Phenomenological Lagrangians

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Some mathematical aspects of the phenomenological description of Goldstone particles are considered. The treatment is given for an arbitrary group G of internal symmetries and for an arbitrary group containing the Poincaré group as a subgroup.

INTRODUCTION

In recent years, the method of phenomenological Lagrangians has been successfully used in elementary-particle physics. This method arose as a simple means of representing the results of chiral algebra for the chiral group $SU(2) \otimes SU(2)$.[1] The method of phenomenological Lagrangians was generalized to the chiral group $SU(3) \otimes SU(3)$,[2] and somewhat later to an arbitrary semisimple group of internal symmetries,[3] and then to groups containing the Poincaré group as a subgroup.[4] The method of phenomenological Lagrangians has now been fairly well developed and has a comparatively simple mathematical technique that provides a standard phenomenological description of the interaction of arbitrary Goldstone particles.[5]

The method of phenomenological Lagrangians is based on the assumption that the elements of the S matrix after the separation of the contribution of the pole terms are smooth functions of the particle momenta in the region of moderate energies and can therefore be obtained from the ordinary Feynman rules with allowance for only tree-type diagrams from a certain Lagrangian containing phenomenological constants. The method bears fruit when additional symmetry requirements must be satisfied, which makes it possible to establish relations between the constants in the phenomenological Lagrangians.

For the case of interacting Goldstone particles the symmetry requirements are decisive. As is well known, these particles arise in systems with a degenerate vacuum whose symmetry is lower than that of the original Lagrangian, which describes certain primary interactions. The appearance of Goldstone particles is usually interpreted as a reaction of the system that should restore the symmetry broken by the vacuum degeneracy.

The method of phenomenological Lagrangians enables one to give a well-defined meaning to the concept of symmetry restoration: The vacuum degeneracy breaks the algebraic symmetry of the system, i.e., the symmetry under which the transformation properties of the operators are determined by linear representations of the corresponding group; the Goldstone particles restore the system's symmetry. The transformation properties of the operators of these particles are determined by nonlinear transformations, and this, in conjunction with the requirement that the phenomenological Lagrangian be invariant under the considered transformations, corresponds to a symmetry of dynamical type. The internal mechanism responsible for the Goldstone particles as well as the mechanism that restores the symmetry remain outside the scope of the method of phenomenological Lagrangians.

The introduction of phenomenological fields of Goldstone particles and the determination of their transformation properties under the considered group of transformations can be carried through in the following standard manner. Let G be a continuous symmetry group of some original Lagrangian that leads to vacuum degeneracy and the appearance of Goldstone particles and let H be the maximal subgroup of G that leaves the vacuum invariant. We represent an arbitrary transformation of G in the form of the product

$$G = K(a)\,H(b), \tag{1}$$

where H(b) is a transformation of the subgroup H; $K(a)$ is a transformation belonging to the left coset of G with respect to H; and a and b are the corresponding parameters determining the parametrization of the group space of G.

Multiplying an element of G from the left by an arbitrary transformation of the same group,

$$G(g)\,K(a)\,H(b) = K(a')\,H(b'), \tag{2}$$

one can determine how the parameters a and b transform under transformations of the group:

$$a^{\alpha\prime} = a^{\alpha\prime}(a^{\alpha}, g); \tag{3}$$

$$b^{m\prime} = b^{m\prime}(b^{m}, a^{\alpha}, g). \tag{3a}$$

It is important that for the parametrization (1) the transformation (3) does not contain the parameters b. Except for the explicit separation in G of the subgroup H(b) and the corresponding coset $K(a)$, the parametrization (1) is completely arbitrary. Each of the parametrizations $K(a)$ corresponds to a definite choice of the coordinates in the space G/H. When the parametrization $K(a)$ is varied, the transformations (3) vary accordingly. However, as the following treatment will show (see Sec. 2), this freedom in the choice of the coordinate system in G/H is not important in the description of the interaction of Goldstone particles in the method of phenomenological Lagrangians.

This method associates with each parameter a^{α} a local field of Goldstone particles $A^{\alpha}(x)$, so that for the local fields $A^{\alpha}(x)$ the transformation law (3) is satisfied:[1)]

$$A^{\alpha\prime}(x) = a^{\alpha\prime}(A^{\alpha}(x), g). \tag{4}$$

The use of linear representations of the subgroup H in conjunction with the transformation (3a) enables one to determine the so-called induced representations for a transformation of the group G (see Secs. 4 and 5). Associating the components of such representations with local fields $\psi^{m}(x)$, we can determine the transformation properties of fields that are not fields of Goldstone particles. If the fields have certain transformation properties under transformations of G, the construction of the phenomenological Lagrangian reduces to finding invariants of these transformations that contain a minimal number of derivatives of the local fields.

Because the transformations (3) and (3a) are nonlinear with respect to the parameters a and b, respectively, of the fields $A^{\alpha}(x)$, the use of the method of phenomenological Lagrangians leads to the establishment of definite relationships for the elements of the S matrix with a different number of interacting particles.

The reviews[6] are devoted to applications of the method of phenomenological Lagrangians in elementary-particle physics for concrete symmetry groups, especially for the chiral groups SU(2) $\otimes$ SU(2) and SU(3) $\otimes$ SU(3).

The main aim of the present review is to consider some general aspects of the method, and also to establish the relationship between the methods and concepts used in the treatment with phenomenological Lagrangians and the methods and concepts of modern differential geometry[2)] which are related to Cartan's ideas and have led in the last decade to the construction of the general theory of fiber spaces[8-10] and connections on them (see, for example, ref. 10). At the same time, we have attempted to avoid modern mathematical terminology as far as possible and restrict ourselves in the main part of the treatment to the conceptual side of the problem. In Secs. 2-4 we consider some properties of phenomenological Lagrangians for semisimple groups of internal symmetries. In Sec. 2 we consider the covariant properties of an S matrix constructed in the approximation of tree-type diagrams under transformations of the local fields $A^{\alpha}(x)$. As a phenomenological Lagrangian we use the quadratic form

$$L = g_{\mu\nu}(A^{\alpha}(x))\,\partial A^{\mu}\,\partial A^{\nu}/2, \tag{4a}$$

where ∂A^{μ} are the derivatives of the fields with respect to the space–time variables. In (4a) and the subsequent equations the indices of these variables are omitted.

The elements of the S matrix obtained by summing tree-type diagrams for the Lagrangian (4a) can be rearranged into an explicitly covariant form with respect to arbitrary transformations of the fields of Goldstone particles. That such a rearrangement is always possible is a consequence of a general theorem of axiomatic field theory[11] and an analogous theorem in the framework of the Lagrangian approach:[12] The elements of the S matrix on the mass shell do not depend on the choice of the interpolating fields.[3)]

If the coefficients of the quadratic form are interpreted as a metric tensor, the corresponding matrix elements are function of the curvature tensor of the space and its covariant derivatives. The property of the matrix elements is illustrated first for the simplest cases of diagrams with three and four external lines. We then consider a general iterative procedure for constructing a matrix element with an arbitrary number of external lines.

The iterative procedure has two important aspects: 1) allowance for all possible reductions of pole diagrams to an effective contact interaction, and 2) the use of a classification of contact interactions in accordance with Young tableaux with respect to permutations of the indices of the field operators. Both these aspects have a simple geometrical interpretation. The first of them corresponds to a transformation of the metric tensor to a new coordinate system, while the separation in the matrix elements of certain Young tableaux corresponds to a definite choice of the coordinate system, known in differential geometry as a normal coordinate system. As a result of this procedure, a reduced matrix element can be represented as a sum of two terms, the first of which is obtained as a result of the explicitly covariant procedure of going over to a normal coordinate system, and the second corresponds to the function of the transformation from the original to the normal coordinate system. This term vanishes on the mass shell of the particles.

In Sec. 3 we investigate restrictions on the metric tensor for the case when the space under consideration has a transitive group of motions. This question belongs to a well studied field of group theory and classical differential geometry. In Sec. 3 we therefore give merely a few equations that illustrate for the example of symmetric spaces the structure of the metric tensor in a normal coordinate system. Because of the general relationship between the theory of symmetric spaces and the theory of semisimple groups, the case of symmetric spaces is of greater interest for physical applications. In particular, the well known chiral symmetries belong to this case.

In Sec. 3 we also prove the equivalence of Adler's principle:[14] The elements of the S matrix must vanish on the mass shell when one of the momenta of the Goldstone particles tends to zero, and the requirement of symmetry of the phenomenological Lagrangian under a transitive group. The proof of this assertion for matrix elements with arbitrarily many external lines is based on allowing for an additional reduction of pole diagrams to contact diagrams that arise when one of the particle momenta tends to zero and is carried through by means of an iterative procedure similar to that employed in Sec. 2.

The results of this section, like those of Sec. 2, have a simple geometrical interpretation: The reduced matrix element can be represented in accordance with the classification by Young tableaux in the form of two terms, the first of which reproduces the well known Killing equation for the invariance of the metric form, and the second, which vanishes on the mass shell, corresponds to an infinitesimally small transformation of the symmetry group.

In Sec. 4 we treat the interactions of Goldstone particles with fields of other particles. The theory of fiber spaces and connections on them[10] enables one to give a rigorous mathematical description of the question studied in this section. In accordance with the general plane of the review, we restrict the treatment to the decisive concepts, of which the fundamental are the holonomy group and a connection with a given holonomy group.

Because the symmetries that are considered in elementary-particle physics are broken symmetries, the geometrical point of view can be useful when one wishes to establish the nature of the possible symmetry breakings. In Sec. 4 we give an example of the determination of the properties of phenomenological Lagrangians from geometrical requirements that are less stringent than a symmetry requirement. In Sec. 5 we consider Cartan forms and we give a list of the main relations between these forms used in the method of phenomenological Lagrangians.

More complicated symmetry groups, containing as subgroups not only groups of internal symmetries but also the Poincaré group, are considered in Sec. 6. It is shown that phenomenological Lagrangians for these groups are constructed naturally not on the basis of an invariant quadratic form, as was the case for groups of internal symmetries, but on the basis of a four-volume that is invariant under transformations of the group in a certain n-dimensional space determined by the structure of the given symmetry group. The use of Cartan forms is also convenient in this more general case. As an example, we consider phenomenological Lagrangians for a group of conformal transformations.

1. COVARIANT PROPERTIES OF THE S MATRIX UNDER FIELD TRANSFORMATIONS

For a given transformation (4), the determination of phenomenological Lagrangians for Goldstone particles reduces to finding invariants of these transformations containing a minimal number of derivatives. Because the space G/H is homogeneous with respect to transformations of G, there do not exist invariant functions that do not contain derivatives of A^{α} and are different from a constant. Therefore, the minimal number of derivatives satisfying the requirement of relativistic invariance is two, and the most general expression for the effective interaction Lagrangian in the long-wave limit has the form

$$L = g_{\mu\nu}(A^{\alpha})\,\partial A^{\mu}\,\partial A^{\nu}/2. \tag{5}$$

Interaction processes between Goldstone particles in the long-wave limit are determined by pole diagrams of tree-type for the S matrix:

$$S = \left[\frac{(i)^n}{n!}\right] \int d^4x_1 \ldots \int d^4x_n T^*(L'(x_1)\ldots L'(x_n)), \tag{6}$$

where L' is the interaction Lagrangian,

$$L' = \sum_{k=1} L^{(k)};$$

$$L^{(k)} = \partial^k_{\cdot\cdot} g_{\mu\nu}(0)\, A^{\cdot}_{\cdot} \ldots A^{\cdot}\, \partial A^{\mu}\,\partial A^{\nu}/2\cdot k! \tag{7}$$

(to simplify the equations in expressions that are symmetric in the operators A^{α}, we replace the indices of the internal variables by dots); T^* is the operator of chronological ordering, which is determined when derivatives of the field operators exist by the relation

$$T^*(\partial A(x), B(y)) = \partial T(A(x), B(y)), \tag{8}$$

where T is the ordinary chronological product. When derivatives are absent, T^* is identical to T.

