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ENERGY LEVELS AND ELECTROMAGNETIC PROPERTIES OF HYDROGEN-LIKE ATOMS

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This paper reviews the present state of quantum theory of the simplest two-particle systems – hydrogen-like atoms. This theory is based on Logunov and Tavkhelidze's three-dimensional quasipotential method in the new relativistically invariant formulation proposed by the author. The paper deals most thoroughly with corrections to the energy levels for nuclear structure and motion effects, the positronium energy levels, and relativistic corrections to the atomic magnetic moment. The theory is compared with the most recent experimental data.

INTRODUCTION

One of the simplest and most fully developed regions of application of quantum mechanics is the theory of hydrogen-like (HL) atoms. These include hydrogen (e^-p), muonium ($e^-\mu^\pm$), positronium (e^-e^+), muonic hydrogen (μ^-p), and pionic hydrogen (π^-p) atoms. All these systems are bound states of two charged particles interacting through an electromagnetic field.

Throughout almost the entire history of quantum mechanics and quantum electrodynamics the hydrogen atom has been of great importance for the testing of basic theoretical postulates. It was used to establish the quantization rules and the Bohr–Sommerfeld correspondence principle. One of the main achivements of the Schrödinger equation was the calculation of the energy levels for the Coulomb potential. A brilliant confirmation of Dirac's relativistic electron theory was the calculation, based on this theory, of the fine structure of the hydrogen energy levels.

The idea of the existence of an anomalous magnetic moment in the electron first arose in the investigation of the hyperfine splitting of the hydrogen levels. The Lamb shift of the energy levels was also first discovered in the hydrogen atom. The interpretation of the anomalous magnetic moment of the electron and the Lamb shift as radiation effects, and their very accurate calculation on the basis of renormalization theory, represent an outstanding achievement of quantum electrodynamics.

Hydrogen-like systems have been the main objects used in the investigation of the relativistic quantum two-body problem on the basis of the Breit, Bethe–Salpeter, and Logunov–Tavkhelidze equations. The various relativistic and quantum-electrodynamic corrections calculated by means of these equations are in good agreement with experiment.

Hydrogen-like atoms owe their privileged position to the fact that, on one hand, the radiation and relativistic corrections to the Coulomb energy levels are small and can be calculated very accurately from perturbation theory. On the other hand, the energy levels of these atoms are accessible to exceedingly accurate experimental investigation. Finally, the theory of HL atoms in external electric and magnetic fields has been fully worked out. All these features make HL atoms ideal systems for testing the correctness of the basic postulates of quantum mechanics and quantum electrodynamics.

A general account of the present state of the theory in this area of research is given below. This review does not make any claim to completeness, since most attention is given to problems which can be resolved most fully. The choice of

such problems is based partially on the scientific interests of the author himself. The main method used to describe the two-particle bound system is the Logunov–Tavkhelidze quasipotential method in the new relativistically invariant formulation introduced by the author.

I. ENERGY LEVELS OF HYDROGEN-LIKE ATOMS

1. Relativistic wave function of a bound two-particle system. The most widely known method of representing the relativistic quantum two-body problem is the explicitly covariant Bethe–Salpeter equation [1]. The wave function of the bound two-particle system in this case has the form

$$\Psi_{BK_B}(x_1, x_2) = \langle 0 \mid T\{\varphi_1(x_1)\varphi_2(x_2)\} \mid M_B, J; \mathbf{K}_B\rangle, \quad (1.1)$$

where $\varphi_{1,2}$ are complete Heisenberg operators of fields 1 and 2; M_B, J, $\mathbf{K}_B$ are the mass, spin, and momentum, respectively, of the bound state; $E_B = (M_B^2 + \mathbf{K}_B^2)^{1/2}$.

The wave function (1.1) differs considerably in its properties from the nonrelativistic wave function conforming to the Schrödinger equation. Firstly, the norm of wave function (1.1) is not positive definite and, hence, the usual probabilistic interpretation is lost; secondly, it depends on two temporal coordinates, and the physical sense of the relative time parameters is not completely clear.

In almost all practical applications of the Bethe–Salpeter equation this additional parameter has been eliminated in some way or another. This has given rise to the question of whether this could not be done systematically from the very start. Such a method was proposed by Logunov and Tavkhelidze [2] and was called the quasipotential method. In this method the wave function of the bound system depends on one temporal parameter and conforms to a Schrödinger-type equation with a complex, energy-dependent, and nonlocal kernel – a quasipotential. Although this still does not completely restore the probabilistic interpretation of the wave function, its physical sense can still be found [3]. The quasipotential method can also be used successfully in scattering problems [4]. Different variants of relativistic three-dimensional equations of the quasipotential type have been examined in [5-12].

We will make a few comments on the equations proposed in [11, 12]. The effective potential model, as Grotch and Yennie call it, is conceptually very close to the quasipotential method. The main difference is that exact solutions of the Dirac equation (for spinor particles) with a modified (due to recoil of the nucleus) Coulomb potential are taken as an initial approximation. This choice has some practical advantages in the case of the hydrogen atom. These advantages, however, are achieved at the expense of the consistent formulation of the problem for bound states. Another significant shortcoming is the disparate consideration of the two particles. Hence, this method is quite inapplicable to such systems as positronium, where the two particles have the same mass. For this reason we will use the more consistent and more widely applicable Logunov–Tavkhelidze method. Our new covariant formulation [14] of the quasipotential method is conceptually close to [7].

In nonrelativistic quantum mechanics the state of a particle system can be represented by a wave function which depends on the coordinates (momenta) of the particles at a particular instant. The most simple and natural relativistically invariant generalization of such a representation is the assignment of the wave function to the time of the system itself. In this case the direction of the time axis must coincide with that of the total momentum vector of the system. Thus, the concept of simultaneity acquires a quite definite and Lorentz-invariant meaning. The times of the particles in the CMS must be the same. This replacement of the individual time coordinates by a common coordinate is the essence of the process of combining: It defines the change in our viewpoint when we consider the system as a whole instead of its constituent parts [13]. According to Eddington's figurative expression [13], "a hydrogen atom is composed of a proton and electron, but a proton today and an electron yesterday do not constitute a hydrogen atom."

Henceforth we will mainly use momentum space and, hence, we define [14] the covariant quasipotential wave function in the following way:

$$(2\pi)^4\delta^4(\mathcal{P} - K_B)\,\Psi_{B\mathcal{P}}(\mathbf{p}) = \int d^4x_1 d^4x_2 e^{ip_1x_1 + ip_2x_2} \times \delta(hx_1 - hx_2)\,\langle 0 \mid \varphi_1(x_1)\varphi_2(x_2) \mid M_B, J; \mathbf{K}_B\rangle, \quad (1.2)$$

where

$$h^\mu = \frac{\mathcal{P}^\mu}{\sqrt{\mathcal{P}^2}}; \quad \mathcal{P} = p_1 + p_2; \quad \mu = 0, 1, 2, 3; \quad px \equiv p^\mu x_\mu,$$

and it is assumed that the coordinates and momenta of the particles are expressed in terms of the corresponding quantities in the CMS:

$$\left.\begin{aligned} x_{1,2} &= L_{\mathcal{J}}(\tau_{1,2};\ \overset{0}{x}_{1,2}) = \tau_{1,2}h + \sum_{i=1}^{3}(n^{(i)}(\mathcal{P})\,\overset{0}{x}{}^{i}_{1,2});\\ p_{1,2} &= L_{\mathcal{J}}(\overset{0}{\varepsilon}_{1,2};\ \overset{0}{p}_{1,2}) = \overset{0}{\varepsilon}_{1,2}h + \sum_{i=1}^{3}(n^{(i)}(\mathcal{P})\,\overset{0}{p}{}^{(i)}_{1,2});\\ n^{(i)\mu} &= L^{\mu}_{\mathcal{J}}\,i = \left(\frac{\mathcal{J}i}{\sqrt{\mathcal{J}^2}};\ \frac{\mathcal{J}i\mathcal{J}j}{\sqrt{\mathcal{J}^2}(E+\sqrt{\mathcal{J}^2})} + \delta_{ij}\right);\\ (n^{(i)}\mathcal{P}) &= 0;\quad (n^{(i)}n^{(j)}) = -\delta_{ij};\\ \overset{0}{\mathbf{p}}_1 &= -\overset{0}{\mathbf{p}}_2 = \overset{0}{\mathbf{p}}; \end{aligned}\right\} \quad (1.3)$$

$L_{\mathcal{J}}$ is the pure Lorentz transformation which converts the CMS to an arbitrary coordinate system. The metric of four-dimensional space is chosen in the form (1, −1, −1, −1). It is obvious from relationships (1.3) that

$$\tau_{1,2} = (h \cdot x_{1,2}),$$

and, thus, the δ function in the integral (1.2) actually ensures the equalization of the particle times in the CMS.

For definiteness we will assume that both particles have spin 1/2. It is convenient [8, 15] to project the wave function (1.2) onto the positive frequency states by means of the relation

$$\Psi^{(+)}_{\mathrm{B}\mathcal{P}}(\overset{0}{\mathbf{p}}) = \frac{\sqrt{\overset{0}{\varepsilon}_1(\mathbf{p})\,\overset{0}{\varepsilon}_2(\mathbf{p})}}{2m_1m_2}\,\bar{u}_1(p_1)\,\bar{u}_2(p_2)\,\Psi_{\mathrm{B}\mathcal{P}}(\overset{0}{\mathbf{p}}), \quad (1.4)$$

where

$$\bar{u} = u^*\gamma_0; \quad \varepsilon_{1,2}(\mathbf{p}) = \sqrt{m^2_{1,2} + \mathbf{p}^2},$$

u(p) is the positive frequency Dirac spinor, normalized by the condition $\bar{u}u = 2m$.

