THE MODEL HAMILTONIAN IN SUPERCONDUCTIVITY THEORY

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A system of fermions with attraction described by the model Hamiltonian in superconductivity theory with separable interaction is considered. Asymptotically exact estimates (as $V \to \infty$) for the minimal eigenvalue of the Hamiltonian, correlation functions, and Green's functions are obtained.

§ 1. Statement of the Problem

The simplest model system considered in superconductivity theory is characterized by a Hamiltonian in which only the interaction between particles having opposite momenta and spins is retained:

$$H = \sum_f T(f) a_f^+ a_f - \frac{1}{2V} \sum_{f, f'} \lambda(f) \lambda(f') a_f^+ a_{-f}^+ a_{-f'} a_{f'}, \tag{1.1}$$

where $f = (\mathbf{p}, s)$, $s = \pm 1$; $\mathbf{p}$ is the momentum vector. For fixed $V = L^3$,

$$p_x = \frac{2\pi}{L} n_x, \quad p_y = \frac{2\pi}{L} n_y, \quad p_z = \frac{2\pi}{L} n_z,$$

n_x, n_y, n_z are integers; $-f = (-\mathbf{p}, -s)$.

Finally, $T(f) = (p^2/2m) - \mu$, where $\mu > 0$ is the chemical potential,

$$\lambda(f) = \begin{cases} J \cdot \varepsilon(s) & \text{for } \left|\frac{\mathbf{p}^2}{2m} - \mu\right| \leqslant \Delta, \\ 0 & \text{for } \left|\frac{\mathbf{p}^2}{2m} - \mu\right| > \Delta; \end{cases}$$

$$\varepsilon(s) = \pm 1, \quad J = \text{const}.$$

The application of the Bardeen–Cooper–Schrieffer method [1] and the method of compensation of dangerous diagrams leads to the identical result in the case given. Moreover, in [2] it was shown that a Hamiltonian of the type (1.1) is of great methodological interest, since here we have one of the very few completely solvable problems in statistical physics.

In the paper mentioned it is established that for this problem we may obtain an asymptotically exact (for $V \to \infty$) expression for the free energy.

This result was found there in the following manner. The Hamiltonian (1.1) was partitioned into two parts H_0 and H_1 in a special manner. The problem with the Hamiltonian H_0 was solved exactly. Perturbation theory was used to consider the effect of H_1. It was shown that any n-th term of the corresponding expansion becomes asymptotically small for $V \to \infty$, in connection with which it was concluded that the effect of H_1 may in general be neglected after the transition in the limit $V \to \infty$. Of course, reasoning of this kind

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Joint Institute for Nuclear Research, Dubna. Translated from Problemy Fiziki Élementarnykh Chastits i Atomnogo Yadra, Vol. 1, No. 2, pp. 301–364, 1971.

cannot pretend to mathematical rigor, but it should nevertheless be underlined that in statistical physics problems still cruder devices are often used. For example, approximate methods based on selective summation of "principal terms" (in some sense) of the perturbation-theory series are very widely used; here the remaining terms are discarded even though they do not vanish even for $V \to \infty$.

Doubts of the validity of the results of [2] also arise in connection with the fact that various attempts at using conventional Feynman diagram techniques (without allowance for "anomalous pairings" $\overline{a_f\, a_{-f}}$, $\overline{a^+_{-f}\, a^+_f}$, to which canonical u-, v-transformation leads) did not yield the expected result. Furthermore, based on the summation of a certain class of Feynman diagrams, Prange [3] obtained a solution which differed in principle from the solution obtained in [1, 2] and assumed that the latter papers were wrong.

In [4] a study was made of a chain of linked equations for the Green's function without the use of perturbation theory. It was shown there that the Green's function for the Hamiltonian H_0 satisfied this entire chain of equations for the exact Hamiltonian $H = H_0 + H_1$ with an error of order $1/V$. This substantiates the results of [2] and reveals the "inefficiency" of the correction H_1.

However, one can also dwell on the purely mathematical point of view. As soon as we have fixed the Hamiltonian, say in the form (1.1), we have an already fully defined mathematical problem which should be solved rigorously without any "physical assumptions." In this case, the approximate expressions satisfy the exact equations with an error of order $1/V$, and we should estimate the difference between the most exact and approximate expressions.

Having in mind complete parity in the problem of the behavior of a dynamic system having the Hamiltonian (1.1), we shall take precisely such a purely mathematical viewpoint in this paper.

We shall study the Hamiltonian (1.1) at a temperature $\theta = 0$ and demonstrate rigorously that the relative difference $(E-E_0)/E_0$ between the lowest energy levels H and H_0, and likewise between the corresponding Green's functions, tends to vanish for $V \to \infty$; we shall obtain estimates for the order of decrease.

Based on methodological concepts it is convenient to consider a somewhat more general Hamiltonian containing terms which represent sources of creation and annihilation of pairs:

$$\mathcal{H} = \sum_f T(f)\, a_f^+ a_f - \nu \sum_f \frac{\lambda(f)}{2}\left(a_{-f}\, a_f + a_f^+ a_{-f}^+\right) - \frac{1}{2V} \sum_{f,\, f'} \lambda(f)\, \lambda(f')\, a_f^+ a_{-f}^+ a_{-f'}\, a_{f'}, \tag{1.2}$$

where ν is a parameter which we shall assume to be greater than or equal to zero.

Let us note that the case $\nu < 0$ need not be considered, since it can be reduced to the case $\nu > 0$ by a trivial change in the gauge of the Fermi operators:

$$a_f \to \mathrm{i}\, a_f; \qquad a_f^+ \to -\mathrm{i}\, a_f^+.$$

Let us emphasize the fact that the case $\nu > 0$ will be considered exclusive of those notions that it is of interest in understanding the situation in the actual case $\nu = 0$.

For the investigation undertaken we shall not need those specific properties of the functions $\lambda(f)$, $T(f)$ of which we spoke above. It will be quite sufficient if the following general conditions are satisfied:

1) the functions $\lambda(f)$ and $T(f)$ are real, piecewise continuous, and have the symmetry conditions

$$\lambda(-f) = -\lambda(f); \qquad T(-f) = T(f);$$

2) $\lambda(f)$ is uniformly bounded throughout the entire space, and $T(f) \to \infty$ for $|f| \to \infty$;

3) $\frac{1}{V} \sum_f |\lambda(f)| \leqslant \text{const}$ for $V \to \infty$;

4) $\lim_{V\to\infty} \frac{1}{2V} \sum_f \frac{\lambda^2(f)}{\sqrt{\lambda^2(f)\,x + T^2(f)}} > 1$ for sufficiently small positive x.

Let us represent $\mathcal{H}$ (1.2) in the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \tag{1.3}$$

where

$$\mathcal{H}_0 = \sum_f T(f)\, a_f^+ a_f - \frac{1}{2} \sum_f \lambda(f) \{(\nu + \sigma^*)\, a_{-f}\, a_f + (\nu + \sigma)\, a_f^+ a_{-f}^+\} + \frac{|\sigma|^2 V}{2}, \tag{1.4}$$

$$\mathcal{H}_1 = -\frac{1}{2V} \left(\sum_f \lambda(f)\, a_f^+ a_{-f}^+ - V\sigma^* \right) \left(\sum_f \lambda(f)\, a_{-f}\, a_f - V\sigma \right). \tag{1.5}$$

Here σ is a certain complex number.

Let us note that if σ is determined from the condition for the minimum of the least eigenvalue $\mathcal{H}_0$, while $\mathcal{H}_1$ is discarded, we arrive at the well-known approximate solution which was considered in [1,2,4]. Here our problem will consist in finding the estimates for the deviation of the minimal eigenvalues $\mathcal{H}_0$, $\mathcal{H}$ and for the deviation of the corresponding Green's functions. Let us show that these deviations will vanish in the process of the transition in the limit $V \to \infty$.*

§ 2. The General Properties of the Hamiltonian

1. In this section we shall establish certain general properties of the model Hamiltonian $\mathcal{H}$ (1.2). Let us consider the occupancy numbers $n_f = a_f^+ a_f$ and let us show that the differences $n_f - n_{-f}$ are integrals of motion. Actually,

$$a_{-f}\, a_f (n_f - n_{-f}) - (n_f - n_{-f})\, a_{-f}\, a_f = 0,$$

and likewise

$$a_f^+ a_{-f}^+ (n_f - n_{-f}) - (n_f - n_{-f})\, a_f^+ a_{-f}^+ = 0,$$

therefore,

$$\mathcal{H} (n_f - n_{-f}) - (n_f - n_{-f})\, \mathcal{H} = 0.$$

Consequently,

$$\frac{d}{dt} (n_f(t) - n_{-f}(t)) = 0. \tag{2.1}$$

2. Let us show that for the wave function $\Phi_{\mathcal{H}}$, corresponding to the least eigenvalue of the Hamiltonian $\mathcal{H}$, we may place

$$(n_f - n_{-f})\, \Phi_{\mathcal{H}} = 0 \tag{2.2}$$

for any f.

In order to prove this let us assume the opposite. Since $(n_f - n_{-f})$ commutates with $\mathcal{H}$ (and with one another) one can always choose $\Phi_{\mathcal{H}}$ in such a way that it is an eigenfunction for all these operators:

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*Recently papers have appeared [7-12] in which new methods have been developed for finding asymptotically exact expressions for multitemporal correlation functions (Green's functions) in the case of arbitrary temperatures θ. Estimates were likewise constructed for finding expressions for the free energies in model systems of the BCS type which are exact for $V \to \infty$. Based on an analysis and generalization of the papers, it was possible to formulate a new principle — the minimax principle [12] — for an entire class of model problems in statistical physics.

$$n_f - n_{-f} = \begin{cases} 1 \\ 0 \\ -1 \end{cases}.$$

Let us use K_0, K_-, K_+, respectively to denote the ensemble of subscripts f for which

$$\begin{aligned} &(n_f - n_{-f})\,\Phi_{\mathcal{H}} = 0, && f \in K_0; \\ &(n_f - n_{-f} - 1)\,\Phi_{\mathcal{H}} = 0, && f \in K_+; \\ &(n_f - n_{-f} + 1)\,\Phi_{\mathcal{H}} = 0, && f \in K_-. \end{aligned}$$

This assumption can be reduced to the proposition that the sets K_+, K_- are not empty and that*

$$\langle \Phi^*_{\mathcal{H}}\, \mathcal{H}\, \Phi_{\mathcal{H}} \rangle \leqslant \langle \varphi^*\, \mathcal{H}\, \varphi \rangle$$

for any function φ.

Further we shall require that φ satisfy the additional conditions

$$(n_f - n_{-f})\,\varphi = 0. \tag{2.3}$$

Let us note now that if $f \in K_+$, then $n_f = 1$, $n_{-f} = 0$, while if $f \in K_-$, then $n_f = 0$; $n_{-f} = 1$. Therefore, $\Phi_{\mathcal{H}}$ may be represented in the form of the direct product

$$\Phi_{\mathcal{H}} = \Phi_{K_0}\, \Phi_{K_+}\, \Phi_{K_-},$$

where

$$\Phi_{K_+} = \prod_{f \in K_+} \delta(n_f - 1)\,\delta(n_{-f}); \quad \Phi_{K_-} = \prod_{f \in K_-} \delta(n_f)\,\delta(n_{-f} - 1),$$

while Φ_{K_0} is a function of only those n_f for which $f \in K_0$:

$$\Phi_{K_0} = F(\ldots n_f \ldots); \quad \langle \Phi^+_{K_0} \Phi_{K_0} \rangle = 1, \quad f \in K_0.$$

Let us note further that

$$a_{-f}\, a_f\, \delta(n_f - 1)\,\delta(n_{-f}) = 0; \quad a_{-f}\, a_f\, \delta(n_f)\,\delta(n_{-f} - 1) = 0;$$

$$a^+_f\, a^+_{-f}\, \delta(n_f - 1)\,\delta(n_{-f}) = 0; \quad a^+_f\, a^+_{-f}\, \delta(n_f)\,\delta(n_{-f} - 1) = 0,$$

and therefore that

$$a_{-f}\, a_f\, \Phi_{K_+} \Phi_{K_-} = 0; \quad a^+_f\, a^+_{-f}\, \Phi_{K_+} \Phi_{K_-} = 0,$$

if $f \in K_+$ for K_-. Consequently,

$$\mathcal{H}\,\Phi_{\mathcal{H}} = \Big\{ \sum_{f \in K_+} T(f) + \sum_{f \in K_-} T(f) + \sum_{f \in K_0} T(f)\, n_f - \frac{\nu}{2} \sum_{f \in K_0} \lambda(f)\,(a_{-f}\, a_f + a^+_f\, a^+_{-f})$$

$$- \frac{1}{2V} \sum_{f \in K_0} \sum_{f' \in K_0} \lambda(f)\,\lambda(f')\, a^+_f\, a^+_{-f}\, a_{-f'}\, a_{f'} \Big\}\, \Phi_{\mathcal{H}}.$$

And thus,

$$\langle \Phi^*_{\mathcal{H}}\, \mathcal{H}\, \Phi_{\mathcal{H}} \rangle = \sum_{f \in K_+} T(f) + \sum_{f \in K_-} T(f) + \Big\langle \Phi^*_{K_0} \Big\{ \sum_{f \in K_0} T(f)\, n_f - \frac{\nu}{2} \sum_{f \in K_0} \lambda(f)\,(a_{-f}\, a_f + a^+_f\, a^+_{-f})$$

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* The symbol $< \Phi^* \Psi >$ will be used to denote the scalar product of the functions Φ and Ψ.

$$-\frac{1}{2V}\sum_{f\in K_0}\sum_{f'\in K_0}\lambda(f)\lambda(f')a_f^+a_{-f}^+a_{-f'}a_{f'}\Big\}\Phi_{K_0}\Big\rangle.$$

Let us now partition the set $K_+ + K_-$ into two sets

$$K_+ + K_- = Q_+ + Q_-$$

in such a way that Q_+ will include those subscripts f from $K_+ + K_-$ for which $T(f) \geq 0$, while Q_- will include those subscripts from $K_+ + K_-$ for which $T(f) < 0$. In view of the symmetry of $T(f) = T(-f)$ the subscript f will always be included in Q_+ and Q_- simultaneously with $-f$.

We have

$$\langle\Phi_{\mathscr{H}}^*\mathscr{H}\Phi_{\mathscr{H}}\rangle = \sum_{f\in Q_+}|T(f)| - \sum_{f\in Q_-}|T(f)|$$

$$+\Big\langle\Phi_{K_0}^*\Big\{\sum_{f\in K_0}T(f)n_f - \frac{\nu}{2}\sum_{f\in K_0}\lambda(f)(a_{-f}a_f + a_f^+a_{-f}^+)$$

$$-\frac{1}{2V}\sum_{f\in K_0}\sum_{f'\in K_0}\lambda(f)\lambda(f')a_f^+a_{-f}^+a_{-f'}a_{f'}\Big\}\Phi_{K_0}\Big\rangle.$$

Let us now construct the function φ likewise in the form of a simple product, having placed

$$\varphi = \Phi_{K_0}\Phi_{Q_+}\Phi_{Q_-},$$

where

$$\Phi_{Q_+} = \prod_{f\in Q_+}\delta(n_f)\,\delta(n_{-f});\qquad \Phi_{Q_-} = \prod_{f\in Q_-}\delta(n_f - 1)\,\delta(n_{-f} - 1).$$

(Here it is precisely essential that f belong to Q_+ or Q_- simultaneously with $-f$.) For such a function

$$\langle\varphi^*\mathscr{H}\varphi\rangle = -2\sum_{f\in Q_-}|T(f)| + \Big\langle\Phi_{K_0}^*\Big\{\sum_{f\in K_0}T(f)n_f - \frac{\nu}{2}\sum_{f\in K_0}\lambda(f)(a_{-f}a_f + a_f^+a_{-f}^+)$$

$$-\frac{1}{2V}\sum_{f\in K_0}\sum_{f'\in K_0}\lambda(f)\lambda(f')a_f^+a_{-f}^+a_{-f'}a_{f'}\Big\}\Phi_{K_0}\Big\rangle - \frac{1}{2V}\sum_{f\in Q_-}\lambda^2(f).$$

As is evident,

$$\langle\Phi_{\mathscr{H}}^*\mathscr{H}\Phi_{\mathscr{H}}\rangle > \langle\varphi^*\mathscr{H}\varphi\rangle.$$

On the other hand, the method of construction of φ satisfies all of the additional conditions (3), and we have arrived at a contradiction with Eq. (2). Thus, our statement has been proved. From the statement (2.2) it follows, in particular, that the total momentum for $\Phi_{\mathscr{H}}$ is equal to zero:

$$\sum_{\mathbf{f}}\mathbf{f}n_{\mathbf{f}}\Phi_{\mathscr{H}} = \frac{1}{2}\sum_{\mathbf{f}}\mathbf{f}(n_f - n_{-f})\Phi_{\mathscr{H}} = 0. \tag{2.4}$$

As is evident from what has been said earlier, the eigenfunction Φ for the least eigenvalue $\mathscr{H}$ may always be sought in the class of functions φ which are governed by the additional conditions (2.3). Let us note that for this special class of φ satisfying the conditions (2.3) the Hamiltonian $\mathscr{H}$ may be expressed in terms of Pauli amplitudes.

Let us consider the operators

$$b_f = a_{-f}a_f;\quad b_f^+ = a_f^+a_{-f}^+.$$

Independently of the additional conditions, we have

$$b_f b_{f'} = b_{f'} b_f; \quad b_f^+ b_{f'}^+ = b_{f'}^+ b_f^+; \quad b_f^2 = 0; \quad b_f^{+2} = 0;$$

$$b_f b_{f'}^+ - b_{f'}^+ b_f = 0; \quad f \neq f'.$$

Moreover, with allowance for the additional conditions we have

$$b_f^+ b_f + b_f b_f^+ = n_f n_{-f} + (1 - n_f)(1 - n_{-f}) = 1,$$

since n_f and n_{-f} are simultaneously either both equal to zero or both equal to unity.

Thus, in the class (2.3) investigated the operators b_f, b_f^+ are Pauli amplitudes. In this class of functions the Hamiltonian (1.2) has the form

$$\mathcal{H} = 2\left\{\sum_{f>0} T(f)\, b_f^+ b_f - \frac{\nu}{2}\sum_{f>0} \lambda(f)\{b_f + b_f^+\} - \frac{1}{V}\sum_{\substack{f>0\\ f'>0}} \lambda(f)\lambda(f')\, b_f^+ b_{f'}\right\}. \tag{2.5}$$

We isolated the class of subscripts $f > 0$ so that all operators b_f would be different, since $b_f = -b_{-f}$. A Hamiltonian of this type was considered in our previous paper [5].

§ 3. The Upper Estimate of the Eigenvalue of the Hamiltonian (1.2)

Let us now consider the problem of the upper estimate of the minimal eigenvalue of the Hamiltonian $\mathcal{H}$. We shall start from the representation of the Hamiltonian $\mathcal{H}$ in the form (1.2). Let us use $E_{\mathcal{H}}$ to denote the least eigenvalue of the Hamiltonian $\mathcal{H}$ (1.2) and $E_0(\sigma)$ to denote the least eigenvalue of the Hamiltonian $\mathcal{H}_0$ (1.4). Note that the operator $\mathcal{H}_1 < 0$, and therefore the minimal eigenvalue of the Hamiltonian $\mathcal{H}_0$, is larger than the minimal eigenvalue of the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$:

$$E_0(\sigma) \geqslant E_{\mathcal{H}} \tag{3.1}$$

for any σ. Thus, the minimal eigenvalues of the Hamiltonian $\mathcal{H}_0$ majorize the minimal eigenvalue of $\mathcal{H}$. The best estimate is obtained for σ which yields $\min E_0(\sigma)$.

Let us now go over to calculating the eigenvalues of the Hamiltonian $\mathcal{H}_0$. Carrying out the appropriate canonical transformation which diagonalizes the quadratic form of $\mathcal{H}_0$ (1.4), we obtain the identity

$$\mathcal{H}_0 = \sum_f \sqrt{\lambda^2(f)(\nu+\sigma^*)(\nu+\sigma) + T^2(f)}\,(a_f^+ u_f + a_{-f} v_f^*)(u_f a_f + v_f a_{-f}^+)$$

$$+ \frac{1}{2} V\left\{\sigma^*\sigma - \frac{1}{V}\sum_f \left[\sqrt{\lambda^2(f)(\nu+\sigma^*)(\nu+\sigma) + T^2(f)} - T(f)\right]\right\}, \tag{3.2}$$

where

$$\left.\begin{aligned} u_f &= \frac{1}{\sqrt{2}}\sqrt{1 + \frac{T(f)}{\sqrt{\lambda^2(f)(\nu+\sigma^*)(\nu+\sigma) + T^2(f)}}}, \\ v_f &= \frac{-\varepsilon(f)}{\sqrt{2}}\sqrt{1 - \frac{T(f)}{\sqrt{\lambda^2(f)(\nu+\sigma^*)(\nu+\sigma) + T^2(f)}}}\;\frac{\sigma+\nu}{|\sigma+\nu|}. \end{aligned}\right\} \tag{3.3}$$

Here

$$\lambda(f) = \varepsilon(f)\,|\lambda(f)|; \quad \varepsilon(f) = \operatorname{sign}\lambda(f). \tag{3.4}$$

Obviously,

$$u(-f)=u(f);\quad v(-f)=-v(f);\quad u^2+|v|^2=1, \tag{3.5}$$

where u is real, and v is complex.

From this it is evident that the amplitudes

$$\left.\begin{aligned}\alpha_f&=u_f\,a_f+v_f\,a^+_{-f};\\ \alpha_f^+&=u_f\,a_f^++v_f^*\,a_{-f}\end{aligned}\right\} \tag{3.6}$$

are fermion amplitudes. Consequently, the expression for $\mathcal{H}_0$ may be rewritten in the form

$$\begin{aligned}\mathcal{H}_0=&\sum\sqrt{\lambda^2(f)(\nu+\sigma^*)(\nu+\sigma)+T^2(f)}\;\alpha_f^+\alpha_f\\ &+\frac{1}{2}V\left\{\sigma^*\sigma-\frac{1}{V}\sum_f\left[\sqrt{(\nu+\sigma^*)(\nu+\sigma)\lambda^2(f)+T^2(f)}-T(f)\right]\right\}.\end{aligned} \tag{3.7}$$

It is obvious that min $\mathcal{H}_0$ will be reached for the occupancy numbers $\overset{+}{\alpha}_f\alpha_f=0$. Consequently, for the ground-state energy of the Hamiltonian $\mathcal{H}_0$ we obtain

$$E_0(\sigma)=\frac{1}{2}V\left\{\sigma^*\sigma-\frac{1}{V}\sum_f\left[\sqrt{\lambda^2(f)(\nu+\sigma^*)(\nu+\sigma)+T^2(f)}-T(f)\right]\right\}. \tag{3.8}$$

In order to improve the upper estimate of $E_{\mathcal{H}}$ it is necessary to take $E_0(\sigma)$.

Let us consider the following cases separately:

1. The case $\nu=0$. Let us place $x=\sigma^*\sigma>0$; then $E_0(\sigma)=1/2\,VF(\sigma^*\sigma)$, where

$$F(x)=x-\frac{1}{V}\sum_f\left\{\sqrt{\lambda^2(f)x+T^2(f)}-T(f)\right\}.$$

In this case, as is evident from the minimum condition, one may determine only the modulus of σ but not its phase. We have

$$F'(x)=1-\frac{1}{2V}\sum_f\frac{\lambda^2(f)}{\sqrt{\lambda^2(f)x+T^2(f)}};$$

$$F''(x)=\frac{1}{4V}\sum_f\frac{\lambda^4(f)}{\left(\sqrt{\lambda^2(f)x+T^2(f)}\right)^3}.$$

As is evident, $F''(x)>0$ in the interval $0\le x\le\infty$, and therefore, $F'(x)$ may have no more than one root in this interval. Taking account of the properties of the functions $\lambda(f)$ and $T(f)$ (see §1), we shall have $F'(0)<0$; $F'(\infty)>0$. And, consequently, in the interval $0<x<\infty$ there exists a single solitary solution of the equation $F'(x)=0$; it is this solution which realizes the absolute minimum. Thus, we finally have

$$\frac{V}{2}\min F(x)\geqslant E_{\mathcal{H}}\qquad(0<x<\infty). \tag{3.9}$$

2. The case $\nu>0$. Let us place $(\nu+\sigma^*)(\nu+\sigma)=x$ (it is obvious that $x>0$), and note that

$$\sigma^*\sigma=x+\nu^2-\nu(\sigma+\nu+\sigma^*+\nu)=(\sqrt{x}-\nu)^2+\nu\left\{2\sqrt{x}-(\sigma+\nu+\sigma^*+\nu)\right\}.$$

Here the root, as always, is assigned the sign +. Then

$$\sigma+\nu=\sqrt{x}\,e^{i\varphi};\quad\sigma^*+\nu=\sqrt{x}\,e^{-i\varphi}$$

and

$$\sigma^*\sigma = (\sqrt{x} - \nu)^2 + 2\nu\sqrt{x}\,(1 - \cos\varphi).$$

Therefore,

$$E_0(\sigma) = \frac{V}{2} F(x) + V\nu\sqrt{x}\,(1 - \cos\varphi), \tag{3.10}$$

where

$$F(x) = (\sqrt{x} - \nu)^2 - \frac{1}{V} \sum_f \{\sqrt{\lambda^2(f)\,x + T^2(f)} - T(f)\}.$$

Further we have

$$F'(x) = 1 - \frac{\nu}{\sqrt{x}} - \frac{1}{2V} \sum_f \frac{\lambda^2(f)}{\sqrt{\lambda^2(f)\,x + T^2(f)}};$$

$$F''(x) = \frac{\nu}{2x^{3/2}} + \frac{1}{4V} \sum_f \frac{\lambda^4(f)}{(\lambda^2(f)\,x + T^2(f))^{3/2}}.$$

Since $F''(x) > 0$, we see that $F'(x)$ may not have more than one root in the interval $[0, \infty]$. But $F'(0) = -\infty$, $F'(\infty) = 1$. Therefore, a x_0 exist in the interval $0 < x_0 < \infty$, for which $F'(x_0) = 0$. It is precisely for this value of x_0 that the function $F(x)$ has an absolute minimum.