We show that the elements of the S matrix defined by (5)-(8) do not depend on the choice of the fields $A^{\alpha}(x)$ if all the particles are on the mass shell $\partial^2 A^{\alpha} = 0$.

Under an arbitrary transformation of the fields,

$$A^{\mu'} = A^{\mu'}(A^{\mu}), \tag{9}$$

the derivatives ∂A^{μ} transform as vectors in a space in which the A^{μ} are coordinates, and $g_{\mu\nu}(A^{\alpha})$ transforms as a tensor and can be identified with the metric tensor of this space.

To prove that the S matrix on the mass shell does not depend on the choice of the fields, it is sufficient to show that it can be represented as a function of quantities explicitly covariant with respect to the transformations (9).

Let us begin with the simplest examples. Processes in which three particles participate are described by diagrams with one vertex:

$$L^{(1)} = \partial_{\alpha} g_{\mu\nu} A^{\alpha}\,\partial A^{\mu}\,\partial A^{\nu}/2. \tag{10}$$

Using the well known relation between the derivatives of the metric tensor and the Christoffel symbol,

$$\partial_{\alpha} g_{\mu\nu} = \Gamma_{\mu\alpha,\,\nu} + \Gamma_{\nu\alpha,\,\mu}, \tag{11}$$

the symmetry properties of these with respect to transposition of the first two subscripts,

$$\Gamma_{\alpha\beta,\,\nu} = \Gamma_{\beta\alpha,\,\nu}, \tag{12}$$

and the conservation of four-momentum at the vertex, which corresponds to vanishing of the derivatives of the product of all the operators in the vertex, we can write (10) in the form

$$L^{(1)} = -\Gamma_{\cdot\cdot,\,\rho} A^{\cdot} A^{\cdot}\,\partial^2 A^{\rho}/2. \tag{13}$$

Because of the presence of the operator $\partial^2 A^{\rho}$ in (13), this expression vanishes if all the particles are on the mass shell.

Processes in which four particles participate are described by a diagram with one vertex,

$$L^{(2)} = \partial^2_{\cdot\cdot} g_{\mu\nu} A^{\cdot} A^{\cdot}\,\partial A^{\mu}\,\partial A^{\nu}/2\cdot 2!, \tag{14}$$

and the pole diagram with two vertices (13).

In the cases when the couplings of the operators in the two different vertices (13) do not contain operators with the derivatives $\partial^2 A^{\rho}$, the corresponding matrix element has a pole because of the presence of the propagator and vanishes on the mass shell because of the presence of the operators $\partial^2 A^{\mu}$ in both vertices.

When there are couplings that contain $\partial^2 A^{\mu}$ in either of the vertices, the pole graph reduces to a certain effective contact interaction; for as can be seen directly from the relation

$$\partial^2 \langle T(A^{\mu}(x) A^{\nu}(y))\rangle = -ig^{\mu\nu}\delta(x-y), \tag{15}$$

the denominator of the pole propagator cancels in this case against the square of the four-momentum in the numerator, which arises as a result of the presence of the square of the derivative.

The resulting contact interaction taken together with the contribution from (14) can be written in the form

$$(\partial^2_{\cdot\cdot} g_{\mu\nu} - \partial_{\rho} g_{\mu\nu}\Gamma^{\rho}_{\cdot\cdot} - 4\partial_{\cdot} g_{\rho\nu}\Gamma^{\rho}_{\cdot\mu} + 2\Gamma_{\mu\cdot\cdot,\,\rho}\Gamma^{\rho}_{\nu\cdot})\, A^{\cdot} A^{\cdot}\,\partial A^{\mu}\,\partial A^{\nu}/4. \tag{16}$$

The expression (16) is not covariant and therefore depends on the choice of the fields. We shall show that this expression can be transformed to a form that contains an explicitly covariant part that is nonvanishing on the mass shell and a noncovariant part that vanishes on the mass shell. For this, we expand the four-index quantity in the brackets in the expression (16) with respect to the Young tableaux , , and with respect to permutations of the corresponding indices. The part of the expression (16) corresponding to the second and the third of these tableaux can, on the basis of the same transformations as we used for the transition from (10) to (13), be represented in a form that contains three operators without derivatives and one operator of the form $\partial^2 A^\rho$. Indeed, for both the Young tableaux considered there always exist three symmetric indices, two of which are contracted against operators without derivatives and one with a first derivative. Because of this symmetry, such an expression can be written in the form of a derivative of the product of three operators, and then, using the conservation of four-momentum, one can transfer this derivative to the remaining operator ∂A^ρ, which leads to the operator $\partial^2 A^\rho$.

For the first of the above Young tableaux, such a transformation is impossible, since the expression obtained as the result of such a transformation, if it were possible, would be symmetric with respect to at least three indices, which contradicts the symmetry of this Young tableau.

When transformed as we have indicated, the expression (16) takes the form

$$-R_{\cdot\mu,\,\cdot\nu} A^{\cdot} A^{\cdot}\, \partial A^{\mu}\, \partial A^{\nu}/6 + [\Gamma^{\rho}_{\cdot\cdot}\Gamma_{\cdot\nu,\,\rho} + 3\Gamma^{\rho}_{\cdot\cdot}\Gamma_{\cdot\rho,\,\nu} - \partial_{\cdot}\Gamma_{\cdot\cdot,\,\nu}]\, A^{\cdot} A^{\cdot} A^{\cdot}\, \partial^2 A^{\nu}/6, \tag{17}$$

where $R_{\alpha\mu,\,\beta\nu}$ is the curvature tensor of the space G/H at the origin:

$$R_{\alpha\mu,\,\beta\nu} = \partial_\beta \Gamma_{\mu\nu,\,\alpha} - \partial_\nu \Gamma_{\mu\beta,\,\alpha} + \Gamma^{\gamma}_{\beta\mu}\Gamma_{\alpha\nu,\,\gamma} - \Gamma^{\gamma}_{\mu\nu}\Gamma_{\alpha\beta,\,\gamma}. \tag{18}$$

The representation (17) solves our problem: Because of the covariance of the curvature tensor, the first term does not vary under transformations that leave the coordinate origin fixed; the second term depends on the coordinate system, but because of the presence of the operators $\partial^2 A^\nu$ it vanishes on the mass shell.

There is no great difficulty in carrying through a similar treatment for a large number of particles. The pole terms in the matrix elements either join explicitly covariant blocks to other such blocks, or they join them to noncovariant terms with $\partial^2 A^\rho$ on the external lines. In this case, the result does not depend on the choice of the fields on the mass shell. At least one of the operators responsible for the pole term contains $\partial^2 A^\rho$, and in this case the pole term reduces to a contact interaction. The total contribution of all the contact interactions with the same number of field operators can be represented as a sum of terms of two types that belong, respectively, to the Young tableau (type I) or to the tableaux and (type II) under permutation of the indices of the field operators. These tableaux exhaust all possible Young tableaux, since only two operators with derivatives occur in all the contact interactions. Terms corresponding to type I give explicitly covariant expressions. Terms of type II reduce to expressions containing $\partial^2 A^\rho$ and either vanish if the particles are on the mass shell or lead to further reductions of pole diagrams to contact interactions for processes with a larger number of particles.

If there is a large number of particles, the above procedure becomes very cumbersome. Therefore, we prove its validity for an arbitrary number of particles without constructing the explicit expressions for the corresponding matrix elements.

Let us consider the general structure of a matrix element with n external lines in the pole approximation under the assumption that all the matrix elements with fewer lines are known. For this we take an arbitrary diagram with n external lines and k + 1 vertices. Accordingly, in this diagram we have k internal lines. We separate the component parts of the diagram in two different ways. The first consists of separating in a diagram any one of the vertices and separating from this vertex the parts connected to it by one internal line. With this method there will be k + 1 different representations of the original diagram corresponding to the number of vertices. In the second method, we divide diagrams into two parts by cutting one of the internal lines. With this method of division there will be k different representations of the original matrix element corresponding to the number of internal lines.

Since each of these representations of the original diagram gives the same matrix element and since the number of vertices is always one more than the number of internal lines, it follows that, taking the sum of all the different representations of the diagrams in the first method and subtracting from it all the representations of the diagram in the second method, we again obtain the matrix element corresponding to the diagram. Summing over all different diagrams, we obtain for the first method of division a sum of diagrams in which one of the vertices of the original Lagrangian is joined to a certain number of internal lines with total matrix elements of the lower approximations. In the second method of division we obtain diagrams in which two matrix elements of lower order are joined by one internal line.

For diagrams with five external lines, a systematic representation of a matrix element as a sum of pole diagrams and as pole diagrams with matrix elements of lower orders as vertices is shown in Figs. 1 and 2. To each of the diagrams of Fig. 2 there correspond definite diagrams of Fig. 1. To the diagrams A, B, and D there correspond one each of the diagrams a, b, and c. To each of the diagrams C and E there correspond two diagrams b and c. Since the diagram E gives a contribution with a minus sign, the total contribution of the diagrams in Fig. 2 is the same as in Fig. 1.

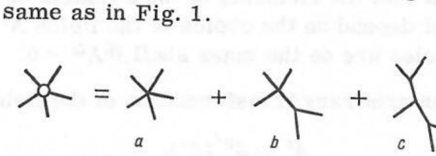


Fig. 1. Representation of matrix element as a sum of pole diagrams.

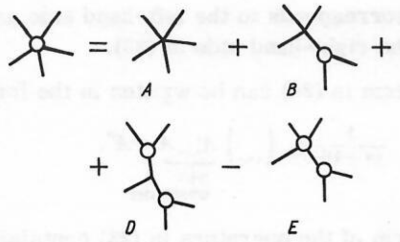


Fig. 2. Representation of matrix element as a sum of pole diagrams with matrix elements of lower orders (with fewer lines) as effective vertices.

Let us now consider the conditions under which a pole diagram can be reduced to a contact interaction. As is evident from the example considered below, to reduce the diagram it is necessary to cancel the denominator of the pole term against the square of the momentum of a virtual particle, which is possible only if the operator $\partial^2 A^\rho$ is present in the corresponding vertex.

Assuming that in the vertices corresponding to the matrix elements of lower orders we have separated the parts in which all the poles are reduced to contact interactions, we decompose these vertices into two terms corresponding to Young tableaux of the types I and II, respectively.

For the reduced matrix elements corresponding to terms of a tableaux of type II one can readily construct a system of equations which enables one to obtain their values in higher approximations from known matrix elements in lower approximations. Indeed, to each matrix element of such a form there corresponds a graph in which one of the vertices contains the operator ∂^2 A on an external line and does not contain such operators on internal lines. We cut all internal lines joined to this vertex; then the parts of the graph that have been cut off must contain the operators ∂^2A on these lines in order to ensure reduction of the poles corresponding to the cut lines, i.e., the cut off parts must correspond to matrix elements of lower orders of the type II. The resulting system of equations can be represented graphically as a sum of all possible diagrams in each of which there is one vertex corresponding to the original Lagrangian with operator ∂^2A on an external line and the remaining lines joining this vertex are either external or joined to matrix elements of type II in lower approximations. For a four-line diagram this method of representing a matrix element for the given case is shown in Fig. 3.