In explicit form we have

$$u^{\lambda}(\mathbf{p}) = \sqrt{\varepsilon(\mathbf{p}) + m}\begin{pmatrix} 1 \\ \dfrac{\sigma\mathbf{p}}{\varepsilon(\mathbf{p}) + m} \end{pmatrix} w^{\lambda}. \quad (1.5)$$

Using Lorentz invariance it can be shown [14] that

$$\Psi^{(+)}_{\mathrm{B}\mathcal{P}}(\overset{0}{\mathbf{p}}) = D^{1/2}_1(R^W_1)\,D^{1/2}_2(R^W_2)\,\Psi^{(+)}_{\mathrm{B}0}(\overset{0}{\mathbf{p}}), \quad (1.6)$$

where $\Psi^{(+)}_{B_0}$ is the wave function in the CMS; $D^{1/2}(R)$ are well-known finite rotation matrices for angular momentum 1/2, and the Wigner rotation is

$$R^W_{1,2} = L^{-1}_{L_{\mathcal{P}}p_{1,2}}L_{\mathcal{P}}L_{p_{1,2}}.$$

The wave function $\Psi^{(+)}_{\mathrm{B}\mathcal{P}}$ satisfies the equation

$$\left(M_{\mathrm{B}} - \sqrt{\overset{0}{\mathbf{p}}{}^2 + m_1^2} - \sqrt{\overset{0}{\mathbf{p}}{}^2 + m_2^2}\right)\Psi^{(+)}_{\mathrm{B}\mathcal{P}}(\overset{0}{\mathbf{p}}) = \frac{1}{(2\pi)^3}\int d\overset{0}{\mathbf{q}}\,V(\overset{0}{\mathbf{p}},\ \overset{0}{\mathbf{q}};\ \mathcal{P})\,\Psi^{(+)}_{\mathrm{B}\mathcal{P}}(\overset{0}{q}). \quad (1.7)$$

The quasipotential V is defined in terms of the scattering amplitude of the two particles off the energy shell:

$$V = T_{(+)}(1 + G^fT_{(+)})^{-1}, \quad (1.8)$$

where the free-particle Green's function is

$$G^f(\overset{0}{\mathbf{p}},\ \overset{0}{\mathbf{q}}) = \frac{(2\pi)^3\,\delta(\overset{0}{\mathbf{p}} - \overset{0}{\mathbf{q}})}{M_{\mathrm{B}} - \varepsilon_1(\overset{0}{\mathbf{p}}) - \varepsilon_2(\overset{0}{\mathbf{p}})} \equiv (2\pi)^3\,\delta(\overset{0}{\mathbf{p}} - \overset{0}{\mathbf{q}})\,F(\overset{0}{\mathbf{p}}). \quad (1.9)$$

Multiplication in equation (1.8) is understood in the operator sense as integration over three-dimensional momentum space. Relationship (1.8) is usually used in the form of an expansion in perturbation theory:

$$\left.\begin{aligned} V &= V^{(1)} + V^{(2)} + \ldots;\quad T_{(+)} = T^{(1)}_{(+)} + T^{(2)}_{(+)} + \ldots;\\ V^{(1)} &= T^{(1)}_{(+)};\quad V^{(2)} = T^{(2)}_{(+)} - T^{(1)}_{(+)}G^fT^{(1)}_{(+)}. \end{aligned}\right\} \quad (1.10)$$

In the most general form the scattering amplitude T is determined [14, 15] by means of a covariant dual-time two-particle Green's function projected onto the positive frequency states [like Eq. (1.4)]:

$$G^fT_{(+)}G^f = G^{(+)} - G^f;\quad G^{(+)} = \bar{u}_1\bar{u}_2Gu_1u_2.$$

However, in some applications (e.g., to obtain the energy levels of the bound system) it is possible to construct the amplitude T by using the elements of the scattering matrix on the mass shell. Henceforth we will mainly use this simpler method. The equivalence of the two approaches was demonstrated in [16]. Thus, in the CMS we have

$$T_{(+)}(\mathbf{p},\ \mathbf{q};\ M_{\mathrm{B}}) =$$

$$\frac{\bar{u}_1(p)\,\bar{u}_2(p)}{2\sqrt{\varepsilon_1(\mathbf{p})\,\varepsilon_2(\mathbf{p})}}\,T(\mathbf{p},\ \mathbf{q};\ E_1,\ E_2)\,\frac{u_1(q)\,u_2(q)}{2\sqrt{\varepsilon_1(\mathbf{q})\,\varepsilon_2(\mathbf{q})}}, \quad (1.11)$$

where

$$E_1 = \frac{M_{\mathrm{B}}^2 - m_2^2 + m_1^2}{2M_{\mathrm{B}}};\quad E_2 = \frac{M_{\mathrm{B}}^2 - m_1^2 + m_2^2}{2M_{\mathrm{B}}};\quad \mathbf{p}^2 \neq \mathbf{q}^2.$$

In the nonrelativistic limit Eq. (1.7) obviously becomes the ordinary Schrödinger equation:

$$\left(W - \frac{p^2}{2\mu}\right)\Psi_B(\mathbf{p}) = \frac{1}{(2\pi)^3}\int d\mathbf{q}V(\mathbf{p},\ \mathbf{q};\ W)\,\Psi_{\mathrm{B}}(\mathbf{q}), \quad (1.12)$$

where

$$W = M_B - m_1 - m_2$$

and $\mu = \frac{m_1m_2}{m_1 + m_2}$ is the reduced mass.

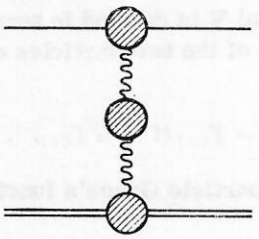

Fig. 1

2. Structure of Coulomb energy levels.

In quantum electrodynamics the interaction of charged particles involves photon exchange. In the lowest approximation we obviously have one-photon exchange, which can be represented by the diagram in Fig. 1.

For particles with spin 1/2 the interaction potential in this approximation, according to Eqs. (1.10) and (1.11), in the CMS is

$$V^{(1)}(\mathbf{p}, \mathbf{q}) = \frac{\langle \mathbf{p}|J_{1\mu}(0)|\mathbf{q}\rangle}{2\sqrt{\varepsilon_1(\mathbf{p})\varepsilon_1(\mathbf{q})}} D^{\mu\nu}(k) \frac{\langle -\mathbf{p}|J_{2\nu}(0)|-\mathbf{q}\rangle}{2\sqrt{\varepsilon_2(\mathbf{p})\varepsilon_2(\mathbf{q})}}; \quad \mathbf{k} = \mathbf{p} - \mathbf{q}; \quad k^0 = 0. \tag{2.1}$$

The one-particle matrix elements of the electromagnetic current operator have the well-known structure

$$\langle \mathbf{p}|J_\mu(0)|\mathbf{q}\rangle = \bar{u}(p)\left[\gamma_\mu \rho(k^2) + \frac{i}{2m}\sigma_{\mu\nu}k^\nu f(k^2)\right]u(q); \quad k^\nu = (p-q)^\nu, \tag{2.2}$$

where $\rho_{1,2}(0) = e_{1,2}$; $f_{1,2} = e_{1,2}\varkappa_{1,2}$; $\varkappa_{1,2}$ is the anomalous magnetic moment (in corresponding magnetons). It is convenient to choose the photon propagation function $D^{\mu\nu}$ in the so-called Coulomb gauge:

$$D^{00}(k) = -\frac{d(k^2)}{\mathbf{k}^2}; \quad D^{ij}(k) = -\frac{d(k^2)}{k^2}\left(\delta_{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right); \quad D^{0i} = D^{i0} = 0.$$

Here vacuum polarization has been taken into account. Using the explicit form of the spinors (1.5), performing the expansion in Eq. (2.1) in $(\mathbf{p}^2/m^2)$ and keeping only first-order terms in this parameter, we obtain

$$V^{(1)}(\mathbf{p}, \mathbf{q}) = \frac{d(k^2)}{\mathbf{k}^2}\left\{G_{E1}(k^2)G_{E2}(k^2) + \rho_1(k^2)\rho_2(k^2)\left[\frac{\mathbf{p}^2+\mathbf{q}^2}{4m_1m_2} + \frac{\mathbf{p}^2\mathbf{q}^2}{m_1m_2\mathbf{k}^2} - \frac{\mathbf{k}^2}{8\mu^2}\right] + \frac{i}{4m_1}[\mathbf{p}\times\mathbf{q}]\sigma_1\rho_2(k^2)\left[\left(\frac{1}{m_1} + \frac{2}{m_2}\right)\rho_1(k^2) + \frac{2}{\mu}f_1(k^2)\right] + \frac{i}{4m_2}[\mathbf{p}\times\mathbf{q}]\sigma_2\rho_1(k^2)\left[\left(\frac{1}{m_2} + \frac{2}{m_1}\right)\rho_2(k^2) + \frac{2}{\mu}f_2(k^2)\right] - \frac{1}{4m_1m_2}G_{M1}(k^2)G_{M2}(k^2)\left|(\sigma_1\sigma_2)\mathbf{k}^2 - (\mathbf{k}\cdot\sigma_1)(\mathbf{k}\cdot\sigma_2)\right|\right\}, \tag{2.3}$$

$$k^2 = -\mathbf{k}^2, \qquad \mu = \frac{m_1m_2}{m_1+m_2},$$

where we have introduced the usual Sachs form factors:

$$G_E(k^2) = \rho(k^2) - \frac{k^2}{4m^2}f(k^2); \quad G_M(k^2) = \rho(k^2) + f(k^2).$$

It is convenient to separate the purely Coulomb potential and put $V^{(1)}$ in the form

$$V^{(1)} = V_C + \Delta V^{(1)}; \quad V_C(k^2) = \frac{e_1e_2}{\mathbf{k}^2}. \tag{2.4}$$

Henceforth we will set

$$e_1 = -e; \quad e_2 = Ze.$$

As an initial approximation it is natural to select the nonrelativistic Schrödinger equation (1.12) with the Coulomb potential:

$$\left(W - \frac{\mathbf{p}^2}{2\mu}\right)\Psi_B(\mathbf{p}) = -\frac{Ze^2}{(2\pi)^3}\int d\mathbf{q}\,\frac{\Psi_B(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2} \tag{2.5}$$

with the only difference that the wave function here is two-component for each of the particles. The solution of this equation is most easily obtained by the Fock method. This gives the well-known Coulomb energy levels

$$W_C = M_B - m_1 - m_2 = -\mu\frac{(Z\alpha)^2}{2n^2}, \quad \alpha = \frac{e^2}{4\pi}, \quad n = 1, 2, \ldots \tag{2.6}$$

and Pauli-type wave functions, which for the S state have the very simple form

$$\Psi_B^{(C)}(\mathbf{p}) = \psi_C(\mathbf{p})w; \quad \psi_C(\mathbf{p}) = \tilde{\psi}_C(0)\frac{8\pi Z\alpha\mu}{(p^2+\mu^2Z^2\alpha^2)^2}; \quad \tilde{\psi}_C(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{p}e^{i\mathbf{p}\mathbf{r}}\psi_C(\mathbf{p}) = \sqrt{\frac{(\mu Z\alpha)^3}{\pi n^3}}e^{-\mu(Z\alpha)r}, \tag{2.7}$$

where w is a two-component spinor.