From (3.10) it is evident that the only possible choice of σ corresponding to the absolute minimum will be

$$x = x_0, \qquad \varphi = 0. \tag{3.11}$$

Thus, we have

$$\sigma + \nu = \sqrt{x}, \quad \sigma = \sqrt{x} - \nu.$$

Thus, in the case given ($\nu > 0$) the phase of σ can also be determined. As we can see, σ must be real. We likewise have

$$\frac{V}{2} \min F(x) \geqslant E_{\mathcal{H}} \qquad (0 < x < \infty). \tag{3.12}$$

The simple concepts used in [2] show that in Eq. (1.3) the additional term $\mathcal{H} - \mathcal{H}_0 = \mathcal{H}_1$ is ineffective for $V \to \infty$. However, the rigorous establishment of this property is complicated by the fact that we have only the upper estimate for $E_{\mathcal{H}}$ and do not have an analogous lower estimate. In general, it would be desirable to cancel the term

$$\Big(\sum_f \lambda(f)\, a_f^+ a_{-f}^+ - V\sigma^*\Big)\Big(\sum_f \lambda(f)\, a_{-f}\, a_f - V\sigma\Big).$$

This could be achieved by making σ the operator

$$L = \frac{1}{V} \sum_f \lambda(f)\, a_{-f}\, a_f$$

rather than a number. But with an operator one cannot perform canonical transformations from a-fermions to α-fermions. However, we shall try to generalize the identity (3.2) for such a case. One need merely establish the order of the operators correctly. It is precisely in this way that we prove the theorem to the effect that using $\mathcal{H}_0$ one may obtain the asymptotically exact solution for $\mathcal{H}$ when $V \to \infty$.

§ 4. The Lower Estimate of the Eigenvalue of the Hamiltonian

In order to obtain the lower estimate of the Hamiltonian (1.2) we first of all generalize the identity (1.3) in such a way that the term $\mathcal{H}_1$ (1.5) vanishes. This may be done by treating σ as a certain operator L rather than as a c-number:

$$L = \frac{1}{V} \sum_f a_{-f}\, a_f\, \lambda(f). \tag{4.1}$$

Instead of the c-number $(\nu + \sigma^*)(\nu + \sigma)$ we introduce the operators

$$K = (L + \nu)(L^+ + \nu) + \beta^2, \quad \widetilde{K} = (L^+ + \nu)(L + \nu) + \beta^2, \tag{4.2}$$

where β is a certain constant.

We now introduce the operators

$$p_f = \frac{1}{\sqrt{2}} \sqrt{\sqrt{K\lambda^2(f) + T^2(f)} + T(f)}\,; \quad p_f = p_f^+;$$

$$q_f = -\frac{\varepsilon(f)}{\sqrt{2}} \sqrt{\sqrt{K\lambda^2(f) + T^2(f)} - T(f)} \cdot \frac{1}{\sqrt{K}}\,(L + \nu). \tag{4.3}$$

Obviously,

$$p_f q_f = -\frac{\lambda(f)}{2}(L + \nu); \tag{4.4}$$

$$p_f^2 = \frac{1}{2}\{\sqrt{K\lambda^2(f) + T^2(f)} + T(f)\}; \tag{4.5}$$

$$q_f^+ q_f = (L^+ + \nu)\,\frac{1}{2K}\{\sqrt{K\lambda^2(f) + T^2(f)} - T(f)\}(L + \nu). \tag{4.6}$$

Taking account of the fact that for any operator ξ the identity

$$\xi^+ F(\xi\xi^+)\xi = \xi^+\xi F(\xi^+\xi) \tag{4.7}$$

is valid, Eq. (4.8) can be written in the form

$$q_f^+ q_f = (L^+ + \nu)(L + \nu)\frac{1}{2\widetilde{K}}\left\{\sqrt{\widetilde{K}\lambda^2(f) + T^2(f)} - T(f)\right\}$$

$$= \frac{1}{2}\left\{\sqrt{\widetilde{K}\lambda^2(f) + T^2(f)} - T(f)\right\} - \frac{\beta^2}{2\widetilde{K}}\left\{\sqrt{\widetilde{K}\lambda^2(f) + T^2(f)} - T(f)\right\}. \tag{4.8}$$

Going on to apply Lemma II [see Appendix, Eqs. (A1.9)-(A1.10)], we write

$$q_f^+ q_f = \frac{1}{2}\left\{\sqrt{\lambda^2(f)\left(K + \frac{2s}{V}\right) + T^2(f)} - T(f)\right\}$$

$$-\frac{1}{2}\left\{\sqrt{\lambda^2(f)\left(K + \frac{2s}{V}\right) + T^2(f)} - \sqrt{\lambda^2(f)\widetilde{K} + T^2(f)}\right\}$$

$$-\frac{\beta^2}{2\widetilde{K}}\left\{\sqrt{\widetilde{K}\lambda^2(f) + T^2(f)} - T(f)\right\}. \tag{4.9}$$

Note that the second term on the right is not negative, while s is the upper estimate of the expression $\frac{1}{V}\sum_f |\lambda(f)|^2$:

$$\frac{1}{V}\sum_f |\lambda(f)|^2 \leqslant s. \tag{4.10}$$

Moreover, we have

$$p_f^2 = \frac{1}{2}\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}+T(f)\right\}$$

$$-\frac{1}{2}\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}-\sqrt{K\lambda^2(f)+T^2(f)}\right\}. \tag{4.11}$$

Let us now consider the equation

$$\Omega = \sum_f (a_f^+ p_f + a_f q_f^+)(p_f a_f + q_f a_{-f}^+). \tag{4.12}$$

Making use of the equation $q_f^+ q_f = q_{-f}^+ q_{-f}$ and (4.4), we obtain

$$\Omega = \sum_f a_f^+ p_f^2 a_f + \sum_f a_f q_f^+ q_f a_f^+$$

$$-\sum_f \frac{\lambda(f)}{2}\{(L^+ + \nu)\, a_{-f} a_f + a_f^+ a_{-f}^+ (L+\nu)\} + R_1, \tag{4.13}$$

where

$$R_1 = \sum_f \frac{\lambda(f)}{2}\{(L^+ a_{-f} - a_{-f} L^+)\, a_f + a_f^+ (a_{-f}^+ L - L a_{-f}^+)\}. \tag{4.14}$$

Note that

$$\sum_f \frac{\lambda(f)}{2}\{(L^+ + \nu)\, a_{-f} a_f + a_f^+ a_{-f}^+ (L+\nu)\} = VL^+L + \frac{V}{2}(\nu L + \nu L^+) \tag{4.15}$$

and, consequently,

$$\Omega + \frac{V}{2}L^+L - \sum_f a_f^+ p_f^2 a_f - \sum_f a_f q_f^+ q_f a_f^+ = -\frac{V}{2}\{L^+L + \nu(L+L^+)\} + R_1. \tag{4.16}$$

Or, by virtue of (4.9) and (4.11), we have

$$\sum_f \{a_f^+ p_f + a_{-f} q_f^+\}\{p_f a_f + q_f a_{-f}^+\}$$

$$+\frac{1}{2}\sum_f a_f^+ \left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}-\sqrt{K\lambda^2(f)+T^2(f)}\right\} a_f$$

$$+\frac{1}{2}\sum_f a_f \left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}-\sqrt{\widetilde{K}\lambda^2(f)+T^2(f)}\right\} a_f^+$$

$$+\frac{1}{2}\sum_f a_f \frac{\beta^2}{2\widetilde{K}}\left\{\sqrt{\widetilde{K}\lambda^2(f)+T^2(f)}-T(f)\right\} a_f^+$$

$$-\frac{1}{2}\sum_f a_f^+\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}+T(f)\right\}a_f$$

$$-\frac{1}{2}\sum_f a_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}-T(f)\right\}a_f^+ + \frac{V}{2}L^+L$$

$$=-\frac{V}{2}\{L^+L+\nu(L+L^+)\}+R_1. \tag{4.17}$$

Let us introduce the notation

$$\Delta_1=\frac{1}{2}\sum_f a_f^+\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-\sqrt{K\lambda^2+T^2}\right\}a_f; \tag{4.18}$$

$$\Delta_2=\frac{1}{2}\sum_f a_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-\sqrt{\widetilde{K}\lambda^2+T^2}\right\}a_f^+; \tag{4.19}$$

$$\Delta_3=\frac{1}{2}\sum_f a_f\frac{\beta^2}{2\widetilde{K}}\left\{\sqrt{\widetilde{K}\lambda^2+T^2}-T\right\}a_f^+. \tag{4.20}$$

Then, by virtue of Lemma II [see Appendix, Eqs. (A1.9), (A1.10)]

$$\Omega\geqslant 0;\quad \Delta_1\geqslant 0;\quad \Delta_2\geqslant 0;\quad \Delta_3\geqslant 0. \tag{4.21}$$

Thus,

$$\Omega+\Delta_1+\Delta_2+\Delta_3-\frac{1}{2}\sum_f a_f^+\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}+T\right\}a_f$$

$$-\frac{1}{2}\sum_f a_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-T\right\}a_f^+ + \frac{V}{2}L^+L$$

$$=-\frac{V}{2}\{L^+L+\nu(L+L^+)\}+R_1. \tag{4.22}$$

Let us set

$$R_2=\frac{1}{2}\sum_f a_f^+\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}\,a_f-a_f\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}\right\}; \tag{4.23}$$

$$R_3=\frac{1}{2}\sum_f a_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}\,a_f^+-a_f^+\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}\right\}. \tag{4.24}$$

Then

$$\Omega+\Delta_1+\Delta_2+\Delta_3-R_2-R_3+\frac{V}{2}L^+L-\frac{1}{2}\sum_f a_f^+a_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}+T\right\}$$

$$-\frac{1}{2}\sum_f a_f a_f^+\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-T\right\}=-\frac{V}{2}\{L^+L+\nu(L+L^+)\}+R_1. \tag{4.25}$$

But

$$\frac{1}{2}\sum_f a_f^+ a_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}+T\right\}$$

$$+\frac{1}{2}\sum_f a_f a_f^+\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-T\right\}$$

$$=\frac{1}{2}\sum_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-T\right\}+\sum_f T(f)\,a_f^+ a_f. \tag{4.26}$$

Consequently,

$$\Omega+\Delta_1+\Delta_2+\Delta_3-R_1-R_2-R_3+\frac{V}{2}(L^+L-LL^+)$$

$$+\frac{1}{2}V\left[LL^+-\frac{1}{V}\sum_f\left\{\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2+T^2}-T\right\}\right]$$

$$=\sum_f T(f)\,a_f^+a_f-\frac{V}{2}\{L^+L+\nu(L+L^+)\}$$

$$=\sum_f T(f)\,a_f^+a_f-\nu\sum_f\frac{\lambda(f)}{2}(a_{-f}a_f+a_f^+a_{-f}^+)$$

$$-\frac{1}{2V}\sum_{ff'}\lambda(f)\lambda(f')\,a_f^+a_{-f}^+a_{-f'}a_{f'}=\mathcal{H}. \tag{4.27}$$

Thus, we finally will have

$$\mathcal{H}=\frac{1}{2}V\left\{LL^+-\frac{1}{V}\sum_f\left[\sqrt{\left(K+\frac{2s}{V}\right)\lambda^2(f)+T^2(f)}-T(f)\right]\right\}$$

$$+\Omega+\Delta_1+\Delta_2+\Delta_3-R_1-R_2-R_3+\frac{V}{2}(L^+L-LL^+). \tag{4.28}$$

Equation (4.28) represents an identical transformation of the Hamiltonian (1.2). The first term of Eq. (4.28) will be treated as the principal term; as far as the terms R_1, R_2, and R_3 are concerned, we shall show that they are asymptotically small, while the terms Ω, Δ_1, Δ_2, and Δ_3 will be dropped. Since they are positive (4.21), we shall obtain the lower estimate for $\mathcal{H}$.

It can easily be shown that when (2.2) is considered, we have

$$-R_1+\frac{V}{2}(L^+L-LL^+)=-\frac{1}{V}\sum_f\lambda^2(f), \tag{4.29}$$

where in accordance with (4.10), we have $\frac{1}{V}\sum_f \lambda^2(f) \leqslant s$. Further, by virtue of Lemma IV (inequality A1.30)

$$|R_2|+|R_3| \leqslant C, \tag{4.30}$$

where

$$C=\frac{4}{\pi}\cdot\frac{1}{V}\sum_f |\lambda(f)|^2\left[1+\frac{|\lambda(f)|\left(\frac{1}{V}\sum|\lambda(f)|+V\right)}{2\frac{1}{V}\sum|\nu(f)|^2+V\frac{T^2(f)}{\lambda^2(f)}}\right]\int_0^\infty \frac{\sqrt{t}}{(1+t)^2}\,dt. \tag{4.31}$$

Thus, for any normalized function Φ the inequality

$$\langle\Phi^*\mathcal{H}\Phi\rangle \geqslant -(s+C)+\frac{1}{2}V\left\langle\Phi^*\left(LL^+-\frac{1}{V}\right.\right.$$
$$\left.\left.\times\sum_f\left[\sqrt{\left\{(L+\nu)(L^++\nu)+\beta^2+\frac{2s}{V}\right\}\lambda^2+T^2(f)}-T(f)\right]\right)\Phi\right\rangle$$

is valid by virtue of (4.21). But s and C do not depend on β. Therefore, performing a transition in the limit $\beta\to 0$, we find

$$\langle\Phi^*\mathcal{H}\Phi\rangle \geqslant -(2s+C)+\frac{1}{2}V\left\langle\Phi^*\left\{LL^*+\frac{2s}{V}\right.\right.$$
$$\left.\left.-\frac{1}{V}\sum_f\left[\sqrt{\left\{(L+\nu)(L^++\nu)+\frac{2s}{V}\right\}\lambda^2(f)+T^2(f)}-T(f)\right]\right\}\Phi\right\rangle. \tag{4.32}$$

Further, we have

$$\left.\begin{aligned} LL^+ &= (L+\nu)(L^++\nu)-\nu\{L+\nu+L^++\nu\}+\nu^2;\\ LL^++\frac{2s}{V} &= \left\{(L+\nu)(L^++\nu)+\frac{2s}{V}\right\}-\nu\{(L+\nu+L^++\nu)\}+\nu^2.\end{aligned}\right\} \tag{4.33}$$

We set

$$(L+\nu)(L^++\nu)+\frac{2s}{V}=X. \tag{4.34}$$

Then

$$LL^++\frac{2s}{V}=(\sqrt{X}-\nu)^2+\nu\{2\sqrt{X}-(L+\nu)-(L^++\nu)\}. \tag{4.35}$$

But according to Lemma I (see A1.1, A1.2), having placed $\xi = L+\nu$; $\xi^+ = L^++\nu$ in the inequality, we obtain

$$2\sqrt{(L+\nu)(L^++\nu)+\frac{s}{V}}-(L+\nu)-(L^++\nu)\geqslant 0, \tag{4.36}$$

and the more so since

$$2\sqrt{X}-(L+\nu)+(L^++\nu)\geqslant 0. \tag{4.37}$$

Let us consider the function F(x) (3.11):

$$F(x)=(\sqrt{x}-\nu)^2-\frac{1}{V}\sum_f\left[\sqrt{x\lambda^2(f)+T^2(f)}-T(f)\right].$$

Then Eq. (4.32) may be written in the form

$$\langle \Phi^* \mathcal{H} \Phi \rangle \geqslant -(2s+C) + \frac{1}{2} V \langle \Phi^* F(X) \Phi \rangle$$

$$+ V \frac{\nu}{2} \langle \Phi^* \{ 2\sqrt{X} - (L+\nu+L^+ +\nu) \} \Phi \rangle$$

$$\geqslant -(2s+C) + \frac{1}{2} V \langle \Phi^* F(X) \Phi \rangle, \tag{4.38}$$

where X is an operator which is determined by Eq. (4.34).

Assume $E_{\mathcal{H}}$ is the least eigenvalue of $\mathcal{H}$; $\Phi_{\mathcal{H}}$ is the corresponding eigenfunction. Assume further that $E_{\mathcal{H}_0}$ is the least eigenvalue of $\mathcal{H}_0$; then with allowance for (3.11), we have $E_{\mathcal{H}_0} = \frac{V}{2} \min F(x)$. Assume the absolute minimum can be reached for F(x) for

$$x = x_0 = C^2. \tag{4.39}$$

We have

$$\frac{V}{2} F(C^2) \geqslant E_{\mathcal{H}} = \langle \Phi^*_{\mathcal{H}} \mathcal{H} \Phi_{\mathcal{H}} \rangle \geqslant -(2s+C)$$

$$+ \frac{1}{2} V \langle \Phi^* F(X) \Phi \rangle \geqslant -(2s+C) + \frac{V}{2} F(C^2). \tag{4.40}$$

From this, taking account of the fact that the energy of the system is proportional to the volume of the system, we obtain the final estimate for the eigenvalues of the Hamiltonian $\mathcal{H}$ (1.2):

$$0 \leqslant \frac{E_{\mathcal{H}_0} - E_{\mathcal{H}}}{V} \leqslant \frac{2s+C}{V}. \tag{4.41}$$

Note that C (4.31) and s remain finite for $V \to \infty$ in accordance with the conditions of §1. Therefore, the difference between the eigenvalues of the approximate Hamiltonian $\mathcal{H}_0$ (1.4) and the exact Hamiltonian $\mathcal{H}$ (1.2), normalized to the system volume, decreases as 1/V for $V \to \infty$. Thus, the solution of the approximate Hamiltonian $\mathcal{H}_0$ (1.4) yields an asymptotically exact solution of the Hamiltonian $\mathcal{H}$ (1.2) for $V \to \infty$.

Let us now show that the operator X (4.34) may be treated as a c-number with asymptotic accuracy (i.e., accurate to 1/V). For this purpose we choose any normalized function Φ such that

$$\langle \Phi^* \mathcal{H} \Phi \rangle - E_{\mathcal{H}} \leqslant C_1 = \text{const}. \tag{4.42}$$

Now on the basis of (4.38), (4.40), and (4.42), we have

$$\langle \Phi^* (F(X) - F(C^2)) \Phi \rangle + \nu \langle \Phi^* (2\sqrt{X} - (L+\nu+L^+ +\nu)) \Phi \rangle \leqslant \frac{l}{V}; \tag{4.43}$$

$$l = 2(2s+C+C_1).$$

Note that both terms in the left side of (4.43) are positive. In particular, in view of the positiveness of the second term in the left side of the inequality (4.43) (see Lemma I, Eqs. A1.1 and A1.2), we obtain

$$\langle \Phi^* (F(X) - F(C^2)) \Phi \rangle \leqslant \frac{l}{V}, \tag{4.44}$$

but

$$F(X)-F(C^2)=\frac{1}{2}F''(\xi)(X-C^2)^2; \tag{4.45}$$

$$F''(x)=\frac{\nu}{2x^{3/2}}+\frac{1}{4}\cdot\frac{1}{V}\sum_f\frac{\lambda^4(f)}{(x\lambda^2(f)+T^2(f))^{3/2}},$$

$$\frac{1}{2}F''(\xi)\geqslant\alpha=\text{const}>0. \tag{4.46}$$

From this we obtain

$$\langle\Phi^*|X-C^2|^2\Phi\rangle\leqslant\frac{l}{\alpha V}. \tag{4.47}$$

From (4.46) it follows that the operator X may be treated as a c-number with asymptotic accuracy.

For the case $\nu>0$ one may obtain more complete information on the mathematical expectation of the operators L, L^+. Specifically, we shall show that the mean-square deviation of the operator L from the quantity C (4.39) is asymptotically small for $V\to\infty$. We have the obvious inequality

$$(\sqrt{X}-C)^2=\frac{(X-C^2)^2}{(\sqrt{X}+C)^2}\leqslant\frac{1}{C^2}(X-C^2)^2. \tag{4.48}$$

From this, making use of (4.47), we obtain

$$\langle\Phi^*(\sqrt{X}-C)^2\Phi\rangle\leqslant\frac{l}{\alpha C^2V}. \tag{4.49}$$

Define

$$\langle\Phi^*\sqrt{X}\Phi\rangle=C_0. \tag{4.50}$$

Then for consideration of (4.49), we have

$$\langle\Phi^*(\sqrt{X}-C_0)^2\Phi\rangle\leqslant\langle\Phi^*(\sqrt{X}-C)^2\Phi\rangle\leqslant\frac{l}{\alpha C^2V}. \tag{4.51}$$

This is actually so, since

$$\langle\Phi^*(\sqrt{X}-C)^2\Phi\rangle=(C-C_0)^2+\langle\Phi^*(\sqrt{X}-C_0)^2\Phi\rangle. \tag{4.52}$$

The bound for the mathematical expectation of the operator X derives from the estimate (4.51) and is given by the relationship

$$\langle\Phi^*X\Phi\rangle-C_0^2\leqslant\frac{l}{\alpha C^2V}. \tag{4.53}$$

Finally, from (4.51) and (4.52), we obtain the estimate for the difference $(C-C_0)^2$:

$$(C-C_0)^2\leqslant\frac{l}{\alpha C^2V}. \tag{4.54}$$

Assume now that

$$\xi=L+\nu;\ \xi^+=L^++\nu; \tag{4.55}$$

then for the mean-square deviation of ξ from C_0 we have the following result when (4.34) is considered:

$$\langle\Phi^*(C_0-\xi)(C_0-\xi^+)\Phi\rangle\leqslant C_0^2+\langle\Phi^*X\Phi\rangle-C_0\langle\Phi^*(\xi+\xi^+)\Phi\rangle. \tag{4.56}$$

Then, using (4.53) and (4.43), we obtain

$$\langle \Phi^* (C_0-\xi)(C_0-\xi^+)\Phi\rangle \leqslant 2C_0^2+\frac{l}{\alpha C^2 V}-C_0\langle\Phi^*(\xi+\xi^+)\Phi\rangle$$

$$=\langle\Phi^*\{2\sqrt{X}-(\xi+\xi^+)\}\Phi\rangle C_0+\frac{l}{\alpha C^2V}\leqslant\frac{lC_0}{\nu V}+\frac{l}{\alpha C^2 V}. \tag{4.57}$$

Thus, for the quantity $\langle\Phi^*(C-\xi)\times(C-\xi^+)\Phi\rangle$ of interest to us, we obtain the following estimate:

$$\langle\Phi^*(C-\xi)(C-\xi^+)\Phi\rangle=\langle\Phi^*(C-C_0+C_0-\xi)(C-C_0+C_0-\xi^+)\Phi\rangle$$

$$\leq 2(C-C_0)^2+2\langle\Phi^*(C_0-\xi)(C_0-\xi^+)\Phi\rangle$$

$$\leq\frac{2l}{\alpha C^2V}+\frac{2lC_0}{\nu V}+\frac{2l}{\alpha C^2 V}\leq\frac{\text{const}}{V}=\frac{I}{V} \tag{4.58}$$

$$(I=\text{const}).$$

Note that the bound obtained is valid only for $\nu > 0$, since ν is included in the denominator of the right side of the inequality (4.58).

Let us discuss the results obtained. Assume $\nu=0$. Then, as we have seen, for states having an average energy which is asymptotically close to the least E_H, the operator L^+L is equal to the c-number C^2 with asymptotic accuracy. These states, however, do not have such properties of the operators L and L^+ proper. Let us consider the state Φ_H having the least energy E_H. In general, a case of degeneration may arise such that we shall have not one state Φ_H but a certain linear manifold $\{\Phi_H\}$ of possible states having the same least energy E_H.

Since the operator $N=\Sigma a_f^+ a_f$, which is the total number of particles in the case $\nu=0$, commutates exactly with H, one can always find within the manifold $\{\Phi_H\}$ a state Φ_f' which is such that N takes a certain value N_0. Then it is obvious that

$$\langle\Phi_H^{*\prime}L\Phi_H'\rangle=0;\quad\langle\Phi_H^{*\prime}L^+\Phi_H'\rangle=0.$$

This means that L cannot take even an approximately definite value in the state Φ_H', since otherwise L^+L for this state would turn out to be approximately equal to 0 rather than to C^2.