The system of equations for the total reduced matrix element, i.e., containing terms of type I and II, is obtained as a consequence of the system of equations considered above. For the system of equations corresponding to Fig. 1 to lead to reduced matrix elements, it is necessary that one of the vertices joined to each internal line

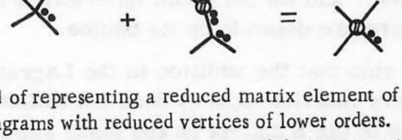


Fig. 3. Method of representing a reduced matrix element of type II as a sum of pole diagrams with reduced vertices of lower orders. The dots on the lines stand for operators containing second derivatives with respect to the spatial variables.

contain the operator ∂^2A, i.e., such an operator must be contained in either the vertex of the original Lagrangian or in the effective matrix element of lower order. However, it is readily seen that as a result of equations of the type shown in Fig. 3, the matrix elements for diagrams for which the reduction holds as a result of the presence of the operators ∂^2A only in the vertices of the original Lagrangian cancel against the matrix elements corresponding to diagrams of the type E in Fig. 2, in which only one of the vertices contains the operator ∂^2A.

The method of representing the system of equations for determining the total reduced matrix element for the case of a five-line diagram is shown in Fig. 4. In the general case, the reduced matrix element for an n-line diagram is given by a sum of all possible diagrams, in each of which there is one vertex of the original Lagrangian joined in all possible manners to reduced diagrams of type II of lower approximations with operators ∂^2A in an internal line minus the sum of all possible diagrams with two reduced vertices of type II with operators ∂^2A from both vertices in an internal line. To construct the elements of the S matrices on the basis of reduced matrix elements, it is necessary to allow for all possible contractions of operators that do not contain the operators ∂^2A.

Note that the equations considered above are valid in the more general case of particles with arbitrary spin and mass; for in deriving these equations we have used only the condition that the denominators of the pole terms cancel against the corresponding factors in the effective matrix elements; the form of the equation of motion of the particles is not important. In Appendix I we derive the equations considered above on the basis of a quasiclassical approach.

We now show that a reduced n-line diagram of the type I does not depend on the choice of the field operators. To do this we compare the procedure we have described above with the well-known procedure for going over from an arbitrary coordinate system in Riemannian space to a so-called normal coordinate system, which is defined by the conditions

$$\partial^n_{(\ldots}\Gamma^{\gamma}_{\alpha\beta)} = 0; \quad n = 0,\ 1,\ 2,\ \ldots \tag{19}$$

(the brackets denote symmetrization with respect to the corresponding indices) and in which the nonvanishing derivatives of the metric tensor are determined solely in terms of explicitly covariant quantities: the curvature tensor and its covariant derivatives.[4]

The conditions (19) are equivalent to the conditions

$$\partial^n_{(\ldots}\Gamma_{\alpha\beta),\,\gamma} = 0; \quad n = 0,\ 1,\ 2,\ \ldots \tag{20}$$

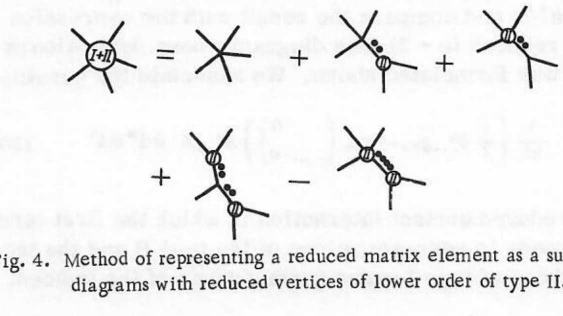


Fig. 4. Method of representing a reduced matrix element as a sum of pole diagrams with reduced vertices of lower order of type II.

for the Christoffel symbols with subscripts.

In terms of the derivatives of the metric tensor, the condition (20) corresponds to the vanishing of their part that transforms under transposition of the indices in accordance with Young tableaux of the type II.

Suppose g_{mn} and $\tilde{g}_{\mu\nu}$ are the metric tensors, respectively, in the arbitrary and the Riemannian normal coordinate system and

$$\left.\begin{aligned} \partial A^m/\partial A^\mu &= \binom{m}{\mu}; \\ \partial^n A^m/\partial A^\cdot\,\partial A^\cdot \ldots \partial A^\mu &= \binom{m}{\ldots\,\mu} \end{aligned}\right\} \tag{21}$$

are the derivatives of the transformation from one coordinate system to the other, i.e.,

$$\tilde{g}_{\mu\nu} = g_{mn}\binom{m}{\mu}\binom{n}{\nu}. \tag{22}$$

Differentiating (22) a total of n times with respect to the coordinates in the Riemannian coordinate system and equating to zero the contributions to the derivatives from the Young tableaux of type II, we can determine the derivatives for $\tilde{g}_{\mu\nu}$ and the coefficients (21) from given g_{mn}.

The derivative of $\tilde{g}_{\mu\nu}$ has the form

$$\begin{aligned} \partial^e \ldots \tilde{g}_{\mu\nu} &= \partial^e \ldots g_{mn}\binom{m}{\mu}\binom{n}{\nu}\binom{\cdot}{\cdot}\ldots\binom{\cdot}{\cdot} + \ldots \\ &+ \Big(e!/\prod_{p=1}^{\infty}\alpha_p!\,(p!)^{\alpha_p}\,n_\mu!\,n_\nu!\Big)\,\partial^k \ldots g_{mn}\overbrace{\binom{\cdot}{\cdot}\ldots\binom{\cdot}{\ldots}}^{k\text{ brackets}}\binom{m}{\ldots\,\mu} \\ &\times\binom{n}{\ldots\,\nu} + \ldots + g_{mn}\binom{m}{\ldots\,\mu}\binom{n}{\nu} + g_{mn}\binom{m}{\mu}\binom{n}{\ldots\,\nu}, \end{aligned} \tag{23}$$

where the k dots in $\partial^k \ldots g_{mn}$ and accordingly the k upper dots in the brackets $(\dot{\ldots})$ denote indices with respect to which there is summation: α_p is the number of brackets $(\dot{\ldots})$ with p subscripts; n_μ and n_ν are the numbers of subscripts exluding μ and ν, in the brackets $\binom{m}{\ldots\mu}$ and $\binom{n}{\ldots\nu}$. The numbers α_p satisfy the relations: $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + n_\mu + n_\nu = l$; $\alpha_1 + \alpha_2 + \alpha_3 + \ldots = k$, $k = 0, 1, 2, \ldots$. The last two terms in (23) refer to representations of the type II under permutations of the free indices and are determined uniquely by the requirement that the contribution of such representations to $\partial^e \ldots g_{\mu\nu}$ vanish.

The matrices $\binom{m}{\mu}$ at the origin are arbitrary, which corresponds to the freedom in the choice of the metric tensor at the origin; we assume that $\binom{m}{\mu}\big|_{A=0} = \delta^m_\mu$. Then all the quantities $\binom{m}{\ldots\mu}$ are determined in the last terms in (23).

We multiply (23) by the field operators $A^\cdot_{\ldots}A^\cdot\partial A^\mu\partial A^\nu$ and by $e!/2$ and compare the result with the expression for the reduced (e + 2)-line diagram whose derivation procedure was formulated above. We associate the quantity

$$\frac{1}{n!}\left(\frac{1}{2}\,\partial^n_{\ldots}\tilde{g}_{\mu\nu} - g_{\mu n}\binom{n}{\ldots\,\nu}\right)A^\cdot_{\ldots}A^\cdot\,\partial A^\mu\,\partial A^\nu \tag{24}$$

with a reduced contact interaction in which the first term corresponds to representations of the type II and the second to those of type I under permutations of the indices. Note that (24) corresponds to the left-hand side and the last terms of the right-hand side in (23).

The last term in (24) can be written in the form

$$\frac{1}{(n+1)!}\,g_{\rho\cdot}\binom{\cdot}{\ldots}\underbrace{A^\cdot_{\ldots}A^\cdot}_{\substack{n+1\\ \text{operators}}}\partial^2 A^\rho. \tag{25}$$

The general term of the operators in (23) containing the derivatives $\partial^k \ldots g_{mn}$ corresponds to a diagram with one vertex (7) and several vertices (25). The brackets $(\dot{\ldots})$ correspond to the case when the operator $\partial^2 A$ in the vertex (25) is associated with an operator A without derivatives in the vertex (7), and the brackets $\binom{m}{\ldots\mu}$ and $\binom{n}{\ldots\nu}$ correspond to analogous couplings, but with the operators ∂A in the vertex (7). The factors g_ρ in the contraction against $g^{\rho\nu}$ in the couplings of the operators (15) go over into Kronecker deltas. The factorials $(p!)^{\alpha_p}$, $n_\mu!$, and $n_\nu!$ in the denominator of the coefficient arise because of the corresponding factors in (25). At the same time allowance is made for the factors $(n_\mu + 1)$ and $(n_\nu + 1)$, which when there is a coupling of an operator with a derivative arise as a result of the differentiation of $n_\mu + 1$ operators without derivatives in (25). The factors $(\alpha_p!)^{-1}$ in the coefficients (23) appear as a result of the presence of $(k!)^{-1}$ in (7) and the factor $k!/\Pi(\alpha_p!)$, which gives the number of possible couplings of (7) with α_1, α_2, etc., identical vertices. The sign of the coefficient is determined by the factors $(i)^n$ in (5) and the factors $(-i)$ for each coupling of the operators (15) and is positive.

Similarly, the terms in (23) that do not contain derivatives of the metric tensor (k = 0) correspond to coupling of two vertices (25), and at the same time two operators $\partial^2 A$ are coupled in each vertex. The contraction of such operators is

$$\partial_x^2\partial_y^2\,\langle T(A^\mu(x),\ A^\nu(y))\rangle = i\partial_x\partial_y g^{\mu\nu}\delta(x-y). \tag{26}$$

When the derivatives are transferred to the corresponding vertices, we find, with allowance for (25), an expression for the terms with k = 0 in (23); however, because of the difference in the signs in (15) and (26), these differ by their sign from the last ones. This corresponds to the previously formulated rule for selecting the sign (see Fig. 4). Thus, we have shown that the problems of finding the derivatives of $\tilde{g}_{\mu\nu}$ in a normal coordinate system and constructing reduced matrix elements with the Lagrangian (10) are mathematically equivalent. Therefore, for constructing the S matrix on the mass shell, irrespective of the original coordinate system, the result always corresponds to a normal coordinate system and does not depend on the choice of the original system.

However, it turns out to be important that the transformation of the coordinate system leaves fixed the point in the neighborhood of which the expansion in powers of the field is made. But if the series expansion is made in powers of $A - A_0$, the S matrix is a function of the curvature tensor and its covariant derivatives at the point A_0 and therefore depends on its choice.

