THEORY OF FIELDS WITH NONPOLYNOMIAL LAGRANGIANS

M. K. Volkov

This work contains a review of contemporary methods of constructing two-point Green's functions and particle-scattering amplitudes, applicable in the quantum theory of fields with rapidly increasing spectral functions. These methods must ensure that the theory is finite in all orders of modified perturbation theory and must be such that the S-matrix is unitary. The nonanalyticity in the coupling constants, arising in the use of these methods, indicates that the usual perturbation theory is not applicable here.

INTRODUCTION

Considerable progress has recently been achieved in the development of methods, applicable in quantum field theory, of describing unrenormalizable interactions between elementary particles. In all these methods, an attempt is made to avoid the usual perturbation-theory methods and their accompanying difficulties (the appearance of an infinite number of undetermined constants in unrenormalizable theories; the impossibility of describing the nonanalytic dependence of amplitudes and Green's functions on the coupling constants in these theories, although some solvable models indicate the presence of such a dependence [1-4], etc.). Many authors have succeeded in solving such problems. At the present, however, the development of such methods is far from complete, although they permit us to obtain descriptions of many unrenormalizable interactions in complete agreement with the basic requirements imposed on field theory by the causality principle and the principle that the S-matrix must be unitary. In the present work we review the contemporary state of all these methods.

Before beginning our review, we briefly sketch the history of the formulation and solution of the problems discussed here.

The first article concerning the construction of two-point Green's functions in theories with Lagrangian (1.1) or (1.3) was published in 1954 by S. Okubo [2]. The method proposed by Okubo is similar to the methods of Group 2 or Group 3 (cf. §2). The technique used by Okubo and the formulas he obtained differ only slightly from contemporary technique and formulas. The only flaw in his work is the incorrect analytic continuation with respect to the coupling constants, which leads to a violation of the condition that the S-matrix must be unitary. Unfortunately this important work did not at first receive the attention it merited, and its value has only been appreciated recently.

The next article to appear was written by Arnowitt and Deser [5], and was published in 1955. The Lagrangian (1.1) was considered, but the method proposed by these authors for the construction of particle-scattering amplitudes and Green's functions was much cruder than Okubo's method. B. M. Barbashov and G. V. Efimov [6] showed in 1962 that the condition that the S-matrix be unitary is also violated in this work. Almost ten years passed before any important new work was published. In 1963, several authors returned simultaneously to the study of the theory of fields with nonpolynomially increasing spectral functions. Independent attempts were made by G. V. Efimov and E. S. Fradkin to construct a finite quantum theory for fields with nonpolynomial Lagrangians [30, 31]. In the same year G. Feinberg, in collaboration with A. Pais, proposed the peratization method, which was used in the theory of weak interactions for the construc-

Joint Institute for Nuclear Research, Dubna. Translated from Problemy Fiziki Élementarnykh Chastits i Atomnogo Yadra, Vol. 2, No. 1, pp. 33-66, 1971.

tion of particle-scattering amplitudes depending nonpolynomially on the amplitude [18]. After the appearance of this article, the number of papers published on work in this direction increased rapidly. An incomplete list of the authors is as follows: Guttinger, M. K. Volkov, B. A. Arbuzov and A. T. Filippov, Fried, Delbourgo, Salam and Strathdee, Lee and Zumino, Lehmann and Pohlmeyer, Keck and Taylor, Budini and Calucci [4, 7-12].

1. EXAMPLES OF UNRENORMALIZABLE INTERACTIONS

We begin by giving several examples of unrenormalizable interactions of elementary particles which have been described by the methods under consideration.

1. A neutral pseudoscalar theory, with pseudovector coupling between scalar and spinor fields [2, 5, 13]:*

$$L(x)=L_0(\Psi(x),\varphi(x))-\mathrm{i}g:\overline{\Psi}(x)\gamma_5\gamma_\nu\Psi(x)\partial^\nu\varphi(x): \tag{1.1}$$

By using Dyson's transformation [15]

$$\Psi'(x)=\exp\{-g\gamma_5\varphi(x)\}\Psi(x) \tag{1.2}$$

for the spinor field, we can get rid of the derivative in the interaction Lagrangian and write (1.1) as

$$L(x)=L_0(\Psi'(x),\varphi(x))-m:\overline{\Psi}'(x)(\exp[-2g\gamma_5\varphi(x)]-1)\Psi'(x): \tag{1.3}$$

(here the normal-product sign refers only to spinor fields).

The two-point Green's function, and also the scalar-particle-scattering amplitude, in the second order with respect to the dimensionless parameters (mg), can be expressed in terms of the function

$$F(x)=\mathrm{i}C(m'g)^2\,\mathrm{Sp}\{S^c(x)S^c(-x)\}\exp\{-\mathrm{i}(2g)^2\Delta^c(x)\}, \tag{1.4}$$

where $S^c(x)$ is the propagator of the free spinor field; $\Delta^c(x)$ is the propagator of the free scalar field; $m'=m\exp\{\mathrm{i}\,2g^2\Delta^c(0)\}$, and C is a constant.

The function F(x) has an essential singularity on the light cone, and its spectral function grows more rapidly than any polynomial in p^2. This indicates that the usual field-theory methods are inapplicable, and we have a typical situation in which a qualitatively new method must be used to describe the interaction.

2. A weak interaction, not conserving parity, between a neutral vector meson and a spinor field [16]:

$$L_{\text{inter}}(x)=G:\overline{\Psi}(x)\gamma^\nu(a+\mathrm{i}b\gamma_5)\Psi(x)W_\nu(x): \tag{1.5}$$

we use Stuekelberg's transformation [17]

$$W_\nu(x)=\varphi_\nu(x)+m_W^{-1}\frac{\partial}{\partial x_\nu}\theta(x), \tag{1.6}$$

where $\theta(x)$ is a scalar field and $\varphi_\nu(x)$ is a vector field with propagator

$$\Delta_{\mu\nu}^{(\varphi)}(p)=\frac{g_{\mu\nu}}{m_W^2-p^2-\mathrm{i}\varepsilon} \tag{1.7}$$

[the zero-spin part of the field $\varphi_\nu(x)$ has a negative metric], to separate the unrenormalizable part of the interaction. It is written

*In our notation we follow the monograph by N. N. Bogolyubov and D. V. Shirkov [14].

$$L_{\text{inter (unre)}}(x) = \frac{G}{m_W} : \overline{\Psi}(x)\,\gamma_\nu\,(a + ib\gamma_5)\,\Psi(x)\,\partial^\nu\theta(x): \tag{1.8}$$

and is similar to the interaction (1.1). Applying Dyson's transformation [15]

$$\Psi'(x) = \exp\left\{-iG\,(a + ib\gamma_5)\frac{\theta(x)}{m_W}\right\}\Psi(x), \tag{1.9}$$

we again obtain an interaction of type (1.3) with a nonpolynomial dependence on $\theta(x)$.

3. A weak four-Fermi interaction

$$L_{\text{inter}}(x) = G j_A(x)\, j_B(x), \tag{1.10}$$

where $j_A(x) = :\overline{\Psi}_A(x)\,\gamma_5\Psi_A(x):$. For the case $m_A = m_B = 0$ and $s = (p_1 + p_2)^2 = 0$, Guttinger [7] found expressions for the Bethe-Salpeter scattering amplitude of Fermi particles in the "ladder" approximation. This amplitude is

$$F_G(x) = a\exp\left\{-\frac{\sqrt{G}}{x^2 - i\varepsilon}\right\} + b\exp\left\{\frac{\sqrt{G}}{x^2 - i\varepsilon}\right\}, \tag{1.11}$$

and also differs only slightly from (1.4).

4. The scattering amplitude of spinor particles, found by Feinberg and Pais in the ladder approximation for a weak interaction between vector mesons and Fermi mesons [18]:

$$L_{\text{inter}}(x) = G{:}\overline{\Psi}(x)\,\gamma^\nu\,(1 + i\gamma_5)\,\Psi(x)\,W_\nu(x): \tag{1.12}$$

has a form qualitatively different from (1.4):

$$F_F^{(\pm)}(x) = -iG^2\,\frac{\left[g_{\mu\nu} + \dfrac{\partial_\mu\partial_\nu}{m_W^2}\right]\Delta^c(x)}{1 \pm i\left(\dfrac{2G}{m_W}\right)^2\Delta^c(x)}, \tag{1.13}$$

but it also leads to a spectral function growing more rapidly than any polynomial. In contrast to (1.4), the solution (1.13) does not satisfy the conditions of interaction localizability [19-22].

5. Chirally symmetric Lagrangians [8, 23-27]. For a typical example we use Weinberg's Lagrangian for π-mesons [26]:

$$L_{\text{inter}}(x) = G\,\frac{(\partial_\nu\varphi(x))^2}{(1 + g\varphi^2(x))^2}. \tag{1.14}$$

Here scattering-amplitude spectral functions also grow more rapidly than any polynomial, and the interaction has a nonlocal character.

There are examples of other interactions of a similar type (Einstein's gravitational Lagrangian, the Yang-Mills theory with massive fields, etc.), for example, in the work of Salam and his collaborators [8]. We confine ourselves to the five examples already described as being the most characteristic. For the demonstration of various methods used in theories with rapidly growing spectral functions, we consider a certain Lagrangian of general form. For simplicity we assume that it depends only on a one-component scalar field, does not contain derivatives, and can be expressed as an infinite series in powers of $\varphi(x)$:

$$L_{\text{inter}}(x) = G\sum_0^\infty \frac{u(n)}{n!}\,:(\varphi(x))^n: = G{:}U(\varphi(x)): \tag{1.15}$$

The coefficients u(n) are proportional to the second coupling constants g^n, which are always present in such theories together with G. In the following, we construct the perturbation theory with respect to G. In each order with respect to G, all orders with respect to g are taken into consideration. We mainly consider second orders with respect to G, since this suffices for the description of characteristic features of the methods.

