THE SELF-CONSISTENT-FIELD METHOD IN NUCLEAR THEORY

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The self-consistent-field method is described in the Bogolyubov formulation. It is shown that this method yields equations for the effective fields in the theory of finite Fermi systems and the secular equations for a model with pairing and multipole forces.

It is difficult to conceive of nuclear physics today without the concept of the self-consistent field of the nucleus. A great amount of experimental information indicates that nucleons in the nucleus behave to a certain extent as independent particles moving in a common potential. For this reason it is natural to construct a nuclear theory based on the concept of a self-consistent field, at least for the low-lying excited states. We describe below the self-consistent-field method in the Bogolyubov formulation [1]. We will use it to derive equations describing the ground and low-lying excited states of the nucleus, and we will show by various examples that the familiar results of the microscopic approach to nuclear structure follows from these equations: secular equations for the case of multipole and spin-multipole forces, equations for pairing-vibration frequencies, and equations for the theory of finite Fermi systems.

We adopt the total Hamiltonian for the system in a quite general form:

$$\left.\begin{array}{c} H=\sum\limits_{ff'} T(f,f')\, a_f^+ a_{f'} - \frac{1}{4} \sum\limits_{f_1 f_2 f_1' f_2'} G(f_1 f_2;\, f_2' f_1')\, a_{f_1}^+ a_{f_2}^+ a_{f_2'} a_{f_1'};\\ T(f,f') = I(f,f') - \lambda \delta_{ff'}, \end{array}\right\} \tag{1}$$

where f is the set of quantum numbers characterizing the one-particle states, a_f^+ and a_f are the Fermion creation and annihilation operators, respectively, λ is the chemical potential, I is the one-particle Hamiltonian, and G is the particle-interaction matrix.

Since the operators a_f^+ and a_f do not commute, and since the Hamiltonian is Hermitian, we see that

$$\left.\begin{array}{c} I(f,f') = I^*(f',f);\\ G(f_1 f_2;\, f_2' f_1') = -G(f_1 f_2;\, f_1' f_2') = -G(f_2 f_1;\, f_2' f_1')\\ = G(f_2 f_1;\, f_2' f_1') = G^*(f_1' f_2';\, f_2 f_1). \end{array}\right\} \tag{2}$$

We will also use the representation $f = q, \sigma$, where $\sigma = \pm 1$ distinguishes between states which are conjugate with respect to time inversion:

$$\hat{T} a_{q\sigma}^+ \hat{T}^{-1} = s_\sigma a_{q-\sigma}. \tag{3}$$

Here $\hat{T}$ is the time-inversion operator, and the coefficients s_σ have the following properties:

$$s_\sigma s_{-\sigma} = -1;\; s_\sigma^2 = 1. \tag{4}$$

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Using Eq. (3) we can show that the invariance of the Hamiltonian with respect to time inversion yields the properties

$$\left.\begin{array}{c} I(q\sigma,\ q'\sigma') = I^*(q-\sigma,\ q'-\sigma')\, s_\sigma s_{\sigma'}; \\ G(q_1\sigma_1,\ q_2\sigma_2;\ q_2'\sigma_2',\ q_1'\sigma_1') = G^*(q_1-\sigma_1,\ q_2-\sigma_2; \\ q_2'-\sigma_2',\ q_1'-\sigma_1')\, s_{\sigma 2} s_{\sigma_1} s_{\sigma_1'} s_{\sigma_2'}. \end{array}\right\} \quad (5)$$

Let us consider the function

$$F(f_1,\ f_2) = \langle a_{f_1}^+ a_{f_2} \rangle;\ \Phi(f_1,\ f_2) = \langle a_{f_1} a_{f_2} \rangle. \quad (6)$$

The averaging is carried over the ground state of the system. We note that the equations of motion yield the following exact relations for the functions F and Φ:

$$\left.\begin{array}{l} i\dfrac{\partial}{\partial t} F(f_1,\ f_2) = \langle [a_{f_1}^+ a_{f_2},\ H] \rangle \equiv \mathfrak{B}(f_1,\ f_2); \\ i\dfrac{\partial}{\partial t} \Phi(f_1,\ f_2) = \langle [a_{f_1} a_{f_2},\ H] \rangle \equiv \mathfrak{A}(f_1,\ f_2). \end{array}\right\} \quad (7)$$

In the self-consistent-field method $\mathfrak{A}$ and $\mathfrak{B}$ may be expressed in terms of F and Φ [1, 2]:

$$\begin{array}{c} \mathfrak{A}(f_1, f_2) = \sum_f \{\xi(f_1, f)\,\Phi(f, f_2) + \xi(f_2, f)\,\Phi(f_1, f)\} \cdot \\ -\dfrac{1}{2} \sum_{ff_1'f_2'} \Phi(f_2', f_1')\,\{G(f_1 f;\ f_2' f_1')\,F(f, f_2) + G(f, f_2;\ f_2' f_1') \\ \times F(f, f_1)\} + \dfrac{1}{2} \sum_{f_1' f_2'} \Phi(f_2', f_1')\,G(f_1 f_2;\ f_2' f_1'); \end{array} \quad (8)$$

$$\begin{array}{c} \mathfrak{B}(f_1, f_2) = \sum_f \{\xi(f_2, f)\,F(f_1, f) - \xi(f, f_1)\,F(f, f_2)\} \\ +\dfrac{1}{2} \sum_{ff_1'f_2'} \{\Phi^*(f_1, f)\,G(f_2 f;\ f_2' f_1')\,\Phi(f_2', f_1') \\ -\Phi(f_2, f)\,G(f_1 f;\ f_2' f_1')\,\Phi^*(f_2', f_1'), \end{array} \quad (9)$$

where

$$\xi(f, f') = T(f, f') - \sum_{f_1 f_2} G(ff_1;\ f_2 f')\,F(f_1, f_2). \quad (10)$$

The functions F and Φ are not independent, being related by

$$\left.\begin{array}{r} F(f_1, f_2) = \sum_f \{F(f_1, f)\,F(f, f_2) + \Phi^*(f, f_1)\,\Phi(f, f_2)\}; \\ 0 = \sum_f \{F(f_1, f)\,\Phi(f, f_2) + F(f_2, f)\,\Phi(f, f_1)\}. \end{array}\right\} \quad (11)$$

If we are not interested in the time-independent ground state, we should solve, instead of Eqs. (3), the following equations [2]:

$$\mathfrak{A}(f_1, f_2) = 0;\ \mathfrak{B}(f_1, f_2) = 0. \quad (12)$$

We denote the solutions for Eqs. (12) by $F_0(f_1, f_2)$ and $\Phi_0(f_1, f_2)$. We can obtain the same results as from a solution of Eqs. (12) by assuming that the ground state is the vacuum state for quasiparticles related to ordinary Fermions by a general Bogolyubov transformation:

$$a_f = \sum_\nu \{u(f, \nu)\alpha_\nu + v(f, \nu)\alpha_\nu^+\}, \quad (13)$$

whose coefficients satisfy

$$\left.\begin{aligned}&\sum_{\nu}(u(f,\nu)u^*(f',\nu)+v(f,\nu)v^*(f',\nu))=\delta_{ff'};\\&\sum_{\nu}(u(f,\nu)v(f',\nu)+u(f',\nu)v(f,\nu))=0.\end{aligned}\right\} \tag{14}$$

If we are interested in the spectrum of elementary excitations related to small oscillations about the ground state, we must consider small increments in the functions F_0 and Φ_0:

$$\left.\begin{aligned}&F(f,f')=F_0(f,f')+\delta F(f,f');\\&\Phi(f,f')=\Phi_0(f,f')+\delta\Phi(f,f').\end{aligned}\right\} \tag{15}$$

Equations can be obtained for δF and $\delta\Phi$ from Eqs. (7):

$$\left.\begin{aligned}&i\frac{\partial}{\partial t}\delta F(f,f')=\delta\mathfrak{B}(f,f');\\&i\frac{\partial}{\partial t}\delta\Phi(f,f')=\delta\mathfrak{B}(f,f').\end{aligned}\right\} \tag{16}$$

Moreover, δF and $\delta\Phi$ are not independent, but are related by auxiliary relations which follow from Eqs. (11):

$$\left.\begin{aligned}&\delta\left\{F(f_1,f_2)-\sum_f F(f_1,f)F(f,f_2)-\sum_f\Phi^*(f,f_1)\Phi(f,f_2)\right\}=0;\\&\delta\left\{\sum_f F(f_1;f)\Phi(f,f_2)+\sum_f F(f_2,f)\Phi(f,f_1)\right\}=0;\end{aligned}\right\} \tag{17}$$

Since δF and $\delta\Phi$ are related, it is more convenient to represent them in terms of new independent unknowns which automatically satisfy Eqs. (17).

