x},x^0),\,\hat{S}^{i0}B(\mathbf{y},x^0)]-\mathrm{i}\,(x^i-y^i)\left[A(\mathbf{x},x^0),\,\frac{\partial}{\partial x^0}B(\mathbf{y},x^0)\right]$$
$$=\mathrm{i}\sum_{k=0}^{n}\frac{1}{k!}\,S^{i0}F^{i_0\ldots i_k}(\mathbf{x},x^0)\,\frac{\partial}{\partial y^{i_1}}\ldots\frac{\partial}{\partial y^{i_k}}\,\delta^3(\mathbf{x}-\mathbf{y}).\qquad(10)$$

In a slightly different but equally general form this transformation was obtained in ref. 8.

The transformation (10) enables us to obtain new commutators from the original (4), but only when the four-dimensional transformation properties of not only A(x) and B(x) but also $F^{i_1\ldots i_k}(x)$ are known. But if only the covariant properties of A(x) and B(x) are known, the relation (10) is not fully defined. The repeated commutators of (10) with $M^{j0}$ lead to new and ever more cumbersome relations, which indicates that the representation for the $F^{i_1\ldots i_k}$ is infinite-dimensional in the general case.

4. To demonstrate the inadequacy of the relation (10) for the solution of the problem posed in Sec. 6.1, let us consider the simplest example of the commutator of two four-scalars that does not contain gradient terms:

$$\left.\begin{aligned}&[A(\mathbf{x},x^0),\,B(\mathbf{y},x^0)]=\mathrm{i}F(\mathbf{x},x^0)\,\delta^3(\mathbf{x}-\mathbf{y});\\&\hat{S}^{\alpha\beta}A(x)=\hat{S}^{\alpha\beta}B(x)=0.\end{aligned}\right\}\qquad(11)$$

In this case, (10) takes the form

$$-\mathrm{i}\,(x^i-y^i)\,[A(\mathbf{x},x^0),\,\partial B(\mathbf{y},x^0)/\partial x^0]=\mathrm{i}\hat{S}^{i0}F(\mathbf{x},x^0)\,\delta(\mathbf{x}-\mathbf{y}).\qquad(12)$$

Let us consider what can be said concerning the four-dimensional tensor-dimensionality of F(x), using (12). It would seem that one can make F a four-scalar by setting the commutator in (12) equal to zero. In Sec. 8 it will be shown that such a conclusion is erroneous, and we shall show there that one of the correct solutions is to equate F(x) to the component $C^0(x)$ of the four-vector $C^{\alpha}(x)$. In this case we have

$$\hat{S}^{i0}F(x)=\hat{S}^{i0}C^0(x)=-\mathrm{i}C^i(x),$$

and it follows from (12) that

$$[A(\mathbf{x},x^0),\,\partial B(\mathbf{y},x^0)/\partial x^0]$$
$$=\mathrm{i}D(\mathbf{x},x^0)\,\delta^3(\mathbf{x}-\mathbf{y})+\mathrm{i}C^j(x)\,\partial\delta^3(\mathbf{x}-\mathbf{y})/\partial y^j,\qquad(13)$$

where D(x) is a new local operator, which, as can be shown, can be set equal to zero without giving rise to a contradiction:

$$D(x)=0.$$

Then, transforming the expression (13) in the same way as in the transition from (4) to (10), we obtain

$$[A(\mathbf{x},x^0),\,\partial B(\mathbf{y},x^0)/\partial y_i]+(x^i-y^i)\,[A(\mathbf{x},x^0),\,(\partial/\partial x^0)^2\,B(\mathbf{y},x^0)]$$
$$=-\mathrm{i}C^0(\mathbf{x},x^0)\,\partial\delta^3(\mathbf{x}-\mathbf{y})/\partial y_i.\qquad(14)$$

This expression still does not make it clear that the system of commutators can be closed under Lorentz transformations. And it is only in knowing the answer that we can understand that in the second term of the right-hand side we must make the transformation

$$(x^i-y^i)\left(\frac{\partial}{\partial x^0}\right)^2B(\mathbf{y},x^0)=(x^i-y^i)\,\Box\,B(\mathbf{y},x^0)$$
$$-g^{jl}\frac{\partial}{\partial y^j}\cdot\frac{\partial}{\partial y^l}\{(x^i-y^i)\,B(\mathbf{y},x^0)\}-2\frac{\partial}{\partial y^i}B(\mathbf{y},x^0),$$

and then use the fact that in accordance with (11)

$$(x^i-y^i)\,[A(\mathbf{x},x^0),\,B(\mathbf{y},x^0)]=0$$

and finally adopt (and we must still show that this does not lead to a contradiction)

$$(x^i-y^i)\,[A(\mathbf{x},x^0),\,\Box\,B(\mathbf{y},x^0)]=0.$$

It is only after this that we find that (14) is a consequence of (11), so that the commutators (11) and (13) form a system that is closed under Lorentz transformations.

## 7. EQUAL-TIME COMMUTATORS IN THE FORMALISM OF UNIVERSAL ALGEBRAS

1. To investigate the relativistic and other properties of equal-time commutators, it is very convenient to

employ the formalism of universal algebras.[12,13] For in this formalism it has been possible to pose and solve[14,15] the problem of making the equal-time commutators covariant.

To go over to the universal algebra, we replace the commutator (4) by an equivalent countable system of relations:

$$(1/i)\int d^3y\,(x^{i_1}-y^{i_1})\ldots(x^{i_k}-y^{i_k})\,[A(\mathbf{x},x^0),B(\mathbf{y},x^0)] = F^{i_1\cdots i_k}(\mathbf{x},x^0), \qquad (15)$$

where k = 0, 1, 2, .... Obviously, the right-hand sides of (15) vanish when k > n. Note that multiplication of (4) by a polynomial in $\mathbf{x}-\mathbf{y}$ is a mathematically allowed operation, since the support of the right-hand side (and hence of the left-hand side) of (4) with respect to this variable is concentrated at a point.

Let us now attempt to simplify the relations (15) without reducing the information they contain. The meaning of each of these relations is that the two local quantities A(x) and B(x) are associated with a third, $F^{i_1\cdots i_k}(x)$, by means of a standard operation, which consists of taking the equal-time commutator and multiplying by $i^{-1}$ and by a polynomial of degree k in $\mathbf{x}-\mathbf{y}$ with subsequent integration with respect to $d^3y$. We denote this operation by $\mu^{i_1\cdots i_k}$.

We now note that the operation $\mu^{i_1\cdots i_k}$ in (15) is local: It is sufficient to assume that all the quantities A(x), B(x), and $F^{\cdots}(x)$ are specified in the neighborhood of the same four-point x. In other words, if the equal-time commutators exist and have the form (4), each of the operations $\mu^{i_1\cdots r_k}$ can be treated as some well-defined nonassociative multiplication of fields at a single four-point. Therefore, without detriment to clarity, one can omit the arguments of the local quantities, i.e., one can write A instead of A(x), etc. Finally, we agree to write the operations of the nonassociative multiplications $\mu\cdots$ to the right of the multiplicands, as is the practice in universal algebras. As a result, the relations (15) take the compact form[14]

$$AB\mu^{i_1\cdots i_k}=F^{i_1\cdots i_k}. \qquad (16)$$

Any universal algebra consists of a set of elements and a set of operations over these elements. The operations are split into binary, unary, and others, which we do not require. In a binary operation every ordered pair of elements is associated with a third. In our algebra the binary operations consist of ordinary addition and all nonassociative multiplications $\mu^{i_1\cdots i_k}$. By definition, a unary operation transforms each element of the algebra into another element. In the given algebra the unary operations are multiplication of elements by complex numbers and differentiation, which we shall denote by $\partial_\alpha$ and also write on the right. Thus, to the derivative $\partial A(x)/\partial x^\alpha$ there corresponds the element $A\partial_\alpha$ in the universal algebra. A formalized and more detailed exposition of the transition to a universal algebra can be found in ref. 14.

2. The convenience of the notation (16) is not only its compactness, but also that with its use the commutator (15) can be split into the composite entities A, B, $\mu\cdots$, and $F\cdots$, which can be studied separately. Thus, in Sec. 8, we shall show that the problem of making the equal-time commutators covariant reduces to decomposing the operations $\mu\cdots$ with respect to the components of irreducible four-tensors.

Besides the binary operations $\mu\cdots$, it is frequently convenient to use quantities of the type $B\mu\cdots$, which are unary operations. The action of such unary operations (of course, on the right) on elements of the algebra is defined by the relations (4). Translated into ordinary language, the unary operation $B\mu^i$ should be written in the form

$$(1/i)\int d^3y\,(x^i-y^i)\,[\ldots(\mathbf{x},x^0),B(\mathbf{y},x^0)]. \qquad (17)$$

Instead of (17) one usually introduces the integral operators $\int d^3yB(\mathbf{y},x^0)$ and $\int d^3yy^iB(\mathbf{y},x^0)$ and studies their commutators with local quantities. However, such integrals may not exist in certain cases (see, for example, refs. 16 and 17), whereas the existence of (17) has a meaning. By the same token, the form of expression $B\mu\cdots$ indicates that one must first take the equal-time commutator and then multiply it by a polynomial and integrate over the three-volume. Note that the product of an arbitrary number of unary operations is associative and is a unary operation. Products of a unary operation and a binary operation, for example, $\partial^0\mu\cdots$ and $\mu\cdots\partial^0$, are binary operations.

3. To have the possibility of working with the relations (16) without recourse to the pre-images (15), we must express the properties of equal-time commutators that are not reflected in (16) in terms of the universal algebra. There are three such properties.

First, any commutator is antisymmetric:

$$[A(x),B(y)]=-[B(y),A(x)]. \qquad (18)$$

For the commutator (4) this property is expressed in the universal algebra by the system of equations[14]

$$AB\mu^{i_1\cdots i_k}=(-1)^{k+1}BA\left(\mu^{i_1\cdots i_k}+\frac{1}{1!}\mu^{i_1\cdots i_k i_{k+1}}\partial_{i_{k+1}}\cdots+\frac{1}{(n-k)!}\mu^{i_1\cdots i_n}\partial_{i_{k+1}}\ldots\partial_{i_n}\right)\quad\text{for } k\leqslant n \qquad (19)$$

and

$$AB\mu^{i_1\cdots i_k}=BA\mu^{i_1\cdots i_k}=0\quad\text{for } k>n.$$

Secondly, the commutators satisfy the Jacobi identity:

$$[[C(z),A(x)],B(y)]-[[C(z),B(y)],A(x)]=[C(z),[A(x),B(y)]]. \qquad (20)$$

From (20) we obtain a system of relations for the commutators of the unary operations:[14]

$$A\mu^{i_1\cdots i_k}B\mu-B\mu A\mu^{i_1\cdots i_k}=AB\mu\mu^{i_1\cdots i_k}; \qquad (21)$$

$$A\mu^{i_1\cdots i_k}B\mu^j-B\mu^j\mu^{i_1\cdots i_k}=AB\mu^j\mu^{i_1\cdots i_k}+AB\mu\mu^{i_1\cdots i_k j}. \qquad (22)$$

The third property of the operation $\mu\cdots$ is obtained by differentiating the commutator (4) with respect to the three-dimensional variable $y^i$ and can be expressed in the form of a relation for the binary operations:[14]

$$\partial^i\mu = 0,\ \partial^i\mu^{i_1\cdots i_k} = g^{ii_1}\mu^{i_2\cdots i_k} + g^{ii_2}\mu^{i_1 i_3\cdots i_k} + \ldots + g^{ii_k}\mu^{i_1\cdots i_{k-1}}. \quad (23)$$

Finally, for completeness we note that under Hermitian conjugation the expression (16) becomes

$$A^+B^+\mu^{i_1\cdots i_k} = (F^+)^{i_1\cdots i_k}. \quad (24)$$

With allowance for (19), (21)-(24) the system of equal-time commutators can be studied in the form (16) without recourse to the original commutators (4). And although in the following section we shall once more require the commutators (4) to derive the conditions of relativistic invariance, in Sec. 10 we shall show that these conditions can be formulated naturally in the framework of the formalism developed here.