2. REMARKS CONCERNING ALL METHODS, AND CONDITIONS IMPOSED ON TWO-POINT FUNCTIONS IN MOMENTUM SPACE

In theories with essentially nonlinear Lagrangian, the S-matrix is constructed as in the renormalizable theories with renormalizable Lagrangians [14]:

$$S = 1 + \sum_{1}^{\infty} \frac{G^n}{n!} S_n, \tag{2.1}$$

where

$$S_n = (\mathrm{i})^n \int \dots \int d^4x_1 \dots d^4x_n T\left(U(\varphi(x_1)) \dots U(\varphi(x_n))\right). \tag{2.2}$$

Elastic and inelastic scattering of scalar particles is possible for the Lagrangian (1.15) in the first order with respect to G. However,this order is not of interest since it is trivial (there is no divergence, and all amplitudes are expressed in terms of constants). We thus turn to the consideration of second-order perturbation theory:

$$S_2 = -\int\int d^4x_1 d^4x_2 \sum_{0}^{\infty} \sum_{0}^{\infty} F^{(2)}_{k_1k_2}(x_1 - x_2):$$

$$:\frac{\varphi^{k_1}(x_1)}{k_1!} \cdot \frac{\varphi^{k_2}(x_2)}{k_2!}:, \tag{2.3}$$

where

$$F^{(2)}_{k_1k_2}(x_1 - x_2) = \sum_{m=0}^{\infty} \frac{u_{m+k_1} u_{m+k_2}}{m!} \left[-\mathrm{i}\Delta^c(x_1 - x_2)\right]^m, \tag{2.4}$$

and

$$\Delta^c(x_1 - x_2) = \frac{1}{(2\pi)^4} \int d^4p \frac{e^{\mathrm{i}p(x_1 - x_2)}}{m^2 - p^2 - \mathrm{i}\varepsilon} \tag{2.5}$$

is the propagator of a free scalar particle with mass m. We shall often consider the interaction of scalar particles with zero rest mass. In this case (2.5) has the simple form

$$\Delta^c(x) = -\frac{\mathrm{i}}{(2\pi)^2 (x^2 - \mathrm{i}\varepsilon)}. \tag{2.6}$$

All Green's functions and scattering amplitudes of scalar particles in the second order with respect to G will be expressed in terms of the two-point function (2.4).

Two difficulties arise in the study of this operator. The first is related to the appearance of ultraviolet divergence at the transition to the momentum space. The form of (2.4) shows that there is a pole of any order on the light cone. In localizable interactions, in particular, this causes the function $F^{(2)}_{k_1k_2}(x)$ to have an essential singularity on the light cone.

Another difficulty is a specific feature of nonlocalizable interactions. It is that the series (2.4), for such interactions, either converges only in a bounded region of x^2, or it does not converge in general for any region of x^2 and is an asymptotic series. Hence, in addition to formerly known difficulties concerning divergent integrals, we are also confronted with the necessity of summing divergent series.

The principal requirement for the solution of these problems is that the S-matrix of the final theory be finite and unitary. The microcausality principle must be satisfied in theories with localizable interactions, and the macrocausality principle must be satisfied in theories with nonlocalizable interactions [21].

How shall we find the boundary between localizable and nonlocalizable interactions? This is most simply done by studying the asymptotic behavior of the operator (2.4) in momentum space, and in particular its imaginary part, which is uniquely determined. For localizable interactions there is a definite limit to the rate of growth of Green's functions in momentum space, first discovered in somewhat different forms by Meiman [19] and Jaffe [20]. We start from the more general condition obtained by Meiman to the effect that for increasing p^2 we have

$$|\tilde{F}^{(2)}_{k_1k_2}(p)| < C \exp\{\sigma \sqrt{|p^2|}\} \tag{2.7}$$

for any $\sigma > 0$ (cf. also [21, 22]).

In particular (2.7) must be satisfied by the imaginary part of the Fourier transform (2.4), whose behavior for $p^2 \to \infty$ is decisive for all functions $\tilde{F}^{(2)}_{k_1k_2}(p)$. Hence, it is natural to start with the determination of this imaginary part.

We also need the imaginary part of $\tilde{F}^{(2)}_{k_1k_2}(p)$ to verify that the S-matrix is unitary. What exactly is this condition? Consider the Fourier transform of the two-point function (2.4):

$$\tilde{F}^{(2)}_{k_1k_2}(p) = i \int d^4x e^{ipx} F^{(2)}_{k_1k_2}(x). \tag{2.8}$$

We require that the function $\tilde{F}^{(2)}_{k_1k_2}(p)$ have completely analytic properties. For definiteness, we consider the elastic scattering of two scalar particles. For $p^2 < 0$ the function (2.8) must be real, for $0 < p^2 < 4m^2$ there must be a simple pole for $p^2 = m^2$, and at the point $p^2 = 4m^2$ there is a branch point from which a cut extends to infinity. The discontinuity at the cut must be expressed in terms of the sum of the phase volumes of the scalar particles:

$$\operatorname{Im} \tilde{F}^{(2)}_{22}(p) = \pi u_3^2 \delta(p^2 - m^2) + \pi \sum_{k=2}^{\left[\frac{\sqrt{p^2}}{m}\right]} \frac{u^2_{k+2}}{k!} \Omega^{(m)}_k(p^2), \tag{2.9}$$

where

$$\Omega^{(m)}_k(p^2) = \frac{1}{(2\pi)^{3(k-1)}} \int \frac{dq_1}{2\omega_1} \cdots \int \frac{dq_k}{2\omega_k} \delta^{(4)}(p - q_1 - \cdots - q_k)$$
$$(\omega_i = \sqrt{q_i^2 + m^2}). \tag{2.10}$$

If the rest mass of scalar particles is zero, the phase volume is simply a power function of p^2:

$$\Omega^{(0)}_k(p^2) = (4\pi)^{-2} \left[\frac{p^2}{(4\pi)^2}\right]^{k-2} \frac{\Theta(p^2)\,\Theta(p^0)}{(k-1)!\,(k-2)!}, \tag{2.11}$$

and the upper limit of the sum in (2.9) is infinity. Concerning the analytic properties $\tilde{F}^{(2)}_{22}(p)$ must possess for mass-free particles, it must be real for $p^2 < 0$, and it must have a pole and a branch point for $p^2 = 0$. There must be a logarithmic cut from the origin to plus infinity [cf. [11] and also formulas (6.26) and (6.32)].

Whatever method is used for constructing the finite function $\tilde{F}^{(2)}_{k_1k_2}(p)$, it must possess the above properties for the S-matrix to be unitary.

We now consider the determination of the boundary between localizable and nonlocalizable interactions. We must find the asymptotic properties of $\mathrm{Im}\,\widetilde{F}_{22}^{(2)}(p)$ for $p^2 \to \infty$. For mass-free particles, we use (2.11) for phase volumes. If the masses are not zero, we use an approximate expression for $\Omega_k^{(m)}(p^2)$, valid for $p^2 \gg m^2$ [28]:

$$\Omega_k^{(m)}(p^2) \approx \frac{p^{\frac{k-3}{2}}(p-km)^{\frac{3k-5}{2}}}{\Gamma(2k)}\varphi(p,\ k), \tag{2.12}$$

where $p = \sqrt{p^2}$, $\Gamma(2k)$ is the gamma function, and $\varphi(p, k)$ is a slowly varying function whose dependence on p can be neglected in the calculation of the asymptotic properties of $\mathrm{Im}\,\widetilde{F}_{22}^{(2)}(p)$ for large p^2. Results for various u_k are given in Table 1.

By using the limitation on the growth of scattering amplitudes in localizable theories specified in (2.7), we easily obtain the corresponding conditions for the u_k:

$$\overline{\lim_{k\to\infty}}\left|\frac{u_k}{\Gamma\left(\frac{k}{2}\right)}\right|^{1/k} = 0 \tag{2.13}$$

or

$$\overline{\lim_{k\to\infty}}\,|c_m(k)|^{1/k} = 0, \tag{2.14}$$

where

$$c_m(k) = \frac{(u_{k+m})^2}{k!}. \tag{2.15}$$

Lagrangians for which the coefficients of the series expansion (1.15) satisfy conditions (2.13) or (2.14) describe localizable interactions. If the above conditions are not satisfied, we have nonlocalizable interactions. There are two subclasses of the latter: the first in which the series (1.15) converges to a definite function of the field $\varphi(x)$, and the second in which this series diverges. Only the first subclass is interesting from the physical point of view. In the second subclass, not only does the series (1.15) diverge, but in most cases the spectral function does not exist for mass-free particles (cf. Table 1).

An interesting difference between localizable and nonlocalizable interactions is the following. The asymptotic behavior of amplitudes for large momentum is identical in local theories for particles of zero and nonzero mass. For nonlocalizable interactions, the asymptotic behavior of scattering amplitudes for particles with mass and without mass is different, as can be seen from Table 1.* Hence, in approximate calculations for large energies in nonlocal theories, we cannot simply neglect the mass and use expressions from the case for mass-free particles. (This error is made in the paper by Salam and Strathdee [9]; for large p^2, they use for the propagator of a massive particle an approximate expression coinciding with the expression for a zero-mass particle.)

We thus give a brief description of problems to be discussed and classify the methods used for the solution of these problems.

We shall study a two-point function F(x), expressed as an infinite series in powers of the propagator of a free scalar particle:

$$F(x) = \sum_1^{\infty} c(n)\,[-i\Delta^c(x)]^n. \tag{2.16}$$

*We obtained this result recently, and it is published for the first time in the present article. For the definition of local and nonlocal theories, and also localizable and nonlocalizable interactions, see the footnote to Table 1.