Let us now consider a manifold $\{\Phi\}$ of states having an energy asymptotically close to E_H. Since L, L^+ approximately commutates with H, it is natural to expect that in $\{\Phi\}$ one may choose a Φ for which L, L^+ take definite values with asymptotic accuracy. And this is the case in reality. For example, the Φ_{H_0}-state having the least energy for H_0 has the property given. In fact, Φ_{H_0} is determined by the relationship $\alpha_f\Phi=0$, where $\alpha_f=u_f\alpha_f-v_f a_{-f}^+$;

$$u_f=\frac{1}{\sqrt{2}}\sqrt{1+\frac{T(f)}{\sqrt{\lambda^2(f)C^2+T^2(f)}}};$$
$$v_f=\frac{\varepsilon(f)}{\sqrt{2}}\sqrt{1-\frac{T(f)}{\sqrt{\lambda^2(f)C^2+T^2(f)}}}.$$

Having expressed L in terms of the fermions α, α^+, we find

$$L=\frac{1}{V}\sum_f\lambda(f)\{u_f^2\,\alpha_{-f}a_f-v_f^2\,\alpha_f^+\alpha_{-f}^+-2u_fv_f\alpha_f^+\alpha_f\}+\frac{1}{V}\sum_f u_fv_f\lambda(f).$$

But

$$\frac{1}{V}\sum_f u_f v_f=\frac{1}{2V}\sum_f\frac{|\lambda(f)|^2C}{\sqrt{\lambda^2(f)C^2+T^2(f)}}=C,$$

and, consequently,

$$\langle \Phi^*_{H_0} (L^+ - C)(L - C) \Phi_{H_0} \rangle \leqslant \frac{\text{const}}{V};$$
$$\langle \Phi^*_{H_0} (L - C)(L^+ - C) \Phi_{H_0} \rangle \leqslant \frac{\text{const}}{V}.$$

For Φ_{H_0}, L and L^+ are approximately equal to C. It is specifically this fact which produced the success of the approximate method in which we replaced the Hamiltonian H with the exact law for conservation of N by H_0 for which N is no longer the exact integral of motion. Now it is likewise evident that the approximate method could be formulated so that the law for conservation of N would not be formally violated. For this purpose it would be necessary to replace the fermions α_f by the amplitude

$$\alpha_f = u_f a_f - v_f \frac{L}{|C|} a^{\pm}_{-f},$$

which satisfy the commutation relations for the Fermi amplitudes with asymptotic accuracy. Then α_f decreases N by one, while α_f^+ increases N by one. Such amplitudes have analogs with the amplitude

$$b_f = \frac{a_0^+}{\sqrt{N_0}} a_f,$$

which were introduced in superfluidity theory [6] during the isolation of the condensate. In general there is a strong analogy between the amplitudes a_0, a_0^+ for a Bose-condensate and the amplitudes L, L^+ in the case considered.

When we include a term with pair sources ($\nu > 0$) in $\mathcal{H}$, the operators L, L^+ immediately begin to take definite values with asymptotic accuracy for the states having an energy close to $\mathcal{H}$. Here we obtain an analogy with the theory of ferromagnetism in an isotropic medium. In the absence of a magnetic field the direction of the magnetization axis is indefinite. For turn-on of a magnetic field which is arbitrarily weak and acts in a definite direction, the magnetization vector is immediately established precisely along this direction. Finally, from the relationships

$$L^+ L \approx C^2 \quad (\nu = 0);$$
$$\nu + L \approx C; \quad \nu + L^+ \approx C \quad (\nu > 0)$$

one can find that the correlation averages

$$\langle \Phi^*_H \ldots a_{f_i}(t_i) \ldots a^+_{f_j}(t_j) \ldots \Phi_H \rangle$$

for the Hamiltonian $\mathcal{H}$ are also asymptotically equal to the corresponding averages for the Hamiltonian $\mathcal{H}_0$. For $\nu > 0$ this applies to all averages of the given type, while for $\nu = 0$ it applies, of course, only to those for which the c-number is equal to the number a^+ (i.e., the averages of those operators which conserve N).

§ 5. The Green's Function for the Case $\nu > 0$

In this section we shall deal with asymptotic estimates for the Green's functions and correlation averages of the case $\nu > 0$. From these estimates it will follow that the solution of the equations for the Green's functions, constructed on the Hamiltonian $\mathcal{H}_0$ (1.4), will be different to an asymptotically small degree from the corresponding solutions for such a model Hamiltonian $\mathcal{H}$ (1.2) for $V \to \infty$.

Let us consider the equations of motion for the operators a_f, a_f^+. Taking account of (1.2), we obtain

$$\left. \begin{aligned} i \frac{da_f}{dt} &= T(f) a_f - \lambda(f) a^{\pm}_{-f} (\nu + L); \\ i \frac{da_f^+}{dt} &= -T(f) a_f^+ + \lambda(f) (\nu + L^+) a_{-f}, \end{aligned} \right\} \qquad (5.1)$$

whence we likewise obtain

$$\left.\begin{aligned} i\frac{da_{-f}}{dt} &= T(f)\,a_{-f} + \lambda(f)\,a_f^+(\nu + L); \\ i\frac{da_{-f}^+}{dt} &= -T(f)\,a_{-f}^+ - \lambda(f)\,(\nu + L^+)\,a_f\,. \end{aligned}\right\} \tag{5.2}$$

Let us set (see Eqs. (3.3) and (3.11))

$$\left.\begin{aligned} u_f &= \frac{1}{\sqrt{2}}\sqrt{1 + \frac{T(f)}{\sqrt{C^2\lambda^2(f) + T^2(f)}}}\,; \\ v_f &= -\frac{\varepsilon(f)}{\sqrt{2}}\sqrt{1 - \frac{T(f)}{\sqrt{C^2\lambda^2(f) + T^2(f)}}} \end{aligned}\right\} \tag{5.3}$$

and let us introduce a new fermion amplitude

$$\alpha_f^+ = u_f a_f^+ + v_f a_{-f}\,. \tag{5.4}$$

We have

$$\begin{aligned} i\frac{d\alpha_f^+}{dt} &= u_f\, i\frac{da_f^+}{dt} + v_f\, i\frac{da_{-f}}{dt} \\ &= u_f\{-T(f)\,a_f^+ + \lambda(f)\,(\nu + L^+)\,a_{-f}\} + v_f\{T(f)\,a_{-f} + \lambda(f)\,a_f^+(\nu + L)\} \\ &= -a_f^+\{T(f)\,u_f - \lambda(f)\,v_f(\nu + L)\} + \{\lambda(f)\,(\nu + L^+) + T(f)\,v_f\}\,a_{-f} \\ &= -a_f^+\{T(f)\,u_f - \lambda(f)\,v_f C\} + \{\lambda(f)\,C u_f + T(f)\,v_f\}\,a_{-f} + R_f\,, \end{aligned}$$

where

$$\left.\begin{aligned} R_f &= R_f^{(1)} + R_f^{(2)}\,; \\ R_f^{(1)} &= u_f\,\lambda(f)\,(L^+ + \nu - C)\,a_{-f}\,; \\ R_f^{(2)} &= v_f\,\lambda(f)\,a_f^+(L + \nu - C). \end{aligned}\right\} \tag{5.5}$$

From the identities

$$\left.\begin{aligned} T(f)\,u_f - \lambda(f)\,v_f C &= \sqrt{C^2\lambda^2(f) + T^2(f)}\,u_f, \\ T(f)\,v_f + \lambda(f)\,u_f C &= -\sqrt{C^2\lambda^2(f) + T^2(f)}\,v_f \end{aligned}\right\} \tag{5.6}$$

it follows that

$$i\frac{d\alpha_f^+}{dt} + \sqrt{C^2\lambda^2(f) + T^2(f)}\,\alpha_f^+ = R_f, \tag{5.7}$$

and this means that

$$i\frac{d\alpha_f}{dt} - \sqrt{C^2\lambda^2(f) + T^2(f)}\,\alpha_f = -R_f^+\,. \tag{5.8}$$

Let us estimate the quantities associated with R, R^+. We have

$$\begin{aligned} &\left\langle \Phi_{\mathscr{H}}^* R_f R_f^+ \Phi_{\mathscr{H}} \right\rangle \leqslant 2\left\langle \Phi_{\mathscr{H}}^* R_f^{(1)} R_f^{(1)+} \Phi_{\mathscr{H}} \right\rangle \\ &+ 2\left\langle \Phi_{\mathscr{H}}^* R_f^{(2)} R_f^{(2)+} \Phi_{\mathscr{H}} \right\rangle = 2u_f^2\lambda^2(f)\,\langle \Phi_{\mathscr{H}}^* (L^+ + \nu - C)\,a_{-f}\,a_{-f}^+(L + \nu \\ &- C)\Phi_{\mathscr{H}}\rangle + 2v_f^2\,\lambda^2(f)\,\langle \Phi_{\mathscr{H}}^* a_f^+ (L + \nu - C)\,(L^+ + \nu - C)\,a_f\,\Phi_{\mathscr{H}}\rangle. \end{aligned}$$

But since $|a_{-f} a_{-f}^{+}| \leq 1$, it follows that

$$\langle \Phi_{\mathcal{H}}^{*} (L^{+} + \nu - C)\, a_{-f} a_{-f}^{+} (L + \nu - C) \Phi_{\mathcal{H}} \rangle$$

$$\leqslant \langle \Phi_{\mathcal{H}}^{*} (L^{+} + \nu - C)(L + \nu - C) \Phi_{\mathcal{H}} \rangle,$$

and further, by virtue of (A1.18), we have

$$\begin{aligned} &\langle \Phi_{\mathcal{H}}^{*} a_f^{+} (L + \nu - C)(L^{+} + \nu - C)\, a_f \Phi_{\mathcal{H}} \rangle \\ &\leqslant \frac{2s}{V} + \langle \Phi_{\mathcal{H}}^{*} a_f^{+} (L^{+} + \nu - C)(L + \nu - C)\, a_f \Phi_{\mathcal{H}} \rangle \\ &= \frac{2s}{V} + \langle \Phi_{\mathcal{H}}^{*} (L^{+} + \nu - C)\, a_f^{+} a_f (L + \nu - C) \Phi_{\mathcal{H}} \rangle \\ &\leqslant \frac{2s}{V} + \langle \Phi_{\mathcal{H}}^{*} (L^{+} + \nu - C)(L + \nu - C) \Phi_{\mathcal{H}} \rangle. \end{aligned}$$

Thus,

$$\langle \Phi_{\mathcal{H}}^{*} R_f R_f^{+} \Phi_{\mathcal{H}} \rangle \leqslant 2\lambda^2(f) \langle \Phi_{\mathcal{H}}^{*} (L^{+} + \nu - C)(L + \nu - C) \Phi_{\mathcal{H}} \rangle + 2\lambda^2(f)\, v_f^2 \frac{2s}{V}. \tag{5.9}$$

Completely analogously we obtain

$$\langle \Phi_{\mathcal{H}}^{*} R_f^{+} R_f \Phi_{\mathcal{H}} \rangle \leqslant 2\lambda^2(f) \langle \Phi_{\mathcal{H}}^{*} (L + \nu - C)(L^{+} + \nu - C) \Phi_{\mathcal{H}} \rangle + 2\lambda^2(f)\, u_f^2 \frac{2s}{V}. \tag{5.10}$$

But as was shown earlier (4.58),

$$\langle \Phi_{\mathcal{H}}^{+} (L + \nu - C)(L + \nu - C) \Phi_{\mathcal{H}} \rangle \leqslant \frac{I}{V};$$

$$\langle \Phi_{\mathcal{H}}^{*} (L + \nu - C)(L^{+} + \nu - C) \Phi_{\mathcal{H}} \rangle \leqslant \frac{I}{V}.$$

Therefore, introducing the constant

$$\gamma = 2(I + 2s), \tag{5.11}$$

one may write

$$\begin{aligned} &\langle \Phi_{\mathcal{H}}^{*} R_f R_f^{+} \Phi_{\mathcal{H}} \rangle \leqslant \frac{\gamma}{V} |\lambda(f)|^2; \\ &\langle \Phi_{\mathcal{H}}^{*} R_f^{+} R_f \Phi_{\mathcal{H}} \rangle \leqslant \frac{\gamma}{V} |\lambda(f)|^2. \end{aligned} \tag{5.12}$$

Let us make several additional more general estimates. Let us consider the operators A_f, each of which is a linear combination of the operators a_f and a_{-f}^{+}:

$$A_f = p_f a_f + q_f a_{-f}^{+} \tag{5.13}$$

with bounded coefficients

$$|p_f|^2 + |q_f|^2 \leqslant \text{const}. \tag{5.14}$$

Let us show that

$$\left.\begin{aligned} &\left|\langle \Phi_{\mathcal{H}}^{*} A_{f_1} \dots A_{f_l} R_f A_{f_{l+1}} \dots A_{f_m} R_f^{+} A_{f_{m+1}} \dots \Phi_{\mathcal{H}} \rangle\right| \leqslant \frac{\text{const}}{V}; \\ &\left|\langle \Phi_{\mathcal{H}}^{*} A_{f_1} \dots A_{f_l} R_f^{+} A_{f_{l+1}} \dots A_{f_m} R_f A_{f_{m+1}} \dots \Phi_{\mathcal{H}} \rangle\right| \leqslant \frac{\text{const}}{V}. \end{aligned}\right\} \tag{5.15}$$

Proof. Note first of all that

$$La_f - a_f L = 0; \quad L^+ a_f^+ - a_f^+ L^+ = 0;$$

$$|La_f^+ - a_f^+ L| \leqslant \frac{2|\lambda(f)|}{V}; \quad |L^+ a_f - a_f L^+| \leqslant \frac{2|\lambda(f)|}{V}.$$

Therefore, for example,

$$\langle \Phi^*_{\mathcal{H}} A_{f_1} \ldots (L+\nu-C) A_{f_j} \ldots (L^+ + \nu - C) A_{f_i} \ldots \Phi_{\mathcal{H}} \rangle$$

$$= Z + \langle \Phi^*_{\mathcal{H}} (L+\nu-C) A_{f_1} \ldots A_{f_n} (L^+ + \nu - C) \Phi_{\mathcal{H}} \rangle,$$

where $|Z| \leq \mathrm{const}/V$. Consequently,

$$|\langle \Phi^*_{\mathcal{H}} A_{f_1} \ldots (L+\nu-C) A_{f_j} \ldots (L^+ + \nu - C) A_{f_i} \ldots \Phi_{\mathcal{H}} \rangle| \leqslant \frac{\mathrm{const}}{V}$$

$$+ |A_{f_1}| \ldots |A_{f_n}| \langle \Phi^*_{\mathcal{H}} (L+\nu-C)(L^+ + \nu - C) \Phi_{\mathcal{H}} \rangle \leqslant \frac{\mathrm{const}}{V}. \quad (5.16)$$

Analogously, it is proved that

$$|\langle \Phi^*_{\mathcal{H}} A_{f_1} \ldots (L^+ + \nu - C) A_{f_j} \ldots (L+\nu-C) A_{f_i} \ldots \Phi_{\mathcal{H}} \rangle| \leqslant \frac{\mathrm{const}}{V}. \quad (5.17)$$

Considering (5.16) and (5.17), we have

$$|\langle \Phi^*_{\mathcal{H}} A_{f_1} \ldots (L+\nu-C) A_{f_j} \ldots (L+\nu-C) A_{f_i} \ldots \Phi_{\mathcal{H}} \rangle|$$

$$\leqslant \frac{\mathrm{const}}{V} + |\langle \Phi^*_{\mathcal{H}} (L+\nu-C) A_{f_1} \ldots A_{f_n} (L+\nu-C) \Phi_{\mathcal{H}} \rangle|$$

$$\leqslant \frac{\mathrm{const}}{V} + \sqrt{\langle \Phi^*_{\mathcal{H}} \{(L+\nu-C) A_{f_1} \ldots A_{f_n} A^+_{f_n} \ldots A^+_{f_1} (L^+ + \nu - C)\} \Phi_{\mathcal{H}} \rangle}$$

$$\times \sqrt{\langle \Phi^*_{\mathcal{H}} (L^+ + \nu - C)(L+\nu-C) \Phi_{\mathcal{H}} \rangle} \leqslant \frac{\mathrm{const}}{V} + |A_{f_1}| \ldots |A_{f_n}|$$

$$\times \sqrt{\langle \Phi^*_{\mathcal{H}} (L+\nu-C)(L^+ + \nu - C) \Phi_{\mathcal{H}} \rangle \, \Phi^*_{\mathcal{H}} (L^+ + \nu - C)(L+\nu-C) \Phi_{\mathcal{H}} \rangle} \leqslant \frac{\mathrm{const}}{V}. \quad (5.18)$$

Analogously, it is proved that

$$\langle \Phi^*_{\mathcal{H}} A_{f_1} \ldots (L^+ + \nu - C) A_{f_j} \ldots (L^+ + \nu - C) A_{f_i} \ldots \Phi_{\mathcal{H}} \rangle| \leqslant \frac{\mathrm{const}}{V}. \quad (5.19)$$

The validity of the proved inequalities (5.15) follows from Eqs. (5.16)-(5.19).

Let us now deal with the estimates for the correlation functions. Using Eq. (5.7), we obtain

$$i \frac{d}{dt} \langle \Phi^*_{\mathcal{H}} \alpha_f^+(t) \alpha_f \Phi_{\mathcal{H}} \rangle = -\sqrt{C^2 \lambda^2(f) + T^2(f)}$$

$$\times \langle \Phi^*_{\mathcal{H}} \alpha_f^+(t) \alpha_f \Phi_{\mathcal{H}} \rangle + \langle \Phi^*_{\mathcal{H}} R_f(t) \alpha_f \Phi_{\mathcal{H}} \rangle, \quad (5.20)$$

where, as always, $\alpha_f(0) = \alpha_f$, $\alpha_f^+(0) = \alpha_f^+$.

Recalling that from the equation

$$i\frac{dJ(t)}{dt} = -\Omega J(t) + R(t)$$

it follows that

$$J(t) = J(0)\,e^{i\Omega t} + e^{i\Omega t}\int_0^t e^{-i\Omega t} R(t)\,dt,$$

we write

$$\langle \Phi^*_{\mathscr{H}} \alpha^+_f(t)\, \alpha_f \Phi_{\mathscr{H}} \rangle = e^{i\sqrt{C^2\lambda^2(f)+T^2(f)}\,t} \langle \Phi^*_{\mathscr{H}} \alpha^+_f \alpha_f \Phi_{\mathscr{H}} \rangle$$

$$+ e^{i\sqrt{C^2\lambda^2(f)+T^2(f)}\,t} \int_0^t e^{-i\sqrt{C^2\lambda^2(f)+T^2(f)}\,t} \langle \Phi^*_{\mathscr{H}} R_f(t)\, \alpha_f \Phi_{\mathscr{H}} \rangle\, dt. \quad (5.21)$$

On the other hand, since $\Phi_{\mathscr{H}}$ is an eigenfunction of $\mathscr{H}$, corresponding to the least eigenvalue, its conventional spectral representation yields

$$\langle \Phi^*_{\mathscr{H}} \alpha^+_f(t)\, \alpha_f \Phi_{\mathscr{H}} \rangle = \int_0^\infty J_f(\nu)\, e^{-i\nu t}\, d\nu, \quad (5.22)$$

where

$$J_f \geqslant 0 \text{ and } \int_0^\infty J_f(\nu)\, d\nu \leqslant 1. \quad (5.23)$$

Let us place

$$h(t) = \int_0^2 \omega^2 (2-\omega)^2 e^{-i\omega t}\, d\omega. \quad (5.24)$$

As is evident, this function is regular on the entire real axis. Using integration by parts, it is not difficult to confirm the fact that for $|t| \to \infty$, h(t) decreases according to the estimate:

$$|h(t)| \leqslant \frac{\text{const}}{|t|^3}. \quad (5.25)$$

Therefore, the integral

$$\int_{-\infty}^{\infty} |t h(t)|\, dt \quad (5.26)$$

turns out to be finite. Let us set

$$\sqrt{C^2\lambda^2(f) + T^2(f)} = \Omega \quad (5.27)$$

and let us note that

$$h(\Omega t) = \frac{1}{\Omega^5} \int_0^{2\Omega} \nu^2 (2\Omega - \nu)^2 e^{-i\nu t}\, d\nu. \quad (5.28)$$

From such a method of construction it is evident that

$$\int_{-\infty}^{\infty} h(\Omega t)\, e^{-i\nu t}\, dt = 0 \text{ for } \nu \geqslant 0, \quad (5.29)$$

and by virtue of (5.22)

$$\int_{-\infty}^{\infty} \langle \Phi^*_{\mathcal{H}} \alpha_f^+(t)\, \alpha_f \Phi_{\mathcal{H}} \rangle h(\Omega t)\, dt = 0. \tag{5.30}$$

Therefore, we have

$$\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \int_{-\infty}^{\infty} e^{i\Omega t} h(\Omega t)\, dt = -\int_{-\infty}^{\infty} h(\Omega t) e^{i\Omega t} \times \left(\int_0^t e^{-i\Omega t'} \langle \Phi^*_{\mathcal{H}} R_f(t')\, \alpha_f \Phi_{\mathcal{H}} \rangle dt' \right) dt \tag{5.31}$$

from (5.21). But

$$\int_{-\infty}^{\infty} e^{i\Omega t} h(\Omega t)\, dt = \frac{2\pi}{\Omega}. \tag{5.32}$$

This means that

$$\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \leqslant \frac{\Omega}{2\pi} \int_{-\infty}^{\infty} |h(\Omega t)| \left\{ \int_0^t |\langle \Phi^*_{\mathcal{H}} R_f(t')\, \alpha_f \Phi_{\mathcal{H}} \rangle| \, dt' \right\} dt. \tag{5.33}$$

But in view of (5.12),

$$|\langle \Phi^*_{\mathcal{H}} R_f \alpha_f \Phi_{\mathcal{H}} \rangle| \leqslant \sqrt{|\langle \Phi^*_{\mathcal{H}} R_f R_f^+ \Phi_{\mathcal{H}} \rangle| \, |\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle|}$$

$$\leqslant \left(\frac{\gamma}{V} \right)^{\frac{1}{2}} |\lambda(f)| \left(\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \right)^{\frac{1}{2}}. \tag{5.34}$$

Consequently,

$$\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \leqslant \frac{\Omega}{2\pi} \int_{-\infty}^{\infty} |h(\Omega t)| \, |t| \, dt \left(\frac{\gamma}{V} \right)^{\frac{1}{2}} \left(\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \right)^{\frac{1}{2}}$$

$$= \frac{1}{2\pi\Omega} \int_{-\infty}^{\infty} |h(\tau)\tau| \, d\tau \left(\frac{\gamma}{V} \right)^{\frac{1}{2}} \left(\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \right)^{\frac{1}{2}} |\lambda(f)|.$$

Thus,

$$\langle \Phi^*_{\mathcal{H}} \alpha_f^+ \alpha_f \Phi_{\mathcal{H}} \rangle \leqslant \frac{|\lambda(f)|^2}{2\pi (C^2 |\lambda(f)|^2 + T^2(f))} \cdot \frac{\gamma}{V} \left(\int_{-\infty}^{\infty} |h(\tau)\tau| \, a\tau \right)^2. \tag{5.35}$$

From this we obtain a number of estimates. In view of the Schwartz inequality, the fact that $|a_f^+ a_f| \leq 1$, and using (5.35), we obtain

$$|\langle \Phi^*_{\mathcal{H}} \alpha_{f_1}^+ \dots \alpha_{f_s}^+ \alpha_{g_e} \dots \alpha_{g_1} \Phi_{\mathcal{H}} \rangle|$$

$$\leqslant \sqrt{\langle \Phi^*_{\mathcal{H}} \alpha_{f_1}^+ \dots \alpha_{f_s}^+ \alpha_{f_s} \dots \alpha_{f_1} \Phi_{\mathcal{H}} \rangle \langle \Phi^*_{\mathcal{H}} \alpha_{g_1}^+ \dots \alpha_{g_i}^+ \alpha_{g_i} \dots \alpha_{g_1} \Phi_{\mathcal{H}} \rangle}$$

$$\leqslant \sqrt{\langle \Phi^*_{\mathcal{H}} \alpha_{f_1}^+ \alpha_{f_1} \Phi_{\mathcal{H}} \rangle \langle \Phi^*_{\mathcal{H}} \alpha_{g_1}^+ \alpha_{g_1} \Phi_{\mathcal{H}} \rangle} \leqslant \frac{\text{const}}{V}. \tag{5.36}$$

