TWO-DIMENSIONAL EXPANSIONS OF RELATIVISTIC AMPLITUDES

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The present paper is a review of research on double expansions of relativistic amplitudes. In the beginning of the review the general theory of the expansion of functions stipulated in both the time-like and space-like domains is presented; then the general theory of the expansion of a scalar function and a function with spin is presented. After that various coordinate systems and matrices for the transition from certain basis functions to other ones are indicated. The review likewise considers certain corollaries deriving from the existence of relativistic expansions.

INTRODUCTION

Very many papers have been devoted to the analytic properties of relativistic amplitudes. Expressions in which the variables s, t, u are chosen as the variables are used as the original representation of the investigations. A natural first step turns out to be the expansion of the amplitudes into partial waves which represent a generalization of the expansions of nonrelativistic amplitudes into spherical ones. Recently, the attention paid to such expansions has increased, especially in connection with the papers by Toller et al. However, in most of the papers expansions in only one of the variables are used, and in this sense such expansions are not, strictly speaking, relativistic.

In 1964 N. Ya. Vilenkin and Ya. A. Smorodinskii produced the theory of two-dimensional expansions which are the relativistic analog of expansions into spherical waves. In this paper the relativistic amplitude is parametrized in such a way that it is determined on the upper fields of a two-cavity hyperboloid in the velocity space. It may be shown (Verdiev [20], Matthews and Feldman) that such a parametrization may also be arrived at by beginning the expansion with the determination and expansion of two-particle amplitudes.

In this survey we have brought together the equations which apply to the expansion of scattering amplitudes for the case of equal masses.

The dependence of the amplitude of the scattering of two particles on two variables (the energy and the scattering angle) allows us to determine the variables in terms of the coordinates of the points on a three-dimensional hyperboloid with allowance for the equation $p_i^2 = m_i^2$, $p_1 + p_2 - p_3 + p_4$ [1] (i.e., for the amplitudes of the four-tail it is sufficient to consider the coordinates of just one of the free ends). From this it follows that the scattering amplitude may be treated as a function stipulated on the upper field of a two-cavity hyperboloid in order to be specific. Then the shift operator on this surface (these operators obviously realize a proper Lorentz group) may be used to convert the scattering amplitudes from one value of the variables to another. We shall call the possibility of such a transition from one value of the scattering amplitude to another by means of shift operators on a hyperboloid (or cone) the extended relativistic-invariance condition. Let us note that the conventional invariance condition consists only in the fact that the amplitude depends on the invariant variables s, t. Such an extension of the definition of invariance,

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which is required for the introduction of the expansions, naturally leads to the conditions governing the behavior of the functions at large values of the argument.

Thus, the problem consists in investigating expansions of the scattering amplitude in representations of the Lorentz group. Let us begin with an investigation of the realization of the representations of the Lorentz group.

I. THE FUNCTIONS ON A HYPERBOLOID

1. The Coordinate System and the Eigenfunctions of the Laplace Operator on a Hyperboloid

From the principle of relativistic invariance and unitarity it follows that the scattering amplitude should be expanded into functions which realize unitary finite-dimensioned representations of the Lorentz group. These functions are solutions of the equations [2, 3]:

$$\Delta f=(-1-\rho^2+\nu^2)f; \tag{1.1}$$

$$\Delta' f=-\nu\rho t, \tag{1.2}$$

where Δ and Δ' are invariant Casimir operators of the Lorentz group which are constructed from infinitesimal operators of the space and hyperbolic rotations **M** and **N** and are respectively equal to:

$$\Delta=\sum_{i=1}^{3}(M_i^2-N_i^2); \tag{1.3}$$

$$\Delta'=\sum_{i=1}^{3}M_iN_i. \tag{1.4}$$

Let us give detailed consideration to the expansion of the scalar function $f(u)$ stipulated on the hyperboloid $u^2 = p^2/m^2 = 1$. In this case the operator Δ is equal to the Laplacian on the hyperboloid, while the operator Δ', which is associated with the spin, is identically equal to zero. Let us likewise indicate the modification of the expansions for the case of nonzero spin.

On the hyperboloid one may introduce various coordinate systems which we shall proceed to describe now.

1. The spherical coordinate system S:

$$\left.\begin{aligned} &u_0=\operatorname{ch} a; \qquad u_2=\operatorname{sh} a\sin\theta\cos\varphi;\\ &u_3=\operatorname{sh} a\cos\theta;\ u_1=\operatorname{sh} a\sin\theta\sin\varphi;\end{aligned}\right\} \tag{1.5}$$
$$0<a<\infty,\ 0<\theta<\pi,\ 0<\varphi<2\pi.$$

In the variables a, θ, φ the Laplace operator takes the form

$$\Delta_L=\frac{1}{\operatorname{sh}^2 a}\cdot\frac{\partial}{\partial a}\operatorname{sh}^2 a\frac{\partial}{\partial a}+\frac{1}{\operatorname{sh}^2 a}\left(\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{1.6}$$

The eigenfunctions corresponding to this operator having the eigenvalue $-(1+\rho^2)$ are equal to

$$\langle\rho lm\,|\,a\theta\varphi\rangle=(\operatorname{sh} a)^{-1/2}P^{-1/2-l}_{-1/2+i\rho}(\operatorname{ch} a)\,Y_{lm}(\theta,\varphi). \tag{1.7}$$

2. The Lobachevskii coordinate system L:

$$\left.\begin{aligned} &u_0=\operatorname{ch} a\operatorname{ch} b\operatorname{ch} c:\quad u_2=\operatorname{ch} a\operatorname{sh} b;\\ &u_3=\operatorname{ch} a\operatorname{ch} b\operatorname{sh} c;\quad u_1=\operatorname{sh} a.\end{aligned}\right\} \tag{1.8}$$

The Laplacian is determined from the following equation in the variables a, b, c:

$$\Delta_L = \frac{1}{\operatorname{ch}^2 a} \frac{\partial}{\partial a} \operatorname{ch}^2 a \frac{\partial}{\partial a} + \frac{1}{\operatorname{ch}^2 a} \left(\frac{1}{\operatorname{sh} b} \frac{\partial}{\partial b} \operatorname{sh} b \frac{\partial}{\partial b} + \frac{1}{\operatorname{sh}^2 b} \frac{\partial^2}{\partial c^2} \right), \tag{1.9}$$

and its eigenfunctions are equal to:

$$\langle \rho q m \mid a, b, c \rangle = (\operatorname{ch} a \sqrt{\operatorname{ch} b})^{-1} P^{i\rho}_{-1/2+iq} (\operatorname{th} a) P^{iq}_{-1/2+im} (\operatorname{th} b) e^{imc}. \tag{1.10}$$

3. The hyperbolic coordinate system H:

$$\left. \begin{array}{ll} u_0 = \operatorname{ch} a \operatorname{ch} b; & u_2 = \operatorname{ch} a \operatorname{sh} b \sin \varphi; \\ u_3 = \operatorname{ch} a \operatorname{sh} b \cos \varphi; & u_1 = \operatorname{sh} a. \end{array} \right\} \tag{1.11}$$

In this system the Laplace operator and the eigenfunctions corresponding to it have the form

$$\Delta_L = \frac{1}{\operatorname{ch}^2 a} \cdot \frac{\partial}{\partial a} \operatorname{ch}^2 a \frac{\partial}{\partial a} + \frac{1}{\operatorname{ch}^2 a} \left(\frac{1}{\operatorname{sh} b} \cdot \frac{\partial}{\partial b} \operatorname{sh} b \frac{\partial}{\partial b} + \frac{1}{\operatorname{sh}^2 b} \cdot \frac{\partial^2}{\partial \varphi^2} \right); \tag{1.12}$$

$$\langle \rho q m \mid a b \varphi \rangle = (\operatorname{ch} a)^{-1} P^{i\rho}_{-\frac{1}{2}+iq} (\operatorname{th} a) P^{m}_{-\frac{1}{2}+iq} (\operatorname{ch} b) e^{im\varphi}. \tag{1.13}$$

4. The cylindrical coordinate system C:

$$\left. \begin{array}{ll} u_0 = \operatorname{ch} b \operatorname{ch} a; & u_2 = \operatorname{sh} b \sin \varphi; \\ u_3 = \operatorname{sh} b \cos \varphi; & u_1 = \operatorname{ch} b \operatorname{sh} a. \end{array} \right\} \tag{1.14}$$