Using canonical transformation (13), we write F, F_0, Φ, and Φ_0 in the form

$$F(f_1,f_2)=\langle a^+_{f_1}a_{f_2}\rangle=\sum_g v^*(f_1,g)v(f_2,g)+\sum_{g_1g_2}\{u^*(f_1,g_1)u(f_2,g_2)\langle\alpha^+_{g_1}\alpha_{g_2}\rangle$$

$$-v^*(f_1,g_1)v(f_2,g_2)\langle\alpha^+_{g_2}\alpha_{g1}\rangle+u^*(f_1,g_1)$$

$$\times v(f_2,g_2)\langle\alpha^+_{g_1}\alpha^+_{g_2}\rangle+v^*(f_1,g_1)u(f_2,g_2)\langle\alpha_{g_1}\alpha_{g_2}\rangle\}; \tag{18}$$

$$F_0(f_1,f_2)=\sum_g v^*(f_1,g)v(f_2,g); \tag{19}$$

$$\Phi(f_1,f_2)=\langle a_{f_1}a_{f_2}\rangle=\sum_g u(f_1,g)v(f_2,g)$$

$$+\sum_{g_1g_2}\{v(f_1,g_1)u(f_2,g_2)\langle\alpha^+_{g_1}\alpha_{g_2}\rangle-u(f_1,g_1)$$

$$\times v(f_2,g_2)\langle\alpha^+_{g_2}\alpha_{g_1}\rangle+u(f_1,g_1)u(f_2,g_2)\langle\alpha_{g_1}\alpha_{g_2}\rangle$$

$$+v(f_1,g_1)v(f_2,g_2)\langle\alpha^+_{g_1}\alpha^+_{g_2}\rangle\}; \tag{20}$$

$$\Phi_0(f_1,f_2)=\sum_g u(f_1,g)v(f_2,g). \tag{21}$$

In the self-consistent-field method, in the nonsteady-state formulation of the problem, the wave function for the nuclear ground state ceases to be a quasiparticle vacuum state. However, the average number of quasiparticles in the ground state is small, so we assume that this number is approximately zero:

$$\langle \alpha^+_{g_1} \alpha_{g_2} \rangle = 0. \tag{22}$$

To characterize deviations of the wave function from that of the quasiparticle vacuum state we introduce the coefficients

$$\mu(g_1, g_2) = \langle \alpha_{g_1} \alpha_{g_2} \rangle, \tag{23}$$

which satisfy

$$\mu(g_1, g_2) = -\mu(g_2, g_1). \tag{24}$$

We express δF and ΦF in terms of μ and μ^*:

$$\delta F(f_1, f_2) = \sum_{g_1 g_2} \{v^*(f_1, g_1) u(f_2, g_2) \mu(g_1, g_2) + u^*(f_1, g_1) v(f_2, g_2) \mu^*(g_2, g_1)\}; \tag{25}$$

$$\delta\Phi(f_1, f_2) = \sum_{g_1 g_2} \{u(f_1, g_1) u(f_2, g_2) \mu(g_1, g_2) + v(f_1, g_1) v(f_2, g_2) \mu^*(g_2, g_1)\}. \tag{26}$$

To obtain equations for μ, we write μ in terms of δF and $\delta\Phi$. Multiplying Eq. (25) by $v(f_1, g)$ and Eq. (26) by $u^*(f_1, g)$, combining them, summing over f_1, and using orthonormalization condition (14), we find

$$\begin{aligned}&\sum_{f_1} \{v(f_1, g)\, \delta F(f_1, f_2) + u^*(f_1, g)\, \delta\Phi(f_1, f_2)\}\\ &= \sum_{f_1 g_1 g_2} \{[v(f_1, g) v^*(f_1, g_1) + u^*(f_1, g) u(f_1, g_1)]\\ &\times u(f_2, g_2) \mu(g_1, g_2) + [v(f_1, g) u^*(f_1, g_1) + u^*(f_1, g) v(f_1, g_1)]\\ &\times v(f_2, g_2) \mu^*(g_2, g_1) = \sum_{g_2} u(f_2, g_2) \mu(g, g_2).\end{aligned} \tag{27}$$

Similarly, we multiply Eq. (25) by $u(f_1, g)$, multiply Eq. (26) by $v^*(f_1, g)$, combine, and sum over f_1 [with account of (14)], finding

$$\sum_{f_1} \{u^*(f_1, g)\, \delta F^*(f_1, f_2) + v(f_1, g)\, \delta\Phi^*(f_1, f_2)\} = -\sum_{g_2} v^*(f_2, g_2) \mu(g, g_2). \tag{28}$$

We again use this procedure: we multiply Eq. (27) by $u^*(f_2, g')$, multiply Eq. (28) by $v(f_2, g')$, subtract Eq. (28) from Eq. (27), and sum over f_2; we find

$$\begin{aligned}\mu(g, g') = \sum_{f_1 f_2} \{&v(f_1, g) u^*(f_2, g')\, \delta F(f_1, f_2) + u^*(f_1, g)\\ &\times u^*(f_2, g')\, \delta\Phi(f_1, f_2) - u^*(f_1, g) v(f_2, g')\, \delta F^*(f_1, f_2)\\ &- v(f_1, g) v(f_2, g')\, \delta\Phi^*(f_1, f_2)\}.\end{aligned}$$

Differentiating this expression with respect to t and taking (16) into account, we find an equation for μ:

$$\begin{aligned}i \frac{\partial}{\partial t} \mu(g_1, g_2) = \sum_{f_1 f_2} \{&u^*(f_1, g_1) u^*(f_2, g_2)\, \delta\mathfrak{A}(f_1, f_2)\\ &+ v(f_1, g_1) v(f_2, g_2)\, \delta\mathfrak{A}^*(f_1, f_2) + v(f_1, g_1) u^*(f_2, g_2)\\ &\times \delta\mathfrak{B}(f_1, f_2) + u^*(f_1, g_1) v(f_2, g_2)\, \delta\mathfrak{B}(f_1, f_2\}.\end{aligned} \tag{29}$$

To find an explicit equation for μ, we express $\delta\mathfrak{A}$ and $\delta\mathfrak{B}$ in terms of μ; from Eqs. (8) and (9) we find

$$\begin{aligned}\delta\mathfrak{A}(f_1, f_2) = \sum_f \{&\delta\xi(f_1, f)\, \Phi_0(f, f_2) + \xi_0(f_1, f)\, \delta\Phi(f, f_2)\\ &+ \delta\Phi(f_1, f)\, \xi_0(f_2, f) + \Phi_0(f_1, f)\, \delta\xi(f_2, f) - \frac{1}{2} \sum_{f f'_1 f'_2} \delta\Phi(f'_2, f'_1)\\ &\times \{G(f_1 f;\, f'_2 f'_1)\, F_0(f, f_2) + G(f f_2;\, f'_2 f'_1)\, F_0(f_1, f)\}\end{aligned}$$

$$-\frac{1}{2}\sum_{ff'_1f'_2}\Phi_0(f'_2, f'_1)\{G(f_1f; f'_2f'_1)\,\delta F(f, f_2)+G(ff_2; f'_2f'_1)$$

$$\times\,\delta F(f, f_1)+\frac{1}{2}\sum_{f'_1f'_2}\delta\Phi(f'_2, f'_1)\,G(f_1f_2; f'_2f'_1); \qquad (30)$$

$$\delta\mathfrak{B}(f_1, f_2)=\sum_f\{\delta\xi(f_2, f)\,F_0(f_1, f)+\xi_0(f_2, f)\,\delta F(f_1, f)$$

$$-\delta\xi(f, f_1)\,F_0(f, f_2)-\xi_0(f, f_1)\,\delta F(f, f_2)\}+\frac{1}{2}\sum_{ff'_1f'_2}G(f_2f; f'_2f'_1)$$

$$\times\{\delta\Phi^*(f_1, f)\,\Phi_0(f'_2, f'_1)+\Phi^*_0(f_1, f)\,\delta\Phi(f'_2, f'_1)\}$$

$$-\frac{1}{2}\sum_{ff'_1f'_2}G(f_1f; f'_2f'_1)\{\delta\Phi(f_2, f)\,\Phi^*_0(f'_2, f'_1)+\Phi_0(f_2, f)\,\delta\Phi^*(f'_2, f'_1)\}, \qquad (31)$$

where

$$\delta\xi(f, f')=-\sum_{f'_1f'_2}G(ff'_1; f'_2f')\,\delta F(f'_1, f'_2);$$

$$\xi_0(f, f')=T(f, f')-\sum_{f'_1f'_2}G(ff'_1; f'_2f')\,F_0(f'_1, f'_2).$$

We substitute Eqs. (30) and (31) into Eq. (29) and use Eqs. (25) and (26). Grouping similar terms and using the orthonormalization relation, we find, after lengthy calculations, the following equations:

$$i\frac{\partial}{\partial t}\mu(g_1, g_2)=\sum_{g'}(\Omega(g_2, g')\,\mu(g_1, g')-\Omega(g_1, g')$$

$$\times\,\mu(g_2, g'))+\sum_{g'_1g'_2}\{X(g_1g_2; g'_1g'_2)\,\mu(g'_1, g'_2)$$

$$+Y(g_1g_2; g'_1g'_2)\,\mu^*(g'_2, g'_1)\}; \qquad (32)$$

$$-i\frac{\partial}{\partial t}\mu^*(g_1, g_2)=\sum_{g'}(\Omega^*(g_2, g')\,\mu^*(g_1, g')-\Omega^*(g_1, g')\,\mu^*(g_2, g'))$$

$$+\sum_{g'_1g'_2}\{X^*(g_1g_2; g'_1g'_2)\,\mu^*(g'_1, g'_2)+Y^*(g_1, g_2; g'_1g'_2)\,\mu(g'_2, g'_1)\}; \qquad (33)$$

$$\Omega(g, g')=\sum_{ff'}\xi_0(f, f')\{u^*(f, g)\,u(f', g')-v^*(f, g)\,v(f', g')\}$$

$$-\sum_{f_1f_2}\{c^0_{f_1f_2}u^*(f_1, g)\,v^*(f_2, g')+c^{0*}_{f_1f_2}v(f_2, g)\,u(f_1, g')\}, \qquad (34)$$

where

$$c^0_{f_1f_2}=\frac{1}{2}\sum_{f'_1f'_2}G(f_1f_2; f'_2f'_1)\,\Phi_0(f'_2, f'_1);$$

$$X(g_1g_2; g'_1g'_2)=-\frac{1}{2}\sum_{f_1f_2f'_1f'_2}G(f_1f_2; f'_2f'_1)$$

$$\times\{u(f_1, g_2)\,u(f_2, g_1)\,u(f'_1, g'_2)\,u(f'_2, g'_1)+v(f'_1, g_1)$$

$$\times\,v(f'_2, g_2)\,v(f_1, g'_1)\,v(f_2, g'_2)+(v(f'_1, g_1)\,u(f_1, g_2)$$

$$-u(f_1, g_1)\,v(f'_1, g_2))\,(v(f_2, g'_1)\,u(f'_2, g'_2)-v(f_2, g'_2)\,u(f'_2, g'_1))\}; \qquad (35)$$

$$Y(g_1g_2;\ g_1'g_2') = -\frac{1}{2}\sum_{f_1f_2f_1'f_2'} G(f_1f_2;\ f_2'f_1')\{u(f_2, g_1)$$
$$\times u(f_1, g_2)\, v(f_2', g_1')\, v(f_1', g_2') + v(f_1', g_1)\, v(f_2', g_2)$$
$$\times u(f_2, g_2')\, u(f_1, g_1') + (v(f_1', g_1)\, u(f_1, g_2) - v(f_1', g_2)$$
$$\times u(f_1, g_1))\,(u(f_2, g_1')\, v(f_2', g_2') - u(f_2, g_2')\, v(f_1', g_1'))\}. \tag{36}$$

We seek solutions of homogeneous equations (32) and (33) in the form

$$\left.\begin{aligned} \mu(g_1, g_2) &= \sum_{\omega} e^{-i\omega t}\, \psi_\omega(g_1, g_2); \\ \mu^*(g_1, g_2) &= \sum_{\omega} e^{-i\omega t}\, \varphi_\omega(g_1, g_2), \end{aligned}\right\} \tag{37}$$

where $\varphi_\omega = \psi^*_{-\omega}$.