Thus, we have defined the action of the abstract generators on the elements of the universal algebra and on the operations $\partial_\alpha$, and this action is consistent with the group structure and the action of the generators on the state vectors in the original field theory.

2. A new and unusual property of our universal algebra is that in it the Poincaré group acts nontrivially on not only the elements, but also on the operations $\mu\cdots$ of the nonassociative multiplications.

To define a consistent action of the Lie algebra of the Poincaré group on the operations $\mu\cdots$, we must commute the operators $P^\gamma$ and $M^{\alpha\beta}$ with (15) and then translate the resulting relations into the language of universal algebra. The commutator of the operator $P^\alpha$ with (15) gives, after we have used the Jacobi identity and gone over to the universal algebra,[14]

$$A\partial^\alpha B\mu^{\cdots} + AB\partial^\alpha\mu^{\cdots} = AB\mu^{\cdots}\partial^\alpha, \quad (30)$$

i.e., the ordinary rule for differentiating a product.

Let us consider which conclusions can be drawn from (30) concerning the action of the generator $P^\alpha$ on the operations $\mu\cdots$. Under an infinitesimally small transformation of the Poincaré group, the relation (16) goes over into $(A + \delta A)(B + \delta B)(\mu\cdots + \delta\mu\cdots) = F\cdots + \delta F\cdots$. The infinitesimal condition of relativistic invariance is that the original equation is unaffected to first order:

$$(\delta A)B\mu^{\cdots} + A(\delta B)\mu^{\cdots} + AB(\delta\mu^{\cdots}) = \delta F^{\cdots}. \quad (31)$$

It can be seen by comparing (31) with (30) that $\delta\mu\cdots = 0$ for four-translations, so that all the binary operations $\mu\cdots$ are translationally invariant. The commutators of the operators $M^{i0}$ and $M^{ij}$ with (15) after the use of (30) and the Jacobi identity lead in the universal algebra to the relations[14,15]

$$AS^{i0}B\mu^{i_1\cdots i_k} + ABS^{i0}\mu^{i_1\cdots i_k} - iAB\partial^0\mu^{i_1\cdots i_k i} = AB\mu^{i_1\cdots i_k}S^{i0}; \quad (32)$$

$$AS^{ij}B\mu^{i_1\cdots i_k} + ABS^{ij}\mu^{i_1\cdots i_k} + iAB(\partial^i\mu^{i_1\cdots i_k j} - \partial^j\mu^{i_1\cdots i_k i}) = AB\mu^{i_1\cdots i_k}S^{ij}. \quad (33)$$

The differentiations on the right-hand side of (33) have been retained for brevity. They can be readily eliminated by means of (23).

It is obvious that the set of relations (32) for different k is none other than the infinitesimal Lorentz transformation (10) rewritten in terms of the universal algebra. But, using the form (32), we can go further. Namely, comparing (32) with (31), we see that if we construct a linear space on the $\mu\cdots$, regarded as basis vectors, the action of the generator $\hat{S}^{i0}$ from (25) on $\mu\cdots$ can be realized by an operator $\hat{K}^{i0}$ in this space:

$$\hat{K}^{i0}\mu^{i_1\cdots i_k} = i\partial^0\mu^{i_1\cdots i_k}. \quad (34)$$

This clearly brings out the advantage of the formalism of universal algebra. The compact equation (34) contains all the information given by the cumbersome relation (10).

Similarly, comparing (33) with (31), we see that the action of the generator $\hat{S}^{ij}$ on $\mu\cdots$ is realized by an operator $\hat{K}^{ij}$ that acts in the linear space spanned by the operations $\mu\cdots$:

$$\hat{K}^{ij}\mu^{i_1\cdots i_k} = i(\partial^j\mu^{i_1\cdots i_k i} - \partial^i\mu^{i_1\cdots i_k j}) = i\{(g^{ji_1}\mu^{ii_2\cdots i_k} - g^{ii_1}\mu^{ji_2\cdots i_k}) + \ldots + (g^{ji_k}\mu^{i_1\cdots i_{k-1}i} - g^{ii_k}\mu^{i_1\cdots i_{k-1}j})\}. \quad (35)$$

## 8. REALIZATION OF THE CONDITIONS OF RELATIVISTIC INVARIANCE OF EQUAL-TIME COMMUTATORS IN THE FRAMEWORK OF AN OUTER AUTOMORPHISM

1. In the language of universal algebra, the problem of making a system of equal-time commutators covariant reduces to finding and investigating the transformation properties of relations of the type (16). We shall initially restrict ourselves to infinitesimal conditions of relativistic invariance. In accordance with what we have said in Sec. 2, we must define an action consistent with (19)-(23) and with (91) (ref. 1, p. 321) of the abstract generators[3)] $M^{\alpha\beta}$ and $P^\gamma$ on elements A, B, ..., on the unary operations $\partial^\alpha$, and on the binary operations $\mu\cdots$.

In the original quantum field theory, the abstract generators of the Poincaré group act on vector states and on the operators of local quantities. When they act on vector states, these generators are realized by the operators $M^{\alpha\beta}$ and $P^\gamma$. The action of the abstract generators on the operator of a local quantity A(x) is realized by the linear operators $\hat{L}^{\alpha\beta} + \hat{S}^{\alpha\beta}$, $i(\partial/\partial x_\gamma)$, specified in a linear space formed by all components of A(x) at all four-points $x^\alpha$. From this point of view, (8) and (9) are consistency conditions for the realizations of the action of the abstract generators on state vectors and on local quantities.

It is now not difficult to define the action of the abstract generators $M^{\alpha\beta}$ and $P^\gamma$ on the elements A and B of the universal algebra.

The generator $P^\gamma$ is associated with the operation $i\partial^\gamma$, and the generator

$$\tilde{S}^{\alpha\beta} = -x^\alpha P^\beta + x^\beta P^\alpha + M^{\alpha\beta} \quad (25)$$

with a new unary operation $S^{\alpha\beta}$, which, in accordance with (8), realizes an infinitesimal Lorentz transformation of the elements of the algebra with respect to the tensor indices. Thus, for the four-scalar A, the four-vector $B^{\gamma}$, and the four-tensor $T^{\gamma\varepsilon}$, we have, respectively,

$$\left.\begin{aligned} &AS^{\alpha\beta}=0;\quad B^{\gamma}S^{\alpha\beta}=i\,(B^{\beta}g^{\alpha\gamma}-B^{\alpha}g^{\beta\gamma});\\ &T^{\gamma\varepsilon}S^{\alpha\beta}=i\,(T^{\beta\varepsilon}g^{\alpha\gamma}-T^{\alpha\varepsilon}g^{\beta\gamma}+T^{\gamma\beta}g^{\alpha\varepsilon}-T^{\gamma\alpha}g^{\beta\varepsilon}).\end{aligned}\right\}\qquad(26)$$

Note that the unary operation $S^{\alpha\beta}$ also acts on the tensor indices that arise when the original field is differentiated. Thus, for the gradient of the four-scalar $A\partial^{\gamma}$ we have

$$A\partial^{\gamma}S^{\alpha\beta}=iA\,(\partial^{\beta}g^{\alpha\gamma}-\partial^{\alpha}g^{\beta\gamma}).$$

The more general expression is

$$[S^{\alpha\beta}\partial^{\gamma}]=i\,(\partial^{\alpha}g^{\gamma\beta}-\partial^{\beta}g^{\alpha\gamma}).\qquad(27)$$

In addition, by the definition of the operation $\partial^{\alpha}$, we have

$$[\partial^{\alpha},\,\partial^{\beta}]=0,\qquad(28)$$

and in accordance with (25) and (26),

$$[S^{\alpha\beta},\,S^{\gamma\varepsilon}]=i\,(g^{\alpha\varepsilon}S^{\beta\gamma}+g^{\alpha\gamma}S^{\varepsilon\beta}-g^{\beta\varepsilon}S^{\gamma\alpha}-g^{\gamma\beta}S^{\alpha\varepsilon}),\qquad(29)$$

so that the unary operations $S^{\alpha\beta}$ and $i\partial^{\gamma}$ satisfy, as one would expect, the commutation relations that characterize the Lie algebra of the Poincaré group.

3. It follows from (34) that the linear space generated by the entities $\mu\cdots$ is noninvariant, i.e., it is not closed under Lorentz transformations. But this space can, of course, be extended to an invariant space. The corresponding extended Lorentz-invariant space will be denoted henceforth by L; it is the linear space with basis system[4)] of vectors:[15] $\Box^{m}\mu^{i_1\cdots i_k}$, $\Box^{l}\partial^{0}\mu^{i_1\cdots i_q}$, where $\Box\equiv\partial^{\alpha}\partial_{\alpha}$ and m, k, $l$, and q range independently over all nonnegative integers. It is not difficult to find the action of the linear operator $\hat{K}^{\alpha\beta}$ on these basis vectors. The unary operation $\Box$ is invariant, so that it can be regarded as a factor that commutes with $\hat{K}^{\alpha\beta}$. It is therefore sufficient to consider the action of $\hat{K}^{\alpha\beta}$ on $\mu^{i_1\cdots i_k}$ and on $\partial^{0}\mu^{i_1\cdots i_k}$. For the first this is given by (34) and (35). To obtain the action of $\hat{K}^{i0}$ on $\partial^{0}\mu\cdots$, it is necessary to replace B in (32) by $B\partial^{0}$ and use (27). It is then necessary to make the transformation

$$(\partial^{0})^{2}=\partial_{0}\partial^{0}=\Box-\partial^{i}\partial_{i},$$

and then "absorb" all the remaining spatial differentiations $\partial^{i}$ by operations $\mu\cdots$ by means of (23). The end result is[15]

$$\begin{aligned}\hat{K}^{i0}\partial^{0}\mu^{i_1\cdots i_k}&=i\,\Box\,\mu^{i_1\cdots i_k i}+i\partial^{i}\mu^{i_1\cdots i_k}-i\partial^{j}\partial_{j}\mu^{i_1\cdots i_k i}\\&=i\,\Box\,\mu^{i_1\cdots i_k i}-i\,(g^{ii_1}\mu^{i_2\cdots i_k}+\ldots+g^{ii_k}\mu^{i_1\cdots i_{k-1}})\\&\quad-2i\,(g^{i_1i_2}\mu^{i_3\cdots i_k i}+\ldots+g^{i_ki_{k-1}}\mu^{i_1\cdots i_{k-2}i}).\end{aligned}\qquad(36)$$

In particular,

$$\hat{K}^{i0}\partial^{0}\mu=i\,\Box\,\mu^{i}.\qquad(37)$$

The operator $\hat{K}^{ij}$ acts on $\partial^{0}\mu\cdots$ in accordance with an equation analogous to (35) since the operation $\partial^{0}$ is invariant under three-rotations.

