SHORT-RANGE REPULSION AND BROKEN CHIRAL SYMMETRY IN LOW-ENERGY SCATTERING*

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The introduction of short-range repulsive "potentials" into the low-energy equations for the lower partial waves makes it possible to eliminate the main difficulties of the pure elastic low-energy (Pele) approximation. In principle, it is then possible to obtain solutions with short s-wave scattering lengths and broad resonances. The use of threshold conditions that follow from chiral symmetry enables one (under certain simple additional conditions) to express the main resonance scattering parameters in terms of the pion decay characteristics. Thus, on the basis of the approximation of broken chiral symmetry and unitarity dispersion equations for low-energy $\pi\pi$ and πN scattering, expressions are obtained for the masses, lifetimes, and coupling constants for p-wave resonances, it being only necessary to specify the pion and nucleon masses and lifetimes and the Fermi coupling constant.

1. PHYSICAL INCOMPLETENESS OF THE LOW-ENERGY REGION

1.1. Physical Content of the Pure Elastic Low-Energy Model

The pure elastic low-energy (Pele) model (see Ch. 2 in [1]) is based on the following fundamental assumptions.

1. Strict analyticity for the partial waves.

2. Two-particle unitarity (i.e., one can neglect the three-particle and higher mass states in the unitarity condition).

3. Neglect of the higher partial waves f_l, $l > l_{max}$ (usually, one considers only s and p waves, i.e., $l_{max} = 1$).

4. Approximate crossing symmetry for the lower partial waves f_l ($l \leq l_{max}$) obtained by combining the dispersion relations for forward and backward scattering (differential approximation).

Assumptions 1 and 3 and also assumption 4, which is based on 3, are good approximations in the low-energy region, in which the contributions of the inelastic (multiparticle) channels are either absent or numerically small and, as a rule, the higher partial waves are also numerically small.

The physical content of the pure elastic low-energy model is basically determined by assumption 2. Because of this assumption, the integral contributions to the direct channel (s channel) are exhausted by the graphs depicted in Fig. 1 (for the illustration we show only the contributions to the $\pi\pi$ and πN scattering).

* This paper is a significantly revised exposition of the Chapter "Short-Range Repulsion" from the English edition [1a] of the book "Dispersion Theories of Strong Interaction at Low Energy" (this chapter was not included in the Russian edition [1]). In contrast to [1a], we introduce the concept of chiral symmetry in the present paper.

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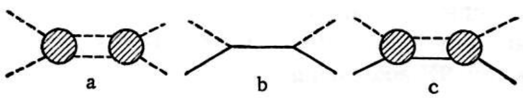

Fig. 1. Contributions from the direct channel to $\pi\pi$ (a) and πN scattering (b, c).

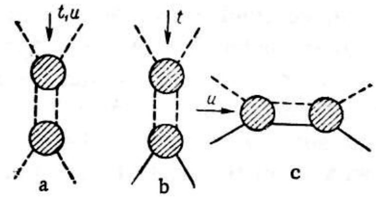

Fig. 2. Integral contributions from the crossed channels to $\pi\pi$ scattering (a) and πN scattering (b, c).

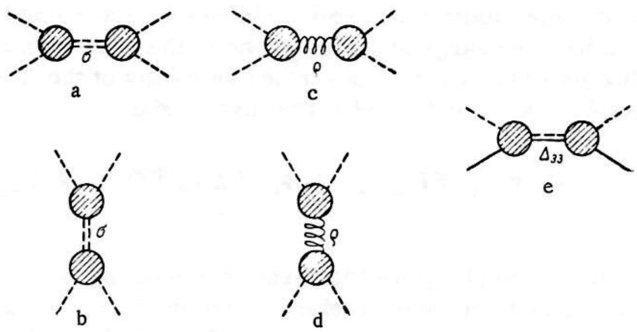

Fig. 3. Resonance contributions to the $\pi\pi$ and πN scattering.

The integral contributions from the crossed channels correspond to the graphs of Fig. 2.

Since the imaginary parts of the amplitudes are approximated by a small number of lower partial waves (by virtue of assumption 3), the integrals are basically saturated by the low-energy resonance contributions from the corresponding two-particle states. For the $\pi\pi$ system these are the contributions from the ρ meson (m = 765 MeV, I = l = 1) and the σ resonance† ($m_\sigma \approx$ 500–1000 MeV, I = l = 0) (Fig. 3a, b); for the πN system they are the Δ_{33} contributions (M_{33} = 1236 MeV, I = l = 3/2) (see Fig. 3c, e).

In other words, in the direct channel the scattering proceeds through the lowest excited resonance states and the contribution of the crossing channels corresponds to exchange of the lowest partial excitations.

The integral contributions from the higher parts of the spectrum are suppressed by conditions 2 and 3 and also by the fact that in the pure elastic low-energy model one considers especially solutions that decrease fairly rapidly with increasing energy (the condition of physical self-consistency).

Thus, in the pure elastic low-energy models, the scattering is almost completely described by an intermediate state with relatively small masses $M_i^*(m_\rho, m_\sigma, M_{33})$ (of order 1 GeV or less). This means that the scattering has a peripheral nature, since the wave functions of these states are concentrated mainly at relatively large distances from the center of mass:

$$R_i^* \sim \frac{\hbar c}{M_i^*} \sim (0.5-1)\,10^{-13}\ \text{cm}. \tag{1.1}$$

If, in agreement with the quantum-mechanical description of scattering, the contributions of the crossed channels are associated with exchange forces, the ranges of the forces are determined by the relation (1.1) and are relatively large. The exchange forces have a long-range nature.

The characteristics of the exchanged quasi-particles are associated with the resonances in the direct channel by conditions that ensure the correct threshold behaviors for the considered (i.e., not small) partial

† The experimental data for the σ meson are discussed in more detail in Sec. 5.4.

waves with $l > 1$. Since the equations that describe the low-energy scattering are integral equations with Cauchy kernels, these conditions are equivalent to the requirement that the partial waves decrease sufficiently rapidly, in accordance with the unitarity condition, at high energies. In the case of $\pi\pi$ scattering, these conditions relate the positions and the widths of the $\pi\pi$ resonances. For the more complicated cases of πN and NN scattering these conditions express the constants (the widths and masses) of the interaction of the unstable mesons (ρ, σ, ω) with the baryons and pions through the characteristics (scattering lengths and resonance widths) of the partial waves of the πN and NN scattering.

This kind of relationship can be interpreted as conditions of low-energy self-consistency: the low-energy resonances arise solely because of forces associated with the exchange of such low-energy resonances. The resulting situation closely resembles the "low-energy bootstrap" program (see Sec. 30.2 in [1]).

1.2. Effective Lagrangian

It is convenient to consider separately intermediate states corresponding to the annihilation t channel. These states do not have a baryon charge and correspond to the ρ and σ mesons. The contributions of these states to the scattering amplitudes can be described by means of the Born approximation (i.e., by means of the pole Feynman graphs) based on the "effective Lagrangian"

$$L_{\text{eff}} = g_{1V}\,\bar{\Psi}\boldsymbol{\tau}\gamma_\mu\Psi_\rho\rho_\mu + \frac{g_{2V}}{M}\bar{\Psi}\sigma_{\mu\nu}\boldsymbol{\tau}\Psi\left(\partial_{\mu\rho}\rho_\nu - \partial_{\nu\rho}\rho_\mu\right) + g_{\sigma NN}\bar{\Psi}\Psi\sigma + g_\rho\rho_\mu\left[\frac{\partial\pi}{\partial x_\mu}\pi\right] + g_\sigma(\pi\pi)\sigma. \tag{1.2}$$

Here Ψ is the operator of the nucleon field, γ_μ are the Dirac matrices, ρ_μ is the operator of the ρ-meson field, a Lorentz four-vector and an isotopic three-vector, τ are the isotopic Pauli matrices, $\sigma_{\mu\nu}$ is the spin matrix tensor of the σ mesons, a Lorentz and isotopic scalar, and π is the operator of the pion field. The expression (1.2) gives the part of the effective Lagrangian that is needed to describe only the $\pi\pi$- and πN-scattering processes. To describe the NN-scattering potential it would be necessary, for example, to consider also the interaction of the nucleons with the ω and A_2 mesons.

Of course, the parameters associated with the effective Lagrangian (1.2), i.e., the coupling constants and the masses of the meson states, are not known a priori. However, considering the interaction processes in the sequence corresponding to the hierarchical scheme, according to which it is impossible to describe meson-baryon scattering theoretically without knowledge of the properties of meson-meson scattering (see Ch. 1 of [1]), these parameters can be determined by consistency considerations for all the reactions ($\pi\pi$, πN, and NN scattering) as a whole. Thus, the constants g_ρ and g_σ and the masses m_ρ and m_σ can be determined from $\pi\pi$ scattering. At the same time, for example, the width of the ρ meson can be expressed in terms of g_ρ and m_ρ as follows:

$$\Gamma_\rho^{\text{tot}} = \frac{g_\rho^2}{6\pi}\cdot\frac{q_\rho^3}{m_\rho^2} = \frac{1}{12}\left(\frac{g_\rho^2}{4\pi}\right)\frac{(m_\rho^2-4\mu^2)^{3/2}}{m_\rho^2}. \tag{1.3}$$

Similarly, the constants g_{1V} and g_{2V} can be determined from the condition of consistency of the contributions to the πN and NN scattering or consistency of the πN scattering with the electromagnetic nucleon form factor.

The effective Lagrangian is also convenient in that it can be used to take into account the additional symmetry principles that are not imposed directly in the dispersion approach. For example, Sakurai's principle of universality of the interaction of the ρ-meson field with the vector current leads to the following relationships between the constants of the ρ-meson interaction:

$$g_{1V} = \frac{1}{2}g_\rho,\quad g_{2V}:g_{1V} = 2\mu_N:1. \tag{1.4}$$

Here $\mu_N = 1.9$ is the anomalous nucleon magnetic moment.

We must emphasize particularly that we require the Lagrangian (1.2) to construct the t-channel contributions to the dispersion-type equations. These contributions have the same poles and residues as the single-particle Feynman graphs. However, in contrast to the latter, they decrease at infinity.* One can

* For the case of unsubtracted dispersion relations.

therefore say that the dispersion contributions are the long-range parts of the corresponding Feynman graphs.

It is shown in Ch. 3 of the monograph [1] that the pure elastic low-energy theory of $\pi\pi$ scattering leads to only qualitative agreement with the experimental data. In particular, it cannot explain the ρ-meson width of ~ 100 MeV. We shall now show that the pure elastic low-energy description of πN scattering also possesses a similar shortcoming. In particular, it strongly contradicts Sakurai's universality principle.

1.3. Long-Range Model of πN Scattering

We shall now obtain the system of equations for the s and p waves of πN scattering using the pure elastic low-energy assumptions formulated above. We shall also make the following simplifying assumptions.

5. The static limit, i.e., the ratio of the pion-nucleon mass, will be assumed to be infinitely small: $\mu/M \ll 1$.

6. We shall approximate the contributions of the annihilation channel $\pi\pi \rightarrow N\bar{N}$ by pole graphs (see Fig. 3b, d) corresponding to the effective Lagrangian (1.2).

Let us now consider the dispersion relations for forward and backward scattering written down for the following four combinations of the structure functions of πN scattering:

$$\varphi^{\pm}(\nu, \cos\theta) = \frac{A^{\pm} + \frac{s-u}{4M} B^{\pm}}{4\pi}; \quad \beta^{\pm}(\nu, \cos\theta) = \frac{B^{\pm}}{8\pi M}.$$

These combinations are convenient in that the lower partial waves go over into them very simply in the static limit:

$$\left.\begin{aligned} \varphi^{\pm} &\simeq \left[s_1 + \begin{pmatrix} 2 \\ -1 \end{pmatrix} s_3\right] + \frac{\cos\theta}{3}\left[p_{11} + 2p_{13}\right. \\ &\quad \left. + \begin{pmatrix} 2 \\ -1 \end{pmatrix}(p_{31} + 2p_{33})\right] \equiv s^{\pm} + \cos\theta\, p^{\pm}; \\ \beta^{\pm}(\nu, \theta) &\simeq \frac{1}{3}\left[h_{11} - h_{13} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}(h_{31} - h_{33})\right] \equiv \beta^{\pm}(\nu). \end{aligned}\right\} \tag{1.5}$$

Here $\nu = q^2$ is the square of the meson momentum in the center of mass system, θ is the scattering angle in the center of mass system, s_{2I} is the s wave of the scattering with isospin I, and $p_{2I,\,2J}$ and $h_{2I,\,2J}$ are p waves with isospin I and total angular momentum J. At the same time

$$s_k = \frac{e^{i\delta_k}\sin\delta_k}{q}, \quad p_{ik} = \frac{e^{i\delta_{ik}}\sin\delta_{ik}}{q} = q^2 h_{ik}. \tag{1.6}$$

We now write down unsubtracted forward dispersion relations for $\varphi^{\pm}$ and $\beta^{\pm}$:

$$\varphi^{+}(\nu, 1) = \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\varphi^{+}(\nu', 1)}{\nu' - \nu}\, d\nu'; \tag{1.7a}$$

$$\varphi^{-}(\nu, 1) = \frac{2f^2}{\omega} + \frac{\omega}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\varphi^{-}(\nu', 1)}{\omega'(\nu' - \nu)}\, d\nu'; \tag{1.7b}$$

$$\beta^{\pm}(\nu, 1) = -\frac{2f^2}{\omega} + \frac{\omega}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\beta^{+}(\nu', 1)\, d\nu'}{\omega'(\nu' - \nu)}; \tag{1.7c}$$

$$\beta^{-}(\nu, 1) = \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\beta^{-}(\nu', 1)\, d\nu'}{\nu' - \nu}. \tag{1.7d}$$

Equations (1.7) differ from the well-known dispersion relations for forward πN scattering in two respects.

First, they are written down in the static limit, i.e., we have used assumption 5. In this approximation the laboratory energy is $E = \omega = \sqrt{\nu + 1}$.

Secondly, and this is very important, they are written down without subtractions, since we assume that the approximate representations 4 and 5 are valid in the whole range of integration; it follows by virtue of 4 that $\mathrm{Im}\varphi^{\pm}$ and $\mathrm{Im}\beta^{\pm}$ decrease sufficiently rapidly and the unsubtracted integrals converge.

The dispersion relations for backward scattering written down without subtractions in the static limit have the form:

$$\varphi^{\pm}(\nu, -1) = \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,\varphi^{+}(\nu', -1)}{\nu' - \nu}\,d\nu' + \frac{\alpha_\sigma}{1+\nu_\sigma+\nu}; \tag{1.8a}$$

$$\varphi^{-}(\nu_1 - 1) = -\frac{2f^2}{\omega} + \frac{\omega\alpha_{\rho 1}}{1+\nu_\rho+\nu} + \frac{\omega}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,\varphi^{-}(\nu', -1)}{\omega'(\nu'-\nu)}\,d\nu'; \tag{1.8b}$$

$$\beta^{+}(\nu, -1) = -\frac{2f^2}{\omega} + \frac{\omega}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,\beta^{+}(\nu', -1)}{\omega'(\nu'-\nu)}\,d\nu'; \tag{1.8c}$$

$$\beta^{-}(\nu, -1) = \frac{\alpha_{\rho 1} + \alpha_{\rho 2}}{2M(1+\nu+\nu_\rho)} + \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,\beta^{-}(\nu', -1)}{\nu'-\nu}\,d\nu'. \tag{1.8d}$$

The last terms on the right sides of Eqs. (1.8a), (1.8b), and (1.8d) are the contributions from the annihilation channel and are approximated by the σ- and ρ-meson exchange graphs by means of the effective Lagrangian (1.2). In this case

$$\alpha_\sigma = \frac{g_{\sigma NN}\, g_\sigma}{8\pi}; \quad \alpha_{\rho_1} = \frac{g_{1V}\, g_\rho}{8\pi}; \quad \alpha_{\rho 2} = \frac{g_{2V}\, g_\rho}{8\pi}, \tag{1.9}$$

and the parameters ν_σ and ν_ρ are related to the masses m_σ and m_ρ by the equations

$$m_\sigma^2 = 4(1+\nu_\sigma), \quad m_\rho^2 = 4(1+\nu_\rho).$$

To go over from the functions $\varphi^{\pm}$ to the partial waves, we must, in accordance with (1.5), consider the half-sums and half-differences of Eqs. (1.7a) and (1.7b) and (1.8a) and (1.8b):

$$s^{\pm}(\nu) = \frac{\varphi^{\pm}(\nu, 1) + \varphi^{\pm}(\nu, -1)}{2}, \quad p^{\pm}(\nu) = \frac{\varphi^{\pm}(\nu, 1) - \varphi^{\pm}(\nu, -1)}{2}.$$

As regards the p-wave combinations $\beta^{\pm}(\nu)$, these must be determined by Eqs. (1.7c) and (1.7d) and (1.8c) and (1.8d). Since we have neglected d waves, a certain ambiguity arises. We shall define the $\beta^{\pm}$ combinations as the half-sum

$$\beta^{\pm}_{(\nu)} = \frac{\beta^{\pm}(\nu, 1) + \beta^{\pm}(\nu, -1)}{2},$$

since the half-sums do not contain d waves.