The functions ψ_ω and φ_ω have the properties

$$\psi_\omega(g_1, g_2) = -\psi_\omega(g_2, g_1),\ \ \varphi_\omega(g_1, g_2) = -\varphi_\omega(g_2, g_1). \tag{38}$$

Substituting Eqs. (37) into Eqs. (32) and (33), we find equations for the spectrum of elementary excitations:

$$\omega\psi_\omega(g_1, g_2) = \sum_{g'}\{\Omega(g_2, g')\,\psi_\omega(g_1, g') - \Omega(g_1, g')$$
$$\times \psi_\omega(g_2, g')\} + \sum_{g_1'g_2'}\{X(g_1g_2;\ g_1'g_2')\,\psi_\omega(g_1', g_2')$$
$$-Y(g_1g_2;\ g_1'g_2')\,\varphi_\omega(g_1', g_2')\}; \tag{39}$$

$$-\omega\varphi_\omega(g_1, g_2) = \sum_{g'}\{\Omega^*(g_2, g')\,\varphi_\omega(g_1, g') - \Omega^*(g_1, g')\,\varphi_\omega(g_2, g')\}$$
$$+ \sum_{g_1'g_2'}\{X^*(g_1g_2;\ g_1'g_2')\,\varphi_\omega(g_1', g_2')$$
$$-Y^*(g_1g_2;\ g_1', g_2')\,\psi_\omega(g_1', g_2')\}. \tag{40}$$

These equations were first derived in [1]. We note that if ψ_ω, φ_ω, and ω are solutions of Eqs. (39) and (40), the transformations

$$\omega \to -\omega,\ \ \psi_\omega \to \varphi^*_\omega,\ \ \varphi_\omega \to \psi^*_\omega$$

again yield the solution of this system.

Equations (39) and (40) were obtained without any assumptions whatsoever about the nature of the particle interaction or the structure of the ground state. We assume below that the functions I and G are real and that the functions u, v, and ξ_0 are both real and diagonal:

$$\left.\begin{aligned} u(f, g) &= u_f\,\delta_{fg};\ \ u_f \equiv u_{q\sigma} = u_q;\ \ u_f^* = u_f; \\ v(f, g) &= v_f\,\delta_{-fg};\ \ v_f \equiv v_{q\sigma} = s_\sigma v_q;\ \ v_f^* = v_f; \end{aligned}\right\} \tag{41}$$

$$\xi_0(f, f') = \xi(f)\,\delta_{ff'},\ \ \xi(f) = \xi^*(f); \tag{42}$$

$$\left.\begin{aligned} c^0_{ff'} &= c^{0*}_{ff'} = c_f\,\delta_{-f,\,f'}; \\ c_f &= \frac{1}{2}\sum_{f'} G(f,\ -f;\ -f',\ f')\, u_{f'} v_{f'}. \end{aligned}\right\} \tag{43}$$

Then we have

$$\Omega(g, g') = \sum_{ff'} \xi(f)\,\delta_{ff'}\,\{u_f\,\delta_{fg}\,u_{f'}\,\delta_{f'g'} - v_f\,\delta_{-fg}\,v_{f'}\,\delta_{-f'g'}\}$$

$$-\sum_{f_1 f_2} c_{f_1} \delta_{-f_1 f_2} \{u_{f_1} \delta_{f_1 g} v_{f_2} \delta_{-f_2 g'} + u_{f_1} \delta_{f_1 g'} v_{f_2} \delta_{-f_2 g}\}$$
$$= \delta_{gg'} \{\xi(g)(u_g^2 - v_g^2) + 2c_g u_g v_g\}.$$

Using results found from the superfluid model for the nucleus, we find

$$\sqrt{\Omega(g, g')} = \delta_{gg'} \varepsilon(g); \quad \varepsilon(g) = \sqrt{c_g^2 + \xi^2(g)}. \tag{44}$$

In this approximation we have

$$\begin{aligned} X(g_1 g_2; g_1' g_2') = &-\frac{1}{2} G(g_1 g_2; g_2' g_1') u_{g_1} u_{g_2} u_{g_2'} u_{g_1'} \\ &-\frac{1}{2} G(-g_1, -g_2; -g_2', -g_1') v_{g_1} v_{g_2} v_{g_1'} v_{g_2'} \\ &-\frac{1}{2} G(g_1, -g_2'; g_1', -g_2) u_{g_1} v_{g_2} u_{g_1'} v_{g_2'} \\ &-\frac{1}{2} G(-g_1, g_2'; -g_1' g_2) v_{g_1} u_{g_2} v_{g_1'} u_{g_2'} \\ &+\frac{1}{2} G(g_1, -g_1'; g_2', -g_2) u_{g_1} v_{g_2} u_{g_2'} v_{g_1'} \\ &+\frac{1}{2} G(-g_1, g_1'; -g_2', g_2) v_{g_1} u_{g_2} v_{g_2'} u_{g_1'}; \end{aligned} \tag{45}$$

$$\begin{aligned} Y(g_1 g_2; -g_1', -g_2') = &-\frac{1}{2} G(g_1 g_2; g_2' g_1') u_{g_1} u_{g_2} v_{g_1'} v_{g_2'} \\ &-\frac{1}{2} G(-g_1, -g_2; -g_2', -g_1') v_{g_1} v_{g_2} u_{g_1'} u_{g_2'} \\ &-\frac{1}{2} G(g_1, -g_2'; g_1', -g_2) u_{g_1} v_{g_2} u_{g_2'} v_{g_1'} \\ &+\frac{1}{2} G(-g_1, g_2'; -g_1', g_2) v_{g_1} u_{g_2} u_{g_1'} v_{g_2'} \\ &-\frac{1}{2} G(g_1, -g_1'; g_2', -g_2) u_{g_1} v_{g_2} u_{g_1'} v_{g_2'} \\ &-\frac{1}{2} G(-g_1, g_1'; -g_2', g_2) v_{g_1} u_{g_2} v_{g_1'} u_{g_2'}; \\ &X^* = X; \quad Y^* = Y. \end{aligned} \tag{46}$$

Substituting Eqs. (44) into Eqs. (39) and (40), we find

$$\begin{aligned} \omega\psi_\omega(g_1, g_2) = &(\varepsilon(g_1) + \varepsilon(g_2))\psi_\omega(g_1, g_2) \\ &+ \sum_{g_1' g_2'} \{X(g_1 g_2; g_1' g_2')\psi_\omega(g_1', g_2') \\ &- Y(g_1, g_2; -g_1', -g_2')\varphi_\omega(-g_1', -g_2')\}; \end{aligned} \tag{47}$$

$$\begin{aligned} -\omega\varphi_\omega(-g_1, -g_2) = &(\varepsilon(g_1) + \varepsilon(g_2))\varphi_\omega(-g_1, -g_2) \\ &+ \sum_{g_1' g_2'} \{X(-g_1, -g_2; -g_1', -g_2')\varphi_\omega(-g_1', -g_2') \\ &- Y(-g_1, -g_2; g_1' g_2')\psi_\omega(g_1', g_2')\}. \end{aligned} \tag{48}$$

We introduce the new unknowns

$$Z^{(\pm)}(g_1, g_2) = \frac{1}{2}\{\psi_\omega(g_1, g_2) \pm \varphi_\omega(-g_1, -g_2)\}, \tag{49}$$

whose equations are

$$\omega Z^{(\mp)}(g_1, g_2) = (\varepsilon(g_1) + \varepsilon(g_2)) Z^{(\pm)}(g_1, g_2)$$

$$+\frac{1}{2}\sum_{g_1' g_2'}\{[X(g_1 g_2; g_1' g_2')+X(-g_1, -g_2; -g_1', -g_2')]$$
$$\mp [Y(+g_1, +g_2; -g_1', -g_2')+Y(-g_1, -g_2; +g_1' g_2')]\}$$
$$\times Z^{(\pm)}(g_1' g_2')+\frac{1}{2}\sum_{g_1' g_2'}\{[X(g_1 g_2; g_1' g_2')$$
$$-X(-g_1, -g_2; -g_1', -g_2')\pm[Y(g_1 g_2; -g_1', -g_2')$$
$$-Y(-g_1, -g_2; g_1' g_2')]\} Z^{(\mp)}(g_1', g_2'), \tag{50}$$