The Laplacian and the eigenfunctions in the C-system are written in the form

$$\Delta_L = \frac{1}{\operatorname{ch} b \operatorname{sh} b} \cdot \frac{\partial}{\partial b} \operatorname{ch} b \operatorname{sh} b \frac{\partial}{\partial b} + \frac{1}{\operatorname{ch}^2 b} \cdot \frac{\partial^2}{\partial a^2} + \frac{1}{\operatorname{sh}^2 b} \cdot \frac{\partial^2}{\partial \varphi^2}; \tag{1.15}$$

$$\langle \rho \tau m \mid a b \varphi \rangle = e^{i(\tau a + m\varphi)} \frac{(\operatorname{sh} b)^m}{(\operatorname{ch} b)^{m+1+i\rho}} {}_2F_1 \left(\frac{m+1+i\rho+i\tau}{2}, \frac{m+1+i\rho-i\tau}{2}, m+1; \operatorname{th}^2 b \right). \tag{1.16}$$

5. The orispheric coordinate system O:

$$\left. \begin{array}{l} u_0 = 1/2\, [e^{-a} + (r+1)\, e^{a}];\ u_2 = r e^{a} \cos \varphi; \\ u_3 = 1/2\, [e^{-a} + (r-1)\, e^{a}];\ u_1 = r e^{a} \sin \varphi. \end{array} \right\} \tag{1.17}$$

In the variables a, r, φ the Laplacian is given by the expressions

$$\Delta_L = e^{-2a} \left[\frac{\partial}{\partial a} e^{2a} \frac{\partial}{\partial a} + \frac{1}{r} \cdot \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \varphi^2} \right], \tag{1.18}$$

and its eigenfunctions are equal to

$$\langle \rho k m \mid b r \varphi \rangle = (kb)\, K_{i\rho} (kb)\, J_m (kr)\, e^{im\varphi}. \tag{1.19}$$

Here $K_i(kb)$ and $J_m(kr)$ are respectively Macdonald and Bessel functions, while $e^{-a} = b$.

2. The Method of Orispheres

Thus, we have constructed the various eigenfunctions. Now it is necessary to orthonormalize these functions or, what amounts to the same thing, to find the equations for the inverse transformation. This can easily be done using the Gel'fand-Oraev equations [4, 5]. Let $f(u)$ be a scalar function which is stipulated on a hyperboloid, while $h(k)$ is a scalar function stipulated on the cone $k^2 = 0$. If

$$h(k)=\int f(u)\,\delta((ku)-1)\frac{d^n u}{u_0}, \tag{1.20}$$

then

$$f(u)=\frac{(-)^{\frac{n-1}{2}}}{2(2\pi)^{n-1}}\int \delta^{(n-1)}((uk)-1)\,h(k)\frac{d^n k}{k_0} \tag{1.21}$$

for $n = 2m + 1$ and

$$f(u)=\frac{(-)^{\frac{n}{2}}\Gamma(u)}{(2\pi)^n}\int (uk-1)^{-n}\,h(k)\frac{d^n k}{k_0} \tag{1.22}$$

for $n = 2m$.

Here n is the dimensionality of the manifold, while $d^n u/u_0$ and $d^n k/k_0$ are invariant measures on the hyperboloid and cone.

Let us expand $h(k)$ into homogeneous components:

$$h(k)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\Phi(k,\sigma)\,d\sigma\,*, \tag{1.23}$$

where

$$\Phi(k,\sigma)=\int_0^\infty h(kt)\,t^{-\sigma-1}\,dt. \tag{1.24}$$

From Eqs. (1.20) and (1.24) it follows that

$$\Phi(k,\sigma)=\int f(u)\,(uk)^\sigma\frac{d^n u}{u_0}. \tag{1.25}$$

Using Eqs. (1.21)-(1.23), we obtain†

$$f(u)=\frac{(-)^{\frac{n-1}{2}}}{2i(2\pi)^n}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\sigma+n-1)}{\Gamma(\sigma)}\int_\Gamma \Phi(k',\sigma)\,(uk')^{-\sigma-n+1}\,d^{n-1}k'\,d\sigma \tag{1.26}$$

for $n = 2m + 1$ and

$$f(u)=\frac{(-)^{\frac{n}{2}-1}}{2i(2\pi)^n}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\sigma+n-1)}{\Gamma(\sigma)}\,\mathrm{ctg}\,\pi\sigma\int_\Gamma \Phi(k',\sigma)\,(uk')^{-\sigma-n+1}\,d^{n-1}k'\,d\sigma. \tag{1.27}$$

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*The quantity δ is chosen in such a way that the poles of the function $\Phi(k,\sigma)$ lie outside the strip

$$0\leqslant \mathrm{Re}\,\delta\leqslant\delta.$$

†Equation (1.26) was obtained for the first time by I. S. Shapiro for $n = 3$ [6].

The integration contour Γ is arbitrary on a cone intersecting all generant cones; $d^{n-1}k'$ is an element of the volume of this contour and is determined by the equation

$$d(tk') = t^{n-3}\,dt\,dk' \qquad (0 < t < \infty).$$

In the unitary case the quantity $\sigma = -\frac{n-1}{2} + i\rho$.

3. Derivation of the Inversion Formulas

Let us derive the inversion formulas which are associated with the eigenfunctions of the Laplacian.

1. The S-System. The integration contour Γ in the S-system is a sphere ($k_0 = 1$). The function $\Phi(k, \sigma)$ is stipulated on the sphere. Let us expand it into a series in spherical harmonics:

$$\Phi(k, \sigma) = \sum_{lm} a_{lm}(\sigma)\, Y_{lm}(\mathbf{k}/|\mathbf{k}|). \tag{1.28}$$

Substituting (1.28) into (1.26) and performing integration with respect to dΩ, we obtain

$$f(u) = \frac{1}{2i(2\pi)^{3/2}} \int \sum_{lm} (-)^l \frac{\Gamma(1-\sigma)}{\Gamma(-\sigma-l-1)} \frac{a_{lm}(\sigma)}{\sqrt{\operatorname{sh} a}} P^{-l-\frac{1}{2}}_{-\sigma-\frac{3}{2}}(\operatorname{ch} a)\, Y_{lm}(\theta, \varphi)\, d\sigma. \tag{1.29}$$

Taking account of Eqs. (1.28) and (1.25) for the coefficients $a_{lm}(\sigma)$, we find the following expression:

$$a_{lm}(\sigma) = \frac{(-)^l (2\pi)^{3/2}\,\Gamma(\sigma+1)}{\Gamma(\sigma-l+1)} \int f(u)\, \operatorname{sh}^{-1/2} a P^{-l-\frac{1}{2}}_{\sigma+\frac{1}{2}}(\operatorname{ch} a)\, Y^*_{lm}(\theta, \varphi) \frac{d^3u}{u_0}. \tag{1.30}$$

Here $\frac{d^3u}{u_0} = \operatorname{sh}^2 a\, da\, \operatorname{sh}\theta\, d\theta\, d\varphi$.

Equations (1.29) and (1.30) are written for the conventional four-dimensional space (n = 3). The functions on multidimensional hyperboloids were investigated [7, 8].

2. The H-System. The expansion in the hyperbolic coordinate system is carried out by the same method as that used in the spherical system. The integration contour Γ consists of two parts Γ_+ and Γ_- corresponding to two cross sections of the cone produced by the planes $k_3 = \pm 1$. The calculations lead to the following results:

$$f(u) = -\frac{(\operatorname{ch} a)^{-1}}{8(2\pi)^4} \sum_m \int_{\delta-i\infty}^{\delta+i\infty} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\Gamma(\sigma+\tau+2)\,\Gamma(\sigma-\tau+1)}{\Gamma(\sigma)\,\Gamma(m-\sigma)}\, \tau \operatorname{ctg} \pi\tau P^m_{-\tau-1}(\operatorname{ch} b)\, e^{im\varphi}$$

$$\times \{a^+_m(\tau, \sigma) P^{-\sigma-1}_{-\tau-1}(-\operatorname{th} a) + a^-_m(\tau, \sigma) P^{-\sigma-1}_{-\tau-1}(\operatorname{th} a)\}\, d\tau\, d\sigma, \tag{1.31}$$

where

$$a^{\pm}(\tau, \sigma) = \frac{\Gamma(\tau)\,\Gamma(-\sigma-\tau-1)\,\Gamma(\tau-\sigma)}{\Gamma(\tau-m+1)\,\Gamma(-\sigma)} \int f(u)\, P^{\sigma+1}_{\tau}(\mp \operatorname{th} a)\, P^m_{\tau}(\operatorname{ch} b)\, e^{-im\varphi} \frac{d^3u}{u_0}. \tag{1.32}$$

The unitary space corresponds to the quantities $\sigma = -1 + i\rho$, $\tau = -1/2 + iq$.