TABLE 1. Asymptotic Properties of $\ln \mathrm{Im}\tilde{F}^{(2)}_{22}(p^2)$ for Large p^2

Section	No.	$u(k)$	$\overline{\lim\limits_{k\to\infty}} \mid c(k) \mid^{1/k}$, $c(k)=\frac{(u_k)^2}{k!}$	$U(\varphi)$	$\ln \mathrm{Im}\tilde{F}^{(2)}_{22}(p)$ $p\to\infty$ $(m\neq 0)$	$\ln \mathrm{Im}\tilde{F}^{(2)}_{22}(p)$ $p\to\infty$ $(m=0)$	Interactions
I	1	$u(k)=0$, $k>N$	0	$\sum\limits_0^N \frac{u(k)}{k!} : \varphi^k(x) :$	$2(N-4)\ln p$	$2(N-4)\ln p$	Localizable
I	2	$\sim [\Gamma(\gamma k)]^{-1}_{(\gamma>0)}$	0	$I_0(\sqrt{g\varphi(x)})_{\gamma=1}$	const $p^{\frac{2}{3+2\gamma}}$	const $p^{\frac{2}{3+2\gamma}}$	Localizable
I	3	$\sim(\text{const})^k$	0	$\exp\{g\varphi(x)\}$	const $p^{2/3}$	const $p^{2/3}$	Localizable
I	4	$\sim\Gamma(\gamma k)$, $\gamma<1/2$	0	$\int\limits_0^\infty dv e^{vg\varphi(x)-v^{1/\gamma}}$	const $p^{\frac{2}{3-2\gamma}}$	const $p^{\frac{2}{3-2\gamma}}$	Localizable
II	5	$\sim\Gamma\left(\frac{k}{2}\right)$	$\sim\frac{1}{2}$	$\exp\{\mp g^2\varphi^2(x)\}$	const p	const p	Nonlocalizable
II	6	$\sim\Gamma(\gamma k)$, $1/2<\gamma\leqslant 1$	∞	$\exp\{\mp g^{2m}\varphi^{2m}(x)\}$, $m\geqslant 2$	const $(2\gamma-1)\,p\ln p$	const $p^{\frac{2}{3-2\gamma}}$	Nonlocalizable
				$\left(\gamma=1-\frac{1}{2m}\right)\ \gamma<1$			Nonlocalizable
II	7	$\sim\Gamma(k)$	∞	$\frac{g\varphi}{\sqrt{1+g^2\varphi^2}}$; $\frac{g\varphi}{1\pm g^2\varphi^2}$	const $p\ln p$	const p^2	Nonlocalizable
III	8	$\sim\Gamma(\gamma k)$, $\gamma>1$	∞	$\sum\limits_0^\infty \Gamma((\gamma-1)k)(g\varphi)^k$	const $(2\gamma-1)\,p\ln p$	const $p^{\frac{2}{3-2\gamma}}$, $\gamma<3/2$; Does not exist if $\gamma>3/2$	Nonlocalizable
III	9	$\sim e^{k^2}$	∞	$\sum\limits_0^\infty e^{n^2}(g\varphi)^n$	const $2p^2$	Does not exist	Nonlocalizable

*An interaction is localizable if it leads to a spectral function satisfying (2.7). If the spectral function grows more rapidly than (2.7) for large values of the momentum, then the interaction is called nonlocalizable.

Local theories are theories in which the microcausality principle holds. A theory is nonlocal if the microcausality principle is violated at small distances, and only the macrocausality principle holds (cf. [21]). Nonlocalizable interactions are always described by nonlocal theories, while localizable interactions can be described by both nonlocal and local theories, depending on the method used in constructing Green's functions and particle-scattering amplitudes [19-22].

We wish to construct the Fourier transform of F(x):

$$\tilde{F}(p) = i \int d^4x e^{ipx} F(x), \tag{2.17}$$

which must be a finite function of p^2 and have completely analytic properties. The function $\tilde{F}(p)$ must have a spectral representation of the form [20, 21]

$$\tilde{F}(p) = \frac{c(1)}{m^2 - p^2 - i\varepsilon} + \int_{4m^2}^{\infty} d\varkappa^2 \frac{\rho(\varkappa^2)}{\varkappa^2 - p^2 - i\varepsilon} V(\varkappa^2, p^2) + W(p^2), \tag{2.18}$$

where $\rho(\varkappa^2)$; $V(\varkappa^2, p^2)$ and $W(p^2)$ are entire functions of $\varkappa^2$ and p^2 in the corresponding complex planes and

$$V(\varkappa^2, \varkappa^2) = 1. \tag{2.19}$$

The existing methods can be divided into the following four groups.

1. Determination of F(x).

2. Determination of $\tilde{F}(p)$.

3. Determination of F(x) and $\tilde{F}(p)$ by using a solution of the corresponding equations for Green's functions or scattering amplitudes.

4. Introduction of nonlocal form-factors.

Nonlocalizable interactions are usually investigated by the first type of method. Here (2.16) is an asymptotic series. We postulate that there is a "regular" function F(x) having no singularity for $x^2 \to 0$, while $\Delta^c(x) \to \infty$. The series (2.16) is the asymptotic expansion of F(x) for $x^2 \to \infty$, while $\Delta(x) \to 0$. The problem is to find this "regular" function.

Methods in the second group are close to the so-called analytic-regularization methods, which are applied in renormalizable theories. By using analytic continuation with respect to the propagator index n in (2.16), the originally divergent integral (2.17) is given a completely definite meaning. At the final stage of the calculation, this intermediate procedure is removed, and we obtain a finite expression for $\tilde{F}(p)$ with the required analytic properties.

In group 3, the equations for the Green's functions or the scattering amplitudes are solved in momentum space in a euclidean region of the momentum. Solutions are obtained for nonfixed values of the coupling constant and are then continued analytically to true values. The solutions of the equations can be used to find F(x) in x-space.

In the last group, form-factors are introduced which permit the avoidance of ultraviolet divergences, but which do not violate the condition that the S-matrix be unitary.

These methods have the following common features. Investigation begins in a euclidean region of the variables, and the resulting functions are continued analytically into the whole range of p^2. Nonuniqueness arises in these methods only in the second order of perturbation theory.

Further nonuniqueness does not arise in higher orders.

3. CONDITIONS IMPOSED ON THE TWO-POINT FUNCTION IN CONFIGURATION SPACE*

Before describing methods in group 1, we stress that all these methods apply to nonlocal theories. Some are theoretically applicable only to nonlocalizable interactions (cf. Secs. II and III of Table 1). Others can also be applied to localizable interactions, but we obtain only nonlocal theories.

*In §3 and 4, and in part of §2, we follow G. V. Efimov's review [29].

What are the conditions that the two-point function must satisfy, in order that transition to momentum space will yield a finite function with regular analytic properties, in conformity with the condition that the S-matrix be unitary? We shall indicate these general conditions for all methods of group 1.

Since (2.16) is an asymptotic series for nonlocalizable interactions, it can be associated with some function F(x) for $x^2 \to -\infty$ ($\Delta^c(x) \to 0$). This function must satisfy the following two conditions:

1) the absence of ultraviolet divergence:

$$\lim_{x^2\to 0} |F(x)| = 0; \tag{3.1}$$

2) the reality of the amplitude in the nonphysical region of the variables $x^2 < 0$ (or $p^2 < 0$). The function F(x) must be real and have no singularities in the interval

$$-\infty < x^2 < 0. \tag{3.2}$$

The second condition ensures the regular analytic properties of the scattering amplitude.

We now assume that conditions 1 and 2 are satisfied, and show that $\widetilde{F}(p)$ has the regular analytic properties corresponding to the unitary property of the S-matrix.

In the euclidean region $p^2 = -q^2 < 0$, the integral (2.17) can be written

$$\widetilde{F}(p) = \int d_e^4 x e^{iqx} F(x) = \widetilde{F}_1(p) + \widetilde{F}_2(p) + \widetilde{F}_3(p), \tag{3.3}$$

where

$$\widetilde{F}_1(p) = \int d_e^4 x e^{iqx} \Theta(a^2 - x^2) F(x); \tag{3.4}$$

$$\widetilde{F}_2(p) = \int d_e^4 x e^{iqx} \Theta(x^2 - a^2) \left\{ F(x) - \sum_1^N C(n) [-i\Delta^c(x)]^n \right\}; \tag{3.5}$$

$$\widetilde{F}_3(p) = \int d_e^4 x e^{iqx} \Theta(x^2 - a^2) \sum_1^N C(n) [-i\Delta^c(x)]^n. \tag{3.6}$$

Here a is a nonzero real parameter, and N is any integer.

Integration over euclidean angles in (3.4) and (3.5) yields

$$\widetilde{F}_1(p) = (2\pi)^2 \int_0^a du u^2 \frac{J_1(u\sqrt{-p^2})}{\sqrt{-p^2}} F(u); \tag{3.7}$$

$$\widetilde{F}_2(p) = (2\pi)^2 \int_a^\infty du u^2 \frac{J_1(u\sqrt{-p^2})}{\sqrt{-p^2}} \left\{ F(u) - \sum_1^N C(n) \left(\frac{mK_1(mu)}{4\pi^2 u} \right)^n \right\}, \tag{3.8}$$

where $J_1(x)$ and $K_1(x)$ are Bessel and Macdonald functions. The integral (3.7) converges for any complex p^2, and defines an entire function of order 1/2 in the p^2 complex plane. The integral (3.8) defines an analytic function of p^2 for

$$\text{Re}\, p^2 < (N+1)^2 m^2, \tag{3.9}$$

since, for large u, the expression in braces decreases like

$$\sim \exp\{-(N+1)mu\}. \tag{3.10}$$

The functions $\widetilde{F}_1(p)$ and $\widetilde{F}_2(p)$ are real. The contribution from the imaginary part of $\widetilde{F}(p)$ yields only $\widetilde{F}_3(p)$. Using the identity

$$[-i\Delta_m^c(x)]^n = -i\int_{(nm)^2}^{\infty} dx^2\Omega_n^{(m)}(\varkappa^2)\,\Delta_\varkappa^c(x), \tag{3.11}$$

where $\Delta_\varkappa^c(x)$ is the causality function for a scalar particle with mass $\varkappa$ and $\Omega_n^{(m)}(\varkappa)$ is the phase volume of n particles of mass m, we reduce the integral (3.6) to the form

$$\widetilde{F}_3(p) = \frac{C(1)}{m^2-p^2-i\varepsilon} + \int_{4m^2}^{\infty} d\varkappa^2 \frac{\rho_N(\varkappa^2)\,d_a(\varkappa^2, p^2)}{\varkappa^2-p^2-i\varepsilon}, \tag{3.12}$$

where

$$\rho_N(\varkappa^2) = \sum_2^N C(n)\,\Omega_n^{(m)}(\varkappa^2), \tag{3.13}$$

and

$$d_a(\varkappa^2, p^2) = \varkappa a\left[J_0\left(a\sqrt{-p^2}\right)K_1(a\varkappa) + \frac{J_1(a\sqrt{-p^2})}{a\sqrt{-p^2}}\,a\varkappa K_0(a\varkappa)\right]; \tag{3.14}$$

$$d_a(\varkappa^2, \varkappa^2) = 1 \tag{3.15}$$

The integral (3.12) converges because $\rho_N(\varkappa^2) \sim \varkappa^{2N}$ and $d_a(\varkappa^2, p^2) \sim \exp\{-a\varkappa\}$. for $\varkappa \to \infty$.