where

$$[X(g_1 g_2; g_1' g_2')+X(-g_1, -g_2, -g_1', -g_2')]\mp[Y(g_1 g_2; -g_1', -g_2')$$
$$+Y(-g_1, -g_2; g_1' g_2')] = -\frac{1}{2}[G(g_1 g_2; g_2' g_1')$$
$$+G(-g_1, -g_2; -g_2', -g_1')]\, v^{(\pm)}_{g_1 g_2} v^{(\pm)}_{g_1' g_2'}$$
$$-\frac{1}{2}\{[G(g_1, -g_2'; g_1', -g_2)\mp G(g_1, -g_1'; g_2', -g_2)]$$
$$+[G(-g_1, g_2'; -g_1', g_2)\mp G(-g_1, g_1'; -g_2', g_2)]\} u^{(\pm)}_{g_1 g_2} u^{(\pm)}_{g_1' g_2'}; \tag{51'}$$

$$[X(g_1 g_2; g_1' g_2')-X(-g_1, -g_2; -g_1', -g_2')]$$
$$\pm[Y(g_1, g_2; -g_1', -g_2')-Y(-g_1, -g_2; g_1', g_2')]$$
$$= -\frac{1}{2}[G(g_1, g_2; g_2', g_1')-G(-g_1, -g_2; -g_2', -g_1')]$$
$$\times v^{(\pm)}_{g_1 g_2} v^{(\mp)}_{g_1' g_2'} - \frac{1}{2}\{[G(g_1, -g_2'; g_1', -g_2)$$
$$\pm G(g_1, -g_1'; g_2', -g_2)]-[G(-g_1, g_2'; -g_1', g_2)$$
$$\pm G(-g_1, g_1'; g_2', g_2)]\} u^{(\pm)}_{g_1 g_2} u^{(\mp)}_{g_1' g_2'}; \tag{51''}$$

$$u^{(\pm)}_{gg'} = u_g v_{g'} \pm u_{g'} v_g, \quad v^{(\pm)}_{gg'} = u_g u_{g'} \mp v_g v_{g'}. \tag{52}$$

The basic equation may thus be written

$$\omega Z^{(\pm)}(g_1, g_2) = (\varepsilon(g_1)+\varepsilon(g_2))\, Z^{(\mp)}(g_1, g_2)$$
$$-\frac{1}{4}\sum_{g_1' g_2'}[G(g_1, g_2; g_2', g_1')+G(-g_1, -g_2; -g_2', -g_1')]$$
$$\times v^{(\mp)}_{g_1 g_2} v^{(\mp)}_{g_1' g_2'} Z^{(\mp)}(g_1', g_2') - \frac{1}{4}\sum_{g_1' g_2'}\{[G(g_1, -g_2'; g_1', -g_2)$$
$$\mp G(g_1, -g_1'; g_2', -g_2)]+[G(-g_1, g_2'; -g_1', g_2)$$
$$\mp G(-g_1, g_1'; -g_2', g_2)]\} u^{(\mp)}_{g_1 g_2} u^{(\mp)}_{g_1' g_2'} Z^{(\mp)}(g_1', g_2')$$
$$-\frac{1}{4}\sum_{g_1' g_2'}[G(g_1 g_2; g_2', g_1')-G(-g_1, -g_2; -g_2', -g_1')]$$
$$\times v^{(\mp)}_{g_1 g_2} v^{(\pm)}_{g_1' g_2'} Z^{(\pm)}(g_1', g_2') - \frac{1}{4}\sum_{g_1' g_2'}\{[G(g_1-g_2'; g_1'-g_2)$$
$$\pm G(g_1-g_1'; g_2'-g_2)]-[G(-g_1 g_2'; -g_1' g_2)$$
$$\pm G(-g_1, g_1'; -g_2' g_2)]\} u^{(\mp)}_{g_1 g_2} u^{(\pm)}_{g_1' g_2'} Z^{(\pm)}(g_1', g_2'). \tag{53}$$

We convert Eq. (53) into the q, σ representation, using relations (5):

$$\omega Z^{(\mp)}(q_1\sigma_1, q_2\sigma_2) = (\varepsilon(q_1)+\varepsilon(q_2))\, Z^{(\pm)}(q_1\sigma_1, q_2\sigma_2) - \frac{1}{4}\times$$

$$\times \sum_{q_1'\sigma_1', q_2'\sigma_2'} G(q_1\sigma_1, q_2\sigma_2; q_2'\sigma_2', q_1'\sigma_1')\left(1+s_{\sigma_1}s_{\sigma_2}s_{\sigma_1'}s_{\sigma_2'}\right)$$
$$\times v^{(\pm)}_{q_1\sigma_1, q_2\sigma_2} v^{(\pm)}_{q_1'\sigma_1', q_2'\sigma_2'} Z^{(\pm)}(q_1'\sigma_1', q_2'\sigma_2') - \frac{1}{4}\sum_{q_1'\sigma_1', q_2'\sigma_2'}$$
$$\times G(q_1\sigma_1, q_2\sigma_2; q_2'\sigma_2', q_1'\sigma_1')\left(1-s_{\sigma_1}s_{\sigma_2}s_{\sigma_1'}s_{\sigma_2'}\right) v^{(\pm)}_{q_1\sigma_1, q_2\sigma_2}$$
$$\times v^{(\mp)}_{q_1'\sigma_1', q_2'\sigma_2'} Z^{(\mp)}(q_1'\sigma_1', q_2'\sigma_2') - \frac{1}{4}$$
$$\times \sum_{q_1'\sigma_1', q_2'\sigma_2'} [G(q_1\sigma_1, q_2' - \sigma_2'; q_1'\sigma_1', q_2 - \sigma_2)$$
$$\mp G(q_1\sigma_1, q_1' - \sigma_1'; q_2'\sigma_2', q_2 - \sigma_2)]\left(1+s_{\sigma_1}s_{\sigma_2}s_{\sigma_1'}s_{\sigma_2'}\right)$$
$$\times u^{(\pm)}_{q_1\sigma_1, q_2\sigma_2} u^{(\pm)}_{q_1'\sigma_1', q_2'\sigma_2'} Z^{(\pm)}(q_1'\sigma_1', q_2'\sigma_2')$$
$$-\frac{1}{4}\sum_{q_1'\sigma_1', q_2'\sigma_2'} [G(q_1\sigma_1, q_2' - \sigma_2'; q_1'\sigma_1', q_2 - \sigma_2)$$
$$\pm G(q_1\sigma_1, q_1' - \sigma_1'; q_2'\sigma_2', q_2 - \sigma_2)]\left(1-s_{\sigma_1}s_{\sigma_2}s_{\sigma_1'}s_{\sigma_2'}\right)$$
$$\times u^{(\pm)}_{q_1\sigma_1, q_2\sigma_2} u^{(\mp)}_{q_1'\sigma_1', q_2'\sigma_2'} Z^{(\mp)}(q_1'\sigma_1', q_2'\sigma_2'). \tag{54}$$

To write Eq. (54) for the cases $\sigma_1 = \sigma_2$ and $\sigma_1 = -\sigma_2$, we use relations (41) and the properties of the coefficients s_σ; it follows from (41) that

$$u^{(\pm)}_{q\sigma, q'\sigma} = s_\sigma (u_q v_{q'} \pm u_{q'} v_q) \equiv s_\sigma u^{(\pm)}_{qq'};$$
$$u^{(\pm)}_{q\sigma, q'-\sigma} = s_{-\sigma} (u_q v_{q'} \mp u_{q'} v_q) \equiv s_{-\sigma} u^{(\mp)}_{qq'};$$
$$v^{(\pm)}_{q\sigma, q'\sigma} = u_q u_{q'} \mp v_q v_{q'} \equiv v^{(\pm)}_{qq'};$$
$$v^{(\pm)}_{q\sigma, q'-\sigma} = u_q u_{q'} \pm v_q v_{q'} \equiv v^{(\mp)}_{qq'}.$$

Using these equations, we find

$$\omega Z^{(\mp)}(q_1\sigma, q_2\sigma) = [\varepsilon(q_1) + \varepsilon(q_2)] Z^{(\pm)}(q_1\sigma, q_2\sigma)$$
$$-\frac{1}{2}\sum_{q_1' q_2' \sigma'} G(q_1\sigma, q_2\sigma; q_2'\sigma', q_1'\sigma') v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} Z^{(\pm)}(q_1'\sigma', q_2'\sigma')$$
$$-\frac{1}{2}\sum_{g_1' g_2' \sigma'} G(q_1\sigma, q_2\sigma; q_2' - \sigma', q_1'\sigma') v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} Z^{(\mp)}(q_1'\sigma', q_2' - \sigma')$$
$$-\frac{1}{2}\sum_{q_1' q_2' \sigma'} [G(q_1\sigma, q_2' - \sigma'; q_1'\sigma', q_2 - \sigma)$$
$$\mp G(q_1\sigma, q_1' - \sigma'; q_2'\sigma', q_2 - \sigma)] s_\sigma s_{-\sigma'} u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} Z^{(\pm)}(q_1'\sigma', q_2'\sigma')$$
$$-\frac{1}{2}\sum_{q_1' q_2' \sigma'} [G(q_1\sigma, q_2^! - \sigma'; q_1'\sigma', q_2 - \sigma)$$
$$\pm G(q_1\sigma, q_1' - \sigma'; q_2' - \sigma', q_2 - \sigma)] s_\sigma s_{-\sigma'}$$
$$\times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} Z^{(\pm)}(q_1'\sigma', q_2' - \sigma'); \tag{55}$$

$$\omega Z^{(\pm)}(q_1\sigma, q_2 - \sigma) = [\varepsilon(q_1) + \varepsilon(q_2)] Z^{(\mp)}(q_1\sigma, q_2 - \sigma)$$
$$-\frac{1}{2}\sum_{q_1' q_2' \sigma'} G(q_1\sigma, q_2 - \sigma; q_2' - \sigma', q_1'\sigma') v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} \times$$

$$\times Z^{(\mp)}(q_1'\sigma', q_2'-\sigma') - \frac{1}{2}\sum_{q_1' q_2' \sigma'} G(q_1\sigma, q_2-\sigma; q_2'\sigma', q_1'\sigma')$$

$$\times v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} Z^{(\mp)}(q_1'\sigma', q_2'\sigma') - \frac{1}{2}$$

$$\times \sum_{q_1' q_2' \sigma'} [G(q_1\sigma, q_2'\sigma'; q_1'\sigma', q_2\sigma) \pm G(q_1\sigma, q_1'-\sigma'; q_2'$$

$$-\sigma', q_2\sigma)] \times s_\sigma s_{\sigma'} u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} Z^{(\mp)}(q_1'\sigma', q_2'-\sigma') - \frac{1}{2}$$

$$\times \sum_{q_1' q_2' \sigma'} [G(q_1\sigma, q_2'-\sigma'; q_1'\sigma', q_2\sigma) \mp G(q_1\sigma, q_1$$

$$-\sigma'; q_2'\sigma', q_2\sigma)] \times s_\sigma s_{-\sigma'} u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} Z^{(\pm)}(q_1'\sigma', q_2'\sigma').$$

In deriving Eqs. (55) and (56), we have made use of the fact that during summation over σ_1' and σ_2' in (54), half of the terms vanish because of the factors $(1 \pm s_{\sigma_2} s_{\sigma_2} s_{\sigma_1'} s_{\sigma_2'})$.