3. The O-System. In the orispheric coordinate system let us choose the cross section of the cone produced by the plane $k_0 - k_3 = 2$.

$$\left.\begin{aligned} k_0 &= (1+\mu^2); & k_2 &= 2\mu\cos\alpha; \\ k_3 &= (-1+\mu^2); & k_1 &= 2\mu\sin\alpha. \end{aligned}\right\} \tag{1.33}$$

Representing the function $\Phi(k, \sigma)$ in the form

$$\Phi(k, \sigma)=\int_0^{2\pi}\int_0^{\infty}\Psi(\varkappa, \theta, \sigma)\, e^{i\varkappa\mu\cos(\theta-\alpha)}\varkappa\, d\theta\, d\varkappa \tag{1.34}$$

and substituting this expression into (1.26), we obtain

$$f(u)=\frac{ib}{(2\pi)^2}\int_{\delta-i\infty}^{\delta+i\infty}\frac{1}{\Gamma(\sigma)}\int_0^{2\pi}\int_0^{\infty}\Psi(\varkappa, \theta, \sigma)\left(\frac{\varkappa}{2}\right)^{\sigma+2} e^{i\varkappa\mu\cos(\theta-\Phi)}K_{-\sigma-1}(b\varkappa)\, d\varkappa\, d\theta\, d\sigma \tag{1.35}$$

after performing the calculations. The coefficients $\Psi(\varkappa, \theta, \sigma)$ are determined according to equation

$$\Psi(\varkappa, \theta, \sigma)=\frac{2}{\pi\Gamma(-\sigma)}\left(\frac{2}{\varkappa}\right)^{\sigma+1}\int f(u)\, e^{-i\varkappa r\cos(\theta-\Phi)}K_{\sigma+1}(b\varkappa)\frac{1}{b^2}\cdot\frac{d^3u}{u_0}. \tag{1.36}$$

In Eqs. (1.35), (1.36) $K_\nu(x)$ is a Macdonald function, $e^{-a}=b$, $d^3u/u_0=r\, dr\, db\, d\Phi$.

4. The C-System. Finally, let us consider the cylindrical coordinate system. The integration contour Γ in this system is the intersection of the hyperboloids with the cylinder $k_0^2-k_1^2=1$. Let us parametrize Γ as follows:

$$\left.\begin{aligned} k_0&=\operatorname{ch} c; & k_2&=\sin\alpha;\\ k_1&=\operatorname{sh} c; & k_3&=\cos\alpha.\end{aligned}\right\} \tag{1.37}$$

Using the Fourier-expansion of $\Phi(k, \sigma)$

$$\Phi(k, \sigma)=\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}a_m(\tau, \sigma)\, e^{i(m\alpha+\tau c)}\, d\tau, \tag{1.38}$$

we obtain the following representations for $f(u)$ after calculation:

$$f(u)=-\frac{1}{16\pi^2 i}\sum_m\frac{\operatorname{th}^m b\, e^{im\varphi}}{\Gamma(m+1)}\int_{\delta-i\infty}^{\delta+i\infty}e^{i\tau a}\int_{-\infty}^{\infty}\frac{a_m(\tau, \sigma)}{\Gamma(\sigma)}\Gamma(A)\Gamma(B)\left(\frac{2}{\operatorname{ch} b}\right)^{\sigma+2}_2 F_1(A, B, m+1; \operatorname{th}^2 b)\, d\sigma\, d\tau. \tag{1.39}$$

Here $A=\frac{1}{2}(m+\sigma+i\tau+2)$; $B=\frac{1}{2}(m+\sigma-i\tau+2)$. The coefficients $a_m(\tau, \sigma)$ in this system have the following form:

$$a_m(\tau, \sigma)=\frac{\Gamma(A')\Gamma(B')}{4\pi\Gamma(m+1)\Gamma(-\sigma)}\int\operatorname{th}^m b\left(\frac{2}{\operatorname{ch} b}\right)^{-\sigma}{}_2F_1(A', B', m+1, \operatorname{th}^2 b)\, e^{i(m\Phi+\tau a)}f(u)\frac{d^3u}{u_0}, \tag{1.40}$$

where $\frac{d^3u}{u_0}=\operatorname{sh}^2 a\, da\, db\, d\Phi$; $A'=\frac{1}{2}(m-\sigma+i\tau)$; $B'=\frac{1}{2}(m-\sigma-i\tau)$.

Thus, we have carried out the expansion of the scattering amplitude in four coordinate systems in functions which realize infinite-dimensioned unitary representations of a class-1 Lorentz group. A detailed derivation of Eqs. (1.29), (1.31), (1.35), and (1.39) was given in the paper by N. Ya. Vilenkin and Ya. A. Smorodinskii [1].

4. Expansion of the Functions Stipulated in the Space-like Domain

In order to carry out the expansion of a function stipulated in the space domain one should use the Gel'fand-Graev equation in the form [5]:

$$f(u)=\frac{(-)}{4i(2\pi)^3}\int\limits_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\int\limits_{\Gamma}F(k,\sigma)\,|(uk)|^{-\sigma-2}\,d^2k\,d\sigma+\frac{2}{\pi^2}\sum_{n=1}^{\infty}2n\int\limits_{\Gamma}F(k,u;2n)\,\delta(uk)\,d^2k. \tag{1.41}$$

Here $f(u)$ is an even quadratically integrable function; $u^2=u_0^2-\mathbf{u}^2=-1$; $k^2=0$; Γ is the integration contour on a cone (a sphere for $k_0=1$). The numbers σ and n are the weights of the representations of the Lorentz groups of the fundamental and discrete series, $\sigma=-1+i\rho$ holding as previously in the unitary case.

In order to write the expansion for an odd function $f(u)=-f(-u)$ $(x_0\to -x_0,\ \ \mathbf{x}\to -\mathbf{x})$, it is required to replace $|(uk)|^{-\sigma-2}$ by $|(uk)|^{-\sigma-2}\sin(uk)$ in Eq. (1.41) and 2n by 2n−1 in the second term (see [9]). If the expansion of the functions $F(k,\sigma)$ and $F(k, u, 2n)$ is carried out, and, for example, the spherical coordinate system is introduced, we have

$$\left.\begin{aligned} &u_0=\operatorname{sh}a; && k_0=1;\\ &u_3=\operatorname{ch}a\cos\theta; && k_3=\cos\theta';\\ &u_2=\operatorname{ch}a\sin\theta\cos\varphi; && k_2=\sin\theta'\cos\Phi;\\ &u_1=\operatorname{ch}a\sin\theta\sin\varphi; && k_1=\sin\theta'\sin\Phi, \end{aligned}\right\} \tag{1.42}$$

then, having carried out integration with respect to $d^2k=d\cos\theta'\,d\Phi$, we obtain [9]

$$\begin{pmatrix} f_{\text{ч}}(u)\\ f_{\text{нч}}(u)\end{pmatrix}=\frac{(-)}{4i(2\pi)^3}\int\limits_{\delta-i\infty}^{\delta+i\infty}\sigma(\sigma+1)\Gamma(-\sigma-1)\sum_{lm}a_{lm}Y_{lm}(\theta,\varphi)=\frac{P_l^{\sigma+1}(\operatorname{th}a)\pm(-)^l P_l^{\sigma+1}(-\operatorname{th}a)}{\operatorname{ch}a}\,d\sigma$$

$$+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\begin{pmatrix}2n\\2n-1\end{pmatrix}\sum_{lm}\frac{4\pi}{2l+1}c_{lm}\frac{1}{\operatorname{ch}a}\begin{pmatrix}P_l^{2n}(\operatorname{th}a)\\P_l^{2n-1}(\operatorname{th}a)\end{pmatrix}Y_{lm}(\theta,\varphi); \tag{1.43}$$

$P_l^{\sigma+1}(\operatorname{th}a)$ and $P_l^{n}(\operatorname{th}a)$ are associated Legendre functions; $f_e(u)$ and $f_o(u)$ are even and odd functions, respectively. The first term in this expression is similar to the expansion of the function on a two-cavity hyperboloid, which was obtained in [1]; the second term is the expansion of the function in discrete representations of the Lorentz group [10]. Problems associated with the analytic continuation of the amplitudes into the space-like domain, and likewise the eigenfunctions of the Laplacian in the space-like domain, were considered in [11-13].