The function $\widetilde{F}_3(p)$ has a simple pole at $p^2 = m^2$, and a cut starting from the point $p^2 = 4m^2$. For $4m^2 < p^2 < (N+1)^2m^2$, we have

$$\operatorname{Im}\widetilde{F}_3(p) = \pi\sum_2^{\left[\frac{\sqrt{p^2}}{m}\right]} C(n)\,\Omega_n^{(m)}(p^2), \tag{3.16}$$

which is in agreement with the unitary condition.

Since N can be arbitrarily large, it may be chosen so that (3.9) always holds, and $\widetilde{F}(p)$ has a simple pole for $p^2 = m^2$ and a cut starting from the point $p^2 = 4m^2$, across which the discontinuity is given by (3.16). There are no other singularities in the finite range of p^2. Hence $\widetilde{F}(p)$ has a representation of the type (2.18), ensuring the regular unitary properties and the observance of the causality principle.

We now consider a question related to the nonuniqueness of these methods. Summation of asymptotic series does not yield a unique result. Different functions obtained by such summations will differ by a function with the following property:

$$\lim_{x^2\to-\infty} \Delta^{-A} f(\Delta) = 0 \quad (f(\Delta) = F_{(1)}(x) - F_{(2)}(x)). \tag{3.17}$$

This function has an essential singularity for $x^2 \to -\infty$, and its contribution to the asymptotic series (2.16) is always zero. The Fourier transform $f(\Delta)$ is an entire analytic function [cf. $W(p^2)$ in (2.18)].

4. METHODS OF DETERMINING F(x)

1. The Efimov-Fradkin Method. This method was proposed independently by G. V. Efimov [30] and E. S. Fradkin [31] in 1963. It was the first attempt to construct a finite quantum theory with an essentially

nonlinear Lagrangian. The authors started from the position that a finite theory cannot be constructed for an arbitrary Lagrangian, and their object was to find the class of Lagrangians for which a finite theory was possible. Their method yields a finite theory for Lagrangians satisfying the following conditions.

a. $U(\alpha)$ is a continuous function with no singularities on the real axis, which can be expanded in the neighborhood of the origin in a Taylor series with a radius of convergence ρ:

$$U(\alpha)=\sum_{0}^{\infty}\frac{u(n)}{n!}\alpha^{n}. \tag{4.1}$$

b. The integral of $|U(\alpha)|^2$ must exist over any bounded region of the complex α-plane.

c. At infinity $U(\alpha)$ satisfies the condition

$$\overline{\lim_{|\alpha|\to\infty}}\frac{U(\alpha)}{\alpha^{2}}=0. \tag{4.2}$$

This method describes nonlocalizable interactions with a Lagrangian of type 7 (see Table 1).

We now describe this method. Using Vik's theorem [32], we write the symbol T for the product in (2.2) in the form

$$T=\exp\left\{-\frac{i}{2}\int\int d^4x_1 d^4x_2\Delta^c(x_1-x_2)\frac{\delta^2}{\delta\varphi(x_1)\delta\varphi(x_2)}\right\}. \tag{4.3}$$

Relation (2.3) now becomes

$$S_2=-\int\int d^4x_1d^4x_2\exp\left\{-\frac{i}{2}\int\int d^4y_1d^4y_2\Delta^c(y_1-y_2)\frac{\delta^2}{\delta\varphi(y_1)\delta\varphi(y_2)}\right\}:U(\varphi(x_1))U(\varphi(x_2)):. \tag{4.4}$$

Our problem is to find an expression of the type

$$S_2(\Delta,\alpha_1,\alpha_2)=\exp\left\{-\frac{i}{2}\Delta^c\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\right\}U(\alpha_1)U(\alpha_2). \tag{4.5}$$

Using the integral representation

$$\exp\left\{-\frac{i}{2}\Delta\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\right\}=\frac{1}{\pi}\int\int_{-\infty}^{\infty}dt_1dt_2\exp\left\{-t_1^2-t_2^2+\sqrt{\frac{\Delta}{2i}}\left[(t_1+it_2)\frac{\partial}{\partial\alpha_1}+(t_1-it_2)\frac{\partial}{\partial\alpha_2}\right]\right\} \tag{4.6}$$

in (4.5) and noting that (4.6) contains a translation operator, we rewrite (4.5) as

$$S_2(\Delta,\alpha_1,\alpha_2)=\frac{1}{\pi}\int\int_{-\infty}^{\infty}dt_1dt_2e^{-t_1^2-t_2^2}U\left(\alpha_1+\sqrt{\frac{\Delta}{2i}}(t_1+it_2)\right)U\left(\alpha_2+\sqrt{\frac{\Delta}{2i}}(t_1-it_2)\right). \tag{4.7}$$

The change of variables

$$t_1=u_1-\frac{1}{\sqrt{-2i\Delta}}(\alpha_1+\alpha_2),$$

$$t_2=u_2-\frac{1}{\sqrt{2i\Delta}}(\alpha_1-\alpha_2), \tag{4.8}$$

yields

$$S_2(\Delta,\alpha_1,\alpha_2)=\frac{1}{\pi}\int\int_{-\infty}^{\infty}du_1du_2U\left(\sqrt{\frac{\Delta}{2i}}(u_1+iu_2)\right)U\left(\sqrt{\frac{\Delta}{2i}}(u_1-iu_2)\right)\exp\left\{-\left[u_1-\frac{\alpha_1+\alpha_2}{\sqrt{-2i\Delta}}\right]^2-\left[u_2-\frac{\alpha_1-\alpha_2}{\sqrt{2i\Delta}}\right]^2\right\}. \tag{4.9}$$

The expansion of this function in powers of α_1 and α_2 shows that F(x) can be expressed as a finite sum of functions

$$f_{k_1k_2}(\Delta)=\frac{1}{\pi}\int\int_{-\infty}^{\infty} du_1\,du_2 u_1^{k_1}u_2^{k_2}e^{-u_1^2-u_2^2}\left|U\left(\sqrt{\frac{\Delta}{2i}}(u_1+iu_2)\right)\right|^2. \tag{4.10}$$

It is clear that for (3.1) to hold (the absence of ultraviolet divergence in the theory), it is necessary that

$$\lim_{|z|\to\infty}|U(z)|=0. \tag{4.11}$$

Condition c follows. But condition b is necessary for the integral (4.10) to exist. This condition implies that U(z) has cuts of order $\gamma < 1$ in the complex z-region, for example

$$U(\varphi)=\frac{1}{[1+g^2\varphi^2(x)]^{\gamma}}. \tag{4.12}$$

We stress the fact that the intermediate stages of the calculation carried out here have no rigorous mathematical basis. For example, there is no mathematical justification for the conversion of (4.5) into (4.7) by the application of the translation operator to the argument of $U(\alpha)$ since, in the region of the translation, this function may have singularities whose intersection can, in general, yield extra contributions to (4.7). It follows that the method does not possess the property of uniqueness.

The method is applicable in all orders of perturbation theory, and it leads to a finite theory with a unitary S-matrix.

2. The Lee-Zumino Method. A direct development from the Efimov-Fradkin method is a procedure described by Lee and Zumino [10]. While the former authors aimed to find a class of Lagrangians which could be used to construct a finite and unitary quantum field theory, Lee and Zumino had the object of describing given Lagrangians obtained, for example, in the chirally symmetric theories [23-27]. They thus considered a Lagrangian, differing slightly from chirally symmetric Lagrangians, but inconsistent with condition b of the above method:

$$U(\varphi)=\frac{\sqrt{\varkappa}\varphi(x)}{1-\varkappa\varphi^2(x)}. \tag{4.13}$$

Here the function (2.16) is

$$F(x)=\sum_0^{\infty}(2n+1)!\varkappa^{2n+1}[-i\Delta^c(x)]^{2n+1}. \tag{4.14}$$

Borel summation is applied to the divergent series (4.14) [33]:

$$F(x)=\int_0^{\infty}dte^{-t}\sum_0^{\infty}\varkappa^{2n+1}(-it\Delta^c(x))^{2n+1}=-i\varkappa\int_0^{\infty}dt\frac{te^{-t}\Delta^c(x)}{1+[t\varkappa\Delta^c(x)]^2}. \tag{4.15}$$

For $p^2 < 0$, the Fourier transform of F(x) is

$$\widetilde{F}_{\varkappa^2}(p)=\frac{4\pi^2\varkappa}{\sqrt{-p^2}}\int_0^{\infty}dtte^{-t}\int_0^{\infty}drr^2J_1(\sqrt{-p^2}r)\frac{\left(\frac{mK_1(mr)}{4\pi^2r}\right)}{1-t^2\varkappa^2\left(\frac{mK_1(mr)}{4\pi^2r}\right)^2}. \tag{4.16}$$

The function $\widetilde{F}_{\varkappa^2}(p)$ is defined for all $\varkappa^2$ except for values on the positive real axis, where it has a discontinuity. But the physical value of $\varkappa^2$ is on this cut. Hence the value of $\widetilde{F}(p)$ for $\varkappa^2 > 0$ must be determined from a combination of the values on the upper and lower sides of this cut, as in [11]:

$$\widetilde{F}(p)=\alpha\widetilde{F}_{\varkappa^2+i0}(p)+\alpha^*\widetilde{F}_{\varkappa^2-i0}(p), \tag{4.17}$$

where $\alpha = 1/2 + i\eta$, with η a real parameter. This choice of α ensures the regular unitary properties of $\widetilde{F}(p)$. The function $\widetilde{F}(p)$ is easily continued analytically from the region $p^2 < 0$ to the whole range $-\infty < p^2 < \infty$. The nonuniqueness of the procedure is evinced by the undefined parameter η. We do not discuss the construction of higher orders.