A system in which Coulomb interaction plays the main role is weakly bound, and the relativistic effects in it are small, since the mean value of $\mathbf{p}^2$ is

$$\langle \mathbf{p}^2\rangle = (Z\alpha)^2\mu^2. \tag{2.8}$$

Now let the mass of one of the particles be much greater than the other, e.g.,

$$m_2 \gg m_1.$$

Examples of such systems are hydrogen and muonium atoms. In this case, keeping only linear terms in (m_1/m_2) and using (2.3), we can put the correction $\Delta V^{(1)}$ in Eq. (2.4) in the form

$$\Delta V^{(1)} = V_{fs}^{(1)} + V_s^{(1)} + V_{hfs}^{(1)} + V_0^{(1)}, \qquad (2.9)$$

where

$$V_{fs}^{(1)} = i\frac{[\mathbf{p}\times\mathbf{q}]\,\boldsymbol{\sigma}_1}{4m_1^2\mathbf{k}^2} d(k^2)\,\rho_2(k^2)\left\{\left(1+2\frac{m_1}{m_2}\right)\rho_1(k^2) + 2\left(1+\frac{m_1}{m_2}\right) f_1(k^2)\right\} - \frac{1}{8\mu^2}\rho_1(k^2)\,\rho_2(k^2); \qquad (2.10)$$

$$V_s^{(1)} = \{d(k^2)\,G_{E1}(k^2)\,G_{E2}(k^2) - e_1e_2\}\frac{1}{\mathbf{k}^2}; \qquad (2.11)$$

$$V_{hfs}^{(1)} = i\frac{[\mathbf{p}\times\mathbf{q}]\,\boldsymbol{\sigma}_2}{2m_1m_2\mathbf{k}^2} d(k^2)\,\rho_1(k^2)\,G_{M2}(k^2) - \frac{d(k^2)}{4m_1m_2} G_{M1}(k^2)\,G_{M2}(k^2)\left\{(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) - \frac{1}{\mathbf{k}^2}(\mathbf{k}\cdot\boldsymbol{\sigma}_1)(\mathbf{k}\cdot\boldsymbol{\sigma}_2)\right\}; \qquad (2.12)$$

$$V_0^{(1)} = \rho_1(k^2)\,\rho_2(k^2)\left\{\frac{\mathbf{p}^2+\mathbf{q}^2}{4m_1m_2\mathbf{k}^2} + \frac{\mathbf{p}^2\mathbf{q}^2}{m_1m_2\mathbf{k}^4}\right\}. \qquad (2.13)$$

In addition, there is a correction to the kinetic energy from the expansion of the square root in Eq. (1.7):

$$V_{kin} = -\left(\frac{\mathbf{p}^4}{8\mu^3} - \frac{3\mathbf{p}^4}{8m_1^3m_2}\right)(2\pi)^3\delta(\mathbf{p}-\mathbf{q}). \qquad (2.14)$$

The shift of the energy level is determined from the usual perturbation theory for the Schrödinger equation, and in first order in perturbation of $\Delta V^{(1)}$ we find

$$\Delta W = \langle B|\Delta V^{(1)}|B\rangle = \frac{1}{(2\pi)^6}\int d\mathbf{p}\,d\mathbf{q}\,\Psi_B^*(\mathbf{p})\,\Delta V^{(1)}(\mathbf{p},\mathbf{q})\,\Psi_B(\mathbf{q}). \qquad (2.15)$$

We now discuss the physical sense of the different terms in Eq. (2.8). The potential $V_{fs}^{(1)}$ is independent of the spin of the second particle and together with the first term in the correction (2.14) to the kinetic energy gives the so-called fine splitting of the Coulomb energy levels (2.7). The term $V_s^{(1)}$ is the correction for particle structure and vacuum polarization. In conjunction with the part of the potential $V_{fs}^{(1)}$ proportional to the form factor $f_1(k^2)$ it leads to the Lamb shift. The term $V_{hfs}^{(1)}$ contains the dependence on the spin of the second particle and causes hyperfine splitting of the energy levels. This splitting is approximately (m_2/m_1) times less than the fine splitting. The last term $V_0^{(1)}$, together with the second term in the correction (2.14) to the kinetic energy, does not give additional splitting of the levels, but merely leads to a general shift of the energy levels [17], which depends on the principal quantum number n.

We now consider in a little more detail the case of the hydrogen atom (particle 1 is an electron, and particle 2 is a proton). Since, as already noted above, the relativistic corrections to a HL system are small, we need only use the expressions, obtained within the framework of quantum electrodynamics [19, 37], for the form factors and functions $d(k^2)$ close to the zero in momentum transfer k^2:

$$d(k^2) = 1 - \left\{\frac{\alpha}{15\pi} + \frac{\alpha^2}{4\pi^2}\left(1+\frac{1}{81}\right)\right\}\frac{k^2}{m_1^2} + \ldots; \qquad (2.16)$$

$$\rho_1(k^2) = -e + k^2\left.\frac{d\rho_1(k^2)}{dk^2}\right|_{k^2=0} + \ldots;$$

$$m_1^2\left.\frac{d\rho_1(k^2)}{dk^2}\right|_{k^2=0} = -e\left\{\frac{\alpha}{3\pi}\left(\ln\frac{m_1}{2\lambda} + \frac{11}{24}\right) + \frac{\alpha^2}{\pi^2}0.470\right\}; \qquad (2.17)$$

$$f_1(k^2) \approx f_1(0) = -e\left(\frac{\alpha}{2\pi} - \frac{\alpha^2}{\pi^2}0{,}32848\right) = -e\varkappa_1; \qquad (2.18)$$

$$G_{E2}(k^2) = Ze + \frac{Ze}{6}\langle r_{E2}^2\rangle k^2 + \ldots. \qquad (2.19)$$

The term of order α^2 in the expression for the function $d(k^2)$ was calculated in [18]. The form factor $\rho_1(k^2)$ contains the infrared divergence [19], which makes it necessary to introduce a cutoff parameter λ for the photon momentum near zero. This is a consequence of the approximation which we made when we regarded the electron as free in calculation of the form factor. If we take into account that the electron is actually bound, the parameter λ is replaced [17] by a quantity of order $W_C \sim (Z\alpha)^2\mu$. We deal now with the correction of order α^2 to the slope of the form factor $\rho_1(k^2)$ when $k^2 = 0$. The analytical calculation of this quantity made in [20] was erroneous. This was first detected by Appelquist and Brodsky [21] by means of a numerical calculation. They obtained a contribution equal to $(\alpha^2/\pi^2)(0.48 \pm 0.07)$. This result was confirmed in [22]. A new calculation of this quantity was subsequently made in [23], which led to the more accurate value indicated in Eq. (2.17).

Substituting now the expression (2.10) and the first term from (2.14) in Eq. (2.15), and using Eqs. (2.7) and (2.6)-(2.19), we obtain a correction leading to fine splitting of the levels (2.6):

a) for the S states, $l = 0$ (l is the orbital quantum number),

$$\Delta W_{fs}^{(1)} = -\frac{(Z\alpha)^4}{2n^3}\mu\left(1-\frac{3}{4n}\right); \quad (2.20)$$

b) for all other states, $l \neq 0$;

$$\Delta W_{fs}^{(1)} = -\frac{(Z\alpha)^4}{2n^3}\mu\left\{\left[\frac{1}{j+\frac{1}{2}}-\frac{3}{4n}\right] - 2\varkappa_1\left[\frac{1}{l+\frac{1}{2}}-\frac{1}{j+\frac{1}{2}}\right]\left(1-\frac{m_1}{m_2}\right)\right\};$$

$$j = l \pm \frac{1}{2}. \quad (2.21)$$

Here we continue to write Z, although Z = 1 for the proton, in order to distinguish the radiation corrections from relativistic effects.

The fine-splitting ΔE of the levels ($2P_{3/2} - 2P_{1/2}$) is of particular interest, since a very accurate experimental value is known for it and it is also insensitive to radiation corrections:

$$\Delta E = \frac{(Z\alpha)^4}{32}\mu\left\{\left(1+\frac{5}{8}(Z\alpha)^2\right) + 2\varkappa_1\left(1-\frac{m_1}{m_2}\right) - 2\frac{\alpha}{\pi}(Z\alpha)^2\ln(Z\alpha)^{-1}\right\}, \quad (2.22)$$

where the term of order $(Z\alpha)^2[fs]$ can be obtained from expansion in $(Z\alpha)^2$ of the exact Dirac equation for the fine structure [17].