We likewise have

$$\left|\langle \Phi^*_{\mathcal{H}} \alpha_{f_1} \ldots \alpha_{f_s} \Phi_{\mathcal{H}} \rangle\right|$$

$$\leqslant \sqrt{\langle \Phi^*_{\mathcal{H}} \alpha_{f1} \ldots \alpha_{fs-1} \alpha^+_{f_{s-1}} \ldots \alpha^+_{f_1} \Phi_{\mathcal{H}} \rangle \langle \Phi^*_{\mathcal{H}} \alpha^+_{f_s} \alpha_{f_s} \Phi_{\mathcal{H}} \rangle} \leqslant$$

$$\leqslant \sqrt{\langle \Phi^*_{\mathcal{H}} \alpha^+_{f_s} \alpha_{f_s} \Phi_{\mathcal{H}} \rangle} \leqslant \frac{\text{const}}{\sqrt{V}} \tag{5.37}$$

and

$$\left|\langle \Phi^*_{\mathcal{H}} \alpha^+_{f_1} \ldots \alpha^+_{f_s} \Phi_{\mathcal{H}} \rangle\right| \leqslant \sqrt{\langle \Phi^*_{\mathcal{H}} \alpha^+_{f_1} \alpha_{f1} \Phi_{\mathcal{H}} \rangle} \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{5.38}$$

Let us now compare the averages

$$\langle \Phi^*_{\mathcal{H}} \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{fs} \Phi_{\mathcal{H}} \rangle$$

(where $\mathfrak{A}_f = a_f$ and a_f^+) with the corresponding averages calculated on the basis of the Hamiltonian $\mathcal{H}_0$, in which $\nu + \sigma = C$. For convenience we shall denote averages of these types by $\langle \mathfrak{A}_{f1} \ldots \mathfrak{A}_{fs} \rangle_{\mathcal{H}}$ and $\langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{fs} \rangle_{\mathcal{H}_0}$, respectively. Let us estimate the magnitude of the difference

$$\langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}} - \langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}_0}. \tag{5.39}$$

Let us say a few words concerning the way in which $\langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}_0}$ is calculated. We make use of the formulas $a_f^+ = u_f \alpha_f^+ - v_f \alpha_{-f}$, $a_f = u_f \alpha_f - v_f \alpha^+_{-f}$ and then reduce the product $\mathfrak{A}_{f_1} \cdots \mathfrak{A}_{f_s}$ to the sum of products of a normal type in which all α^+ precede α. Since all terms of the type

$$\langle \alpha^+ \ldots \alpha^+ \rangle_{\mathcal{H}_0},\ \langle \alpha \ldots \alpha \rangle_{\mathcal{H}_0},\ \langle \alpha^+ \ldots \alpha \rangle_{\mathcal{H}_0} \tag{5.40}$$

are equal to zero, we obtain the expression for calculating $\langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}_0}$. Let us apply this same procedure to the calculation of $\langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}}$. As is evident, the difference (5.39) is caused by terms which are proportional to

$$\langle \alpha^+ \ldots \alpha^+ \rangle_{\mathcal{H}},\ \langle \alpha \ldots \alpha \rangle_{\mathcal{H}},\ \langle \alpha^+ \ldots \alpha \rangle_{\mathcal{H}}, \tag{5.41}$$

and which, unlike (5.40), are in general not equal to zero. But for the quantities (5.41) the estimates (5.36)-(5.38) exist. Therefore, we have

$$\left|\langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}} - \langle \mathfrak{A}_{f_1} \ldots \mathfrak{A}_{f_s} \rangle_{\mathcal{H}_0}\right| \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{5.42}$$

Let us now consider the bitemporal correlation functions and show that the differences

$$\langle \mathcal{B}_{f_1}(t) \ldots \mathcal{B}_{f_l}(t) \mathfrak{A}_{f_m}(\tau) \ldots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathcal{H}} - \langle \mathcal{B}_{f_1}(t) \ldots \mathcal{B}_{f_l}(t) \mathfrak{A}_{f_m}(\tau) \ldots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathcal{H}_0}, \tag{5.43}$$

where $\mathfrak{A}_f$, $\mathcal{B}_f$ are equal to either a_f or α_f^+, will all be bounded in modulus by quantities of order $1/\sqrt{V}$. Note that on the one hand

$$\langle \alpha^+_{f_1}(t) \ldots \alpha_{f_j}(t) \mathfrak{A}_{f_m}(\tau) \ldots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathcal{H}_0} = 0, \tag{5.44}$$

while on the other hand

$$\left|\langle \alpha^+_{f_1}(t) \ldots \alpha_{f_j}(t) \mathfrak{A}_{f_m}(\tau) \ldots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathcal{H}}\right|$$

$$\leqslant \sqrt{\langle \alpha^+_{f_1}(t) \alpha_{f_1}(t) \rangle_{\mathcal{H}} \langle \omega^+ \omega \rangle_{\mathcal{H}}} = \sqrt{\langle \alpha^+_{f_1} \alpha_{f_1} \rangle_{\mathcal{H}} \langle \omega^+ \omega \rangle_{\mathcal{H}}}, \tag{5.45}$$

where $\omega = \alpha_{f_j}(t)\mathfrak{A}_{f_m}(\tau)\dots\mathfrak{A}_{f_n}(\tau)$, so that

$$\left|\langle\alpha^+_{f_1}(t)\dots\alpha_{f_j}(t)\mathfrak{A}_{f_m}(\tau)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}}\right| \leqslant \sqrt{\langle\alpha^+_{f_1}\alpha_{f_1}\rangle} \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{5.46}$$

Therefore, we need only establish the fact that differences of the type

$$\langle\alpha_{f_1}(t)\dots\alpha_{f_l}(t)\mathfrak{A}_{f_m}(\tau)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}} - \langle\alpha_{f_1}(t)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}_0}$$

are bounded in modulus by quantities of order $1/\sqrt{V}$. Let us set

$$\langle\alpha_{f_1}(t)\dots\alpha_{f_l}(t)\mathfrak{A}_{f_m}(\tau)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}} = \Gamma(t-\tau). \tag{5.47}$$

By virtue of (5.8), we have

$$i\frac{d\Gamma(t-\tau)}{dt} - \{\Omega(f_1)+\dots+\Omega(f_l)\}\Gamma(t-\tau) = \Delta(t-\tau), \tag{5.48}$$

where $\Omega(f) = \sqrt{C^2\lambda^2(f) + T^2(f)}$ and

$$\Delta(t-\tau) = \Delta_1(t-\tau) + \dots \Delta_l(t-\tau);$$
$$\Delta_1(t-\tau) = -\langle R^+_{f_1}(t)\alpha_{f_2}(t)\dots\alpha_{f_l}(t)\mathfrak{A}_{f_m}(\tau)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}};$$
$$\dots$$
$$\Delta_l(t-\tau) = -\langle\alpha_{f_1}(t)\dots\alpha_{f_{l-1}}(t)R^+_{f_l}(t)\mathfrak{A}_{f_m}(\tau)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}}.$$

But

$$\begin{aligned}|\Delta_s(t-\tau)| &\leqslant \sqrt{\langle\alpha_{f_1}(t)\dots\alpha_{f_{s-1}}(t)R^+_{f_s}(t)\dots\alpha_{f_l}(t)\alpha^+_{f_l}(t)\dots R_{f_s}(t)\dots\alpha^+_{f_1}(t)\rangle_{\mathcal{H}}} \\ &\quad\times\sqrt{\langle\mathfrak{A}^+_{f_n}(\tau)\dots\mathfrak{A}^+_{f_m}(\tau)\mathfrak{A}_{f_n}(\tau)\dots\mathfrak{A}_{f_n}(\tau)\rangle_{\mathcal{H}}} \\ &= \sqrt{\langle\alpha_{f_1}\dots\alpha_{f_{s-1}}R^+_{f_s}\dots\alpha_{f_l}\alpha^+_{f_l}R_{f_s}\dots\alpha^+_{f_1}\rangle_{\mathcal{H}}\langle\mathfrak{A}^+_{f_n}\dots\mathfrak{A}_{f_n}\rangle_{\mathcal{H}}} \\ &\leqslant \sqrt{\langle\alpha_{f_1}\dots\alpha_{f_{s-1}}R^+_{f_s}\dots\alpha_{f_l}\alpha^+_{f_l}R_{f_s}\dots\alpha^+_{f_1}\rangle_{\mathcal{H}}},\end{aligned}$$

and therefore, taking account of (5.15), we have

$$|\Delta_s(t-\tau)| \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{5.49}$$

Consequently,

$$|\Delta(t-\tau)| \leqslant \frac{s}{\sqrt{V}}, \text{ where } s = \text{const}. \tag{5.50}$$

But from (5.48), we have

$$\Gamma(t-\tau) = \Gamma(0)e^{-i\{\Omega(f_1)+\dots+\Omega(f_l)\}(t-\tau)} + \exp\left[\{-i[\Omega(f_1)+\dots+\Omega(f_l)](t-\tau)\}\right.$$

$$\left.\times\left\{\int_0^{t-\tau}\exp^{i[\Omega(f_1)+\dots+\Omega(f_l)]\omega}\Delta(\omega)\,d\omega\right\}\right], \tag{5.51}$$

whence by virtue of (5.50), we have

$$\left|\Gamma(t-\tau) - \Gamma(0)e^{-i\{\Omega(f_1)+\dots+\Omega(f_l)\}(t-\tau)}\right| \leqslant \frac{s}{\sqrt{V}}|t-\tau|. \tag{5.52}$$

But, on the other hand,

$$\langle \alpha_{f_1}(t) \dots \alpha_{f_l}(t) \dots \mathfrak{A}_{f_m}(\tau) \dots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathscr{H}_0}$$

$$= e^{-i\{\Omega(if_1)+\dots+\Omega(f_l)\}(t-\tau)} \langle \alpha_{f_1} \dots \alpha_{f_l} \mathfrak{A}_{f_m} \dots \mathfrak{A}_{f_n} \rangle \mathscr{H}_0. \quad (5.53)$$

Thus, according to (5.52), (5.53), we have

$$D \equiv |\langle \alpha_{f_1}(t) \dots \alpha_{f_l}(t) \mathfrak{A}_{f_m}(\tau) \dots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathscr{H}} - \langle \alpha_{f_1}(t) \dots \alpha_{f_l}(t) \mathfrak{A}_{f_m}(\tau) \dots$$
$$\dots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathscr{H}_0}| \leqslant \frac{s}{\sqrt{V}} |t-\tau| + |\langle \alpha_{f_1} \dots \alpha_{f_l} \mathfrak{A}_{f_m} \dots \mathfrak{A}_{f_n} \rangle_{\mathscr{H}}$$
$$- \langle \alpha_{f_1} \dots \alpha_{f_l} \mathfrak{A}_{f_m} \dots \mathfrak{A}_{f_n} \rangle_{\mathscr{H}_0}|.$$

But the difference between the simultaneous averages appears in the second term of the right side, and as was shown previously (see Eq. (5.42)), it is majorized by a term of order $1/\sqrt{V}$. Thus, let us establish the fact that for the bitemporal averages

$$|\langle \mathscr{B}_{f_1}(t) \dots \mathscr{B}_{f_l}(t) \mathfrak{A}_{f_m}(\tau) \dots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathscr{H}}$$

$$- \langle \mathscr{B}_{f_1}(t) \dots \mathscr{B}_{f_l}(t) \mathfrak{A}_{f_m}(\tau) \dots \mathfrak{A}_{f_n}(\tau) \rangle_{\mathscr{H}_0}|$$

$$\leqslant \frac{G_1}{\sqrt{V}} |t-\tau| + \frac{G_2}{\sqrt{V}}, \quad G_1,\ G_2 = \text{const.} \quad (5.54)$$

These bounds may also be generalized for the case of s-time correlation averages

$$\left.\begin{aligned} &\langle \mathfrak{P}_s(t_s) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1) \rangle; \\ &\mathfrak{P}_j(t) = \mathfrak{A}_1^{(j)}(t) \dots \mathfrak{A}_l^{(j)}(t), \end{aligned}\right\} \quad (5.55)$$

where $\mathfrak{A}_s^{(j)}(t)$ is equal to $a_f(t)$ and $a_f^+(t)$. Let us show that

$$|\langle \mathfrak{P}_s(t_s) \dots \mathfrak{P}_1(t_1) \rangle_{\mathscr{H}} - \langle \mathfrak{P}_s(t_s) \dots \mathfrak{P}_1(t_1) \rangle_{\mathscr{H}_0}| \leqslant \frac{(K_s|t_s - t_{s-1}| + \dots + K_2|t_2 - t_1| + Q_s)}{\sqrt{V}}, \quad (5.56)$$

where

$$K_j = \text{const}, \quad Q_s = \text{const}. \quad (5.57)$$

The proof can easily be carried out by the method of induction. Let us assume that these relationships are true for (s−1)-time averages, and let us prove them for s-time averages. Reasoning as in the two-dimensional case, we see that it is sufficient to prove (5.56) for $\mathfrak{P}_s(t) = \alpha_{f_1}(t) \dots \alpha_{f_l}(t)$. But then

$$\langle \mathfrak{P}_s(t_s) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1) \rangle_{\mathscr{H}_0} = \exp\{-i(\Omega_{f_1} + \dots$$

$$+ \Omega_{f_l})(t_s - t_{s-1})\} \langle \mathfrak{P}_s(t_{s-1}) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1) \rangle_{\mathscr{H}_0}. \quad (5.58)$$

On the other hand, on the basis of (5.8) and (5.15) and the reasoning by means of which the inequality (5.52) was established we verify the fact that

$$|\langle \mathfrak{P}_s(t_s) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1) \rangle_{\mathscr{H}} - \exp\{-i(\Omega_{f_1} + \dots$$

$$+ \Omega_{f_l})(t_s - t_{s-1})\} \langle \mathfrak{P}_s(t_{s-1}) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1) \rangle_{\mathscr{H}}|$$

$$\leqslant \frac{K_1^{(s)} |t_s - t_{s-1}|}{\sqrt{V}}, \quad (5.59)$$

where $K_1^{(s)} = \text{const}$.

Thus,

$$|\langle \mathfrak{P}_s(t_s) \dots \mathfrak{P}_1(t_1)\rangle_{\mathcal{H}} - \langle \mathfrak{P}_s(t_s) \dots \mathfrak{P}_1(t_1)\rangle_{\mathcal{H}_0}|$$

$$\leqslant \frac{K_1^{(s)} |t_s - t_{s-1}|}{\sqrt{V}} + |\langle \mathfrak{P}_s(t_{s-1}) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1)\rangle_{\mathcal{H}}$$

$$- \langle \mathfrak{P}_s(t_{s-1}) \mathfrak{P}_{s-1}(t_{s-1}) \dots \mathfrak{P}_1(t_1)\rangle_{\mathcal{H}_0}|. \tag{5.60}$$

But the second term in the right side represents the difference between correlation averages with (s−1)-times for which the adopted estimates are established by assumption. For this reason they are also true for the s-time averages. Thus, $\mathcal{H}_0$ yields the asymptotic approximation for all correlation averages of the type $\langle \mathfrak{P}_s(t_s) \dots \mathfrak{P}_1(t_1)\rangle$. Consequently, the same statement is also valid for the Green's functions constructed on the basis of the operators considered.

Comment. Note that in the estimate of the degree of approximation we would have been able to obtain const/V instead of the const/$\sqrt{V}$ obtained throughout if we had replaced C by C_1 in determining u_f, v_f, $\mathcal{H}_0$:

$$C_1 = \langle L + \nu\rangle_{\mathcal{H}} = \langle L^+ + \nu\rangle_{\mathcal{H}}. \tag{5.61}$$

Since it is obvious (see Eq. (4.58)) that

$$(C - C_1)^2 \leqslant \frac{\text{const}}{V},$$

it follows that all estimates of the type (5.12), (5.15), and (5.35) remain valid. Additional new useful relationships are added:

$$\left.\begin{aligned} &|\langle A_{f_1} \dots R_f \dots A_{f_n}\rangle_{\mathcal{H}}| \leqslant \frac{\text{const}}{V}; \\ &|\langle A_{f_1} \dots R_f^+ \dots A_{f_n}\rangle_{\mathcal{H}}| \leqslant \frac{\text{const}}{V}. \end{aligned}\right\} \tag{5.62}$$

For their proof it is sufficient to expand the expressions

$$\langle A_{f_1} \dots R_f \dots A_{f_n}\rangle_{\mathcal{H}};\ \langle A_{f_1} \dots R_f^+ \dots A_{f_n}\rangle_{\mathcal{H}}, \tag{5.63}$$

having expressed all a and a^+ in terms of α and α^+. Then Eqs. (5.33) may be represented by a sum of terms of the type $\langle \alpha^+ \dots \alpha\rangle_{\mathcal{H}}$:

$$\left.\begin{aligned} &\langle (L + \nu - C_1) \dots \alpha\rangle_{\mathcal{H}};\ \langle \alpha^+ \dots (L + \nu - C_1)\rangle_{\mathcal{H}}; \\ &\langle (L^+ + \nu - C_1) \dots \alpha\rangle_{\mathcal{H}};\ \langle \alpha^+ \dots (L^+ + \nu - C_1)\rangle_{\mathcal{H}}; \\ &\text{const}\ \langle L + \nu - C_1\rangle_{\mathcal{H}} \equiv 0;\ \text{const}\ \langle L^+ + \nu - C_1\rangle_{\mathcal{H}} \equiv 0 \end{aligned}\right\} \tag{5.64}$$

and commutation terms of order 1/V. (The vanishing of the latter two Eqs. (5.64) derives from (5.61).) Applying the inequality

$$|\langle AB\rangle| \leqslant \sqrt{|\langle AA^+\rangle|}\ \sqrt{|\langle B^+B\rangle|}$$

to (5.64) along with (5.35), we see that all quantities will be of order 1/V, and this proves (5.62).

Let us now make use of these additional relationships. Let us consider the expression $\langle \alpha_{f_1}^+ \dots \alpha_{f_n}^+\rangle$ which evidently does not depend on t. Therefore,

$$\frac{d}{dt}\langle \alpha_{f_1}^+ \dots \alpha_{f_n}^+\rangle_{\mathcal{H}} = \left\langle \frac{d\alpha_{f_1}^+}{dt} \dots \alpha_{f_n}^+\right\rangle_{\mathcal{H}} + \dots + \left\langle \alpha_{f_1}^+ \dots \frac{d\alpha_{f_n}^+}{dt}\right\rangle_{\mathcal{H}} = 0. \tag{5.65}$$

Consequently, from (5.7), we obtain

$$(\Omega(f_1)+\ldots+\Omega(f_n))\ \langle \alpha^+_{f_1}\ldots\alpha^+_{f_n}\rangle_{\mathcal{H}}=\langle R_{f_1}\ldots\alpha^+_{f_n}\rangle_{\mathcal{H}}+\ldots+\langle\alpha^+_{f_1}\ldots R_{f_n}\rangle_{\mathcal{H}}. \tag{5.66}$$

But by virtue of (5.62)

$$\left|\langle R_{f_1}\ldots\alpha^+_{f_n}\rangle+\ldots+\langle\alpha^+_{f_1}\ldots R_{f_n}\rangle\right|\leqslant\frac{D}{V},\ D=\text{const}, \tag{5.67}$$

and therefore

$$\left|\langle\alpha^+_{f_1}\ldots\alpha^+_{f_n}\rangle\right|\leqslant\frac{D}{V(\Omega(f_1)+\ldots+\Omega(f_n))}, \tag{5.68}$$

whence, going over to conjugate quantities, we have

$$\left|\langle\alpha_{f_1}\ldots\alpha_{f_n}\rangle\right|\leqslant\frac{D}{V(\Omega(f_1)+\ldots+\Omega(f_n))}. \tag{5.69}$$

Having made use of the new inequalities (5.68) and (5.69) instead of the old ones (5.37) and (5.38), and having retained (5.36), it can be shown that the inequality

$$\left|\langle\mathfrak{A}_{f_1}\ldots\mathfrak{A}_{f_s}\rangle_{\mathcal{H}}-\langle\mathfrak{A}_{f_1}\ldots\mathfrak{A}_{f_s}\rangle_{\mathcal{H}_0}\right|<\frac{\text{const}}{V} \tag{5.70}$$

holds instead of the inequality (5.42). Analogous improvements of the estimates may be fulfilled for all correlation averages of the types considered earlier. We shall not present a general proof here. Let us restrict our analysis to estimating the difference

$$\langle\alpha_{f_1}(t)\ldots\alpha_{f_l}(t)\,\alpha^+_{g_1}(\tau)\ldots\alpha^+_{g_r}(\tau)\rangle_{\mathcal{H}}-\langle\alpha_{f_1}(t)\ldots\alpha^+_{g_r}(\tau)\rangle_{\mathcal{H}_0}. \tag{5.71}$$

We have

$$\Gamma_{\substack{\mathcal{H}\\ \mathcal{H}_0}}(t-\tau)=\langle\alpha_{f_1}(t)\ldots\alpha^+_{g_r}(\tau)\rangle_{\substack{\mathcal{H}\\ \mathcal{H}_0}}. \tag{5.72}$$

We have

$$i\,\frac{\partial\Gamma_{\mathcal{H}}(t-\tau)}{\partial t}=(\Omega(f_1)+\ldots+\Omega(f_l))\,\Gamma_{\mathcal{H}}(t-\tau)+\Delta_{\mathcal{H}}(t-\tau), \tag{5.73}$$

where

$$\Delta_{\mathcal{H}}=-\sum_j\langle\alpha_{f_1}(t)\ldots R^+_{f_j}(t)\ldots\alpha_{f_l}(t)\,\alpha^+_{g_1}(\tau)\ldots\alpha^+_{g_r}(\tau)\rangle_{\mathcal{H}}. \tag{5.74}$$

Differentiating (5.74) with respect to τ, we find

$$i\,\frac{\partial\Delta_{\mathcal{H}}(t-\tau)}{\partial\tau}\quad-(\Omega(g_1)+\ldots+\Omega(g_r))\,\Delta_{\mathcal{H}}(t-\tau)+\zeta(t-\tau), \tag{5.75}$$

where

$$\zeta(t-\tau)=-\sum_{j,s}\langle\alpha_{f_1}(t)\ldots R^+_{f_j}(t)\,\alpha^+_{g_1}(\tau)\ldots R_{f_s}(\tau)\ldots\alpha^+_{g_r}(\tau)\rangle_{\mathcal{H}}. \tag{5.76}$$

But in view of (5.15)

$$|\zeta(t-\tau)|\leqslant\frac{Q}{V},\ \text{where } Q=\text{const}. \tag{5.77}$$

Therefore, from (5.75), we obtain the following expression in conventional fashion (see, for example, (5.48)–(5.72)):

$$|\Delta_{\mathscr{H}}(t-\tau)-\Delta_{\mathscr{H}}(0)\exp\{i[\Omega(g_1)+\ldots+\Omega(g_r)](\tau-t)\}|\leqslant\frac{Q}{V}|t-\tau|. \tag{5.78}$$

But by virtue of (5.62) and (5.74), we have

$$|\Delta_{\mathscr{H}}(0)|\leqslant\frac{Q_1}{V},\quad Q_1=\text{const}. \tag{5.79}$$

This means that

$$|\Delta_{\mathscr{H}}(t-\tau)|\leqslant\frac{Q_1+Q|t-\tau|}{V}. \tag{5.80}$$

Let us substitute this estimate into (5.73). We find that

$$|\Gamma_{\mathscr{H}}(t-\tau)-\Gamma_{\mathscr{H}}(0)\exp\{i[\Omega(f_1)+\ldots+\Omega(f_l)](\tau-t)\}|$$

$$\leqslant\frac{Q_1|t-\tau|+Q|t-\tau|^2\frac{1}{2}}{V}. \tag{5.81}$$

On the other hand,

$$\Gamma_{\mathscr{H}_0}(t-\tau)=\Gamma_{\mathscr{H}_0}(0)\exp\{i[\Omega(f_1)+\ldots+\Omega(f_l)](\tau-t)\}. \tag{5.82}$$

Consequently,

$$|\Gamma_{\mathscr{H}}(t-\tau)-\Gamma_{\mathscr{H}_0}(t-\tau)|\leqslant|\Gamma_{\mathscr{H}}(0)-\Gamma_{\mathscr{H}_0}(0)|+\frac{Q_1|t-\tau|+Q|t-\tau|^2\frac{1}{2}}{V}. \tag{5.83}$$

But from (5.70), we have

$$|\Gamma_{\mathscr{H}}(0)-\Gamma_{\mathscr{H}_0}(0)|=|\langle\alpha_{f_1}\ldots\alpha_{f_l}\alpha^+_{g_1}\ldots\alpha^+_{g_r}\rangle_{\mathscr{H}}$$

$$-\langle\alpha_{f_1}\ldots\alpha_{f_l}\alpha^+_{g_1}\ldots\alpha^+_{g_r}\rangle_{\mathscr{H}_0}|\leqslant\frac{Q_2}{V},\quad Q_2=\text{const}. \tag{5.84}$$

Thus,

$$|\langle\alpha_{f_1}(t)\ldots\alpha_{f_l}(t)\alpha^+_{g_1}(\tau)\ldots\alpha^+_{g_r}(\tau)\rangle_{\mathscr{H}}-\langle\alpha_{f_1}(t)\ldots\alpha^+_{g_r}(\tau)\rangle_{\mathscr{H}_0}|$$

$$\leqslant\frac{Q_2+Q_1|t-\tau|+Q|t-\tau|^2\frac{1}{2}}{V}.$$

Reasoning further, it is not difficult to raise the order $1/\sqrt{V}$ to $1/V$ throughout in the previously obtained estimate according to this scheme.