5. The Expansion of Fields

We have considered the expansion of a scalar function in eigenfunctions of the Laplace operator on two-cavity and one-cavity hyperboloids. In order to expand a function having spin s and a projection σ onto the direction of the momentum* (the spiral state; see [14]) it is necessary to modify the kernel of the integral transformation—namely, one should write $(uk)^{-1-i\rho}$ in Eq. (1.26) instead of the kernel $(uk)^{-1-i\rho}\mathscr{D}^s(R)$, where R is a certain rotation which considers the spin requantization; $\mathscr{D}^s(R)$ is a Wigner function. Thus, the expansion of the functions $\psi_{s\sigma}(u)$ having a stipulated spin on a hyperboloid in eigenfunctions $\Phi_{\rho\nu}(k)$, which are transformed according to representations (ρ,ν) that are irreducible relative to the proper Lorentz group, is now determined by the following expression [15, 16, 17]:

$$\Psi_{s\sigma}(u)=\frac{1}{2(2\pi)^3}\sum_{\nu=-s}^{s}\int\limits_{-\infty}^{\infty}d\rho\,(\rho^2+\nu^2)\int\limits_{\Gamma}(uk)^{-1-i\rho}\,\mathscr{D}^s_{\sigma\nu}(R)\,\Phi_{\nu\rho}(k)\,d^2k, \tag{1.44}$$

where Γ is the integration contour (a sphere for the cross section of the cone $k^2=0$ produced by the plane $k_0=1$); $d^2k=d\Omega$; R is a rotation in the $(\mathbf{k},\mathbf{u})$ plane through the angle $\pi-\theta'$. If the spherical coordinate system $\mathbf{n}=\frac{\mathbf{k}}{|\mathbf{k}|}=(\cos\theta,\ \sin\theta\cos\varphi,\ \sin\theta\sin\varphi)$, is introduced on the cone, then (see [15-17])

*In this section σ is the spin projection and not at all the number characterizing the representation of the Lorentz group.

$$\cos\theta' = -\frac{u_0\cos\theta - |\mathbf{u}|}{u_0 - (\mathbf{un})}. \tag{1.45}$$

Assume the function $\Phi_{\rho\nu}(\mathbf{k})$ is represented in the form

$$\Phi_{\rho\nu}(k) = \sum_{JM} a_{JM}(\rho)(2J+1)\mathscr{D}^J_{M\nu}\left(\frac{\mathbf{k}}{|\mathbf{k}|}\right). \tag{1.46}$$

Then, substituting Eq. (1.46) into Eq. (1.44) and carrying out integration with respect to dΩ, we obtain [14]

$$\Psi_{s\sigma}(u) = \frac{1}{2(2\pi)^3}\sum_{\nu=-s}^{s}\int_{-\infty}^{+\infty} d\rho\,[\rho^2+\nu^2]\sum_{JM} a^{\nu}_{JM}(\rho)\, I^{s\sigma}_{\rho\nu JM}(u). \tag{1.47}$$

Here

$$I^{s\sigma}_{\rho\nu JM}(u) = I^{s\sigma}_{\rho\nu J}(a)\,\mathscr{D}^J_{M\sigma}(\varphi, \theta, -\varphi); \tag{1.48}$$

$$I^{s\sigma}_{\rho\nu J}(a) = 2\pi\sum_{r,j} G_{J\sigma\nu j}G_{s\sigma\nu r}(-)^{r+j}\exp(\sigma-\nu-1-i\rho+2r-2s)\,a\,\frac{\Gamma(J+s-\mu+1)\,\Gamma(\mu+1)}{\Gamma(J+s+2)}$$

$$\times\, {}_2F_1(1+i\rho+s,\ J+s-\mu+1,\ J+s+r;\ 1-e^{-2a}); \tag{1.49}$$

$$G_{J\sigma\nu j} = i^{\sigma-\nu}\frac{[(J-\sigma)!\,(J+\sigma)!\,(J-\sigma)!\,(J+\nu)!]^{1/2}}{j!\,(J-\sigma-j)!\,(J+\sigma-j)!\,(\sigma-\nu+j)!}; \tag{1.50}$$
$$\mu = \sigma-\nu+r+j;$$

$\mathscr{D}^J_{M\sigma}(\Phi, \theta, -\Phi)$ is the matrix for the finite rotation of the rotation group (see [18]); $u=(u_0, \mathbf{u}) = (\operatorname{ch} a, \operatorname{sh} a\cos\theta, \operatorname{sh} a\sin\theta\cos\Phi\ \operatorname{sh} a\sin\theta\sin\Phi)$.

The function (1.49) constitutes the matrix of a finite hyperbolic rotation of the Lorentz group [17, 19-21].

At the end of this chapter it should be noted that in order to construct functions which realize the representations of the Lorentz group (ρ, ν) we have used the integral method. Other methods of constructing relativistic spherical functions have been expounded in [22-23].

6. The Transition from Channel to Channel for Spiral Scattering Amplitudes

In modern elementary-particle theory the universality principle which allows the amplitudes of different channels to be associated with one and the same diagram plays an important role. When, for example, the contribution to the scattering amplitude in a given channel caused by the unitarity relation in the cross channel needs to be determined, it is necessary to make a transition from channel to channel. In the case of the scattering of spinless particles such a transition can be reduced to the substitution s ⇌ t or the substitution of the variables u, t for s, t. However, in the case of the scattering of particles with spin it is necessary, in addition to performing the analytic continuation of the amplitude, to perform requantization of spins because of the transition from channel to channel.

The problem of the transition from channel to channel was investigated in [24-28] for the spin case. The simplest method of transition was proposed in [24]. However, in this paper the spins in the t-channel were quantized in different directions; this led to nonstandard equations and was corrected by E. L. Surkov [28]. In the present exposition we shall basically follow [28].

Let us consider a reaction of the type (s-channel)

$$1+2\to 3+4$$

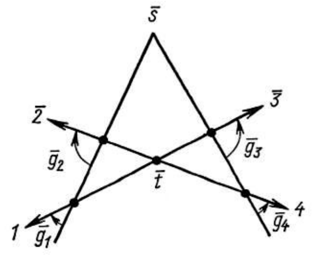

Fig. 1. Kinematic diagram of the $1 + 2 \rightarrow 3 + 4$ process.

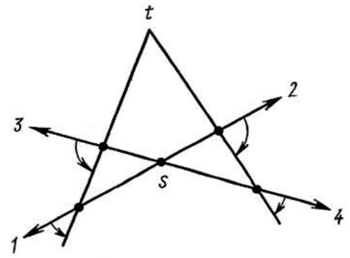

Fig. 2. Kinematic diagram of the $1 + \overline{3} \rightarrow \overline{2} + 4$ process.

(the numbers denote particles). The law of energy-momentum conservation in the s-channel has the form

$$p_1 + p_2 = p_3 + p_4.$$

The spiral amplitude corresponding to such a transition is usually denoted by $\langle p_3\lambda_3;\ p_4\lambda_4 \,|\, M \,|\, p_1\lambda_1;\ p_2\lambda_2\rangle$. Here λ_i are the spiralities of particles which participate in the reaction. The spiral amplitudes of the reaction $1 + 2 \rightarrow 3 + 4$ are understood to mean the amplitudes defined in the center of inertia system. After this the quantization axes turn out to be rigidly fixed—they must always be directed from the center of inertia (point s in Fig. 1).

The spiral amplitudes depend on the invariants:

$$s = (p_1 + p_2)^2;\ t = (p_1 - p_3)^2;\ u = (p_1 - p_4)^2.$$

Figure 1 depicts the kinematic diagrams [24] corresponding to the reaction $1 + 2 \rightarrow 3 + 4$ in the velocity space. Points 1 and 2 are the velocities of the particles before scattering; points 3 and 4 are the velocities of the particles after scattering.

The point s is the center of inertia of the reaction and corresponds to the vector $S = p_1 + p_2$; the point t corresponds to the vector $T = p_1 - p_3$ (here we have depicted the case when the vector T, just as the vector S, is time-like, which is possible in the case of unequal particle masses).