Note also that the addition to the Lagrangian (4) of an arbitrary function $\psi(A(x))$ does not violate the covariant properties of the S matrix on the mass shell. The presence of the additional term subject to the condition $\partial_\mu\varphi(0) =$

0 leads, first, to the appearance of a mass term in the equations of motion of a free particle,

$$\partial^2 A^\mu + M^\mu_\nu A^\nu = 0, \tag{27}$$

where

$$M^\mu_\nu = -g^{\mu\rho}\varphi_{,\,\rho\nu}(0), \tag{28}$$

and in the elements of the S matrix off the mass shell; and, second, to the appearance of new vertices not containing the derivatives ∂A^μ and depending only on the derivatives of the function φ in the normal coordinate system:

$$\frac{1}{n!}\,\partial^n_{\cdot\cdot\cdot}\widetilde{\varphi}A^{\cdot}_{\cdot}\ldots A^{\cdot}, \quad (n \geqslant 3). \tag{29}$$

To prove this assertion, it is sufficient to consider the expression for the transformation of the n-th derivative of a scalar function under a variation of the coordinate system:

$$\frac{1}{n!}\,\partial^n_{\cdot\cdot\cdot}\widetilde{\varphi} = \sum_k \frac{1}{\prod\limits_p \alpha_p!\,(p!)^{\alpha_p}}\,\partial^k_{\cdot\cdot\cdot}\varphi\,\underbrace{\binom{\cdot}{\cdot}\ldots\binom{\cdot}{\cdot}}_{\alpha_1}\,\underbrace{\binom{\cdot}{\cdot\cdot}\ldots\binom{\cdot}{\cdot\cdot}}_{\alpha_2}\ldots \tag{30}$$

and to compare this expression with the result that is obtained from a perturbation-theory calculation for graphs of the form shown in Fig. 4. These graphs lead to the sum (30), in which the summation index is $k \geq 3$. The missing term in the sum (30) with $k = 2$ can be written in the form of two terms:

$$\partial^2_{\cdot\cdot}\varphi\binom{\cdot}{\cdot}\underbrace{\binom{\cdot}{\cdots}}_{n-1} + \partial^2_{\cdot\cdot}\varphi\binom{\cdot}{\cdots}\binom{\cdot}{\cdots}, \tag{31}$$

the first of which corresponds to the addition of a mass term to the function (25), and the second arises because of the presence of a mass term in graphs of the form shown in Fig. 4, with minus sign.

Because the new vertices that arise from the expansion of the function φ in powers of the field do not contain derivatives of the field operators with respect to the spatial variables, the presence of such vertices does not affect the form of the transformation coefficients and they correspond as before to a transition to a normal coordinate system, which proves our assertion.

2. SYMMETRY OF PHENOMENOLOGICAL LAGRANGIANS AND ADLER'S PRINCIPLE

We now study the consequences of symmetry of the phenomenological Lagrangians with respect to transformations of the group G. We shall first study the simplest but, evidently, most important case for physical applications when the space G/H is a symmetric space,[10] i.e., a space for which at all points

$$R_{\alpha\beta\gamma\delta,\,\mu} = 0, \tag{32}$$

where $R_{\alpha\beta\gamma\delta,\mu}$ is the covariant derivative of the curvature tensor.

Symmetric spaces are intimately related to the geometry of semisimple Lie groups. For an arbitrary semisimple Lie group G one can always distinguish a subgroup H such that the following dispersion relations hold:

$$\left.\begin{aligned} [X_\alpha, X_\beta] &= ib^c_{\alpha\beta}Y_c; \\ [X_\alpha, Y_\beta] &= ia^\gamma_{\alpha\beta}X_\gamma; \\ [Y_a, Y_b] &= ic^c_{ab}Y_c, \end{aligned}\right\} \tag{33}$$

where Y_a are the generating operators of H. In the space G/H in this case one can always introduce a metric in such a manner that it is a symmetric space. In particular, the symmetry of chiral groups leads to the study of symmetric spaces.

For symmetric spaces, as a consequence of (32), the metric tensor in a normal coordinate system is uniquely determined by the value of the curvature tensor at the origin and has the form

$$g_{\mu\nu}(A^\rho) = g_{\mu\nu} + \sum_{k=1}^{\infty} \frac{(-1)^k\, 2^{2k+1}}{(2k+2)!}\,(m^k)^\rho_\mu\, g_{\rho\nu}, \tag{34}$$

where

$$m^\nu_\mu = R^\nu_{\mu\alpha\beta}A^\alpha A^\beta; \tag{35}$$

the curvature tensor is expressed in terms of the structure constants in the commutation relations (33):

$$R^\nu_{\mu\alpha\beta} = -b^c_{\mu\alpha}a^\nu_{\beta c}. \tag{36}$$

The relations (28)–(31) completely determine the effective interaction Lagrangian (4) for the given symmetry type.

The elements of the S matrix, calculated on the basis of the phenomenological Lagrangian (5) defined in terms of (34), have the following property, which does not depend on the number of particles that participate in an interaction process.

If all the particles are on the mass shell, then when the four-momentum of one of the particles tends to zero, the element of the S matrix vanishes (Adler's principle[16]). In what follows we shall make this definition somewhat more precise.

For the case of four interacting particles, in accordance with (17) and (34),

$$L^{(2)} = -R_{\cdot\mu,\,\cdot\nu}A^{\cdot}A^{\cdot}\,\partial A^\mu\,\partial A^\nu/6. \tag{37}$$

If one of the particles has zero momentum, the expression (35) can be written in the form

$$L^{(2)} = R_{\cdot\mu,\,\cdot\nu}A^{\cdot}A^{\cdot}\overline{A}^\mu\partial^2 A^\nu/3, \tag{38}$$

where $\overline{A}^\mu$ is the operator of the particle with zero momentum, and it therefore vanishes on the mass shell.

The transition from (37) to (38) is associated with the fact that the symmetry of the curvature tensor under permutations of the indices corresponds, as we have shown above, to the Young tableau $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; & \; \\ \hline\end{array}$. Since $\partial\overline{A}^\mu = 0$, the operator $\overline{A}^\mu$ can be eliminated when one considers permutations of the indices; then the three remaining operators transform in accordance with the Young tableau $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$, and they can therefore be transformed to a form in which one of the operators contains a second derivative. A similar situation obtains for a larger number of particles. After the operator A^μ has been separated, the pole terms can be reduced to contact interactions in such a way that the sum of the original and the reduced contact interactions can be transformed to

$$L^{(2n)} = [2^{2n}/(2n)!]\, B_n\, (m^n)^\rho_\mu\, g_{\rho\nu} \bar{A}^\mu \partial^2 A^\nu, \tag{39}$$

where B_n is the Bernoulli number and therefore vanishes on the mass shell.

We shall not dwell on the derivation of (39), since below we shall consider in detail the equivalent of Adler's principle to invariance of the phenomenological Lagrangians, without restricting ourselves to the class of symmetric spaces. Note that (39) is not the complete element of the S matrix, but merely some effective vertex. To construct the complete matrix element, it is necessary to consider the contractions of the operators in the vertices (34) and (39) that do not contain the operators $\partial^2 A$.

The requirement that a phenomenological Lagrangian be invariant under transformations of the group G can be written in the form

$$\xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\alpha} \partial_\nu \xi^\alpha + g_{\nu\alpha} \partial_\mu \xi^\alpha = 0 \tag{40}$$

(Killing's equation) where ξ^α is a vector field on G/H associated with any of the generating operators of the group by the relation

$$X = i\xi^\alpha \partial_\alpha. \tag{41}$$

We shall consider the transformations (41) corresponding to displacements of the origin. For such transformations, the ξ^α can be written in the form

$$\xi^\alpha = \varepsilon^\alpha + \varepsilon^\beta f^\alpha_\beta, \tag{42}$$

where ε^α is a constant vector. The components of ε^α can be chosen arbitrarily, since all the generating operators of the cosets can be represented in the form (42), and there are as many such independent operators as there are dimensions of the space G/H.

Differentiating (40) a total of n times, we obtain

$$\begin{aligned} \partial^{n+1}_{\dots\alpha} g_{\mu\nu} + \dots + \frac{n!}{m!\,(n-m)!}\, \partial^m_{\dots} f^\beta_\alpha \partial^{n-m+1}_{\dots\beta} g_{\mu\nu} \\ + \dots + \frac{n!}{m!\,(n-m)!}\, \partial^{m+1}_{\dots\nu} f^\beta_\alpha \partial^{n-m}_{\dots} g_{\mu\beta} + \dots \\ + \frac{n!}{m!\,(n-m)!}\, \partial^{m+1}_{\dots\mu} f^\beta_\alpha \partial^{n-m}_{\dots} g_{\nu\beta} + \dots \\ + g_{\mu\beta} \partial^{n+1}_{\dots\nu} f^\beta_\alpha + g_{\nu\beta} \partial^{n+1}_{\dots\mu} f^\beta_\alpha = 0. \end{aligned} \tag{43}$$

The relations (43) are necessary and sufficient for Adler's principle to hold. Its fulfilment means that an element of the S matrix with fixed number of external lines can be represented as a sum of pole diagrams with reduced matrix elements as vertices and one vertex of the form

$$\underset{n-2}{f^{(n)}_{\dots\alpha,\,\rho}}\, A^{\cdot} A^{\cdot}_{\cdot} \dots A^{\cdot} \bar{A}^\alpha \partial^2 A_\rho, \tag{44}$$

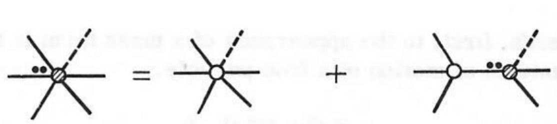


Fig. 5. Representation of the reduction of diagrams of lower order leading to Adler's principle: The dashed line represents the line corresponding to the particle with zero momentum.

containing the operator $\bar{A}^\alpha$ of the particle with zero momentum. Since all the vertices are assumed to be reduced, the couplings of operators corresponding to poles do not contain the operators $\partial^2 A$. Since the vertex (44) contains the operator $\partial^2 A$ for an external particle, it follows, if all the particles are on the mass shell, that this vertex vanishes. If on the transition to the limit the denominators of the poles do not vanish, then the complete matrix element vanishes in this limit.

We now show that if the reduced vertices containing the operator $\bar{A}^\alpha$ with n and fewer external lines have the form (44), it follows from the symmetry requirement for the Lagrangian (40) that the reduced vertices with n + 1 external lines also have the form (44). As will be seen directly from the proof, the converse is also true, i.e., if the reduced vertices containing the operator $\bar{A}$ have the form (44) for an arbitrary number of external lines, then the relation (40) holds, and the Lagrangian is invariant under a certain transitive group.

For simplicity we shall use a normal coordinate system. In this case the reduced vertices that do not contain $\bar{A}$ coincide with the derivatives of the metric tensor. To find a reduced matrix element with n+1 external lines, we must allow for all possible reductions of the pole terms. Since only the expressions (44) contain the operators $\partial^2 A$, all possible diagrams leading to reduction have the form shown in Fig. 5, in which the dashed line represents the operator of the particle with zero momentum. In the general case, the diagrams leading to a reduction have the form of pole diagrams with two vertices, one of which corresponds to the terms of the expansion of the Lagrangian (7) in powers of the field A, and the other has the form (44). At the same time, the contraction corresponding to the pole contains the operator $\partial^2 A$ in (44).

We multiply (43) by $\bar{A}^\alpha A^{\cdot}_{\cdot} \dots A^{\cdot} \partial A^\mu \partial A^\nu / 2 \cdot n!$; then the contribution of these diagrams coincides with the expression (43) without the last two terms, provided that

$$f^k_{\dots\alpha,\,\rho} = \partial^{k-z}_{\dots} f^\beta_\alpha g_{\beta\rho} / (k-2)!. \tag{45}$$

At the same time we have the following corresponding for the terms in (43): the first to the interaction (7), the second and the third, and also the fourth to a pole diagram in which the operator $\partial^2 A$ in (44) is coupled, respectively, to A or to ∂A. The presence of the last two terms in (43) means that when symmetry is present, the sum of the diagrams under consideration can be transformed to the form (44), i.e., Adler's principle is also satisfied for matrix elements with n + 1 particles.

Conversely, if Adler's principle is satisfied and (44) holds, this means that in the total matrix element there are no representations with a Young tableau of type I with respect to permutations of the indices of the operators of the particles with nonvanishing momentum and, as a result, the relations (43) hold. Thus, Adler's principle is equivalent to requiring invariance of the Lagrangian under transformations of G.