By direct verification one can show that the linear operator $\hat{K}^{\alpha\beta}$ defined in accordance with (34)-(36) satisfies the commutation relations for the generators of the Lorentz group. By the same token the vectors of the infinite-dimensional space transform in accordance with some representation of the Lorentz group. The problem posed in Secs. 6.1 and 6.2 of making the commutator (4) covariant now reduces to decomposing the linear representation of the Lorentz group that is found into irreducible representations.

Thus, the transition to the universal algebra has made it possible to reduce the problem of the transformation properties of the equal-time commutator to the corresponding properties of one of the component parts of this commutator: the binary operation $\mu\cdots$.

4. It can be seen from (35)-(37) that the representation of the Lorentz group realized in the space L is infinite-dimensional. From the mathematical point of view, the decomposition of the representation in the space L into irreducible representations leads to certain difficulties of a fundamental nature, which have been pointed out in Sec. 5.7, since such an expansion for infinite-dimensional nonunitary representations is highly ambiguous.

However, the situation is greatly simplified if it is assumed that in (4) not only the quantities A(x) and B(x), but also all the $F^{i_1\cdots i_k}(x)$ transform in accordance with finite-dimensional representations of the Lorentz group. It should be emphasized that this assumption is independent of the assumption that there is a finite number of terms on the right-hand side of (4). In the language of universal algebra the new assumption means that for every given product $AB\mu\cdots$ the representation for $\mu\cdots$ must be invariantly truncated to a finite-dimensional representation. A consistent mathematical treatment with the derivation of all possible truncations and listing of the remaining finite-dimensional representations can be found in ref. 15. Here we shall content ourselves with a exposition that is mathematically lax, but physically clearer.

5. We consider a different-time commutator of general form, expressed as

$$[A\,(x),\,B\,(x-z)]=iF\,(x,\,z).\qquad(38)$$

If equal-time commutators exist for A and B and all derivatives with respect to the time, they can all be obtained by expanding (38) in a Taylor series in $z^{0}$. We shall make this expansion in the formalism of universal algebra. To go over from (38) to the universal algebra, we must throughout omit the letter x and introduce the binary operation $\Lambda(z)$, which realizes multiplication by $-i$, commutation, and a shift of the argument of B(x) by $-z$:

$$AB\Lambda\,(z)=F\,(z).\qquad(39)$$

It follows from (38) that

$$\partial\Lambda\,(z)/\partial z^{\alpha}=-\partial_{\alpha}\Lambda\,(z),\qquad(40)$$

so that the expansion of $\Lambda$ in a Taylor series in $z^0$ has the form

$$\Lambda(z)=\sum_{n=0}^{\infty}\frac{(-z^0\partial_0)^n}{n!}\widetilde{\mu}(z^i). \qquad (41)$$

Here, for each given product, $AB\widetilde{\mu}(z^i)$ is a polynomial in the three-dimensional $\delta$ function and its derivative. This means that the corresponding Fourier transform

$$\mu(p_i)=\int d^3z\exp(ip_iz^i)\widetilde{\mu}(z)/(2\pi)^3 \qquad (42)$$

for each product $AB\mu(p)$ is a polynomial whose coefficients are equal to $\mu^{i_1\cdots i_k}$:

$$\mu(p)=\sum_k\frac{i^k}{k!}p_{i_1}\cdots p_{i_k}\mu^{i_1\cdots i_k}. \qquad (43)$$

Thus, the operations $(\partial_0)^n\mu^{i_1\cdots i_k}$ for all possible nonnegative n and k, which also form a basis of the space L, comprise the set of coefficients of the Taylor series for the "partial" Fourier transform $\widetilde{\lambda}(z^0, p_i)$ of the operation $\Lambda(z)$:

$$\widetilde{\lambda}(z^0, p_i)=\int d^3p\exp(ip_iz^i)\Lambda(z)/(2\pi)^3. \qquad (44)$$

Note that (23) follows from (40) for the derivatives with respect to the spatial components. This means that the space L itself can be realized by the set of functions $\Lambda(z)$ that allow the expansion (44). The generator $\hat{K}^{\alpha\beta}$ in this realization has the usual form

$$\hat{K}^{\alpha\beta}=i(z^\alpha\partial/\partial z_\beta-z^\beta\partial/\partial z_\alpha). \qquad (45)$$

The representation of the Lorentz group described by the generator (45) decomposes for the desired class of functions into irreducible representations by means of equation (131) of ref. 1 (see p. 328) (see also ref. 15):

$$\lambda(p)=[(2\pi)^3/i]\sum_{k=0}^{\infty}\frac{i^k}{k!}p_{\alpha_1}\cdots p_{\alpha_k}\int_0^{\infty}ds\rho^{\alpha_1\cdots\alpha_k}(s)D_s(p_\alpha), \qquad (46)$$

where $D_s(p_\alpha)$ is the Pauli function

$$D_s(p_\alpha)=[i/(2\pi)^3]\delta(p^\alpha p_\alpha-s)\operatorname{Sgn}p_0. \qquad (47)$$

$\lambda(p)$ is the Fourier transform of $\Lambda(z)$:

$$\lambda(p)=[1/(2\pi)^4]\int d^4z\exp(ip_\alpha z^\alpha)\Lambda(z). \qquad (48)$$

The operations $\rho^{\cdots}(s)$ are symmetric with respect to all indices and have vanishing contractions with respect to any pair of indices:

$$g_{\alpha_1\alpha_2}\rho^{\alpha_1\alpha_2\cdots\alpha_k}(s)=\ldots=0. \qquad (49)$$

For the existence of the expansions (41) and (43) there are moments $\eta^{\cdots}_{(n)}$ of all powers in s for $\rho^{\cdots}(s)$:

$$\eta^{\alpha_1\cdots\alpha_k}_{(n)}=i\int_0^{\infty}dss^n\rho^{\alpha_1\cdots\alpha_k}(s). \qquad (50)$$

The operations $\eta^{\alpha_1\cdots\alpha_k}_{(n)}$ are also symmetric with respect to all indices and have vanishing contractions with respect to any pair of indices. Thus, each of these operations transforms in accordance with a $(k+1)^2$-dimensional irreducible representation of the Lorentz group. The question of the completeness of the decomposition of the representation in the space L into irreducible representations $\eta^{\alpha_1\cdots\alpha_k}_{(n)}$ is considered in ref. 15.

Multiplication of the binary operations $\eta^{\cdots}_{(n)}$ from the right by the unary operation $\partial^\alpha$ can be obtained from (40). If k = 0,

$$\partial^\alpha\eta_{(n)}=-\eta^\alpha_{(n+1)}/4, \qquad (51)$$

and if k > 0,

$$\partial^\alpha\eta^{\alpha_1\cdots\alpha_k}_{(n)}=(g^{\alpha\alpha_1}\eta^{\alpha_2\cdots\alpha_k}_{(n)}+\cdots+g^{\alpha\alpha_k}\eta^{\alpha_1\cdots\alpha_{k-1}}_{(n)}-\frac{1}{k}(g^{\alpha_1\alpha_2}\eta^{\alpha\alpha_3\cdots\alpha_k}_{(n)}+\cdots+g^{\alpha_{k-1}\alpha_k}\eta^{\alpha\alpha_1\cdots\alpha_{k-2}}_{(n)})-\frac{1}{2(k+2)}\eta^{\alpha\alpha_1\cdots\alpha_k}_{(n+1)}. \qquad (52)$$

It follows from (52), in particular, that

$$\partial_{\alpha_k}\eta^{\alpha_1\cdots\alpha_k}_{(n)}=\frac{(k+1)^2}{k}\eta^{\alpha_1\cdots\alpha_{k-1}}_{(n)}; \qquad (53)$$

$$\Box\eta^{\alpha_1\cdots\alpha_k}_{(n)}=-\eta^{\alpha_1\cdots\alpha_k}_{(n+1)}. \qquad (54)$$

6. For the binary operation $\Lambda(z)$ of different-time commutation there are two decompositions: with respect to the operations $(\partial_0)^n\mu^{i_1\cdots i_k}$ of equal-time commutation, and with respect to the relativistically covariant operations $\eta^{\alpha_1\cdots\alpha_k}_{(n)}$. Obviously, to solve the problem of making the equal-time commutators covariant we must decompose the operations $\Box^m\mu^{i_1\cdots i_k}$ and $\Box^k\partial^0\mu^{i_1\cdots i_q}$ with respect to $\eta^{\cdots}_{(n)}$. And, in accordance with (54), it is sufficient to find these decompositions for $\mu^{\cdots}$ and for $\partial^0\mu^{\cdots}$. To obtain the latter, it is necessary to take the moments of zeroth and first order of (46) with respect to the variable $p_0$, make the expansion (43) on the left-hand side, and then equate the coefficients of equal powers of $p_i$. Referring the reader to ref. 15 for the details of these simple but rather tedious calculations, we give the practically important expansions for the operations of the lowest three-tensor dimensionalities:

$$\mu=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\eta^{\overbrace{0\cdots0}^{2k+1}}_{(k)}=\eta^0_{(0)}-\frac{1}{3!}\eta^{000}_{(1)}+\frac{1}{5!}\eta^{00000}_{(2)}-\ldots; \qquad (55)$$

$$\mu^i=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\eta^{\overbrace{0\cdots0i}^{2k+1}}_{(k)}=\eta^{0i}_{(0)}-\frac{1}{3!}\eta^{000i}_{(1)}+\frac{1}{5!}\eta^{00000i}_{(2)}-\ldots; \qquad (56)$$

$$\mu^{ij}=\eta^{0ij}_{(0)}-\frac{1}{3}g^{ij}\eta^{000}_{(0)}-\frac{1}{6}\eta^{000ij}_{(1)}+\frac{1}{30}g^{ij}\eta^{00000}_{(1)}+\ldots; \qquad (57)$$

$$\mu^{ijl}=\eta^{0ijl}_{(0)}-\frac{1}{3}(g^{ij}\eta^{000l}_{(0)}+\text{sym})-\frac{1}{6}\eta^{000ijl}_{(1)}+\frac{1}{30}(g^{ij}\eta^{00000l}_{(1)}+\text{sym})+\ldots; \qquad (58)$$

$$\mu^{i_1i_2i_3i_4}=\eta^{0i_1i_2i_3i_4}_{(0)}-\frac{1}{3}(g^{i_1i_2}\eta^{000i_3i_4}_{(0)}+\text{sym})+$$

$$+\frac{1}{15}(g^{i_1i_2}g^{i_3i_4}+\text{sym})\,\eta_{(0)}^{00000}+\ldots; \quad (59)$$

$$\ldots\ldots\ldots$$

$$\partial^0\mu=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\eta_{(k)}^{\overbrace{0\ldots0}^{2k}}=\eta_{(0)}-\frac{1}{2!}\eta_{(1)}^{00}+\frac{1}{4}\eta_{(2)}^{0000}-\ldots; \quad (60)$$