Continuing the spiral amplitudes of the physical domain of the annihilation channel (the t-channel: $1 + \overline{3} \rightarrow \overline{2} + 4$), we obtain amplitudes which are quantized as before relative to the point s (the spirality does not change under these conditions). The center of inertia vector of the t-channel now becomes the vector $\overline{T} = p_1 + p_3$, while the point $\overline{s}$ turns out to be outside the kinematic quadrangle ($p_2 \rightarrow -\overline{p}_2$, $p_3 \rightarrow -\overline{p}_3$) (Fig. 2). If we wish the spiral amplitudes to be quantized relative to the center of inertia in the t-channel itself, then we must carry out a Lorentz transformation from the point $\overline{s}$ to the point $\overline{t}$. Under these conditions the amplitudes are transformed by means of the matrices $D^{s}_{\lambda\mu}(\overline{g})$ according to the equation

$$\langle \mu_4;\ \mu_{\overline{2}} \,|\, \hat{M} \,|\, \mu_1;\ \mu_{\overline{3}}\rangle = (-)^{\lambda_2-\lambda_3} D^{*s_4}_{\lambda_4\mu_4}(\overline{g}_4)\, D^{*s_2}_{\lambda_2\mu_2}(\overline{g}_2) < \lambda_3;$$

$$\lambda_4 \,|\, \hat{M} \,|\, \lambda_1;\ \lambda_2 > D^{s_1}_{\lambda_1\mu_1}(\overline{g}_1)\, D^{s_3}_{\lambda_3\mu_3}(\overline{g}_4). \qquad (1.51)$$

Let us note that $\overline{g}_i \equiv (0, \overline{\theta}_i, 0)$. Using the Lobachevskii trigonometry formulas it is very easy to determine the rotation angles θ_i. From Fig. 2 we find that:

$$\left.\begin{aligned}
&\cos\theta_1 = \frac{\operatorname{ch}(1, \overline{2})\operatorname{ch}(1, \overline{3}) - \operatorname{ch}(\overline{2}, \overline{3})}{\operatorname{sh}(1, 2)\operatorname{sh}(1, \overline{3})} = \cos(\overline{2}\,1\,\overline{3});\\
&\cos\theta_2 = -\cos(1\,\overline{2}\,4);\ \cos\overline{\theta}_3 = -\cos(1\,\overline{3}\,4);\\
&\cos\theta_4 = \cos(\overline{3}\,4\,\overline{2});\\
&\cos(abc) = \frac{\operatorname{ch}(ab)\operatorname{ch}(bc) - \operatorname{ch}(ac)}{\operatorname{sh}(ab)\operatorname{sh}(bc)}.
\end{aligned}\right\} \qquad (1.52)$$

Let us write out the explicit expression for cos $\bar{\theta}_1$:

$$\cos\theta_1 = \frac{(\bar{s}-M_1^2-M_2^2)(M_1^2+M_2^2-\bar{t})-2M_1^2(M_3^2+M_2^2-\bar{u})}{\{[\bar{s}-(M_1+M_2)^2][\bar{s}-(M_1-M_2)^2][\bar{t}-(M_1+M_3)^2][t-(M_1-M_3^2)]\}}\,.$$

We consider the case when the vectors T, S are time-like. But S may also be space-like (for example, in the case of equal masses). In this case the rotations can be determined similarly, since it is not the point s which is important to us but the direction to it.

II. FUNCTIONS ON A CONE

1. The Coordinate System, the Complete Sets of Quantum Numbers, and the Basis Functions on a Cone

Let us consider the realization of representations of a Lorentz group using functions stipulated on a cone ($k^2 = 0$) and let us calculate the matrices for a transformation between representations, which correspond to reduction into various subgroups [29]. It is obvious that the transition coefficients indicated do not depend on the method of realization of the representations, and therefore they may be used to obtain basis functions of the Lorentz group from the functions (1.48), which correspond to reduction into the subgroups O(2.1), E(2), etc. The papers [30, 32] are likewise devoted to the calculation of the transformation coefficients for a Lorentz group.

Let us choose the infinitesimal operators **M** and **N** to be in the form [33]:

$$\begin{aligned}
M_1 &= -\mathrm{i}\,[\mathbf{k}\nabla]_1 + \lambda\frac{k_1}{k_0+k_3}\,;\\
M_2 &= -\mathrm{i}\,[\mathbf{k}\nabla]_2 + \lambda\frac{k_2}{k_0+k_3}\,;\\
M_3 &= -\mathrm{i}\,[\mathbf{k}\nabla]_3 + \lambda;\\
N_1 &= \mathrm{i}k_0\frac{\partial}{\partial k_1} + \lambda\frac{k_2}{k_0+k_3}\,;\\
N_2 &= \mathrm{i}k_0\frac{\partial}{\partial k_2} - \lambda\frac{k_1}{k_0+k_3}\,;\\
N_3 &= \mathrm{i}k_0\frac{\partial}{\partial k_3}\,,
\end{aligned}$$

where $k = (k_0, \mathbf{k})$ is a four-dimensional vector lying on a cone (i.e., the momentum of a particle having zero rest mass). The basis functions of the Lorentz group are eigenfunctions of the Casimir operators Δ and Δ' defined by Eqs. (1.3) and (1.4) and of the two other commutating operators defined by choosing the subgroups.* Let us introduce the coordinate systems on a cone and let us write out the corresponding diagonal operators and their eigenfunctions.

1. The S-System: the subgroup $O(3) \supset O(2)$

$$\left.\begin{aligned}
&k_0 = e^a; \quad k_2 = e^a\sin\theta\cos\varphi;\\
&k_1 = e^a\sin\theta\sin\varphi; \quad k_3 = e^a\cos\theta,
\end{aligned}\right\}$$

where $-\infty < a < \infty$; $0 < \theta < \pi$; $0 < \varphi < 2\pi$. The diagonal operators are

$$M^2 = M_1^2+M_2^2+M_3^2 = -\frac{\partial^2}{\partial\theta^2} - \mathrm{ctg}\,\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{2\lambda}{1+\cos\theta}\left(\mathrm{i}\frac{\partial}{\partial\varphi}+\lambda\right); \quad M_3 = \mathrm{i}\frac{\partial}{\partial\varphi}+\lambda$$

having the eigenvalues $J(J + 1)$ and M, respectively. The eigenfunctions are Wigner D-functions [18] $D^J_{M\lambda}(\varphi, \theta, -\varphi)$.

*These operators are the Casimir operator of the subgroup and the operator of its simple Abelian subgroup (here O(2) throughout).

2. The H-System: the subgroup $O(2, 1) \supset O(2)$

$$\left.\begin{aligned} k_0 &= e^a \operatorname{ch} \beta; \quad k_2 = e^a \operatorname{sh} \beta \cos \varphi; \\ k_1 &= e^a \operatorname{sh} \beta \sin \varphi; \quad k_3 = \varepsilon e^a, \end{aligned}\right\}$$

where $-\infty < a < \infty$, $-\infty < \beta < \infty$, $0 < \varphi < 2\pi$, and

$$\varepsilon = \begin{cases} +1 & \text{for} \quad k_3 > 0; \\ -1 & \text{for} \quad k_3 < 0. \end{cases}$$

The diagonal operators are:

$$H^2 = M_3^2 - M_1^2 - N_2^2 = \frac{\partial^2}{\partial \beta^2} + \operatorname{cth} \beta \frac{\partial}{\partial \beta} + \frac{1}{\operatorname{sh}^2 \beta} \cdot \frac{\partial^2}{\partial \varphi^2} + \frac{2\varepsilon\lambda}{\varepsilon + \operatorname{ch} \beta} \left(i \frac{\partial}{\partial \varphi} + \lambda \right)$$

and $M_3 = i \frac{\partial}{\partial \varphi} + \lambda$ having the eigenvalues $(q^2 + 1/4)$ and M, respectively (here q is real for unitary representations). The eigenfunctions H^2 and M_3 are

$$T^q_{M\lambda}(\varphi, \beta, -\varphi) = e^{iM\varphi} d^{q(\varepsilon)}_{M\lambda}(\operatorname{ch} \beta)\, e^{i\lambda\varphi},$$

where $d^{q(\varepsilon)}_{M\lambda}(\operatorname{ch} \beta)$ can be expressed as follows in terms of the Jacobi functions [34] $\mathfrak{P}^q_{M\lambda}(\operatorname{ch} \beta)$:

$$\begin{aligned} d^q_{M\lambda}(\operatorname{ch} \beta) &= \mathfrak{P}^q_{M\lambda}(\operatorname{ch} \beta) \quad \text{for} \quad \varepsilon = +1; \\ d^q_{M\lambda}(\operatorname{ch} \beta) &= \mathfrak{P}^q_{-M\lambda}(\operatorname{ch} \beta) \quad \text{for} \quad \varepsilon = -1. \end{aligned}$$

3. The O-System: the subgroup $E(2) \supset O(2)$

$$\begin{aligned} k_0 &= e^a \frac{1 + r^2}{2}; \quad k_2 = e^a r \cos \varphi; \\ k_1 &= e^a r \sin \varphi; \quad k_3 = e^a \frac{-1 + r^2}{2}, \end{aligned}$$

where $-\infty < a < \infty$; $0 < r < \infty$; $0 < \varphi < 2\pi$.