The "structural" correction (2.11) leads to an additional (Lamb) shift of the levels:

$$\Delta W_s^{(1)} = -|\widetilde{\psi}_C(0)|^2\left\{d(0)\,G_{E2}(0)\frac{d\rho_1}{dk^2}\Big|_{k^2=0} + \frac{1}{4m_1^2}f_1(0)\,d(0)\,G_{E2}(0) + \rho_1(0)\,G_{E2}(0)\times\frac{dd(k^2)}{dk^2}\Big|_{k^2=0} + \rho_1(0)\,d(0)\frac{dG_{E2}}{dk^2}\Big|_{k^2=0}\right\}. \quad (2.23)$$

Elimination of the infrared divergence by explicit allowance for the boundedness of the system leads to the so-called Bethe logarithm [17], the numerical values of which are well known. As a result, using Eqs. (2.10) and (2.18), we obtain the shift of the S levels ($l = 0$) in the form

$$\Delta W_{Ls}^{(1)} = \frac{4\alpha(Z\alpha)^4\mu^3}{3\pi n^3 m_1^2} \times\left\{\ln(Z\alpha)^{-2} + \ln\frac{(Z\alpha)^2\mu}{2\Delta\varepsilon_{n,0}} + \frac{19}{30} + \ln\left(1+\frac{m_1}{m_2}\right) - \frac{3\alpha}{4\pi}\left[0.328 - 4\cdot 0.443 + \left(1+\frac{1}{81}\right)\right]\right\} + 2\frac{(Z\alpha)^4}{3n^3}\mu^3\langle r_{E2}^2\rangle, \quad (2.24)$$

where $\Delta\varepsilon_{n,l}$ is the mean excitation energy introduced by Bethe [17]. For all other states with $l \neq 0$, combining the term dependent on l in Eq. (2.21) and the contribution from expression (2.23), we obtain after elimination of the infrared divergence

$$\Delta W_{Ls}^{(1)} = \frac{4\alpha(Z\alpha)^4\mu^3}{3\pi n^3 m_1^2}\left\{\ln\frac{(Z\alpha)^2\mu}{2\Delta\varepsilon_{n,l}} + \frac{3}{8}\left(1-2\frac{\alpha}{\pi}0.328\right)\left[\frac{1}{l+\frac{1}{2}}-\frac{1}{j+\frac{1}{2}}\right]\times\left(1+\frac{m_1}{m_2}\right)\right\}, \quad l \neq 0. \quad (2.25)$$

Equations (2.24) and (2.25) give theoretical expressions for the radiation shift of the energy levels, usually called the Lamb shift, in the one-phonon exchange approximation with nuclear motion taken into account.

Potential (2.13) in conjunction with the second term from the correction (2.14) leads to the total shift of the levels [17]:

$$\Delta W_0^{(1)} = -\frac{(Z\alpha)^4\mu^3}{8n^4 m_1 m_2}, \quad (2.26)$$

which is independent of j and l.

We proceed now to the part of the potential leading to hyperfine splitting. We will not give here the well-known equation for hyperfine splitting of an arbitrary level [17], but will discuss in more detail the splitting of the S levels. In this case, substituting expression (2.12) in Eq. (2.15) and putting $d(k^2) \approx 1$; $G_{M1}(k^2) \approx -e(1+\varkappa_1)$, we have

$$\Delta W_{hfs}^{(1)} = \frac{2\pi Z\alpha}{3m_1 m_2}(1+\varkappa_1)(1+\varkappa_2)\langle\sigma_1\cdot\sigma_2\rangle\left\{|\widetilde{\psi}_C(0)|^2 + \int\frac{dp\,dq}{(2\pi)^6}\psi_C^*(p)\left[\frac{1}{Ze(1+\varkappa_2)}G_{M2}(k^2)-1\right]\psi_C(q)\right\}, \quad k^2 = -(p-q)^2. \quad (2.27)$$

The first term in Eq. (2.27) is the well-known Fermi relation and the second is the correction due to the electromagnetic structure of the proton (nucleus). We cannot use here, however, the expansion of the form factor in powers of k^2, since the corresponding integrals diverge. Nevertheless, the main contribution to the integral is made by relatively small k^2 and hence the second term in Eq. (2.27) is of the same order of magnitude as the contribution from two-photon exchange. In fact, since the integrand becomes zero when $k^2 = 0$,

we can use an approximate expression for the Coulomb wave function (2.7) of the S state:

$$\psi_C(\mathbf{p}) = \tilde{\psi}_C(0)\,[(2\pi)^3\,\delta(\mathbf{p}) + F(\mathbf{p})\,V_C(\mathbf{p}) + \ldots], \quad (2.28)$$

which is an expansion in powers of $(Z\alpha)$. After substitution of this expression, Eq. (2.27) becomes

$$\Delta W^{(1)}_{hfs} = \frac{2\pi(Z\alpha)}{3m_1m_2}(1+\varkappa_1)(1+\varkappa_2)\,|\tilde{\psi}_C(0)|^2\,\langle\sigma_1\cdot\sigma_2\rangle \times\left\{1 + 2\int\frac{d\mathbf{k}}{(2\pi)^3}V_C(\mathbf{k})\,F(\mathbf{k})\,F(\mathbf{k}) \times\left[\frac{1}{Ze(1+\varkappa_2)}G_{M2}(k^2) - 1\right]\right\}, \quad (2.29)$$

where $F(\mathbf{k}) = [W_C - (\mathbf{k}^2/2\mu)]^{-1}$.

Thus, we can compare the second term in Eq. (2.29) with the difference in two diagrams, in which there is an exchange of one Coulomb and one transverse (magnetic) photon. We will return to this equation in the next section.

If instead of nonrelativistic Coulomb functions we use the more accurate wave functions, we can also obtain [12, 24] corrections of order $(Z\alpha)^2 \times (m_1/m_2)[hfs]$, which depend on the state (i.e., on the principal quantum number n).

3. Corrections for two-photon exchange. The scattering amplitude corresponding to two-photon exchange between particles of a system can be represented by a Feynman diagram (Fig. 2).

We have neglected the contribution of vacuum polarization, since it is an effect of higher order. In the case of HL systems, where one of the particles is much heavier than the other ($m_2 \gg m_1$), it is convenient to consider separately the corrections to the fine and hyperfine splitting, since the latter is m_2/m_1 times less than the former.

We begin with the corrections to the fine splitting. These corrections in turn can be divided into two categories. The first includes relativistic and radiation corrections, which remain when $m_2 \to \infty$ and have relative order $(Z\alpha)^2[fs]$, $\alpha(Z\alpha)\cdot[fs]$, and $\alpha(Z\alpha)^2\ln^k(Z\alpha)[fs]$, $k = 0, 1, 2$. Terms of order $(Z\alpha)^2[fs]$ can easily be obtained from the expansion of the exact Dirac equation for the fine structure [17]. The other terms have been fully discussed in Erickson and Yennie's excellent paper [25]. We will not dwell in detail on these corrections, since they do not reflect the characteristic features of the two-particle problem – nuclear (proton) recoil effects.

A secondary category of corrections is associated with the finiteness of the nuclear (proton) mass. Since we have considered the effects of structure and the anomalous magnetic moment of the proton in the one-photon term, we can now consider the proton as a point Dirac particle. Thus, we must consider the contribution to the potential from the diagrams shown in Fig. 3.

A calculation of these effects on the basis of the Bethe–Salpeter equation was first carried out by Salpeter [26] and was subsequently confirmed by Fulton and Martin [27]. These calculations were later corroborated by Grotch and Yennie [12], using the effective-potential model (see Introduction). Their results agreed with the results of the previous papers. Here we give only the final expressions for the corrections to the 2S and 2P levels of most practical interest:

$$\Delta W^{(2)}_{fs}(2S) = \frac{(Z\alpha)^5\mu^3}{8\pi m_1m_2}\left\{\frac{8}{3}\ln\frac{Z\alpha m_1}{2\Delta\varepsilon_{2,0}} - \ln(Z\alpha)^{-2} + \frac{187}{18} - \frac{2}{m_2^2-m_1^2}\left[m_1^2\ln\frac{\mu}{m_2} - m_2^2\ln\frac{\mu}{m_1}\right]\right\}; \quad (3.1)$$

$$\Delta W^{(2)}_{fs}(2P) = \frac{(Z\alpha)^5\mu^3}{8\pi m_1m_2}\left\{\frac{8}{3}\ln\frac{(Z\alpha)^2 m_1}{2\Delta\varepsilon_{2,1}} - \frac{7}{18}\right\}. \quad (3.2)$$

These expressions were obtained without assuming $m_1/m_2 \ll 1$. In the case where this ratio is small (hydrogen and muonium atoms), the term with the square brackets in Eq. (3.1) is proportional to an additional factor (m_1/m_2) and can be neglected. Corrections (3.1) and (3.2) contribute to Lamb splitting of the levels, since they depend on l. More accurate calculations of the two-photon corrections to the fine structure with realistic proton form factors taken into account have not been carried out so far. Corrections for polarizability of the proton, as was shown in [28], are negligible.

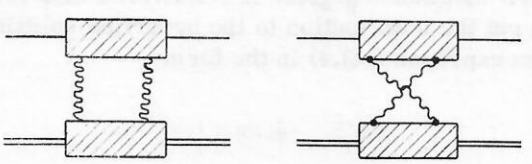

Fig. 2

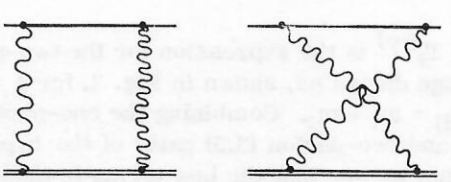

Fig. 3

We now consider in a little more detail the scheme of calculation [15] of corrections to hyperfine splitting in the hydrogen (or muonium) atom on the basis of Eq. (1.7). As in the case of corrections to the fine splitting, radiation and relativistic effects can be treated separately by considering the interaction with the static magnetic moment of the proton [29]. Hence, we will concentrate our attention on structure and nuclear (proton or muon) recoil effects.