3. INTERACTION OF GOLDSTONE PARTICLES WITH OTHER PARTICLES

We now consider how the method of phenomenological Lagrangians enables one to describe the interaction of Goldstone particles with other particles. Having in mind an application of the method not only to exact but also to broken symmetries, we consider first the way in which an interaction with other fields can be included for the case when the phenomenological Lagrangian of the Goldstone particles has the form (5) with an arbitrary metric form, and only then do we turn to consider the case of exact symmetry.

Accordingly, for the metric form (5) we can introduce at each point of space A^{α} ($\alpha = 1, 2, \ldots, n$) a certain orthogonal basis. Specifying in each basis as new independent variables the components ψ_i of some representation of the group SO, we turn to the study of the extended, so-called fiber space with coordinates A^{α} and ψ_i, which can be used to describe both Goldstone and other particles.

Defining parallel transport and covariant differentiation in some manner, for example, by introducing an ordinary Riemannian connection, we can include in a covariant manner terms in the phenomenological Lagrangian containing any representation of SO(n) and its derivatives. This procedure is analogous to the well-known procedure for including spinors in general relativity.

For the method of phenomenological Lagrangians the following circumstance is important. The introduction of an orthogonal basis at different points of space is not a unique procedure. On the transition to a new basis the covariance of the expressions in the phenomenological Lagrangian is preserved, but the explicit functional dependence of these expressions on A^{α} has a different form in the different bases in such a way that the elements of the S matrix are also different in these bases.

One can show (see ref. 4) that if all the particles are on the mass shell, the elements of the S matrix obtained by summing tree-type diagrams can be rearranged into a form that does not depend on the choice of the basis. The elements of the S matrix off the mass shell contain additional terms that do depend on the choice of the basis and which are related in a simple manner to the function of the transformation from an arbitrarily chosen basis to a basis defined by parallel transport along geodesics of the basis at the origin. The procedure of rearranging the matrix elements used to prove these assertions is similar to that considered in Sec. 2 and is based on expansion or products of field operators in Young tableaux and a reduction of pole terms for the fields ψ_i, and we shall not dwell on it in detail.

More important from the physical point of view is the fact that multiplets of other particles may not allow their classification in accordance with representations of SO(n), but allow only a restricted classification with respect to representations of one of its subgroups. For example, for the case of broken SU(3) ⊗ SU(3) symmetry, the form (5) corresponds to an eight-dimensional Riemannian space, but the known multiplets of baryons and other particles do not allow classification according to the representations of SO(8), but only a restricted classification according to the representations of SU(3), which is a subgroup of SO(8). In this case, the requirement that there exist covariant derivatives for individual SU(3) multiplets is a strong restriction on the structure of the Riemannian space.

In the general case, for the introduction of covariant derivatives for each representation of a subgroup H of SO(n), it is necessary and sufficient that the space be such that when one passes around any closed loop the different geometrical entities undergo a transformation of the subgroup H, i.e., that the group H be the so-called holonomy group of the space.[5] When a definite holonomy group H is present, the covariant derivatives for all multiplets of H are determined in a basis obtained by parallel transport along the curves of the basis at the origin, and in the general case they have the form

$$D\psi_i = \partial\psi_i + (\lambda_m)_i^k \, \theta^m (\partial A) \, \psi_k, \tag{46}$$

where the matrices λ^m are generators of H for the given representation. In the method of phenomenological Lagrangians the first term in (46) defines the ordinary derivative with respect to the space–time variables, and the second defines the correction due to the change of the basis on the transition from the point $A^{\alpha}(x)$ to the point $A^{\alpha}(x) + dx\, \partial A^{\alpha}(x)$.[6]

Under a transformation of the basis the covariant derivative (46) transforms in the same way as ψ_i. For a given holonomy group, quantities that are covariant with respect to local transformations of the bases correspond to invariants of the group H composed of ψ_i, $D\psi_i$, and the components $\omega_i(\partial A)$ of the vector ∂A^{α} in the basis under consideration.

We shall show that on the transition to the case when the space with the metric (5) is a homogeneous space with symmetry group G, the invariants with respect to the transformations of the subgroup H also become invariants with respect to transformations of G; for if we suppose that local bases defined by a parallel transport are rigidly connected to the space and we make a transformation of the space corresponding to the group G, the new bases at an arbitrary point will differ by a certain rotation from the old basis at the same point. It is easy to show that the resulting revolution is a transformation of the holonomy group.[7]

For different points of space the transformations of the bases are different. Because the components of the representation are referred to basis vectors at each point of space, the procedure described above defines a transformation of G for every representation of the holonomy group H. It also follows from the above procedure that any H-invariant combination constructed from components of different representations of H is also invariant under transformations of G defined in the above manner.

4. CARTAN FORMS

In the foregoing section we have shown that in the construction of phenomenological Lagrangians it is convenient to use the mathematical apparatus of differential geometry. Here we give some equations that enable us to obtain in a simple manner the fundamental relations of the method of phenomenological Lagrangians by using Cartan forms.

The description of the geometry of Riemannian spaces and spaces with affine connection on the basis of differential forms was introduced in the classical works of Cartan (see, for example, ref. 10 and 11).

The basis of such a description is the Cartan structure equations

$$\left.\begin{aligned}(\omega^i)' &= \omega^k \Lambda \omega_k^i - \tfrac{1}{2} T_{jk}^i \omega^j \Lambda \omega^k;\\ (\omega_i^j)' &= \omega_i^k \Lambda \omega_k^j - \tfrac{1}{2} R_{ikl}^j \omega^k \Lambda \omega^e,\end{aligned}\right\} \tag{47}$$

which establish the relation of the differential forms ω^i and ω_i^k to the curvature tensor R_{ikl}^j and the torsion tensor T_{jk}^i. The prime in (47) denotes outer differentiation and the sign Λ, the outer product of differential forms.

In a space with coordinates a^α in a certain basis that depends continuously on the parameters b^i, the differential form ω^i defines a vector da^α, i.e.,

$$\omega^i = \omega^i(a,\ b,\ da), \tag{48}$$

and the form

$$\omega_i^k = \omega_i^k(a,\ b,\ da,\ db) \tag{49}$$

defines a transformation of the components of a vector under a change of the basis:

$$dB^i(a,\ b) = -\omega_k^i(a,\ b,\ 0,\ db)\, B^k \tag{50}$$

and the covariant differential of the vector:

$$DB^i = \frac{\partial B^i}{\partial a}\, da + \omega_k^i(a,\ b,\ da,\ 0)\, B^k. \tag{51}$$

As a consequence of (47) and the definitions (50) and (51) the basis DB^i transforms in accordance with (50).

If the holonomy group of the space (see Sec. 4) is a subgroup of an affine group, the forms ω_k^i are a linear combination of the generators of the holonomy group:

$$\omega_k^i(a,\ b,\ da,\ db) = (\lambda_\alpha)_k^i\, \theta^\alpha(a,\ b,\ da,\ db), \tag{52}$$

and accordingly the transformation of the vector components (50) and the definition of the covariant derivatives (51) can be given for any representation of the holonomy group.

If the space under consideration is a homogeneous space with a transitive symmetry group, the Cartan forms can be expressed in terms of the differential forms ω^i and θ^α, which determine the structure of the group:[10,11]

$$\left.\begin{aligned}(\omega^i)' &= C_{k\alpha}^i \omega^k \Lambda \theta^\alpha + \tfrac{1}{2} C_{kl}^i \omega^k \Lambda \omega^l;\\ (\theta^\alpha)' &= \tfrac{1}{2} C_{\beta\gamma}^\alpha \theta^\beta \Lambda \theta^\gamma + \tfrac{1}{2} C_{kl}^\alpha \omega^k \Lambda \omega^l,\end{aligned}\right\} \tag{53}$$

where $C_{k\alpha}^i$, C_{kl}^i, C_{kl}^α, and $C_{\beta\gamma}^\alpha$ are the structure constants of the group, whose generators satisfy the commutation relations

$$\left.\begin{aligned}[Y_\alpha,\ Y_\beta] &= i\, C_{\alpha\beta}^\gamma Y_\gamma;\\ [X_k,\ Y_\alpha] &= i\, C_{k\alpha}^i X_i;\\ [X_k,\ X_l] &= i\, C_{kl}^\alpha Y_\alpha + i\, C_{kl}^m X_m.\end{aligned}\right\} \tag{54}$$

In the relations (54) the operators Y_α referring to the subgroup H are separated explicitly, and the generators X_i refer to definite representations of H. When the elements of the group are parametrized by the representation:

$$G(a,\ b) = K(a)\, H(b) \tag{55}$$

the differential forms ω^i and θ^α are defined as follows:

$$G^{-1}(a,\ b)\, dG(a,\ b) = i\, \omega^i(a,\ b,\ da)\, X_i + i\, \theta^\alpha(a,\ b,\ da,\ db)\, Y_\alpha. \tag{56}$$

The relations (53) are a consequence of the definition (56) and the commutation relations (54).

The structure equations of the group (53) can always be written in the form of Eqs. (47). As an example, let us consider an arbitrary semisimple group when there is chosen in it a subgroup such that the structure constants with their Latin indices vanish:

$$C_{ik}^l = 0. \tag{57}$$

Comparing Eqs. (47) and (53), we obtain

$$\omega_k^i = C_{k\alpha}^i \theta^\alpha \tag{58}$$

and

$$T_{jk}^i = 0;\quad R_{ikl}^j = -C_{i\alpha}^j C_{kl}^\alpha. \tag{59}$$

Using the definition of the covariant derivative and the well-known properties of the structure constants, we can readily prove that

$$R_{ikl;\,m}^j = 0, \tag{60}$$

i.e., the space is symmetric.

Because the matrices $C_{k\alpha}^i$ realize a representation of the subgroup H [see (52)], H is the holonomy group of the symmetric space under consideration. As we have noted above, the covariant derivatives exist in this case for every representation of H.

In (47)-(58) we have retained the dependence on the parameters b, which characterize the choice of the local basis. This is convenient when one is considering the symmetry properties of the forms ω^i and θ^α; for if one applies an arbitrary element of G to (55) from the left, the expression (56) is not affected and therefore under transformations of the group the forms ω^i and θ^α are invariant, i.e.,

$$\omega^i(a,\ b,\ da) = \omega^i(a',\ b',\ da'); \tag{61}$$

$$\theta^\alpha(a,\ b, da,\ db) = \theta^\alpha(a',\ b',\ da',\ db'). \tag{62}$$

At the same time, it is important that the right-hand sides of (61) and (62) contain the transformed values of the parameters b and db.

In their turn, the relations (61) and (62) define transformations of the group:

$$a' = a'(a, g); \tag{63}$$

$$b' = b'(a, b, g). \tag{64}$$

The relation (63) defines a transformation of the coordinates under transformations of the group; the relation (64) shows that the basis at the point with coordinates a is transformed to the new basis.

For a fixed choice of the basis, for example, for b = 0, the differential forms ω^i and θ^α are not invariant. Their transformation under infinitesimally small transformations of the group can be determined on the basis of the structure equations (53).

Indeed, in this case (64) can be written in the form

$$b' = \delta b(a, 0, g). \tag{65}$$

The forms ω^i being independent of the differentials db, as a consequence of (53), we obtain

$$\delta\omega^i = -C^i_{h\alpha}\theta^\alpha(a, 0, 0, \delta b)\,\omega^h(a, 0, da), \tag{66}$$

i.e., under transformations of G the forms ω^i transform as representations of H with transformation parameters that depend on the coordinates a.