The diagonal operators are:

$$O^2 = (N_1 + M_2)^2 + (N_2 - M_1)^2 = -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \cdot \frac{\partial}{\partial r} - \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \varphi^2} + \frac{4\lambda}{r^2} \left(i \frac{\partial}{\partial \varphi} + \lambda \right);$$

$$M_3 = i \frac{\partial}{\partial \varphi} + \lambda.$$

The eigenvalues O^2 and M_3 are $\varkappa^2$ and M, respectively, where $\varkappa$ is real. The eigenfunctions

$$I^{\varkappa}_{M\lambda}(\varphi, r, -\varphi) = e^{-iM\varphi} J_{M+\lambda}(\varkappa r)\, e^{i\lambda\varphi}.$$

Here $J_{M+\lambda}(\varkappa r)$ are Bessel functions.

4. The C-System: the subgroup $O(2) \otimes O(1, 1)$

$$\begin{aligned} k_0 &= e^a \operatorname{ch} \beta; \quad k_2 = e^a \cos \varphi; \\ k_1 &= e^a \sin \varphi; \quad k_3 = e^a \operatorname{sh} \beta, \end{aligned}$$

where $-\infty < a < \infty$, $-\infty < \beta < \infty$, $0 < \varphi < 2\pi$. The diagonal operators are M_3 and N_3 in this case and have the eigenvalues M and τ. Their eigenfunctions are

$$I^{\tau}_{M\lambda}(\varphi,\ \beta,\ -\varphi) = e^{-iM\varphi}e^{-i\tau\beta}e^{i\lambda\varphi}.$$

Calculating the Casimir operators Δ and Δ' in all of the coordinate systems on the cone, we find that they have an identical form:

$$\Delta = \frac{\partial^2}{\partial a^2} + 2\frac{\partial}{\partial a} + \lambda^2; \tag{2.1}$$

$$\Delta' = i\lambda\left(1 + \frac{\partial}{\partial a}\right). \tag{2.2}$$

Solving Eqs. (1.1) and (1.2) with allowance for (2.1) and (2.2), we find that for unitary representation $\lambda = \nu$, while $\exp(-1 + i\rho)a$ is the portion of the basis functions depending on a.

2. The Matrix Elements

In various physics problems it is required to know the matrix elements of the transformation from the basis functions corresponding to the reduction of the Lorentz group to one chain of subgroups, to basis functions corresponding to a different reduction. In view of the fact that such matrix elements do not depend on the method of realizing the representations, they are simplest to calculate by means of the functions on a cone.

1. The Matrix Elements of the S-C Transition. Assume the basis functions in the S-system are $\langle a\theta\varphi \mid p\nu JM\rangle$, while in the C-system they are $\langle b\zeta\varphi \mid p\nu\tau M\rangle$. The matrix elements of the transition from the S-system to the C-system are determined by the following integrals:

$$\langle p\nu\tau M \mid p\nu JM'\rangle = \int \frac{d^3k}{k_0}\langle p\nu\tau M \mid b\zeta\varphi\rangle\langle a\theta\varphi \mid p\nu JM\rangle.$$

Having expressed the variables in the C-system in terms of the variables in the S-system, $e^b = e^a\sin\theta$, $\operatorname{th}\zeta = \cos\theta$, and having substituted the explicit form of the basis functions, we obtain

$$\langle p\nu\tau M \mid p\nu JM'\rangle = N_S N_C^* \int\limits_{-\infty}^{+\infty} e^{2a}\,da \int\limits_0^{\pi}\sin\theta\,d\theta\int\limits_0^{2\pi}d\varphi\ e^{(-1-ip)b}e^{(-1+ip)a}e^{-i\tau\zeta}e^{-i(\nu-M')\varphi}e^{i(\nu-M)\varphi}P^J_{M\nu}(\cos\theta)$$

$$= N_S N_C^*\int\limits_0^{\pi}\sin\theta\,d\theta\,(\sin\theta)^{-1+ip}\left(\operatorname{tg}\frac{\theta}{2}\right)^{i\tau}P^J_{M\nu}(\cos\theta), \tag{2.3}$$

where N_S and N_C are the normalization factors of the functions $\langle a\vartheta\varphi \mid p\nu JM\rangle$ and $\langle b\zeta\varphi \mid p\nu\tau M\rangle$ respectively.

After calculating the integral (2.3) and substituting the values of N_S and N_C, we finally obtain

$$\langle p\nu\tau M \mid p\nu JM'\rangle = \delta_{MM'}2^{ip-1}i^{(M-\nu)}\sqrt{\frac{(2J+1)(J-\nu)!\,(J+M)!}{2(J+\nu)!\,(J-M)!}}\ \frac{\Gamma\left(\frac{1+M-\nu+i\tau+ip}{2}\right)\Gamma\left(\frac{1+M+\nu-i\tau+ip}{2}\right)}{\Gamma(M-\nu+1)\,\Gamma(1+M+ip)}$$

$$\times\,{}_3F_2\begin{pmatrix}J+M+1, & M-J, & \frac{1}{2}(1+M-\nu+i\tau+ip)\\ M-\nu+1, & 1+M+ip; & \end{pmatrix}.$$

The matrix elements of the transition are calculated analogously [29] in the remaining cases; here we shall present only the final results.

2. The Matrix Elements S–O:

$$\langle p\nu\varkappa M \mid p\nu JM'\rangle = \delta_{MM'}\frac{i^{M-\nu}}{\pi^2}\left(\varkappa\frac{(2J+1)(J-M)!\,(J+M)!}{2(J+M)!\,(J+\nu)!}\right)^{1/2}\sum_{n=0}^{J-H}\frac{(-)^n(J+M+n)!\left(\frac{\varkappa}{2}\right)^{M+ip+n}}{(M-\nu+n)!\,(J-M-n)!\,n!\,\Gamma(M+n+ip+1)}K_{\nu-ip-n}(\varkappa).$$

Here $K_{\nu-ip-n}(\varkappa)$ is a Macdonald function.

3. The Matrix Elements S—H:

$$\langle p\nu JM \mid p\nu qM'\rangle = \delta_{MM'}\,[J(\nu) + (-1)^{2(J-M-\nu)}\, J(-\nu)],$$

where $J(\pm\nu)$ denotes the following expression:

$$J(\pm\nu) = \pi^2 \mathrm{i}^{M\mp\nu}\, 2^{J+\mathrm{i}q-1/2}\, \frac{\Gamma(1/2-\mathrm{i}q\pm\nu)}{\Gamma\left(\frac{1}{2}+M-\mathrm{i}q\right)\Gamma(J+M+1)\,\Gamma(J\mp\nu+1)\,\Gamma\left(\frac{1}{2}+\mathrm{i}q-M\right)}$$

$$\times \frac{1}{\Gamma\left(\frac{1}{2}+\mathrm{i}q\pm\nu\right)\Gamma\left(\frac{1}{2}+\mathrm{i}q+\mathrm{i}p\right)}\sqrt{\frac{(J\mp\nu)!\,(J+M)!}{(J\pm\nu)!\,(J-M)!}}\sum_{nn'}(-1)^n \frac{\Gamma(J+M+n+1)\,\Gamma(J\mp\nu+n+1)\,\Gamma\left(\frac{1}{2}-M+\mathrm{i}q+n'\right)\Gamma\left(\frac{1}{2}+\mathrm{i}q\pm\nu+n'\right)}{\Gamma(M\mp\nu+n+1)\,n!\,n!\,\Gamma(1-M\pm\nu+n')\,\Gamma(M\mp\nu+n+n')}$$

$$\times\Gamma\left(\frac{3}{2}+M+\mathrm{i}q+\mathrm{i}p\mp\nu+n+n'\right){}_2F_1\left(\frac{3}{2}+J+M+\mathrm{i}q\mp\nu+n+n',\ \frac{1}{2}+\mathrm{i}q+\mathrm{i}p,\right.$$

$$\left.\frac{3}{2}+M+\mathrm{i}q+\mathrm{i}p+n+n';\ -1\right).$$

4. The Matrix Elements O—C:

$$\langle p\nu\varkappa M \mid p\nu\tau M'\rangle = \delta_{MM'}\,\frac{2^{-1+\mathrm{i}\tau}\,\Gamma\left(\frac{M+\nu+\mathrm{i}p+\mathrm{i}\tau+1}{2}\right)}{\varkappa^{\frac{1}{2}+\mathrm{i}\tau}\,\Gamma\left(\frac{M+\nu-\mathrm{i}p-\mathrm{i}\pi+1}{2}\right)}.$$