As a result, we obtain a system of equations for the s and p waves:

$$s^{+}(\nu) = \frac{\alpha_\sigma}{2(1+\nu_\sigma+\nu)} + \frac{1}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,s^{+}(\nu')\,d\nu'}{\nu'-\nu}; \tag{1.10a}$$

$$s^{-}(\nu) = \frac{\omega\alpha_{\rho 1}}{2(1+\nu_\rho+\nu)} + \frac{\omega}{\pi}\int_0^{\infty} \frac{\mathrm{Im}\,s^{-}(\nu')\,d\nu'}{(\nu'-\nu)\,\omega'}; \tag{1.10b}$$

$$p^+(\nu) = -\frac{\alpha_\sigma}{2(1+\nu_\sigma+\nu)} + \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\, p^+(\nu')\, d\nu'}{\nu'-\nu}; \tag{1.10c}$$

$$p^-(\nu) = \frac{2f^2}{\omega} - \frac{\omega\alpha_{\rho 1}}{2(1+\nu_\rho+\nu)} + \frac{\omega}{\pi}\int_0^\infty \frac{\text{Im}\, p^-(\nu')\, d\nu'}{\omega'(\nu'-\nu)}; \tag{1.10d}$$

$$\beta^+(\nu) = -\frac{2f^2}{\omega} + \frac{\omega}{\pi}\int_0^\infty \frac{\text{Im}\, \beta^+(\nu')\, d\nu'}{\omega'(\nu'-\nu)}; \tag{1.10e}$$

$$\beta^-(\nu) = \frac{\alpha_{\rho 1}+\alpha_{\rho 2}}{4M(1+\nu_\rho+\nu)} + \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\, \beta^-(\nu')\, d\nu'}{\nu'-\nu}. \tag{1.10f}$$

In conjunction with the elastic unitarity conditions for the partial waves, Eqs. (1.10) form a complete system of equations for the s and p waves of πN scattering derived in the pure elastic low-energy model for the static case.

$$\text{Im}\, s_i = q\,|s_i|^2, \quad \text{Im}\, p_{ik} = q\,|p_{ik}|^2, \quad 0 \leqslant q < \infty.$$

We shall not solve this system. Our aim is to verify its internal consistency and agreement with the main experimental data of low-energy πN scattering.

Consider first the threshold behaviors of the $p^\pm$ combinations of the p waves. In accordance with the quantum-mechanical threshold conditions

$$p_{ik}(q) \simeq a_{ik}\, q^2 \quad \text{as} \quad q^2 \to 0,$$

where the constants a_{ik} are the p-wave scattering lengths. Thus, the combinations $p^\pm$ must vanish at the threshold. Equations (1.10c) and (1.10d) yield

$$-\frac{\alpha_\sigma}{2(1+\nu_\sigma)} + \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\, p^+\, d\nu}{\nu} = 0; \tag{1.11a}$$

$$2f^2 - \frac{\alpha_{\rho 1}}{2(1+\nu_\rho)} + \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\, p^-\, d\nu}{\omega\nu} = 0. \tag{1.11b}$$

Let us now turn to the asymptotic behavior of the functions $\beta^\pm$. By virtue of (1.5) and (1.6), they must decrease not slower than q^{-3} as $q \to \infty$.

With allowance for (1.10e) and (1.10f), this yields the two conditions

$$-2f^2 - \frac{1}{\pi}\int_0^\infty \frac{\text{Im}\, \beta^+}{\omega}\, d\nu = 0; \tag{1.12a}$$

$$\frac{\alpha_{\rho 1}+\alpha_{\rho 2}}{4M} - \frac{1}{\pi}\int_0^\infty \text{Im}\, \beta^-\, d\nu = 0. \tag{1.12b}$$

One can readily show that the asymptotic conditions (1.12) are threshold conditions for the functions $q^2\beta^\pm$, which are, in accordance with (1.5), linear combinations of the p waves in the usual normalization (1.6). At the same time, it must be assumed that unsubtracted dispersion relations hold for the functions $q^2\beta^-$ and $(q^2/\omega)\beta^+$.

We shall now assume that the solutions of the system (1.10) include a solution that agrees closely with the experimental data in the low-energy region, i.e., all the s and p waves with the exception of p_{33} are small and the p_{33} pass through a resonance. Accordingly, we shall neglect all the imaginary parts with the exception of $\mathrm{Im} p_{33}$, which we represent in the form

$$\mathrm{Im}\, p_{33} = \pi \nu_{33} \Gamma_{33} \delta(\omega - \omega_{33}). \tag{1.13}$$

Substituting (1.13) into (1.12a), we obtain

$$\Gamma_{33} = \frac{3}{2} f^2. \tag{1.14}$$

Equations (1.11b) and (1.12b) yield

$$\alpha_{\rho 1} = 0, \quad \frac{\alpha_{\rho 2}}{4M} = \frac{2}{3} \Gamma_{33} \omega_{33}. \tag{1.15}$$

Equation (1.11a) leads to the relationship

$$\frac{\alpha_\sigma}{m_\sigma^2} = \frac{4}{3} \Gamma_{33} \omega_{33}. \tag{1.16}$$

The energy dependence of the s waves is described by the exchange terms. With allowance for (1.14)-(1.16), we obtain

$$s^+(\nu) = \frac{4f^2 \omega_{33}}{1 + 4\frac{q^2}{m_\sigma^2}}, \quad s^- = 0. \tag{1.17}$$

Let us discuss the results obtained. The relation (1.14) is known from the Chew-Goldberger-Low-Nambu theory [3]. This is not surprising, since Eqs. (1.11b) and (1.12a) used to derive (1.14) do not contain contributions from the annihilation channel. However, the relationship (1.14) is numerically unsatisfactory. Substituting the generally adopted value $f^2 = 0.08$, we obtain $\Gamma_{33} = 0.12$.

In accordance with the experimental data the phase δ_{33} passes through a resonance at $M_3 = \sqrt{s_{33}} =$ 1236 MeV. The width of the resonance in its proper system is 120 MeV. In the laboratory system this corresponds to $E_{33}^{\mathrm{lab.kin}} = 190$ MeV. The total width in the laboratory system calculated from $\sin^2 \delta_{33}$ is $\gamma_{\mathrm{lab}}^{\mathrm{tot}} = 170$ MeV. Going over to the center of mass system by means of the Jacobian $\partial\omega_{\mathrm{sms}}/\partial E_{\mathrm{lab}}$, we obtain

$$\omega_{33,\,\mathrm{exp}} = 265 \text{ MeV} = 1.92 \quad \mu, \quad \gamma_{\mathrm{sms,\ exp}}^{\mathrm{tot}} = 75 \text{ MeV} \simeq 0.54\ \mu,$$

which, for the "reduced half-width" in (1.13), yields

$$\Gamma_{33,\,\mathrm{exp}} = \frac{\gamma^{\mathrm{tot}}}{2q_{33}^2} = 0.088.$$

At the same time, a calculation of the integrals $\int \mathrm{Im}\, h_{33} d\omega$ and $\int \mathrm{Im} h_{33} \omega d\omega$ from the experimental phases leads to the effective values

$$\Gamma_{33.\,\mathrm{exp}}^{\mathrm{eff}} = 0.055, \quad \omega_{33.\,\mathrm{exp}}^{\mathrm{eff}} \simeq 1.9. \tag{1.18}$$

The difference between these numbers and those calculated by means of the Jacobian (in the δ approximation) is due to the fairly large width of the 33 resonance.

The relationship (1.15) strongly contradicts the universality principle (1.4), which is satisfied experimentally (see [4]):

$$\alpha_{\rho 1}^{\mathrm{exp}} \simeq 0.7 - 0.5, \quad \alpha_{\rho 2}^{\mathrm{exp}} \simeq 2.65 - 1.90. \tag{1.19}$$

The second of the relations (1.15) enables one to express ω_{33} in terms of Γ_{33} and $\alpha_{\rho 2}$. Setting $\Gamma_{33} = 0.12$ and using (1.19), we find $\omega_{33} \approx 1.25$, which is smaller than the experimental value (1.18) by a factor of 1.5.

Finally, the expressions for the s waves also strongly contradict the experimental data, according to which

$$s^{+}(0) = a^{+} = 0: \quad s^{-}(0) = a^{-} = 0.08 - 0.09. \tag{1.20}$$

We, therefore, conclude that the pure elastic low-energy description of the low-energy πN scattering based on allowance for only the long-range effects fails to give quantitative agreement with the experimental results for the p waves and fails to give even qualitative agreement for the s waves.

As we have already mentioned, a similar situation obtains for $\pi\pi$ scattering.

1.4. Need for Allowance for Short-Range Effects

Let us now analyze the various means by which the pure elastic low-energy approximation could be improved. Of course, assumptions 5 and 6 used to study the πN system introduce definite errors, but it is known that, although they are important, the relativistic corrections to the static limit can change the numerical results by $\leq 50\%$ in the low-energy region. In addition, assumption 5 does not affect the $\pi\pi$ problem.

Equally, assumption 6 can be improved by the introduction of the f_0 resonance in the two-pion system with I = 0, $l = 2$, ($m_{f_0} = 1260$ MeV, $\Gamma_{f_0}^{tot} = 120$ MeV) and also by allowance for the complex structure of the so-called σ resonance (see Sec. 5.4). However, the corresponding corrections are also small.

Let us return to the original approximations of the pure elastic low-energy model, i.e., the assumptions 2–4. One can show (see, for example, Sec. 17.2 of [1]) that allowance for the nearest neglected partial wave yields small numerical effects. On the other hand, an analysis of the experimental data (phase analysis of πN scattering and the approximate equality of the cross sections at the ρ and f_0 maxima and the geometric cross sections) indicates that the nearest inelastic corrections (the contributions of the three- and four-particle states) to the imaginary parts of the partial amplitudes are also numerically small. Thus, the defect of the pure elastic low-energy model is most probably to be found in the neglect of the influence of the high-energy region on the low-energy region, i.e., the assumption that the low-energy region is physically complete, with its consequence that only long-range effects are important.

Let us illustrate this by taking the example of the pure elastic low-energy model of πN scattering that we have just considered. In this approximation, the scattering lengths a_1 and a_3 (as also in $\pi\pi$ scattering) are strongly overestimated; thus, we obtained $a_1 \approx a_3 \approx 5f^2 \approx 0.4$ (the experimental values are $a_1 \approx 0.16$, $a_3 \approx -0.08$). It is obvious that the theory ignores effects that make important negative contributions to the scattering lengths. To understand the source responsible for these effects, let us consider the function $\varphi^{+}(\nu, 1)$.

It follows from Eq. (1.7a) and the definition (1.5) for φ^{+} that $a^{+} > 0$ for arbitrary solutions [since all f and p waves make positive contributions to Im $\varphi^{+}(\nu, 1)$]. Thus, only the trivial solution can be compatible with $a^{+} = 0$. However, such a solution requires that all the inhomogeneous terms in the system (1.10) vanish, and is clearly meaningless.

Equation (1.7a) corresponds to a forward dispersion relation for the crossing-even combination

$$\varphi^{+}(\nu, 1) = \frac{T(\pi^{+}p) + T(\pi^{-}p)}{2} \tag{1.21}$$

and differs from it by the use of the approximations 2–5 and also by the absence of a subtraction. It is precisely because of the subtraction that the rigorous dispersion relation for (1.21) agrees with the experimental data and, in particular, with the first of Eqs. (1.20). It is therefore clear that we could improve the situation by adding to the right side of Eq. (1.7a) the negative subtraction constant

$$\varphi^{+}(\nu, 1) = \frac{1}{\pi}\int \frac{\operatorname{Im}\varphi^{+}(\nu', 1)}{\nu' - \nu}\, d\nu' - v,$$

where

$$v=\frac{1}{\pi}\int_{0}^{\infty}\frac{\operatorname{Im}\varphi^{+}(\nu,1)}{\nu}\,d\nu\simeq\frac{8}{3}\,\omega_{33}\,\Gamma_{33}>0.$$

This constant reflects the fact that we approximate the original dispersion relation, which does not exist unsubtracted, by the unsubtracted equation (1.7a). Thus, the constant v reflects the properties of the original dispersion relation of the high-energy region, i.e., it has a short-range nature.

Let us elucidate what we have said in more detail by considering the model example of the scattering of neutral spinless particles.

1.5. Analysis of the Short-Range Contributions

Let us consider the dispersion relation for the forward scattering of neutral spinless particles:

$$A(x,1)=A(0,1)+\frac{x}{\pi}\int_{1}^{\infty}\frac{\operatorname{Im}A(x',1)\,dx'}{x'(x'-x)}. \tag{1.22}$$

Here $A(x, 1)$ is the forward scattering amplitude considered as a function of the variable

$$x=\left(\frac{s-2m^2}{2m^2}\right)^2=(2\nu+1)^2;$$

in accordance with the optical theorem

$$\operatorname{Im}A(x,1)=\frac{\sqrt{x-1}}{8\pi}\,\sigma_{\text{tot}}\,(x),$$

and a subtraction has been made at the point $x = 0$, which is a symmetry point in the original variable

$$z=\frac{s-2m^2}{2m^2}=-\frac{u-2m^2}{2m^2}.$$

Let us follow in detail the procedure for obtaining the equation for the lowest partial wave from the dispersion relation (1.22). We split the range of variation $(1, \infty)$ of the energy variable x into two parts: 1) the low-energy region $1 < x < \Lambda$; 2) the high-energy region $\Lambda < x < \infty$.

In the low-energy region we shall assume that the forward scattering amplitude can be well-approximated by the elastic s wave:

$$A(x,1)\simeq A_0(x),\quad \operatorname{Im}A_0(x)=K(x)|A_0(x)|^2;\quad 1<x<\Lambda. \tag{1.23}$$

Here $K(x)$ is the ratio of the momentum to the energy:

$$K(x)=\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)^{1/2}.$$

In the high-energy region all the partial waves are important. We shall assume that in this region the s-wave partial cross section is negligibly small compared with the total cross section:

$$\operatorname{Im}A_0(x)\ll\operatorname{Im}A(x,1)=\frac{\sqrt{x-1}\,\sigma_{\text{tot}}}{8\pi};\quad \Lambda<x<\infty. \tag{1.24}$$

Under real physical conditions the low-energy region defined by (1.23) and the high-energy region defined by (1.24) are separated by a region of intermediate energies in which there is a smooth transition from the low- to the high-energy conditions. The first approximation consists of assuming that this region is infinitely narrow (or rather sufficiently narrow for one to be able to neglect its integral contributions to the low-energy region).