$$\partial^0\mu^i=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\eta_{(k)}^{\overbrace{0\ldots0i}^{2k}}=\eta_{(0)}^i-\frac{1}{2!}\eta_{(1)}^{00i}+\frac{1}{4!}\eta_{(2)}^{0000i}-\ldots; \quad (61)$$

$$\partial^0\mu^{ij}=\eta_{(0)}^{ij}-g^{ij}\eta_{(0)}^{00}-\frac{1}{2}\eta_{(1)}^{00ij}+\frac{1}{6}g^{ij}\eta_{(1)}^{0000}+\ldots; \quad (62)$$

$$\partial^0\mu^{ijl}=\eta_{(0)}^{ijl}-(g^{ij}\eta_{(0)}^{00l}+\text{sym})-\frac{1}{2}\eta_{(1)}^{00ijl}$$

$$+\frac{1}{6}(g^{ij}\eta_{(1)}^{0000l}+\text{sym})+\ldots; \quad (63)$$

$$\partial^0\mu^{i_1i_2i_3i_4}=\eta_{(0)}^{i_1i_2i_3i_4}-(g^{i_1i_2}\eta_{(0)}^{00i_3i_4}+\text{sym})$$

$$+\frac{1}{3}(g^{i_1i_2}g^{i_3i_4}+\text{sym})\,\eta_{(0)}^{0000}+\ldots; \quad (64)$$

$$\ldots\ldots$$

The symbols «+sym» in the braces stand for the addition of the terms needed to symmetrize the expressions in the braces with respect to all three-dimensional indices with allownace for the fact that the tensors $g^{i_1i_2}$ and $\eta_{(n)}^{i_1\cdots i_k}$ themselves have this symmetry. For example,

$$(g^{i_3i_4}\eta_{(0)}^{00i_1i_2}+\text{sym})=g^{i_3i_4}\eta^{00i_1i_2}+g^{i_3i_1}\eta_{(0)}^{00i_2i_4}$$
$$+g^{i_3i_2}\eta_{(0)}^{00i_1i_4}+g^{i_1i_4}\eta_{(0)}^{00i_2i_3}+g^{i_2i_4}\eta_{(0)}^{00i_1i_3}+g^{i_1i_2}\eta_{(0)}^{00i_3i_4}.$$

All the series (55)-(64) are only formally infinite, because in accordance with the assumption that we have made that the representations for A, B, and $F^{\cdots}$ in (4) are finite-dimensional, the representation for $\mu^{\cdots}$ in (16) must also be finite-dimensional. Therefore, when the corresponding expansion of $\mu^{\cdots}$ with respect to $\mu_{(n)}^{\cdots}$ is substituted into (16) only a finite number of terms remain on the right-hand side in all cases.

7. Equations (55)-(64) completely solve the problem of finding all possible ways of making the relations (16) covariant, i.e., ultimately all possible ways of augmenting the commutator (4) to a system of equal-time commutators that are invariant under Lorentz transformations.

The method we have found is very simple and reduces to this: First, one must go over from the original commutator (4) to the system of equations (16); secondly, in (16) the operations $\mu^{i_1\cdots i_k}$ must be replaced by one or several (in principle, any) terms from the corresponding expansions (55)-(64); thirdly, one must form four-tensor equations with a minimal number of components that go over into the equations just found for corresponding values of the indices. The resulting four-tensor relations are the desired "covariantizations", in which one can return to the operations $\mu^{\cdots}$, i.e., to equal-time commutators.

Let us illustrate this method by the example of the commutator (11), whose covariant properties we attempted to study in Sec. 6 by means of the ordinary formalism. In the universal algebra, (11) goes over into the system

$$AB\mu=F;\ AB\mu^{i_1\cdots i_k}=0\quad\text{for}\quad k>0. \quad (65)$$

Then, in accordance with the procedure we have just described we must replace $\mu$ in the first of Eqs. (65) by one or several of the terms from the right-hand side of (55). We shall consider two possible substitutions of this kind in order to demonstrate the ambiguity of the procedure for covariantization. The simplest one is to leave only the first term on the right-hand side of (55):

$$AB\eta_{(0)}^0=F. \quad (66)$$

In (66), the left-hand side is the zeroth component of a four-vector. Hence, F on the right must also be the component $C^0$ of some four-vector $C^\alpha$: $F = C^0$. The four-tensor equation corresponding to (66) obviously has the form

$$AB\eta_{(0)}^{\alpha}=C^{\alpha}. \quad (67)$$

The spatial components (67) yield a new relation, which, using (61), can again be expressed in terms of the $\mu$ operation:

$$AB\partial^0\mu^i=C^i. \quad (68)$$

The pair of relations (66) and (68) comprise a system that is closed with respect to Lorentz transformations if it is equivalent to (67), i.e., if all but the first terms in the expansions (55) and (61) vanish:

$$AB\eta_{(1)}^{\alpha\beta\gamma}=0;\quad AB\eta_{(2)}^{\alpha_1\cdots\alpha_s}=0\ldots \quad (69)$$

From (69) we conclude that it is valid to ignore the series of terms that arise under the successive infinitesimal Lorentz transformations in Sec. 6.5. It can be immediately seen from (55) that F(x) in (11) cannot be a four-scalar, since there is no scalar $\eta$ operation on the right-hand side of (55).

Instead of (68) we can write down the equal-time commutator

$$[A(\mathbf{x},x^0),\,\partial B(\mathbf{y},x^0)/\partial x^0]=iC^i(x)\frac{\partial}{\partial y^i}\delta^3(\mathbf{x}-\mathbf{y}). \quad (70)$$

However, it is more accurate to write down the corresponding relation (15) for k = 1, since, in accordance with (60), the nongradient term $iD(x)\,\delta^3(\mathbf{x}-\mathbf{y})$, where D(x) is a four-scalar, can be present on the right-hand side of (70).

Another possible way to make (65) covariant is to replace $\mu$ by the second term on the right-hand side of (56). In this case, writing $\Box B(x) = -J(x)$ and using (54), we can rewrite the first of these relations (65) in the form

$$AJ\eta_{(0)}^{000}/6=-F. \quad (71)$$

Therefore, F must here be assumed to be the component of a symmetric four-tensor $-6G^{\alpha\beta\gamma}$ with vanishing contractions:

$$F=-G^{000}/6. \quad (72)$$

The corresponding four-tensor relation is then

$$AJ\eta_{(0)}^{\alpha\beta\gamma}=G^{\alpha\beta\gamma}. \quad (73)$$

In this case one obtains a different system of equal-time commutators, which is also closed with respect to Lorentz transformations. One of the commutators of this system is obtained for $\alpha = 0$, $\beta = i$, and $\gamma = j$ and, in accordance with (57), is defined by $AJ\mu^{ji} = G^{0ij}$, i.e., it has the form

$$[A(\mathbf{x}, x^0),\ J(\mathbf{y}, x^0)] = \frac{i}{2} G^{0ij}(x) \frac{\partial}{\partial y^i} \cdot \frac{\partial}{\partial y^j} \delta^3(\mathbf{x}-\mathbf{y}).$$

It can be seen from this example that in the formalism of universal algebra the derivation of relativistically invariant systems of equal-time commutators reduces to the simple and familiar calculations of a four-tensor algebra. In Sec. 10 below we shall see that for equal-time commutators containing the components of conserved currents, there exist additional nontrivial relations between the $\dot{\mu}$ operations and the $\eta$ operations.

8. In this section the unary operations $S^{\alpha\beta}$ and $\partial^\gamma$ have been introduced from outside, and not expressed in terms of local quantities and $\mu$ or $\eta$ operations. This means that the action of the Poincaré group on the universal algebra is realized by an outer automorphism.

In Lagrangian field theory this automorphism is an inner automorphism. It is believed by some that the restriction to inner automorphisms is not necessary to ensure relativistic invariance of quantum theory. In Sec. 9 we shall adduce some arguments to counter this opinion and in Sec. 11 we shall show how one can make a natural transition to the inner automorphism in the covariant form of universal algebra we have just developed.

## 9. MICROCOVARIANCE AND MICROCAUSALITY

1. In the last decade there has been a marked quickening of interest in the properties of the energy–momentum tensor $T^{\alpha\beta}(x)$ in quantum field theory. We shall show here that this tendency is in no sense fortuitous, but is stimulated by the gradual understanding of the fundamental importance of this physical quantity for any (and not only a Lagrangian) relativistic quantum theory.

On the other hand, the fundamental importance of the energy–momentum tensor and the need to introduce it into theory are by no means yet generally recognized. The opinion is widespread that this tensor exists of necessity only in Lagrangian theories and, therefore, the assertion that it must exist from the point of view of axiomatic theories is by no means as well justified as the other consequences of field perturbation theory. The situation is aggravated by the fact that at the present time there are several different definitions of $T^{\alpha\beta}(x)$, and in the majority of investigations precise definitions are not given. It is not uncommon for different definitions to be used in different parts of the same paper without warning. Therefore, let us begin by listing and discussing the different definitions.

2. The main properties of the energy–momentum tensor are its conservation,

$$\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0, \tag{74}$$

and its relation to the operators of the four-momentum and four-angular momentum:

$$P^\alpha = \int T^{\alpha 0}(x)\, d^3(x); \tag{75}$$

$$M^{\alpha\beta} = \int \{x^\alpha T^{\beta 0}(x) - x^\beta T^{\alpha 0}(x)\}\, d^3(x). \tag{76}$$

In a number of papers the relations (74)-(76) are tacitly adopted as the definition of $T^{\alpha\beta}(x)$. In such an approach the existence of $T^{\alpha\beta}$ appears quite fortuitous.

In classical Lagrangian field theory there always exists a canonical energy–momentum tensor, $T^{\text{can}}_{\alpha\beta}$, which remains formally on the transition to the corresponding quantum theory (see, for example, the monographs 18 and 19). The canonical tensor may be asymmetric. But it can readily be symmetrized by means of a correction that does not violate (74) and (75) and leads to (76). There is freedom in the choice of this correction. The canonical tensor is composed of mathematically incorrect products of fields at one point and, as a rule, has very bad properties even for free fields (see, for example, refs. 20 and 21).

Apart from the canonical tensor, there also exists in classical field theory the gravitational Lagrangian energy–momentum tensor, $T^{\text{gr.L}}_{\alpha\beta}(x)$, which is obtained as follows.[22] The expression for the action is generalized to the case when there is present an external gravitational field described by the metric tensor $g^{\alpha\beta}(x)$. Then $T^{\text{gr.L}}_{\alpha\beta}(x)$ is defined to within a factor as the variational derivative of the action $W[g]$ with respect to the metric tensor:

$$T^{\text{gr.L}}_{\alpha\beta}(x) = (1/\sqrt{-g})\,(\delta W[g]/\delta g^{\alpha\beta}(x)). \tag{77}$$

In the final result one can return to the Euclidean metric. The tensor $T^{\text{gr.L}}_{\alpha\beta}(x)$ is uniquely defined, and it coincides with the canonical when the symmetrizing correction is chosen appropriately. Therefore, $T^{\text{gr.L}}_{\alpha\beta}(x)$ in quantum theory has the same "bad" mathematical properties as $T^{\text{can}}_{\alpha\beta}(x)$. Nevertheless, $T^{\text{gr.L}}_{\alpha\beta}(x)$ is very frequently used to obtain different properties, especially commutation properties, of the energy–momentum tensor; this line of attack was initiated by Schwinger.[23]

Obviously, if one eschews the Lagrangian approach, the canonical and gravitational energy–momentum tensors cease to be obligatory for the theory (like canonical momenta).