The correction to the energy levels, according to the usual perturbation theory, has the form

$$\Delta W_B^{(2)} = \langle B | V^{(2)} | B \rangle + \sum_{n \neq B} \langle B | \Delta V^{(1)} | n \rangle \frac{1}{W_B - W_n} \langle n | \Delta V^{(1)} | B \rangle. \quad (3.3)$$

We will be interested subsequently in the hyperfine splitting of only the S levels. Hence, we will neglect all relativistic corrections of order $(p^2/m_1^2) \sim (Z\alpha)^2$. In view of this, the sum over the complete set of intermediate states of the system in Eq. (3.3) can be replaced, without reducing the accuracy of approximation [30], by the sum over the free-particle states. After this substitution Eq. (3.3) can be written schematically in the form

$$\Delta W_B^{(2)} = \langle V^{(2)} \rangle - \langle \Delta V^{(1)} G' \Delta V^{(1)} \rangle.$$

Substituting in this equation the expression for $V^{(2)}$ from Eq. (1.10) and using (2.4), we finally obtain

$$\Delta W_B^{(2)} = \langle T^{(2\gamma)} \rangle - \langle V_C G' \Delta V^{(1)} \rangle - \langle \Delta V^{(1)} G' V_C \rangle - \langle V_C G' V_C \rangle. \quad (3.4)$$

The last term in Eq. (3.4) gives no corrections to the hyperfine splitting. Taking into account the above-mentioned neglect of relativistic effects, we can put the contribution to the hyperfine splitting from expression (3.4) in the form

$$\Delta W_{hfs}^{(2)} = |\tilde{\psi}_C(0)|^2 \left\{ \langle T_0^{(2\gamma)} \rangle - 2 \frac{2\pi Z\alpha}{3 m_1 m_2} \langle \sigma_1 \cdot \sigma_2 \rangle \int \frac{d\mathbf{k}}{(2\pi)^3} V_C(\mathbf{k}) F(\mathbf{k}) \frac{1}{Ze} G_{M2}(k^2) \right\}, \quad (3.5)$$

where $T_0^{(2\gamma)}$ is the expression for the two-photon exchange diagrams, shown in Fig. 2, for $\mathbf{p} = \mathbf{q} = 0$ and $M_B = m_1 + m_2$. Combining the one-photon (2.29) and two-photon (3.5) parts of the hyperfine splitting, we see that the last terms in these equations cancel each other [31], and we obtain the sum

$$\Delta W_{hfs} = |\tilde{\psi}_C(0)|^2 \left\{ \frac{2\pi (Z\alpha)}{3 m_1 m_2} (1 + \varkappa_1)(1 + \varkappa_2) \langle \sigma_1 \cdot \sigma_2 \rangle + \langle T_0^{(2\gamma)} \rangle - \frac{4\pi (Z\alpha)}{3 m_1 m_2} (1 + \varkappa_2) \langle \sigma_1 \cdot \sigma_2 \rangle \int \frac{d\mathbf{k}}{(2\pi)^3} F(\mathbf{k}) V_C(\mathbf{k}) \right\}. \quad (3.6)$$

The splitting of the triplet and singlet S energy levels can now be written as

$$\nu_{hfs} = W(^3S_1) - W(^1S_0) = \nu_F (1 + \varkappa_1 + \delta). \quad (3.7)$$

where

$$\nu_F = \frac{8 (Z\alpha)^4 \mu^3}{3 n^3 m_1 m_2} (1 + \varkappa_2) \quad (3.8)$$

is the well-known Fermi equation, and δ denotes corrections for structure and motion of the nucleus (proton or muon) and in the general case [32] is

$$\delta = \frac{(Z\alpha) m_1}{\pi (1 + \varkappa_2) m_2} \left\{ \frac{3 m_2^2}{i\pi^2} \int \frac{d^4k}{k^4} N_{\mu\nu}^1(k) N_2^{\mu\nu}(k) - 8 m_2 (1 + \varkappa_2) \int_0^\infty \frac{dk}{\mathbf{k}^2 - 2\mu W_C} \right\}, \quad (3.9)$$

where

$$N_{\mu\nu}^{1,2} = \frac{1}{4} \mathrm{Tr} \left[C_{\mu\nu}^{1,2} \frac{1 + \gamma_0}{2} \gamma_z \gamma_5 \right];$$

$C_{\mu\nu}^{1,2}(\mathbf{k})$ is the amplitude of virtual Compton forward scattering.

We note that only the spin-dependent part of this amplitude contributes to the hyperfine splitting. The simplest result is obtained if the two particles are point Dirac particles, as is the case, for instance, of muonium and positronium. The Compton amplitudes then reduce to simple pole diagrams (Fig. 4). The expression (3.9) in this case reduces to

$$\delta_m = -\frac{3 (Z\alpha) m_1 m_2}{\pi (m_2^2 - m_1^2)} \ln \frac{m_2}{m_1}. \quad (3.10)$$

The situation is much more complicated in the case of the hydrogen atom. The amplitude of the virtual Compton effect on the electron is given, as before, by the diagram in Fig. 4. The amplitude of virtual Compton scattering on the proton, however, is not actually known owing to strong interaction effects. From this amplitude we separate the contribution of the one-nucleon pole term and illustrate it graphically (Fig. 5).

This can be done most consistently by writing the dispersion relations for the invariant amplitudes [31-33]. The pole term is completely un-

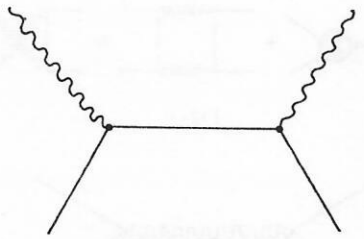

Fig. 4

known in this case, since it is expressed in terms of the electromagnetic form factors of the real proton.

As regards the contribution of many-particle intermediate states, its imaginary part can be determined in principle from experiments on inelastic scattering of electrons by protons at high energies. Such experiments have been carried out and widely discussed [34]. The difficulty, however, lies in the fact that obtaining the spin-dependent part of the scattering amplitude requires experiments with polarized electrons and protons, which have not yet been carried out.

Various models have been used to assess the contribution of inelastic intermediate states [31-33]. An interesting detail was discovered. It was found that it was convenient to separate from the integral in the dispersion relations the term which together with the contribution from the nucleon pole gives an expression for the modified Feynman pole diagram. We have in mind that, as before, the form factors of the real proton are at the vertices of this diagram. Then, as calculations show, the remaining part of the dispersion integral usually gives a small contribution of the order $(3-5)\cdot 10^{-6}$ in comparison with the contribution of the modified pole term, which is $(35-40)\cdot 10^{-6}$ (depending on the particular expression chosen for the form factors).

This can probably be attributed to the fact that the presence of an electron and two photon propagators ensures fairly good convergence of the integrals with respect to photon momentum, even in the case of a relatively weak falloff of the Compton amplitude. Hence, the main contribution to the integral is made by small momentum values: $k^2 \ll m_2^2$. At the same time, the contribution of the low-momentum region of the subtracted dispersion integral is reduced owing to the validity of the low-energy theorem for the corresponding invariant amplitude. This question has been discussed more fully in [31, 32].

As an illustration we give the expression for δ obtained in [35] by considering only a modified pole term and taking the form factor as

$$\rho^2(k_2) = \frac{1}{1+\varkappa_2} f_2(k^2) = e_2 \frac{\Lambda^2}{\Lambda^2 - k^2}; \quad \varkappa_2 = 1.79,$$

where k^2 is the square of the 4-momentum. Then we have

$$\sigma_N = -\frac{3(Z\alpha)m_1}{\pi m_2}\left\{\frac{4-\varkappa_2^2}{4(1+\varkappa_2)}\ln\frac{\Lambda m_2}{m_1} - \frac{\varkappa_2^3}{16(1+\varkappa_2)} + \frac{8\Lambda^{-2}+\varkappa_2-1}{\sqrt{4\Lambda^{-2}-1}}\operatorname{arctg}\sqrt{4\Lambda^{-2}-1}\right\}, \quad Z=1. \tag{3.11}$$

where the parameter Λ is expressed in proton masses. It follows from the data for the root-mean-square radius of the proton that $\Lambda = 0.64$. Substituting this value in expression (3.11), we obtain

$$\delta_N = -34\cdot 10^{-6}. \tag{3.12}$$

Estimates of the remaining part of the proton polarization effect [31-33] give

$$|\delta_{pol}| = (1-6)\cdot 10^{-6}. \tag{3.13}$$

The relativistic and radiation corrections of order $(Z\alpha)^2[hfs]$, $\alpha(Z\alpha)[hfs]$, and $\alpha(Z\alpha)^2[hfs]$ have also been calculated. A detailed discussion of these corrections and a bibliography can be found in Brodsky and Erickson's paper [29].

This leads to the following expression for the splitting of the triplet and singlet S levels of the hydrogen atom:

$$\nu_{hfs} = \nu_F[1 + \varkappa_1 + \delta_N + \delta_{pol} + \varepsilon],$$

where

$$\varepsilon = \frac{3}{2}(Z\alpha)^2 + \alpha(Z\alpha)\left(\ln 2 - \frac{5}{2}\right) -$$

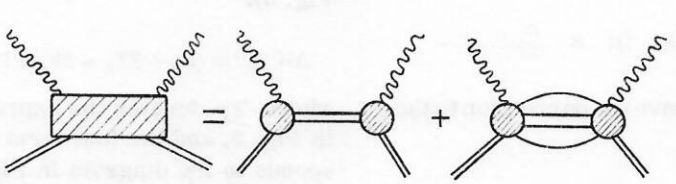

Fig. 5

$$-\frac{8\alpha}{3\pi}(Z\alpha)^2\ln(Z\alpha)\left[\ln(Z\alpha)-\ln 4+\frac{281}{480}\right]$$
$$+\frac{\alpha}{\pi}(Z\alpha)^2(18.4\pm 5). \qquad (3.14)$$

This equation is valid for muonium, too, if δ is given by expression (3.10). In addition, corrections of order $(m_1/m_2)(Z\alpha)^2\ln(Z\alpha)$ [hfs] were recently calculated for muonium with the aid of the Bethe–Salpeter equation [36]:

$$\Delta\delta_m=\frac{9m_1m_2}{2(m_1+m_2)^2}(Z\alpha)^2\ln(Z\alpha)^{-1},\quad Z=1. \qquad (3.15)$$

This result is obtained if the dependence on the external momenta [see Eq. (3.4)] is retained in the expression for the two-photon exchange amplitude.

4. Positronium energy levels.

Positronium is a HL system consisting of an electron and a positron. It follows from the CP invariance of electromagnetic interactions that the total spins s of positronium, which takes values of 0 and 1, is a conserved quantity. In accordance with this, all the positronium energy levels can be divided into singlet levels with s = 0 (parapositronium) and triplet levels with s = 1 (orthopositronium). Since the masses of the electron and the positron are the same, the fine and hyperfine splitting in positronium are of the same order of magnitude. An exception is the S levels, for which there is only hyperfine splitting into triplet and singlet levels.

Another specific feature of positronium is the presence of additional exchange interaction due to virtual annihilation of the electron and positron.