Let us rewrite identically Eq. (1.22):

$$A(x, 1) = \lambda + \frac{x}{\pi}\int_1^\infty \frac{\operatorname{Im} A_0(x')\,dx'}{x'(x'-x)} + \frac{x}{\pi}\int_1^\infty \frac{\operatorname{Im}[A(x', 1) - A_0(x')]}{x'(x'-x)}\,dx'. \tag{1.25}$$

Here,

$$\lambda = A(0, 1). \tag{1.26}$$

Using (1.23) and (1.24), we now rewrite (1.25) for the region $1 < x < \Lambda$ in the form

$$A_0(x) = \frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} A_0(x')\,dx'}{x'-x} + V(x), \tag{1.27}$$

where

$$V(x) = \lambda - \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im} A_0(x)\,dx}{x} + \frac{x}{8\pi^2}\int_\Lambda^\infty \frac{\sqrt{x'-1}\,\sigma(x')\,dx'}{x'(x'-x)} \tag{1.28}$$

is the short-range contribution to the low-energy region. In the region $x < \Lambda$ this short-range effective potential can be approximated by a polynomial of the first degree:

$$V(x) \simeq -v + xI, \tag{1.29}$$

where

$$v = \frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} A_0(x')}{x'}\,dx' - \lambda = \lambda_0 - \lambda, \quad I = \frac{1}{8\pi^2}\int_\Lambda^\infty \frac{\sqrt{x-1}\,\sigma(x)\,dx}{x}. \tag{1.30}$$

Equation (1.27) differs from the pure elastic low-energy equation for the s wave by the presence of the potential $V(x)$. Equation (1.27) describes low-energy scattering in the field of the potential $V(x)$. Let us investigate this potential in more detail.

The first important property of $V(x)$ is that it is negative ($v > 0$). The potential $V(x)$ is repulsive. This property is common to the different processes ($\pi\pi$, KN, πN, etc.) and can be established by fairly general considerations related to the behavior of the forward scattering amplitude (or rather its imaginary part) in the high-energy region.

To illustrate this, let us consider the inverse function of the forward scattering amplitude $H(x, 1) = A^{-1}(x, 1)$. Its discontinuity across the cut $1 \le x \le \infty$ is

$$2i \operatorname{Im} H(x, 1) = -\frac{2i \operatorname{Im} A(x, 1)}{|A(x, 1)|^2}, \tag{1.31}$$

and, as $x \to \infty$,

$$\operatorname{Im} H(x, 1) < -\frac{8\pi}{\sqrt{x}\,\sigma(\infty)}.$$

It follows that the spectral integral that represents $H(x, 1)$ converges. The spectral representation for $H(x, 1)$ has the form

$$H(x, 1) = \lambda^{-1} - \frac{x}{\pi}\int_1^\infty \frac{\operatorname{Im} A(x', 1)}{|A(x', 1)^2} \cdot \frac{dx'}{x'(x'-x)} - \frac{x\alpha_0}{x_0(x_0-x)} - \sum_i \frac{x\beta_i}{x_i(x_i-x)}. \tag{1.32}$$

It follows from the representation (1.22) that $A(x, 1)$ is an R function in the complex plane of X. It follows that $H(x, 1)$ is also an R function, i.e., it cannot have poles in the complex plane, with the possible exception of the real axis. Therefore, $\alpha_0 > 0$ and $\beta_i > 0$. In (1.32) x_0 is the possible position of a zero that lies in front of the reaction threshold, $x_0 < 1$, and x_i are possible zeros in the physical region $x_i > 1$. A necessary condition for Im A to increase as $\sqrt{x}$ as $x \to \infty$ is $H(x, 1) \to 0$ as $x \to \infty$. This gives

$$\lambda^{-1} = \frac{-1}{\pi} \int \frac{\operatorname{Im} A}{|A|^2} \cdot \frac{dx}{x} - \frac{\alpha_0}{x_0} - \sum \frac{\beta_i}{x_i}. \tag{1.33}$$

It can be seen from Eq. (1.33) that $\lambda < 0$ when the amplitude A (x, 1) does not have a zero for $x < 0$, i.e.,

$$\lambda < \lambda_0 = \frac{1}{\pi} \int \frac{\operatorname{Im} A_0}{x} dx \text{ and } v = \lambda_0 - \lambda > 0.$$

However, if $\lambda > 0$, then we must have $\alpha_0 \neq 0$ and $x_0 < 0$, i.e., the function Re A passes through zero in the region $x < 0$. Consequently, we must augment the unsubtracted representation for $A_0(x)$ (1.27) with a negative term to ensure that for $x < 0$ the function Re A (x, 1) $\approx$ Re $A_0(x)$ passes through zero; hence, in this case, too, $v > 0$.

This argument is based solely on the very general fact that the imaginary part of the forward scattering amplitude (which is positive because of the unitarity condition) increases, albeit weakly, in the high-energy region:

$$\operatorname{Im} A(x, 1) \geqslant c(\ln x)^{1+\varepsilon}, \quad \varepsilon > 0.$$

Numerical estimates for the values of v in a number of cases can be obtained from the experimental values for the s-wave scattering lengths and the low-energy total cross sections that occur in the crossing-even forward scattering amplitude, which requires subtractions. We shall see below that the values of v in both $\pi\pi$ and πN scattering are $\sim \mu^{-1}$.

The range of the forces associated with the potential V(x) can be estimated by means of a numerical estimate of the coefficient of I in the linear term in (1.29). This linear term becomes important for

$$x \sim x_0 = I^{-1}.$$

Setting, for example, for $\pi\pi$ scattering

$$\Lambda = \left(\frac{s - 2\mu^2}{2\mu^2}\right)^2 \bigg|_{s \sim 1 \text{GeV}^2} \simeq (21.5)^2$$

and $\sigma_{tot} = 0.8\mu^{-2}$ (which corresponds to 16 mb), we obtain

$$x_0 = 5\pi^2 \sqrt{\Lambda} \simeq 1060, \tag{1.34}$$

which corresponds to $s_0 \simeq 2\mu^2(\sqrt{x_0} + 1) \simeq 68\mu^2 \simeq (1.15 \text{ GeV})^2$. Therefore, the "range" of the repulsive forces is smaller by a factor of 10 than the pion Compton length.

Thus, the high-energy effects lead to the appearance in the low-energy region of an effective repulsion with an intensity of ~ 1 (in reciprocal pion masses) and a range of ~ 0.1 of the reciprocal pion mass.

Note that the very fact of a small change of the potential in the low-energy region ($x \ll 1000$) is due solely to the fact that at high energies the cross sections of all processes are less than 0.1 b.

Our next aim is to construct a solvable model for the lower partial waves taking into account this short-range repulsion.

2. NEUTRAL MODEL

2.1. Choice of the Model

The simplest such model can be obtained by the approximate replacement of the potential (1.28) by the pole expression

$$V(x) \to v(x) = -\frac{v}{1 + \frac{x}{p}}; \quad p = \frac{v}{I}. \tag{2.1}$$

The "approximate potential" v(x) possesses the following properties.

A. In the physical region of low energies it is numerically very close to the approximate expression (1.28), having two identical terms in the Laurent expansion.

B. It is finite in the whole of the physical region ($1 < x < \infty$).

C. It has a singularity (a pole at the unphysical partial point $x = -p$).

By virtue of property B the equation for the neutral case with the potential (2.1)

$$A_0(x) = -\frac{v}{1+\frac{x}{p}} + \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im} A_0(x')\,dx'}{x'-x} \tag{2.2}$$

is compatible with the unitarity condition

$$\operatorname{Im} A_0(x) = K(x)\,|A_0(x)|^2$$

in the whole of the interval $1 < x < \infty$ and, hence, can be solved analytically. Of course, among the solutions of Eq. (2.2) one must take those for which $A_0(x)$ is small for $x > \Lambda$. It is only subject to this condition that the solution has physical meaning in the low-energy region for $x < \Lambda$.

By virtue of property C a partial amplitude that satisfies Eq. (2.2) has a pole at $x = -p$. This violates strict analyticity (Sec. 1.1).

However, because of the short-range nature of the phenomenological potential we have introduced, the value of p is large [cf. with (1.34)]. It follows that the analytic properties are violated only in the high-energy part of the complex plane.

It follows that proposition 1 in Sec. 1.1 is replaced by: 1a) the correct analyticity properties of the partial waves in the low-energy part of the complex energy plane (in a circle of radius $|x| \approx \Lambda$).

Here it should be noted that, in accordance with assumptions 2-4, we also have departures from strict crossing symmetry and unitarity in the high-energy region.

2.2. Solution of the Equation

The analytic solution of Eq. (2.2) can be represented in the form

$$A_0(x) = \frac{N(x)}{D(x)}, \tag{2.3}$$

where

$$N(x) = \lambda + \frac{x\beta}{x+p} \tag{2.4}$$

and

$$D(x) = 1 - x\sum \frac{\alpha_i}{x_i(x_i - x)} - cx - \lambda I(x) - \beta x\frac{I(x) - I(-p)}{x+p}. \tag{2.5}$$

Here $I(x) = \frac{x}{\pi}\int_1^{\infty} \frac{K(x')\,dx'}{x'(x'-x)}$.

As in the case of the pure elastic low-energy approximation, the solution (2.3)-(2.5) is multiparametric. It depends on the parameters λ, α_i, and x_i. The value of β can be determined from the condition that at the point $-p$ the function A_0 has a residue equal to $-v$:

$$\beta = \frac{v}{1 - vpI'(-p)}\left\{1 + p\sum \frac{\alpha_i}{x_i(x_i+p)} + pc - \lambda I(-p)\right\}. \tag{2.6}$$

Since p is large,

$$\beta \simeq \frac{v}{1-\frac{v}{\pi}}\left\{1+p\sum \frac{\alpha_i}{x_i(x_i+\rho)}+pc+\frac{\lambda \ln p}{\pi}\right\}. \tag{2.7}$$

It is obvious that for $\lambda > 0$ and $v < \pi$ the function N(x) [i.e., $A_0(x)$] has a zero at the point

$$x_0 = -\frac{\lambda p}{\lambda+\beta}.$$

For fixed λ, α_i, and x_i the zero at the point x_0 is nearer the reaction threshold the larger is v. As $v \to 0$, $\beta \to 0$, and $x_0 \to -\rho$, the solution (2.3) goes over into the pure elastic low-energy solution. The presence of this zero means that the solution we have obtained is essentially different from the solution of the pure elastic low-energy approximation, its proximity to the subthreshold region being a measure of the influence of the high energy contribution. For a fixed value of v the value of λ is bounded in the interval $\lambda_{min} \le \lambda \le \lambda_{max}$. Here λ_{min} is determined by the equation $\lambda_{min} + \beta = 0$, when $\alpha_i = 0$, reflecting the fact that the zero in the function N need not lie beyond the pole at the point $-p$. This yields

$$\lambda_{min} = -\frac{1}{\frac{1}{v}-\frac{1}{\pi}+\frac{\ln p}{\pi}} < 0;$$

λ_{max} (as in the pure elastic low-energy approximation) is bounded by the condition that there are no bound states, $D(1) = 0$, in the interval $0 < x < 1$. For $p \gg 1$

$$\lambda_{max} \simeq \frac{1-\sum \frac{\alpha_i}{x_i(x_i-1)}-c-\frac{v}{\pi-v}\left\{1+p\sum \frac{\alpha_i}{x_i(x_i+p)}+cp\right\}\frac{\ln p}{p}}{I(1)+\frac{v}{\pi p}\cdot\frac{\ln^2 p}{\pi-v}}.$$

It can be seen from Eqs. (2.5) and (2.4) that there exist two types of solution. If the zero x_0 is sufficiently distant, i.e., $\beta \ll p$, then $\lambda > 0$ and the solution in the low-energy region is virtually the same as the corresponding solution in the pure elastic low-energy approximation. However, if the point x_0 is not far from the threshold, i.e., $\beta \sim p$, then $|\lambda| \sim (\ln p)^{-1}$; the scattering length is $|a_0| \sim (\ln p)^{-1}$ and the solution can have a low-energy resonance since $\beta \ln p/\pi\tilde{p}x_r^{-1}$ [see the last term in (2.5)] can be finite. In this case the width of the resonance is fairly small, $\sim (\ln p)^{-1}$, but it is very difficult to approximate the expression for the s wave by a Breit-Wigner type formula. The s wave can be approximated better by the expression

$$A_0 \simeq \frac{\lambda\left(1+\frac{x}{x_0}\right)}{1-\frac{x}{x_r}-i\lambda\left(1+\frac{x}{x_0}\right)K(x)}. \tag{2.8}$$

In order to get an idea of the order of the quantities that restrict the constant λ and the scattering length for solutions that depend strongly on the high-energy contribution, we have given the relevant quantities in Table 1.

TABLE 1. Dependence of λ_{min} and a_{min} on the Position of the Pole p in the Repulsive Potential

p	100	1 000	10 000
$W(p)$	660 MeV	1,1 GeV	2 GeV
$-\lambda_{min}$	1,1	0,61	0,41
$-a_{min}$	0,49	0,36	0,28

The quantity W(p) in the second row characterizes the distance which separates the pole p from the threshold in the energy scale $W = 280\left[\frac{\sqrt{p}+1}{2}\right]^{1/2}$ MeV.

Table 1 shows that even if the pole is at an appreciable distance solutions are possible with a negative s-wave scattering length that is not small.

When the conditions $(v/\pi)\ln p \ll 1$ and $\lambda \ll 1$ are satisfied, the resonance solution can be represented conveniently in the form

$$\mathrm{Re}\, A_0(x) = \frac{\lambda + v}{1 - x/x_r}; \quad \mathrm{Im}\, A_0 = \pi x_r \Gamma \delta(x - x_r); \qquad (2.9)$$

$$\Gamma = \lambda + v = \lambda_0.$$

The integral in Eq. (2.2) can be well-approximated by the resonance contributions. In what follows, we shall exploit this fact to make an approximate analysis of the equations that describe the scattering of real particles.

From our investigation we conclude that the scattering amplitude in the low-energy part of the complex plane $|x| \ll \Lambda$ begins to vanish (or is very nearly equal to zero) at certain points because of the influence of the high-energy contributions. Thus, the threshold conditions for the partial waves with $l \geq 1$ cannot be satisfied without correct allowance for the short-range contributions. On the other hand, the properties of the low-energy amplitude, including the distribution of its zeros, can be obtained by a completely different approximation, namely, the massless pion approximation.

3. CURRENT ALGEBRA AND CHIRAL SYMMETRY

3.1. The Algebra of Charges and the Chiral Group

This section is devoted to an extremely condensed exposition of the main features of the method of current algebra. The reader who is unacquainted with this method should consult the works listed in the bibliography and also the review literature [5].

The appearance of the current-algebra method [6] opened up a new field of activity in the physics of elementary particles. The main quantities in this method, the vector j_A^μ and axial j_{5A}^μ currents, are isotopic three-vectors (A = 1, 2, 3) and a Lorentz four-vector and pseudovector, respectively ($\mu = 0, 1, 2, 3$). It is most convenient to represent j_A^μ and j_{5A}^μ as the currents associated with the variation of a Lagrangian under unitary infinitesimal transformations of the fields:

$$\Psi(x) \to \Psi'(x) = U\Psi(x)U^{-1}; \quad U = 1 - i\,\alpha_A(x)\,Q^A - i\,\beta_B(x)\,Q_5^B. \qquad (3.1)$$

The currents and their divergences are defined in terms of the derivatives of the variation of the total Lagrangian [7]:

$$j_A^\mu = -\frac{\partial \delta L(x)}{\partial(\partial_\mu \alpha_A(x))}; \quad j_{5B}^\mu = -\frac{\partial \delta L(x)}{\partial(\partial_\mu \beta_B(x))}; \qquad (3.2)$$

$$\partial_\mu j_A^\mu(x) = -\frac{\partial \delta L(x)}{\partial \alpha_A(x)}; \quad \partial_\mu j_{5B}^\mu(x) = -\frac{\partial \delta L(x)}{\partial \beta_B(x)}. \qquad (3.3)$$

The transformation properties of the currents are determined by the properties of the operator generators Q_A and Q_B of the transformation (3.1). If the Lagrangian is invariant under (3.1) with parameters α_A and β_B that do not depend on the coordinates $[\partial_\mu \alpha_A(x) = \partial_\gamma \beta_B(x) = 0]$, (3.3) yields continuity equations for the currents:

$$\partial_\mu j_A^\mu(x) = \partial_\nu j_{5B}^\nu(x) = 0.$$

In this case, the "charges"

$$Q_A(x^0) = \int d^3x\, j_A^0(x), \quad Q_{5B}(x^0) = \int d^3x\, j_{5B}^0(x) \qquad (3.4)$$

do not depend on the time and must be identified with the generators of the transformation (3.1):

$$Q_A(x^0) = Q_A, \quad Q_{5B}(x^0) = Q_{B5}.$$

One of the most important physical hypotheses of the method is that the currents j_A^μ and j_{5B}^μ describe simultaneously both the weak and the electromagnetic interactions of the hadrons. This hypothesis enables one to establish a relationship between the characteristics of the strong interactions (such as the form factors) and the parameters of the weak interactions of the hadrons (the weak decay lifetimes).