3. In 1963 the present author gave a new unique definition of the energy–momentum tensor directly in the framework of axiomatic quantum field theory without recourse to the Lagrangian formalism.[24] This definition is a natural generalization of the definition due to Bogolyubov of currents in terms of variational derivatives of the S matrix with respect to external fields.[18,25]

We make the physical assumption that an S-matrix description (on the mass shell) also exists in the presence of a weak external gravitational field that vanishes sufficiently rapidly as infinity is approached in all directions. Then the S matrix is a functional of the metric tensor:

$$S = S[g^{\alpha\beta}(x)]. \tag{78}$$

To an infinitesimally small change of the metric

$$g^{\alpha\beta}(x) = g^{\alpha\beta} + \delta g^{\alpha\beta}(x), \quad (79)$$

there corresponds a transformation of the S matrix:

$$S = \left(1 - \int d^4x \delta g^{\alpha\beta}(x)\, \delta/\delta g^{\alpha\beta}(x)\right) S', \quad (80)$$

which can be rewritten in the form

$$S = S' \left(1 - i/2 \int d^4(x) \sqrt{-g}\, \delta g^{\alpha\beta}(x)\, T_{\alpha\beta}(x)\right), \quad (81)$$

where

$$T_{\alpha\beta}(x) = \frac{2}{i} S^{+} \frac{\delta S}{\delta g^{\alpha\beta}} \cdot \frac{1}{\sqrt{g}}. \quad (82)$$

Instead of the tensor $T_{\alpha\beta}(x)$ it is frequently more convenient to consider the corresponding tensor density $\tau_{\alpha\beta}(x)$:

$$\tau_{\alpha\beta}(x) = \sqrt{-g}\, T_{\alpha\beta}(x). \quad (83)$$

In the flat-space limit, $\sqrt{-g} = 1$, of course, so that $\tau_{\alpha\beta} = T_{\alpha\beta}$.

As is shown in ref. 24, $T_{\alpha\beta}(x)$ has all the properties of an energy–momentum tensor. First, it satisfies a covariant conservation law:

$$T^{\beta}_{\alpha;\,\beta} \equiv \frac{\partial}{\partial x^{\gamma}} \left(g^{\beta\gamma} T_{\alpha\beta} \sqrt{-g}\right) + \frac{1}{2} \cdot \frac{\partial g^{\gamma\beta}}{\partial x^{\alpha}} T_{\beta\gamma} \sqrt{-g} = 0, \quad (84)$$

which goes over into (74) for flat four-space. Secondly, (75) and (76) hold in flat space.

Thus, from the existence of quantum theory in the presence of a weak external gravitational field it follows directly that there must exist an energy–momentum tensor defined in accordance with (82). Below we shall use only this definition.

4. The tensor $T_{\alpha\beta}(x)$ introduced in Sec. 9.3 is also the gravitational tensor, though it is not the Lagrangian tensor but the axiomatic tensor. In accordance with the equivalence principle, the tensor $T_{\alpha\beta}(x)$ determines both the gravitational and the inertial properties of matter. Here, we shall not study the gravitational properties, assuming that they are weak. In contrast, the inertial properties of particles are manifested significantly in all processes in which elementary particles participate. Therefore, we shall consider the functional derivatives with respect to the metric tensor only in the limit of a flat metric.

Putting it crudely, by going over to a metric that is only slightly nonplanar and is different at different four-points, we label, as it were, all the points of space without essentially disturbing the investigated processes.

A detailed discussion of the meaning of such a treatment can be found in ref. 24 and can be summarized by the following conclusions. The conditions of invariance of a physical theory under the Poincaré group formulated in Sec. 2 of ref. 1 reflect only the property that space–time is pseudo-Euclidean at infinity, far from all particles.[5)] It is for this reason that Poincaré covariance is called the condition of macrocovariance in ref. 24.

In contrast, the fulfillment of the conditions for the existence of the tensor $T_{\alpha\beta}(x)$ in (82), which satisfies the conservation law (84), ensures in the flat limit that the four-space is pseudo-Euclidean at all points; in ref. 24, they are therefore called the conditions of microcovariance of space–time. Obviously, the conditions of microcovariance are more stringent than those of macrocovariance, and include the latter as a special case.

We now formulate conditions of microcovariance without departing from a flat metric. The first condition of microcovariance is obtained by going over directly to a flat metric in (82) and (84): If space–time is everywhere pseudo-Euclidean, there exists a local operator $T_{\alpha\beta}(x)$ satisfying (74) and (76).

Further conditions of microcovariance can be obtained by differentiating (84) once or several times with respect to the functional variable $g^{\alpha\beta}(x)$ and then going over to a flat metric. Thus, the first differentiation leads to the relation

$$2 \frac{\partial}{\partial y^{\lambda}} \left\{ g^{\lambda\varepsilon}(x) \frac{\delta \tau_{\alpha\beta}(x)}{\delta g^{\gamma\varepsilon}(y)} \right\}$$

$$= \left\{ \tau_{\alpha\gamma}(x) \frac{\partial}{\partial x^{\beta}} + \tau_{\gamma\beta}(x) \frac{\partial}{\partial x^{\alpha}} + \frac{\partial \tau_{\alpha\beta}(x)}{\partial x^{\gamma}} \right\} \delta^4(x-y), \quad (85)$$

which in flat four-space takes the form

$$\frac{\partial}{\partial y_{\varepsilon}} \left\{ \frac{\delta \tau_{\alpha\beta}(x)}{\delta g^{\gamma\varepsilon}(y)} \right\} = \frac{1}{2} \left\{ T_{\alpha\gamma}(x) \frac{\partial}{\partial x^{\beta}} \right.$$

$$\left. + T_{\beta\gamma}(x) \frac{\partial}{\partial x^{\alpha}} + \frac{\partial T_{\alpha\beta}(x)}{\partial x^{\gamma}} \right\} \delta^4(x-y) \quad (86)$$

and is called the second condition of microcovariance. The derivation we have given of condition (86) is proposed in ref. 26. This condition was first obtained in a different manner in ref. 24.

The variational derivative $\delta \tau_{\alpha\beta}(x)/\delta g^{\gamma\varepsilon}(y)$ may be naturally called the local gravitational polarizability, since it describes the change in the distribution of matter and stresses at the point under the influence of the external gravitational field.

5. Using the axiomatic gravitational energy–momentum tensor, it is natural to formulate the condition of microcausality, which is this: Any perturbation at a given four-point affects the matter distribution only in the forward light cone at this point. As such a perturbation it is natural to take a change in the metric, since it affects the motion of all forms of matter. Therefore, the most general condition of microcausality has the form

$$\frac{\delta \tau_{\alpha\beta}(x)}{\delta g^{\lambda\sigma}(y)} = \frac{\delta T_{\alpha\beta}(x)}{\delta g^{\lambda\sigma}(y)} = 0 \quad (87)$$

in the forward light cone of a point y, i.e., for

$$y^0 > x^0, \quad (y-x)^2 > 0. \quad (88)$$

Three remarks concerning the condition (87) are appropriate.

First, this condition has the form of Bogolyubov's causality condition,[18,25] but it is written down for the gravitational current $T_{\alpha\beta}(x)$, which gives (87) universal applicability, since all forms of matter have inertial properties.

Second, in the condition (87) essential use is made of the in-basis for state vectors, since in this basis the development of events is assumed to occur forward in time.[6)]

In the out-basis the definition (82) is replaced by

$$T^{\text{out}}_{\alpha\beta}(x)=ST_{\alpha\beta}(x)S^{+}=\frac{2}{i}\cdot\frac{\delta S}{\delta g^{\alpha\beta}(x)}S^{+}\frac{1}{\sqrt{-g}}, \quad (89)$$

and the condition of microcausality takes the form

$$\delta\tau^{\text{out}}_{\alpha\beta}(x)/\delta g^{\gamma\varepsilon}(y)=\delta T^{\text{out}}_{\alpha\beta}(x)/\delta g^{\gamma\varepsilon}(y)=0 \quad (90)$$

in the backward cone of the point y, i.e., for

$$y^0<x^0,\ (y-x)^2=0. \quad (91)$$

Note that in the Lagrangian gravitational approach the basis is not fixed and therefore the condition of microcausality cannot be formulated. It follows from (77) [cf. the equation before (12) in ref. 21] that

$$\delta\tau^{\text{gr.L}}_{\alpha\beta}(x)/\delta g^{\gamma\varepsilon}(y)=\delta\tau^{\text{gr.L}}_{\gamma\varepsilon}(y)/\delta g^{\alpha\beta}(x), \quad (92)$$

which is clearly incompatible with a condition of microcausality of the type (87).

Thirdly, since, in accordance with (82) and (83),

$$2\{\delta\tau_{\alpha\beta}(x)/\delta g^{\lambda\sigma}(y)-\delta\tau_{\lambda\sigma}(y)/\delta g^{\alpha\beta}(x)\}=i[\tau_{\alpha\beta}(x),\ \tau_{\gamma\varepsilon}(y)], \quad (93)$$

we obtain from the condition of microcausality the weaker condition of local commutativity:

$$[T_{\alpha\beta}(x),\ T_{\gamma\varepsilon}(y)]=0 \quad \text{for} \quad (x-y)^2>0. \quad (94)$$

At the same time the polarizabilities on the left-hand side of (92) contain quasilocal terms, i.e., terms proportional to the four-dimensional $\delta$ function and its derivative, which can, in principle, be retained in the commutator, which contradicts the assumption that equal-time commutators exist. The condition that the quasilocal terms in (93) are compensated imposes definite restrictions on the structure of the polarizabilities.[7)] A generalization when this compensation is not present is discussed in Sec. 11.3.

It is intuitively clear from the definition (82) that $T_{\alpha\beta}(x)$ is a local tensor of second rank. We shall show that to prove this assertion we require the second condition of microcovariance and the condition of microcausality. We integrate (86) with respect to $d^4y$ over a convex four-volume containing the point x. Using (90) and (93), we obtain

$$\partial T_{\alpha\beta}(x)/\partial x^{\lambda}=i\int[T_{\alpha\beta}(x),\ T_{\lambda\varepsilon}(y)]\,d\sigma^{\varepsilon};$$

where $d\sigma^{\varepsilon}$ is an element of a three-hypersurface, and the integral is over only that part of the hypersurface that lies within the backward light cone of the point x. Of course, this integral can be extended to an infinite spatial hypersurface, which leads to the relation (9) for $T_{\alpha\beta}(x)$:

$$[P^{\gamma},T^{\alpha\beta}(x)]=\frac{1}{i}\cdot\frac{\partial T^{\alpha\beta}(x)}{\partial x_{\gamma}}. \quad (95)$$

Similarly, one can obtain for $T_{\alpha\beta}(x)$ commutation relations of the type (8):

$$[M^{\alpha\beta}T^{\gamma\varepsilon}(x)]=i(-x^{\alpha}\partial T^{\gamma\varepsilon}/\partial x_{\beta}+x^{\beta}\partial T^{\gamma\varepsilon}/\partial x_{\alpha}-g^{\alpha\gamma}T^{\beta\varepsilon}+g^{\beta\gamma}T^{\alpha\varepsilon}-g^{\alpha\varepsilon}T^{\gamma\beta}+g^{\beta\varepsilon}T^{\gamma\alpha}). \quad (96)$$

6. The structure of the operator $T_{\alpha\beta}(x)$ is such that its vacuum expectation value can have a nonvanishing value

$$\langle 0|T_{\alpha\beta}(x)|0\rangle=ag_{\alpha\beta}, \quad (97)$$

where $a\geq 0$. Physically, this means that vacuum loops of different types with definite mean density are distributed in space and lead to the presence of an invariant energy $a$ in each unit of three-volume. In diagrammatic technique, vacuum energy is created by diagrams of the "tadpole" type, which have only one external line, and that a gravitational one. Such diagrams are not forbidden by any conservation laws, so that there are no justifications for eliminating the constant $a$.