Using relations (4) and (5), we can show that the coefficients

$$\left.\begin{array}{l}
\sum_\sigma G(q_1\sigma, q_2\sigma; q_2'\sigma', q_1'\sigma');\\
\sum_\sigma G(q_1\sigma, q_2\sigma; q_2'-\sigma', q_1'\sigma') s_{\sigma'};\\
\sum_\sigma s_\sigma G(q_1\sigma, q_2-\sigma; q_2'\sigma', q_1'\sigma');\\
\sum_\sigma s_\sigma G(q_1\sigma, q_2-\sigma; q_2'-\sigma', q_1'\sigma') s_{\sigma'};\\
\sum_\sigma G(q_1\sigma, q_2'\sigma'; q_1'\sigma', q_2\sigma);\\
\sum_\sigma G(q_1\sigma, q_2'-\sigma'; q_1'\sigma'; q_2\sigma) s_{-\sigma'};\\
\sum_\sigma s_{-\sigma} G(q_1\sigma, q_2'\sigma'; q_1'\sigma', q_2-\sigma);\\
\sum_\sigma s_{-\sigma} G(q_1\sigma, q_2'-\sigma'; q_1'\sigma', q_2-\sigma) s_{-\sigma'}
\end{array}\right\} \quad (57)$$

do not depend on σ'. For example, we have

$$\sum_\sigma s_{-\sigma} G(q_1\sigma, q_2'\sigma'; q_1'\sigma', q_2-\sigma) = \sum_\sigma s_{-\sigma} s_\sigma s_{-\sigma} s_{\sigma'}^2$$

$$\times G(q_1-\sigma, q_2'-\sigma'; q_1'-\sigma', q_2\sigma)$$

$$= \sum_\sigma s_\sigma G(q_1-\sigma, q_2'-\sigma'; q_1'-\sigma', q_2\sigma)$$

$$= \sum_\sigma s_{-\sigma} G(q_1\sigma, q_2'-\sigma'; q_1'-\sigma', q_2-\sigma).$$

Accordingly, we introduce a new notation for coefficients (57) which reflect this property:

$$\left.\begin{array}{l}
\frac{1}{2}\sum_\sigma G(q_1\sigma, q_2\sigma; q_2'\sigma', q_1'\sigma')\\
\qquad \equiv G^\xi(q_1+, q_2+; q_2'+, q_1'+);\\
\frac{1}{2}\sum_\sigma G(q_1\sigma, q_2\sigma; q_2'-\sigma', q_1'\sigma') s_{\sigma'}\\
\qquad \equiv G^\xi(q_1+, q_2+; q_2'-, q_1'+);\\
\frac{1}{2}\sum_\sigma s_\sigma G(q_1\sigma, q_2-\sigma; q_2'\sigma', q_1'\sigma')\\
\qquad \equiv G^\xi(q_1+, q_2-; q_2'+, q_1'+);
\end{array}\right\}$$

$$\left.\begin{aligned}
&\frac{1}{2}\sum_{\sigma} s_{\sigma} G(q_1\sigma, q_2-\sigma;\ q_2'-\sigma',\ q_1'\sigma')\, s_{\sigma'} \\
&\qquad \equiv G^{\xi}(q_1+, q_2-;\ q_2'-,\ q_1'+); \\
&\frac{1}{2}\sum_{\sigma} G(q_1\sigma,\ q_2'\sigma';\ q_1'\sigma',\ q_2\sigma) \\
&\qquad \equiv G^{\omega}(q_1+,\ q_2+;\ q_2'+, q_1'+); \\
&\frac{1}{2}\sum_{\sigma} G(q_1\sigma,\ q_2'-\sigma';\ q_1'\sigma',\ q_2\sigma)\, s_{-\sigma'} \\
&\qquad \equiv G^{\omega}(q_1+,\ q_2+;\ q_2'-,\ q_1'+); \\
&\frac{1}{2}\sum_{\sigma} s_{-\sigma} G(q_1\sigma, q_2'\sigma';\ q_1'\sigma',\ q_2-\sigma) \\
&\qquad \equiv G^{\omega}(q_1+, q_2-;\ q_2'+, q_1'+); \\
&\frac{1}{2}\sum_{\sigma} s_{-\sigma} G(q_1\sigma, q_2'-\sigma;\ q_1'\sigma', q_2-\sigma)\, s_{-\sigma'} \\
&\qquad \equiv G^{\omega}(q_1+, q_2-;\ q_2'-,\ q_1'+).
\end{aligned}\right\} \tag{58}$$

We sum Eq. (55) over σ and multiply Eq. (56) by s_{σ} and sum it over σ; we find

$$\begin{aligned}
&\omega \sum_{\sigma} Z^{(\mp)}(q_1\sigma, q_2\sigma) = (\varepsilon(q_1)+\varepsilon(q_2)) \sum_{\sigma} Z^{(\pm)}(q_1\sigma, q_2\sigma) \\
&- \sum_{q_1' q_2'} G^{\xi}(q_1+, q_2+;\ q_2'+, q_1'+)\, v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} \sum_{\sigma} Z^{(\pm)}(q_1'\sigma, q_2'\sigma) \\
&- \sum_{q_1' q_2'} G^{\xi}(q_1+, q_2+; q_2'-, q_1'+)\, v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} \sum_{\sigma} s_{\sigma} Z^{\mp}(q_1'\sigma, q_2'-\sigma) \\
&- \sum_{q_1' q_2'} [G^{\omega}(q_1+, q_2-; q_2'-, q_1'+) \mp G^{\omega}(q_1+, q_2-;\ q_1'-, q_2'+)] \\
&\qquad \times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} \sum_{\sigma} Z^{(\pm)}(q_1'\sigma, q_2'\sigma) \\
&\quad - \sum_{q_1' q_2'} [G^{(\omega)}(q_1+, q_2-;\ q_2'+, q_1'+) \pm G^{\omega}(q_1+, q_2-; \\
&\quad q_1'+,\ q_2'+)] \times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} \sum_{\sigma} s_{\sigma} Z^{\mp}(q_1'\sigma, q_2'-\sigma);
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\omega \sum_{\sigma} s_{\sigma} Z^{(\pm)}(q_1\sigma, q_2-\sigma) = (\varepsilon(q_1)+\varepsilon(q_2)) \sum_{\sigma} s_{\sigma} Z^{\mp}(q_1\sigma, q_2-\sigma) \\
&- \sum_{q_1' q_2'} G^{\xi}(q_1+, q_2-; q_2'-, q_1'+)\, v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} \sum_{\sigma} s_{\sigma} Z^{(\mp)} \times (q_1'\sigma, q_2'-\sigma) \\
&\quad - \sum_{q_1' q_2'} G^{\xi}(q_1+, q_2-;\ q_2'+,\ q_1'+)\, v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} \\
&\times \sum_{\sigma} Z^{(\pm)}(q_1'\sigma, q_2'\sigma) - \sum_{q_1' q_2'} [G^{\omega}(q_1+, q_2+; q_2'+,\ q_1'+) \\
&\pm\ G^{\omega}(q_1+, q_2'+;\ q_1'+, q_2'+)]\, u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} \sum_{\sigma} s_{\sigma} Z^{(\mp)}(q_1'\sigma, q_2'-\sigma) \\
&- \sum_{q_1' q_2'} [G^{\omega}(q_1+, q_2+; q_2'-, q_1'+) \mp G^{\omega}(q_1+, q_2+;\ q_1'-,\ q_2'+)] \\
&\qquad \times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} \sum_{\sigma} Z^{(\pm)}(q_1'\sigma, q_2'\sigma).
\end{aligned} \tag{60}$$