At the same time, the currents, which are the main operator quantities in this method, satisfy definite commutation relations. The form of the commutation relations for the current densities j_A^μ and j_{5B}^ν depends essentially on the specific physical properties attributed to the structure of the elementary particles or the structure of the strong interactions. For example, the commutation relations for the vector currents that follow from the quark particle model [6, 8] differ appreciably from the commutation relations obtained on the assumption of vector dominance [9].

However, irrespective of the concrete commutation relations between the current densities, the commutation relations between the corresponding charges have a very simple form:

$$[Q_A, Q_B] = i\,\varepsilon_{ABC} Q_C; \tag{3.5a}$$

$$[Q_A, Q_{5B}] = i\,\varepsilon_{ABC} Q_{5C}. \tag{3.5b}$$

Here ε_{ABC} is the antisymmetric unit tensor of third rank.

The relations (3.5a) reflect the fact that the three quantities Q_1, Q_2, and Q_3 are the generators of the isotopic group SU(2), i.e., they are the operators of the isospin system,*

$$\mathbf{Q} = \mathbf{I}.$$

The relations (3.5b) indicate that the triplet Q_{51}, Q_{52}, and Q_{53} behaves as a vector under isotopic transformations.

It is usually assumed that the commutator of the axial charges has the form [6]

$$[Q_{5A}, Q_{5B}] = i\,\varepsilon_{ABC} Q_C. \tag{3.5c}$$

This relation, which closes the charge algebra, is the main model assumption. It defines the structure of the theory.

To establish this structure it is convenient to introduce left- and right-polarized charges:

$$\mathbf{Q}^\pm = \frac{\mathbf{Q} \pm \mathbf{Q}_5}{2}.$$

The commutation relations (3.1) for the charges take the form

$$\begin{aligned} [Q_A^\pm, Q_B^\pm] &= i\,\varepsilon_{ABC} Q_C^\pm; \\ [Q_A^+, Q_B^-] &= 0. \end{aligned} \tag{3.7}$$

Thus, the components Q^+ and Q^- form individual algebras of isotopic type [SU(2) algebras]. The charges Q and Q_5 must therefore be regarded as generators of the group SU(2) × SU(2).

Under the influence of the parity operator P the polarized charges go over into one another:

$$P\mathbf{Q}^\pm P^{-1} = \mathbf{Q}^\mp.$$

Since the properties of the charges $Q^\pm$ have a great resemblance to those of chiral projection operators (and, for example, also contain them explicitly in the quark realization) the algebra that is obtained is also known as the chiral SU(2) × SU(2) algebra.

The invariance of the Lagrangian under the corresponding group of transformations generated by Q and Q_5 is known as chiral symmetry.

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* Here and in what follows, the symbol **a** always denotes an isotopic three-vector. Its components satisfy

$$[Q_A, Q_B] = i\,\varepsilon_{ABC} a_C. \tag{3.6}$$

3.2. The PCAC Hypothesis and the Massless Pion Approximation

Another important hypothesis in the treatment of processes in which pions participate is that of the partially conserved axial current (the PCAC hypothesis), which defines explicitly the divergence of the axial current [7, 10]

$$\partial_\mu j_5^\mu = f_\pi \mu^2 \pi. \tag{3.8}$$

Using this operator relationship one can express the amplitudes of processes with n + 1 external pions in terms of the amplitude of processes with n pions and, in particular, one can relate the amplitude for scattering of a pion by a particle H with the pion form factor of this particle. Here an important role is played by the procedure for extrapolation in the square of the external pion mass from the point $m_\pi^2 = 0$ [the PCAC relation (3.8) is used at this point] to the physical point $m_\pi^2 = \mu^2 = 0.02$ GeV^2.

It is obvious that in the massless pion approximation ($m_\pi^2 = 0$) both the vector and the axial currents are conserved [see (3.8)], i.e., the charges Q and Q_5 commute with the Hamiltonian and we have exact chiral symmetry. In this limit, as in quantum electrodynamics, there exist low-energy theorems for the scattering amplitudes (considered at the unphysical point $m_\pi^2 = 0$). One goes onto the mass shell by analytic continuation in m_π^2. In the case of scattering of pions by a heavy target (pion-baryon scattering) $\pi + B \to \pi + B$, when the "natural" unit of the mass scale is large ($M_B^2 \gg \mu^2$) extrapolation from $m_\pi^2 = 0$ to $m_\pi^2 = \mu^2$ is completely justified. However, the situation is not so favorable in the case of pion-meson scattering. Therefore, in particular, the threshold characteristics of $\pi\pi$ scattering can only be explained in the framework of broken chiral symmetry.

3.3. Different Realization of the Chiral Group

In contrast to, for example, isotopic symmetry, chiral symmetry is a dynamic symmetry. This means that the Lagrangian is symmetric only as a whole, i.e., the Lagrangian of the free fields and the interaction Lagrangian are not symmetric individually.

It has been shown in a number of investigations [11] that all the results of current algebra are equivalent to calculations by means of the lowest orders of perturbation theory for a chiral-symmetric Lagrangian. At the same time, it is necessary to calculate only the contributions corresponding to tree diagrams, which do not contain divergences.

The problem now reduces to the construction of this effective Lagrangian, in which two possibilities arise.

In the first variant the Lagrangian is constructed in terms of field operators that are multiplets or linear realizations of the chiral group SU(2) × SU(2) [12]. In this case one must introduce bosons of opposite parity and specify their quantum numbers. The infinitesimal increments of the fields under the action of the group generators are here linear functions of these fields. In the general case they are given by the corresponding infinitesimal commutators:

$$\delta_A \Phi = -i\,[Q_A, \Phi]\,\alpha_A, \quad \bar{\delta}_B \Phi = -i\,[Q_{5B}, \Phi]\,\beta_B.$$

(Here and in what follows, the symbol $\bar{\delta}$ denotes the increment associated with the transformation generated by the axial charge; α_A and β_B are infinitesimally small parameters and there is no summation on the right sides.)

In this case the invariant Lagrangian is a polynomial function of the field operators and their first derivatives.

The second possibility is based on the fact that an invariant Lagrangian can be constructed in the form of a complicated (nonpolynomial!) function of a small number of fields, for example, the pion and nucleon fields. The pion field π and the nucleon field Ψ, being multiplets of the isotopic group, do not belong in this case to any linear representation of the chiral group as a whole, and their infinitesimal increments depend nonlinearly on the π field [see (4.9) and (6.5)]. In this case, we have a nonlinear realization of the chiral group.

We shall show below how one constructs the Lagrangian and which are the most important properties of the Born–Feynman graphs of this Lagrangian for $\pi\pi$ and πN scattering [13].

We shall apply this approximation to obtain the threshold characteristics for the low-energy equations of $\pi\pi$ and πN scattering. We shall show that the equations can be reconciled with the massless pion approximation provided a short-range repulsion is introduced in the equations and that the simplest solutions of these equations without Castillejo-Dalitz-Dyson terms (i.e., that do not contain additional parameters) already possess a strong and even resonance p-wave interaction in a reasonable energy range. This clearly indicates that in the framework of the theory that is nonlinear in the π field we obtain dynamical resonances with higher spins that phenomenologically realize the chiral group SU(2) × SU(2) linearly.

4. MASSLESS PION APPROXIMATION IN THE $\pi\pi$ INTERACTION

4.1. Simplest Linear Realization of Chiral Symmetry (Sigma Model)

Since the chiral transformations (3.1) do not conserve parity, the simplest linear realization that contains a pion must also contain a scalar particle σ. If the isotopic spin of this particle is, like that of the pion, equal to 1, the (π, σ) multiplet can be constructed [14] from the linear representations (0, 1) and (1, 0) of the chiral group. This case is not of interest physically since experiments have not revealed even a trace of the σ particle (I = 1). If one assumes that the isotopic spin of the σ particle vanishes, then a multiplet can be constructed from the linear representation (1/2, 1/2) and the corresponding model is known as the Gell-Mann–Levy sigma model [7].

The importance of this model is also due to the fact that it is intimately related to the PCAC approximation.

In this model the variations of the π and σ fields are given by the equations

$$\left.\begin{aligned} \bar{\delta}\pi_A &\equiv -i\left[\beta_B Q_5^B, \pi_A\right] = \beta_A \sigma; \\ \bar{\delta}\sigma &\equiv -i\left[\beta_B Q_5^B, \sigma\right] = -\beta_A \pi_A; \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} \delta\pi_A &\equiv -i\left[\alpha_B Q^B, \pi_A\right] = -\varepsilon_{ABC}\alpha_B \pi_C; \\ \delta\sigma &\equiv 0. \end{aligned}\right\} \tag{4.2}$$

The chiral-symmetric Lagrangian corresponding to (4.1) and (4.2) has the form

$$L = \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{m^2}{2}(\pi^2 + \sigma^2) + \sum_{n \geqslant 2} \lambda_n (\pi^2 + \sigma^2)^n. \tag{4.3}$$

This Lagrangian leads to conservation of the axial current and equality of the masses of the π and σ mesons. We shall now show that the addition to the Lagrangian of the term

$$L_{\text{br}} = f_\pi \mu^2 \sigma, \tag{4.4}$$

which violates the conservation of the axial current, automatically leads to mass inequality. To see this, we calculate the corresponding variation of the Lagrangian:

$$\partial_\mu j_5^\mu = -\frac{\partial(L + L_{\text{br}})}{\partial\beta} = f_\pi \mu^2 \pi, \tag{4.5}$$

i.e., the PCAC condition (3.4). On the other hand, the interaction (4.4) leads to the appearance of Feynman graphs in which there exists a $\sigma \rightarrow$ vacuum transition ("tadpole" graphs) [15]. Such graphs can be eliminated by means of the shift transformation

$$\sigma = \sigma_0 + \sigma'; \quad \sigma_0 = \langle 0 | \sigma | 0 \rangle.$$

Restricting ourselves in the Lagrangian (4.3) to the term n = 2, we obtain the condition for such graphs to compensate each other:

$$-m^2 \sigma_0 + 4\lambda\sigma_0^3 + f_\pi \mu^2 = 0.$$

At the same time the interaction term ($\sim \lambda$) leads to mass renormalization:

$$\mu^2 = m^2 - 4\lambda\sigma_0^2;$$
$$m_\sigma^2 = m^2 - 12\lambda\sigma_0^2.$$

The condition of compatibility of these relations is

$$\sigma_0 = f_\pi.$$

The constant λ can be expressed in terms of the physical masses μ and m_σ as follows:

$$\lambda = -\frac{m_\sigma^2 - \mu^2}{8f_\pi^2}. \tag{4.6}$$

Note that in the limit $\mu^2 \to 0$ the symmetry of the Lagrangian is restored. The $\pi\pi$ scattering lengths in the Born approximation have the form

$$a_0 = \frac{1}{4\pi}\left(5\lambda + \frac{24\lambda^2 f_\pi^2}{m_\sigma^2 - 4\mu^2} + \frac{16\lambda^2 f_\pi^2}{m_\sigma^2}\right);$$
$$a_2 = \frac{1}{4\pi}\left(2\lambda + \frac{16\lambda^2 f_\pi^2}{m_\sigma^2}\right). \tag{4.7}$$

As $\mu^2 \to 0$, (4.7) yields [with allowance for (4.6)]

$$a_0 = a_2 = 0. \tag{4.8}$$

Thus, we have seen that in the limit in which chiral symmetry is satisfied ($\mu^2 \to 0$) the simplest linear model that satisfies the PCAC condition gives nonvanishing scattering lengths.

4.2. Nonlinear Realization of the Chiral Group*

The same result can be obtained without introducing the σ meson explicitly. To do this one must regard the pion field as a nonlinear realization of the chiral group. The nonlinear realization is defined by the functions of the π-meson field that characterize the increment of the pion field under the infinitesimal transformation associated with Q_5. This increment has the form

$$\bar{\delta}\pi_A = -i[\beta_B Q_5^B, \pi_A] = \beta_A f(\pi) + \pi_A(\beta\pi) g(\pi^2). \tag{4.9}$$

To satisfy the Jacobi identity

$$[Q_5^A, [Q_5^B, \pi^C]] - [Q_5^B, [Q_5^A, \pi^C]] = [[Q_5^A, Q_5^B], \pi^C],$$

where the right side is calculated by means of (3.5c) and (3.6), it is necessary that the functions f and g satisfy

$$1 + 2f'(x) f(x) = g(x)\{f(x) - 2xf'(x)\}, \quad x = \pi^2, \quad f' = df/dx. \tag{4.10}$$

Depending on the choice of f, this equation gives the function g and, consequently, specification of f completely characterizes the transformation law (4.9). One might get the impression that Eq. (4.9) defines a complete class of inequivalent realizations of the chiral group. However, this is not so. One can show [13, 16] that the indeterminacy in the choice of f is entirely transferred to the indeterminacy associated with the redefinition of the pion field:

$$\tilde{\pi}_A = \pi_A \Phi(\pi^2)$$

and, as a result, $\tilde{f}(\pi^2) = f(\pi^2)\Phi(\pi^2)$. It follows that a concrete specification of f is equivalent to a definite specification of the physical π field that describes the real π meson.

* Here we follow the exposition of Weinberg [13].

Let us now turn to the construction of the corresponding invariant Lagrangian. It can be seen from (4.9) that the variation of the gradient $\overline{\delta\partial}_\mu\pi$ is proportional to the gradient itself. As a result, one introduces the so-called covariant derivative

$$D_\mu \pi_A = d_{AB}(\pi)\,\partial_\mu \pi_B$$

for the invariant generalization of the kinetic term $(1/2)(\partial_\mu\pi)^2$.

The function d_{AB} must be chosen so as to ensure that the commutation relations have the form:

$$\begin{aligned} &[Q_5^A, D_\mu \pi^C] = -iV^{AB}(\pi)\,\varepsilon_{BCD} D_\mu \pi^D; \\ &[Q^A; D_\mu \pi^C] = -i\varepsilon_{ABC} D_\mu \pi_\beta. \end{aligned} \qquad (4.11)$$

The presence on the right sides of these equations of antisymmetric tensors ensures invariance of the generalized kinetic term. Using the Jacobi identities and the commutation relations of the operators Q^A and Q_5^B, we obtain

$$V_{AB}(\pi) = \varepsilon_{ABC}\pi_C W(\pi^2), \qquad (4.12a)$$

where

$$W(x) = \left[f(x) + \sqrt{f^2(x) + x}\right]^{-1}, \qquad (4.12b)$$

and also

$$D_\mu \pi \sim \frac{\partial_\mu \pi}{[f^2(\pi^2)+\pi^2]^{1/2}} - \frac{f'(\pi^2) + \frac{1}{2}W(\pi^2)}{f^2(\pi^2)+\pi^2}\,\pi\partial_\mu \pi^2. \qquad (4.13)$$

The following special cases are most widely known.

A. Weinberg's definition [13] of the π field: $g = \lambda$, $f(\pi^2) = (1-\lambda^2\pi^2)/2\lambda$. From (4.12) and (4.13) we also have $W = \lambda$.

Using (4.9) one can show that it is impossible to construct an invariant expression that depends only on the field π (but not on its gradients). The Lagrangian therefore has the form

$$L = \frac{1}{2}(D_\mu \pi)^2 = \frac{1}{2}\cdot\frac{(\partial_\mu \pi)^2}{(1+\lambda^2\pi^2)^2}.$$

In this expression the normalization factor omitted in (4.13) is chosen in such a manner that the usual expression is obtained in the limit $\lambda \to 0$.

B. The definition of the π field corresponding to the sigma model [17], $g = 0$ [cf. with (4.1)], $f = \sqrt{f_0^2 - \pi^2}$.

$$W = \left(f_0 + \sqrt{f_0^2 - \pi^2}\right)^{-1}$$

and

$$L = \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}\cdot\frac{(\pi\partial_\mu \pi)^2}{f_0^2 - \pi^2}.$$

This Lagrangian corresponds to the Lagrangian (4.3) to within additive c numbers if one makes the identification

$$\sigma = \sqrt{f_0^2 - \pi^2}.$$

It is now clear that, since the Lagrangians have a purely kinetic form, their Born terms in $\pi\pi$ scattering are proportional to the scalar derivatives of the four-momenta. Since the mass of the π meson vanishes, the scattering lengths also vanish.