§ 6. The Green's Function for the Case $\nu = 0$

In the previous section we obtained all of the necessary asymptotic estimates for the Green's function in the case $\nu > 0$. Since certain estimates [see Eq. (4.58)] which we used in § 5 are meaningless for $\nu = 0$, the results of that section cannot be carried over directly to the case $\nu = 0$. This case requires special consideration.

Since L and L^+ now do not take definite values in the lowest energy state $\Phi_{\mathscr{H}}$, with asymptotic accuracy, we shall, unlike the case considered previously, work with the amplitudes

$$\alpha_f = u_f a_f + v_f a^+_{-f} \frac{L}{C}, \tag{6.1}$$

where

$$\left.\begin{aligned} u_f &= \frac{1}{\sqrt{2}} \sqrt{1 + \frac{T(f)}{\sqrt{C^2\lambda^2(f) + T^2(f)}}}; \\ v_f &= -\frac{\varepsilon(f)}{\sqrt{2}} \sqrt{1 - \frac{T(f)}{\sqrt{C^2\lambda^2(f) + T^2(f)}}}. \end{aligned}\right\} \tag{6.2}$$

These amplitudes satisfy the commutation relations for Fermi amplitudes only in an asymptotic approximation rather than exactly. In order to obtain the estimates it will be necessary to establish a number of inequalities.

First of all we consider the expression $\Sigma \Omega(f) \alpha_f^+ \alpha_f$ in which

$$\Omega(f) = \sqrt{C^2\lambda^2(f) + T^2(f)}. \tag{6.2'}$$

With allowance for Eq. (6.1), we have

$$\begin{aligned} \sum \Omega(f) \alpha_f^+ \alpha_f &= \sum \Omega(f) \left\{ u_f a_f^+ + v_f \frac{L^+}{C} a_{-f} \right\} \left\{ u_f a_f + v_f a^+_{-f} \frac{L}{C} \right\} \\ &= \sum \Omega(f) \left\{ u_f^2 a_f^+ a_f + v_f^2 \frac{L^+}{C} a_{-f} a^+_{-f} \frac{L}{C} + \right. \\ &\quad \left. + u_f v_f \frac{L^+}{C} a_{-f} a_f + u_f v_f a_f^+ a^+_{-f} \frac{L}{C} \right\}. \end{aligned}$$

But

$$\begin{aligned} \sum \Omega(f) v_f^2 \frac{L^+}{C} a_{-f} a^+_{-f} \frac{L}{C} &= -\sum \Omega(f) v_f^2 \frac{L^+}{C} a_f^+ a_f \frac{L}{C} \\ + \sum \Omega(f) v_f^2 \frac{L^+ L}{C^2} &= -\sum \Omega(f) v_f^2 a_f^+ \frac{L^+ L}{C^2} a_f + \sum \Omega(f) v_f^2 \frac{L^+ L}{C^2}. \end{aligned}$$

Further, since $u_f v_f = C\lambda(f)/2\Omega(f)$ it follows that

$$-\sum \Omega(f) \left\{ u_f v_f \frac{L^+}{C} a_{-f} a_f + u_f v_f a_f^+ a^+_{-f} \frac{L}{C} \right\} = V L^+ L.$$

This means that

$$\begin{aligned} \sum \Omega(f) \alpha_f^+ \alpha_f &= \sum_f \Omega \left\{ u_f^2 a_f^+ a_f - v_f^2 a_f^+ \frac{L^+ L}{C^2} a_f \right\} \\ + \sum \Omega(f) v_f^2 \frac{L^+ L}{C^2} - V L^+ L &= \sum_f \Omega(f) (u_f^2 - v_f^2) a_f^+ a_f \\ - \sum_f \Omega(f) v_f^2 a_f^+ \frac{L^+ L - C^2}{C^2} a_f &+ \sum \Omega(f) v_f^2 \frac{L^+ L}{C^2} - V L^+ L. \end{aligned}$$

But $\Omega(f)\ (u_f^2 - v_f^2) = T(f)$, and therefore

$$\begin{aligned} H = \Sigma T(f) a_f^+ a_f - V \frac{L^+ L}{2} &= \sum \Omega(f) \alpha_f^+ \alpha_f + \frac{V L^+ L}{2} \\ - \sum \Omega(f) v_f^2 \frac{L^+ L}{C^2} &+ \sum \Omega(f) v_f^2 a_f^+ \frac{L^+ L - C^2}{C^2} a_f, \end{aligned}$$

and, consequently,

$$\begin{aligned} H = \sum \Omega(f) \alpha_f^+ \alpha_f &+ \frac{V}{2} \left\{ C^2 - \frac{2}{V} \sum \Omega(f) v_f^2 \right\} \\ + \frac{V(L^+ L - C^2)}{2} - \sum \Omega(f) v_f^2 \frac{L^+ L - C^2}{C^2} &+ \sum \Omega(f) v_f^4 \frac{L^+ L - C^2}{C^2} \end{aligned}$$

$$+\sum \Omega(f)\, v_f^2 \left\{ a_f^+ \frac{L^+L-C^2}{C^2} a_f - v_f^2 \frac{L^+L-C^2}{C^2} \right\}.$$

On the other hand, we have

$$C^2 \frac{V}{2} - \sum \Omega(f)\, v_f^2 + \sum \Omega(f)\, v_f^4 = C^2 \frac{V}{2} - \sum \Omega(f)\, u_f^2\, v_f^2$$

$$= \frac{V}{2} \left\{ C^2 - \frac{1}{2V} C^2 \sum \frac{\lambda^2(f)}{\sqrt{C^2 \lambda^2(f) + T^2(f)}} \right\} = \frac{V}{2} C^2 \mathcal{F}'(C^2),$$

where $\mathcal{F}(C^2) = C^2 - \frac{2}{V} \sum \Omega(f)\, v_f^2$. Therefore,

$$H = \sum \Omega(f)\, \alpha_f^+ \alpha_f - w + \frac{V}{2} (L^+ L - C^2)\, \mathcal{F}'(C^2) + \mathcal{F}(C^2). \tag{6.3}$$

Here

$$w = -\sum \Omega(f)\, v_f^2 \left\{ a_f^+ \frac{L^+L-C^2}{C^2} a_f - v_f^2 \frac{L^+L-C^2}{C^2} \right\}. \tag{6.4}$$

But, by definition, C^2 is the root of the equation (see Eq. (3.8)) $\mathcal{F}'(x) = 0$. Furthermore,

$$\langle \Phi_H^* H \Phi_H \rangle \leqslant \mathcal{F}(C^2).$$

Consequently,

$$\langle \Phi_H^* \sum \Omega(f)\, \alpha_f^+ \alpha_f \Phi_H \rangle \leqslant \langle \Phi_H^* w \Phi_H \rangle. \tag{6.5}$$

Let us now undertake to estimate the average value w. We resort to the definition of the amplitudes α (6.1). We have

$$\alpha_f^+ = w_f\, a_f^+ + v_f \frac{L^+}{C} a_{-f};$$

$$\alpha_{-f} = -v_f\, a_f^+ \frac{L}{C} + w_f\, a_{-f},$$

whence

$$u_f\, \alpha_f^+ - v_f \frac{L^+}{C} \alpha_{-f} = u_f^2\, a_f^+ + v_f^2 \frac{L^+}{C} a_f^+ \frac{L}{C} = a_f^+ \left(u_f^2 + v_f^2 \frac{L^+L}{C^2} \right).$$

Let us set

$$\left.\begin{aligned} \eta_f^+ &= a_f^+ v_f^2 \frac{C^2 - L^+L}{C^2} = v_f^2 \frac{C^2 - L^+L}{C^2} a_f^+ + \frac{2\lambda(f)\, L^+}{VC^2} a_{-f}; \\ \eta_f &= v_f^2 \frac{C^2 - L^+L}{C^2} a_f = v_f^2\, a_f \frac{C^2 - L^+L}{C^2} + \frac{2\lambda(f)}{VC^2} a_{-f}^+ L. \end{aligned}\right\} \tag{6.6}$$

Then

$$\left.\begin{aligned} a_f^+ &= u_f\, \alpha_f^+ - v_f \frac{L^+}{C} \alpha_{-f} + \eta_f^+; \\ a_f &= u_f\, \alpha_f - v_f\, \alpha_{-f}^+ \frac{L}{C} + \eta_f. \end{aligned}\right\} \tag{6.7}$$

Let us now turn to Eq. (6.4); we write:

$$w = w_1 + w_2 + w_3;$$

$$w_1 = \sum \Omega(f) v_f^2 u_f \alpha_f^+ \frac{C^2 - L^+ L}{C^2} a_f = \sum \Omega(f) v_f^2 u_f \alpha_f^+ a_f \frac{C^2 - L^+ L}{C^2} + \sum \Omega(f) v_f^2 u_f \alpha_f^+ \frac{2\lambda(f)}{VC^2} a_{-f}^+ L;$$

$$w_2 = \sum \Omega(f) v_f^2 \eta_f^+ \frac{C^2 - L^+ L}{C^2} a_f = \sum \Omega(f) v_f^2 \eta_f^+ a_f \frac{C^2 - L^+ L}{C^2} + \sum \Omega(f) v_f^2 \eta_f^+ \frac{2\lambda(f)}{VC^2} a_{-f}^+ L;$$

$$\begin{aligned} w_3 = {} & \sum \Omega(f) v_f^2 \left\{ -v_f \frac{L^+}{C} \alpha_{-f} \frac{C^2 - L^+ L}{C^2} a_f - v_f^2 \frac{C^2 - L^+ L}{C^2} \right\} \\ = {} & -\sum \Omega(f) v_f^3 \frac{L^+}{C} \left\{ \frac{L^+ L}{C^2} \alpha_{-f} - \alpha_{-f} \frac{L^+ L}{C^2} \right\} a_f \\ & + \sum \Omega(f) v_f^2 \left\{ -v_f \frac{L^+}{C} \left(\frac{C^2 - L^+ L}{C^2} \right) (\alpha_{-f} a_f + a_f \alpha_{-f}) \right. \\ & \left. - v_f^2 \frac{C^2 - L^+ L}{C^2} \right\} + \sum \Omega(f) v_f^3 \frac{L^+}{C} \left(\frac{C^2 - L^+ L}{C^2} \right) a_f \alpha_{-f}. \end{aligned}$$

Let us estimate the averages w_1, w_2, and w_3; for w_1 we use the previously proved inequality (4.47) which we write in the form

$$\left. \begin{aligned} & \left\langle \Phi_H^* \left(\frac{C^2 - L^+ L}{C^2} \right) \Phi_H \right\rangle \leqslant \frac{G}{V}, \text{ where } G = \text{const}; \\ & \left\langle \Phi_H^* \left(\frac{C^2 - L L^+}{C^2} \right) \Phi_H \right\rangle \leqslant \frac{G}{V}. \end{aligned} \right\} \tag{6.8}$$

We have

$$\begin{aligned} |\langle \Phi_H^* w_1 \Phi_H \rangle| \leqslant {} & \sum \Omega(f) v_f^2 u_f \left| \left\langle \Phi_H^* \alpha_f^+ a_f \frac{C^2 - L^+ L}{C^2} \Phi_H \right\rangle \right| \\ & + \sum \Omega(f) v_f^2 u_f \left| \langle \Phi_H^* \alpha_f^+ a_{-f}^+ L \Phi_H \rangle \right| \frac{2|\lambda(f)|}{VC^2} \\ \leqslant {} & \sum \Omega(f) v_f^2 u_f \sqrt{\langle \Phi_H^* \alpha_f^+ a_f a_f^+ \alpha_f \Phi_H \rangle} \\ & \times \sqrt{\left\langle \Phi_H^* \left(\frac{C^2 - L^+ L}{C^2} \right)^2 \Phi_H \right\rangle} + \sum \Omega(f) v_f^2 u_f \\ & \times \sqrt{\langle \Phi_H^* \alpha_f^+ \alpha_f \Phi_H \rangle \langle \Phi_H^* L^+ a_{-f} a_{-f}^+ L \Phi_H \rangle} \frac{2|\lambda(f)|}{VC^2} \\ \leqslant {} & \sum \Omega(f) v_f^2 u_f \left(\frac{G}{V} \right)^{\frac{1}{2}} \sqrt{\langle \Phi_H^* \alpha_f^+ \alpha_f \Phi_H \rangle} \\ & + \frac{1}{V} \sum \Omega(f) v_f^2 u_f \frac{2|\lambda(f)|}{C^2} |L| \sqrt{\langle \Phi_H^* \alpha_f^+ \alpha_f \Phi_H \rangle} \\ \leqslant {} & \sqrt{\langle \Phi_H^* \sum \Omega(f) \alpha_f^+ \alpha_f \Phi_H \rangle} \left\{ \sqrt{\frac{G}{V} \sum_f \Omega(f) v_f^4 u_f^2} \right. \\ & \left. + \frac{2|L|}{C^2 \sqrt{V}} \sqrt{\frac{1}{V} \sum_f \Omega(f) v_f^4 u_f^2 |\lambda(f)|^2} \right\}. \end{aligned}$$

Consequently,

$$|\langle \Phi_H^* w_1 \Phi_H \rangle| \leqslant R_1 \sqrt{\langle \Phi_H^* \sum_f \Omega(f) \alpha_f^+ \alpha_f \Phi_H \rangle}, \quad R_1 = \text{const}.$$

We find $|\langle \Phi_H^* w_2 \Phi_H \rangle| \leq R_2$, where $R_2 = \text{const}$, in a completely analogous manner. Let us go over to w_3. Note that

$$\alpha_{-f} a_f + a_f \alpha_{-f} = \left(-v_f a_f^+ \frac{L}{C} + u_f a_{-f}\right) a_f$$

$$+ a_f\left(-v_f a_f^+ \frac{L}{C} - u_f a_{-f}\right) = -\frac{v_f}{C}(a_f^+ L a_f + a_f a_f^+ L)$$

$$= -\frac{v_f}{C}(a_f^+ a_f + a_f a_f^+) L = -\frac{v_f}{C} L.$$

This means (see the expression for w_3) that

$$\Delta \equiv \sum_f \Omega(f) v_f^2 \left\{-v_f \frac{L^+}{C}\left(C^2 - \frac{L^+ L}{C^2}\right)(\alpha_{-f} a_f + a_f \alpha_{-f}) - v_f^2 \frac{C^2 - L^+ L}{C^2}\right\}$$

$$= \sum_f \Omega(f) v_f^2 \left\{v_f^2 \frac{L^+}{C}\left(\frac{C^2 - L^+ L}{C^2}\right)\frac{L}{C} - v_f^2 \frac{C^2 - L^+ L}{C^2}\right\} =$$

$$= \sum_f \Omega(f) v_f^4 \frac{L^+}{C^2}\left(\frac{LL^+ - L^+ L}{C^2}\right) L - \sum_f \Omega(f) v_f^4 \left(\frac{C^2 - L^+ L}{C^2}\right)^2,$$

and therefore (see Eq. II, (1.18))

$$\langle \Phi_H^* \Delta \Phi_H \rangle \leqslant \sum_f \Omega(f) v_f^4 \left\langle \Phi_H^* \frac{L^+}{C}\left(\frac{LL^+ - L^+ L}{C^2}\right)\frac{L}{C}\Phi_H \right\rangle$$

$$= \frac{2}{V^2 C^2} \sum_{f, f'} \Omega(f) v_f^4 \lambda^2(f') \left\langle \Phi_H^* \frac{L^+}{C}(1 - a_{f'}^+ a_{f'} - a_{f'}^+ a_{-f'})\frac{L}{C}\Phi_H \right\rangle$$

$$\leqslant 2\frac{|L|^2}{C^4} \cdot \frac{1}{V}\sum_f \Omega(f) v_f^4 \frac{1}{V}\sum_{f'} \lambda^2(f') \leqslant \text{const.}$$

We likewise find

$$\sum_f \Omega(f) |v_f|^3 \left\langle \Phi_H^* \frac{L^+}{C}\left(\frac{L^+ L}{C^2}\alpha_{-f} - \alpha_{-f}\frac{L^+ L}{C^2}\right) a_f \Phi_H \right\rangle \leqslant \text{const};$$

$$\sum_f \Omega(f) |v_f|^3 \left\langle \Phi_H^* \frac{L^+}{C}\left(\frac{C^2 - L^+ L}{C^2}\right) a_f \alpha_{-f} \Phi_H \right\rangle$$

$$\leqslant R_3 \sqrt{\sum_f \langle \Phi_H^* \alpha_f^+ \alpha_f \Phi_H \rangle \Omega(f)}.$$

Thus, combining the expressions for w_1, w_2, and w_3, we have

$$\langle \Phi_H^* w \Phi_H \rangle \leqslant \gamma_1 \sqrt{\left\langle \Phi_H^* \sum_f \Omega(f) \alpha_f^+ \alpha_f \Phi_H \right\rangle} + \gamma_2,$$

where $\gamma_1 = \text{const}$, $\gamma_2 = \text{const}$.

Substituting this inequality into (6.5), we obtain

$$\left\langle \Phi_H^* \sum_f \Omega(f) \alpha_f^+ \alpha_f \Phi_H \right\rangle \leqslant \gamma_1 \sqrt{\left\langle \Phi_H^* \sum_f \Omega(f) \alpha_f^+ \alpha_f \Phi_H \right\rangle} + \gamma_2.$$

Let us place $x = \sqrt{\left\langle \Phi_H^* \sum_f \Omega(f) \alpha_f^+ \alpha_f \Phi_H \right\rangle}$. Then $x^2 - \gamma_1 x \le \gamma_2$;

$$\left(x - \frac{\gamma_1}{2}\right)^2 \leqslant \gamma_2 + \frac{\gamma_1^2}{4} \text{ and } x < \frac{\gamma_1}{2} + \sqrt{\gamma_2 + \frac{\gamma_1^2}{4}}.$$

Thus,

$$\left\langle \Phi_H^* \frac{1}{V}\sum_f \Omega(f) \alpha_f^+ \alpha_f \Phi_H \right\rangle \leqslant \frac{R}{V}, \tag{6.9}$$

where

$$R = \left(\frac{\gamma_1}{2} + \sqrt{\gamma_2 + \frac{\gamma_1^2}{4}}\right)^2 = \text{const.}$$

One can now go over to a consideration of the equations of motion.

For $\nu = 0$, we have

$$\left.\begin{aligned} i\frac{da_f}{dt} &= T(f)\,a_f - \lambda(f)\,a_{-f}^{+} L; \\ i\frac{da_{-f}}{dt} &= T(f)\,a_{-f} + \lambda(f)\,a_f^{+} L \end{aligned}\right\} \quad (6.10)$$

from Eqs. (5.1) and (5.2). Therefore,

$$\begin{aligned} i\frac{dL}{dt} &= \frac{1}{V}\sum\lambda(f)\{T(f)\,a_{-f} + \lambda(f)\,a_f^{+} L\}\,a_f \\ &\quad + \frac{1}{V}\sum\lambda(f)\,a_{-f}\{T(f)\,a_f - \lambda(f)\,a_{-f}^{+} L\} \\ &= \frac{2}{V}\sum\lambda(f)\,T(f)\,a_{-f}a_f + \frac{1}{V}\sum\lambda^2(f)\,(a_f^{+}a_f - a_{-f}a_{-f}^{+})\,L \\ &= \frac{2}{V}\sum\lambda(f)\,T(f)\,a_{-f}a_f + \frac{1}{V}\sum\lambda^2(f)\,(a_f^{+}a_f - a_f a_f^{+})\,L \\ &= \frac{2}{V}\sum\lambda(f)\,T(f)\,a_{-f}a_f + \frac{1}{V}\sum\lambda^2(f)\,(2a_f^{+}a_f - 1)\,L. \end{aligned}$$

Note now that

$$\begin{aligned} &-2\lambda(f)\,T(f)\,u_f v_f\frac{L}{C} + \lambda^2(f)\,(2v_f^2 - 1)\,L \\ &= \lambda^2(f)\frac{T(f)}{\Omega(f)}L + \lambda^2(f)\left(1 - \frac{T(f)}{\Omega(f)} - 1\right)L = 0, \end{aligned}$$

and consequently,

$$\left.\begin{aligned} &i\frac{dL}{dt} = D_1 + D_2; \\ &D_1 = \frac{2}{V}\sum\lambda(f)\,T(f)\left\{a_{-f}a_f + u_f v_f\frac{L}{C}\right\}; \\ &D_2 = \frac{2}{V}\sum\lambda^2(f)\,(a_f^{+}a_f - v_f^2)\,L. \end{aligned}\right\} \quad (6.11)$$

But with allowance for (6.7),

$$\begin{aligned} a_{-f}a_f + u_f v_f\frac{L}{C} &= \left(u_f\alpha_{-f} + v_f\alpha_f^{+}\frac{L}{C}\right)\left(u_f\alpha_f - v_f\alpha_{-f}^{+}\frac{L}{C}\right) \\ &\quad + \eta_{-f}a_f + a_{-f}\eta_f - \eta_{-f}\eta_f + u_f v_f\frac{L}{C} = u_f^2\alpha_{-f}\alpha_f - \\ &\quad - v_f^2\alpha_f^{+}\frac{L}{C}\alpha_{-f}^{+}\frac{L}{C} - u_f v_f\alpha_{-f}\alpha_{-f}^{+}\frac{L}{C} + u_f v_f\alpha_f^{+}\frac{L}{C}\alpha_f \\ &\quad + \eta_{-f}a_f + a_{-f}\eta_f - \eta_{-f}\eta_f + u_f v_f\frac{L}{C} = u_f^2\alpha_{-f}\alpha_f - \\ &\quad - u_f v_f(\alpha_{-f}\alpha_{-f}^{+} + \alpha_{-f}^{+}\alpha_{-f} - 1)\frac{L}{C} - v_f^2\alpha_f^{+}\frac{L}{C}\alpha_{-f}^{+}\frac{L}{C} \\ &\quad + u_f v_f\left(\alpha_f^{+}\frac{L}{C}\alpha_f + \alpha_{-f}^{+}\alpha_{-f}\frac{L}{C}\right) + \eta_{-f}a_f + a_{-f}\eta_f - \eta_{-f}\eta_f. \end{aligned} \quad (6.12)$$

Further, we have

$$\alpha_{-f}\alpha_{-f}^{+} + \alpha_{-f}^{+}\alpha_{-f} - 1 = \left(-v_f a_f^{+}\frac{L}{C} + u_f a_{-f}\right)$$

$$\times\left(-v_f\frac{L^+}{C}a_f+u_fa^+_{-f}\right)+\left(-v_f\frac{L^+}{C}a_f+u_fa^+_{-f}\right)$$
$$\times\left(-v_fa_f^+\frac{L}{C}+u_fa_{-f}\right)-1=v_f^2a_f^+\frac{LL^+}{C^2}a_f+u_f^2a_{-f}a^+_{-f}$$
$$-u_fv_fa_f^+\frac{L}{C}a^+_{-f}-u_fv_fa_{-f}\frac{L^+}{C}a_f+v_f^2\frac{L^+}{C}a_fa_f^+\frac{L}{C}$$
$$+u_f^2a^+_{-f}a_{-f}-u_fv_fa^+_{-f}a_f^+\frac{L}{C}-u_fv_f\frac{L^+}{C}a_fa_{-f}-1$$
$$=v_f^2a_f^+\frac{LL^+-L^+L}{C^2}a_f+v_f^2\frac{L^+L}{C^2}+u_f^2-1$$
$$-u_fv_fa_f^+\left(\frac{L}{C}a^+_{-f}-a^+_{-f}\frac{L}{C}\right)-u_fv_f\left(a_{-f}\frac{L^+}{C}-\frac{L^+}{C}a_{-f}\right)a_f,$$

and therefore,

$$\alpha_{-f}\alpha^+_{-f}+\alpha^+_{-f}\alpha_{-f}-1=\frac{2}{V^2}\sum_{(g)}v_f^2a_f^+\frac{\lambda^2(g)}{C^2}\left(1-a_g^+a_g-a^+_{-g}a_{-g}\right)a_f$$

$$+v_f^2\frac{L^+L-C^2}{C^2}+u_fv_fa_f^+a_f\frac{2\lambda(f)}{V}+u_fv_fa_f^+a_f\frac{2\lambda(f)}{V}. \tag{6.13}$$

Consequently, we have

$$\langle\Phi_H^*D_1D_1^+\Phi_H\rangle=\frac{2}{V}\sum_f\lambda(f)T(f)u_f^2\langle\Phi_H^*\alpha_{-f}\alpha_fD_1^+\Phi_H\rangle$$
$$-\frac{2}{V}\sum_f\lambda(f)T(f)\left\langle\Phi_H^*\alpha_f^+\left\{v_f^2\frac{L}{C}\alpha^+_{-f}\frac{L}{C}\right.\right.$$
$$\left.\left.-u_fv_f\left(\frac{L}{C}\alpha_f+\alpha_f\frac{L}{C}\right)\right\}D_1^+\Phi_H\right\rangle-\frac{4}{V^3}\sum_{fg}\lambda(f)T(f)u_fv_f^3\frac{\lambda^2(g)}{C^2}$$
$$\times\left\langle\Phi_H^*a_f^+\left(1-a_g^+a_g-a^+_{-g}a_{-g}\right)a_f\frac{L}{C}D_1^+\Phi_H\right\rangle$$
$$+\frac{2}{V}\sum_f\lambda(f)T(f)\langle\Phi_H^*(\eta_{-f}a_f-\eta_fa_{-f}+[a_{-f}\eta_f+\eta_fa_{-f}]$$
$$-\eta_{-f}\eta_f)D_1^+\Phi_H\rangle-\frac{2}{V}\sum_f\lambda(f)T(f)\left\langle\Phi_H^*\left\{v_f^2\frac{L^+L-C^2}{C^2}\right.\right.$$
$$\left.\left.+2u_fv_fa_f^+a_f\frac{2\lambda(f)}{V}\right\}\frac{L}{C}D_1^+\Phi_H\right\rangle.$$