The presence of this constant means that the total energy is infinite for all states. Therefore, when $a>0$ the tensor $T_{\alpha\beta}(x)$ splits into two parts:[21]

$$T_{\alpha\beta}(x)=\overline{T}_{\alpha\beta}(x)+ag_{\alpha\beta}. \quad (98)$$

Obviously, the vacuum expectation value for $\overline{T}_{\alpha\beta}(x)$ vanishes. Replacement of $T_{\alpha\beta}$ by $\overline{T}_{\alpha\beta}$ does not alter the left-hand sides of any commutators containing the energy–momentum tensor, but it may lead to the appearance of c-number terms on the right-hand sides. The presence of (97) and (98) means one must use care when employing the operator $\overline{T}_{\alpha\beta}(x)$, since the axiomatic definition (82) refers to $T_{\alpha\beta}(x)$, for which the corresponding integrals (75) and (76), which define the generators $P^{\gamma}$ and $M_{\alpha\beta}$, diverge. Therefore, when $a\neq 0$, $T_{\alpha\beta}$ in (75) and (76) must be replaced by $\overline{T}_{\alpha\beta}$. Under such a substitution the derivation[24] mentioned in Sec. 9.3 of the conditions (91) of ref. 1 of macrocovariance from the first condition of microcovariance ceases to hold, but the more fundamental relations (95) and (96) remain valid.

As is noted in ref. 24, it is natural to impose the condition that the energy density be positive on the energy–momentum tensor:

$$\langle\phi|T_{00}|\phi\rangle\geq 0 \quad (99)$$

at all points and for all states in which the left-hand side of this condition is defined. The relation (99) may be called the microspectral condition. When $a>0$, this condition takes the form

$$\langle\phi|\overline{T}_{00}(x)|\phi\rangle\geq -a, \quad (100)$$

so that the density of the energy "above the vacuum" may also be negative within bounded limits.

7. Let us consider the formulation of the conditions of microcovariance and microcausality for local quanti-

ties, i.e., for fields, currents, and their derivatives of different orders. The conditions of microcovariance of an arbitrary local quantity A(x) can be expressed in infinitesimal form by the conditions (8) and (9), to which there correspond in universal algebra the set of relations (25), (27)-(29). Note that the conditions of microcovariance in universal algebra are more general, since one does not require the existence of the operators $M^{\alpha\beta}$ and $P^{\gamma}$ for them to hold.

By their physical meaning, local quantities describe the state of a physical system at definite points of space. Therefore, whereas macrocovariance is the most important thing for global quantities like the S matrix, we may expect that the important thing for a local quantity is microcovariance, i.e., the geometric properties in the neighborhood of the point at which this quantity is investigated.

We obtain a condition of microcovariance for a scalar field A(x) following ref. 27. If a weak gravitational field is present, A(x) is a functional, $A[x, g^{\alpha\beta}(y)]$, of the metric tensor, which we shall assume goes over into a pseudo-Euclidean tensor at infinity. To an infinitesimally small coordinate transformation

$$x^{\alpha}=x'^{\alpha}+\xi^{\alpha}(x') \tag{101}$$

there corresponds a transformation of the metric tensor (see for example, ref. 22)

$$g^{\alpha\beta}(x)=g'^{\alpha\beta}(x)+g'^{\alpha\gamma}\partial\xi^{\beta}/\partial x^{\gamma}+g'^{\gamma\beta}\partial\xi^{\alpha}/\partial x^{\gamma} -(\partial g'^{\alpha\beta}/\partial x^{\gamma})\,\xi^{\gamma}\equiv g'^{\alpha\beta}+\delta g^{\alpha\beta}. \tag{102}$$

Under the action of (101), the operator A(x) is transformed with respect to both the coordinates $x^{\alpha}$ and the functional variable $g^{\alpha\beta}(y)$. The resulting transformation must be the identity transformation, since the value of a four-scalar at a certain point does not depend on the choice of a coordinate system:

$$A[x, g^{\alpha\beta}(y)]=[1+\xi^{\alpha}(x)\,\partial/\partial x^{\alpha}] \times\left[1-\int d^4y\,\delta g^{\alpha\beta}(y)\,\delta/\delta g^{\alpha\beta}(y)\right]A[x, g]. \tag{103}$$

The condition of microvariance of A(x) follows from the fact that $\xi^{\alpha}(x)$ is arbitrary, and this condition has the form

$$\frac{\partial}{\partial x_{\beta}}\cdot\frac{\delta A(x)}{\delta g^{\alpha\beta}(y)}=-\frac{1}{2}\,\delta^4(x-y)\,\frac{\partial A(x)}{\partial x^{\alpha}}. \tag{104}$$

At the same time, in accordance with Bogolyubov's condition of microcausality,[18,25] in the in-basis we have

$$\delta A(x)/\delta g^{\alpha\beta}(y)=0, \tag{105}$$

if x lies outside the forward light cone of y (88) and

$$\delta/\delta g^{\alpha\beta}(y)\{SA(x)S^{+}\}=0, \tag{106}$$

if x lies outside the backward light cone (91) of y.

In the limit of flat space, conditions of macrocovariance follow from (104), but again only with the conditions (105) and (106) of microcausality. Thus, the condition (9) of translational invariance

$$[P^{\alpha}, A(x)]=\frac{1}{i}\cdot\frac{\partial A(x)}{\partial x_{\alpha}} \tag{107}$$

is obtained by integrating (104) over the four-volume between the hyperplanes $x^0+\varepsilon=0$ and $x^0-\varepsilon=0$ with the use of (105), (106), (82), and (85).

It is not difficult to write down conditions of microcovariance for vector and tensor local operators. But on the transition to spinor local quantities we encounter a difficulty, because spinors do not have an affine nature but a metric nature, so that the transformation of a spinor to curvilinear coordinates cannot be defined independently of a metric.

The condition of microcovariance (104) is much stronger than the conditions of macrocovariance (8) and (9). Thus, the in- and the out-field are macrocovariant, but not microcovariant, i.e., they are not truly local quantities.

The difference between micro- and macrocovariance of local quantities can be illustrated by the following simple example. From the covariant quantity A(x) we form the new B(x):

$$B(x)=\int d^4y\,F(x-y)\,A(y),$$

where F is an invariant function of the variable x − y. It is readily seen that B(x) satisfies (8) and the expressions (9), but not (104).

## 10. ENERGY–MOMENTUM TENSOR AND CONSERVED CURRENT IN UNIVERSAL ALGEBRA

1. The investigation of the structure of equal-time commutators of the components $T^{\alpha\beta}(x)$ with one another and the components of other local quantities was begun 10 years ago. In 1962 Dirac[28] and Schwinger[23] showed that if the operators $M^{\alpha\beta}$ and $P^{\gamma}$ are formed from $T^{\alpha\beta}(x)$ in accordance with (75) and (76), fairly stringent conditions must be imposed on the possible form of the equal-time commutators $[T^{00}, T^{00}]$, $[T^{00}, T^{0i}]$, and various others if the conditions (95) and (96) of covariance of $T^{\alpha\beta}(x)$ are to be satisfied. They wrote down equal-time commutators with a minimal number of gradient terms satisfying these restrictions. The complete system of equal-time commutators between the componenets $T^{\alpha\beta}(x)$ that follow from (75) and (76) have been written down by Boulware and Deser.[21] They discuss the question of possible additions to the commutators with a minimal number of gradient terms. It was found from the spectral condition that in the equal-time commutators $[T^{00}, T^{0i}]$ and $[T^{0i}, T^{jl}]$ there must exist nonzero gradient terms with the third derivative of the three-dimensional δ function. Otherwise, the form of the corrections remains arbitrary. From the existence of this freedom and also the fact that the system obtained in ref. 21 does not contain the commutator $[T^{ij}, T^{lk}]$, Boulware and Deser concluded that their system [Eq. (7) of ref. 21] "does not form an algebra". We shall see below that this conclusion is incorrect.

In ref. 29 the equal-time commutators $[T^{0\alpha}, J^0]$, $[T^{00}, J^i]$, and $[T^{00}, \partial_{\alpha}J^{\alpha}]$ between the components of $T^{\alpha\beta}$ and the current $J^{\gamma}$ were obtained similarly.

Equal-time commutators with the participation of $T^{\alpha\beta}$ have been constructed and investigated in some other papers.[8,20,30–42] In them, as a rule, additional models or other arguments based on Lagrangian theory are made. For our purposes it is only important to note that in none of these papers is the problem even posed of constructing systems of equal-time commutators that contain $T^{\alpha\beta}$ and are closed under transformations of the Poincaré group. This problem was posed and solved in ref. 15 with the use of the universal algebra formalism developed in Secs. 7 and 8.

2. Let us formulate in the language of universal algebra the first condition of microcovariance for equal-time commutators.

To the tensor $T^{\alpha\beta}(x)$ in universal algebra[8)] there corresponds an element $T^{\alpha\beta}$, and the conservation law (74) takes the form

$$T^{\alpha\beta}\partial_\beta = 0. \tag{108}$$

Substituting (75) into (9) and going over from (15) to (16), we obtain

$$AT^{\alpha 0}\mu = A\partial^\alpha. \tag{109}$$

Since A is arbitrary, we obtain the following equation for unary operations:[14]

$$T^{\alpha 0}\mu = \partial^\alpha. \tag{110}$$

Similarly, we obtain expressions for the components of the unary operation $S^{\alpha\beta}$ in terms of[14] $T^{\gamma\varepsilon}$:

$$S^{i0} = -\mathrm{i}T^{00}\mu^i; \tag{111}$$

$$S^{ij} = \mathrm{i}(T^{i0}\mu^j - T^{j0}\mu^i). \tag{112}$$

The relations (108), (110), and (112) in conjunction with the condition that follows from the second condition of microcovariance (96), and the third of the relations (26),

$$T^{\gamma\varepsilon}S^{\alpha\beta} = \mathrm{i}\,(T^{\beta\varepsilon}g^{\alpha\gamma} - T^{\alpha\varepsilon}g^{\beta\gamma} + T^{\gamma\beta}g^{\alpha\varepsilon} - T^{\gamma\alpha}g^{\beta\varepsilon}) \tag{113}$$

form a complete system of conditions imposed on the energy–momentum tensor by the conditions of relativistic invariance alone. For example, the system of commutators (7) from ref. 21 reduces to (113) and the relation

$$T^{\beta\gamma}T^{\alpha 0}\mu = T^{\beta\gamma}\partial^\alpha,$$

which is obtained by multiplying (110) by $T^{\beta\gamma}$ from the left.