The transformation (66) applies to any representation of H if the matrices $C^i_{k\alpha}$ in it are replaced by matrices of the corresponding representation, i.e.,

$$\delta\psi = -\lambda_\alpha\theta^\alpha(a, 0, 0, \delta b)\,\psi. \tag{67}$$

The covariant derivatives for the representation ψ are determined by

$$D\psi = d\psi + \lambda_\alpha\theta^\alpha(a, 0, da, 0)\,\psi. \tag{68}$$

As a consequence of Eqs. (68), $D\psi$ transforms under transformations of G in the same way as ψ, i.e., according to Eq. (67). Thus, to write down a Lagrangian that is invariant under a transformation of G, it is sufficient to form from ω^i, ψ, and $D\psi$ all possible combinations that are invariant under the subgroup H.[8)]

We find ω^i and θ^α in the expression for the covariant derivatives for the case of symmetric spaces. In the normal coordinate system corresponding to the exponential parametrization (55), $x^\mu = a^\mu t$, Eqs. (47) can be written in the form[11]

$$\left.\begin{aligned}\partial\bar{\omega}^i/\partial t &= da^i + a^h\omega^i_h;\\ \partial\omega^i_e/\partial t &= -R^i_{ejh}a^j\bar{\omega}^h,\end{aligned}\right\} \tag{69}$$

where $\omega^i = a^i dt + \bar{\omega}^i$ for t = 0; $\bar{\omega}^i(t, a^j; da^k)|_{t=0}$ and $\omega^i_e\cdot(t, a^j; da^k)|_{t=0} = 0$.

On account of the vanishing of the covariant derivatives of the curvature tensor and the fact that the basis is determined by the operation of parallel transport, the components R^i_{ejk} are constants and Eq. (69) can be integrated readily:

$$\omega^i|_{t=1} = \sum_n \frac{(-1)^n}{(2n+1)!}(m^n)^i_k\,da^k; \tag{70}$$

$$\omega^i_k|_{t=0} = \sum_n \frac{(-1)^n}{(2n+2)!}R^i_{kje}a^j(m^n)^l_p\,da^p, \tag{71}$$

where $m^i_e = R^i_{kne}a^k a^n$.

Note that Eq. (29) from Sec. 3 is a consequence of (70) and the relation

$$g_{\mu\nu}\,\partial a^\mu\,\partial a^\nu = g_{ik}\omega^i\omega^k. \tag{72}$$

5. PHENOMENOLOGICAL LAGRANGIANS FOR SYMMETRY GROUPS CONTAINING THE POINCARÉ GROUP AS A SUBGROUP

The method of invariant Cartan forms can also be used for symmetry groups whose transformations correspond not only to internal variables, but also to space–time coordinates and spin variables. Let G be such a symmetry group. We represent an arbitrary element of G in the form

$$G = \exp(iPx)\,K(a)\,H(b)\,L(l), \tag{73}$$

where P is the energy–momentum operator; x are the space–time coordinates; L(l) is a finite transformation of the proper Lorentz group; H(b) is a subgroup of the transformations of internal symmetries that leaves the vacuum invariant. The product exp (iPx)K(a) parametrizes the left coset of the group G with respect to the subgroup HL.

The representation (73) corresponds to the previously considered representation (1) with explicitly distinguished transformations corresponding to the Poincaré group. It follows directly from the representation (73) that the coordinates x transform in the usual manner when a transformation of the Poincaré group is applied from the left to the representation.

Acting on (73) from the left with an arbitrary element of the group G, we find that the coordinates x transform by themselves or together with the parameters a depending on whether or not the product KHL is a subgroup or not. For the construction of phenomenological Lagrangians, the second possibility differs in no fundamental respect from the first if one discounts purely psychological difficulties associated with the circumstance that on the replacement of the parameters a by the fields of Goldstone particles A(x), the coordinates x' under transformations of the group G are functions of the old coordinates x and the fields A(x).

To find a phenomenological Lagrangian for the group (73) we construct the invariant Cartan forms (56). For the generators corresponding to the first two factors these forms have the form

$$\omega^i(x, a, dx, da, b, l). \tag{74}$$

As follows from Sec. 5, any combination of the forms (5) that is invariant for fixed values of b and l with respect to transformations of the subgroup HL does not depend on the parameters b and l and is an invariant of G. For one to be able to use such a combination to construct a phenomenological Lagrangian, it is necessary and sufficient to represent it as a product of a certain function and d^4x, the element of four-volume for the space–time variables. The simplest combination of such a form corresponds to the outer product of the four forms (74); for when the parameters a in (74) are replaced by the fields A(x) of the Goldstone particles, the differentials da are replaced by the differentials $(\partial A/\partial x)\,dx$, as a result of which the antisymmetrized product of the four forms ω^1 becomes proportional to d^4x and the factor in front of d^4x is a function that depends on the coordinates x, the fields A(x), and their derivatives. Note that none of the factors is invariant under G; only the product as a whole is invariant.

The outer product of the four forms ω^1 has a simple geometric meaning and corresponds to an infinitesimally small four-volume that is invariant with respect to transformations of G in the n-dimensional space with coordinates x and a.

More complicated invariants of G can be constructed as follows. Let W_k be an arbitrary outer product (not necessarily invariant with respect to G) of the four forms (74). Each of these forms is proportional to d^4x. From the forms W_k we form an expression that is invariant under transformations of the subgroup HL and is homogeneous of degree unity with respect to the element d^4x. Such an expression is invariant under transformations of G and is proportional to d^4x and can therefore be used to construct phenomenological Lagrangians.

After the forms W_i, the simplest expressions of such type that contain a minimal number of the derivatives of the fields have the form W_iW_i/W_0, where W_i are the forms obtained by antisymmetrization of the three forms corresponding to the energy-momentum operators and the single form corresponding to the generators in $K(a)$; W_0 is an invariant volume constructed from the four forms corresponding to the energy–momentum operators. It is readily seen that for the case of a group G that is a direct product of the Poincaré group and the group of internal symmetries, the expression W_iW_i/W_0 is identical with the previously considered expression (72).

The use of differential forms associated with generators in H and L enables one to define in a standard manner covariant differentiation for an arbitrary representation of the subgroup HL and to include an interaction with other fields in the scheme. At the same time the components of such fields can be added in the form of factors to expressions of the form W_iW_i/W_0, the only requirement being that the resulting expressions, which define the structure of the phenomenological Lagrangian, be invariant under HL. The covariant differentials of the additional fields are treated on an equal footing with the differential forms (74) and occur as structure units in the outer products of the W_i.

If the parameters a or the additional fields include representations of the Lorentz group with half-integral spin, the phenomenological Lagrangians for the corresponding fields are constructed by means of anticommuting variables.

Note that in the construction of phenomenological Lagrangians by the method considered above, the number of fields of Goldstone particles may be less than the number of parameters a. This is because a phenomenological Lagrangian is defined to within total derivatives with respect to the space–time coordinates. Therefore, if one of the fields of Goldstone particles or, equivalently, one of the parameters a^{α} occurs only in a total spatial derivative, the presence of such a field does not affect the physical properties of the system. But because terms of the form of total derivatives transform under transformations of G together with part of the phenomenological Lagrangians for the other fields of Goldstone particles, the structure of the phenememological Lagrangian depends nontrivially on such terms. In what follows we shall illustrate this observation by the example of a phenomenological Lagrangian that is invariant under a conformal group.

Above, we have considered the general procedure for constructing the phenomenological Lagrangian for a group G with parametrization (73), and we have explicitly separated the Lorentz group and the group of internal symmetries H, with respect to which we have used linear representations. We here mention the interesting possibility of replacing the product HL in the representation (73) by a more general expression that does not reduce to a direct product of the Lorentz group and the group of internal symmetries. As an example, we give the group SL(6, C) and other relativistic generalizations of SU(6).

If we apply a transformation of such a group to the representation (73) from the right, the coordinates x transform together with the fields of the Goldstone particles, so that this approach differs significantly from the usual procedures for relativization of SU(6) symmetry and may be free of the difficulties inherent in such procedures.

As an example of the use of the above procedure, we construct a phenomenological Lagrangian for the symmetry of a conformal group.[9]

The generators of the conformal group satisfy the relations

$$\left.\begin{array}{c}[M_{\mu\nu},\ D]=0;\\ [P_{\mu},\ D]=iP_{\mu};\\ [M_{\mu\nu},\ K_{\rho}]=-i(g_{\rho\mu}K_{\nu}-g_{\rho\nu}K_{\mu});\\ [P_{\mu},\ K_{\nu}]=2i(g_{\mu\nu}D-M_{\mu\nu});\\ [K_{\mu},\ K_{\nu}]=0;\\ [D,\ K_{\mu}]=iK_{\mu},\end{array}\right\} \qquad (75)$$

where P_μ are the energy–momentum operators; $M_{\mu\nu}$, D, and K_μ are, respectively, the generators of the Lorentz group, scale transformations, and conformal transformations. For brevity, we have omitted in (75) the well-known commutation relations between the operators P_μ and $M_{\mu\nu}$, which define the Poincaré group.

In accordance with the representation (73), we write an arbitrary element of the conformal group in the form

$$G=\exp(iPx)\exp(iK\varphi)\exp(iD\eta), \qquad (76)$$

where φ^μ and η are a vector and a scalar field of Goldstone particles.

For the considered parametrization, the expression $G^{-1}dG$, which defines the Cartan forms in accordance with (56), has the form

$$G^{-1}dG = L^{-1}(\exp(-\mathrm{i}D\eta)\exp(-\mathrm{i}K\varphi)\,\mathrm{i}P\,dx \times \exp(\mathrm{i}K\varphi)\exp(\mathrm{i}D\eta) + \exp(-\mathrm{i}D\eta)\,\mathrm{i}K\,d\varphi\exp(\mathrm{i}D\eta) + \mathrm{i}D\,d\eta)\,L + L^{-1}dL. \quad (77)$$

Using (75), we can readily find the differential forms corresponding to the different generators:

$$\omega(P)^\mu = dx^\mu \exp(-\eta); \quad (78)$$

$$\omega(D) = d\eta - 2\varphi\,dx; \quad (79)$$

$$\omega(K)^\mu = (d\varphi^\mu + 2\varphi^\mu(\varphi\,dx) - dx^\mu\varphi^2). \quad (80)$$

In the expressions (78), the notation in the brackets for the differential form indicates the generator to which the given form corresponds in the expansion (56). In this expression, we have omitted for brevity the Lorentz matrices in (77) and the form $\omega(M)^{\mu\nu}$, which are not important for what follows.

The invariant outer products of the four forms ω that do not contain derivatives of the fields η and φ^μ of higher than second order have the form

$$W(P, P, P, P) = \exp(-4\eta)\,d^4x; \quad (81)$$

$$W(K, P, P, P) = (\partial\varphi - 2\varphi^2)\exp(-2\eta)\,d^4x; \quad (82)$$

$$W(K, K, P, P) = [(\partial\varphi)^2 - \partial_\mu\varphi^\nu\,\partial_\nu\varphi^\mu - 2\partial_\mu(\varphi^\mu\varphi^2)]\,d^4x, \quad (83)$$

where the notation of the generators in the brackets corresponds to the forms in the outer product.