Since the coefficients $\sum_\sigma Z^{(\pm)}(q\sigma, q'\sigma)$ are antisymmetric with respect to interchange of the indices q and q', while the coefficients $\sum_\sigma s_\sigma Z^{(\pm)}(q\sigma, q'-\sigma)$ are symmetric with respect to this interchange, and since for each type of excitation for fixed q and q' only one of the coefficients $\sum_\sigma Z^{(\pm)}(q\sigma, q'\sigma)$ and $\sum_\sigma s_\sigma Z^{(\pm)}(q\sigma, q'-\sigma)$ is nonvanishing, we see that we can significantly simplify Eqs. (59) and (60). First, we may omit terms containing $\pm G^\omega$ on the basis of the symmetry properties of the coefficients. Second, we can replace the unknowns $\sum_\sigma Z^{(\pm)}(q\sigma, q'\sigma)$ and $\sum_\sigma s_\sigma Z^{(\pm)}(q\sigma, q'-\sigma)$ in the summation over q and q' by their sum, since in each case only one of the terms is nonvanishing. Instead of Eqs. (59) and (60) we then find

$$\begin{aligned}\omega \sum_\sigma Z^{(\mp)}(q_1\sigma, q_2\sigma) &= (\varepsilon(q_1)+\varepsilon(q_2))\sum_\sigma Z^{(\pm)}(q_1\sigma, q_2\sigma)\\ &- \sum_{q_1' q_2'}\left[G^\xi(q_1+, q_2+; q_2'+, q_1'+) + G^\xi(q_1+, q_2+; q_2'-, q_1'+)\right]\\ &\times v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'}\left[\sum_\sigma Z^{(\pm)}(q_1'\sigma, q_2'\sigma) + \sum_\sigma s_\sigma Z^{(\mp)}(q_1'\sigma, q_2'-\sigma)\right]\\ &-2\sum_{q_1' q_2'}\left[G^\omega(q_1+, q_2-; q_2'-, q_1'+) + G^\omega(q_1+, q_2-; q_2'+, q_1'+)\right]\\ &\times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'}\left[\sum_\sigma Z^{(\pm)}(q_1'\sigma, q_2'\sigma) + \sum_\sigma s_\sigma Z^{(\mp)}(q_1'\sigma; q_2'-\sigma)\right];\end{aligned} \quad (61)$$

$$\begin{aligned}\omega \sum_\sigma s_\sigma Z^{(\pm)}(q_1\sigma, q_2-\sigma) &= [\varepsilon(q_1)+\varepsilon(q_2)]\sum_\sigma s_\sigma Z^{(\mp)}(q_1\sigma, q_2-\sigma)\\ &- \sum_{q_1' q_2'}\left[G^\xi(q_1+, q_2-; q_2'-, q_1'+) + G^\xi(q_1+, q_2-; q_2'+, q_1'+)\right]\\ &\times v^{(\pm)}_{q_1 q_2} v^{(\mp)}_{q_1' q_2'}\left[\sum_\sigma Z^{(\pm)}(q_1'\sigma, q_2'\sigma) + \sum_\sigma s_\sigma Z^{(\mp)}(q_1'\sigma, q_2'-\sigma)\right]\\ &-2\sum_{q_1' q_2'}\left[G^\omega(q_1+, q_2+; q_2'+, q_1'+) + G^\omega(q_1+, q_2+; q_2'-, q_1'+)\right]\\ &\times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'}\left(\sum_\sigma Z^{(\pm)}(q_1'\sigma, q_2'\sigma) + \sum_\sigma s_\sigma Z^{(\mp)}(q_1'\sigma, q_2'-\sigma)\right).\end{aligned} \quad (62)$$

Combining Eqs. (61) and (62), introducing the new unknowns

$$R^{(\pm)}(q, q') = \sum_\sigma Z^{(\pm)}(q\sigma, q'\sigma) + \sum_\sigma s_\sigma Z^{(\mp)}(q\sigma, q'-\sigma)$$

and introducing the notation

$$\begin{aligned}G^\xi(q_1 q_2; q_2' q_1') &\equiv G^\xi(q_1+, q_2+; q_2'+, q_1'+)\\ &+ G^\xi(q_1+, q_2+; q_2'-, q_1'+) + G^\xi(q_1+, q_2-; q_2'+, q_1'+)\\ &+ G^\xi(q_1+, q_2-; q_2'-, q_1'+);\\ G^\omega(q_1 q_2; q_2' q_1') &\equiv G^\omega(q_1+, q_2+; q_2'+, q_1'+)\\ &+ G^\omega(q_1+, q_2+; q_2'-, q_1'+) + G^\omega(q_1+, q_2-; q_2'+, q_1'+)\\ &+ G^\omega(q_1+, q_2-; q_2'-, q_1'+),\end{aligned}$$

we find equations for the new unknowns:

$$\begin{aligned}\omega R^{(\mp)}(q_1, q_2) &= [\varepsilon(q_1)+\varepsilon(q_2)]\, R^{(\pm)}(q_1, q_2)\\ &- \sum_{q_1' q_2'} G^\xi(q_1 q_2; q_2' q_1')\, v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} R^{(\pm)}(q_1', q_2') -\end{aligned}$$

$$-2\sum_{q_1'q_2'} G^{\omega}(q_1q_2;\ q_2'q_1')\, u^{(\pm)}_{q_1q_2} u^{(\mp)}_{q_1'q_2'} R^{(\pm)}(q_1', q_2'). \tag{63}$$

The interaction in the particle-particle channel affects the properties of the collective states through the terms of Eq. (63) which are proportional to $v^{(\pm)}_{qq'}$. The contribution of the interaction and the particle-hole channel is contained in terms proportional to $u^{(\pm)}_{qq'}$.

In studying the properties of low-lying nuclear states it should be kept in mind that the $C(g_1g_2;\ g_2'g_1')$ interaction is used for various momenta of the colliding particles. Some of the collective effects associated with quadrupole, octupole, etc., correlations in the particle-hole channel are governed by the interaction with small momentum transfer [in this case, this is $G^0(q_1q_2;\ q_2'q_1')$]. Other effects are related to pairing correlations of the superconducting type. These effects are governed by the interaction with a vanishing net colliding-particle momentum [$G^{\xi}(q_1q_2;\ q_2'q_1')$]. Generally speaking, these two interactions should be considered independent.

In deriving Eqs. (63) we treated the interparticle interaction in general form. We know, however, that the appearance of vibrational levels in nuclei is due primarily to an interaction in the particle-hold channel, which makes a coherent contribution. For this reason we consider the case in which the effect of the interaction in the particle-particle channel on the properties of the vibrational states may be neglected. We write the interaction in the particle-hole channel as a sum of multipole and spin-multipole interactions:

$$G^{\omega}(q_1q_2;\ q_2'q_1') = \varkappa_f f(q_1, q_2) f(q_1', q_2') + \varkappa_t t(q_1, q_2)\, t(q_1', q_2'), \tag{64}$$

where $f(q, q')$ and $t(q, q')$ are the one-particle matrix elements of the operators corresponding to the multipole and spin-multipole moment, respectively.

In this case Eqs. (63) become, when account is taken of the symmetry properties of the coefficients $R^{(\pm)}(q, q')$,

$$\begin{aligned}\omega R^{(-)}(q_1, q_2) &= [\varepsilon(q_1)+\varepsilon(q_2)]\, R^{(+)}(q_1, q_2)\\ &-2\varkappa_f f(q_1, q_2)\, u^{(+)}_{q_1q_2}\sum_{q_1'q_2'} f(q_1', q_2')\, u^{(+)}_{q_1'q_2'} R^{(+)}_{(q_1'q_2')};\end{aligned} \tag{65}$$

$$\begin{aligned}\omega R^{(+)}(q_1, q_2) &= [\varepsilon(q_1)+\varepsilon(q_2)]\, R^{(-)}(q_1, q_2)\\ &-2\varkappa_t - t(q_1, q_2)\, u^{(-)}_{q_1q_2}\sum_{q_1'q_2'} t(q_1', q_2')\, u^{(-)}_{q_1'q_2'} R^{(-)}(q_1', q_2').\end{aligned} \tag{66}$$

There are no terms in Eqs. (65) and (66) containing

$$\sum_{q_1'q_2'} t(q_1', q_2')\, u^{(+)}_{q_1'q_2'} R^{(+)}(q_1', q_2'),\quad \sum_{q_1'q_2'} f(q_1', q_2')\, u^{(-)}_{q_1'q_2'} R^{(-)}(q_1', q_2'),$$

since these sums vanish due to the symmetry of the coefficients f, t, $R^{(\pm)}$, and $u^{(\pm)}$ with respect to interchange of the indices q_1' and q_2'. The coefficients $R^{(\pm)}(q_1, q_2)$ are antisymmetric with respect to interchange of indices if identical σ_1 and σ_2 correspond to the given q_1 and q_2 for this type of excitation. Otherwise, these coefficients are symmetric with respect to interchange of indices. These symmetry properties are the same as those of the coefficient f and are opposite the symmetry properties of the coefficients t.

To transform Eqs. (65) and (66), we introduce the notation

$$\left.\begin{aligned} V^{(+)} &= 2\varkappa_f \sum_{qq'} f(q, q')\, u^{(+)}_{qq'} R^{(+)}(q, q');\\ V^{(-)} &= 2\varkappa_t \sum_{qq'} t(q, q')\, u^{(-)}_{qq'} R^{(-)}(q, q')\end{aligned}\right\} \tag{67}$$

and we rewrite Eqs. (65) and (66):

$$(\varepsilon(q_1)+\varepsilon(q_2))\,R^{(\pm)}(q_1,\ q_2)-\omega R^{(\mp)}(q_1,\ q_2)=\begin{Bmatrix} f(q_1,\ q_2) \\ t(q_1,\ q_2)\end{Bmatrix} u^{(\pm)}_{q_1 q_2} V^{\pm}. \tag{68}$$

We find

$$\left.\begin{aligned} R^{(+)}(q_1,\ q_2)&=\frac{f(q_1,\ q_2)\,u^{(+)}_{q_1 q_2}(\varepsilon(q_1)+\varepsilon(q_2))\,V^{(+)}+t(q_1,\ q_2)\,u^{(-)}_{q_1 q_2}\,\omega V^{(-)}}{(\varepsilon(q_1)+\varepsilon(q_2))^2-\omega^2};\\ R^{(-)}(q_1,\ q_2)&=\frac{t(q_1,\ q_2)\,u^{(-)}_{q_1 q_2}(\varepsilon(q_1)+\varepsilon(q_2))\,V^{(-)}+f(q_1,\ q_2)\,u^{(+)}_{q_1 q_2}\,\omega V^{(+)}}{(\varepsilon(q_1)+\varepsilon(q_2))^2-\omega^2}.\end{aligned}\right\} \tag{69}$$

Substituting Eqs. (69) into Eqs. (67) and setting the determinant of the resulting system of linear equations equal to zero, we find the secular equation, which yields the collective-vibration frequencies:

$$\left(1-2\varkappa_f\sum_{qq'}\frac{f^2(q,\ q')\,u^{(+)2}_{qq'}(\varepsilon(q)+\varepsilon(q'))}{(\varepsilon(q)+\varepsilon(q'))^2-\omega^2}\right)\left(1-2\varkappa_t\sum_{qq'}\frac{t^2(q,\ q')\,u^{(-)2}_{qq'}(\varepsilon(q)+\varepsilon(q'))}{(\varepsilon(q)+\varepsilon(q'))^2-\omega^2}\right)$$

$$=4\varkappa_t\varkappa_t\left\{\sum_{qq'}\frac{f(q,\ q')\,t(q,\ q')\,u^{(+)}_{qq'}\,u^{(-)}_{qq'}\,\omega}{(\varepsilon(q)+\varepsilon(q'))^2-\omega^2}\right\}^2.$$

This equation was studied in [3] in a treatment of quadrupole states in deformed nuclei. Setting $\varkappa_t = 0$, we find the well-known secular equation for the multipole-multipole interaction:

$$1=2\varkappa_f\sum_{qq'}\frac{f^2(q,\ q')\,u^{(+)2}_{qq'}(\varepsilon(q)+\varepsilon(q'))}{(\varepsilon(q)+\varepsilon(q'))^2-\omega^2}.$$

The roots of this equation are the energies of the vibrational states. Solutions of equations of this type were found in a study of the vibrational states in spherical [4] and deformed nuclei [5].