The commutators (27)-(29) for the unary operations $S^{\alpha\beta}$ and $\partial^\gamma$ follow from the relations we have just listed; for example,

$$\begin{aligned}[S^{i0}, \partial^0] &= -\mathrm{i}\,(T^{00}\mu^i T^{00}\mu - T^{00}\mu T^{00}\mu^i)\\ &= -\mathrm{i}T^{00}T^{00}\mu\mu^i = -\mathrm{i}T^{00}\partial^0\mu^i\\ &= \mathrm{i}T^{0j}\partial_j\mu^i = \mathrm{i}T^{0i}\mu = \mathrm{i}\partial^i.\end{aligned} \tag{114}$$

In deriving (114) we have used the Jacobi identity (21) and the property (23) for k = 1.

We see that when $T^{\alpha\beta}$ is introduced, the unary operations $S^{\alpha\beta}$ and $\partial^\gamma$ in the universal algebra are not introduced as independent, but are constructed from the element $T^{\alpha\beta}$ and the nonassociative multiplications $\mu$ and $\mu^i$. In such a formulation, the conditions of relativistic invariance can be replaced by the set of relations (108), (113), and the definitions (110), (111), and (112).

3. In Eq. (110), the right-hand side is a four-vector, while the left-hand side is, when the expansion (55) for $\mu$ is substituted into it, apparently a quantity that transforms in accordance with a complicated representation whose dimensionality depends on the choice of the invariant truncation. We shall show that when the conservation law (108) holds, the decomposition of the unary operation $T^{\alpha 0}\mu$ with respect to the Lorentz covariant unary operations $T^{\alpha\beta}\eta^{\cdots}_{(k)}$ allows closed summation of the right-hand sides. This summation is not related to the specific properties of the energy–momentum tensor and is possible for all conserved local quantities; for example, for the conserved current $J_\alpha(x)$ we have

$$J_\alpha\partial^\alpha = 0. \tag{115}$$

In this case, the unary operation

$$Q = J_0\mu \tag{116}$$

is an invariant of the Poincaré group, since, in accordance with (23) and (30),

$$Q\partial^\alpha - \partial^\alpha Q = J_0\partial^\alpha\mu = J_0\partial^0\mu g^{\alpha 0} = -J_i\partial^i\mu g^{\alpha 0} = 0,$$

and, in accordance with (23), (25), (32), and (33), $QS^{\alpha\beta} - S^{\alpha\beta}Q = 0$. It is therefore to be expected that the right-hand side of (116) can be written in the form of a four-scalar. To obtain this form, we consider (75) for $k = 2n+2$, $\alpha_1 = \ldots = \alpha_{2n+2} = 0$:

$$\partial^\alpha \eta_{(n)}^{\overbrace{0\ldots0}^{2n+2}} = (2n+2)\, g^{\alpha 0}\eta_{(n)}^{\overbrace{0\ldots0}^{2n+1}} - \frac{2n+1}{2}\eta_{(n)}^{\alpha\overbrace{0\ldots0}^{2n}} - \frac{1}{4}\eta_{(n+1)}^{\alpha\overbrace{0\ldots0}^{2n+2}}. \tag{117}$$

Multiplying (117) from the left by $[(-1)^n/(2n+2)!]J_\alpha$ and summing over n from zero to infinity, we find that the left-hand side vanishes on account of (116), and summation of the right-hand side with allowance for (55) leads to the desired manifestly invariant result

$$Q = J^0\mu = \frac{1}{4}J_\alpha\eta^\alpha_{(0)}, \tag{118}$$

which is obviously true for any invariant termination of the series (55).

Similarly, the relation (110) can be summed to give the manifestly covariant equation

$$\partial^\alpha = \frac{1}{4}T^\alpha_\beta\,\eta^\beta_{(0)}, \tag{119}$$

and the relations (111) and (112), to give the single covariant equation

$$S^{\alpha\beta} = \frac{1}{6}\,(T^\alpha_\gamma\,\eta^{\gamma\beta}_{(0)} - T^\beta_\gamma\eta^{\gamma\alpha}_{(0)}). \tag{120}$$

4. It follows from (119) and (120) that the system of well-known commutators[21] of $T^{\alpha\beta}$, obtained by substituting (111) and (112) into the third of the relations (26):

$$T^{\gamma\varepsilon}(T^{\alpha 0}\mu^i\delta_i^\beta - T^{\beta 0}\mu^i\delta_i^\alpha) = i\,(T^{\beta\varepsilon}g^{\alpha\gamma} - T^{\alpha\varepsilon}g^{\beta\gamma} + T^{\gamma\beta}g^{\alpha\varepsilon} - T^{\gamma\alpha}g^{\beta\varepsilon}), \quad (121)$$

is closed under Lorentz transformations. In this sense the following systems of commutators are closed:

$$T^{\alpha\beta}T^{\gamma 0}\mu = T^{\alpha\beta}\partial^\gamma; \quad (122)$$

$$T^{\alpha\beta}T^0_{\beta}\mu = 0, \quad (123)$$

these being obtained by multiplying (110) from the left by $T^{\alpha\beta}$. We must emphasize that the relations (121)-(123) represent equations of the type (15), i.e., in contrast to refs. 21 and 23 and others they do not refer to the equal-time commutators as a whole, but only to individual terms on the right-hand sides. With regard to the other commutation properties of the tensor $T^{\alpha\beta}(x)$, in particular, with regard to the form of the commutators $[T^{ij}, T^{kl}]$ and gradient terms with third and higher derivatives of the $\delta$ function, only restrictive conditions of a fairly general form can be obtained from the requirements of relativistic invariance alone.

5. The example of the set of relations (121)-(123) shows us that the summations (118)-(120) appreciably extend the possibility of making systems of equal-time commutators covariant, when they contain components of conserved quantities. We shall demonstrate these possibilities by the example of the derivation of the simplest "covariantization" of the current algebra (1). In universal algebra one automatically writes down stronger commutators than (1); in them one integrates only one component of the current:

$$\left[J_a^0(\mathbf{x}, x^0),\ \int d^3y J_b^0(\mathbf{y}, x^0)\right] = iG_{ab}^c J_c^0(\mathbf{x}, x^0).$$

Such a commutator is a special case of (15) and thus allows a transition to (16):

$$J_a^0 J_b^0\mu = G_{ab}^c J_c^0. \quad (124)$$

We now carry through the covariantization (124), without recourse to model arguments, in particular, without invoking currents as bilinear combinations of fields. The method for doing this depends on whether or not conservation laws are satisfied for the currents.

If there is no conservation, then the minimal (in the sense of the number of components and dimensionality of the four-tensors) relativization of (124) is obtained in accordance with (55) by the replacement $\mu \to \eta^0_{(0)}$:

$$J_a^0 J_b^0 \eta^0_{(0)} = G_{ab}^c J_c^0 g^{00}$$

and it leads to the covariant relations

$$(J_a^\alpha J_b^\beta \eta^\gamma_{(0)} + \text{sym}) = 3G_{ab}^c\,(J_c^\alpha g^{\beta\gamma} + \text{sym}), \quad (125)$$

where in each bracket there is an expression that is symmetric with respect to $\alpha$, $\beta$, and $\gamma$. For fixed $a$ and b, Eq. (125) has twenty independent components, of which all, except one, are assertions, in addition to (124), about the form of the equal-time commutators of the components of the currents. All these new relations can be obtained by making the substitution $\eta^0_{(0)} \to \mu$, $\eta^i_{(0)} \to \partial^0\mu^i$ in (125) and then going over from (16) to (15).

If the currents are conserved,

$$J_a^\alpha \partial_\alpha = 0 \quad \text{for all} \quad a, \quad (126)$$

then, in accordance with (118), the unary operation

$$Q_b = J_b^0\mu = \frac{1}{4}J_{b\alpha}\eta^\alpha_{(0)} \quad (127)$$

is a four-scalar. However, in (124) we cannot directly use analogous properties of the components $J^0_a$ and $J^0_c$. But this can be done by multiplying (124) from the right by $\mu$ and applying to the left-hand side the Jacobi identity (21) for k = 0:

$$J_a^0 J_b^0\mu\mu = J_a^0\mu J_b^0\mu - J_b^0\mu J_a^0\mu. \quad (128)$$

As a result we obtain the manifestly covariant relation

$$Q_a Q_b - Q_b Q_a = G_{ab}^c Q_c, \quad (129)$$

which is analogous to (1) but is written down not for integrated charges, but for the unary operations that correspond to them in accordance with (17). Therefore, for (110) to hold one does not require the existence of charge operators, which are not always available (see, for example, refs. 16 and 17).

The covariant form of expressions (127) and (129) does not lead to the appearance of new commutators additional to the original relation (124). However, these new commutators do appear as soon as we fix a restriction to a finite number of terms in the expansion of $\mu$ in (55). Thus, in the simplest case,

$$J_b^0\mu \to J_b^0\eta^0_{(0)}, \quad (130)$$

we obtain from (127)[9])

$$J_b^0\eta^0_{(0)} = \frac{1}{4}J_{b\gamma}\eta^\gamma_{(0)}g^{00}$$

with the obvious covariant generalization

$$J^\alpha_\beta\eta^\beta_{(0)} + J^\beta_b\eta^\alpha_{(0)} = \frac{1}{2}g^{\alpha\beta}J_{b\gamma}\eta^\gamma_{(0)}. \quad (131)$$

Equation (131) leads to new commutators, which differ from (124). For example, setting $\alpha$ = i and $\beta$ = j in (131) and going back from $\eta^{\cdot\cdot}_{(0)}$ to $\mu\cdots$, we obtain

$$J_b^i\partial^0\mu^j + J_b^j\partial^0\mu^i = 2g^{ij}J_b^0\mu. \quad (132)$$

We multiply this equation from the left by $J^0_a$ and use (124):

$$J_a^0 J_b^i\partial^0\mu^j + J_a^0 J_b^j\partial^0\mu^i = 2g^{ij}G_{ab}^c J_c^0. \quad (133)$$

In accordance with (4), (15), and (16), the meaning of (133) is this: The sum of commutators $g^{jl}[J^0_\alpha(\mathbf{x}, x^0), (\partial/\partial x^0) J^i_b(\mathbf{y}, x^0)] + g^{il}[J^0_\alpha(\mathbf{x}, x^0), (\partial/\partial x^0) J^j_b(\mathbf{y}, x^0)]$ contains the gradient term $2ig^{ij}G^c_{ab}J^0_c(\mathbf{x}, x^0)(\partial/\partial y_l)\delta^3(\mathbf{x}-\mathbf{y})$. It can be seen from the above examples that the covariantization of any system of equal-time commutators is a purely technical problem, which can, however, be rather lengthy.

## 11. PROSPECTS AND POSSIBLE GENERALIZATIONS OF THE COVARIANT FORMALISM IN UNIVERSAL ALGEBRA

1. A direct and, if we may say so, the most immediate field of application for the method of making equal-time commutators covariant that we have developed in Secs. 8 and 10 is the construction of covariant current algebras with gradient terms and the subsequent derivation of experimentally verifiable relations of the type of sum rules. Without denying the importance of this direction, we should like to point out that from the point of view developed here it is not the only possible one and it may not even be the principal field of application of the method expounded in this paper.