3. The Direct Summation Method. We name this method after its author, G. V. Efimov [21, 29, 34]. It is similar to those of group 4, except that the form-factor used is introduced in the definition of F(x) and not in the interaction Lagrangian or in the free-field propagator, as in group 4. This is related to the definition of the T-product.

We shall demonstrate how this is done. If the representation

$$U(\varphi)=\int_{-\infty}^{\infty} d\beta\widetilde{U}(\beta)\, e^{i\beta\varphi(x)} \tag{4.18}$$

is used for the interaction Lagrangian, where

$$\int_{-\infty}^{\infty} d\beta\, |\beta^n\widetilde{U}(\beta)|<\infty \tag{4.19}$$

for any n, and the T-product is defined by (4.3), then the n-th order S-matrix is

$$S_n(x_1, \ldots, x_n)=\int_{-\infty}^{\infty}\cdots\int\prod_1^n d\beta_k\widetilde{U}(\beta_k)\, e^{i\beta_k\alpha_k}\prod_{1\leq i<j\leq n} e^{i\Delta_{ij}\beta_i\beta_j}, \tag{4.20}$$

where $\alpha_i\equiv\varphi(x_i)$ and $\Delta_{ij}\equiv\Delta^c(x_i-x_j)$. We easily see that the resulting expression does not satisfy condition (3.1) (the absence of ultraviolet divergence). To satisfy this condition we used the following regularization. We assume that the chronological ordering operation (4.3), acting under the integration sign in (4.2), yields

$$e^{-i\Delta_{ij}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j}}e^{i(\beta_i\alpha_i+\beta_j\alpha_j)}\Rightarrow e^{i(\beta_i\alpha_i+\beta_j\alpha_j)}\, e^{i\Delta_{ij}\beta_i\beta_j}\Theta(1+\lambda\Delta_{ij}^2\beta_i^2\beta_j^2), \tag{4.21}$$

where

$$\Theta(u)=\begin{cases}1 & (u>0)\\ 0 & (u<0)\end{cases} \tag{4.22}$$

and λ is a positive parameter (here and everywhere in this section we are considering euclidean space, where $-i\Delta^c(x) > 0$). By using (4.21) and (4.20) we can write

$$F^{(n)}_{k_1\ldots k_n}(x_1\ldots x_n)=\int_{-\infty}^{\infty}\cdots\int\prod_1^n d\beta_s\widetilde{U}(\beta_s)(i\beta_s)^{k_s}\prod_{1\leq i\leq j<n} e^{i\Delta_{ij}\beta_i\beta_j}\Theta(1+\lambda\Delta_{ij}^2\beta_i^2\beta_j^2). \tag{4.23}$$

This formula yields the usual expression for the coefficient functions in the case of renormalizable interactions of the type $U(\varphi) = \varphi^3(x)$ or $U(\varphi) = \varphi^4(x)$; this is in agreement with the regularity of the procedure.

The procedure is nonunique. Not only is the parameter λ undetermined, but the Θ-function in (4.21) can be defined in various ways. Thus the Θ-function can be $\Theta(1-\lambda(-i\Delta_{ij})^a\beta_i^2)\times\Theta(1-\lambda(-i\Delta_{ij})^a\beta_j^2)$ or $\Theta(1-\lambda(-i\Delta_{ij})^a\beta_i\beta_j)$, where a is a new parameter. But the theory is finite and unitary for any of the indicated T-products.

The last method is close to methods of group 4. Since methods of this group have already been described in a review by G. V. Efimov, we shall only touch on them briefly.

One of these methods uses the introduction of nonlocal properties in the interaction Lagrangian [35]. This is done by redefining the field in the interaction Lagrangian:

$$\Phi(x)=\int d^4yA(x-y)\varphi(y)=A(\Box)\varphi(x)=\sum_0^\infty \frac{a_n}{(2n)!}\Box^n\varphi(x), \tag{4.24}$$

where $A(x-y)$ is a nonlocal generalized function from the appropriate space of nonlocal distributions [21].

In another method, a nonlocal form-factor $V(\varkappa^2, p^2)$ is used in (2.18):

$$V(\varkappa^2, p^2)=\frac{\widetilde{V}(p^2)}{\widetilde{V}(\varkappa^2)}, \tag{4.25}$$

where $\widetilde{V}(p^2)$ is an entire function with the same behavior when $p^2 \to \infty$ or $p^2 \to -\infty$ [35]. This method is applicable to any interaction Lagrangian in the table. The S-matrix is unitary.

5. DETERMINATION OF SCATTERING AMPLITUDES BY SOLUTION OF THE EQUATIONS

1. The Peratization of Feinberg and Pais. A method differing only slightly from methods of the first group was proposed by Feinberg and Pais [18]. They also determine a function F(x) whose Fourier transform has no ultraviolet singularities, but do not start from an essentially nonlinear Lagrangian; they consider the interaction of leptons through a vector boson. Solving the Bethe-Salpeter equation for lepton scattering amplitudes in the ladder approximation and retaining only terms involving higher powers of the variable of integration, they obtain an interative procedure yielding (1.13) for the amplitude in the space coordinate.

We note that the retention of only higher powers of the variable of integration in the equation is equivalent in the language of the Feinman diagram to the collapse, in the ladder diagram, of the ends of all boson lines into two points, one of which lies on one Fermi line and the other on the other. At the same time the propagators of boson lines are converted into propagators of ordinary scalar particles. Hence, in the iteration procedure, we obtain an infinite set of diagrams of the type corresponding to the series (2.16). By means of the equation, this series is summed to the expression (1.13).

Peratization is the specification of a definite method for constructing the Fourier transform from the amplitude (1.13). The first step of this program involves discarding an integral around a closed contour, and the retention of the integral along a radius from 0 to ∞. The second step involves an intermediate regularization (the Pauli-Villars regularization). It is assumed that the limiting transition $M \to \infty$ can be performed under the sign of integration with respect to r although, in general, the order of integration and regularization cannot be changed.

Another difficulty is encountered here which is not completely dealt with by Feinberg and Pais. The case in which the function $F_F^{(+)}(x)$ in the integrand of the integral with respect to r has a pole (for $q^2 < 0$) is avoided by Feinberg and Pais by assuming that the mass has a small imaginary part. It would be more correct to take the average of the circuits above and below (cf. [4, 11], where a similar procedure is used with a cut, but not a pole).*

2. The Arbuzov-Filippov Method. This method is similar to methods of the second group, and has as its object the determination of $\widetilde{F}(p)$ [4]. Its authors consider an interaction Lagrangian of the type (1.1), but with a scalar field. Dyson's transformation [15] is used to convert it to a free Lagrangian, and so the S-matrix is the unit matrix. However, the Green's function of the spinor field, the vertex function, and some

*A. Slavnov and A. Shabad [36] study in more detail the problem caused by the violation of the unitary condition for the S-matrix in the Feinberg-Pais method.

other quantities, have a nontrivial form of the type (1.4). Hence, it is of interest to find how we can use the solution of the equations to determine the Fourier transform of the function (1.4).

For mass-free particles, the Green's spinor function is

$$G(p) = -\frac{\hat{p}}{p^2 + i\varepsilon} f(p^2 + i\varepsilon). \tag{5.1}$$

Then starting from the Dyson-Swinger equation and using the Ward identity, we obtain the following equation for $f(p^2 + i\varepsilon)$ in the euclidean momentum region $p^2 < 0$:

$$x^3 f'''(x) + 3x^2 f''(x) + \lambda^2 x f(x) = 0 \tag{5.2}$$

with the boundary conditions

$$x f(x) \underset{x\to\infty}{\longrightarrow} 0; \quad f(x) \underset{x\to 0}{\longrightarrow} \text{const}. \tag{5.3}$$

Here $x = -p^2$; $\lambda = g/2\pi$. The boundary conditions (5.3) are necessary in order that the sign of λ^2 be negative. This corresponds to a nonphysical imaginary value of the coupling constant g. Hence, we have the following variant for finding the actual function $f(x)$.

First find $f(x, \lambda^2)$ for negative values of λ^2. This function has a logarithmic cut with respect to λ^2 from 0 to ∞. The actual function $f(x)$ is expressed in terms of a combination of the functions on the upper and lower sides of the cut:*

$$f(x) = \alpha f(x, \lambda^2 e^{i\pi}) + \alpha^* f(x, \lambda^2 e^{-i\pi}), \tag{5.4}$$

where $\alpha = 1/2 + i\eta$ and η is an arbitrary real parameter.

Here we first consider a method leading to the local theory [19]. We have

$$f(x, \lambda^2) = G_{03}^{20}(\lambda^2 x \mid 1, 0, -1), \tag{5.5}$$

where $G_{03}^{20}(\lambda^2 x \mid 1, 0, -1)$ is Meyer's function [37]. For large x, this function increases faster than

$$\sim \exp\{3(\lambda p)^{2/3}\}, \tag{5.6}$$

i.e., condition (2.7) is satisfied.

The nonuniqueness of the method is shown by the involvement of the parameter η.

In conclusion we make a few comments on the Feinberg-Pais method. At the beginning of the section we made a criticism concerning the construction of the Fourier transform of F(x) by the peratization method which leads to the violation of the unitary condition for the S-matrix. This critism touches the part of Feinberg-Pais work which is especially close to the theme of the present work. However, we have another criticism referring to the derivation of the equation for F(x). As shown by B. A. Arbuzov and A. T. Filippov [4], using the exact solution for a model, retention only of terms involving higher powers of the variable of integration in the Bethe-Salpeter equation causes a considerable change in the asymptotic behavior of the scattering amplitude for large p, as compared with the exact solution. In particular, this behavior is not in agreement with the local property of the theory [19, 21], while the behavior of the exact solution is.

In support of the method we note that, if the Stuekelberg and Dyson transformations are applied to the Lagrangian (1.12) considered by Feinberg and Pais, and then the perturbation theory is constructed by our method (see below) for the unrenormalizable part of the Lagrangian with the form (1.3), an expression for

---

*Here we have departed from the description given by the authors of the method, who take α to be purely real and equal to 1/2 ($\eta = 0$).

the scattering amplitude is obtained which satisfies the local property in each order of perturbation theory (with respect to mG).