Taking into account the fact that

$$\langle\Phi_H^*\alpha_{-f}\alpha_fD_1^+\Phi_H\rangle=\langle\Phi_H^*\alpha_{-f}D_1^+\alpha_f\Phi_H\rangle$$
$$+\langle\Phi_H^*\alpha_{-f}(\alpha_fD_1^+-D_1^+\alpha_f)\Phi_H\rangle,$$

one can verify the fact that with allowance for (6.8) and (6.9),

$$\langle\Phi_H^*D_1D_1^+\Phi_H\rangle\leqslant\frac{\Gamma_1}{V},\quad\Gamma_1=\text{const.} \tag{6.14}$$

In the same way one may obtain

$$\langle\Phi_H^*D_1^+D_1\Phi_H\rangle\leqslant\frac{\Gamma_2}{V},\quad\Gamma_2=\text{const.} \tag{6.15}$$

Let us now go over to the expression D_2. We have

$$a_f^+a_f-v_f^2=a_f^+\eta_f+\eta_f^+a_f-\eta_f^+\eta_f+\left(u_fa_f^+-v_f\frac{L^+}{C}\alpha_{-f}\right)$$

$$\times\left(u_f\alpha_f - v_f\alpha_{-f}^{+}\frac{L}{C}\right) - v_f^2 = u_f^2\,\alpha_f^{+}\alpha_f + v_f^2\,\frac{L^{+}}{C}\alpha_{-f}\alpha_{-f}^{+}\frac{L}{C}$$

$$-v_f^2 - u_f v_f \alpha_f^{+}\alpha_{-f}^{+}\frac{L}{C} - u_f v_f\frac{L^{+}}{C}\,\alpha_{-f}\alpha_f + a_f^{+}\eta_f +$$

$$+\eta_f^{+}a_f - \eta_f^{+}\eta_f = u_f^2\,\alpha_f^{+}\alpha_f + v_f^2\,\frac{L^{+}}{C}\,(\alpha_{-f}\alpha_{-f}^{+} + \alpha_{-f}^{+}\alpha_{-f} - 1) - \frac{L}{C}$$

$$-v_f^2\,\frac{C^2 - L^{+}L}{C^2} - v_f^2\,\frac{L^{+}}{C}\,\alpha_{-f}^{+}\alpha_{-f}\frac{L}{C} - u_f v_f \alpha_f^{+}\alpha_{-f}^{+}\frac{L}{C}$$

$$-u_f v_f\frac{L^{+}}{C}\,\alpha_{-f}\alpha_f + a_f^{+}\eta_f + \eta_f^{+}a_f - \eta_f^{+}\eta_f\,. \qquad (6.16)$$

Starting from (6.16) and considering the inequalities (6.8) and (6.9), we establish the fact that

$$\left.\begin{aligned}&\langle\Phi_H^{*}D_2D_2^{+}\Phi_H\rangle \leqslant \frac{\Gamma_3}{V},\ \Gamma_3 = \text{const};\\ &\langle\Phi_H^{*}D_2^{+}D_2\Phi_H\rangle \leqslant \frac{\Gamma_3}{V}.\end{aligned}\right\} \qquad (6.17)$$

From (7.10) we now have

$$\left.\begin{aligned}&\left\langle\Phi_H^{*}\left(\frac{dL}{dt}\right)^{+}\frac{dL}{dt}\,\Phi_H\right\rangle \leqslant \frac{\Gamma}{V},\ \Gamma = \text{const};\\ &\left\langle\Phi_H^{*}\frac{dL}{dt}\left(\frac{dL}{dt}\right)^{+}\Phi_H\right\rangle \leqslant \frac{\Gamma}{V}.\end{aligned}\right\} \qquad (6.18)$$

Let us go back again to the equations of motion (6.11). Considering (6.1), (6.2), we obtain

$$\begin{aligned}\mathrm{i}\,\frac{d\alpha_f^{+}}{dt} &= \mathrm{i}\,\frac{d}{dt}\left(u_f a_f^{+} + v_f\frac{L^{+}}{C}\,a_{-f}\right) = u_f\,\mathrm{i}\,\frac{da_f^{+}}{dt} + v_f\,\frac{L^{+}}{C}\,\mathrm{i}\,\frac{da_{-f}}{dt}\\ &+ v_f\,\mathrm{i}\,\frac{dL^{+}}{dt}\cdot\frac{a_{-f}}{C} = u_f\left\{-T(f)\,a_f^{+} + \lambda(f)\,L^{+}a_{-f}\right\}\\ &+ v_f\frac{L^{+}}{C}\left\{T(f)\,a_{-f} + \lambda(f)\,a_f^{+}L\right\} + v_f\,\mathrm{i}\,\frac{dL^{+}}{dt}\cdot\frac{a_{-f}}{C}\\ &= -a_f^{+}\left\{T(f)\,u_f - \lambda(f)\,v_f\,\frac{L^{+}L}{C}\right\} + \frac{L^{+}}{C}\left\{u_f\lambda(f)\,C + T(f)\,v_f\right\}a_{-f}\\ &+ v_f\,\mathrm{i}\,\frac{dL^{+}}{dt}\cdot\frac{a_{-f}}{C} = -a_f^{+}\left\{T(f)\,u_f - \lambda(f)\,v_fC\right\}\\ &+ \frac{L^{+}}{C}\left\{u_f\lambda(f)\,C + T(f)\,v_f\right\}a_{-f} - a_f^{+}\lambda(f)\,v_f\,\frac{C^2 - L^{+}L}{C} + v_f\,\mathrm{i}\,\frac{dL^{+}}{dt}\cdot\frac{a_{-f}}{C}.\end{aligned}$$

But (see Eq. (5.6))

$$\left.\begin{aligned}&u_f\lambda(f)\,C + T(f)\,v_f = -\Omega(f)\,v_f;\\ &T(f)\,u_f - \lambda(f)\,v_fC = \Omega(f)\,u_f,\end{aligned}\right\} \qquad (6.19)$$

and therefore,

$$\mathrm{i}\,\frac{d\alpha^{+}}{dt} + \Omega(f)\,\alpha_f^{+} = R_f, \qquad (6.20)$$

where

$$R_f = -a_f^{+}\lambda(f)\,v_f\,\frac{C^2 - L^{+}L}{C} + v_f\,(D_1^{+} + D_2^{+})\,\frac{a_{-f}}{C}\,.$$

Further, we have

$$\langle\Phi_H^* R_f^+ R_f \Phi_H\rangle \leqslant 2\left\langle\Phi_H^* \frac{C^2 - L^+ L}{C} a_f a_f^+ \frac{C^2 - L^+ L}{C} \Phi_H\right\rangle \lambda^2(f) v_f^2$$

$$+ 2\left\langle\Phi_H^* \frac{a_{-f}^+}{C}(D_1 + D_2)(D_1^+ + D_2^+)\frac{a_{-f}}{C}\Phi_H\right\rangle v_f^2$$

$$\leqslant 2\lambda^2(f) v_f^2 \left\langle\Phi_H^* \frac{(C^2 - L^+ L)^2}{C^2}\Phi_H\right\rangle$$

$$+ 2v_f^2\left\langle\Phi_H^*\left\{\frac{a_{-f}^+}{C}(D_1 + D_2)(D_1^+ + D_2^+)\frac{a_{-f}}{C}\right.\right.$$

$$\left.\left. - (D_1 + D_2)\frac{a_{-f}^+ a_{-f}}{C^2}(D_1^+ + D_2^+)\right\}\Phi_H\right\rangle$$

$$+ 2\langle\Phi_H^*(D_1 + D_2)(D_1^+ + D_2^+)\Phi_H\rangle\frac{v_f^2}{C^2}$$

and likewise

$$\langle\Phi_H^* R_f R_f^+ \Phi_H\rangle \leqslant 2\left\langle\Phi_H^* a_f^+\left(\frac{C^2 - L^+ L}{C}\right)^2 a_f \Phi_H\right\rangle \lambda^2(f) v_f^2$$

$$+ \frac{2v_f^2}{C}\langle\Phi_H^*(D_1^+ + D_2^+)(D_1 + D_2)\Phi_{\mathscr{H}}\rangle$$

$$= 2\lambda^2 v_f^2\left\langle\Phi_H^*\left\{a_f^+\left(\frac{C^2 - L^+ L}{C}\right)\left(\frac{C^2 - L^+ L}{C}\right)a_f\right.\right.$$

$$\left.\left. - \left(\frac{C^2 - L^+ L}{C}\right)a_f^+ a_f\left(\frac{C^2 - L^+ L}{C}\right)\right\}\Phi_H\right\rangle$$

$$+ 2\lambda^2 v_f^2\left\langle\Phi_H^*\left(\frac{C^2 - L^+ L}{C}\right)^2\Phi_H\right\rangle$$

$$+ 2\frac{v_f^2}{C^2}\langle\Phi_H^*(D_1^+ + D_2^+)(D_1 + D_2)\Phi_H\rangle.$$

From this it follows that

$$\left.\begin{aligned}&\langle\Phi_H^* R_f R_f^+ \Phi_H\rangle \leqslant v_f^2\frac{S}{V}, \text{ where } S = \text{const};\\ &\langle\Phi_H^* R_f^+ R_f \Phi_H\rangle \leqslant v_f^2\frac{S}{V}.\end{aligned}\right\} \tag{6.21}$$

Having Eqs. (6.20) and the inequalities (6.21), one may repeat word-for-word our reasoning from the previous section where we considered the case $\nu > 0$. We now obtain (see Eq. (5.35))

$$\langle\Phi_H^* \alpha_f^+ \alpha_f \Phi_H\rangle \leqslant \frac{S}{V}\cdot\frac{v_f^2}{2\pi\Omega^2(f)}\left(\int_{-\infty}^{+\infty}|h(\tau)\tau|\,d\tau\right)^2. \tag{6.21'}$$

In comparison with the inequality (6.9) we have substantial progress here.

The inequality (6.9) shows that $<\Phi_H^* \alpha_f^+ \alpha_f \Phi_H>$ is a quantity of order 1/V on the average for f. However, the inequality (6.21') shows that this expression will be of order 1/U for each f.

From (6.21') we may immediately obtain the estimates for the averages which apply to one time. Assume $\mathfrak{A}_f$ is equal to a_f or a_f^+. Let us consider those operators $\mathfrak{A}_{f_1}\mathfrak{A}_{f_2}\cdots\mathfrak{A}_{f_k}$ which conserve the number of particles. Let us show that

$$\left|\langle\mathfrak{A}_{f_1}\mathfrak{A}_{f_2}\ldots\mathfrak{A}_{f_k}\rangle_H - \langle\mathfrak{A}_{f_1}\mathfrak{A}_{f_2}\ldots\mathfrak{A}_{f_k}\rangle_{H_0}\right| \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{6.22}$$

Now note that Φ_H and Φ_{H_0} satisfy the conditions (2.3)

$$(a_f^+ a_f - a_{-f}^+ a_{-f})\Phi = 0.$$

Therefore,

$$\langle \mathfrak{A}_{f_1} \mathfrak{A}_{f_2} \mathfrak{A}_{f_3} \ldots \mathfrak{A}_{f_h} \rangle$$

may be reduced to the sum of terms of the type

$$\langle \ldots a_f^+ a_f \ldots a_g^+ a_{-g}^+ \ldots a_{-h} a_h \ldots \rangle,$$

where $\pm f$, $\pm g$, $\pm h$ are all different. Of course, the number of subscripts g is equal to the number of subscripts h here. It is likewise obvious that

$$\langle \ldots a_f^+ a_f \ldots a_g^+ a_{-g}^+ \ldots a_{-h} a_h \ldots \rangle_{H_0} = \prod_f v_f^2 \prod_g (-u_g v_g) \prod_h (-u_h v_h).$$

Consequently, we need merely establish the fact that

$$\left| \langle \ldots a_f^+ a_f \ldots a_g^+ a_{-g}^+ \ldots a_{-h} a_h \ldots \rangle_H \right| - \prod_f v_f^2 \prod_g (-u_g v_g) \prod_h (-u_h v_h) \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{6.23}$$

On the basis of what was said previously (see Eqs. (6.12), (6.13), and (6.16)) we note that that

$$\begin{aligned} a_{-h} a_h + u_h v_h \frac{L}{C} &= u_h^2 \alpha_{-h} \alpha_h - u_h v_h \Big\{ \frac{2}{V^2} \sum_{(f)} v_h^2 \alpha_h^+ \frac{\lambda^2(f)}{C^2} \\ &\times (1 - a_f^+ a_f - a_{-f}^+ a_{-f})\, a_h + v_h^2 \frac{L^+ L - C^2}{C^2} \\ &+ u_h v_h a_h^+ a_h \frac{4\lambda(h)}{V} \Big\} \frac{L}{C} - v_h^2 \alpha_h^+ \frac{L}{C} \alpha_{-h}^+ \frac{L}{C} + u_h v_h \\ &\times \left(\alpha_h^+ \frac{L}{C} \alpha_h + \alpha_{-h}^+ \alpha_{-h} \frac{L}{C} \right) + \eta_{-h} a_h + a_{-h} \eta_{-h} - \eta_{-h} \eta_h; \end{aligned} \tag{6.24}$$

$$\begin{aligned} a_f^+ a_f - v_f^2 &= u_f^2 \alpha_f^+ \alpha_f - v_f^2 \frac{L^+}{C} \alpha_{-f}^+ \alpha_{-f} \frac{L}{C} - u_f v_f \alpha_f^+ \alpha_{-f}^+ \frac{L}{C} \\ &- u_f v_f \frac{L^+}{C} \alpha_{-f} \alpha_f + v_f^2 \frac{L^+}{C} \Big\{ \frac{2}{V^2} \sum_g v_f^2 a_f^+ \frac{\lambda^2(g)}{C^2} \\ &\times (1 - a_g^+ a_g - a_{-g}^+ a_{-g})\, a_f + v_f^2 \frac{L^+ L - C^2}{C^2} \\ &+ u_f v_f \frac{4\lambda}{V} a_f^+ a_f \Big\} + a_f^+ \eta_f + \eta_f^+ a_f - \eta_f^+ \eta_f \end{aligned} \tag{6.25}$$

We shall now shift α^+ toward the left parenthesis, α to the right parenthesis, and L^+L-C^2/C^2 (for example, those included in η, η^+) to one of them, no matter to which. Since the subscripts $\pm f$, $\pm g$, $\pm L$ are all different, it follows that the commutators which develop in the process of these commutations will be quantities of order $1/V$. Considering the constantly used inequality $|\langle AB \rangle| \leq \sqrt{\langle AA^+ \rangle \langle B^+B \rangle}$, we see that as soon as α^+ "touches" the left parenthesis or α touches the right parenthesis, or L^+L-C^2 arrives at either of them, we obtain a quantity having an order of smallness of at least const/$\sqrt{V}$ at that instant. Consequently,

$$|\langle \ldots a_f^+ a_f \ldots a_g^+ a_{-g}^+ \ldots a_{-h} a_h \ldots \rangle_H$$

$$\Pi_f v_f^2 \left\langle \ldots (-u_g v_g) \frac{L^+}{C} (-u_h v_h) \frac{L}{C} \ldots \right\rangle_H \Big| \leqslant \frac{\text{const}}{\sqrt{V}}. \tag{6.26}$$

But the number g is equal to the number h, and the commutation of L with L^+ is carried out with an accuracy up to terms of 1/V. Therefore, $\left\langle \ldots u_g v_g \frac{L^+}{C} \ldots u_h v_h \frac{L}{C} \ldots \right\rangle_H$ differs from $\prod_g u_g v_g \prod_h u_h v_h \left\langle \left(\frac{L^+ L}{C^2} \right)^l \right\rangle_H$ by terms of order 1/V. On the other hand,

$$\left\langle \left(\frac{L^+ L}{C^2} \right)^l \right\rangle_H$$

differs from unity by a quantity at least of order $1/\sqrt{V}$.

Thus, the validity of the inequalities (6.23) (and consequently (6.22) as well) has been proved. Let us now go over to the bitemporal correlation averages and show that in general

$$\begin{gathered} |\langle \mathfrak{B}_{f_1}(t) \ldots \mathfrak{B}_{f_l}(t);\ \mathfrak{A}_{g_1}(\tau) \ldots \mathfrak{A}_{g_k}(\tau) \rangle_H \\ -\langle \mathfrak{B}_{f_1}(t) \ldots \mathfrak{B}_{f_l}(t);\ \mathfrak{A}_{g_1}(\tau) \ldots \mathfrak{A}_{g_k}(\tau) \rangle_{H_0} | \leqslant \frac{K(t-\tau)+K_1}{\sqrt{V}}, \\ K = \text{const},\ K_1 = \text{const}. \end{gathered} \tag{6.27}$$

Here $\mathfrak{P}_f$, $\mathfrak{A}_g$ are equal to a or a^+. We assume, as always in such a case, that the operator $\mathfrak{P}_{f_1} \ldots \mathfrak{A}_{g_k}$ conserves the number of particles.

By virtue of the additional conditions mentioned earlier, which are satisfied by Φ_H and Φ_{H_0}, the investigated averages may be reduced to a sum of terms of the type

$$\begin{gathered} \langle \ldots a_f^+(t) a_f(t) \ldots a_g^+(t) a_{-g}^+(t) \ldots a_{-h}(t) a_h(t) \ldots \\ \ldots a_k^+(t) \ldots a_g(t) \ldots a_{f'}^+(\tau) a_{f'}(\tau) \ldots a_{g'}^+(\tau) a_{-g'}^+(\tau) \ldots \\ \ldots a_{-h'}(\tau) a_{h'}(\tau) \ldots a_{k'}(\tau) \ldots a_{q'}^+(\tau) \ldots \rangle, \end{gathered} \tag{6.28}$$

by establishing the "proper order" of the operators; here the number of operators a and a^+ is also identical, where the subscripts $\pm f$, $\pm$g, $\pm$h, $\pm$k, $\pm$q and $\pm f'$, $\pm$g', $\pm$h', $\pm$k', $\pm$q' are all different.

In view of the indicated possibility of reduction it is sufficient for us to prove the inequalities (6.27) for averages of the type (6.28). For the "pairs" a^+a, a^+a^+, aa we make use of Eqs. (6.24) and (6.25), while for "singles" a, a^+ we make use of Eqs. (6.7). We shall now shift $\alpha^+(f)$ and $L^+(t)L(t)-C^2$ to the left, while $\alpha(\tau)$ and $L^+(\tau)L(\tau)-C^2$ are shifted to the right. In view of the difference emphasized above between the subscripts, the commutators which appear (all commutations are carried out only between amplitudes which apply to the same time) will yield quantities of order 1/V. Note that as soon as $\alpha^+(t)$ or $L^+(t)L(t)-C^2$ "touch" the right parenthesis, we immediately obtain quantities at least of order $1/\sqrt{V}$. Consequently, it remains merely for us to show that an inequality of the type (6.27) holds for averages of the form

$$\Gamma(t-\tau) = \langle \alpha_{f_1}(t) \ldots \alpha_{f_l}(t) L^k(t)\, L^{+q}(t) L^{+q_1}(\tau) L^{k_1}(\tau) \alpha_{g_1}^+(\tau) \ldots \alpha_{g_r}^+(\tau) \rangle. \tag{6.29}$$

Let us now make use of the equations of motion (6.19) and the relationships (6.11), (6.18), (6.20), and (6.21) which yield the required estimates. We find

$$i \frac{\partial \Gamma_H(t-\tau)}{\partial t} - \{\Omega(f_1) + \ldots + \Omega(f_l)\} \Gamma_H(t-\tau) = \Delta(t-\tau),$$

for the condition that $|\Delta(t-\tau)| \leqslant \frac{G}{\sqrt{V}}$, where G = const. From this, since

$$\Gamma_H(t-\tau) = e^{-i\{\Omega(f_1)+\ldots+\Omega(f_l)\}(t-\tau)}\Gamma_H(0)$$

$$+e^{-i\{\Omega(f_1)+\ldots+\Omega(f_l)\}(t-\tau)}\int_0^{t-\tau} e^{-i\{\Omega(f_1)+\Omega(f_2)+\ldots+\Omega(f_l)\}z}\Delta(z)\,dz,$$

we obtain

$$\left|\Gamma_H(t-\tau) - e^{-i\{\Omega(f_1)+\ldots+\Omega(f_l)\}(t-\tau)}\Gamma_H(0)\right| \leqslant \frac{G|t-\tau|}{\sqrt{V}}. \tag{6.30}$$

On the other hand,

$$\Gamma_{H_0}(t-\tau) = e^{i\{\Omega(f_1)+\ldots+\Omega(f_l)\}(t-\tau)}\Gamma_{H_0}(0), \tag{6.31}$$

since

$$\Gamma_{H_0}(t-\tau) = \left\langle \alpha_{f_1}(t)\ldots\alpha_{f_l}(t)\alpha^+_{g_1}(\tau)\ldots\alpha^+_{g_r}(\tau)\right\rangle C^{k+q+q_1+k_1}. \tag{6.32}$$

Thus,

$$|\Gamma_H(t-\tau) - \Gamma_{H_0}(t-\tau)| \leqslant |\Gamma_H(0) - \Gamma_{H_0}(0)|$$

$$+\frac{G|t-\tau|}{\sqrt{V}} = \left|\left\langle \alpha_{f_1}\ldots\alpha_{f_l}L^k(L^+)^{q+q_1}L^{k_1}\alpha^+_{g_1}\ldots\alpha^+_{g_r}\right\rangle_H\right.$$

$$\left.-C^{k+k_1+q+q_1}\left\langle\alpha_{f_1}\ldots\alpha_{f_l}\alpha^+_{g_1}\ldots\alpha^+_{g_r}\right\rangle_{H_0}\right| + \frac{G(t-\tau)}{\sqrt{V}}. \tag{6.33}$$

Assume that among the subscripts $f_1, \ldots, f_l$ there is a pair of identical ones. Then, nothing that

$$\alpha_f^2 = \left(u_f a_f + v_f a^+_{-f}\frac{L}{C}\right)\left(u_f a_f + v_f a^+_{-f}\frac{L}{C}\right)$$

$$= v_f^2\, a^+_{-f}\frac{L}{C}a^+_{-f}\frac{L}{C} + u_f v_f\left\{a^+_{-f}\frac{L}{C}a_f + a_f a^+_{-f}\frac{L}{C}\right\}$$

$$= \frac{v_f^2 a^+_{-f}}{C}(La^+_{-f} - a^+_{-f}L)L$$

$$-u_f v_f\{a^+_{-f}a_f + a_f a^+_{-f}\}\frac{L}{C} = -2\frac{v_f^2\lambda(f)}{C^2V}a^+_{-f}a_f L \tag{6.34}$$

will be of order 1/V, we see that $<\ldots>_H$ will also be of the same order. The corresponding $<\ldots>_{H_0}$ are simply equal to zero. The same situation naturally develops in the case in which among the subscripts $g_1, \ldots, g_r$ there is just one pair of identical ones.

Assume further that among the subscripts $f_1, \ldots, f_l$ there is at least one subscript f_j which is not included among $g_1, \ldots, g_r$. Then we may shift α_{f_j} to the right parenthesis in $<\ldots>_H$, obtaining (along the way) commutators of order 1/V; thus we verify the results that $<\ldots>_H$ in the case given turns out to be a quantity having an order of smallness no lower than $1/\sqrt{V}$. The average $<\ldots>_{H_0}$, however, is exactly equal to zero. An analogous situation arises if among $g_1, \ldots, g_r$ there is just one subscript which is not included in $f_1, \ldots, f_l$.

Thus, it remains for us to consider the case in which 1) all $f_1, \ldots, f_l$ are different; 2) the ensemble $g_1, \ldots, g_r$ is the same ensemble $f_1, \ldots, f_l$, but, perhaps, is numbered in a different order.

Now note that in the right side of (6.33) one can establish the "proper order" and replace $\alpha_{g_1}^+ \ldots \alpha_{g_r}^+$ by $\alpha_{f_l}^+ \ldots \alpha_{f_1}^+$. It is natural that in $< \ldots >_{H_0}$ we carry out such a substitution exactly, while in $< \ldots >_H$ we carry it out with an error in the adopted order which is asymptotically small. Since the operators within $< \ldots >$ conserve the number of particles, $k + k_1$ must equal $q + q_1$.