The invariant expressions of the form W_iW_i/W_0, which contain a minimal number of derivatives of the fields and are independent of (82) and (83), have the form

$$\frac{W^\mu(D, P, P, P)\,W_\mu(D, P, P, P)}{W(P, P, P, P)} = (\partial\eta - 2\varphi)^2\exp(-2\eta)\,d^4x; \quad (84)$$

$$\frac{W(K, P, P, P)\,W(K, P, P, P)}{W(P, P, P, P)} = (\partial\varphi - 2\varphi^2)^2\,d^4x; \quad (85)$$

$$\frac{W_{\mu\nu}(K, P, P, P)\,W^{\mu\nu}(K, P, P, P)}{W(P, P, P, P)} = \frac{1}{4}(\partial_\mu\varphi_\nu - \partial_\nu\varphi_\mu)(\partial^\mu\varphi^\nu - \partial^\nu\varphi^\mu)\,d^4x. \quad (86)$$

The expressions (85) and (86) contain two forms that contain the generators K, P, P, and P of scalar and tensor type depending on the contractions of the indices of the corresponding differential forms. The phenomenological Lagrangian for the fields η and φ^μ can be represented as a combination of the forms (82)-(86) with arbitrary coefficients.

Let us now consider whether we can construct a conformally invariant phenomenological Lagrangian using the field η alone. The kinetic term for such a field is contained in (84), but the derivatives of the field η enter this expression together with the field φ_μ. To eliminate the latter, we note that using (84) and (82) we can form the linear combination

$$[0.5(\partial\eta - 2\varphi)^2\exp(-2\eta) + (\partial\varphi - 2\varphi^2)\exp(-2\eta)]\,d^4x = [0.5(\partial\eta)^2\exp(-2\eta) + \partial(\varphi\exp(-2\eta))]\,d^4x, \quad (87)$$

in which the field φ^μ enters only through the total derivative.

Thus, the phenomenological Lagrangian for the field η can be taken in the form (87). Note that the second term cannot be entirely ignored, since it is only the expression (87) as a whole that is an invariant under transformations of the conformal group.

In the general case when fields are included that interact with the field η, the condition that the field φ^μ occur only in the total derivative is equivalent to the well-known condition[16] under which a Lagrangian that is invariant under scale transformations also becomes conformally invariant.

Finally, we note that the possibility considered above of reducing the number of fields of Goldstone particles is due, first, to the fact that there exist terms of the type (82) having the form of outer products and, secondly, that the eliminated fields correspond to the generators of a certain abelian subgroup.[10)]

APPENDIX

We show that the equations for the reduced matrix element formulated on the basis of Feynman diagrams can be obtained by using a quasiclassical approach. Such an approach for the treatment of pole diagrams has been used by Nambu.[13] Here we reproduce some results of ref. 13, and we also consider a generalization of the approach used by Nambu to obtain the relevant equation. In the interaction representation, the S matrix can be represented in the form

$$S = T\left[\exp\left(\mathrm{i}\int L_{\mathrm{int}}(\varphi)\,d^4x\right)\right] \quad (\mathrm{I}.1)$$

or, in the form of a normal product,

$$S = N\left[\exp\left(\frac{1}{2}\int\int\frac{\delta}{\delta\varphi(x)}\Delta(x-x')\frac{\delta}{\delta\varphi(x')}\,d^4x\,d^4x'\right) \times \exp\left(\mathrm{i}\int L_{\mathrm{int}}(\varphi)\,d^4x\right)\right]. \quad (\mathrm{I}.2)$$

In (I.1) and (I.2) we have used the neutral scalar field φ:

$$\Delta(x-x') = \langle T\varphi(x)\,\varphi(x')\rangle \quad (\mathrm{I}.3)$$

is the propagator of the free field.

The generalization to an arbitrary number of Bose and Fermi fields can be made without difficulty. As an auxiliary quantity we introduce the function S_η:

$$S_\eta = N\left[\exp\left(\frac{1}{2}\eta\int\int\frac{\delta}{\delta\varphi(x)}\Delta(x-x')\frac{\delta}{\delta\varphi(x')}\,d^4x\,d^4x'\right) \times \exp\left(\mathrm{i}\int L_{\mathrm{int}}(\varphi)\,d^4x\right)\right], \quad (\mathrm{I}.4)$$

i.e.,

$$S_\eta|_{\eta=1} = S. \quad (\mathrm{I}.5)$$

Representing S_η in the form

$$S_\eta = \exp[\mathrm{i}A(\eta, \varphi)], \quad (\mathrm{I}.6)$$

where $A(\eta, \varphi)$ corresponds to the sum of all connected diagrams, and differentiating (I.6) and (I.4) with respect to η, we obtain an equation for the function $A(\eta, \varphi)$:

$$\frac{\partial A}{\partial \eta}-\frac{i}{2}\int\int \frac{\delta A}{\delta\varphi(x)}\Delta(x-x')\frac{\delta A}{\delta\varphi(x')}d^4x d^4x' -\frac{1}{2}\int\int\Delta(x-x')\frac{\delta^2 A}{\delta\varphi(x)\,\delta\varphi(x')}d^4x d^4x'=0 \qquad (I.7)$$

together with the "initial" condition

$$A(0, \varphi)=\int L_{\rm int}(\varphi)\,d^4x. \qquad (I.8)$$

The relation (I.7) is exact. The transition to the quasiclassical approximation corresponds to ignoring the last term in (I.7); for A in the equation is a dimensionless quantity expressed in units of $\hbar$. The second term contains an extra power of A compared with the third term, so that the latter tends to zero if $\hbar$ does.

It is readily seen that when $A(\eta, \varphi)$ is expanded in powers of η, neglect of the third term in (I.7) corresponds to allowing for pole diagrams and omitting diagrams with closed loops. An approximate equation can be written in the form of the Hamilton–Jacobi equation

$$\partial A/\partial\eta+H(\delta A/\delta\varphi(x))=0 \qquad (I.9)$$

with Hamiltonian

$$H(\pi(x))=-\frac{i}{2}\int\int\pi(x)\,\Delta(x-x')\,\pi(x')\,dx\,dx' \qquad (I.10)$$

and generalized momenta

$$\pi(x)=\delta A/\delta\varphi(x). \qquad (I.11)$$

Equation (I.9) can be readily integrated in a general form for arbitrary initial conditions. In what follows we shall simplify the notation by assuming that there is a finite number of variables φ^i and π^i (i = 1, 2, ...). Accordingly, we rewrite the equations in the form

$$\partial A/\partial\eta+H(\pi)=0; \qquad (I.12)$$

$$H(\pi)=d^{ik}\pi_i\pi_k/2;$$
$$\pi_i=\partial A/\partial\varphi^i;\qquad A(0, \varphi)=A_0(\varphi). \qquad (I.13)$$

The action A for the Hamiltonian H is defined by the well-known relation

$$A=\int_{\varphi_0,\ \eta=0}^{\varphi,\ \eta} L(d\varphi/d\eta)\,d\eta, \qquad (I.14)$$

where $L(d\varphi/d\eta)$ is the Lagrange function corresponding to the Hamiltonian (I.13)

$$L(d\varphi/d\eta)=(d^{-1})_{ik}(d\varphi^i/d\eta)(d\varphi^k/d\eta)/2, \qquad (I.15)$$

and the integration in (I.14) is along the trajectory of a particle,

$$\varphi^i=\alpha^i\eta+\varphi_0^i, \qquad (I.16)$$

which leads to the following expression for $A(\eta, \varphi)$:

$$A(\eta, \varphi)=\frac{1}{2}(d^{-1})_{ik}\,\alpha^i\alpha^k\eta=\frac{1}{2}(d^{-1})_{ik}\frac{(\varphi^i-\varphi_0^i)(\varphi^k-\varphi_0^k)}{\eta}. \qquad (I.17)$$

To satisfy the initial conditions, we construct on the basis of the expression (I.17) a general integral of Eq. (I.12) (see, for example, ref. 22). To do this, we add to (I. 17) an arbitrary constant of integration; we shall assume that this, in turn, is an arbitrary function of the φ_0, and that the φ are functions of φ and η defined by the conditions

$$\partial A/\partial\varphi_0^i=0, \qquad (I.18)$$

i.e.,

$$A(\eta, \varphi)=\frac{1}{2}(d^{-1})_{ik}\frac{(\varphi^i-\varphi_0^i)(\varphi^k-\varphi_0^k)}{\eta}+C(\varphi_0); \qquad (I.19)$$

$$-(d^{-1})_{ik}\frac{(\varphi^k-\varphi_0^k)}{\eta}+\frac{\partial C}{\partial\varphi_0^i}=0 \qquad (I.20)$$

or

$$A(\eta, \varphi)=C(\varphi_0)+\frac{1}{2}d^{ik}\frac{\partial C}{\partial\varphi_0^i}\cdot\frac{\partial C}{\partial\varphi_0^k}\eta; \qquad (I.21)$$

$$\varphi_0^i=\varphi^i-d^{ik}\frac{\partial C}{\partial\varphi_0^k}\eta. \qquad (I.22)$$

It follows from (I.19) that when $\eta = 0$,

$$A(0, \varphi)=C(\varphi), \qquad (I.23)$$

i.e., the function C is determined by the initial conditions and can be expressed in terms of the interaction Lagrangian.

Differentiating (I.21) with respect to φ, we readily see that

$$\partial A(\eta, \varphi)/\partial\varphi=\partial C(\varphi_0)/\partial\varphi_0. \qquad (I.24)$$

Substituting (I.24) into (I.21), we obtain

$$A(1, \varphi)=A_0[\varphi^i-d^{ik}\partial A(1, \varphi)/\partial\varphi^k]+d^{ik}(\partial A(1, \varphi)/\partial\varphi^i)\cdot(\partial A(1, \varphi)/\partial\varphi^k)/2. \qquad (I.25)$$

The terms of the expansion of $A(1, \varphi)$ in powers of φ can be represented graphically by the diagrams shown in Fig. 2; for the expansion of $A_0(\varphi)$ in a series, gives, in accordance with (I.8), the vertices of the original interaction Lagrangian. Replacement of one or several powers of φ^i by the second term in the argument of A_0 in (I.25) leads to graphs in which the vertices of the interaction Lagrangian are joined to one or several effective matrix elements (recall that d^{ik} plays the role of propagators on the transition to field variables). The last term can be associated with a diagram with two effective matrix elements joined by an internal line.

The above derivation is taken from ref. 13. We now consider a generalization of (I.25), which would allow for a possible reduction of pole diagrams to a contact interaction because of the presence in certain vertices of the operators $\partial^2 A$ [or $(\partial^2 - m^2)A$ when a mass term is present]. Our aim is to obtain an equation analogous to (I.25) for the reduced matrix elements and an equation relating these to the elements of the S matrix.

We formulate the problem as follows: Suppose there is a procedure for splitting the operators φ into two classes φ_1 and φ_2, the procedure satisfying the condition that the interaction Lagrangian is a linear function in the operators φ_2. For example, in the case in which we are interested the operators φ to which the sign of the second derivative with respect to the spatial coordinates is applied belong to the class φ_2.

In accordance with this splitting, all the propagators are also split into two classes: those containing one or two of the operators φ_2 and those that do not contain such operators. We calculate the S matrix in the quasiclassical approximation in two stages. First, we sum the contribution of all diagrams containing only propagators with at least one operator φ_2 and then, using the expression obtained as a result of such summation and taking into account the remaining propagators with two operators φ_1, we find an expression for the sum of all possible diagrams.