Namely, the theory of equal-time commutators (and its natural generalization discussed below) opens up a new direction in the investigation of the possibility of constructing correct dynamical equations for quantum fields. Dashen and Sharp[43] have noted that the algebra of unintegrated currents, if it is constructed, may play the role of a dynamical theory. However, the actual treatment in ref. 43 is restricted to very simple (nonrelativistic and Lagrangian) models. An embodiment of Dashen and Sharp's idea is Sugawara's model,[44] which contains a system of equal-time commutators with the participation of SU(3) currents and an energy–momentum tensor. However, in Sugawara's model there are not only equal-time commutators [i.e., the nonassociative multiplications (16)], but also ordinary, mathematically incorrect multiplications of fields at a point. This of course, leads to a number of difficulties, which are pointed out in ref. 45.

A different formulation of the problem of constructing a dynamical theory on the basis of an algebra of equal-time commutators has been proposed by the author in ref. 14. In a somewhat modified form, this formulation consists of the following assumptions.

Assumption I. Equal-time commutators exist for all local quantities and the derivatives of finite orders.

Assumption II. Each equal-time commutator has the form (4), and the local quantities A, B, and F ⋯ transform in accordance with finite-dimensional representations of the Lorentz group.

Assumption III. There exists a conserved local operator $T^{\alpha\beta}$ that satisfies (110)-(113).

Assumption IV. There exists a finite set of operators[10] $A(\mathbf{x}, x^0)$ that forms a complete system for fixed $x^0$.

Assumption V. Any local quantity $B(\mathbf{x}, x^0)$ that does not belong to the complete system can be obtained for fixed $x^0$ from the operators of the complete system by applying a finite number of additions, multiplications by complex numbers, and nonassociative local multiplications (16).

The problem arises of obtaining a nontrivial system of equal-time commutators that satisfies the assumptions I-V (which we shall denote henceforth by S). A nontrivial system of commutators satisfying S may play the role of the dynamical equations of a theory.

2. Let us discuss the main properties of the system S. First, this system does not have a place for (ordinary) multiplication of fields at a four-point. In this, S differs significantly from Lagrangian theory and the Dashen–Sharp and Sugawara models. It should be emphasized that this distinction from Lagrangian theories requires assumption V, for the equations and commutation relations of a Lagrangian theory with a polynomial interaction can be replaced by an equivalent system of equal-time commutators. Thus, for the Lagrangian

$$L(x)=\frac{1}{2}\cdot\frac{\partial A}{\partial x_\alpha}\cdot\frac{\partial A}{\partial x^\alpha}-gA^4$$

the canonical commutation relations can be written in the form

$$AA\partial^0\mu=1;\quad AA\mu^{i_1\cdots i_k}=0;\quad A\partial^0\mu^{i_1\cdots i_k}=0; \tag{134}$$

and the equation of motion

$$\Box A(x)=4A^3(x) \tag{135}$$

can be replaced in the universal algebra by the system

$$A\Box=J;\quad JA\partial^0\mu=12j,\quad jA\partial^0\mu=24A. \tag{136}$$

We have not written out the obvious zeroth nonassociative products. Adding to (134)-(136) the relation (25) with $S^{\alpha\beta}$ from (111) and (112), we obtain a system of relations of the universal algebra that is equivalent to the canonical scheme. However, assumption V excludes this possibility, since, in accordance with (134), one can obtain only c-numbers in the canonical formalism from the original fields by means of the operations of the universal algebra; thus, one cannot obtain the operator $T^{\alpha\beta}$. This is why $T^{\alpha\beta}$ had to be formed by means of the other (and, moveover, "bad") operations and had the "bad" singular properties.

We should also emphasize that what we have said here also applies to the free fields, which also do not satisfy S. This circumstance may be discouraging, since the free field is valued as the only, albeit trivial example that satisfies the axioms of quantum field theory. However, if one adopts the arguments given in Sec. 9.3, the free theory can be regarded as bad, since it leads to an energy–momentum tensor with unit form factors, i.e., to a point particle.

The assumption V requires that the energy–momentum tensor be obtained from the fields by valid operations of the universal algebra. This means (see Sec. 2 in ref. 1 and the end of Sec. 8) that relativistic invariance must be treated in terms of an inner automorphism of the theory.

Another important property of the system S is that the condition of local commutativity is automatically satisfied in S. In this sense, the transition from the ordinary axioms of quantum field theory to S makes it possible to satisfy the condition of local commutativity.

The difficulty of constructing concrete examples of universal algebras that satisfy S is associated with the complexity of the conditions (19) of transposition imposed upon it, of the fulfillment of the Jacobi identities (21) and (22) and others, and of the equations for differentiating binary operations (23) [or, in covariant form, (52)]. It would be highly desirable to satisfy some, at least, of these conditions in a general form.

3. The assumption that there exist equal-time commutators for local quantities and their derivatives of all orders may be too stringent. We shall show that the formalism developed in Secs. 7 and 8 allows a natural generalization to the case when the equal-time commutators need not exist, but local commutativity is preserved.

Namely, instead of (46) we can postulate

$$\lambda(p_\alpha) = \lambda_{\text{adv}}(p) - \lambda_{\text{ret}}(p) + \Pi(p), \qquad (137)$$

where

$$\lambda_{\text{adv}}(p) = \frac{(2\pi)^3}{i} \sum_{k=0}^{\infty} \frac{i^k}{k!} p_{\alpha_1} \ldots p_{\alpha_k} \int_0^\infty ds \rho_{\text{adv}}^{\alpha_1 \ldots \alpha_k}(s) D_s^{\text{adv}}(p); \qquad (138)$$

$$\lambda_{\text{ret}}(p) = \frac{(2\pi)^3}{i} \sum_{k=0}^{\infty} \frac{i^k}{k!} p_{\alpha_1} \ldots p_{\alpha_k} \int_0^\infty ds \rho_{\text{ret}}^{\alpha_1 \ldots \alpha_k}(s) D_s^{\text{ret}}(p); \qquad (139)$$

$$D_s^{\text{adv}}(p) = \frac{1}{(2\pi)^4} \cdot \frac{1}{p^\alpha p_\alpha - s - i0p_0}; \; D_s^{\text{ret}} = \frac{1}{(2\pi)^4} \cdot \frac{1}{p^\alpha p_\alpha - s + i0p_0}; \qquad (140)$$

$\Pi(p)$ is a polynomial in $p_\alpha$,

$$\Pi = \sum_{k=0}^{n} \frac{i^k}{k!} p_{\alpha_1} \ldots p_{\alpha_k} \xi^{\alpha_1 \ldots \alpha_k}, \qquad (141)$$

whose coefficients are tensor binary operations. This polynomial corresponds to the possible presence in the commutator (38) of quasilocal terms proportional to the four-dimensional $\delta$ function $\delta^4(z)$ and its derivatives. The operations $\xi^{\cdots}$ are symmetric with respect to all indices, but their contractions with respect to pairs of indices are not equal to zero in the general case. We emphasize that if the expansion (137) does not reduce to (46), then even when $\Pi(p)$ is not present in the original commutator (38) it appears in the commutators for the derivatives of the fields $A(x)$ and $B(y)$.

In the case of the general expansion (137) one must consider not the equal-time commutators, but their limits from above and below, i.e., as $z^0 \to \pm 0$. Accordingly, the complete system of local binary operations will contain all operations of the form $\square^m \mu_+^{i_1 \cdots i_k}$, $\square^m \mu_-^{i_1 \cdots i_k}$, $\square^m \partial^0 \mu_+^{i_1 \cdots i_k}$, $\square^m \partial^0 \mu_-^{i_1 \cdots i_k}$, and $\xi^{\alpha_1 \cdots \alpha_k}$. Accordingly, the complete system of covariant binary operations will consist of $\eta_{(n)+}^{\alpha_1 \cdots \alpha_k}$, $\eta_{(n)-}^{\alpha_1 \cdots \alpha_k}$, $\xi^{\alpha_1 \cdots \alpha_k}$.

The expansion (137) is the most general possible if one requires that the vacuum expectation values of single or repeated commutators in the x representation do not have essential singularities at zero coordinate differences.

4. An expansion of the type (137) (with vanishing advanced or retarded term) can be used to obtain a system of finite-dimensional local quantities from the Bogolyubov variational derivatives[18,25] of the axiomatic currents $J_A(x)$:

$$J_A(x) = \frac{1}{i} S^+ \frac{\delta S}{\delta A(x)} \qquad (142)$$

with respect to the external fields. For example, by analogy with (15) one can assume the existence of limits of the type

$$\lim_{x^0 - y^0 \to +0} \int \frac{\delta J_A(x)}{\delta B(y)} (x^{i_1} - y^{i_1}) \ldots (x^{i_k} - y^{i_k}) d^3y = F_+^{i_1 \cdots i_k}(x). \qquad (143)$$

It would be very interesting to construct and investigate the corresponding covariant expansion for the electromagnetic polarizability $\delta j_\alpha(x)/\delta A^\beta(y)$, where $j_\alpha(x)$ is the electromagnetic current.

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1)The first indication that such terms must be introduced came in 1955 in ref. 2. In 1959 an analogous result was obtained by Schwinger.[3] The terminology "gradient term" was proposed by Källen.[4]

2)Note that the violation of the Jacobi identity noted in ref. 11 refers to a definite choice of the passage to the limit in the derivation of equal-time commutators in Lagrangian theory and therefore does not apply to the axiomatic approach developed here.

3)The abstract generators $M^{\alpha\beta}$ and $P^\gamma$ must be distinguished from the particular realizations of these operators (denoted by the same letters) by operators acting on state vectors.

4)The vectors of the space L and its subspaces are to be distinguished from standard three- and four-vectors.

5)Putting it more precisely, it only follows from the fulfillment of these conditions that far from particles the space is one of the homogeneous spaces on which the Poincaré group acts effectively and transitively. All such spaces can be obtained from the 10-dimensional space of the parameters of the group $\mathcal{P}$ as factor spaces with respect to different noninvariant subgroups.

6)Let us explain this not immediately apparent circumstance by taking the example of the Cauchy problem in classical field theory. If the initial conditions are specified at the time $x^0 = 0$, the solution is described by retarded potentials for $x^0 > 0$ and advanced, for $x^0 < 0$. Therefore, the polarizability (87) will be zero for $x^0 > 0$, $y^0 > 0$ in the forward cone of the point y and for $x^0 < 0$, $y^0 < 0$ in the backward cone of this point. But if $x^0 y^0 < 0$, then the answer depends on whether the test perturbation affects the initial conditions.

7)I am indebted to Yu. S. Burkhanov for this observation.

8)In this section we shall use the tensor $\overline{T}^{\alpha\beta}$ from (98) and denote it for brevity simply by $T^{\alpha\beta}$. Thus, either $\alpha = 0$ or the right-hand sides of the commutators may contain c-number terms.

9)We emphasize that the truncation (130) is made for a unary operation and is therefore very strong and is hardly a realistic assumption. One must rather expect that for the different local quantities A the truncations for the products $Aj_b^0\mu$ are different.

10)These operators are generalized functions of the spatial coordinates.

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