Further, in $< \alpha_{f_1} \ldots \alpha_{f_l} L^k (L^+)^{k+k_1} L^{k_1} \alpha_{f_l}^+ \ldots \alpha_{f_1}^+ >$ we carry out the substitution $L^k (L^+)^{k+k_1} L^{k_1} \to (L^+ L)^{k+k_1}$ and transfer it to the right parenthesis. Under these conditions we produce an error of order 1/V. Note further that

$$\left| \langle \alpha_{f_1} \ldots \alpha_{f_l} \alpha_{f_l}^+ \ldots \alpha_{f_1}^+ (L^+ L)^{k+k_1} \rangle_H - \langle \alpha_{f_1} \ldots \alpha_{f_l} \alpha_{f_l}^+ \ldots \alpha_{f_1}^+ \rangle_{H_0} C^{2(k+k_1)} \right| \leqslant \frac{\text{const}}{\sqrt{V}}. \quad (6.35)$$

Thus, from (6.33), we obtain

$$\left| \Gamma_H (t - \tau) - \Gamma_{H_0} (t - \tau) \leqslant \frac{G |t - \tau|}{\sqrt{V}} + \frac{k}{\sqrt{V}} \right.$$

$$+ C^{2(k+k_1)} \left| \langle \alpha_{f_1} \ldots \alpha_{f_l} \ldots \alpha_{f_l}^+ \ldots \alpha_{f_1}^+ \rangle_H - \langle \alpha_{f_1} \ldots \alpha_{f_l} \alpha_{f_l}^+ \ldots \alpha_{f_1}^+ \rangle_{H_0} \right|. \quad (6.36)$$

But since all f are different,

$$\langle \alpha_{f_1} \ldots \alpha_{f_l} \alpha_{f_l}^+ \ldots \alpha_{f_1}^+ \rangle_{H_0} = \langle \alpha_{f_1} \alpha_{f_1}^+ \rangle_{H_0} \langle \alpha_{f_2} \alpha_{f_2}^+ \rangle_{H_0} \ldots \langle \alpha_{f_l} \ldots \alpha_{f_l}^+ \rangle_{H_0} = 1.$$

In $< \ldots >_H$ such a distribution may likewise be achieved, but, of course, not exactly but with the allowed asymptotic error.

Thus, our proof has been completed. Just as in the case $\nu > 0$, we could have obtained analogous estimates of the degree of asymptotic approximation for multitemporal correlation functions also. We shall not dwell on this here. The reader may now carry out all of the calculations involved in this himself using the schemes developed above. As in the case of $\nu > 0$, the order of smallness in the case considered may be raised from const/$\sqrt{V}$ to const/V if in the Hamiltonian H_0 the constant C is replaced by $C_1 = \sqrt{\langle L^+ L \rangle_H}$, which differs from C by a quantity of order $1/\sqrt{V}$.

We shall not prove this remark here.

APPENDIX I

In the present section we present the proofs of certain relationships used in this paper.* All of the operators considered here are assumed to be totally continuous, and we deal only with this kind of operator in the main text.

Lemma I. Assume that the operator ξ satisfies the condition

$$| \xi \xi^+ - \xi^+ \xi | \ll \frac{2s}{V}, \quad (A1.1)$$

where s is a number; $\varepsilon = 1$ or $\varepsilon = -1$. Then the inequality

$$2 \sqrt{\xi^+ \xi + \frac{s}{V}} - \varepsilon (\xi + \xi^+) \geqslant 0 \quad (A1.2)$$

holds.

*Let us arbitrarily denote the norm of the function by $\| \Phi \| = \sqrt{< \Phi^* \Phi >}$ and the norm of the operator by $| \mathfrak{A} | = \sup \| \mathfrak{A} \Phi \|$, where $\| \Phi \| = 1$.

Proof. Let us assume the opposite; then one can find a normalized function φ which is such that

$$\left\{2\sqrt{\xi^+\xi+\frac{s}{V}}-\varepsilon(\xi+\xi^+)\right\}\varphi=-\rho\varphi,$$

where $\rho > 0$. From this we have

$$\left(2\sqrt{\xi^+\xi+\frac{s}{V}}+\rho\right)\varphi=\varepsilon(\xi+\xi^+)\varphi. \tag{A1.3}$$

Now let us take into account the fact that $A\varphi = B\varphi$ and A and B are self-conjugate operators, then

$$\langle\varphi^* A^2\varphi\rangle=\langle\varphi^* B^2\varphi\rangle. \tag{A1.4}$$

Considering (A1.4) and (A1.1), we shall have

$$\langle\varphi^*(2\sqrt{\xi^+\xi+s/V}+\rho)^2\varphi\rangle=\langle\varphi^*(\xi+\xi^+)^2\varphi\rangle=2\langle\varphi^*(\xi\xi^++\xi^+\xi)\varphi\rangle$$

$$-\langle\varphi^*(\xi^+-\xi)(\xi-\xi^+)\varphi\rangle\leqslant 2\langle\varphi^*(\xi\xi^++\xi^+\xi)\varphi\rangle$$

$$\leqslant(2\varphi^*(\xi^+\xi+2s/V+\xi^+\xi)\varphi)=4\left\langle\varphi^*\left(\xi^+\xi+\frac{s}{V}\right)\varphi\right\rangle, \tag{A1.5}$$

which is impossible for $\rho > 0$. Thus, the inequality (A1.2) has been proved.

Corollary. We likewise have, having transposed ξ and ξ^+,

$$2\sqrt{\xi\xi^++\frac{s}{V}}-\varepsilon(\xi+\xi^+)\geqslant 0. \tag{A1.6}$$

The following inequalities are also proved analogously:

$$2\sqrt{\xi\xi^++\frac{s}{V}}-\varepsilon\frac{(\xi-\xi^+)}{i}\geqslant 0; \tag{A1.7}$$

$$2\sqrt{\xi^+\xi+\frac{s}{V}}-\varepsilon\frac{(\xi-\xi^+)}{i}\geqslant 0. \tag{A1.8}$$

Lemma II. Assume ξ satisfies the condition

$$|\xi\xi^+-\xi^+\xi|\leqslant 2s/V. \tag{A1.9}$$

Then

$$\sqrt{\xi\xi^++\frac{2s}{V}+A^2}-\sqrt{\xi^+\xi+A^2}\geqslant 0, \tag{A1.10}$$

where A is a real c-number.

Proof. Let us prove the converse. Then one can find a normalized function φ which is such that

$$\left\{\sqrt{\xi\xi^++\frac{2s}{V}+A^2}-\sqrt{\xi^+\xi+A^2}\right\}\varphi=-\rho\varphi. \tag{A1.11}$$

From this we have

$$\left\{\sqrt{\xi\xi^++\frac{2s}{V}+A^2}+\rho\right\}\varphi=\sqrt{\xi^+\xi+A^2}\,\varphi, \tag{A1.12}$$

and using (A1.4), we obtain

$$\left\langle\varphi^*\left(\sqrt{\xi\xi^++\frac{2s}{V}+A^2}+\rho\right)^2\varphi\right\rangle=\langle\varphi^*(\xi^+\xi+A^2)\varphi\rangle\leqslant$$

$$\leqslant \left\langle \varphi^* \left(\xi\xi^+ + \frac{2s}{V} + A^2 \right) \varphi \right\rangle, \tag{A1.13}$$

which is impossible for $\rho > 0$.

Corollary. Changing the role of the operators ξ and ξ^+, we obtain

$$\sqrt{\xi^+\xi + \frac{2s}{V} + A^2} - \sqrt{\xi\xi^+ + A^2} \geqslant 0. \tag{A1.14}$$

If α, λ are real c-numbers, then we have

$$\sqrt{\lambda^2 \left(\xi\xi^+ + \frac{2s}{V} + \alpha^2 \right) + A^2} - \sqrt{\lambda^2 (\xi^+\xi + \alpha^2) + A^2} \geqslant 0; \tag{A1.15}$$

$$\sqrt{\lambda^2 \left(\xi^+\xi + \frac{2s}{V} + \alpha^2 \right) + A^2} - \sqrt{\lambda^2 (\xi\xi^+ + \alpha^2) + A^2} \geqslant 0. \tag{A1.16}$$

Appendix to Lemma II. Let us assume that

$$\xi = \frac{1}{V} \sum_f \lambda(f)\, a_{-f} a_f + \nu \equiv L + \nu. \tag{A1.17}$$

Then

$$\xi\xi^+ - \xi^+\xi = \frac{2}{V^2} \sum_f \lambda^2(f) \left(1 - a_f^+ a_f - a_{-f}^+ a_{-f} \right). \tag{A1.18}$$

Assume $\lambda(f)$ satisfies the condition $\frac{1}{V} \sum_f \lambda^2(f) \leqslant s$; then $|\xi\xi^+ - \xi^+\xi| \leq 2s/V$. Consequently,

$$\sqrt{\lambda^2(f) \{ (L+\nu)(L^+ + \nu) + \alpha^2 + 2s/V \} + T^2(f)} - \sqrt{\lambda^2(f) \{ (L^+ + \nu)(L+\nu) + \alpha^2 \} + T^2(f)} > 0. \tag{A1.19}$$

Lemma III (Generalization of Lemma II). Assume again that $|\xi\xi^+ - \xi^+\xi| \leq 2s/V$. Let us consider the operators $\mathfrak{A}$, $\mathfrak{A}^+$ having the norm $|\mathfrak{A}| < 1$; $|\mathfrak{A}^+| \leq 1$, which are such that

$$|\mathfrak{A}\,\xi^+\,\xi\mathfrak{A}^+ - \xi^+\,\mathfrak{A}\mathfrak{A}^+\,\xi| \leqslant 2l/V. \tag{A1.20}$$

Then

$$2\sqrt{\xi\xi^+ + \frac{s+l}{V}} - \varepsilon(\xi\,\mathfrak{A}^+ + \mathfrak{A}\,\xi^+) \geqslant 0, \tag{A1.21}$$

where $\varepsilon = 1$ or $\varepsilon = -1$.

Proof. Let us assume the converse; then one can find a normalized φ which is such that

$$\left\{ 2\sqrt{\xi\xi^+ + \frac{s+l}{V}} - \varepsilon(\xi\,\mathfrak{A}^+ + \mathfrak{A}\,\xi^+) \right\} \varphi = -\rho\varphi, \quad \rho > 0. \tag{A1.22}$$

From this we have

$$\left(2\sqrt{\xi\xi^+ + \frac{s+l}{V}} + \rho \right) \varphi = \varepsilon(\xi\,\mathfrak{A}^+ + \mathfrak{A}\,\xi^+)\varphi. \tag{A1.23}$$

Consequently, according to (A1.14),

$$\left\langle \varphi^* \left(2\sqrt{\xi\xi^+ + \frac{s+l}{V}} + \rho \right)^2 \varphi \right\rangle = \langle \varphi^* (\xi\,\mathfrak{A}^+ + \mathfrak{A}\,\xi^+)^2 \varphi \rangle =$$

$$=2\langle\varphi^*\{\xi\mathfrak{A}^+\mathfrak{A}\xi^+ + \mathfrak{A}\xi^+\xi\mathfrak{A}^+\}\varphi\rangle - \langle\varphi^*(\xi\mathfrak{A}^+ - \mathfrak{A}\xi^+).$$

$$\times(\mathfrak{A}\xi^+ - \xi\mathfrak{A}^+)\varphi\rangle \leqslant 2\langle\varphi^*\{\xi\mathfrak{A}^+\mathfrak{A}\xi^+ + \mathfrak{A}\xi^+\xi\mathfrak{A}^+\}\varphi\rangle. \qquad \text{(A1.24)}$$

But since by convention $|\mathfrak{A}| \leq 1$, $|\mathfrak{A}^+| \leq 1$, we have $|\mathfrak{A}^+\mathfrak{A}| \leq 1$, and consequently,

$$\langle\varphi^*\xi\mathfrak{A}^+\mathfrak{A}\xi^+\varphi\rangle \leqslant \langle\varphi^*\xi\xi^+\varphi\rangle. \qquad \text{(A1.25)}$$

Further, taking account of (A1.20) and (A1.25), we have

$$\langle\varphi^*\ \mathfrak{A}\xi^+\xi\mathfrak{A}^+\varphi\rangle = \langle\varphi^*\xi^+\mathfrak{A}\mathfrak{A}^+\xi\varphi\rangle$$

$$+\langle\varphi^*\{\mathfrak{A}\xi^+\xi\mathfrak{A}^+ - \xi^+\mathfrak{A}\mathfrak{A}^+\xi\}\varphi\rangle \leqslant \langle\varphi^*\xi^+\mathfrak{A}\mathfrak{A}^+\xi\varphi\rangle$$

$$+\frac{2l}{V} \leqslant \langle\varphi^*\xi^+\xi\varphi\rangle + \frac{2l}{V} \leqslant \langle\varphi^*\xi\xi^+\varphi\rangle + \frac{2(l+s)}{V}$$

$$= \left\langle\varphi^*\left(\xi\xi^+ + \frac{2(l+s)}{V}\right)\varphi\right\rangle. \qquad \text{(A1.26)}$$

Therefore, taking account of (A1.24), we may write

$$\left\langle\varphi^*\left(2\sqrt{\xi\xi^+ + \frac{s+l}{V}} + \rho\right)^2\varphi\right\rangle \leqslant 4\left\langle\varphi^*\left(\xi\xi^+ + \frac{s+l}{V}\right)\varphi\right\rangle. \qquad \text{(A1.27)}$$

But such an inequality is impossible for $\rho > 0$, which proves the statement (A1.21) of Lemma III.

Appendix to Lemma III. Let us assume $\xi = L + \nu$; $\mathfrak{A} = a_g$. Then

$$|\mathfrak{A}\xi^+\xi\mathfrak{A}^+ - \xi^+\mathfrak{A}\mathfrak{A}^+\xi| = |a_g(L^+ + \nu)(L+\nu)a_g^+ - (L^+ + \nu)a_g a_g^+(L+\nu)|$$

$$= |a_g(L^+ + \nu)(L+\nu)a_g^+ - (L^+ + \nu)a_g(L+\nu)a_g^+$$

$$+(L^+ + \nu)a_g(L+\nu)a_g^+ - (L^+ + \nu)a_g a_g^+(L+\nu)|$$

$$\leqslant (|L| + \nu)\{|La_g^+ - a_g^+L| + |a_gL^+ - L^+a_g|\} \leqslant (|L| + \nu)\frac{4}{V}|\lambda(g)|,$$

where (see the identity (A1.17)) $|L| \leqslant \frac{1}{V}\sum_f|\lambda(f)|$, since $|a_f| \leq 1$. Therefore, in accordance with (A1.21), we have

$$2\sqrt{(L+\nu)(L^+ + \nu) + \frac{1}{V}\{s + (|L| + \nu)2|\lambda(g)|\}}$$

$$-\varepsilon\{(L+\nu)a_g^+ + a_g(L^+ + \nu)\} \geqslant 0. \qquad \text{(A1.28)}$$

Having placed $\mathfrak{A} = i a_g$, we likewise obtain

$$2\sqrt{(L+\nu)(L^+ + \nu) + \frac{1}{V}\{s + (|L| + \nu)2|\lambda(g)|\}}$$

$$-\varepsilon\left\{\frac{(L+\nu)a_g^+ - a_g(L^+ + \nu)}{i}\right\} \geqslant 0. \qquad \text{(A1.29)}$$

Lemma IV. Assume β is a real number $\alpha^2 = \beta^2 + 2s/V$ and $\nu \geq 0$. Then

$$\left|\sqrt{\{(L+\nu)(L^+ + \nu) + \alpha^2\}\lambda^2(f) + T^2(f)}\,a_f\right.$$

$$-a_f \sqrt{\{(L+\nu)(L^+ +\nu)+\alpha^2\}\lambda^2(f)+T^2(f)} \leqslant \frac{S_f}{V}, \tag{A1.30}$$

where S_f is bounded for $V \to \infty$. (The same inequality holds if in (A1.30) we take a_f^+ instead of a_f.)

Proof. Let us consider an arbitrary normalized function φ and let us formulate the expression

$$\langle \varphi^* \{ \sqrt{(Q+\alpha^2)\lambda^2(f)+T^2(f)}\,(a_f+a_f^+)$$

$$-(a_f+a_f^+)\sqrt{(Q+\alpha^2)\lambda^2(f)+T^2(f)}\}\,\varphi\rangle = \mathscr{E}, \tag{A1.31}$$

where $Q = (L+\nu)(L^+ + \nu)$. In order to consider the expression (A1.31) let us use the following identical relationship:

$$\sqrt{Z}-\sqrt{Z_0} = \frac{1}{\pi}\int_0^\infty \left\{\frac{1}{Z_0+\omega}-\frac{1}{Z+\omega}\right\}\sqrt{\omega}\,d\omega,$$

where Z_0 is an arbitrary positive number. Note likewise that

$$-\frac{1}{A}B+B\frac{1}{A} = \frac{1}{A}(AB-BA)\frac{1}{A},$$

where A and B are operators. Then we have

$$\mathscr{E} = \frac{1}{\pi}\int_0^\infty \left\langle \varphi^* \frac{\lambda^2(f)}{(Q+\alpha^2)\lambda^2(f)+T^2(f)+\omega}\{Q(a_f+a_f^+)-(a_f+a_f^+)Q\}\right.$$
$$\left.\times\frac{1}{(Q+\alpha^2)\lambda^2(f)+T(f)+\omega}\varphi\right\rangle\sqrt{\omega}\,d\omega.$$

But

$$Qa_f-a_fQ=(L+\nu)\{L^+a_f-a_fL^+\};$$
$$L^+=\frac{1}{V}\sum_f \lambda(f)\,a_f^+a_{-f}^+;\quad L^+a_f-a_fL^+=-\frac{2}{V}\lambda(f)\,a_{-f}^+,$$

and consequently,

$$Q(a_f+a_f^+)-(a_f+a_f^+)Q=-\frac{2}{V}\lambda(f)(L+\nu)\,a_{-f}^+ + \frac{2}{V}\lambda(f)\,a_{-f}(L^+ +\nu).$$

Therefore

$$|\mathscr{E}| = \left|\frac{\mathscr{E}}{i}\right| = \frac{2|\lambda(f)|^3}{\pi}\int_0^\infty \left\langle \varphi^* \right| \frac{1}{(Q+\alpha^2)\lambda^2(f)+T^2(f)+\omega}$$
$$\left.\times\frac{(L+\nu)\,a_{-f}^+-a_{-f}(L^+ +\nu)}{i}\times\frac{1}{(Q+\alpha^2)\lambda^2(f)+T^2(f)+\omega}\varphi\right\rangle\sqrt{\omega}\,d\omega\Bigg|.$$

From this, considering (A1.29) and introducing a new integration variable, we obtain

$$|\mathscr{E}| \leqslant \frac{4|\lambda(f)|^2}{\pi V}\int_0^\infty \left\langle \varphi^* \frac{\sqrt{Q+\frac{1}{V}(s+2|\lambda(f)|)(|L|+\nu)}}{\left(Q+\alpha^2+\frac{T^2(f)}{\lambda^2(f)}+\tau\right)^2}\varphi\right\rangle\sqrt{\tau}\,d\tau.$$

But by convention of the lemma we have $\alpha^2 = \beta^2 + 2s/V$, and therefore

$$\sqrt{Q+\frac{1}{V}(s+2|\lambda(f)|)(|L|+\nu)} < \sqrt{Q+\alpha^2+\frac{T^2(f)}{\lambda^2(f)}+\frac{2|\lambda(f)|(|L|+\nu)}{V}} =$$

$$=\sqrt{Q+\alpha^2+\frac{T^2(f)}{\lambda^2(f)}}\cdot\sqrt{1+\frac{2|\lambda(f)|(|L|+\nu)}{VQ+V\alpha^2+V\frac{T^2(f)}{\lambda^2(f)}}}$$

$$<\sqrt{Q+\alpha^2+\frac{T^2(f)}{\lambda^2(f)}}\cdot\sqrt{1+\frac{|\lambda(f)|(|L|+\nu)}{s+\frac{1}{2}V\frac{T^2(f)}{\lambda^2(f)}}}$$

$$<\left(1+\frac{|\lambda(f)|(|L|+\nu)}{2s+V\frac{T^2(f)}{\lambda^2(f)}}\right)\sqrt{Q+\alpha^2+\frac{T^2(f)}{\lambda^2(f)}}.$$

Let us place $\Lambda=Q+\alpha^2+\frac{T^2(f)}{\lambda^2(f)}\geqslant\alpha^2$. Then

$$|\mathscr{E}|=\frac{4|\lambda(f)|^2}{\pi V}\left(1+\frac{|\lambda(f)|(|L|+\nu)}{2s+V\frac{T^2(f)}{\lambda^2(f)}}\right)\int_0^\infty\left\langle\varphi^*\frac{\sqrt{\Lambda}}{(\Lambda+\tau)^2}\varphi\right\rangle\sqrt{\tau}\,d\tau.$$

Now let us expand the function φ in eigenfunctions of the operator Λ: $\varphi=\Sigma C_\Lambda\varphi_\Lambda$; $\Sigma|C_\Lambda|^2=1$. We obtain

$$\int_0^\infty\left\langle\varphi^*\frac{\sqrt{\Lambda}}{(\Lambda+\tau)^2}\varphi\right\rangle\sqrt{\tau}\,d\tau=\sum_\Lambda|C_\Lambda|^2\int_0^\infty\frac{\sqrt{\Lambda\tau}\,d\tau}{(\Lambda+\tau)^2}=\sum_\Lambda|C_\Lambda|^2\int_0^\infty\frac{\sqrt{t}\,dt}{(1+t)^2}=\int_0^\infty\frac{\sqrt{t}\,dt}{(1+t)^2}.$$

Thus, for an arbitrary normalized function φ we have

$$|\mathscr{E}|=\left|\left\langle\varphi^*\left[\frac{\sqrt{(Q+\alpha^2)\lambda^2+T^2};a_f+a_f^+}{i}\right]\varphi\right\rangle\right|\leqslant S_f,$$

where

$$S_f=\frac{4|\lambda(f)|^2}{\pi V}\left(1+\frac{|\lambda(f)|\left(\sum|\lambda(f)|\frac{1}{V}+\nu\right)}{2\frac{1}{V}\sum|\lambda(f)|^2+V\frac{T^2(f)}{\lambda^2(f)}}\right)\int_0^\infty\frac{\sqrt{t}\,dt}{(1+t)^2}.$$

But the operator

$$\left[\frac{\sqrt{(Q+\alpha^2)\lambda^2+T^2};\ a_f+a_f^+}{i}\right]$$

is a Hermite operator, and consequently, $\left|\left[\sqrt{(Q+\alpha^2)\lambda^2+T^2};\ a_f+a_f^+\right]\right|\leqslant S_f$. In a completely analogous manner we prove that $\left|\left[\sqrt{(Q+\alpha^2)\lambda^2+T^2};\ a_f-a_f^+\right]\right|\leqslant S_f$. But $|\mathfrak{A}|+|\mathfrak{B}|\geq|\mathfrak{A}+\mathfrak{B}|$, and consequently, $\left|\left[\sqrt{(Q+\alpha^2)\lambda^2+T^2};\ a_f\right]\right|\leqslant S_f$, which is what it was required to prove.

From $|\mathfrak{A}|\leq S_f$ it follows that $|\mathfrak{A}^+|\leq S_f$, whence the validity of the supplementary statement of the lemma is evident.

APPENDIX 2

The principle of weakening the correlations between particles for systems in the state of statistical equilibrium is formulated as follows.

The correlation functions

$$\langle\mathfrak{A}_1(x_1,t_1)\ldots\mathfrak{A}_s(x_s,t_s)\ldots\mathfrak{A}_n(x_n,t_n)\rangle,\qquad\text{(A2.1)}$$

where $\mathfrak{A}_s(x_s,t_s)$ is the field function $\psi(x_s,t_s)$ or $\psi^+(t_s,x_s)$, can be decomposed into the product of correlation functions

$$\langle \mathfrak{A}_1(x_1, t_1) \ldots \mathfrak{A}_{s-1}(x_{s-1}, t_{s-1})\rangle \langle \mathfrak{A}_{s+1}(x_{s+1}, t_{s+1}) \ldots \mathfrak{A}_n(x_n, t_n)\rangle, \quad \text{(A2.2)}$$

if the ensemble of points $x_1, \ldots, x_s$ is placed infinitely far from the ensemble of points $x_{s+1}, \ldots, x_n$ at fixed times $t_1, \ldots, t_s, \ldots, t_n$. Note that for the case in which the numbers of the creation and annihilation operators are not equal in the correlation functions, the averaging $<\ldots>$ should be understood in the sense of quasiaverages.