To carry through this procedure we introduce as a factor for each propagator the parameters η_1 and η_2 in accordance with whether the propagator contains only the operators φ_1 or also one or two of the operators φ_2. Repeating arguments similar to those adduced above, we arrive at the following Hamilton–Jacobi equations with two "times" for $A = A(\varphi_1, \varphi_2, \eta_1, \eta_2)$, which is identical with the previously considered $A = A(\varphi, \eta)$ when $\varphi_1 = \varphi_2 = \varphi$ and $\eta_1 = \eta_2 = \eta$:

$$\partial A/\partial\eta_1 + \frac{1}{2} d^{ik}\,\partial A/\partial\varphi_1^i \cdot \partial A/\partial\varphi_1^k/2 = 0; \tag{I.26}$$

$$\partial A/\partial\eta_2 + d^{ik}\,\partial A/\partial\varphi_1^i \cdot \partial A/\partial\varphi_2^k + \frac{1}{2} d^{ik}\,\partial A/\partial\varphi_2^i \cdot \partial A/\partial\varphi_2^k = 0 \tag{I.27}$$

with the initial conditions

$$A(\varphi_1, \varphi_2, 0, 0) = A_0(\varphi_1, \varphi_2). \tag{I.28}$$

When $\varphi_1 = \varphi_2 = \varphi$, $A_0(\varphi_1, \varphi_2)$ is identical with the previously introduced $A_0(\varphi)$, which corresponds to the interaction Lagrangian.

The two stages of finding the elements of the S matrix correspond to successive integration of, first, Eq. (I.27) with the initial conditions (I.28) and then Eq. (I.26) with the initial conditions specified by the function $A(\varphi_1, \varphi_2, 0, 1)$.

Integration of Eq. (I.27) leads to the result

$$A(\varphi_1, \varphi_2, 0, 1) = A_0(\varphi_{10}, \varphi_{20}) + d^{ik} \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{10}^i} \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{20}^k} + \frac{1}{2} d^{ik} \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{20}^i} \cdot \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{20}^k}, \tag{I.29}$$

where

$$\varphi_{10}^i = \varphi_{20}^i - d^{ik} \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{20}^k}; \tag{I.30}$$

$$\varphi_{20}^i = \varphi_2^i - d^{ik} \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{20}^k} - d^{ik} \frac{\partial A_0(\varphi_{10}, \varphi_{20})}{\partial\varphi_{10}^k}. \tag{I.31}$$

The relations (I.28) are obtained from (I.21) by allowing for the matrix structure of the Hamiltonian H, which is associated with the additional indices 1 and 2 of the operators φ_1 and φ_2.

The assumption that $A_0(\varphi_1, \varphi_2)$, the interaction Lagrangian, is linear in the operators φ_2 enables us to reduce the relations (I.29) to a form that does not contain the derivatives $\partial A/\partial\varphi_1^i$ explicitly; for writing the linearity condition in the form

$$A_0(\varphi_1, \varphi_2) = A_1(\varphi_1, 0) + \left[\frac{\partial A_0(\varphi_1, \varphi_2)}{\partial\varphi_2}\right]\varphi_2, \tag{I.32}$$

substituting (I.32) into (I.29)-(I.31), and noting that, as a consequence of (I.24),

$$\partial A_0/\partial\varphi_{10} = \partial A/\partial\varphi_1; \quad \partial A_0/\partial\varphi_{20} = \partial A/\partial\varphi_2, \tag{I.33}$$

we obtain

$$A(\varphi_1, \varphi_2, 0, 1) = A_0(\varphi_1^i - d^{ik}\,\partial A/\partial\varphi_2^k,\ \varphi_2^i - d^{ik}\,\partial A/\partial\varphi_2^k) + d^{ik}\,\partial A/\partial\varphi_2^k \cdot \partial A/\partial\varphi_2^i/2. \tag{I.34}$$

The expansion of the relations (I.34) in powers of φ can be represented graphically by the diagrams shown in Fig. 4, in the same way as this was done for the relations (I.25).

Note that the structure of the relations (I.34) enables us to avoid the explicit construction of the function $A_0(\varphi_1, \varphi_2)$ and enables us to separate the operators φ into the classes φ_1 and φ_2 directly in the argument of the function $A(\varphi_1, \varphi_2, 0, 1)$; for when $\varphi_1 = \varphi_2 = \varphi$ the relation (I.34) can be written in the form

$$A(\varphi_1, \varphi_2, 0, 1) = A_0(\varphi^i - d^{ik}\,\partial A(\varphi_1, \varphi_2)/\partial\varphi_2^k) + d^{ik}\,\partial A(\varphi_1, \varphi_2)/\partial\varphi_2^i \cdot \partial A(\varphi_1, \varphi_2)/\partial\varphi_2^k/2. \tag{I.35}$$

After the calculation of the n-th power of φ on the left-hand side of Eq. (I.35), the division of the operators between φ_1 and φ_2 can be specified arbitrarily and then used for substitution into the right-hand side of the equation for calculating the (n + 1)-th and higher powers. It is only necessary to require that $A(\varphi_1, \varphi_2, 0, 1)$ be a linear function in φ_2. If this condition is satisfied, a function $A_0(\varphi_1, \varphi_2)$ satisfying the requirement of linearity in φ_2 always exists as a consequence of (I.35) and it can be defined from given $A(\varphi_1, \varphi_2, 0, 1)$.

Because it does not contain the derivatives $\partial A/\partial\varphi_2$, Eq. (I.26) can be integrated in the same way as Eq. (I.12). The result can be represented in the form of an ordinary expansion of perturbation theory with the Lagrangian $A(\varphi_1, \varphi_2, 0, 1)$, in which φ_2 is regarded as some given external field.

APPENDIX II

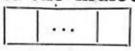
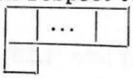
In a space with affine connection straight lines are replaced by geodesics defined by the equations

$$d^2x^\mu/ds^2 + \Gamma^\mu_{\nu\lambda}(dx^\nu/ds)(dx^\lambda/ds) = 0. \tag{II.1}$$

Under a transformation of the coordinates these equations are unaffected, and the Christoffel symbols transform in accordance with the equation

$$\Gamma^{\mu'}_{\nu'\lambda'} = A^{\mu'\nu\lambda}_{\mu\nu'\lambda'}\Gamma^\mu_{\nu\lambda} + A^{\mu'}_\rho \partial_{\nu'} A^\rho_{\lambda'}. \tag{II.2}$$

The presence of the last term in (II.2) enables one to use the freedom in the choice of the coordinate system to impose certain additional conditions on the symbols $\Gamma^\lambda_{\mu\nu}$.

As such additional conditions we require that linear combinations of the derivatives of the Christoffel symbol[11] $\Gamma_{\mu\nu,\lambda}$ (or, which is the same thing, the derivatives of the metric tensor $g_{\mu\nu}$) corresponding to symmetrization of the indices with respect to Young tableaux of the form  and , vanish at the origin.

We shall show that this additional condition is satisfied in a normal coordinate system, in which, by definition, the solution of Eqs. (II.1) has the form[10,23]

$$x^{\mu}=a^{\mu}s. \tag{II.3}$$

It follows from (II.3) that

$$\Gamma^{\mu}_{\nu\lambda}a^{\nu}a^{\lambda}=0 \tag{II.4}$$

or, after multiplication by s^2,

$$\Gamma^{\mu}_{\nu\lambda}x^{\nu}x^{\lambda}=0. \tag{II.5}$$

An analogous relation holds for the Christoffel symbols with subscripts:

$$\Gamma_{\nu\lambda,\,\mu}x^{\nu}x^{\lambda}=0 \tag{II.6}$$

and, as a consequence of this expression, for a certain combination of the derivatives of the metric tensor,

$$(\partial_{\lambda}g_{\mu\nu}+\partial_{\nu}g_{\mu\lambda}-\partial_{\mu}g_{\nu\lambda})\,x^{\nu}x^{\lambda}=0. \tag{II.7}$$

We expand the expression in the brackets in (II.7) in a series in powers of x^{α}:

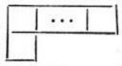
$$\frac{1}{n!}\,(2\partial_{(\alpha\ldots}g_{\gamma)\mu}-\partial_{\mu(\alpha\ldots}g_{\gamma\cdot)})\,x^{\alpha}\ldots x^{\gamma}, \tag{II.8}$$

where there is symmetrization with respect to the indices in the curly brackets. Since x is arbitrary, it follows from (II.8) that

$$2\partial_{(\alpha\ldots}g_{\gamma)\mu}-\partial_{\mu(\alpha\ldots}g_{\gamma\cdot)}=0. \tag{II.9}$$

Multiplying (II.8) by x^{μ} we obtain similarly that the combination of derivatives corresponding to the completely symmetric Young tableau vanishes, i.e.,

$$\partial_{(\alpha\ldots}g_{\gamma\mu)}=0. \tag{II.10}$$

It follows from the relations (II.8) and (II.9) that the combinations of the derivatives of the metric tensor corresponding to Young tableaux of the form , also vanish.

Indeed, separating explicitly the subscript μ in (II.10), we write (II.10) in the form

$$2\partial_{(\alpha\ldots}g_{\gamma)\mu\cdot}+n\partial_{\mu(\alpha\ldots}g_{\gamma\cdot)}=0, \tag{II.11}$$

where n is the rank of the derivative.

As a consequence of (II.9) and (II.11)

$$\partial_{(\mu\ldots}g_{\nu)\lambda}=0; \tag{II.12}$$

$$\partial_{\mu(\ldots}g_{\nu\lambda)}=0, \tag{II.13}$$

i.e., when we carry out symmetrization with respect to arbitrary indices except for one fixed index, the corresponding combinations of derivatives vanish. The combinations corresponding to the Young tableaux that we have considered are obtained from (II.12) or (II.13) by antisymmetrization of μ with one of the indices in the brackets and therefore they also vanish.

Our arguments can be readily reversed, i.e., one can show that (II.5) and (II.6) follow from the vanishing of the combination of derivatives of the metric tensor corresponding to the considered Young tableaux, i.e., the coordinate system satisfying this requirement is normal.

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1) Note that there is not always a one-to-one correspondence between the parameters a^{α} and the local fields $A^{\alpha}(x)$. Thus, in the study of symmetry groups for which the Poincaré group is a subgroup (see Sec. 6) some of the parameters a^{α} determine the space–time coordinates. In addition, if a symmetry group has abelian subgroups, the number of fields of Goldstone particles may be less than the number of parameters a^{α} (see Sec. 6).

2) A geometrical approach to the method of phenomenological Lagrangians has been developed in refs. 5 and 7.

3) This theorem was proved for diagrams of tree-type by Nambu[13] on the basis of a semiclassical treatment and by Callan et al.[3] as a consequence of the homogeneity of these diagrams with respect to a scale transformation of the fields.

4) See Appendix II.

5) Note that for the above example of broken SU(3) ⊗ SU(3) symmetry the requirement that there exist a Riemannian connection with holonomy group SU(3) uniquely defines the structure of the Riemannian space. For the broken group SU(2) ⊗ SU(2) an analogous requirement does not lead to any restrictions.[17]

6) The definition of the "covariant derivative" (46) coincides with the ordinary definition of the covariant differential in the geometry of Riemannian spaces and spaces with affine connection if allowance is made for the dependence of the fields A^{α} on the spatial coordinates x. Note that in (46) the fields are assumed to be independent of A^{α}. In the case when such a dependence arises, for example, on the transition to another basis, one must add to (46) the term $(d\psi/dA^{\alpha})dA^{\alpha}$ (see Sec. 5).

7) For this it is sufficient to determine the bases by parallel transport along the curve A'BA, where A is the initial point, A' is the end point, and B is a fixed point of the transformation; A'B and BA are geodesics joining the corresponding points.

8) This result agrees with that obtained in ref. 3.

9) A phenomenological Lagrangian for a conformal group was first considered in ref. 4 (see also the reviews 15).

10) An example of the elimination of fields of Goldstone particles by an analogous procedure has recently been considered for the case of spin waves in ref. 17.

11) On the transition from the Christoffel symbols $\Gamma^{\gamma}_{\alpha\beta}$ to $\Gamma_{\alpha\beta,\gamma}$ we assume that the space is a Riemannian space.

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