5. The Matrix Elements H—C:

$$\langle p\nu\tau M \mid p\nu qM'\rangle = \delta_{MM'}\,\frac{2^{-\mathrm{i}p}}{(2\pi)^3}\cdot\frac{\Gamma\left(\frac{1}{2}+\nu+\mathrm{i}q\right)\Gamma\left(\frac{1}{2}-\mathrm{i}q+\mathrm{i}p\right)}{\Gamma(\nu-M+1)\,\Gamma\left(\frac{1}{2}+M+\mathrm{i}q\right)}\,\frac{\Gamma\left(\frac{1+\nu-M-\mathrm{i}p+\mathrm{i}\tau}{2}\right)}{\Gamma\left(1-\mathrm{i}q+\frac{\nu-M+\mathrm{i}p-\mathrm{i}\tau}{2}\right)}$$

$$\times\,{}_3F_2\begin{pmatrix}1/2-M-\mathrm{i}q, & \frac{1}{2}+\nu-\mathrm{i}q, & \frac{1+\nu-M-\mathrm{i}p+\mathrm{i}\tau}{2} & \\ \nu-M+1, & 1-\mathrm{i}q+\frac{\nu-M+\mathrm{i}p-\mathrm{i}\tau}{2}; & 1 & \end{pmatrix}.$$

6. The Matrix Elements O—H:

$$\langle p\nu\varkappa M \mid p\nu qM'\rangle = \delta_{MM'}\,2\pi\sqrt{\varkappa}\,[J_1(\varkappa)+J_2(\varkappa)],$$

where J_1 and J_2 are expressed as follows in terms of the Meyer G-function:

$$J_1(\varkappa) = \frac{1}{2}\left(\frac{2}{\varkappa}\right)^{M+\nu}\frac{\Gamma\left(\frac{1}{2}-\nu+\mathrm{i}q\right)\Gamma\left(\frac{1}{2}-\mathrm{i}q-\mathrm{i}p\right)}{\Gamma\left(\frac{1}{2}+M+\mathrm{i}q\right)\Gamma\left(\frac{1}{2}+M-\mathrm{i}q\right)\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}q\right)}\sum_n\frac{\Gamma\left(\frac{1}{2}+M-\mathrm{i}q+n\right)\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}q+n\right)}{\Gamma(M+\nu+n+1)\,n!}$$

$$\times\, G^{11}_{13}\left(\frac{\varkappa^2}{4}\,\middle|\,\begin{matrix}M+\nu\\ 0,\ -\frac{1}{2}+\mathrm{i}q+\mathrm{i}p-n,\ -n\end{matrix}\right);$$

$$J_2(\varkappa) = \frac{1}{2}\left(\frac{2}{\varkappa}\right)^{M+\nu}\frac{\Gamma\left(\frac{1}{2}-\mathrm{i}q+\mathrm{i}p\right)\Gamma\left(\frac{1}{2}+\nu+\mathrm{i}q\right)}{\Gamma\left(\frac{1}{2}+M+\mathrm{i}q\right)\Gamma\left(\frac{1}{2}-M-\mathrm{i}q\right)\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}q\right)}\sum_n\frac{\Gamma\left(\frac{1}{2}-M-\mathrm{i}q+n\right)\Gamma\left(\frac{1}{2}+\nu-\mathrm{i}q+n\right)}{\Gamma(\nu-M+n+1)\,n!}$$

$$\times\, G^{20}_{13}\left(\frac{\varkappa^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2}+M+\nu-\mathrm{i}q-n\\ M+\nu+n-\mathrm{i}p,\ 0,\ M+\nu\end{matrix}\right).$$

Let us likewise note here that for unitary representations the matrix element of the inverse transformation is evidently defined by a Hermite-conjugate matrix; thus, for example, from the matrix elements $\langle S|O\rangle$ and $\langle O|C\rangle$ one can obtain $\langle S|O\rangle$: $\langle S|O\rangle = \langle S|C\rangle\langle C|0\rangle$, etc.

III. THE CONSEQUENCES OF RELATIVISTICALLY INVARIANT EXPANSIONS

1. Asymptotic Integral Representation of the Amplitude and Quadratic Integrability

In order to obtain the asymptotic expansions we shall start by writing the amplitudes in the form

$$f(E,\theta,\varphi)=\frac{1}{2(2\pi)^3}\int_{-\infty}^{\infty}\sum_{lm}(-)^l\rho^2\frac{a_{lm}(\rho)\,\Gamma(-i\rho)\,P_{-1/2-i\rho}^{-l-\frac{1}{2}}(\operatorname{ch}a)}{\Gamma(-i\rho-l)\sqrt{\operatorname{sh}a}}Y_m(\theta,\varphi)\,d\rho, \tag{3.1}$$

where $E=\operatorname{ch}a$; θ is the scattering angle; the angle φ has been written for completeness; ρ characterizes the unitary representation of the Lorentz group; the coefficient $a_{lm}(\rho)$ is determined by the expression

$$a_{lm}(\rho)=\frac{(-)^l(2\pi)^{3/2}\,\Gamma(i\rho)}{2\Gamma(i\rho-l)}\int f(a,\theta,\varphi)\frac{P_{-1/2+i\rho}^{-1/2-l}(\operatorname{ch}a)}{\sqrt{\operatorname{sh}a}}Y_m(\theta,\varphi)\operatorname{sh}^2 a\,da\,d\Omega. \tag{3.2}$$

The function for which the direct and inverse transformations (3.1) and (3.2) are valid must satisfy the following conditions:

a) it must be quadratically integrable

$$\int|f(x)|^2dx<\infty, \tag{3.3}$$

where x is the ensemble of variables a, θ, φ, and $dx=\operatorname{sh}^2a\,da\,d\Omega$ is a volume element;

b) it must realize a regular representation T_g of the group of motions on a hyperboloid, which is isomorphic in the Lorentz group [34]:

$$T_gf(x)=f(g^{-1}x). \tag{3.4}$$

Assume $E\to\infty$ for fixed θ. If the integral and the integrand function are uniformly converging functions, then the limit symbol may be carried over into the integrand (which is usually what is done; see [35, 36]). Using the asymptotic representation of the associated Legendre function $P_\nu^\mu(z)$:

$$P_{-1/2-i\rho}^{-1/2-l}(z)\approx\frac{1}{\sqrt{2\pi z}}\left\{\frac{\Gamma(-i\rho)\,e^{-i\rho\ln 2z}}{\Gamma(-i\rho+l+1)}+\frac{\Gamma(i\rho)\,e^{i\rho\ln 2z}}{\Gamma(i\rho+l+1)}\right\}, \tag{3.5}$$

we obtain

$$f_{\text{asym}}(E,\theta)\approx\frac{\text{const}}{\sqrt{E\sqrt{E^2-1}}}\int_{-\infty}^{\infty}A(\theta,\rho)\,e^{i\rho\ln 2E}\,d\rho, \tag{3.6}$$

where

$$A(\theta,\rho)=\sum_l(-)^l\frac{|\Gamma(i\rho)|^2a_l(\rho)\,\rho^2}{\Gamma(-i\rho-l)\,\Gamma(i\rho+l+1)}P_l(\theta), \tag{3.7}$$

where $A(\theta, -\rho) \neq A^*(\theta, \rho)$. It should be noted that in other coordinate systems a Fourier expansion of the type (3.6) is likewise obtained. Since $\ln 2E \to \infty$, for fixed θ, it follows that due to oscillations in the exponent integration with respect to ρ in (3.6) must actually be limited to the range of small values of ρ. Taking this into account, as well as the cross-symmetry conditions in the form [37]:

$$\tilde{f}(E+i0, \theta) = f(-E-i0, \theta) = f^*(-E+i0, \theta), \tag{3.8}$$

it is not difficult to obtain [38, 29] a relationship between the scattering amplitudes $f(E, \theta)$ and $\tilde{f}(E, \theta)$ of a particle and antiparticle, respectively, on the same target: namely,

$$|f(E, \theta)|_{asym} = |\tilde{f}(E, \theta)|_{asym}, \tag{3.9}$$

which represents the contents of the Pomeranchuk theorem [35] in the case $\theta = 0$. Quadratic integrability of $f(E, \theta)$ requires that $|f(E, \theta)|$ decrease more rapidly than $1/E$ at infinity. If quadratic integrability of the amplitude $f(E, \theta)$ is rejected, then the partial amplitudes $A(\rho, \theta)$ become generalized functions (for example, δ-functions and their derivatives [38]).