Putting the electron and positron form factors equal to their values at zero and $\varkappa_1=\varkappa_2=\varkappa$, $m_1=m_2=m$, $\mu=m/2$, $e_2=-e_1=e$, we can write expression (2.3) for the one-photon exchange potential in the form

$$V^{(1)}(\mathbf{p},\mathbf{q})=-e^2\frac{d(k^2)}{\mathbf{k}^2}-e^2\left\{\frac{\mathbf{p}^2+\mathbf{q}^2}{4m^2\mathbf{k}^2}+\frac{\mathbf{p}^2\mathbf{q}^2}{m^2\mathbf{k}^4}-\frac{1}{2m^2}\right\}$$
$$-\frac{e^2}{2m^2}i[\mathbf{p}\times\mathbf{q}]\mathbf{s}(3+4\varkappa)+\frac{e^2}{2m^2}(1+\varkappa)^2\left[\mathbf{s}^2-1-\frac{1}{\mathbf{k}^2}(\mathbf{k}\cdot\mathbf{s})^2\right], \qquad (4.1)$$

where

$$\mathbf{s}=\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2);\quad \mathbf{s}^2=s(s+1);\quad \varkappa=\frac{\alpha}{2\pi}+\ldots.$$

In addition, as before, we have the correction to the kinetic energy:

$$\Delta V_{kin}=-\frac{\mathbf{p}^4}{4m^3}(2\pi)^3\delta(\mathbf{p}-\mathbf{q}). \qquad (4.2)$$

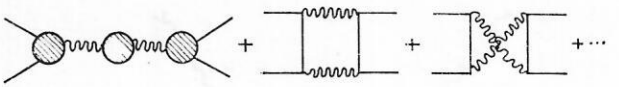

Fig. 6

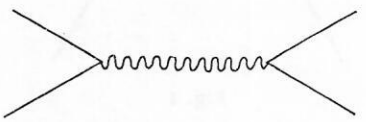

Fig. 7

The exchange interaction can be represented by the diagrams in Fig. 6. In the lowest order of perturbation theory we have only one diagram (Fig. 7), which makes the contribution

$$V_a\approx\frac{e^2}{4m^2}\mathbf{s}^2. \qquad (4.3)$$

We will not dwell on the fine splitting of the positronium energy levels in the lowest approximation, since it has been thoroughly analyzed elsewhere, in [19] for instance. We will concentrate our attention instead on the hyperfine splitting of the S levels, since this quantity for the ground level (n = 1) has been measured very accurately experimentally and provides another test for quantum electrodynamics. This quantity was first calculated by Karplus and Klein [30] on the basis of the Bethe–Salpeter equation. Here we will use the quasipotential equation (1.7).

The contribution to the hyperfine splitting from the one- and two-photon exchange (direct-interaction) diagrams is easily obtained from the results of the preceding sections. Substituting expression (3.7) in Eq. (3.6), we find for the ground level (n = 1)

$$\nu^a_{pos}=\Delta W^a_{pos}(1^3S_1-1^1S_0)=\frac{1}{3}\alpha^4m\left(1-\frac{\alpha}{2\pi}\right). \qquad (4.4)$$

We turn now to exchange interaction. The calculation procedure is similar to that discussed in the previous section, and we will not dwell on the details of the calculations, but merely give the final results [30, 38]. We consider first the contribution from the one-photon annihilation diagrams (see Fig. 6):

$$\Delta W^{ex(1)}_{pos}=\langle V_a+2T_\Gamma-2V_CG'V_a+[d(4m^2)-1]V_a\rangle, \qquad (4.5)$$

where T_Γ denotes the expression for the diagram in Fig. 8, and the last term in Eq. (4.5) corresponds to the diagram in Fig. 9. Using the explicit expression for $d(k^2)$ it is easy to find [37]

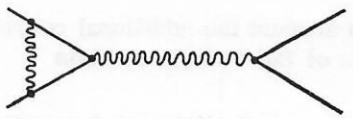

Fig. 8

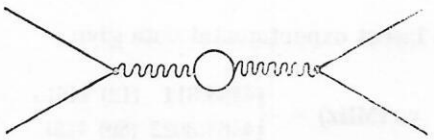

Fig. 9

$$d(4m^2)-1=-\frac{8\alpha}{9\pi}. \qquad (4.6)$$

As a result we obtain

$$\Delta W_{pos}^{ex(1)}=\frac{\pi\alpha}{m^2}|\tilde{\psi}_C(0)|^2\left(1-\frac{44\alpha}{9\pi}\right)\langle s^2\rangle. \qquad (4.7)$$

The contribution from the two-photon annihilation diagrams (see Fig. 6) is

$$\Delta W_{pos}^{ex(2)}=\frac{\pi\alpha}{m^2}|\tilde{\psi}_C(0)|^2\frac{\alpha}{\pi}[2(1-\ln 2)+i\pi]\langle s^2-2\rangle. \qquad (4.8)$$

The imaginary part in expression (4.8) represents inelastic processes in the system, which lead to a complex quasipotential. In this case this imaginary part gives the probability of the decay of parapositronium into two photons:

$$w_{2\gamma}=-2\,\mathrm{Im}\,\Delta W=\frac{1}{2}\alpha^5 m=0.804\times 10^{10}\ \mathrm{sec}^{-1}.$$

Collecting expressions (4.4), (4.7), and (4.8), we obtain the splitting of the singlet and triplet ground levels of positronium, accurate to terms of order α^5 inclusively. The Bethe–Salpeter equation was recently used [39] to calculate the correction of order $\alpha^6 \ln \alpha$, which can be obtained from Eq. (3.12) by putting $m_1 = m_2 = m$. Thus, we finally have

$$\nu_{pos}=\Delta W(1^3S_1-1^1S_0)$$

$$=\frac{1}{2}\alpha^4 m\left\{\frac{7}{6}-\frac{\alpha}{\pi}\left(\frac{16}{9}+\ln 2\right)+\frac{3}{4}\alpha^2\ln\alpha^{-1}\right\}. \qquad (4.9)$$

5. Comparison of theory and experiment. The value of the fine-structure constant was recently obtained for the first time from experiments [40, 42] completely independent of quantum electrodynamics. By means of a method based on the use of the Josephson tunneling effect in superconductors, the ratio e/h, where h is Planck's constant, was measured. Combining this ratio with other fundamental constants gives [40]

$$\alpha^{-1}=137.03608\ (26)\ (1.9\ ppm). \qquad (5.1)$$

The figures in the parentheses indicate the uncertainty in the last figures of the main number, and 1 ppm = 10^{-6}. This new, more accurate value of α (see also [42]) allows a more rigorous and complete test of theory [34, 40, 41].

We begin with the Lamb splitting of the $2S_{1/2}-2P_{1/2}$ levels in the hydrogen atom. The theoretical value of this quantity can be obtained from Eqs. (2.24), (2.25), and (3.1) with the inclusion of the radiation corrections discussed in [25]. We will not write out this cumbersome expression here (it is given in [40]), but will merely give its numerical value when the new value (2.17) of the slope of the electron form factor at zero is used:

$$\mathfrak{L}_{\mathrm{H}}^{th}=\Delta W(2S_{1/2}-2P_{1/2})=1057.91\ (9)\ \mathrm{MHz}. \qquad (5.2)$$

A detailed analysis of the numerical contribution of the different terms can be found in [25, 41].

The experiment of Lamb et al. with the inclusion of new values of the physical constants and some additional effects [40] gives for hydrogen

$$\mathfrak{L}=1057.86\ (6)\ \mathrm{MHz}\ (60\ ppm). \qquad (5.3)$$

Recently Robiscoe et al. measured this quantity experimentally by a new method, using the level-crossing technique. They obtained the value [40]

$$\mathfrak{L}_{\mathrm{H}}=1057.90\ (6)\ \mathrm{MHz}. \qquad (5.4)$$

These two values are in good agreement with the theoretical prediction (5.2). Thus, the new calculations made in [22, 23] removed the discrepancies between theory and experiment which have existed for a long time.

We will not discuss here the Lamb shift in other atoms, since it is much more difficult to take into account the effects of nuclear structure and recoil in them. It would be extremely interesting to measure the Lamb shift in muonic hydrogen, since in this case vacuum polarization and proton structure effects make a much greater contribution [19, 41].

We turn now to the fine structure of the levels. There are accurate experimental data for the fine splitting of the $[2P_{3/2}-2P_{1/2}]$ levels. The theoretical value of this quantity (for hydrogen) is

$$\Delta E_{\mathrm{H}}^{th}=10969.026\ (42)\ \mathrm{MHz}. \qquad (5.5)$$

The direct experimental measurement of this quantity by Metcalf et al. [40] gives

$$\Delta E_{\mathrm{H}} = 10969.127\ (95)\ \mathrm{MHz}\ (8.7\ ppm). \qquad (5.6)$$

This value can also be obtained by combining the experimental value of the $[2P_{3/2} - 2S_{1/2}]$ splitting with the Lamb shift:

$$\Delta E = \Delta W\,(2P_{3/2} - 2S_{1/2}) + \mathcal{L}.$$

The quantity $(\Delta E_{\mathrm{H}} - \mathcal{L}_{\mathrm{H}})$ has been measured in several investigations. If for $\mathcal{L}_{\mathrm{H}}$ we take the value (5.4), which is regarded as most reliable at present, we find [40] that

$$\Delta E_{\mathrm{H}}\,(\mathrm{MHz}) = \begin{cases} 10969.274\ (68)\ \text{[Kaufman, Lamb, et al.]}, \\ 10969.109\ (86)\ \text{[Shyn, Robiscoe, et al.]}, \\ 10969.061\ (76)\ \text{[Vorburger and Cosens]}. \end{cases} \qquad (5.7)$$

The last two values are in reasonable agreement with the theoretical prediction (5.5).