In addition to the vibrational levels, whose appearance is due primarily to the interaction in the particle-hole channel, collective states exist in the nuclei whose properties are governed primarily by the interaction in the particle channel. Pairing vibrations are an example of such states. To examine the properties of the pairing vibrational states, we set

$$\left.\begin{aligned} G^{\omega}(q_1 q_2;\ q_2' q_1')&=0;\\ G^{\xi}(q_1 q_2;\ q_2' q_1')&=G\delta_{q_1 q_2}\delta_{q_1' q_2'}.\end{aligned}\right\} \tag{70}$$

Here Eqs. (63) become

$$\omega R^{(\mp)}(q,\ q)=2\varepsilon(q)\,R^{(\pm)}(q,\ q)-Gv^{(\pm)}_{qq}\sum_{q'}v^{(\pm)}_{q'q'}R^{(\pm)}(q',\ q'). \tag{71}$$

We introduce the notation

$$d^{(\pm)}=G\sum_q v^{(\pm)}_{qq}R^{(\pm)}(q,\ q). \tag{72}$$

Then Eqs. (71) may be written

$$2\varepsilon(q)\,R^{(\pm)}(q,\ q)-\omega R^{(\mp)}(q,\ q)=v^{(\pm)}_{qq}d^{(\pm)}, \tag{73}$$

from which it follows that

$$R^{(\pm)}(q,\ q)=\frac{2\varepsilon(q)\,v^{(\pm)}_{qq}d^{(\pm)}+\omega v^{(\mp)}_{qq}d^{(\mp)}}{4\varepsilon^2(q)-\omega^2}. \tag{74}$$

Substituting Eq. (74) into Eq. (72) and setting the determinant of the resulting system of linear equations equal to zero, we find an equation for the pairing-vibration frequencies:

$$\left\{\sum_q \frac{2\varepsilon(q)\, v_{qq}^{(+)2}}{4\varepsilon^2(q)-\omega^2}-\frac{1}{G}\right\}\left\{\sum_q \frac{2\varepsilon(q)}{4\varepsilon^2(q)-\omega^2}-\frac{1}{G}\right\}=\omega^2\left\{\sum_q \frac{v_{qq}^{(+)}}{4\varepsilon^2(q)-\omega^2}\right\}^2. \tag{75}$$

This type of equation was obtained for spherical nuclei in [6] and for deformed nuclei in [7]. These equations form the basis for the theory of pairing vibrations [8, 9].

We consider the more general case in which

$$G^{(\omega)}(q_1 q_2;\, q_2' q_1')=\varkappa f(q_1,\, q_2)\, f(q_1',\, q_2');$$
$$G^{\xi}(q_1 q_2;\, q_2' q_1')=G\delta_{q_1 q_2}\delta_{q_1' q_2'}.$$

Then Eqs. (63) become

$$\left.\begin{aligned}\omega R^{(-)}(q_1,\, q_2)&=(\varepsilon(q_1)+\varepsilon(q_2))\, R^{(+)}(q_1,\, q_2)-\varkappa f(q_1,\, q_2)\, u_{q_1 q_2}^{(+)}\\ &\times\sum_{q_1' q_2'} f(q_1',\, q_2')\, u_{q_1' q_2'}^{(+)} R^{(+)}(q_1',\, q_2')-Gv_{q_1 q_1}^{(+)}\delta_{q_1 q_2}\\ &\times\sum_{q'} v_{q' q'}^{(+)} R^{(+)}(q',\, q');\\ \omega R^{(+)}(q_1,\, q_2)&=(\varepsilon(q_1)+\varepsilon(q_2))\, R^{(-)}(q_1,\, q_2)-G\delta_{q_1 q_2}\\ &\sum_{q'} R^{(-)}(q',\, q')\, v_{q' q'}^{(-)}.\end{aligned}\right\} \tag{76}$$

We introduce the notation

$$\left.\begin{aligned}V^{(+)}&=\varkappa\sum_{q,\, q'} f(q,\, q')\, u_{qq'}^{(+)} R^{(+)}(q,\, q');\\ d^{(\pm)}&=G\sum_q v_{qq}^{(\pm)} R^{(\pm)}(q,\, q).\end{aligned}\right\} \tag{77}$$

Then Eqs. (76) become

$$\left.\begin{aligned}(\varepsilon(q_1)+\varepsilon(q_2))\, R^{(+)}(q_1,\, q_2)-\omega R^{(-)}(q_1,\, q_2)\\ =f(q_1,\, q_2)\, u_{q_1 q_2}^{(+)} V^{(+)}+v_{q_1 q_1}^{(+)}\delta_{q_1 q_2} d^{(+)};\\ -\omega R^{(+)}(q_1,\, q_2)+(\varepsilon(q_1)+\varepsilon(q_2))\, R^{(-)}(q_1,\, q_2)=\delta_{q_1 q_2} d^{(-)},\end{aligned}\right\} \tag{78}$$

from which it follows that

$$\left.\begin{aligned}R^{(+)}(q_1,\, q_2)&=\frac{(\varepsilon(q_1)+\varepsilon(q_2))\, f(q_1,\, q_2)\, u_{q_1 q_2}^{(+)} V^{(+)}+2\varepsilon(q_1)\, u_{q_1 q_1}^{(+)}\delta_{q_1 q_2} d^{(+)}+\omega\delta_{q_1 q_2} d^{(-)}}{(\varepsilon(q_1)+\varepsilon(q_2))^2-\omega^2};\\ R^{(-)}(q_1,\, q_2)&=\frac{\omega f(q_1,\, q_2)\, u_{q_1 q_2}^{(+)} V^{(+)}+\omega v_{q_1 q_1}^{(+)}\delta_{q_1 q_2} d^{(+)}+2\varepsilon(q_1)\,\delta_{q_1 q_2} d^{(-)}}{(\varepsilon(q_1)+\varepsilon(q_2))^2-\omega^2}.\end{aligned}\right\} \tag{79}$$

Substituting Eqs. (79) into Eqs. (77) and setting the determinant of the resulting system of linear equations equal to zero, we find an equation for the collective-excitation energies:

$$\det\begin{vmatrix} \left(\varkappa\sum_{qq'}\frac{\varepsilon(qq')f^2(qq')\overset{(+)}{U}{}^2_{qq'}}{\varepsilon^2(qq')-\omega^2}-1\right) & \frac{G}{2}\sum_q\frac{f(qq)\overset{(+)}{U}_{qq}\,\omega}{4\varepsilon^2(q)-\omega^2} & \frac{G}{2}\sum_q\frac{2\varepsilon(q)\overset{(+)}{U}_{qq}\overset{(+)}{V}_{qq}f(qq)}{4\varepsilon^2(q)-\omega^2} \\ \varkappa\sum_q\frac{\omega\overset{(+)}{U}_{qq}f(qq)}{4\varepsilon^2(q)-\omega^2} & \left(\frac{G}{2}\sum_q\frac{2\varepsilon(q)}{4\varepsilon^2(q)-\omega^2}-1\right) & \frac{G}{2}\sum_q\frac{\omega\overset{(+)}{V}_{qq}}{4\varepsilon^2(q)-\omega^2} \\ \varkappa\sum_q\frac{2\varepsilon(q)\overset{(+)}{U}_{qq}\overset{(+)}{V}_{qq}f(qq)}{4\varepsilon^2(q)-\omega^2} & \frac{G}{2}\sum_q\frac{\omega\overset{(+)}{V}_{qq}}{4\varepsilon^2(q)-\omega^2} & \left(\frac{G}{2}\sum_q\frac{2\varepsilon(q)\overset{(+)}{V}{}^2_{qq}}{4\varepsilon^2(q)-\omega^2}-1\right) \end{vmatrix}=0$$

$$\varepsilon(qq')\equiv\varepsilon(q)+\varepsilon(q')\ .$$

Similar equations were obtained in [8] for quadrupole states of deformed nuclei.