6. DETERMINATION OF THE INTEGRAL $\tilde{F}(p)$

The methods of finding the function $\tilde{F}(p)$ can be combined under the title "analytic regularization methods." A characteristic feature of all these methods is the consideration of the degrees of particle propagators joining different vertices in the Feinman diagram (as complex numbers). Their real parts are chosen so that all integrals over intermediate momenta or coordinates converge, after which the actual values of the degrees of propagators are used by applying an analytic continuation procedure [38].

1. Speer's Analytic-Continuation Method. In considering the second group, we first describe a method which does not, in general have a direct relation to theories with essentially nonlinear Lagrangians. This method was proposed by Speer for the study of theories with polynomial Lagrangians [39, 40]. In its main features, however, it is quite like methods of the second group.*

We describe the procedure briefly. Let G be a mathematical expression corresponding to any Feinman diagram in momentum space. The corresponding regularized expression G_R is constructed as follows.

1. For the k-th interior line of a particle with mass m_i each propagator

$$\Delta_{m_i}(p_k) = (m_i^2 - p_k^2 - i\varepsilon)^{-1} \tag{6.1}$$

is replaced by

$$\Delta_{m_i}^{\lambda}(p_k) = \frac{(m_i)^{2\lambda} f_i(\lambda)}{(m_i^2 - p_k^2 - i\varepsilon)^{1+\lambda}}, \tag{6.2}$$

where λ is an arbitrary, generally complex,number, and $f_i(\lambda)$ is any regular function of λ such that $f_i(0) = 1$.

2. By taking λ sufficiently large, we can arrange that no integrals with respect to intermediate-sized moments have ultraviolet divergences. We can use the integral representation

$$\frac{1}{(A - i\varepsilon)^{1+\lambda}} = \frac{e^{i\frac{\pi}{2}(1+\lambda)}}{\Gamma(1+\lambda)} \int_0^{\infty} d\alpha\, \alpha^{\lambda} e^{-i\alpha(A - i\varepsilon)}. \tag{6.3}$$

3. Finally the regularized expression G_R is obtained by using the following analytic-continuation procedure:

$$G_R(\lambda_1^0 \ldots \lambda_L^0) = \frac{1}{L!} \sum_Q \frac{1}{(2\pi i)^L} \oint_{C_{Q(1)}} \frac{d\lambda_1}{\lambda_1} \cdots \oint_{C_{Q(L)}} \frac{d\lambda_L}{\lambda_L} C(\lambda_1 \ldots \lambda_L), \tag{6.4}$$

where the $C_{Q(i)}$ are circles of radius r_i with center the origin corresponding to the integration variable

$$r_i > \sum_1^{i-1} r_j \quad (1 \leqslant i \leqslant L),$$

the summation being over all permutations of these circles.

In contrast to the well-known older Pauli-Villars regularization, introducing "ghost" states and violating the unitary condition for the S-matrix, only arbitrary but finite constants appear here.

*Speer's method, used in renormalizable field theory, does not yield essentially new results in comparison with the well known method due to N. N. Bogolyubov and O. S. Parasyuk [41]. These two methods differ only in the technique used.

2. Guttinger's Method. This method differs only slightly from the preceding [7]. There a transformation of the type (6.2) is applied to the propagator, but in coordinate space.

Guttinger defines $\widetilde{F}(p)$ by the independent integration of each term of the sum (2.16). Consider the n-th term

$$[-i\Delta^c(x)]^n. \tag{6.5}$$

On the light cone this term has a singularity of the type

$$\sim \frac{1}{(x^2)^n} \quad \text{or} \quad \sim \frac{(\ln x^2)^m}{(x^2)^n} \ (m < n)$$

(the first for mass-free particles, the second for particles with mass). It is clear that the Fourier transform of this expression, starting with n = 2, is an integral which diverges more strongly for larger n.

To obtain a finite expression for the Fourier transform, Guttinger proposed the following method. Instead of (6.5), consider the quantity

$$a_n^z [-i\Delta^c(x)]^{n+z}, \tag{6.6}$$

where z is a complex number and a_n is an undetermined parameter with the dimensions of a squared length. Assume that Re z < 2 − n. Then the Fourier transform of (6.6) exists in the usual sense:

$$f_n(p, z) = i a_n^z \int d^4x e^{ipx} [-i\Delta^c(x)]^{n+z} \tag{6.7}$$

It is a function of z, regular for Re z < 2 − n, and having poles at the integers on the real axis for Re z ≥ 2 − n.

The regular function $f_n(p)$ is defined as follows:

$$f_n(p) = \frac{1}{2\pi i} \oint_C \frac{dz}{z} f_n(p, z), \tag{6.8}$$

where C is a circle with radius smaller than unity. The expression for $f_n(p)$ for mass-free particles is very simple:

$$f_n^{(0)}(p) = \frac{(-1)}{p^2 + i\varepsilon} \left(\frac{p^2+i\varepsilon}{(4\pi)^2}\right)^{n-1} \frac{1}{(n-1)!(n-2)!} \left\{\ln\left[a_n \frac{p^2+i\varepsilon}{(4\pi)^2} e^{-i\pi}\right] - \Psi(n) - \Psi(n-1)\right\}, \tag{6.9}$$

where $\Psi(n)$ is Euler's function.

By using (6.9), we obtain the expression

$$\widetilde{F}(p) = -\frac{C_1}{p^2+i\varepsilon} - \frac{1}{p^2+i\varepsilon} \sum_1^\infty \frac{C(n+1)}{n!(n-1)!} \left(\frac{p^2+i\varepsilon}{(4\pi)^2}\right)^n \left\{\ln\left[a_{n+1} \frac{p^2+i\varepsilon}{(4\pi)^2} e^{-i\pi}\right] - \Psi(n) - \Psi(n+1)\right\} \tag{6.10}$$

for the case of mass-free particles. The regular imaginary part in the physical range $p^2 > 0$ satisfies (2.9).

We now discuss the disadvantages of this method. First of all, since the Fourier transform of the function (6.6) is defined for Re z < 2 − n, it would appear that the contour of the integral (6.8) should contain at least a small part of this region. However, if C contains a part of the real axis from 0 to Re z < 2 − n, then the expression (6.9) contains a further polynomial of degree n = 3 in p^2. Hence, it must be prescribed that C may include a part of the region Re z < 2 − n, but only if this part contains no singularities. Secondly, the method involves an infinite number of undetermined constants. Thirdly, the final expression has no obvious nonanalytic dependence on the coupling constant, which is so characteristic of unrenormalizable theories. Finally, Guttinger attempts to apply the procedure developed for determining the Fourier transform of the propagator in a finite degree n to the construction of the spectral representation of $\tilde{F}(p)$:

$$\tilde{F}(p)=\frac{1}{\pi}\oint_C\frac{dz}{z}\int_0^{\infty}dm^2\,\frac{\rho(m^2)\,(am^2)^z}{p^2-m^2+i\varepsilon}. \tag{6.11}$$

However, this formula does not define a finite expression, since the spectral function $\rho(m^2)$ increases more rapidly than any polynomial in m^2, and no negative power of m^2 can compensate for this growth.

3. The Integral-Representation Method. This method, like the two foregoing methods, is based on the fact that the degrees n of the propagator in F(x) are converted into complex numbers z. For the series (2.16) there is an integral representation in which Re $z < 2$. The Fourier transform F(p) can be formed with no difficulties concerning ultraviolet divergences, after which we again return from the integral representation to a series [11, 42].

We rewrite (2.16) in a somewhat different form, separating out the coupling constant g^2 from the coefficients C(n):

$$F(x)=\sum_1^{\infty}a(n)\,[-ig^2\Delta^c(x)]^n. \tag{6.12}$$

There are two variants of this method, one of which is applicable to nonlocalizable interactions, and the other to localizable interactions. We start by considering the first.

a. Let the coefficients $a(n)$ satisfy the condition

$$\overline{\lim_{n\to\infty}}\, n^{-b}\,|a(n)|^{1/n}=A, \tag{6.13}$$

where A is a nonzero constant and the values of b satisfy

$$2>b\geqslant 0. \tag{6.14}$$

Theories in which the coefficients $a(n)$ satisfy (6.13) describe almost all physically interesting cases of nonlocalizable interactions. In the table this includes all interactions of II and even those in III for which $\tilde{F}(p)$ has an imaginary part for mass-free particles ($\gamma < 3/2$).

Condition (6.13) can be written in a more rigorous form; namely, we can require that the entire function $\chi(z)$, which can be expressed in the form

$$\chi(z)=\sum_1^{\infty}(-1)^n\frac{a(n)}{(2n)!}z^n, \tag{6.15}$$

be bounded in some sector $|\varphi|\leqslant\delta,\ \delta\geqslant 0,\ z=re^{i\varphi}$. Condition (6.15) implies (6.13). We require that the latter condition hold.

Using (3.11), we can convert the expression for F(x) into

$$F'(x)=-i\sum_1^{\infty}(g^2)^{n+1}a(n+1)\int_{[(n+1)\,m]^2}^{\infty}d\varkappa^2\Omega_{n+1}^{(m)}(\varkappa^2)\,\Delta_{\varkappa}^{c}(x). \tag{6.16}$$

Here we have discarded the first term in F(x), since for it the conversion to momentum space is trivial.

We consider further the case of mass-free particles, because it is simpler and better for explaining our procedure. The generalization to particles with mass is described in [43].

If m = 0, then (6.16) becomes

$$F'_\lambda(x)=i\lambda\sum_{1}^{\infty}(-1)^n\int_0^{M^2}\frac{d\varkappa^2}{\varkappa^2}f(n,\varkappa^2)\Delta^c_\varkappa(x), \tag{6.17}$$

where

$$f(n,\varkappa^2)=\left[\lambda\left(\frac{\varkappa}{4\pi}\right)^2\right]^n\frac{a(n+1)}{\Gamma(n)\Gamma(n+1)}. \tag{6.18}$$

Here $\lambda = -g^2$; $\Gamma(n)$ is the gamma-function, and $M^2 < \infty$ is a cut-off introduced as an intermediate regularizer. It will be used in the following. We assume that $\lambda > 0$. At the end of the calculation we return to physical values $\lambda < 0$.