An intriguing challenge to theoretical physics is the hyperfine splitting of the hydrogen ground level, which has now been measured with exceedingly high accuracy [43] and is the most accurately known physical quantity:

$$\nu_{\mathrm{H}} = 1420{,}405{,}751.7667\ (10)\ \mathrm{Hz}. \qquad (5.8)$$

The theoretical value obtained from Eq. (3.11) is

$$\nu_{\mathrm{H}}^{th} = 1420.4023\,(1 + \delta_{pol}) \pm 0.0057\ \mathrm{MHz}\,(4\ ppm), \qquad (5.9)$$

where the uncertainty is due mainly to the error in α. Comparing Eqs. (5.8) and (5.9) we have

$$\delta_{pol} = (2.5 \pm 4.0)\cdot 10^{-6}.$$

This conclusion is consistent with the expected small contribution (3.10) of proton "polarization" effects (see the discussion of this question in Sec. 3).

The difficulties associated with consideration of the contribution of strong interactions can be removed if a purely electrodynamic system like muonium is considered. In this case the value of δ is given by expression (3.7). Until recently, however, there has been a rather large uncertainty in the value of the magnetic moment of the muon in Bohr magnetons, which has prevented accurate comparison of theory and experiment. Recently published papers [44, 45] give a more accurate value of this quantity. The ratio of the magnetic moment of the muon to the magnetic moment of the proton was measured and found to be

$$a = \frac{\mu_m}{\mu_p} = \begin{cases} 3.183347\ (9)\ \ (2.8ppm)\ [44], \\ 3.183337\ (13)\ (4.2ppm)\ [45]. \end{cases} \qquad (5.10)$$

Taking into account the additional correction (3.12) on the basis of Eq. (3.11), we obtain

$$\nu_m^{th}\ (\mathrm{MHz}) = \begin{cases} 4463.323\ (19)\ \text{for}\ a\ [44], \\ 4463.309\ (22)\ \text{for}\ a\ [45]. \end{cases} \qquad (5.11)$$

The latest experimental data give

$$\nu_m\ (\mathrm{MHz}) = \begin{cases} 4463.311\ \ (12)\ [46], \\ 4463.3022\ (89)\ [45]. \end{cases} \qquad (5.12)$$

On the whole there is good agreement of theory and experiment.

For positronium the best experimental value of the ground-level splitting is [46]

$$\nu_{pos} = 203{,}403\ (12)\ \mathrm{MHz}\ (59\ ppm), \qquad (5.13)$$

which is in good agreement with the theoretical prediction (4.9):

$$\nu_{pos}^{th} = 203{,}415\ (9)\ \mathrm{MHz}. \qquad (5.14)$$

Thus, the above analysis shows that the quantum theory of bound states plus quantum electrodynamics give excellent results in describing two-particle HL systems. In the field of atomic physics at present there is not a single serious discrepancy between theory and experiment.

II. ELECTROMAGNETIC PROPERTIES OF HYDROGEN-LIKE ATOMS

6. Energy levels of a bound system in a weak external electromagnetic field. In the absence of an external field it was sufficient to conduct the whole treatment in the CMS. We now turn to the other problem, where wave functions in an arbitrary coordinate system are required. Let the HL system be situated in a weak, external, time-independent electromagnetic field. The Hamiltonian of the interaction with this field is

$$H_I(t) = \int d\mathbf{x} J_\mu(x) A^\mu(\mathbf{x}). \qquad (6.1)$$

We calculate the change in energy of the bound state from perturbation theory by expansion in powers of the external field.

In the lowest approximation, linear in the external field, we obtain

$$\left.\begin{aligned} &(2\pi)^3 \delta(\Delta)\, 2E_B \delta E_B \approx \langle A \mid H_I(t) \mid B \rangle; \\ &\mathbf{K}_A = \mathcal{P},\ \mathbf{K}_B = \mathbf{Q};\ \Delta = \mathcal{P} - \mathbf{Q}; \\ &E_B = \sqrt{M^2 + \mathcal{P}^2};\ M_A = M_B = M. \end{aligned}\right\} \qquad (6.2)$$

It is convenient to change to a Breit coordinate system by putting

$$\mathcal{P} = -Q = \frac{\Delta}{2}; \quad E_A = E_B = E = \sqrt{M^2 + \frac{\Delta^2}{4}}. \quad (6.3)$$

Here

$$\delta E_B \approx \frac{M}{E}\delta M.$$

The, using translational invariance and integrating both parts of Eq. (6.2) with respect to Δ, we have [8, 48]

$$\delta M = \int dx j_\mu^{AB}(x) A^\mu(x),$$

where

$$j_\mu^{AB}(x) = \frac{1}{2M}\int \frac{d\Delta}{(2\pi)^3} e^{-i\Delta x} J_\mu^{AB}(\Delta);$$
$$J_\mu^{AB}(\Delta) = \langle A | J_\mu(0) | B\rangle. \quad (6.4)$$

If the external field varies slowly in space, we can use the usual multipole expansion. In particular, in the case where a neutral bond system is situated in homogeneous electric and magnetic fields [8, 48] we have

$$\delta M = -(\mathfrak{D}\cdot\mathcal{E}) - (\mathfrak{M}\cdot\mathcal{H}), \quad (6.5)$$

where

$$\mathfrak{D} = \int dx x j_0^{AB}(x) = -\frac{i}{2M}\left\{\frac{\partial}{\partial\Delta} J_0^{AB}(\Delta)\right\}_{\Delta=0}; \quad (6.6)$$

$$\mathfrak{M} = \frac{1}{2}\int dx [x \times j^{AB}(x)] = -\frac{i}{4M}\left[\frac{\partial}{\partial\Delta}\times J^{AB}(\Delta)\right]_{\Delta=0}; \quad (6.7)$$

$\mathcal{E}$ and $\mathcal{H}$ are the electric and magnetic field strengths. Equations (6.5)-(6.7) contain a description of the linear Stark and Zeeman effects. For this we need to know the matrix elements of the electromagnetic current operator between the bound states of a two-particle system.

These matrix elements can be expressed [8, 14, 48] in terms of the bound-state wave functions in the following way:

$$\langle A | J_\mu(0) | B\rangle = \frac{1}{(2\pi)^6}\int d\overset{0}{p}\, d\overset{0}{q}\, \overline{\Psi}^{(+)}_{A\mathcal{P}}(\overset{0}{p})\, \Gamma_\mu(\overset{0}{p}, \overset{0}{q}; \mathcal{P}, Q)\, \Psi^{(+)}_{BQ}(\overset{0}{q}). \quad (6.8)$$

The parameters $\overset{0}{p}$ and $\overset{0}{q}$ are defined, as in Eq. (1.3), by means of the vectors $h_{\mathcal{P}}$, $n^{(i)}(\mathcal{P})$ and h_Q, $n^{(i)}(Q)$ respectively. The generalized vertex function Γ is expressed in terms of a five-point Green's-like function $R^{(+)}$, projected onto the positive frequency states [as for relationship (1.4)]:

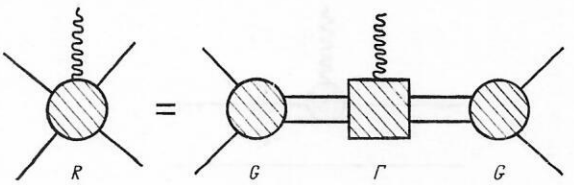

Fig. 10

$$\Gamma_\mu = [G^{(+)}]^{-1} R_\mu^{(+)} [G^{(+)}]^{-1};$$
$$R_\mu^{(+)} = \overline{u}_1\overline{u}_2 R_\mu u_1 u_2. \quad (6.9)$$

The definition of (6.9) is shown graphically in Fig. 10.

The vertex function Γ can usually be calculated only approximately [8, 14] in the form of an expansion of the interaction between the particles:

$$\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \ldots, \quad R = R_{(0)} + R_{(1)} + \ldots,$$
$$[G^{(+)}]^{-1} = [G^f]^{-1} - V^{(1)} - \ldots;$$
$$\Gamma^{(0)} = [G^f]^{-1} R_{(0)}^{(+)} [G^f]^{-1};$$
$$\Gamma^{(1)} = [G^f]^{-1} R_{(1)}^{(+)} [G^f]^{-1} - V^{(1)} G^f \Gamma^{(0)} - \Gamma^{(0)} G^f V^{(1)}. \quad (6.10)$$

Function $R_{(0)}$ corresponds to particles which do not interact with one another (impulse approximation), and it can be represented by the diagram in Fig. 11. In this approximation the function Γ has a particularly simple form [14] (in the Breit system):

$$\Gamma_\mu^{(0)}(\overset{0}{p}, \overset{0}{q}) = \frac{\langle p_1 | J_{1\mu}(0) | q_1\rangle}{2\sqrt{\varepsilon_1(\overset{0}{p})\,\varepsilon_1(\overset{0}{q})}}(2\pi)^3 \frac{\varepsilon_2(p_2)\,\delta(p_2 - q_2)}{\sqrt{\varepsilon_2(\overset{0}{p})\,\varepsilon_2(\overset{0}{q})}} + (1 \leftrightarrow 2), \quad (6.11)$$

where

$$p_{1,2} = \varepsilon_{1,2}(\overset{0}{p})\, h_{\mathcal{P}} + \sum_{i=1}^{3}(n^{(i)}(\mathcal{P})\cdot \overset{0}{p}{}^i_{1,2});$$
$$q_{1,2} = \varepsilon_{1,2}(\overset{0}{q})\, h_Q + \sum_{i=1}^{3}(n^{(i)}(Q)\cdot \overset{0}{q}{}^i_{1,2});$$
$$\mathcal{P} = -Q = \frac{\Delta}{2}; \quad \overset{0}{p}_1 = -\overset{0}{p}_2 = \overset{0}{p}; \quad \overset{0}{q}_1 = -\overset{0}{q}_2 = \overset{0}{q};$$
$$\varepsilon_{1,2}(p) = \sqrt{m_{1,2}^2 + p^2}.$$

The matrix element of the operator $J_{1\mu}$ is taken between the free one-particle states, and in the case of particles with spin 1/2 its structure is given by Eq. (2.2).