Having written down equations for the natural vibrations of the system, we turn now to vibrations under the influence of weak external fields; for this purpose we add to Hamiltonian (1) the term

$$\sum_{ff'}\delta I(f,\ f')\,a_f^+\,a_{f'}\ , \tag{80}$$

where the function $\delta I(f, f') = \delta I^*(f', f)$ characterizes the external field. The expressions $\delta\mathfrak{A}(f, f')$ and $\delta\mathfrak{B}(f, f')$ should be supplemented by the terms

$$\delta\mathfrak{A}_{ex}(f_1,\ f_2)=\sum_f\{\delta I(f_1,\ f)\,\Phi(f,\ f_2)+\delta I(f_2,\ f)\,\Phi(f_1,\ f)\}; \tag{81}$$

$$\delta\mathfrak{B}_{ex}(f_1,\ f_2)=\sum_f\{\delta I(f_2,\ f)\,F(f_1,\ f)+\delta I(f,\ f_1)\,F(f,\ f_2)\}. \tag{81'}$$

Using

$$\delta I(f,\ f')=\sum_\omega e^{-i\omega t}\,\delta I_\omega(f,\ f');\quad \delta I^*(f,\ f')=\sum_\omega e^{-i\omega t}\,\delta I^*_{-\omega}(f,\ f'), \tag{82}$$

we write Eqs. (39) and (40) (in the presence of an external field) as [1]

$$\begin{aligned}\omega\psi_\omega(g_1,\ g_2)&=\sum_{g'}\{\Omega(g_2,\ g')\,\psi_\omega(g_1,\ g')-\Omega(g_1,\ g')\,\psi_\omega(g_2,\ g')\}\\ &+\sum_{q_1'q_2'}\{X(g_1,\ g_2;\ g_1',g_2')\,\psi_\omega(g_1',\ g_2')-Y(g_1,g_2;\ g_1',g_2')\,\varphi_\omega(g_1',g_2')\}\\ &+\sum_{ff'}\{v(f',\ g_1)\,u^*(f,\ g_2)-u^*(f,\ g_1)\,v(f',\ g_2)\}\,\delta I_\omega(f,\ f');\end{aligned} \tag{83}$$

$$\begin{aligned}-\omega\varphi_\omega(g_1,\ g_2)&=\sum_{g'}\{\Omega^*(g_2,\ g')\,\varphi_\omega(g_1,\ g')-\Omega^*(g_1,\ g')\,\varphi_\omega(g_2,\ g')\}\\ &+\sum_{q_1'q_2'}\{X^*(g_1\,g_2;\ g_1'\,g_2')\,\varphi_\omega(g_1',\ g_2')-Y^*(g_1\,g_2;\ g_1'g_2')\,\psi_\omega(g_1',\ g_2')\}\\ &+\sum_{ff'}\{v^*(f',\ g_1)\,u(f,\ g_2)-u(f,\ g_1)\,v^*(f',\ g_2)\}\,\delta I^*_{-\omega}(f,\ f').\end{aligned} \tag{84}$$

We rewrite Eqs. (83) and (84) in approximation (41)-(43). We introduce the functions $R^{(\pm)}(q, q')$ and carry out calculations like those involved in the derivations of Eqs. (63); we find

$$\omega R^{(\mp)}(q_1,\ q_2)=(\varepsilon(q_1)+\varepsilon(q_2))\,R^{(\pm)}(q_1,\ q_2)-\sum_{q_1'q_2'}G^{\xi}(q_1\,q_2;\ q_2'\,q_1')$$

$$\times v^{(\pm)}_{q_1 q_2} v^{(\pm)}_{q_1' q_2'} R^{(\pm)}(q_1', q_2') - 2 \sum_{q_1' q_2'} G^{\omega}(q_1, q_2; q_2', q_1')$$
$$\times u^{(\pm)}_{q_1 q_2} u^{(\pm)}_{q_1' q_2'} R^{(\pm)}_{q_1' q_2'} - \frac{1}{2} u^{(\pm)}_{q_1 q_2} \left[\delta I_{\omega}(q_1, q_2) \pm \delta I^{*}_{-\omega}(q_1, q_2)\right], \tag{85}$$

where

$$\delta I_{\omega}(q_1, q_2) = \sum_{\sigma} (\delta I_{\omega}(q_1 \sigma, q_2 \sigma) - s_{\sigma} \delta I_{\omega}(q_1 \sigma, q_2 - \sigma));$$
$$\delta I^{*}_{\omega}(q_1, q_2) = \sum_{\sigma} (\delta I^{*}_{\omega}(q_1 \sigma, q_2 \sigma) - s_{\sigma} \delta I^{*}_{-\omega}(q_1 \sigma, q_2 - \sigma)).$$

To convert Eqs. (85) to a form similar to that in the theory of finite Fermi systems [10], we introduce

$$d^{(\pm)}(q_1, q_2) = \sum_{q_1' q_2'} G^{\xi}(q_1 q_2; q_2' q_1') v^{(\pm)}_{q_1' q_2'} R^{(\pm)}(q_1', q_2'); \tag{86}$$

$$V^{(\pm)}(q_1, q_2) = \sum_{q_1' q_2'} G^{\omega}(q_1 q_2; q_2' q_1') u^{(\pm)}_{q_1' q_2'} R^{(\pm)}(q_1', q_2') + V_0^{(\pm)}(q_1, q_2), \tag{87}$$

$$V_0^{(\pm)}(q_1, q_2) = \frac{1}{2} (\delta I_{\omega}(q_1, q_2) \pm \delta I^{*}_{-\omega}(q_1, q_2)). \tag{88}$$

Then we have

$$(\varepsilon(q_1) + \varepsilon(q_2)) R^{(\pm)}(q_1, q_2) - \omega R^{(\mp)}(q_1, q_2)$$
$$= v^{(\pm)}_{q_1 q_2} d^{(\pm)}(q_1, q_2) + u^{(\pm)}_{q_1 q_2} V^{(\pm)}(q_1, q_2). \tag{89}$$

from which it follows that

$$R^{(\pm)}(q_1, q_2) = \left[(\varepsilon(q_1) + \varepsilon(q_2))^2 - \omega^2\right]^{-1} \{(\varepsilon(q_1) + \varepsilon(q_2))$$
$$\times \left[u^{(\pm)}_{q_1 q_2} V^{(\pm)}(q_1, q_2) + v^{(\pm)}_{q_1 q_2} d^{(\pm)}(q_1, q_2)\right]$$
$$+ \omega \left[u^{(\mp)}_{q_1 q_2} V^{(\mp)}(q_1, q_2) + v^{(\mp)}_{q_1 q_2} d^{(\mp)}(q_1, q_2)\right]\}. \tag{90}$$

Substituting (90) into (86) and (87), we find

$$V^{(\pm)}(q_1, q_2) = V_0^{(\pm)}(q_1, q_2) + 2 \sum_{q_1' q_2'} G^{\omega}(q_1 q_2; q_2' q_1') u^{(\pm)}_{q_1' q_2'}$$
$$\times \left[(\varepsilon(q_1') + \varepsilon(q_2'))^2 - \omega^2\right]^{-1} \left\{ (\varepsilon(q_1') + \varepsilon(q_2')) \left[u^{(\pm)}_{q_1' q_2'} V^{(\pm)}(q_1', q_2')\right.\right.$$
$$\left.\left. + v^{(\pm)}_{q_1' q_2'} d^{(\pm)}(q_1', q_2')\right] + \omega \left[u^{(\mp)}_{q_1' q_2'} V^{(\mp)}(q_1', q_2') + v^{(\mp)}_{q_1' q_2'} d^{(\mp)}_{q_1' q_2'}\right]\right\}; \tag{91}$$

$$d^{(\pm)}(q_1, q_2) = \sum_{q_1', q_2'} G^{\xi}(q_1 q_2; q_2' q_1') \frac{v^{(\pm)}_{q_1' q_2'}}{(\varepsilon(q_1') + \varepsilon(q_2'))^2 - \omega^2}$$
$$\times \left\{(\varepsilon(q_1') + \varepsilon(q_2')) \left[u^{(\pm)}_{q_1' q_2'} V^{(\pm)}(q_1', q_2') + v^{(\pm)}_{q_1' q_2'} d^{(\pm)}(q_1', q_2')\right]\right.$$
$$\left. + \omega \left[u^{(\mp)}_{q_1' q_2'} V^{(\mp)}(q_1', q_2') + v^{(\mp)}_{q_1' q_2'} d^{(\mp)}(q_1', q_2')\right]\right\}. \tag{92}$$

We have obtained a system of equations for the four unknowns $V^{(\pm)}$ and $d^{(\pm)}$, but only two equations are independent, as Eqs. (63) show. For this reason it is more convenient to solve system (63) rather than (91) and (92).

We have thus derived from the equations of the self-consistent-field method the equations of the theory of finite Fermi systems usually obtained by a Green's-function technique. Although the equations of the self-consistent-field method are written for the distribution functions $\langle\Psi^+(t, r_1)\Psi(t, r_2)\rangle$ and $\langle\Psi(t, r_1)\Psi(t, r_2)\rangle$, this result cannot be considered unexpected: it is a consequence of the general theorem on the variation of the average value of a dynamic quantity [11]:

$$\delta \langle A(t)\rangle = \langle A(t)\rangle_{H+\delta H} - \langle A(t)\rangle_H$$
$$= 2\pi \{e^{-iEt} \langle\langle A, B\rangle\rangle_E \delta\xi + e^{iEt} \langle\langle A, B\rangle\rangle^*_{-E} \delta\xi^*\},$$

where A(t) is some dynamic quantity in the Heisenberg picture, B is an operator which does not explicitly depend on the time, $\delta\xi$ is an infinitesimal C number, and $\langle\langle A, B\rangle\rangle_E$ is the Green's function in the E picture. This theorem relates variations in the distribution functions to the corresponding Green's function. Using this theorem, introducing weak external fields into the Hamiltonian, and varying with respect to the small parameter, we can always find equations for the Green's functions from equations for the distribution functions.

We have shown that of the mathematical methods available in microscopic nuclear theory the self-consistent-field method is the most general. With certain assumptions, its basic equations yield both equations for the effective field of the theory of finite Fermi systems and secular equations for the model with pairing and multipole forces. Even the self-consistent-field method, however, is not free of limitations. The fact that we use simple rules for splitting up the averages of products of four Fermi operators and the fact that we set the matrix elements $\langle\alpha_g^+\alpha_{g'}\rangle$ equal to zero mean that we have neglected nonlinear effects. For this reason the self-consistent-field method is actually equivalent to the quasiboson approximation. Moreover, from the purely practical point of view, it is more convenient to use nuclear wave functions as in the method of approximate second quantization, rather than averages of operators as in the self-consistent-field method.

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