We now use the values of $f(n, \varkappa^2)$ at the sequence of points n = 1, 2, 3, ... to determine an analytic function $f(z, \varkappa^2)$, regular in the right half-plane Re z < 0 and satisfying the following conditions ($z = x + iy$) [44]:

$$\left.\begin{aligned}&\text{a)}\ \ |f(z,\varkappa^2)|<Be^{A|z|},\quad \operatorname{Re} z>0;\\ &\text{b)}\ \ |f(iy,\varkappa^2)|<Be^{(\pi-\delta)|y|},\quad -\infty<y<\infty,\ \delta>0.\end{aligned}\right\} \tag{6.19}$$

Conditions (6.19) ensure, on the one hand, the uniqueness of $f(z, \varkappa^2)$, and on the other hand permits us to write the sum (6.17) as a Mellin-Burns (Sommerfeld-Watson) integral with a contour such that Re z < 1 in the whole range of integration:

$$F'_{\lambda\,(\mathrm{reg})}(x)=\frac{\lambda}{2}\int_{\alpha+i\infty}^{\alpha-i\infty}\frac{dz}{\sin\pi z}\int_0^{M^2}\frac{d\varkappa^2}{\varkappa^2}f(z,\varkappa^2)\Delta^c_\varkappa(x), \tag{6.20}$$

with $0 < \alpha < 1$. This solves the problem stated at the beginning of this section. Conditions (6.19) are compatible with the condition (6.13) imposed on the coefficients $a(n)$ [11].

It is now simple to transfer to momentum space. Since all integrals involved in the application of the Fourier transform to (6.20) are absolutely convergent, this operation can be transferred directly to $\Delta^c_\varkappa(x)$ with the intermediate regularization omitted:

$$\widetilde{F}'_\lambda(p)=i\frac{\lambda}{2}\int_{\alpha+i\infty}^{\alpha-i\infty}\frac{dz}{\sin\pi z}\cdot\frac{\left(\frac{\lambda}{(4\pi)^2}\right)^z a(z+1)}{\Gamma(z)\Gamma(z+1)}\int_0^{\infty}d\varkappa^2\frac{(\varkappa^2)^{z-1}}{\varkappa^2-p^2-i\varepsilon}. \tag{6.21}$$

The resulting expression can be taken as the spectral representation of $\widetilde{F}'_\lambda(p)$. All integrals in (6.21) are easily calculated, and we have

$$\widetilde{F}'_\lambda(p)=\frac{\lambda}{p^2+i\varepsilon}\sum_{1}^{\infty}\left(-\lambda\frac{p^2+i\varepsilon}{(4\pi)^2}\right)^n\frac{a(n+1)}{n!\,(n-1)!}\left\{\ln\left[\lambda\frac{p^2+i\varepsilon}{(4\pi)^2}e^{-i\pi}\right]+(\ln a(n+1))'-\Psi(n)-\Psi(n+1)\right\}. \tag{6.22}$$

The function $\widetilde{F}'_\lambda(p)$ has a logarithmic cut for negative values of λ. Hence, in order to obtain the actual function $\widetilde{F}'(p)$, we use the values of $\widetilde{F}'_\lambda(p)$ on both sides of the cut:

$$\widetilde{F}'(p)=\alpha\widetilde{F}'_{\lambda e^{i\pi}}(p)+\beta\widetilde{F}'_{\lambda e^{-i\pi}}(p), \tag{6.23}$$

where α and β satisfy

$$\alpha+\beta=1;\quad \operatorname{Re}(\alpha-\beta)=0. \tag{6.24}$$

The last condition is a consequence of the fact that the S-matrix is unitary. Hence,

$$\alpha = 1/2 + i\eta; \quad \beta = \frac{1}{2} - i\eta, \tag{6.25}$$

where η is an arbitary parameter. The final result is

$$\widetilde{F}'(p) = \frac{(-g^2)}{p^2 + i\varepsilon} \sum_1^{\infty} \left(g^2 \frac{p^2 + i\varepsilon}{(4\pi)^2}\right)^n \frac{a(n+1)}{n!(n-1)!} \left\{\ln\left(g^2 \frac{p^2 + i\varepsilon}{(4\pi)^2} e^{-i\pi + 2\pi\eta}\right) + (\ln a(n+1))' - \Psi(n) - \Psi(n+1)\right\}. \tag{6.26}$$

The introduction of the parameter λ was necessary to ensure the existence of the integral (6.21). For negative λ the integral with respect to z is divergent.

There is no spectral representation of the type (6.21) for $\widetilde{F}'(p)$, but for $\widetilde{F}'(p, \gamma)$, which is equal to $\widetilde{F}'(p)$ for $\gamma = 1$:

$$\widetilde{F}'(p, \gamma) = \frac{g^2}{2i} \int_{\alpha+i\infty}^{\alpha-i\infty} dz \frac{\cos \pi z}{\sin \gamma\pi z} \cdot \frac{\left(\frac{g}{4\pi}\right)^{2z} a(z+1)}{\Gamma(z)\Gamma(z+1)} \int_0^{\infty} d\varkappa^2 \frac{(\varkappa^2)^{z-1}}{\varkappa^2 - p^2 - i\varepsilon}. \tag{6.27}$$

Here γ must be larger than 2 (more precisely $\gamma > 2 - b/2$).

b. Now consider localizable interactions, for which

$$\overline{\lim} |a(n)|^{1/n} = 0. \tag{6.28}$$

We confine ourselves to localizable interactions for which

$$\overline{\lim_{n\to\infty}} n^k |a(n)|^{1/n} = A, \; A > 0, \; 0 < k < 2. \tag{6.29}$$

In the nonphysical momentum range $p^2 < 0$, we can convert to the euclidean metric in the integral (2.17) and take the integral with respect to the angles:

$$\widetilde{F}'_{\lambda(\text{reg})}(p) = -\frac{\lambda}{|p|} \sum_1^{\infty} \left(-\frac{\lambda}{(2\pi)^2}\right)^n a(n+1) \int_l^{\infty} dr r^{-2n} J_1(|p|r); \; (|p| = \sqrt{-p^2}). \tag{6.30}$$

Here an intermediate regularization is introduced by truncating the integral with respect to r for $r < l$. Using a procedure similar to the above, we rewrite (6.30) as

$$\widetilde{F}'_{\lambda}(p) = i \frac{\lambda}{2|p|} \int_{\alpha+i\infty}^{\alpha-i\infty} \frac{dz}{\sin \pi z} \left(\frac{\lambda}{(2\pi)^2}\right)^z a(z+1) \int_0^{\infty} dr r^{-2z} J_1(|p|r). \tag{6.31}$$

Here the intermediate regularization has been removed, since the integral with respect to r converges at zero. The integral with respect to z is also absolutely convergent if $0 < k < 2$. Integration with respect to r and z yields

$$\widetilde{F}'_{\lambda}(p) = -\frac{\lambda}{|p|^2} \sum_1^{\infty} \left[\lambda\left(\frac{|p|^2}{4\pi}\right)\right]^n \frac{a(n+1)}{n!(n-1)!} \left\{\ln\left(\lambda \frac{|p|^2}{(4\pi)^2}\right) + (\ln a(n+1))' - \Psi(n) - \Psi(n+1)\right\}. \tag{6.32}$$

This expression is easily continued into the physical region $p^2 > 0$, and (6.23) yields the actual function $\widetilde{F}'(p)$.

The first term of the sum (2.16) is easily included in an integral representation of the type (6.27). Then the complete function $\widetilde{F}(p, \gamma)$ is given by the formula

$$\widetilde{F}(p,\gamma)=\mathrm{i}(2\pi)^3\int\limits_{\alpha+\mathrm{i}\infty}^{\alpha-\mathrm{i}\infty}dz\,\mathrm{ctg}\,\pi z\,\frac{e^{-\mathrm{i}\pi z}}{\sin\gamma\pi z}\cdot\frac{C(z)}{\Gamma(z-1)\,\Gamma(z)}(p^2+\mathrm{i}\varepsilon)^{z-2}. \tag{6.33}$$

The value of γ depends here on the behavior of the coefficients C(n) for large n. In the final result we must put $\gamma = 1$.

The representation (6.33) can be used to show that there are no ultraviolet divergencies in any order of perturbation theory. In fact consider the n-th order. In it, n vertices are joined by $n(n-1)/2$ interior lines of the type (6.33). The common multiplicity of the integral with respect to interior moments is $[2n(n-1)-4(n-1)]$. The common degree of the momentum in the integrand is $2\sum\limits_{1}^{n(n-1)/2}(\mathrm{Re}\,z_i-2)$. Thus the condition that there be no ultraviolet singularities can be written

$$2(n-1)(n-2)+2\sum_{1}^{\frac{n(n-1)}{2}}(\mathrm{Re}\,z_i-2)<0. \tag{6.34}$$

Hence, making all $\mathrm{Re}\,z_i$ equal, we obtain

$$\alpha=\mathrm{Re}\,z<\frac{4}{n}. \tag{6.35}$$

But since $0 < \alpha < 1$ and α can be arbitrarily close to zero, condition (6.35) can always be satisfied [11].

A spectral representation of the form (6.27) ensures that the S-matrix is unitary in all orders of perturbation theory.

In the study of particles with mass, we must keep in mind the considerable difference in the asymptotic behavior of amplitudes for high momenta for the nonlocal theories for massive and mass-free particles. Moreover, the analytic properties of $\widetilde{F}(p)$ become much more complicated. In particular, the nonanalyticity with respect to the coupling constant is not of a simple logarithmic type, but of a more essential kind. The examinations of amplitudes we have carried out for low momenta show that, for massive particles, the function $\widetilde{F}(p)$ can be expressed as follows:

$$\widetilde{F}(p)=\frac{g^2}{m^2}\sum_{n=0}^{\infty}(gm)^{2n}a(n+1)\left[f_n\left(\frac{p^2}{m^2}\right)+\ln(gm)\,P_{n-1}^{(n)}\left(\frac{p^2}{m^2}\right)+(\ln(gm))^2\,P_{n-2}^{(n)}\left(\frac{p^2}{m^2}\right)+\ldots+(\ln(gm))^n\,P_0^{(n)}\left(\frac{p^2}{m^2}\right)\right], \tag{6.36}$$

where the $P_k^{(n)}(p^2/m^2)$ are polynomials in p^2/m^2 of degree k, and the $f_n(p^2/m^2)$ are functions of p^2/m^2. The above series shows that the nonanalytic dependence of the amplitude on the coupling constant g is of a very involved nature.