A system having a model Hamiltonian is one of the rare cases in which direct calculations can verify the validity of the principle of correlation weakening. Below, based on the previous asymptotic estimates, we shall show precisely this. Let us consider "vacuum" averages which are formulated from the products of field functions in the spatial representation:

$$\left.\begin{aligned} \Psi_-(t, x) &= \frac{1}{\sqrt{V}} \sum_{(f<0)} a_f(t)\, e^{i(f\cdot x)}; \\ \Psi_+(t, x) &= \frac{1}{\sqrt{V}} \sum_{(f>0)} a_f^+(t)\, e^{-i(f\cdot x)}. \end{aligned}\right\} \quad \text{(A2.3)}$$

Here f represents the aggregate of a momentum and a spin $(\mathbf{k}, \sigma)$, the summation $f > 0$, $f < 0$ denoting summation over $\mathbf{k}$ for fixed $\sigma = \pm$, $(f \cdot x) = (\mathbf{k} \cdot \mathbf{r})$. We have, for example,

$$\langle \Psi_{\sigma_1}(t, x)\Psi_{\sigma_2}^+(t, x')\rangle_{H_0} = \frac{1}{V}\sum_{(f>0)} |u_f|^2 e^{if\cdot(x-x')}\,\delta(\sigma_1-\sigma_2)$$

$$= \left\{\frac{1}{V}\sum_{(f>0)} e^{if\cdot(x-x')} - \frac{1}{V}\sum_{(f>0)} |v_f|^2 e^{if\cdot(x-x')}\right\}\delta(\sigma_1-\sigma_2), \quad \text{(A2.4)}$$

where u_f and v_f are the coefficients of the canonical transformation. As is evident, the term

$$\frac{1}{V}\sum_{(f>0)} |v_f|^2 e^{if\cdot(x-x')}$$

approaches the following integral for $V\to\infty$:

$$\frac{1}{(2\pi)^3}\int |v_f|^2 e^{if\cdot(x-x')}\,d\mathbf{k}.$$

This integral is absolutely convergent, since

$$\int |v_f|^2\,d\mathbf{k} = \frac{1}{2}\int \{\sqrt{T^2(f)+\lambda^2(f)C^2} - T(f)\}^2 \frac{d\mathbf{k}}{T^2(f)+\lambda^2(f)C^2} < \infty.$$

About the expression $\frac{1}{V}\sum_{(f>0)} e^{if\cdot(x-x')}$ we say that it approaches a "delta-function" $\frac{1}{(2\pi)^3}\int e^{if\cdot(x-x')}$ when $V\to\infty$.

However, at present we, of course, ascribe to the words "limit," "convergence of functions" a different meaning: namely, the meaning adopted in the theory of generalized functions.

Let us recall here what the relationship

$$f_V(x_1, \ldots, x_l) \underset{V\to\infty}{\to} f(x_1, \ldots, x_l), \quad \text{(A2.5)}$$

where $f(x_1, \ldots, x_l) = \lim_{V\to\infty} f_V(x_1, \ldots, x_l)$, means in this theory.

Let us consider the class $C(q, r)$ (q, r are positive numbers) of continuous and unboundedly differentiable functions $h(x_1, \ldots, x_l)$ which are such that throughout the space E_l of points $\{x_1, \ldots, x_l\}$ we have

$$\{|x_1|+\ldots+|x_l|\}^{\alpha}_{\alpha=0,1,\ldots,r} |h(x_1,\ldots,x_l)| \leqslant \text{const}; \; \{|x_1|+\ldots+|x_l|\}^{\alpha}_{\alpha=0,1,\ldots,r} \times \left|\frac{\partial^{s_1+\ldots+s_l}h}{\partial x_1^{s_1}\cdots\partial x_l^{s_l}}\right| \leqslant \text{const}.$$

$$s_1+\ldots+s_l=0,1,\ldots,q$$

Then, if the positive numbers q, r can be fixed in such a way that for any function h from the class C(q, r) one may write

$$\int h(x_1,\ldots,x_l) f_V(x,\ldots x_l)\,dx,\ldots,dx_l \to \int h(x_1,\ldots,x_l) f(x_1,\ldots,x_l)\,dx_1\ldots dx_l,$$

we shall say that the generalized limit relation (A2.5) holds. As we have just seen, the averages of the products $\Psi(t, x)$, $\Psi^+(t, x)$ may contain generalized functions. Therefore, the corresponding limit relationships for $V\to\infty$ should be understood in the sense of the theory of generalized functions.

Let us consider the expression

$$\langle \Psi_{\sigma_1}(t_1,x_1)\Psi^+_{\sigma_2}(t_1,x_2)\rangle = \frac{1}{V}\sum_{(f>0)} \langle a_f(t_1)\,a_f^+(t_1)\rangle\, e^{if\cdot(x_1-x_2)}\,\delta(\sigma_1-\sigma_2).$$

We have

$$\int h(x_1-x_2)\langle \Psi_{\sigma_1}(t_1,x_1)\Psi^+_{\sigma_2}(t_2,x_2)\rangle\,dx_1 = \frac{1}{V}\sum_{(f>0)}\langle a_f(t_1)\,a_f^+(t_2)\rangle\,\tilde{h}(f)\,\delta(\sigma_1-\sigma_2),$$

where

$$\tilde{h}(f)=\int h(x)\,e^{i(f\cdot x)}\,dx.$$

Having taken the numbers q, r in the class C(q, r) to which h(x) belongs, one may achieve a situation in which $h(f)$ decreases more rapidly than any power of $|f|\to\infty$ for $1/|f|$. It is sufficient merely to ensure that $\frac{1}{V}\sum_f|\tilde{h}(f)|\leqslant K=\text{const}$.

Then, noting that in accordance with (6.36) we have

$$\left|\langle a_f(t_1)\,a_f^+(t_2)\rangle_H - \langle a_f(t_1)\,a_f^+(t_2)\rangle_{H_0}\right| \leqslant \frac{s_1|t_1-t_2|+s_2}{\sqrt{V}},$$

$$s_1,\, s_2=\text{const},$$

we shall have

$$\begin{aligned}&\left|\int h(x_1-x_2)\left\{\langle\Psi_{\sigma_1}(t_1,x_1)\Psi^+_{\sigma_2}(t_2,x_2)\rangle_H - \langle\Psi_{\sigma_1}(t_1,x_1)\Psi^+_{\sigma_2}(t_2,x_2)\rangle_{H_0}\right\}dx_1\right|\\ &\leqslant \frac{1}{V}\sum_f\left|\langle a_f(t_1)\,a_f^+(t_2)\rangle_H-\langle a_f(t_1)\,a_f^+(t_2)\rangle_{H_0}\right|\left|\tilde{h}(t)\right|\\ &\leqslant K\frac{s_1|t_1-t_2|+s_2}{V}\underset{V\to\infty}{\to}0.\end{aligned}$$

Consequently, the generalized limit relation

$$\langle\Psi_{\sigma_1}(t_1,x_1)\Psi^+_{\sigma_2}(t_2,x_2)\rangle_H - \langle\Psi_{\sigma_1}(t_1,x_1)\Psi^+_{\sigma_2}(t_2,x_2)\rangle_{H_0}\to 0 \tag{A2.6}$$

holds. But by direct calculation we verify the fact that

$$\langle\Psi_{\sigma_1}(t_1,x)\Psi^+_{\sigma_2}(t_2,x_2)\rangle_{H_0} = \frac{1}{V}\sum_{(f>0)}|u_f|^2\,e^{-i\Omega(f)(t_1-t_2)+if\cdot(x_1-x_2)}\,\delta(\sigma_1-\sigma_2),$$

and therefore likewise in the generalized sense we have

$$\langle \Psi_{\sigma_1}(t_1, x_1)\, \Psi^+_{\sigma_2}(t_2, x_2)\rangle_H$$

$$-\int |u_f|^2 \exp\left\{-i\Omega(f)(t_1-t_2)+if\cdot(x_1-x_2)\right\} dk\, \delta(\sigma_1-\sigma_2) \underset{V\to\infty}{\to} 0. \tag{A2.7}$$

From (A2.6) and (A2.7) we finally have

$$\lim_{V\to\infty} \langle \Psi_{\sigma_1}(t_1, x_1)\, \Psi^+_{\sigma_2}(t_2, x_2)\rangle_H$$

$$=\int |u_f|^2 \exp\{-i\Omega(f)(t_1-t_2)+if\cdot(x_1-x_2)\}\, dk\, \delta(\sigma_1-\sigma_2)$$

$$=\{\Delta(t_1-t_2, x_1-x_2)-F(t_1-t_2, x_1-x_2)\}\, \delta(\sigma_1-\sigma_2); \tag{A2.8}$$

$$\left.\begin{aligned} \Delta(t, x) &= \int \exp\{-i\Omega(f)\, t+if\cdot x\}\, dk; \\ F(t, x) &= \int |v_f|^2 \exp\{-i\Omega(f) t+if\cdot x\}\, dk. \end{aligned}\right\} \tag{A2.9}$$

In a fully analogous manner we obtain*

$$\lim_{V\to\infty} \langle \Psi^+_{\sigma_2}(t_2, x_2)\, \Psi_{\sigma_1}(t_1, x_1)\rangle = F(t_2-t_1, x_1-x_2)\, \delta(\sigma_1-\sigma_2). \tag{A2.10}$$

Let us now consider the binary expressions $<\Psi(t_1, x_1)\Psi(t_2, x_2)\Psi^+(t_2', x_2') \times \Psi^+(t_1', x_1')>$. We have

$$\langle \Psi(t_1, x_1)\, \Psi(t_2, t_2)\, \Psi^+(t_2', x_2')\, \Psi^+(t_1', x_1')\rangle$$

$$=\frac{1}{V^2}\sum \langle a_{f_1}(t_1)\, a_{f_2}(t_2)\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle$$

$$\times\{if_1\cdot x_1+if_2\cdot x_2-ig_2\cdot x_2'-ig_1\cdot x_1'\}. \tag{A2.11}$$

Since the total momentum is conserved, while for Φ_H and Φ_{H_0} it is equal to zero, it follows that the expressions

$$\langle a_{f_1}(t_1)\, a_{f_2}(t_2)\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle \tag{A2.12}$$

may be nonvanishing only if

$$f_1+f_2=g_2+g_1. \tag{A2.13}$$

Let us now recall that from (2.1) and (2.2) we have $n_f(t)-n_{-f}(t)$, where $n_f = a_f^+ a_f$ is the integral of motion, and Φ_H (and Φ_{H_0}) satisfies the supplementary relations $(n_f-n_{-f})\Phi = 0$. Note finally that $(n_f-n_{-f})a_h = a_h\{(n_f-n_{-f})-\delta(f-h)+\delta(f+h)\}$. Therefore (for any f)

$$\langle a_{f_1}(t_1)\, a_{f_2}(t_2)\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle$$

$$\langle\{1+n_f-n_{-f}\}\, a_{f_1}(t_1)\, a_{f_2}(t_2)\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle$$

$$=\langle\{1+n_f(t_1)+n_{-f}(t_1)\}\, a_{f_1}(t_1)\, a_{f_2}(t_2)\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle$$

$$=\langle a_{f_1}(t_1)\{1+n_f(t_1)-n_{-f}(t_1)-\delta(f-f_1)+\delta(f+f_1)\}\, a_{f_2}(t_2)$$

$$\times a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle=\langle a_{f_1}(t_1)\{1+n_f(t_2)-n_{-f}(t_2)-\delta(f-f_1)+\delta(f+f_1)\}$$

$$\times a_{f_2}(t_2) a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle=\langle a_{f_1}(t_1)\, a_{f_2}(t_2)\{1+n_f(t_2)-n_{-f}(t_2)$$

$$-\delta(f-f_1)+\delta(f+f_1)-\delta(f-f_2)+\delta(f+f_2)\}\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')\rangle=\dots$$

$$=\langle a_{f_1}(t_1)\, a_{f_2}(t_2)\, a^+_{g_2}(t_2')\, a^+_{g_1}(t_1')$$

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* This limit relation likewise holds in the conventional sense in view of the absolute convergence of the integral which defines F(t, x).

$$\times\{1+n_f-n_{-f}-\delta(f-f_1)+\delta(f+f_1)+\delta(f+f_2)-\delta(f-f_2)$$
$$+\delta(f-g_2)-\delta(f+g_2)+\delta(f-g_1)-\delta(f+g_1)\}\rangle$$
$$=\{1-\delta(f-f_1)+\delta(f+f_1)-\delta(f-f_2)+\delta(f+f_2)$$
$$+\delta(f-g_2)-\delta(f+g_2)+\delta(f-g_1)-\delta(f+g_1)\}$$
$$\times\langle a_{f_1}(t_1)\,a_{f_2}(t_2)\,a^+_{g_2}(t'_2)\,a^+_{g_1}(t'_1)\rangle.$$

This identity shows that the quantities (A2.12) may be nonvanishing only if for any f the relation

$$-\delta(f-f_1)+\delta(f+f_1)-\delta(f-f_2)+\delta(f+f_2)$$
$$+\delta(f-g_2)-\delta(f+g_2)+\delta(f-g_1)-\delta(f+g_1)=0$$

holds. The latter relation together with (A2.13) is fulfilled only in the following cases:

$$f_1-f_2=0,\quad g_1+g_2=0; \tag{A2.14}$$

$$f_1=g_1,\quad f_2=g_2; \tag{A2.15}$$

$$f_1=g_2,\quad f_2=g_1. \tag{A2.16}$$

Moreover, in (A2.15) and (A2.16) it may always be assumed that $g_1\neq g_2$, since

$$a^+_g(t'_2)\,a^+_g(t'_1)\,\Phi_H=0. \tag{A2.17}$$

Actually,

$$(n_g-n_{-g})\,a^+_g(t'_2)\,a^+_g(t'_1)\,\Phi_H=a^+_g(t'_2)\,a^+_g(t'_1)\,(n_g-n_{-g}+2)\,\Phi_H=2a^+_g(t'_2)\,a^+_g(t'_1)\,\Phi_H.$$

But the possible eigenvalues n_g-n_{-g} are only ± 1 and 0, and consequently, the latter equation is possible only when (A2.17) is fulfilled. Thus, one can reduce (A2.11) to the form

$$\langle\Psi(t_1,x_1)\,\Psi(t_2,x_2)\,\Psi^+(t'_2,x'_2)\,\Psi(t'_1,x'_1)\rangle$$
$$=\sum_{f,g}\frac{1}{V^2}\langle a_{-f}(t_1)\,a_f(t_2)\,a^+_g(t'_2)\,a^+_{-g}(t'_1)\rangle\exp\{if\cdot(x_2-x_1)-ig\cdot(x'_2-x'_1)\}$$
$$+\sum_{\substack{f,g\\ \left(\substack{f\neq g\\ f+g\neq 0}\right)}}\frac{1}{V^2}\langle a_f(t_1)\,a_g(t_2)\,a^+_g(t'_2)\,a^+_f(t'_1)\rangle\exp\{if\cdot(x_1-x'_1)+ig\cdot(x_2-x'_2)\}$$
$$+\sum_{\substack{f,g\\ \left(\substack{f\neq g\\ f+g\neq 0}\right)}}\frac{1}{V^2}\langle a_f(t_1)\,a_g(t_2)\,a^+_f(t'_2)\,a^+_g(t'_1)\rangle\exp\{if\cdot(x_1-x'_2)+ig\cdot(x_2-x'_1)\}. \tag{A2.18}$$

Let us now deal with the transition in the limit $V\to\infty$. Let us consider the class $C(q,r)$ of functions $h(x,y)$, and let us fix q, r in such a way that $\frac{1}{V^2}\sum_{f,g}|\tilde h(f,g)|\leqslant\text{const}$, where

$$\tilde h(f,g)=\int h(x,y)\,e^{i(fx+gy)}\,dx\,dy.$$

Since (see (5.56)) for fixed t_1, t_2, t'_2, t'_1,

$$|\langle a_f(t_1)\,a_g(t_2)\,a^+_f(t'_2)\,a^+_g(t'_1)\rangle_H-\langle a_f(t_1)\,a_g(t_2)\,a^+_f(t'_2)\,a^+_g(t'_1)\rangle_{H_0}|\leqslant\frac{\text{const}}{\sqrt{V}},$$

we have

$$\left|\int h(x,y)\{\Gamma_H(t_1,t_2,t'_2,t'_1\,|\,x,y)-\Gamma_{H_0}(t_1,t_2,t'_2,t'_1\,|\,x,y)\}\,dx\,dy\right|\leqslant\frac{\text{const}}{\sqrt{V}}\xrightarrow[V\to\infty]{}0,$$

where

$$\Gamma(t_1, t_2, t_2', t_1' \mid x, y) = \frac{1}{V^2} \sum_{\substack{f, g \\ \left(\substack{f \neq g \\ f+g \neq 0}\right)}} \langle a_f(t_1) a_g(t_2) a_f^+(t_2') a_g^+(t_1') \rangle e^{i(f \cdot x + g \cdot y)}.$$

We obtain the generalized limit relations

$$\Gamma_H(t_1, t_2, t_2', t_1', x_1 - x_2', x_2 - x_1') - \Gamma_{H_0}(t_1, t_2, t_2', t_1', x_1 - x_2', x_2 - x_1') \xrightarrow[V \to \infty]{} 0.$$

But direct calculation, just as in the case of (A2.4), yields

$$\Gamma_{H_0}(t_1, t_2, t_2', t_1', x_1 - x_2', x_2 - x_1')$$

$$= -\frac{1}{V^2} \sum_{\substack{f, g \\ \left(\substack{f \neq g \\ f+g \neq 0}\right)}} |u_f|^2 |u_g|^2 e^{-i\Omega(f)(t_1 - t_2') - i\Omega(g)(t_2 - t_1')}$$

$$\times e^{i(f(x_1 - x_2') + g(x_2 - x_1'))} \rightarrow -\{\Delta(t_1 - t_2', x_1 - x_2')$$

$$-F(t_1 - t_2', x_1 - x_2')\}\{\Delta(t_2 - t_1', x_2 - x_1')$$

$$-F(t_2 - t_1', x_2 - x_1')\}\delta(\sigma_1 - \sigma_2')\delta(\sigma_2 - \sigma_1'), \tag{A2.19}$$

where $\Omega(f)$ is determined by the relationship (6.2'), while $\Delta(t, x)$ and $F(t, x)$ are determined by the relation (A2.9).

Consequently,

$$\lim_{V \to \infty} \Gamma_H(t_1, t_2, t_2', t_1', x_1 - x_2', x_2 - x_1')$$
$$= -\{\Delta(t_1 - t_2', x_1 - x_2') - F(t_1 - t_2', x_1 - x_2')\}$$
$$\times \{\Delta(t_2 - t_1', x_2 - x_1') - F(t_2 - t_1', x_2 - x_1')\}\delta(\sigma_1 - \sigma_2')\delta(\sigma_2 - \sigma_1').$$

We also deal in a completely analogous manner with the terms which are included in the right side of Eq. (A2.18). Now let us place

$$\Phi_\sigma(t, x) = -\int u_f v_f e^{-i\Omega(f)t - if \cdot x} d\mathbf{k} = \int \frac{C\lambda(f)}{2\Omega(f)} e^{-i\Omega(f)t - if \cdot \mathbf{k}} d\mathbf{k}. \tag{A2.20}$$

Then we may write the generalized limit relation in the form

$$\lim_{V \to \infty} \langle \Psi_{\sigma_1}(t_1, x_1) \Psi_{\sigma_2}(t_2, x_2) \Psi^+_{\sigma_2'}(t_2', x_2') \Psi^+_{\sigma_1'}(t_1', x_1') \rangle_H$$
$$= \Phi_{\sigma_2}(t_1 - t_2, x_1 - x_2) \Phi_{\sigma_2'}(t_2' - t_1', x_2' - x_1') \delta(\sigma_1 + \sigma_2) \delta(\sigma_1' + \sigma_2')$$
$$+ \delta(\sigma_1 - \sigma_1') \delta(\sigma_2 - \sigma_2') \{\Delta(t_1 - t_1', x_1 - x_1') - F(t_1 - t_1', x_1 - x_1')\}$$
$$\times \{\Delta(t_2 - t_2', x_2 - x_2') - F(t_2 - t_2', x_2 - x_2')\} - \delta(\sigma_1 - \sigma_2') \delta(\sigma_2 - \sigma_1')$$
$$\times \{\Delta(t_1 - t_2', x_1 - x_2') - F(t_1 - t_2', x_1 - x_2')\} \{\Delta(t_2 - t_1', x_2 - x_1')$$
$$-F(t_2 - t_1', x_2 - x_1')\}. \tag{A2.21}$$

Completely analogous formulas are also obtained for other arrangement orders of the operator functions Ψ, Ψ.*

* The function $\Delta(t, x)$ itself is a generalized function. This relation likewise holds in the generalized sense.

Using the example of the formula (A2.21) which has been derived, one may illustrate the principle of correlation weakening. One need merely note that for fixed t

$$\left.\begin{array}{l} F(t,\ x)\rightarrow 0\ |x|\rightarrow\infty; \\ \Phi(t,\ x)\rightarrow 0\ |x|\rightarrow\infty; \end{array}\right\} \tag{A2.22}$$

$$\Delta(t,\ x)\rightarrow 0^{*}\ |x|\rightarrow\infty. \tag{A2.23}$$

Let us fix the times t_1, t_2, t_2', t_1' and the spatial differences x_1-x_1', x_2-x_2'. The remaining spatial differences x_1-x_2, $x_1'-x_2'$, x_1-x_2', x_2-x_1' are made to tend to infinity. Then the function considered

$$\lim_{V\rightarrow\infty}\langle\Psi_{\sigma_1}(t_1,\ x_1)\,\Psi_{\sigma_2}(t_2,\ x_2)\,\Psi^{+}_{\sigma_2'}(t_2',\ x_2')\,\Psi^{+}_{\sigma_2'}(t_1',\ x_1')\rangle_H \tag{A2.24}$$

will decompose into the product

$$\{\Delta(t_1-t_1',\ x_1-x_1')-F(t_1-t_1',\ x_1-x_1')\}\,\{\Delta(t_2-t_2',\ x_2-x_2')$$
$$-F(t_2-t_2',\ x_2-x_2')\}\,\delta(\sigma_2-\sigma_1')\,\delta(\sigma_2-\sigma_2'),$$

which is equal to [see (A2.8)]

$$\lim_{V\rightarrow\infty}\langle\Psi_{\sigma_1}(t_1,\ x_1)\,\Psi^{+}_{\sigma_1'}(t_1',\ x_1')\rangle_H\lim_{V\rightarrow\infty}\langle\Psi_{\sigma_2}(t_2,x_2)\,\Psi^{+}_{\sigma_2'}(t_2',\ x_2')\rangle_H. \tag{A2.25}$$

Let us now consider another method of correlation weakening. Let us fix the times t_1, t_2, t_2', t_1 and the spatial differences x_1-x_2, $x_1'-x_2'$ anew. The remaining spatial differences x_1-x_1', x_2-x_2', x_1-x_2', x_2-x_1' are made to tend to infinity. Then the function (A2.24) considered decomposes into the product

$$\Phi(t_1-t_2,\ x_1-x_2)\,\Phi(t_2'-t_1',\ x_2'-x_1')\,\Phi_{\sigma_2}\Phi_{\sigma_2'}\,\delta(\sigma_1+\sigma_2)\,\delta(\sigma_1'+\sigma_2'). \tag{A2.26}$$

For $\nu > 0$,

$$\left.\begin{array}{l} \Phi_\sigma(t_1-t_2,\ x_1-x_2)=\lim\limits_{V\rightarrow\infty}\langle\Psi_{-\sigma}(t_1,\ x_1)\,\Psi_\sigma(t_2,\ x_2)\rangle_H; \\ \Phi_\sigma(t_2'-t_1',\ x_2'-x_1')=\lim\limits_{V\rightarrow\infty}\langle\Psi^{+}_\sigma(t_2',\ x_2')\,\Psi^{+}_{-\sigma}(t_1',\ x_1')_H, \end{array}\right\} \tag{A2.27}$$

so that the function (A2.24) decomposes into the product of averages

$$\lim_{V\rightarrow\infty}\langle\Psi_{\sigma_1}(t_1,\ x_1)\,\Psi_{\sigma_2}(t_2,\ x_2)\lim_{V\rightarrow\infty}\left\langle\Psi^{+}_{\sigma_2'}(t_2',\ x_2')\,\Psi^{+}_{\sigma_1'}(t_1',\ x_1')\right\rangle. \tag{A2.28}$$

In the case given the relationships (A2.25) or (A2.28) which have been found constitute the expression of the principle of correlation weakening (A2.2). For $\nu = 0$ we have $<\Psi(t_1,\ x_1)\times\Psi(t_2,\ x_2)>_H = 0$, and the relationships (A2.27) have no place. In this case, however, we may introduce the "quasiaverages"

$$\langle\Psi_{\sigma_1}(t_1,\ x_1)\,\Psi_{\sigma_2}(t_2,\ x_2)\rangle_H=\lim_{\substack{\nu>0\\ \nu\rightarrow 0}}\lim_{V\rightarrow\infty}\langle\Psi_{\sigma_1}(t_1,\ x_1)$$

$$\times\,\Psi_{\sigma_2}(t_2,\ x_2)\rangle=\Phi_{\sigma_2}(t_1-t_2,\ x_1-x_2)\ \delta(\sigma_1+\sigma_2) \tag{A2.29}$$

and take the corresponding product of "quasiaverages" instead of the product of averages (A2.28). The relationships derived above illustrate the general principle of correlation weakening.

I take this opportunity to express sincere thanks to D. N. Zubarev, S. V. Tyablikov,* Yu. A. Tserkovnikov, and E. N. Yakovlev for their discussion.

*Deceased.

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