4.3. Broken Symmetry in the Nonlinear Case

Massless pions correspond to exact symmetry; broken symmetry corresponds to pions with a mass. We shall describe the symmetry breaking by adding a term of the form $\Phi(\pi^2)$ to the Lagrangian. The linear term of the expansion of this function yields the mass term:

$$\pi^2 \Phi'(0) = -\frac{\mu^2}{2}\pi^2,$$

and the quadratic term the contact $\pi\pi$ interaction:

$$\frac{1}{2}(\pi^2)^2 \Phi''(0) = \lambda (\pi^2)^2.$$

In the Born approximation the isotopic combinations

$$2A_0(s, t, u) - 5A_2 \text{ and } A_1(s, t, u)$$

do not therefore contain terms of order λ, i.e., they do not depend on the form of the symmetry breaking function $\Phi(\pi^2)$. Moreover, one can show that they depend only on the single parameter f_π, which is determined by the lifetime of the $\pi^\pm$ mesons. To show this, let us consider the contact $\pi\pi$ scattering terms that follow from the Lagrangian

$$L = \frac{1}{2}(D_\mu \pi)^2 + \Phi(\pi^2). \tag{4.14}$$

It is obvious from (4.13) that to do this we must specify the first two terms of the expansion of f:

$$f(x) = f_0 + x f_1.$$

Using Eq. (4.10), one can express f_1 in terms of f_0 and $g_0 = g(0)$:

$$f_1 = \frac{g_0}{2} - \frac{1}{2f_0}.$$

Then the expansion of the Lagrangian (4.14) in the π fields becomes

$$L = \frac{(\partial_\mu \pi)^2}{2} - \frac{\mu^2 \pi^2}{2} - \frac{g_0}{2f_0}\pi^2 (\partial_\mu \pi)^2 + \left(\frac{1}{2f_0^2} - \frac{g_0}{f_0}\right)(\pi\, \partial_\mu \pi)^2 + \lambda (\pi^2)^2 + \ldots \tag{4.15}$$

Calculating the Born terms, we obtain

$$2A_0(s, t. u) - 5A_2(s, t, u) = \frac{3(2s - t - u)}{32\pi f_0^2}; \tag{4.16}$$

$$A_1(s, t, u) = \frac{t - u}{32\pi f_0^2}. \tag{4.17}$$

Using this Lagrangian to calculate the axial current,

$$j^A_{5\mu} = -\frac{\delta L}{\delta \partial_\mu \beta_A},$$

we note that the coefficient of $\partial_\mu \pi$ is exactly equal to f_0. It is precisely this term that is responsible for the decay of the charged pion $\pi \to \mu + \nu$. It follows that f_0 can be calculated from the lifetime of the

TABLE 2. Dependence of the s-Wave $\pi\pi$ Scattering Lengths on the Type of Chiral Symmetry Breaking

Type of symmetry breaking	$\xi = a_0/a_2$	a_0	a_2
Equation (4.20), $g_0 f_\pi = 0{,}5$	$-1/2$	0,057	$-0{,}105$
The same, $g_0 = 0$	∞	0,315	0
" " $g_0 = -\infty$	5/2	$+\infty$	$+\infty$
Equation (4.21), $N = 1$	$-7/2$	0,184	$-0{,}0525$
The same, $N = 2$	∞	0,315	0
" " $N = 3$	95/14	0,495	0,73
" " $N \to \infty$	5/2	$+\infty$	$+\infty$

π meson. However, it is usually determined by means of the Goldberger–Treiman relation [18] (see Sec. 6.1):

$$f_0 = f_\pi = -\frac{g_A}{g_V} \cdot \frac{M}{g} = 0.615\ \mu = 86\ \text{MeV}. \tag{4.18}$$

Equations (4.16) and (4.17) yield the scattering lengths*

$$2a_0 - 5a_2 = 0.63; \quad A_1' = 0.035\, g^2. \tag{4.19}$$

The ratio of the scattering lengths a_0 and a_2 can be obtained by fixing a definite form of the chiral symmetry breaking.

Thus, setting $\mu \neq 0$ and $\lambda = 0$ in (4.15), i.e., breaking the symmetry only through the mass term, we obtain

$$\frac{a_0}{a_2} = \frac{5}{2} - \frac{3}{2g_0 f_\pi}. \tag{4.20}$$

Another breaking mechanism, which may be called algebraic, is to attribute definite transformation properties to the term Φ (4.14) under transformations associated with the chiral group. Weinberg [13] shows that if the term $\Phi(\pi^2)$ belongs to the representation $(N/2, N/2)$, the symmetry-breaking contact interaction has the form

$$\frac{N(N+2)+2}{40 f_\pi^2}\, \mu^2 (\pi^2)^2.$$

For the ratio of the scattering lengths this gives

$$\frac{a_0}{a_2} = \frac{5}{2} + \frac{30}{N(N+2)-8}. \tag{4.21}$$

In Table 2 we give the numerical values of a_0 and a_2 calculated by means of (4.19)-(4.21) for a number of specific cases of symmetry breaking, including those considered above.

The fourth row of the table (N = 1) corresponds to Weinberg's well-known result [19]: $a_0 \approx 0.2$, $a_2 \approx -0.06$, which he first obtained in an investigation of the soft pion approximation in current algebra. It follows that the case N = 1 corresponds to the PCAC formula (4.5). The other scattering lengths given in Table 2 correspond to expressions for $\partial_\mu j_5^\mu$ in which expansion terms of higher orders in the powers of the π field are added on the right side of (4.5).

Let us note a further interesting fact. In the framework of the linear realization of chiral symmetry (sigma model) the Weinberg scattering lengths can be obtained by letting the mass m_σ of the σ meson tend to infinity in Eqs. (4.7) and (4.6) [11].

* In calculating (4.18) we have used the new value $g_A/g_V = -1.23$. In [13] $f_\pi = 0.59\,\mu$ was obtained by means of the old value, viz. -1.18. Note also that the calculation from the pion lifetime yields $0.68\,\mu = 95$ MeV. As a result, the reliability of the numerical values on the right sides of (4.19) are evidently $\sim \pm 10\%$.

5. PION SCATTERING

5.1. Equations

The dispersion relations for $\pi\pi$ forward scattering can be considered conveniently for the following combinations of the structure amplitudes:

$$\left.\begin{aligned} T_0 &= B + \frac{A+C}{3}; \\ T_1 &= A - C; \\ T_2 &= A + C. \end{aligned}\right\} \qquad (5.1)$$

If s and u are transposed, they transform as follows:

$$T_I(s, u) = (-1)^I T_I(u, s). \qquad (5.2)$$

Here the index 1 corresponds to the isospin in the t channel.

To discuss the high-energy behavior of these combinations we shall use the concept of Regge poles. For the leading asymptotic term for t = 0, we obtain

$$T_0 \sim s, \quad T_1 \sim s^{\alpha_\rho(0)} \sim \sqrt{s}. \qquad (5.3)$$

At the present time, no single resonance with isospin 2 and mass less than 1.5 GeV has been detected. It is therefore natural to assume that even if there exists a corresponding Regge trajectory it lies much lower, i.e.,

$$s^2 T_2 \to 0 \quad \text{as} \quad s \to \infty. \qquad (5.4)$$

Thus, a subtraction is required for only the single dispersion relation for T_0 or, equivalently, the dispersion relation for the crossing-even structure function B. It follows that the vacuum singularities in the low-energy equations can be handled in exactly the same way as the neutral case. As we shall see below, the repulsion parameter is $v \sim 0.5$. The contribution from the high-energy region to T_1 due to the ρ-meson trajectory has the form zv_1 (5.2). The parameter v_1 can be estimated on the basis of the universality hypothesis for the ρ-meson interaction [20]:

$$v_1 = \frac{2}{8\pi^2} \int_{z_H}^{\infty} \frac{\sigma_{+-} - \sigma_{++}}{z^2} \sqrt{z^2 - 1}\, dz \sim 0.02 \quad \text{for} \quad z_H \simeq 25. \qquad (5.5)$$

In what follows we shall neglect this contribution.

For the forward scattering dispersion relation, the relationship (5.4) gives the so-called superconvergent sum rule [21]:

$$\frac{2}{\pi} \int \left\{ \frac{\operatorname{Im} A_0}{3} - \frac{\operatorname{Im} A_1}{2} + \frac{\operatorname{Im} A_2}{6} \right\} z\, dz = 0. \qquad (5.6)$$

In low-energy physics, this sum rule plays the role of a bootstrap type condition: in the resonance approximation it relates the masses and widths of the resonances. At the same time, the presence of the factor z in the integrand means that this sum rule is very sensitive to resonances with large mass and to the nonresonance background environment. In other words, it depends strongly on the cross sections in the intermediate energy range. Unfortunately, these cross sections are virtually unknown for $\pi\pi$ scattering and we cannot actually use Eq. (5.6) to solve the low-energy system of equations. Nevertheless, using the low-energy characteristics of the scattering amplitude and (5.6), one can estimate the contributions from the intermediate region.

Restricting ourselves to s and p waves and using assumptions 2-4 of Sec. 1.1, we obtain the following system of equations for the two s waves $A_0 \equiv A^{l=0}_{I=0}$ and $A_2 \equiv A^{l=0}_{I=2}$ and the p wave $A_1 \equiv A^{l=1}_{I=1}$:

$$A_i(z) = -\alpha_i v(z) + \frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} A_i}{z'-z}\,dz' + \frac{b_{ik}}{\pi}\int_1^\infty \frac{\operatorname{Im} A_k\,dz'}{z'+z}, \tag{5.7}$$

where b_{ik} is the crossing symmetry matrix (see Sec. 13.1 in [1])

$$b_{ik} = \begin{pmatrix} 1/3 & -3 & 5/3 \\ -1/9 & 1/2 & 5/18 \\ 1/3 & 3/2 & 1/6 \end{pmatrix},$$

$\alpha = (1, 1/3, 1)$, and the potential of the short-range repulsion v(z) can be represented in the form

$$v(z) = v - z^2\frac{v}{p^2} \simeq \frac{v}{1+z^2/p^2}. \tag{5.8}$$

The threshold condition for the p wave takes the form

$$\frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} A_1}{z-1}\,dz + \frac{1}{\pi}\int \frac{dz}{z+1}\left[-\frac{\operatorname{Im} A_0}{9} + \frac{\operatorname{Im} A_1}{2} + \frac{5\operatorname{Im} A_2}{18}\right] = \frac{v(1)}{3}. \tag{5.9}$$

Unfortunately, it is very difficult to obtain analytically an exact solution of the system (5.7). Like the neutral model it has many solutions. This is already evident in an investigation of the analogous system of equations without repulsion (see Ch. 3 in [1]). As is well known, the characteristic features of the solution in this case are a small width of the ρ meson and large positive scattering lengths a_0 and a_2. Now, after the introduction of the repulsion, it can be seen from the threshold condition (5.9) that the parameter v(1) increases the ρ-meson width (increases the integral contribution to the threshold condition from the ρ wave). At the same time the scattering lengths decrease.

To obtain a crude estimate of v one can use Eq. (5.8), according to which $v \approx v(1)$ for $p \gg 1$.

Identifying the expansion term in (5.8) that is proportional to $z^2 = x$ with the linear term of a formula analogous to (1.28), we obtain

$$v = \frac{p^2}{4\pi}\int_{z_H}^\infty \frac{dz\,\sqrt{z^2-1}}{z^3}\,\sigma_B(z) \simeq \frac{p^2\sigma_\infty^B}{4\pi^2 z_H};$$

now, setting

$$p \sim z_H \geqslant 25\,(W_H = 1\,\text{GeV} \text{ and } \sigma_B(\infty) \simeq \frac{0.8}{\mu^2}(\sim 16\text{ mb}),$$

we find

$$v \gtrsim 0.5.$$

This estimate shows that the high-energy effects exert an important influence on the s-wave scattering lengths and the ρ-meson width.

We shall now consider in more detail the influence of the short-range repulsion on the threshold characteristics of the s and p waves.

5.2. Threshold Analysis

We recall that the expressions for the combinations $2A_0 - 5A_2$ and A_1 [(4.16) and (4.17)] obtained in the investigation of the phenomenological Lagrangians do not depend on the method chosen to break the chiral symmetry. At the same time, we considered only the phenomenological approximation [16], i.e., the approximation that takes into account only tree graphs (Feynman graphs without internal loops). It is therefore natural to assume that such an approximation works sufficiently well in the analyticity region of the scattering amplitude in the neighborhood of the point z = 0. In other words, Eqs. (4.16) and (4.17) define expansions in z of the solutions of Eqs. (5.7) for small

$$2A_0-5A_2=\frac{3(3z+1)}{16\pi f_\pi^2}; \tag{5.10a}$$

$$A_1=\frac{z-1}{48\pi f_\pi^2}\cdot \tag{5.10b}$$

Let us now expand Eq. (5.7) in powers of z. Retaining terms of order z, we have

$$A_i(z)\approx-\alpha_i v+\frac{1}{\pi}\int\frac{dz'}{z'}\,[\mathrm{Im}\,A_i+b_{ik}\,\mathrm{Im}\,A_k]+\frac{z}{\pi}\int\frac{dz'}{z'^2}\,[\mathrm{Im}\,A_i-b_{ik}\,\mathrm{Im}\,A_k]+\ldots \tag{5.11}$$

The expansions (5.11) have a radius of convergence z = 1. However, if the Im $A_i(z')$ for $z'\sim 1$ are small, the first terms of such expansions can also be used for $z\sim 1$. Therefore, we shall use Eq. (5.11) to analyze solutions with short scattering lengths.

Comparing Eqs. (5.10) and (5.11), we obtain

$$3v+\frac{1}{\pi}\int\frac{dz}{z}\left[\mathrm{Im}\,A_0-\frac{27}{2}\,\mathrm{Im}\,A_1-\frac{5}{2}\,\mathrm{Im}\,A_2\right]=\frac{3}{16\pi f_\pi^2}\cdot \tag{5.12}$$

Now, defining v in terms of integrals of Im A_1 from the threshold condition for the p wave

$$3v=\frac{27}{2\pi}\int\frac{\mathrm{Im}\,A_1}{z}\,dz+\frac{1}{\pi}\int\frac{dz}{z}\left(-\mathrm{Im}\,A_0+\frac{5}{2}\,\mathrm{Im}\,A_2\right)+\frac{1}{\pi}\int\frac{dz}{z^2}\left[\mathrm{Im}\,A_0+\frac{9}{2}\,\mathrm{Im}\,A_1-\frac{5}{2}\,\mathrm{Im}\,A_2\right]$$

in the approximation (5.11), we obtain the following expression from (5.12):

$$\frac{1}{\pi}\int\frac{dz}{z^2}\left\{\mathrm{Im}\,A_0+\frac{9}{2}\,\mathrm{Im}\,A_1-\frac{5}{2}\,\mathrm{Im}\,A_2\right\}=\frac{3}{16\pi f_\pi^2}\,. \tag{5.13}$$

One can show that (5.13) ensures that both the equations (5.10) are satisfied at the symmetry point z = 0 and also on the threshold z = 1 to within terms $\sim\int dz z^{-3}$. The accuracy with which (5.10) is satisfied can be increased by requiring

$$\frac{1}{\pi}\int\frac{dz}{z^3}\left\{\mathrm{Im}\,A_0-\frac{27}{2}\,\mathrm{Im}\,A_1-\frac{5}{2}\,\mathrm{Im}\,A_2\right\}=0. \tag{5.14}$$

The conditions (5.13) and (5.14) ensure that (5.10) is satisfied to within terms $\sim\int dz z^{-4}$.