The presence of the δ function in Eq. (6.11) leads to an equation for $\overset{0}{q}$ (or $\overset{0}{p}$), the solution of which has the form

$$\overset{0}{p} - \overset{0}{q} = \Delta\frac{E}{M^2}\left[\varepsilon_2(\overset{0}{p}) - \frac{1}{2E}(\Delta\cdot\overset{0}{p})\right]; \quad E = \sqrt{M^2 + \frac{\Delta^2}{4}}. \quad (6.12)$$

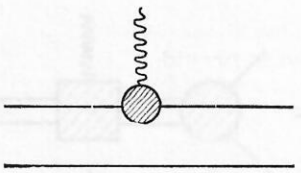

Fig. 11

We will not discuss here the general theory of Stark and Zeeman effects, since they are dealt with in detail in [17, 49], but will concentrate our attention on the diagonal matrix elements of the magnetic dipole moment operator [50].

7. Magnetic moment of hydrogen-like atoms. We consider an expression for the magnetic moment of a HL atom which would include relativistic and radiation corrections of order $(Z\alpha)^2(m_1/m_2)^2$ and $\alpha(Z\alpha)^2(m_1/m_2)$. For this purpose we need the next term in the expansion (6.10) of function Γ, the spatial part of which we given in approximate form [51]:

$$\Gamma^{(1)} \approx \frac{u_1^*(p_1)\,u_2^*(p_2)}{2\sqrt{\overset{0}{\varepsilon}_1(\mathbf{p})\,\overset{0}{\varepsilon}_2(\mathbf{p})}} \cdot \frac{e_1}{2m_1}\{\alpha_1 \Lambda_1^{(-)}(p_1')\,\hat{V}(\mathbf{q}_2-\mathbf{p}_2) + \hat{V}(\mathbf{q}_2-\mathbf{p}_2)\,\Lambda_1^{(-)}(q_1')\,\alpha_1\} \frac{u_1(q_1)\,u_2(q_2)}{2\sqrt{\overset{0}{\varepsilon}_1(\mathbf{q})\,\overset{0}{\varepsilon}_2(\mathbf{q})}} + (1 \leftrightarrow 2), \quad (7.1)$$

where

$$\hat{V}(\mathbf{k}) = \left\{\left(1+\frac{\varkappa_1}{2m_1}\gamma_1\cdot\mathbf{k}\right)\left(1-\frac{\varkappa_2}{2m_2}\gamma_2\cdot\mathbf{k}\right) - \left(\alpha_1+\frac{\varkappa_1}{2m_1}\beta_1 i\,[\Sigma_1\times\mathbf{k}]\right)\left(\alpha_2-\frac{\varkappa_2}{2m_2}\beta_2 i\,[\Sigma_2\times\mathbf{k}]\right)\right\}\frac{e_1e_2}{\mathbf{k}^2};$$

$$\Lambda^{(-)}(p) = \frac{1}{2\varepsilon(\mathbf{p})}[\varepsilon(\mathbf{p}) - (\boldsymbol{\alpha}\cdot\mathbf{p}+m\beta)] \approx \frac{1}{2}(1-\beta) - \frac{\boldsymbol{\alpha}\cdot\mathbf{p}}{2m},$$

$$\mathbf{p}_1' = \mathbf{p}_1 - \Delta; \quad \mathbf{q}_1' = \mathbf{q}_1 + \Delta; \quad \Sigma = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}.$$

As Grotch [52] showed, with the exception of the anomalous magnetic moment of the electron none of the other radiation corrections makes a contribution to the magnetic moment with the required accuracy. It is sufficient to know the D functions in expression (1.6), for the wave function in an arbitrary coordinate system, in the approximate form

$$D^{1/2}(R^W) \approx 1 + i\sigma\cdot\frac{[\overset{0}{\mathcal{P}}\times\mathbf{p}]}{4mM}. \quad (7.2)$$

Substituting Eq. (7.1), (6.11), and (7.2) in relationship (6.8) and using the definition (5.7), we arrive at the following expression for the magnetic moment:

$$\mathfrak{M}_B = \frac{1}{(2\pi)^3}\int d\mathbf{p}\,\overline{\Psi}_{B0}^{(+)}(\mathbf{p})\frac{e_1}{2\varepsilon_1(\mathbf{p})} \times \left\{\sigma_1(1+\varkappa_1) + (1+4\varkappa_1)\frac{\mathbf{p}\times[\sigma_1\times\mathbf{p}]}{4m_1^2} + (1+\varkappa_2)\frac{\mathbf{p}\times[\sigma_2\times\mathbf{p}]}{2m_1m_2} - \frac{\varepsilon_2(\mathbf{p})}{M}\left(1+\frac{M-\varepsilon_1(\mathbf{p})-\varepsilon_2(\mathbf{p})}{m_2}\right) i\left[\mathbf{p}\times\frac{\partial}{\partial\mathbf{p}}\right] + \frac{1}{4m_1M}\mathbf{p}\times\left[\mathbf{p}\times\left(\frac{\sigma_1}{m_1}-\frac{\sigma_2}{m_2}\right)\right]\right\}\Psi_{B0}^{(+)}(\mathbf{p}) + (1\leftrightarrow 2). \quad (7.3)$$

For S states expression (7.3) becomes much simpler and reduces to

$$\mathfrak{M}_B = \frac{1}{2}g_1(B)\frac{e_1}{2m_1}\langle\sigma_1\rangle + \frac{1}{2}g_2(B)\frac{e_2}{2m_2}\langle\sigma_2\rangle, \quad (7.4)$$

where the g factors of the bound states are

$$g_1(B) = g_1\left\{1-\frac{\langle\mathbf{p}^2\rangle}{3m_1^2}\left[1-\frac{3\varkappa_1}{2(1+\varkappa_1)}\right] + \frac{e_2}{e_1}\frac{\langle\mathbf{p}^2\rangle}{3m_2^2} - \frac{\langle\mathbf{p}^2\rangle}{6(1+\varkappa_1)m_1(m_1+m_2)}\left[1-\frac{e_2}{e_1}\cdot\frac{m_1}{m_2}\right]\right\}; \quad (7.5)$$

$$g_2(B) = g_1(B,\ 1\leftrightarrow 2); \quad \frac{1}{2}g_{1,2} = 1+\varkappa_{1,2}. \quad (7.6)$$

In the case of the hydrogen atom (1 = electron; 2 = proton), where $m_1 \ll m_2$, $\varkappa_1 \approx \alpha/2\pi$, and the value of $\langle\mathbf{p}^2\rangle$ is given by Eq. (2.8), expressions (7.5) and (7.6) take the form

$$g_1(\mathrm{H}) = g_1\left\{1-\frac{1}{3}(Z\alpha)^2\left[1-\frac{3m_1}{2m_2}+\frac{3m_1^2}{2m_2^2}(1+Z)\right] + \frac{\alpha}{4\pi}(Z\alpha)^2\left(1-\frac{5m_1}{3m_2}\right)\right\}; \quad (7.7)$$

$$g_2(\mathrm{H}) = g_2\left\{1-\frac{\alpha}{3}(Z\alpha)\left[1-\frac{2m_1}{m_2}+\frac{m_1^2}{2m_2^2}(6-Z)\right] - \frac{\alpha(Z\alpha)m_1}{6(1+\varkappa_2)m_2}\left[1+\frac{m_1}{m_2}(4Z-3)\right]\right\}. \quad (7.8)$$

These results were confirmed in [53]. From the experimental viewpoint the following two ratios are interesting:

$$r_{(1)}^{th} = \frac{g_1(\mathrm{H})}{g_2(\mathrm{H})} = \frac{g_1}{g_2}\cdot\left\{1+\frac{\alpha^3}{4\pi}-\frac{\varkappa_2}{6(1+\varkappa_2)}\alpha^2\frac{m_1}{m_2}\right\} \approx \frac{g_1}{g_2}(1+0.028ppm); \quad (7.9)$$

$$r_{(2)}^{th} = \frac{g_1(\mathrm{H})}{g_1(\mathrm{D})} = 1+\alpha^2\frac{m_1}{4m_2}-\frac{\alpha^3}{\pi}\cdot\frac{5m_1}{24m_2}-\alpha^2\frac{m_1^2}{4m_2^2} \approx 1+7.3\times10^{-9}, \quad (7.10)$$

where D is deuterium. The ratio $r_{(1)}$ is used for the experimental determination of the proton magnetic moment in Bohr magneton units. The quantity

$$\frac{m_2}{m_1} r_{(1)} = 658.21073\ (10)\ (0.15ppm). \qquad (7.11)$$

has been measured experimentally [40], Hence, using expression (7.9) and the experimental value of the electron g-factor [40], we obtain

$$\frac{\mu_p}{\mu_B} = g_2 \frac{m_1}{m_2} = 0.00152103214\ (23)\ (0.15ppm). \qquad (7.12)$$

We note that in [40] an erroneous theoretical expression for $r_{(1)}$ was used and, hence, a result differing slightly from (7.12) was obtained.

The ratio $r_{(2)}$ ("isotope effect") was measured for the first time fairly recently:

$$r_{(2)} = \begin{cases} 1 + (7.2 \pm 1.2) \times 10^{-9}\ [54]; \\ 1 + (9.4 \pm 1.4) \times 10^{-9}\ [55]. \end{cases} \qquad (7.13)$$

The theoretical value (7.10) is in excellent agreement with the result of [54] and in rather poorer agreement with the data of [55]. Thus, further experimental investigations with higher accuracy are required.

As a whole we must note that precision measurements of the electromagnetic properties of atoms (the atomic g factors of the electron and proton) give very valuable information and constitute a new and very promising field for tests of quantum electrodynamics and the quantum theory of bound states.

CONCLUSIONS

The above discussion shows that the quasipotential method is an effective means of calculating the energy levels and electromagnetic properties of HL atoms. In some cases the calculations are much simpler than in the formalism using the Bethe–Salpeter equation. This applies in particular to the introduction of corrections due to nuclear structure and motion. The covariant formation of the quasipotential method allows a consistent description of the behavior of a HL system in external electromagnetic fields. This formulation also allows, in principle, a consideration of scattering by bound systems.

In recent years there has been an appreciable development of interest in the physics of the simplest atomic systems. This has found expression in three major international conferences on this topic [56-58]. This is not surprising, since these systems continue to provide very valuable information on the validity of quantum electrodynamics and the relativistic quantum theory of bound states. The theory of HL systems has a fairly reliable basis, which cannot be said of many other solutions of elementary particle theory.

A review of the present state of quantum electrodynamics as a whole and electromagnetic interactions can be found in [34, 41].

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