We note in conclusion that Salam and Strathdee have recently published an interesting article devoted to a detailed description of the method described above [9]. They attempt to consider high orders in perturbation theory and describe the case of massive particles. They study the problem of the S-matrix being unitary in high orders. In addition to the deficiency in this particle mentioned at the beginning of the present review, we note the not completely correct assumption made concerning the structure of $\widetilde{F}(p)$ in the case of massive particles. The authors assume that, for massive particles, $\widetilde{F}(p)$ has only a simple logarithmic singularity with respect to the coupling constant (cf. p. 33 in [9]).

CONCLUSIONS

Following the above compressed description of the many methods used in quantum field theory with spectral functions of rapid growth, we shall attempt to give an over-all estimate of the value of these methods and to discuss briefly some questions arising concerning higher orders of modified perturbation theory, and also to indicate the main problems remaining open.

We first outline the main characteristics of contemporary methods. In our opinion, the Efimov-Fradkin method is the most firmly based and interesting of the group-1 methods. This method modernized and improved in subsequent articles by G. V. Efimov [21], is applicable to a wide class of nonpolynomial Lagrangians. Efimov has shown that his method yields unitary S-matrices satisfying the macrocausality condition.

Of methods of the second group, we prefer the integral-representation method. This method, like the Efimov-Fradkin method, can be applied to a large class of nonpolynomial Lagrangians. It enables us to construct two-point Green's functions in momentum space with the correct analytic properties, completely compatible with the S-matrix being unitary. It also yields integral representations for two-point functions which greatly simplify the examination of higher orders of perturbation theory with respect to the "principal" coupling constant.

Finally, from methods of the third group we choose the Arbuzov-Filippov method. This method can also be used for the description of many unrenormalizable interactions, and leads to expressions for Green's functions and scattering amplitudes such that the S-matrix is unitary. It is interesting that methods differing as widely as our integral-representation methods and the Arbuzov-Fillipov method yield identical results when applied to the same interactions [4, 11].

In this sense, a good method of checking a method is to apply it in the description of a model which can be solved exactly. Here the above methods yield compatible results and correctly describe the models investigated.

We now discuss some questions arising in the investigation of higher orders of perturbation theory with respect to the principal coupling constant. One of the main problems is that of ensuring that the S-matrix is unitary. To prove that this condition is satisfied in an arbitrary order of perturbation theory is rather difficult. However, if Kutovskii's theorem can be proved in any order of perturbation theory, then the S-matrix is unitary in this theory. G. V. Efimov [21], and also Salam and Strathdee [9] have proved that Kutkovskii's theorem is true in high orders of perturbation theory. In [11], we verified the unitary condition in the third order of perturbation theory and derived a spectral representation for two-point Green's functions which will, it is hoped, maintain the unitary property of the S-matrix in higher orders if these functions are used.

The following problem is connected with the finiteness of high-order perturbation theory with respect to the principal coupling constant. In [11] we show that the theory remains finite in any order, if the integral-representation method is used to calculate the relevant quantities. The Efimov-Fradkin method also ensures that the theory is finite in any order of perturbation theory.

We call attention to the following unsolved problems.

1. Uniqueness of a Method. We believe that this problem must be solved by introducing further physical principles. For example the Arbuzov-Filippov and integral-representation methods become unique if we require that

$$\lim_{p^2 \to \infty} \frac{\operatorname{Re} \widetilde{F}(p)}{\operatorname{Im} \widetilde{F}(p)} = 0.$$

This condition was first suggested by A. T. Filippov [4]. It has not yet any physical basis.

2. The Second Problem Concerning the Nonpolynomial Growth of Amplitudes in Every Order of Perturbation Theory with Respect to the Coupling Constant. The methods described here can be used for calculation in low orders of perturbation theory with respect to the principal coupling constant only for small momenta. To draw conclusions concerning the asymptotic behavior of amplitudes for high energies in these theories, we must consider a series of perturbation theories with respect to the principal coupling constant (situations analogous to those in which properties are obtained for the usual renormalizable theories).

The above problems are difficult, but the author believes that they will soon be solved.

Other problems arises in the description of charge fields and in taking account of the gradient invariance of theories. However, these questions have already been solved [12].

LITERATURE CITED

1. Transactions of the International Conference on Nonlocal Quantum Field Theory, Preprint Joint Institute for Nuclear Research R2-3590, Dubna (1967).
2. S. Okubo, Progr. Theor. Phys., 11, 80 (1954).
3. T. D. Lee, Phys. Rev., 128, 899 (1962).
4. B. A. Arbuzov and A. T. Filippov, Zh. Éksperim. i Teor. Fiz., 49, 990 (1965); Nuovo Cimento, 38, 796 (1965); Yadernaya Fizika, 8, 365 (1968); A. T. Filippov, Preprint Joint Institute for Nuclear Research E2-4189 (1968).
5. R. Arnowitt and S. Deser, Phys., Rev., 100, 349 (1955).
6. B. M. Barbashov and G. V. Efimov, Zh. Éksperim. i Teor. Fiz., 43, 1057 (1962).
7. W. Guttinger, Fortsch. Phys., 14, 483 (1966).
8. R. Delbourgo, A. Salam, and J. Strathdee, ICTP Trieste, Preprint IC/69/17.
9. A. Salam and J. Strathdee, Phys. Rev. D, 1, 12, 3296 (1970).
10. B. W. Lee and B. Zumino, Nucl. Phys. B13, 671 (1969).
11. M. K. Volkov, Ann. Phys., 49, 202 (1968).
12. H. M. Fried, Nuovo Cimento, 52A, 1333 (1967); B. W. Keck and J. G. Taylor, Preprint, University of Southampton, April (1970); P. Budini and G. Calucci, Preprint ICTP IC/70/37, Trieste (1970); H. Lehmann and K. Pohlmeyer, Commun. Math. Phys., 20, 101 (1971).
13. M. K. Volkov, Preprint ITF 69-5, Kiev (1969).
14. N. N. Bogolyubov and D. V. Shirkov, Introduction to Quantum Field Theory [in Russian], Moscow, Gostekhizdat (1957).
15. F. J. Dyson, Phys. Rev., 73, 929 (1948).
16. T. D. Lee, Nuovo Cimento, 59, 579 (1968).
17. E. C. Stueckelberg, Helv. Phys. Acta, 11, 225, 229 (1938).
18. G. Feinberg and A. Pais, Phys. Rev., 131, 2724 (1963); 133B, 477 (1964).
19. N. N. Meiman, Zh. Éksperim. i Teor. Fiz., 47, 1966 (1964).
20. A. M. Jaffe, Ann. Phys., 32, 127 (1965); J. Math. and Phys., 5, 1174 (1965); Phys. Rev. Lett., 17, 661 (1966).
21. G. V. Efimov, Preprint ITF-68-52 (1968); 68-54 (1968); 68-55 (1968); Commun. Math. Phys., 7, 138 (1968).
22. N. N. Bogolyubov, A. A. Logunov, and I. T. Todorov, Elements of the Axiomatic Approach to Quantum Field Theory [in Russian], Moscow, Nauka (1969).
23. F. Gursey, Nuovo Cimento, 16, 230 (1960).
24. J. Schwinger, Phys. Rev. Lett., 24B, 473 (1967).
25. S. Coleman, I. Wess, and B. Zumino, Phys. Rev., 177, 2239 (1969); 177, 2247 (1969).
26. S. Weinberg, Phys. Rev. Lett., 18, 188 (1967).
27. B. V. Struminskii, Preprint Joint Institute for Nuclear Research F2-3554, Dubne (1967).
28. V. A. Kolkunov, Zh. Éksperim. i Teor. Fiz., 43, 1448 (1962).
29. G. V. Efimov, Preprint CERN, Geneva, TH-1087 (1969).
30. G. V. Efimov, Zh. Éksperim. i Teor. Fiz., 44, 2107 (1963); Nuovo Cimento, 32, 2046 (1964); Zh. Éksperim. i Teor. Fiz., 48, 596 (1965).
31. E. S. Fradkin, Nucl. Phys., 49, 624 (1963).
32. S. Hori, Progr. Theor. Phys., 6, 578 (1952).
33. G. H. Hardy, Divergent Series [Russian translation], Izd-vo Inostr. Lit. (1951).
34. G. V. Efimov, Nucl. Phys., 74, 657 (1965).
35. G. V. Efimov, Preprint Joint Institute for Nuclear Research R2-4546, Dubna (1969); R2-4472 (1969).
36. A. Slavnov and A. Shabad, Yadernaya Fizika, 1, 721 (1965).
37. T. Bateman and A. Erdeli, Higher Trascendental Functions, Vol. 1 [Russian translation], Nauka, Moscow (1965).
38. I. M. Gel'fand and G. E. Shilov, Generalized Functions, Vol. 1 [in Russian], Fizmatgiz, Moscow (1958).
39. E. J. Speer, Math. Phys., 9, 1404 (1968).
40. P. Breitenlohner and H. Mitter, Nucl. Phys., B7, 443 (1968).
41. N. N. Bogolyubov and O. S. Parasyuk, Acta Math., 97, 227 (1968).
42. M. K. Volkov, Yadernaya Fizika, 6, 2200 (1967); 7, 448 (1968).
43. M. K. Volkov, Commun. Math. Phys., 15, 69 (1969); Teor. i Mat. Fiz., 2, 197 (1970); 6, 21 (1971).
44. M. A. Evgrafov, Asymptotic Estimates and Entire Functions [in Russian], Fizmatgiz, Moscow (1962).