The scattering lengths have the form

$$a_0=\frac{5}{2\pi}\int\frac{dz}{z}\left\{\mathrm{Im}\,A_0-\frac{9}{2}\,\mathrm{Im}\,A_1+\frac{1}{2}\,\mathrm{Im}\,A_2\right\}+\frac{1}{16\pi f_\pi^2}; \tag{5.15}$$

$$a_2=\frac{2}{3\pi}\int\frac{dz}{z}\left\{\mathrm{Im}\,A_0-\frac{9}{2}\,\mathrm{Im}\,A_1+\frac{1}{2}\,\mathrm{Im}\,A_2\right\}-\frac{1}{8\pi f_\pi^2}\,. \tag{5.16}$$

We expect that the system of equations has a solution with a resonance in the p wave whose characteristics are very similar to the experimental data. In addition, we shall assume that the partial wave A_2 is small over a wide energy range (right up to 1 GeV) and we shall therefore neglect its imaginary part. To make a quantitative estimate of (5.15) and (5.16), we shall use the fact that at the present time the position and width of ρ can be assumed to be fairly well established:

$$m_\rho=765\ \mathrm{MeV},\quad \Gamma_\rho^{\mathrm{tot}}=120\ \mathrm{MeV}. \tag{5.17}$$

TABLE 3. Dependence of the Parameter of Strong Repulsion v and the s-Wave Contribution to the Threshold Conditions for the p Wave and the Ratio of the $\pi\pi$ Scattering Lengths on the Way in Which the Chiral Symmetry Is Broken

Method of symmetry breaking	Ratio of scattering lengths	$\frac{1}{\pi}\int_1^\infty \frac{dz}{z}\,\mathrm{Im}\,A_0$	v
$g_0 f_0 = 0{,}5$	$-1/2$	0,747	0,56
1 (PCAC)	$-7/2$	0,796	0,54
2	$\pm\infty$	0,906	0,51

It follows that the ρ meson is fairly narrow and that the imaginary part of the corresponding partial wave can be approximated by a δ function:

$$\mathrm{Im}\,A_1(z) = \pi\gamma_1 z_1\,\delta(z - z_1). \tag{5.18}$$

Here,

$$z_1 = \frac{m_\rho^2 - 2\mu^2}{2\mu^2} = 14.0; \quad \gamma_1 = \frac{m_\rho \Gamma_\rho^{\mathrm{tot}}}{m_\rho^2 - 2\mu^2} \simeq 0.17.$$

The integral from the p wave in (5.18) makes a contribution to (5.15) that is numerically equal to 1.28 of the pion Compton wavelength. At the same time $3/16\pi f_\pi^2 = 0.158$ [see (4.18)]. Therefore, short scattering lengths corresponding to weak chiral symmetry breaking [small N in Eq. (4.21)] are possible only when the integral of Im A_0 in (5.15) and (5.16) is compensated by the integral of Im A_1, i.e., one must have approximately

$$\frac{1}{\pi}\int_1^\infty \mathrm{Im}\,A_0 \frac{dz}{z} \simeq \frac{9}{2\pi}\int \mathrm{Im}\,A_1 \frac{dz}{z} = 0.756.$$

Table 3 can serve as an illustration of this result.

Note that if such compensation occurs the parameter of the high-energy repulsion is strongly nonzero. In this case we can express the integral in (5.9), which contains Im A_0, in terms of a_2, obtaining

$$v = 3\gamma_1 - \frac{a_2}{2}.$$

However, if v = 0, such compensation is impossible, and we arrive at long scattering lengths:

$$a_0 \simeq \frac{15}{\pi}\int \mathrm{Im}\,A_1 \frac{dz}{z} \simeq 15\gamma_1 \simeq \frac{5}{2}\,a_2 = 2.5.$$

Such a situation corresponds to large N (> 10) in Eq. (4.21).

Expressing the integral of Im A_0 in terms of the repulsion parameter, we obtain simple formulas for the scattering lengths:

$$a_0 = -5v + 2.84; \quad a_2 = -2v + 1.02. \tag{5.19}$$

It follows from (5.19) that v ~ 0.5 when a_2 is small, which corresponds to the estimate from the high-energy integrals (see Sec. 5.1). This is an independent indication that the system of equations (5.7) has a solution with the experimental p wave and short scattering lengths.

Thus, the introduction of a short-range repulsion in the low-energy dispersion theory for the low partial waves enables one to reconcile the theory with the approximation based on a small pion mass. It turns out that the strength of the repulsion v is a very convenient parameter for making a quantitative description of $\pi\pi$ scattering in the low-energy region. It enables one to relate the method of symmetry breaking to the high-energy scattering characteristics due to the vacuum contributions.

5.3. Solvable Model $A_2 = 0$

In our above threshold analysis we have shown that short s-wave scattering lengths arise only if there is a definite compensation of the large integral contributions from Im A_1 and Im A_0 (5.17). One can exploit the idea of this compensation and, using Eqs. (5.7), construct a solvable model for the partial waves A_0 and A_1, whose solution in the region up to 1 GeV gives a reasonable description of the energy dependence of the δ_0^0 and δ_1^1 phases and also corresponds to (5.10).

We shall base such a model on the assumption that in the considered physical region both Im A_2 and Re A_2 may be assumed to vanish. Then, using the third of Eqs. (5.7) to express the repulsion potential v(z) in terms of the crossing integrals of Im A_0 and Im A_1, and substituting the resulting expressions into the first two equations of (5.7), we obtain a solvable model for the waves A_0 and A_1:

$$A_0(z) = \frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} A_0(z')}{z'-z}\,dz' - \frac{9}{2\pi}\int_1^\infty \frac{\operatorname{Im} A_1(z')}{z'+z}\,dz';$$

$$A_1(z) = \frac{1}{\pi}\int_1^\infty \frac{\operatorname{Im} A_1(z')}{z'-z}\,dz' - \frac{2}{9\pi}\int_1^\infty \frac{\operatorname{Im} A_0(z')}{z'+z}\,dz'. \tag{5.20}$$

This model has the simple crossing symmetry

$$A_0(-z) = -\frac{9}{2}A_1(z). \tag{5.21}$$

It follows from (5.21) that the wave A_0 has a zero on the crossing threshold for $z = -1$. The solution obtained by the transition to the inverse function has the form

$$H(z) = A_0^{-1}(z) = -\frac{2}{9}A_1^{-1}(-z) = \frac{1}{\lambda} - \frac{R_0 z}{z+1} - I(z) + \frac{2}{9}I(-z) - \sum_i \frac{z\alpha_i}{z_i(z_i - z)} - cz. \tag{5.22}$$

Here R_0, α_i, $c > 0$, $|z_i| > 1$, and also

$$I(z) = \frac{z}{\pi}\int \sqrt{\frac{z'-1}{z'+1}}\cdot\frac{dz'}{z'(z'-z)}.$$

The absence of bound states, i.e., the absence of poles in the interval $-1 < z < 1$, leads to an upper bound on the possible values of λ:

$$\lambda < \lambda_{\max} = \left|\frac{R_0}{2} + I(1) - \frac{2}{9}I(-1) + \sum \frac{\alpha_i}{z_i(z_i - 1)} + c\right|^{-1}. \tag{5.23}$$

The scattering length a_0 of the wave A_0 can take only positive values. This follows from the solution. It also follows directly from Eq. (5.20); for, in accordance with this equation, $dA_0/dz > 0$ for $-1 < z < 1$, and also $A_0(-1) = 0$.

Using similar arguments one can show that the p-wave scattering length is also positive [in the solution (5.22) this corresponds to $R = (4/9)\,a_1 > 0$].

Let us consider the simplest solution without Castillejo-Dalitz-Dyson terms, i.e., for $c = 0$ and $\alpha_i = 0$). For large z we have the asymptotic behavior

$$(A_0)^{-1} \simeq \frac{7}{9\pi}\ln z, \quad (A_1)^{-1} \simeq -\frac{7}{2\pi}\ln z. \tag{5.24}$$

Since a_1 is positive and A_1 has no zeros in the physical region it now follows that the phase δ_1^1 must definitely pass through a resonance. The position and width of this resonance are determined by the two parameters λ and R_0. If we now determine these parameters by means of the expansion (5.10b) for the p wave:

$$R_0 = \frac{1}{\lambda} = \frac{32\pi}{3} f_\pi^2, \tag{5.25}$$

the mass and width of the ρ meson are determined by the f_π and the pion mass μ by equations which, for $m_\rho^2 \gg \mu^2$, have the form

$$\frac{192}{7} f_\pi^2 = m_\rho^2 \left(\frac{1}{\pi} \ln \frac{m_\rho^2}{\mu^2} - \frac{1}{2} \right); \tag{5.26}$$

$$\frac{\Gamma_\rho}{m_\rho} = \left(\frac{7}{2\pi} + \frac{96\pi f_\pi^2}{m_\rho^2} \right)^{-1}. \tag{5.27}$$

In accordance with (4.18), we set $f_\pi = 0.615\,\mu$ ($R_0 = 12.7$), obtaining $m_\rho \sim 930$ MeV and $\Gamma_\rho^{tot} \sim 230$ MeV. It is interesting to note, going over from Γ_ρ to g_ρ by means of (1.3), that (5.27) is replaced by

$$\frac{2 f_\pi^2 g_\rho^2}{m_\rho^2} = \left(1 + \frac{7 m_\rho^2}{192\pi^2 f_\pi^2} \right)^{-1}. \tag{5.27a}$$

This formula can be regarded as a generalization of the well-known Kawarabayashi-Suzuki-Fayyazuddin-Riazzudin relation [22, 23] which takes into account unitary corrections.

5.4. Scattering Phases in the Region up to 1 GeV

Curve 1 in Fig. 4 gives the energy dependence of the p-wave phase for the case (5.25).

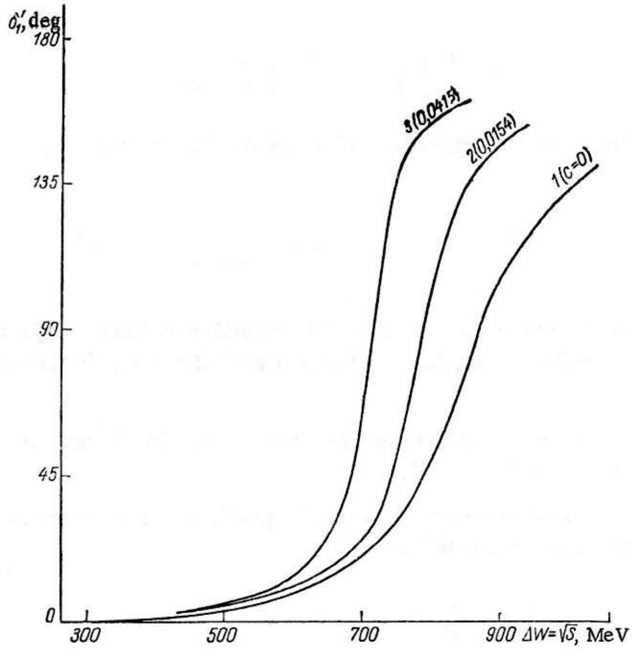

Fig. 4. Energy dependence of the δ_1^1 phase for different c.

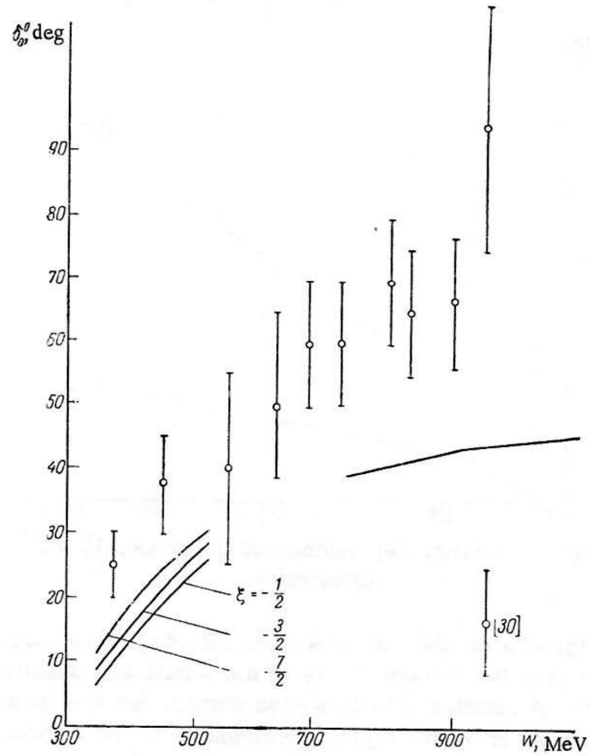

Fig. 5. Energy dependence of the phase δ_0^0 for different ξ.

To calculate the energy dependence of the s phase, we note that the first of Eqs. (5.23) is sufficiently good only in regions far from the threshold. In the region around the threshold it must be corrected by using the expressions (5.10a) and introducing the parameter $\xi = a_0/a_2$, which characterizes the manner in which the symmetry is broken. Then one can readily find that A_0 in the neighborhood of $z = 0$ can be represented in the form

$$A_0 \simeq \frac{z+z_0}{8\pi f_\pi^2},$$

where

$$z_0 = \frac{2\xi+5/2}{\xi-5/2}. \tag{5.28}$$

If there is only a small amount of symmetry breaking (see Table 2), the position of the zero z_0 is a long way from $z = 0$ and it is therefore to be expected that the corrections that take into account unitarity shift the zero by a small amount. Consequently, A_0 in the neighborhood of the threshold in the physical region can be approximated by the expression (cf. [24])

$$A_0 = \frac{1}{\dfrac{3R_0}{4(z+z_0)} - I(z)}. \tag{5.29}$$

For $z \gtrsim 10$ one can neglect A_2 and it is therefore more correct to use Eq. (5.22). The dependence of the s phase δ_0^0 on the energy for different values of ξ is shown in Fig. 5 ($R_0 = 12.7$).

It can be seen from Fig. 4 (curve 1) and Fig. 5 that the values of δ_0^0 lie below the experimental data and that, at the same time, ρ-meson parameters (m_ρ, Γ_ρ) are overestimated. This disagreement with the experimental data was to be expected, since our treatment above is merely a first approximation intended to give only a qualitative picture of the $\pi\pi$ interaction.

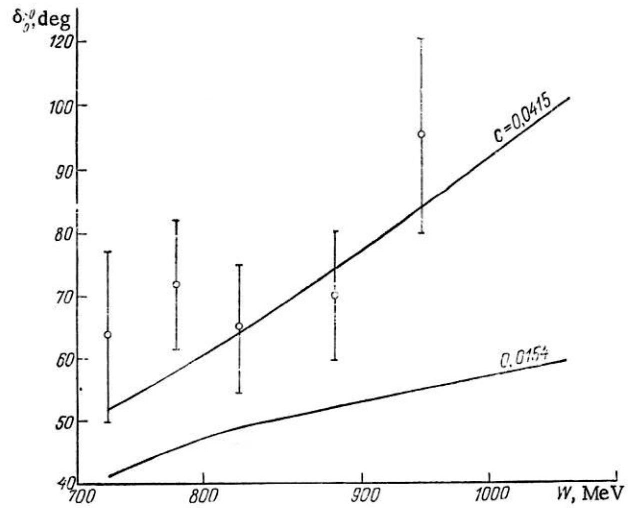

Fig. 6. Energy dependence of the phase δ_0^0 for different c.

We have, for example, ignored a number of important effects: first, we have restricted ourselves to s and p waves, but it is known that the d wave (I = 0) is not small and passes through a resonance in the region of 1.2 GeV (the f_0 meson). A similar situation also obtains for the isotopic amplitude A_1. The result is that the integral contributions from the higher resonances in the region 0.5–1 GeV cannot be neglected. Secondly, we have already estimated the contribution from the ρ-meson trajectory to the real part of the scattering amplitude at the threshold (5.5). At this point it was found to be small. However, in the region of the ρ meson this contribution is of the order of a few tenths of the meson Compton wavelength and it must also be taken into account. It turns out that the s phase δ_0^0 is then increased by 5–10°. Finally, we must not forget that the amplitude A_2 is also nonvanishing.

The most important effect is that due to the contributions from the intermediate region (1–2 GeV). To take these into account we must formulate the problem for the s, p, d, and f partial waves and consider the solution of the resulting equations. However, at this juncture we shall not study this system. We shall only mention that one can also take into account the influence of the higher waves on the s and p solutions by assuming that the parameter c in the expression (5.22) is nonvanishing.