2. The Relationship Between the Lorentz- and Regge-Amplitudes

Following [39, 40], let us consider the expansion of the amplitude in representations of the Lorentz group which correspond to reduction to the subgroup O(2.1). Previous papers [12, 41] have been devoted to analogous problems. This may be done most simply if the indicated expansions realized on basis functions of the Lorentz group, which are defined on a cone [42], are considered. In order to obtain the parametrization corresponding to these expansions it is convenient to construct the isotropic vector:

$$k_{i0} = p_i - p_0 e^{-A(t)}, \tag{3.10}$$

where $\operatorname{sh} A(t) = \frac{(p_i p_0)}{m^2}$, while

$$p_i = m(\operatorname{ch} a \operatorname{ch} \beta, \operatorname{ch} a \operatorname{sh} \beta, 0, \operatorname{sh} a)$$

if the momentum of one of the particles participating in the reaction ($i = 1, 2, 3, 4$); $p_0 = m(0, 0, 0, 1)$ is the momentum of the origin of the Breit system. Having placed $\operatorname{ch} a = e^{\alpha}$, we obtain

$$k_{i0} = me^{\alpha}(\operatorname{ch} \beta, \operatorname{sh} \beta, 0, 1).$$

The corresponding equations for the expansion of the amplitude in irreducible representations of the Lorentz group have the form* [42]

$$f(\alpha, \beta) = \int_{\delta-i\infty}^{\delta+i\infty} d\sigma (\sigma+1) \int_{L-i\infty}^{L+i\infty} A(\sigma, l) e^{-\sigma\alpha} (2l+1) \operatorname{ctg} \pi l P_l(\operatorname{ch} \beta)\, dl. \tag{3.11}$$

Let us recall that the unitary case corresponds to $\sigma = -1 + i\rho$ and $l = -1/2 + ia$. The Mandelstam variables s and t are related to α and β in the following manner:

$$e^{\alpha} = \sqrt{1 - \frac{t}{4m^2}}; \quad \operatorname{ch} \beta = -1 - \frac{2s}{t - 4m^2}.$$

*Since the amplitude is scalar, it follows that the expansion occurs only in degenerate representations having $\nu = 0$.

It may be shown [12, 42] that the Regge representation of the amplitude [12, 43] in the t channel is the analytic continuation of the portion of the expansion (3.11) corresponding to expansion in the subgroup O(2.1) from the s-channel to the t-channel. Taking account of this, we find that the partial amplitude $a(l, t)$ in the t-channel is related to the Lorentz-amplitude $A(\sigma, l)$ by a Laplace transform (a Fourier transform in the unitary case) [42]:

$$a(l, t) = 2i \cos \pi l \int_{\delta - i\infty}^{\delta + i\infty} d\sigma (\sigma + 1) e^{-\sigma\alpha} A(\sigma, l). \tag{3.12}$$

Going over to the unitary case ($\delta = -1$, $\sigma = -1 + i\rho$) in (3.12) and integrating zero to infinity with respect to ρ (see [5]), we find the form of the Lorentz-amplitude which generates a pole of the function $a_l(t)$:

$$A(\rho, l) = \frac{1}{\rho} \exp\{\pm i\rho f(l)\}, \tag{3.13}$$

where $f(l)$ is an arbitary function† having $\operatorname{Im} f(l) \gtrless 0$.

3. Lorentz- and Regge-Poles

In the paper by I. S. Shapiro [43] it was shown that the Regge asymptotic behavior of the amplitude $s^{l(t)}$ corresponds to the pole of the partial Lorentz-amplitude in the ρ-plane.

D. V. Volkov and V. N. Gribov [44] found whole families of Regge trajectories. Then Toller et al. [39] proved in the case t = 0 that the pole in the Lorentz-amplitude generates an entire series of equidistant Regge-poles (daughter trajectories). The case of an infinitely small transferred momentum was considered by Salam [40]. Other papers [45] have likewise been devoted to an investigation of daughter trajectories; in these papers a classification of Regge poles is given in certain models according to the Lorentz-poles. The simplest method of showing the development of daughter trajectories consists in the following. Let us consider the scattering amplitude $f(s, t)$ of two scalar particles having identical masses m. The variables s, t shall be defined as follows:

$$\left.\begin{aligned} s &= (p_1 + p_2)^2 = 2m^2 (1 + \operatorname{ch} a_{12}); \\ t &= (p_1 - p_3)^2 = 2m^2 (1 - \operatorname{ch} a_{13}), \end{aligned}\right\} \tag{3.14}$$

where a_{ik} is the distance between the points i, k lying on a hyperboloid having a radius m (i.e., $p^2 = m^2$). Hereafter we shall consider the case t = 0.

Let $p_i \in \{\mathcal{P}_4 : p_0^2 - p_3^2 - p_2^2 - p_1^2 = m^2\}$. Since $f(s, 0)$ is a Lorentz-invariant amplitude (in the sense $T_g f(s, 0) = f(g^{-1}s, 0) = f(s, 0)$, we shall expand it in Lorentz-invariant functions, namely, in elementary spherical functions which are defined as

$$\sqrt{\frac{2}{\pi}} \cdot \frac{\sin \rho a_{12}}{\rho \operatorname{sh} a_{12}} = \sum_{lm} \Psi^*_{\rho l m}\left(\frac{p_1}{m}\right) \Psi_{\rho l m}\left(\frac{p_2}{m}\right), \tag{3.15}$$

where $\Phi_{\rho l m}\left(\frac{p_i}{m}\right)$ is the function (1.7). (Here the subscript i denotes the number of particles.)

Therefore, one may write the following expansion for $f(s, 0)$ (see [22, 43]):

$$f(s, 0) = \int_0^\infty A(\rho) \sqrt{\frac{2}{\pi}} \cdot \frac{\sin \rho a}{\rho \operatorname{sh} a} \rho^2 d\rho \equiv \int_0^\infty a(\rho) \frac{\sin \rho a}{\operatorname{sh} a} \rho \, d\rho. \tag{3.16}$$

Here $a_{12} = a$. On the other hand, it is well known that for scattering of spinless particles the spatial portion of the momentum is determined completely by the two components; in other words, it may be assumed that

†In (3.13) $A(\rho, l)$ should be understood in the sense of the generalized function.

$p_i \subset \{\mathcal{P}_3 : p_0^2 - p_2^2 - p_1^2 = m^2\}$ (this means that $f(s, 0)$ should be expanded in a scalar product of functions which realize the representation of weight l of the O(2.1) (i.e., it should be expanded in Legendre functions):

$$P_l(\operatorname{ch} a) = \sum_m \Phi^*_{lm}\left(\frac{p_1}{m}\right)\Phi_{lm}\left(\frac{p_2}{m}\right), \tag{3.17}$$

where $\Phi_{lm}\left(\frac{p_i}{m}\right) = P_l^m(\operatorname{ch} a_i)\, e^{im\Phi_i}$, while l is a complex number. Thus,

$$f(s, 0) = \frac{1}{8\pi i}\int\limits_{\varepsilon - i\infty}^{\varepsilon + i\infty} dl a(l) P_{-l-1}(\operatorname{ch} a)\, l \operatorname{ctg} \pi l, \tag{3.18}$$

where

$$a(l) = \int\limits_0^\infty f(s, 0) P_l(\operatorname{ch} a)(\operatorname{sh} a)\, da \tag{3.19}$$

for the unitary case $l = -1/2 + iq$. Let us substitute Eq. (3.16) into (3.19), and then we obtain the following result after performing integration with respect to a:

$$a(l) = \frac{1}{2\pi i}\int\limits_0^\infty d\rho a(\rho)\beta(l, \rho), \tag{3.20}$$

where

$$\beta = -\frac{\sin \pi l}{2\pi}\left|\Gamma\left(\frac{l+1+i\rho}{2}\right)\right|^2\left|\Gamma\left(-\frac{l+i\rho}{2}\right)\right|^2. \tag{3.21}$$

If it is assumed that poles exist for the function $a(\rho)$ then one may obtain

$$a(l) = -\sum_k \frac{\sin \pi l}{2\pi}\Gamma\left(\frac{1+l+i\rho_k}{2}\right)\Gamma\left(\frac{1+l-i\rho_k}{2}\right)\Gamma\left(-\frac{l+i\rho_k}{2}\right)\Gamma\left(-\frac{l-i\rho_k}{2}\right) + \int_L d\rho a(\rho)\beta(l, \rho)$$

by shifting the contour. Thus, from Eq. (3.22) it is evident that each ρ-pole produces a series of equidistant poles in the l-plane as poles of Γ-functions.

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