This can be readily seen by considering the example of the scattering of neutral pions (2.2). Representing the unitarity condition in the form

$$\operatorname{Im} A = K\,|\,A(z)\,|^2 + 2\varphi(z)\operatorname{Re} A(z) + \eta(z),$$

where the functions φ and η are determined by the d wave, we solve Eq. (2.2). We can then show that the solution of this equation without Castillejo-Dalitz-Dyson terms corresponds to the solution (2.3)-(2.5) with purely elastic unitarity, but with a Castillejo-Dalitz-Dyson term with c ≠ 0 in the low-energy region if the functions φ and η are essentially nonvanishing at energies greater than 1 GeV.

The solutions (2.22) for two values of c ≠ 0 are shown in Fig. 4 (curves 2 and 3) and Fig. 6. It can be seen that at small c the mass of the resonance in the p wave and its width approach the experimental values and the s phase increases. A resonance in the region of 1 GeV becomes possible. This resonance has a fairly large width. To obtain a crude estimate of this width in the δ approximation one can use the threshold condition for the second of Eqs. (5.20).

This gives $\gamma_\sigma = 9/2\gamma_\rho$ or

$$\Gamma_\sigma = \frac{9m_\sigma}{2m_\rho}\,\Gamma_\rho \simeq 700 \text{ MeV} \quad (\text{for } m_\sigma \sim 1 \text{ GeV}).$$

Of course, such a broad resonance can only be conventionally called a quasiparticle (σ meson). It would be very difficult to distinguish in the bipion mass spectrum (for example, in the region $\pi + N \to \pi' + \pi'' + N$) from the effects of the interaction of the particles in the final state when this interaction is fairly strong but is not necessarily a resonance interaction.

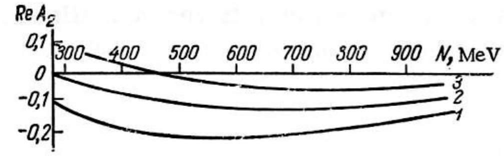

Fig. 7. Energy dependence of the amplitude A_2 for different a_2.

Finally, a few words about the wave A_2. Using the definition (5.8) and the numerical values of A_0 and A_1 obtained from the third equation of (5.7), one can calculate the small wave A_2. Here, the main indeterminacy is associated with the strength of the repulsion v. It can be eliminated by specifying the scattering length a_2 [see the second equation of (5.19)]. The behavior of the phase of this wave for three different values of a_2, namely, +0.10, 0, and −0.10, is shown in Fig. 7 (the details are discussed in [25]).

This figure shows that the phase δ_2^0 is small and negative over a wide energy range. It is therefore clear that the original approximation Im $A_2 = 0$ is consistent with this behavior of the phase.

6. PION-NUCLEON SCATTERING

6.1. Massless Pion Approximation in πN Scattering

The approximation in which symmetry under the chiral group SU(2) × SU(2) is satisfied gives results that are close to the experimental values of the threshold characteristics. Therefore, as in the case of $\pi\pi$ scattering, the consequences of the massless pion approximation can be regarded as boundary conditions for the solutions of the system of equations for the πN scattering partial waves.

As in Sec. 4, we obtain consequences of chiral symmetry for πN scattering from the Born terms of the corresponding effective Lagrangian. As we have already mentioned, this Lagrangian can be constructed in two ways.

The first method is based on the assumption that there exists a certain set of elementary particles that realize linear representations of SU(2) × SU(2). Let us suppose that the nucleon field belongs to the representation (1/2, 1/2). The variations of this field under the action of the generators Q_5^A and Q^A are given by the expressions

$$\delta\Psi = i\,\frac{\boldsymbol{\tau}\boldsymbol{\alpha}}{2}\,\Psi; \quad \bar{\delta}\Psi = i\,\frac{\boldsymbol{\tau}\boldsymbol{\beta}}{2}\,\gamma_5\,\Psi. \tag{6.1}$$

Of course, the mass term in the Lagrangian is not invariant under such a transformation. However, one can ensure that the total Lagrangian is invariant by introducing an interaction with the π and σ fields, which also belong to the representation (1/2, 1/2) and transform in accordance with Eqs. (4.1).

The simplest invariant Lagrangian (the Gell-Mann–Levy sigma model [7, 12]) has the form

$$L = \overline{\Psi}\,(i\gamma_\mu\partial_\mu - M)\,\Psi + \frac{1}{2}\,\{(\partial\sigma)^2 + (\partial\boldsymbol{\pi})^2\} - \lambda\left\{\sigma^2 - \frac{2M}{g}\,\sigma + \boldsymbol{\pi}^2\right\} + g\overline{\Psi}\,[\sigma + \gamma_5\boldsymbol{\pi}\boldsymbol{\tau}]\,\Psi. \tag{6.2}$$

Note that the σ field in (6.2) differs from the $\bar{\sigma}$ field in (4.3) by the additive constant $\sigma = \tilde{\sigma} + M/g$ and the coupling constant g is expressed in terms of the previously introduced (Sec. 1) constant f by the well-known relation

$$f^2 = \frac{g^2}{4\pi}\left(\frac{\mu}{2M}\right)^2.$$

The Born terms of the Lagrangian (6.2) describe only the amplitude T^+. The corresponding scattering length defined in terms of the nucleon and σ-meson exchange graphs have the form

$$a^+ = -\frac{g^2}{4\pi M} + \frac{2\lambda M}{\pi m_\sigma^2}.$$

Now it follows from (6.2) that $m_\sigma^2 = 8M^2\lambda/g^2$; therefore

$$a^+ = 0. \tag{6.3}$$

To obtain the amplitude T^- we must augment (6.2) with terms of the interaction with vector particles. The most suitable candidates are the ρ and A_2 mesons. The corresponding fields, which realize the representation (1, 0) + (0.1), transform as follows:

$$\delta\rho_A^\mu = -\varepsilon_{ABC}\alpha_B\rho_C^\mu; \quad \bar{\delta}\rho_A^\mu = -\varepsilon_{ABC}\beta_B A_{2C}^\mu;$$
$$\delta A_{2A}^\mu = -\varepsilon_{ABC}\alpha_B A_{2C}^\mu; \quad \bar{\delta} A_{2A}^\mu = -\varepsilon_{ABC}\beta_B\rho_C^\mu.$$

Since G parity is conserved, only the ρ-meson exchange makes a contribution to the Born term of the amplitude T^-:

$$\varphi^- = \frac{2g_{1V}g_\rho}{m_\rho^2 - t}.$$

It follows from the universality of the interaction of vector mesons that $2g_{1V} = g_\rho$. Therefore (in units of μ^{-1}),

$$a^- = \frac{g_\rho^2}{4m_\rho^2\pi}. \tag{6.4a}$$

Taking the experimental value of the ρ-meson width (5.17) as our starting point and using (1.3), we can obtain $g_\rho^2/4\pi = 2.5$, from which it follows that $a^- = 0.084$.

The other method for constructing the effective Lagrangian is based on the assumption that the pion field is a nonlinear representation of the chiral group [13]. To obtain an invariant Lagrangian one does not need to introduce other fields. One need only specify the commutators

$$[Q^A, \Psi] = -\frac{\tau^A}{2}\Psi; \quad [Q_5^A, \Psi] = V_{AB}(\pi)\frac{\tau^B}{2}\Psi, \tag{6.5}$$

which determine the variation of the nucleon field Ψ.

The find the function V_{AB} one must use the Jacobi identities and the group relations (3.5c). This gives the same expression for V_{AB} as Eq. (4.12), from which the transformation law for the covarient derivative $D_\mu\pi$ is found. It follows that the nucleon mass term in the Lagrangian is invariant under transformations of the chiral group.

The invariant interaction Lagrangian has the form

$$L_{int} = \frac{ig}{2M}\bar{\Psi}\gamma_\mu\gamma_5\tau\Psi D_\mu\pi, \tag{6.6}$$

where $D_\mu\pi$ is defined in (4.13).

To construct the total Lagrangian one must also determine the covariant derivative $D_\mu\Psi$, which transforms in the same way as the field Ψ (6.5).

Standard manipulation with the Jacobi identities yields

$$D_\mu\Psi = \partial_\mu\Psi + \frac{iV(\pi^2)}{\sqrt{f^2(\pi^2)+\pi^2}}\frac{\tau}{2}(\pi\,\partial_\mu\pi)\Psi. \tag{6.7}$$

The last term yields the πN interaction that augments (6.6). Thus, the total chiral-symmetric Lagrangian of the πN system is

$$L = \bar{\Psi}(i\partial_\mu\gamma_\mu - M)\Psi + \frac{v(\pi^2)}{\sqrt{f^2(\pi^2)+\pi^2}}\bar{\Psi}\gamma_\mu\frac{\tau}{2}\Psi(\pi\,\partial_\mu\pi) + \frac{ig}{2M}\bar{\Psi}\gamma_\mu\gamma_5\tau\Psi D_\mu\pi + \frac{1}{2}(D_\mu\pi)^2. \tag{6.8}$$

Using (6.8) to determine the expressions for the axial vector current and considering the coefficient in front of $\overline{\Psi}\gamma_\mu\gamma_5\Psi$, we can obtain the Goldberger-Treiman relation:

$$\left(\frac{g_A}{g_V}\right) = -\frac{f_\pi}{M} g. \tag{4.18}$$

Calculating the Born terms, we obtain

$$A^+ = \frac{g^2}{M}, \quad A^- = 0$$

and

$$B^+ = g^2\left\{\frac{1}{M^2-s} - \frac{1}{M^2-u}\right\};$$
$$B^- = -\frac{g^2}{2M^2} + \frac{1}{2f_\pi^2} + g^2\left\{\frac{1}{M^2-s} + \frac{1}{M^2-u}\right\}. \tag{6.9}$$

It follows that

$$a^+ = 0 \tag{6.3}$$

and

$$a^- = \frac{1}{8\pi f_\pi^2}. \tag{6.4b}$$

For $f_\pi = 0.615$ [see (4.18)] we obtain $a^- = 0.10$. Comparing (6.4a) and (6.4b) and using (4.18), we obtain a relationship between the characteristics of the ρ meson and $g_{\pi NN}$ [22, 23]:

$$a_{\rho 1} = \frac{g_{1V}\, g_\rho}{8\pi} = \left(\frac{g_V}{g_A}\right)^2 \frac{m_\rho^2}{8M^2}\cdot\frac{g^2}{4\pi}. \tag{6.10}$$

Below we shall use Eqs. (6.3), (6.4), and (6.10) as additional conditions imposed on the system of the s- and p-wave equations with repulsion.

6.2. Introduction of Repulsion

As we have already indicated in Secs. 1 and 4, one must introduce a short-range repulsion $V^+(\nu)$ in the low-energy equation for the function $\varphi^+(\nu, 1)$. By analogy with (2.1), we shall approximate this potential by a pole expression of form-factor type:

$$V^+(\nu) \simeq v^+(\nu) = -\frac{v}{1+\frac{\nu}{\rho^2}}. \tag{6.11}$$

The parameter v is related to the subtraction constant and is the strength of the repulsion; the quantity p^2 characterizes the range of the repulsion. If we assume that the nominal boundary of the low-energy region occurs at $E_{lab} \approx 500$ MeV, then $p^2 \gtrsim 50$.

Equation (1.7a) now takes the form

$$\varphi^+(\nu, 1) = \frac{1}{\pi}\int_0^\infty \frac{\operatorname{Im}\varphi^+(\nu', 1)}{\nu'-\nu}\, d\nu' - \frac{v}{1+\frac{\nu}{p^2}}. \tag{6.12}$$

Using the term of the short-range repulsion, which is proportional to v, we can now obtain small (including vanishing) values of the scattering length a^+. It follows that the introduction of the repulsive term is needed to achieve agreement between the dispersion approach and the consequences of chiral symmetry.

We must also modify the third of Eqs. (1.7). The point is that at high-energies the spin-flip scattering amplitude f_2 is less than the amplitude without spin flip, i.e., $f_2 < f_1$. It follows that Im $\nu_L B^+/4\pi$ tends

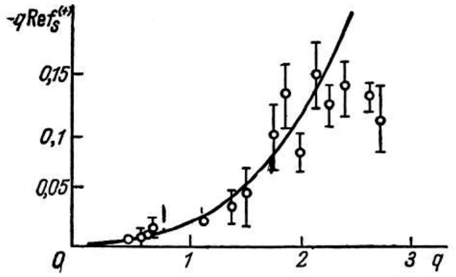

Fig. 8. Energy dependence of $-q\,\mathrm{Re}\,f_S^{(+)}$ on q for $m_\sigma^2 = 50$.

1. Suppose a^- is determined entirely by the effective reductions of the ρ-meson mass. In Eqs. (6.15) and (6.16) we make the substitution $m_\rho^2 \to \tilde{m}^2 < m_\rho^2$. Then, using the relationship between $\alpha_{\rho 1}$ and f^2 from Eq. (6.10), in which m_ρ^2 is the physical ρ-meson mass, we find that (6.18) is replaced by

$$\Gamma_{33} = \frac{3}{2} f^2 \left[1 - \frac{1}{2}\left(\frac{g_V}{g_A} \cdot \frac{m_\rho}{\tilde{m}}\right)^2\right];$$

$$\omega_{33} = \frac{m_\rho^2}{8M}\left(\frac{g_V}{g_A}\right)^2 \frac{\varkappa}{1 - \frac{1}{2}\left(\frac{g_V}{g_A} \cdot \frac{m_\rho}{\tilde{m}}\right)^2}.$$

Instead of the second equation (6.21), we have

$$a^- = \left(\frac{g_V}{g_A}\right)^2 \left(\frac{m_\rho}{\tilde{m}}\right)^2 f^2.$$

For satisfactory values of a^- and Γ_{33} these equations yield an excessively large mass of the 33 resonance. For example, setting $\left(\frac{m_\rho}{\tilde{m}}\right)^2 = 1.5$, and taking the value $\left(\frac{g_V}{g_A}\right) = 1.23$, we obtain

$$a^- = f^2 = 0{,}08, \quad \Gamma_{33} = \frac{3}{4} f^2 \simeq 0{,}06, \quad \omega_{33} = \frac{m_\rho^2 \varkappa}{6M} \simeq 3{,}6. \tag{6.22}$$

To calculate a^+ and v we must know the value of S. Here, it is necessary to remember that the σ meson is a rather complicated formation and, even if it corresponds to a resonance, the latter is broad (see Sec. 5.4); it follows that the Lagrangian description in the form (1.2) is very conditional and the corresponding coupling constants and the "mass" m_σ are certain averaged quantities. Of course, the averaged value of the mass m_σ and the coupling constant g_σ can be extracted from the $\pi\pi$ scattering characteristics. However, to determine the constant $g_{\sigma NN}$ we have no simple theoretical arguments at our disposal.

We shall therefore determine the constant S by means of the first equation (6.21) from the condition $a^+ = 0$ (6.3). With allowance for (6.19), this gives S = 0.125 and v = 0.50.

From Eqs. (6.15) we now obtain expressions that describe the energy dependence of the s waves; we shall assume that the phases are small, i.e., the integrals of the imaginary parts can be neglected:

$$s^+ = -\frac{1}{m_\sigma^2} \cdot \frac{q^2}{1 + 4q^2/m_\sigma^2}; \quad s^- = \frac{a^-}{1 + 4q^2/m_\rho^2}. \tag{6.23}$$

The behavior of the s^+ wave depends on the parameter m_σ^2. Setting $m_\sigma^2 = 50$ (which corresponds to an energy of 1 GeV) and $p^2 = 50$, we obtain the curve shown in Fig. 8. The corresponding curve for the case $m_\sigma^2 = 30$ passes above this at a height that is increased by a factor of ~ 1.5, i.e., along the upper edge of the experimental errors. Thus, for $m_\sigma^2 \approx 30$–50 the first equation (6.23) gives a good description of the experimental data right up to q ~ 2.5.

It should be noted that these rather high values of the averaged mass correspond approximately to the experimental curve of the phase δ_0^0 of the $\pi\pi$ scattering shown in Fig. 6.

Equation (6.23) for s^- leads to a curve that lies below the experimental points (Fig. 9). This means that the effective ρ-meson mass is seriously underestimated.

2. We shall assume that the correction from the annihilation channel can be described sufficiently well by a ρ-meson pole term in which the ρ meson has the physical mass. Then the contributions from the integrals over the intermediate and the high-energy regions are important in a calculation of a^-.

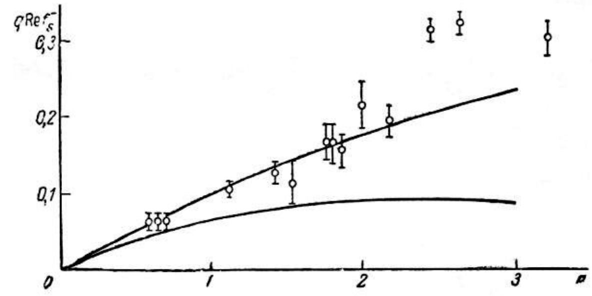

Fig. 9. Energy dependence of $-q\,\mathrm{Re}\,f_S^{(-)}$ on the energy q. The lower curve corresponds to $(m_\rho/m)^2 = 1.5$.

Equations (6.15b) and (6.16b) must be replaced by

$$s^- = \frac{\omega\alpha_{\rho 1}}{2(1+\nu_\rho+\nu)} + \frac{\omega c}{2\left(1+\frac{\nu}{r^2}\right)} + \frac{\omega}{\pi}\int \frac{\mathrm{Im}\, s^-}{\omega'(\nu'-\nu)}\, d\nu';$$

$$p^- = \frac{2f^2}{\omega} - \frac{\omega\alpha_{\rho 1}}{2(1+\nu_\rho+\nu)} + \frac{\omega c}{2(1+\nu/r^2)} + \frac{\omega}{\pi}\int \frac{\mathrm{Im}\, p^-}{\omega'(\nu'-\nu)}\, d\nu'. \quad (6.24)$$

Here, c and r^2 are phenomenological parameters that describe the strength and the range of the forces due to the neglected contributions. One can readily show that this system of equations, too, has a solution with only a large 33 wave. Using the relations obtained in the massless pion approximation (6.4a) and (6.10), we obtain

$$c = R = \frac{1}{2}\left(\frac{g_V}{g_A}\right)^2 f^2.$$

Instead of (4.18) and (4.19), we have

$$\left.\begin{aligned} \Gamma_{33} &= \frac{3}{2} f^2 = 0.12; \\ \omega_{33} &= \frac{m_\rho^2}{8M}\left(\frac{g_V}{g_A}\right)^2 \varkappa = 1.8; \\ a^- &= 2\left(\frac{g_V}{g_A}\right)^2 f^2 = \frac{4}{3} f^2 = 0.10. \end{aligned}\right\} \quad (6.25)$$

In this case (taking satisfactory values of the scattering length a^- and the position of the Δ_{33} resonance) we obtain a slightly large width of the Δ_{33} resonance. The parameter of the high-energy repulsion can be found, as in (6.19), by the normalization $a^+ = 0$ (6.3):

$$v = \frac{f^2 m_\rho^2}{2M}\left(\frac{g_V}{g_A}\right)^2 \varkappa = 0.57, \quad S = 0.14. \quad (6.26)$$

The behavior of the s waves is given by the expressions:

$$s^+ = -\frac{q^2}{m_\sigma^2\left(1+4q^2/m_\sigma^2\right)}; \quad s^- = \frac{4}{3} f^2 \frac{m_\rho^2+2q^2}{m_\rho^2+4q^2}. \quad (6.27)$$

The curves corresponding to the second of Eqs. (6.27) are shown in Fig. 9. In the region q < 1.5, it passes somewhat higher than the experimental points.

Summing up, we conclude that there is indeed an important effective reduction of the ρ-meson mass, and that one must take into account the effects from the regions of higher energy.

6.4. System of Equations for the p Waves

We now come to the p waves. Combining Eqs. (6.16) and (6.24), we obtain a system of equations for the p waves:

$$h_i(\omega) = \frac{3f^2}{\omega^2}(\Lambda_1)_i + \Phi_i(\omega) + \frac{1}{\pi}\int_1^\infty \left\{\frac{\mathrm{Im}\, h_i(\omega')}{\omega'-\omega} + A_{ik}\frac{\mathrm{Im}\, A_k(\omega')}{\omega'+\omega}\right\} d\omega'. \quad (6.28)$$

Here, Φ_i are the p-wave potentials defined by the expressions:

$$3\Phi_i(\omega)=\frac{2v}{M_p^2+4q^2}+\frac{8S}{m_\sigma^2+4q^2}+\frac{Rm_\rho^2\varkappa}{M\left(m_\rho^2+4q^2\right)}(\Lambda_2)_i+\frac{8\omega R(\Lambda_3)_i}{m_\rho^2+4q^2}+\frac{2\omega c}{r^2+q^2}(\Lambda_3)_i-\frac{2\omega v_B}{M_p^2+4q^2}(\Lambda_1+\Lambda_3)_i; \tag{6.29}$$

$$M_p^2=4p^2,\quad h_i=e^{i\delta_i}\sin\delta_i/q^3,$$
$$i=\{(1.1),\ (1.3),\ (3.1),\ (3.3)\},$$

and also

$$A_{ik}=\frac{1}{9}\begin{pmatrix}1 & -4 & -4 & 16\\ -2 & -1 & 8 & 4\\ -2 & 8 & -1 & 4\\ 4 & 2 & 2 & 1\end{pmatrix};\quad \Lambda_1=\begin{pmatrix}-4\\ -1\\ -1\\ 2\end{pmatrix};$$
$$\Lambda_2=\begin{pmatrix}4\\ -2\\ -2\\ 1\end{pmatrix};\quad \Lambda_3=\begin{pmatrix}2\\ 2\\ -1\\ -1\end{pmatrix}. \tag{6.30}$$

The system (6.28) differs from the Chew-Goldberger-Low-Nambu system by the presence of the potentials Φ_i. These potentials, like the corresponding expressions in Eqs. (6.15) for the s waves, contain short-range repulsion terms as well as the σ- and ρ-meson exchange terms. It should be noted that the low-energy properties of the equations are virtually independent of the large parameters M_p^2 and r^2. In the equations for the s waves only the constant terms v and c "work"; these normalize the combinations s^+ and s^- to the correct scattering length $a^+=0$ and $a^-\approx f^2$. In Eqs. (6.28) for the p waves, the corresponding terms in the region of low energies proportional to p^{-2} and r^{-2} are negligibly small. In the high-energy region they make a contribution to the asymptotic behavior of h_{ik} proportional to v and c. Accordingly, these terms play an important role in ensuring the quantum-mechanical threshold conditions for the p waves p_{ik}. As we have already seen, it is only after the introduction of these terms that the system of equations (6.28) has a solution with a sensible 33 wave.

At low energies the potentials Φ_i can be approximated by the expression

$$3\Phi_i(\omega)=f^2\left(\frac{g_V}{g_A}\right)^2\left[\frac{m_\rho^2\varkappa}{M\left(m_\sigma^2+4q^2\right)}+\frac{(\Lambda_2)_i}{2M}\cdot\frac{m_\rho^2\varkappa}{\left(m_\rho^2+4q^2\right)}+\frac{4\omega(\Lambda_3)_i}{m_\rho^2+4q^2}\right]. \tag{6.31}$$

In this approximation a calculation of the p-wave scattering lengths yields

$$a_{33}=0.271;\ a_{13}=-0.033;$$
$$a_{11}=-0{,}099;\ a_{31}=0{,}039.$$

Thus, the use of the relations that follow from the massless pion approximation give a satisfactory qualitative description of N scattering even in the static limit. To achieve this, we had to introduce a short-range repulsion whose strength could be estimated from either the high-energy integrals or their low-energy theorems, the same result being obtained in each case. As in the case of $\pi\pi$ scattering, we have seen that the solution of the equations for the low-energy πN scattering gives a strong p-wave interaction that is even resonant in the 33 wave in a reasonable region. Ultimately, using the massless pion approximation and the unitarity equations for the low-energy $\pi\pi$-and πN-scattering processes, we have obtained the masses of the p-wave resonance, their lifetimes, and the main coupling constants; to do this we needed to specify only the masses of the pion and nucleon and their lifetimes and the Fermi constant.

Of course, our theory can only be regarded as a first approximation. The accuracy with which the main characteristics are obtained is only ~25%. To increase the accuracy we must allow more accurately for the relativistic corrections. In the system of low-energy equations we must introduce equations for the higher partial waves, which take into account inelastic effects and also make a more detailed allowance for the high-energy contributions. We are convinced that in a realistic theory that takes into account all these corrections it will not be necessary to introduce additional parameters.

7.1. Universality of the Repulsion

As we have just seen, the introduction of a short-range repulsion term into the low-energy equations for $\pi\pi$ and πN scattering is needed to obtain quantitative agreement between the solutions of the dispersion equations and the consequences of chiral symmetry in the low-energy region.

As Weinberg has shown [27], a decrease of the combination of the total cross section corresponding to $I = 1$ in the t channel for all scattering processes in which pions participate ensures that the current commutation relations of the chiral group are satisfied. In Secs. 5.2 and 6.2 we have shown that allowance for the short-range "vacuum" contributions corresponding to the nondecreasing combinations of the total cross sections ($I = 0$ in the t channel) are essential to achieve agreement with the requirement of chiral symmetry in the low-energy region. As we have noted in Sec. 5.2, a reduction of the vacuum repulsion is equivalent to an increase in the chiral symmetry breaking and the limit $v \to 0$ corresponds to maximal symmetry breaking [$N \to \infty$ in (4.21)].

The strengths and ranges of the repulsion potentials are parameters that can be used to give a phenomenological description of the influence of the high-energy contributions on the low-energy region. It follows that the existence of any simple dependences between the scattering amplitudes of the various processes in the high-energy region must lead to corresponding relationships between the parameters of the repulsion potentials.

Let us see how such relationships arise in a simple example. We shall assume that in the high-energy region a certain linear combination of the scattering amplitudes of various processes

$$T = \sum_i c_i T_i \tag{7.1}$$

decreases fairly rapidly, and that this combination satisfies an unsubtracted dispersion relation. Of course, the question immediately arises of the arguments of the various terms on the right side of (7.1), i.e., the question of the determination of an energy scale common to the various physical processes.

It is natural to associate this scale with the invariant square of the total energy and choose it in such a way that it explicitly reflects the crossing symmetry properties for forward scattering. The variable

$$x = s - m_i^2 - M_i^2 \tag{7.2}$$

satisfies these conditions; here m_i is the mass of the incident particle and M_i is the mass of the target for the i-th reaction.

We shall assume that each of the T_i is the crossing-even half-sum of the amplitudes for the scattering of a particle and antiparticle by the target [i.e., an amplitude of the type (1.21)]; it then follows from the optical theorem that the imaginary part of T_i is associated with a nondecreasing cross section. The dispersion relation for each of the T_i separately definitely requires a subtraction, and the low-energy model for the corresponding partial waves needs the introduction of a repulsion. The dispersion relation for the function (7.1) written down without a subtraction in the variable x yields

$$T(0) = \frac{2}{\pi} \int \sum_i c_i \operatorname{Im} T_i \, (x + m_i^2 + M_i^2) \frac{dx}{x}. \tag{7.3}$$

The integral on the right side of (7.3) includes pole terms and possible subthreshold unphysical regions. Suppose that for $x > x_q$ the value of Im T vanishes in a sufficiently good approximation. Then (7.3) can be rewritten in the form

$$\sum_i c_i v_i (x_q) = 0, \tag{7.4}$$

where

$$v_i (x_q) = \frac{2}{\pi} \int_0^{x_q} \operatorname{Im} T_i \frac{dx}{x} - T_i (0). \tag{7.5}$$

If we now identify x_q with the boundary of the low-energy region $2m_\pi^2\sqrt{\Lambda} = x_\Lambda$, then the quantities $v_i(x_q)$ are none other than the strengths of the short-range repulsion [cf. (1.30)]. Equation (7.4) yields linear relations between the repulsion strengths for the different processes.

Combinations of the form (7.1) that decrease fairly rapidly at infinity can be obtained on the basis of various models of high-energy scattering. At the present time the most widely known are the model of quark additivity [28] and also the model of Regge asymptotic behavior. Both of these models lead to approximately the same combinations of the form (7.1) (in which the coefficients c_i are numbers of order unity) for all the main hadron scattering processes. It follows that the parameters $v_i(x_q)$ for the various processes are also numbers of the same order. In this case the parameter x_q corresponds to the lower boundary of the asymptotic high-energy region, i.e., the region in which the corresponding Regge or quark relations for T_i are satisfied fairly accurately.

Of course, the values of x_q obtained in this manner are appreciably higher than the values x_Λ corresponding to the upper limits of the low-energy regions. However, as we see in the examples of $\pi\pi$ and πN scattering considered earlier, the simple relationships between $v_i(x_q)$ in fact remain in force for the strengths of the short-range repulsion $v_i(x_\Lambda)$. One can show that there is similar agreement for NN and KN scattering [29]. Thus, we see that the short-range repulsion is a general property of all low-energy processes. The repulsion parameters for the various processes may be related to one another. In this sense one may say that the repulsion is universal.

7.2. General Physical Picture

We shall now attempt to sketch the general picture of the various low-energy scattering processes in terms of a single energy variable which can be used to define the low- and high-energy regions simultaneously for all reactions.

The simplest candidate for this universal variable is the quantity x introduced in (7.2). As follows from Fig. 10, it is also a fairly successful candidate. In this figure we have represented schematically the main characteristics of the $\pi\pi$-, πN-, KN-, and NN-scattering processes in the region of low and intermediate energies plotted on the scale of x. We have indicated the positions of the well-defined resonances that basically have a single-channel two-particle decay, i.e., play an important role in the elastic two-particle scattering. It can be seen that the purely elastic resonances [ρ, Δ_{33}, and also Λ (1405)] are situated in the region $x < 1$ GeV^2. These are all p-wave resonances. It is therefore natural to separate the low-energy region $x < 1$ GeV^2 in which the p (and possibly the s) waves are important and in which the inelastic effects can be neglected.

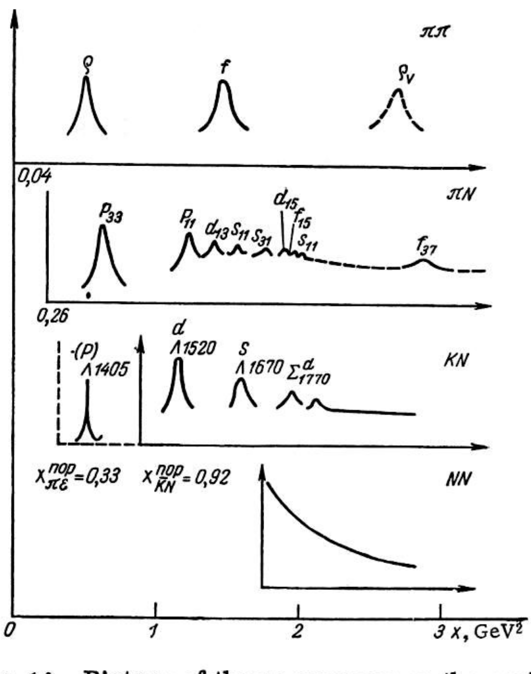

Fig. 10. Picture of the resonances on the scale $x = s - m_1^2 - M_2^2$.

In the region $1 < x < 2$ GeV^2 the d- and f-wave resonances are also manifested. Here, inelasticity becomes important. Above $x = 2$ GeV^2 the total cross sections become smooth functions and many partial waves are manifested; the scattering gradually acquires a diffraction character. It is well known that NN scattering, whose threshold corresponds to $x = 1.76$ GeV^2, does not contain resonances.

Thus, there is a well-defined "elastic low-energy region" $x < 1$ GeV^2 in which only the s and p waves are important. It is precisely in this region that the low-energy assumptions made in Sec. 1 are valid. In this region the scattering can be quantitatively described by a system of equations for the s and p waves, though, of course, the system must include the contributions to the real parts of the partial waves from the region of intermediate and high energies.

The region $1 < x < 2$ GeV^2 may be termed the "inelastic low-energy region." In this region assumption 2 of Sec. 1 is still valid ($l_{max} = 4$), but inelastic processes become important. The system of low-energy equations for this region, which describes the s, p, d, and *f* waves, must take into account not only the high-energy contributions to the real parts of the partial waves, but also the inelastic contributions to their imaginary parts.

In the region above $x = 2\ GeV^2$ the low-energy assumptions clearly cease to have even an approximate validity. It follows that $x = 2\ GeV^2$ is the upper limit for the low-energy dispersion schemes of such a kind.

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